A family of embedding spaces

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Abstract

Let $\mathrm{Emb}(S^j, S^n)$ denote the space of C^∞-smooth embeddings of the j-sphere in the n-sphere. This paper considers homotopy-theoretic properties of the family of spaces $\mathrm{Emb}(S^j, S^n)$ for $n \geq j > 0$. There is a homotopy-equivalence $\mathrm{Emb}(S^j, S^n) \simeq SO_{n+1} \times_{SO_{n-j}} \mathcal{K}_{n,j}$ where $\mathcal{K}_{n,j}$ is the space of embeddings of $\mathbb{R}^j$ in $\mathbb{R}^n$ which are standard outside of a ball. The main results of this paper are that $\mathcal{K}_{n,j}$ is $(2n-3j-4)$-connected, the computation of $\pi_{2n-3j-3}\mathcal{K}_{n,j}$ together with a geometric interpretation of the generators. A graphing construction $\Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ is shown to induce an epimorphism on homotopy groups up to dimension $2n-2j-5$. The graphing construction turns out to be a variant of Litherland's 'deform-spinning.' This gives a new proof of Haefliger's theorem that $\pi_0\mathrm{Emb}(S^j, S^n)$ is a group for $n-j > 2$. The proof given is analogous to the proof that the braid group has inverses. Relationship between the graphing construction and actions of operads of cubes on embedding spaces are developed. The paper ends with a brief survey of what is known about the spaces $\mathcal{K}_{n,j}$, focusing on issues related to iterated loop-space structures.

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1 Introduction

Haefliger proved that the isotopy classes of smooth embeddings of S^j in S^n form a group provided $n-j>2$, with the connect-sum as multiplication. This paper starts with a new proof of Haefliger's result, showing not only that $\pi_0\mathrm{Emb}(S^j,S^n)$ is a group, but the *reason* it is a group is that every element is *spun*. The inverse of a spun knot is its mirror-reflection, as in braid groups. The key strategy revolves around a pseudo-isotopy fibre-sequence $\mathcal{K}_{n+1,j+1}\to\mathcal{P}_{n,j}\to\mathcal{K}_{n,j}$. The fact that the pseudo-isotopy embedding space $\mathcal{P}_{n,j}$ is connected implies the result. In his dissertation, Tom Goodwillie [22] gave a very detailed study of (general) pseudo-isotopy embedding spaces. His results include that $\mathcal{P}_{n,j}$ is at least $(2n-2j-5)$-connected. This allows for the computation of the first non-trivial homotopy groups of $\mathcal{K}_{n,j}$ and $\mathrm{Emb}(S^j,S^n)$ provided $2n-3j-3\geq 0$. The 2-fold spinning construction $\pi_2\mathcal{K}_{4,1}\to\pi_0\mathcal{K}_{6,3}=\pi_0\mathrm{Emb}(S^3,S^6)\simeq\mathbb{Z}$ is shown to be an isomorphism, answering a question posed in [7]. This also allows for a new construction of explicit generators of $\pi_{2n-3j-3}\mathcal{K}_{n,j}$ for all n,j such that $2n-3j-3\geq 0$.

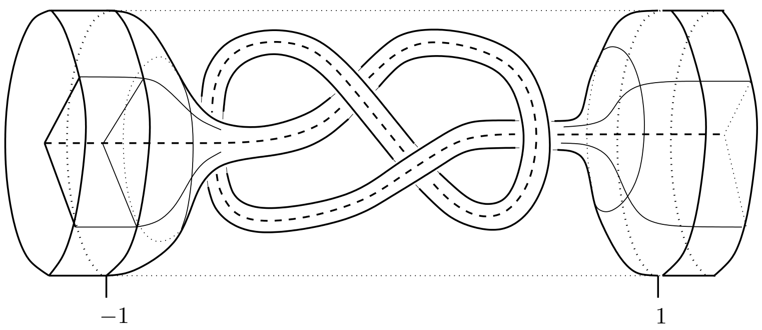
Definition 1.1

- $D^n:=\{x\in\mathbb{R}^n : |x|\leq 1\}$ is the unit n-disc, with $S^{n-1}=\partial D^n$ the $(n-1)$-sphere.
- $\mathbf{I}=[-1,1]=D^1$ is the standard interval.
- Given a topological space (resp. smooth manifold) X with base-point, denote the space of continuous (resp. smooth) functions $f:\mathbb{R}\to X$ such that $f(\mathbb{R}\setminus\mathbf{I})=*$ by ΩX.
- $\mathrm{Emb}(D^j,D^n)$ denotes the space of embeddings $f:D^j\to D^n$ which are 'neat' in the sense that $f(D^j)\cap S^{n-1}=f(S^{j-1})$ and f intersects S^{n-1} transversely.
- The space of smooth embeddings of a j-sphere in an n-sphere is denoted $\mathrm{Emb}(S^j,S^n)$.
- $\mathcal{K}_{n,j}$ denotes the space of 'long' embeddings of $\mathbb{R}^j$ in $\mathbb{R}^n$. This is the space of all smooth embeddings $f:\mathbb{R}^j\to\mathbb{R}^n$ such that $f(t_1,t_2,\cdots,t_j)=(t_1,t_2,\cdots,t_j,0,\cdots,0)$ provided $(t_1,\cdots,t_j)\notin\mathbf{I}^j$ and $f(\mathbb{R}^j)\cap\partial\mathbf{I}^n=\partial\mathbf{I}^j\times\{0\}^{n-j}$. If $f\in\mathcal{K}_{n,j}$, let $\mathcal{K}_{n,j}(f)$ denote the path-component of $\mathcal{K}_{n,j}$ containing f.
- Let $\mathcal{P}_{n,j}$ denote the space of embeddings $f:\mathbb{R}^j\to\mathbb{R}^n$ such that:
  - $f(t_1,t_2,\cdots,t_j)=(t_1,t_2,\cdots,t_j,0,\cdots,0)$ for $(t_1,\cdots,t_j)\notin[-1,\infty)\times\mathbf{I}^{j-1}$
  - there is a $g\in\mathcal{K}_{n-1,j-1}$ such that for all $(t_1,\cdots,t_j)\in[1,\infty)\times\mathbb{R}^{j-1}$, $f(t_1,t_2,\cdots,t_j)=(t_1,g(t_2,\cdots,t_j))$.
  - $f(\mathbb{R}^j)\cap\partial\mathbf{I}^n=f(\partial\mathbf{I}^j)\times\{0\}^{n-j}$.

  In the literature, $\mathcal{P}_{n,j}$ is sometimes given the notation $PE(D^{j-1},D^{n-1})$, $C(D^{j-1},D^{n-1})$ or $cemb(D^{j-1},D^{n-1})$, and is either called a pseudoisotopy embedding space, or concordance embedding space respectively. Here it will be called the pseudoisotopy embedding space.
- $\mathrm{EC}(j,M)$ is defined to be the space of embeddings $f:\mathbb{R}^j\times M\to\mathbb{R}^j\times M$ such that $supp(f)\subset\mathbf{I}^j\times M$, where, $supp(f)=\{x\in\mathbb{R}^j\times M : f(x)\neq x\}$. 'EC' stands for 'cubically-supported embeddings'. These embeddings are not required to send boundary

to boundary.

$$f \in \mathrm{EC}(1, D^2)$$



- $\mathrm{PEC}(j, M)$ is the space of embeddings $f : \mathbb{R}^j \times M \to \mathbb{R}^j \times M$ such that $supp(f) \subset [-1,\infty) \times \mathbf{I}^{j-1} \times M$ and there exists some $g \in \mathrm{EC}(j-1, M)$ such that $f(t_1, t_2, \cdots, t_j, m) = (t_1, g(t_2, \cdots, t_j, m))$ for all $(t_1, t_2, \cdots, t_j, m) \in [1, \infty) \times \mathbb{R}^{j-1} \times M$. The letters 'PEC' stand for 'cubically-supported embedding pseudo-isotopy space.'
- A diagram of two maps $A \to B \to C$ is a homotopy fibre sequence if there exists a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ F \arrow[r] & E \arrow[r] & B \end{tikzcd}$$

  such that $F \to E \to B$ is a fibration and the vertical maps are homotopy-equivalences.
- $\mathrm{Diff}(D^n)$ denotes the space of smooth diffeomorphisms of D^n which restrict to the identity on the boundary. $\mathrm{Diff}(S^n)$ is the group of diffeomorphisms of S^n.

All embedding spaces are endowed with the weak C^∞-topology [33], sometimes also called the Whitney topology. Many classical results on the homotopy properties of embedding spaces that will be repeatedly used in this paper appear in Cerf's [15] paper, such as the fibration properties of restriction maps, and the homotopy-classification of spaces of tubular neighbourhoods.

In the definition of $\mathcal{K}_{n,j}$ replacing the cubes $\mathbf{I}^n$ and $\mathbf{I}^j$ with discs D^n and D^j gives a homotopy-equivalent space. Similarly for the definition of $\mathrm{Diff}(D^n)$ and $\mathrm{EC}(j, M)$. The proof is a typical argument when one deals with these spaces, see for example [7] Corollary 6.

Section 2 briefly covers the most elementary relationships between the spaces defined above: $\mathcal{K}_{n,j}$, $\mathrm{Emb}(S^j, S^n)$, $\mathrm{Emb}(S^j, \mathbb{R}^n)$, $\mathrm{Emb}(D^j, D^n)$, $\mathcal{P}_{n,j}$, $\mathrm{EC}(j, D^{n-j})$ and $\mathrm{PEC}(j, D^{n-j})$. This section also includes a generalisation of an observation of Goodwillie and Sinha [73] concerning the Smale-Hirsch map $\mathcal{K}_{n,j} \to \Omega^j V_{n,j}$. The Goodwillie-Sinha result is that this map is null-homotopic for $j = 1$. The generalisation that appears here is that the map factors as a composite $\mathcal{K}_{n,j} \to \Omega^j V_{n-1,j-1} \to \Omega^j V_{n,j}$ where the map $\Omega^j V_{n-1,j-1} \to \Omega^j V_{n,j}$ is the j-fold loop of the fibre inclusion in the Stiefel fibration $V_{n-1,j-1} \to V_{n,j} \to S^{n-1}$.

Section 3 is the heart of the paper. A proof of Haefliger's theorem, that for $n - j > 2$ $\pi_0\mathrm{Emb}(S^j, S^n)$ is a group is given. The proof permutes some of the main concepts of Haefliger's original argument. It has two essential steps: 1) The construction of a homotopy-equivalence $\mathrm{Emb}(S^j, S^n) \simeq \mathrm{SO}_{n+1} \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n,j}$ together with fibrations $\mathcal{P}_{n,j} \to \mathrm{Emb}(D^j, D^n) \to V_{n,j}$ and

$\mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$ reduces the problem to 2) proving that $\mathrm{Emb}(D^j, D^n)$ is connected. Thus, the argument boils down to showing the monoid $\pi_0\mathcal{K}_{n,j}$ is a group because it is the image of the group $\pi_1\mathcal{K}_{n-1,j-1}$. Further, it is shown that the 'boundary map' $\mathrm{gr}_1 : \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ has a geometric interpretation as a variant of Litherland 'deform spinning.' In this case it is given by the formula

$$(\mathrm{gr}_1 f)(t_0, t_1, \cdots, t_{j-1}) = (t_0, f(t_0)(t_1, \cdots, t_{j-1})).$$

In Proposition 3.9, Goodwillie's dissertation is used to prove that $\mathrm{gr}_1 : \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ induces an epimorphism of the on homotopy groups π_i for $i \leq 2n-2j-5$. By comparing with the work of Turchin and Sinha this allows the computation of $\pi_{2n-3j-3}\mathcal{K}_{n,j}$. An enumerative-geometry argument is used to construct a cohomology class $\nu_2 \in H^{2n-6}(\mathcal{K}_{n,1};\mathbb{Z})$, which is used to find an explicit generator of $\pi_{2n-6}\mathcal{K}_{n,1} \simeq \mathbb{Z}$. The generator can be thought of as the resolutions of a long immersion of $\mathbb{R}$ in $\mathbb{R}^n$ having two regular double points, corresponding to the $\bigotimes$ chord diagram. The generators of the groups $\pi_0\mathcal{K}_{n,j}$ for $2n-3j-3=0$ are constructed as iterated graphs of the generator of $\pi_{j-1}\mathcal{K}_{n-j+1,1}$.

Section 4 investigates the extent to which the fibration $\mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$, and its framed analogue are equivariant with respect to natural actions of operads of cubes. $\mathrm{PEC}(j,M)$ is shown to have an action of the operad of j-cubes, the map $\mathrm{PEC}(j,M) \to \mathrm{EC}(j-1,M)$ is shown to be equivariant with respect to the j-cubes action defined in [7]. The graphing construction $\Omega\mathrm{EC}(j-1,M) \to \mathrm{EC}(j,M)$ is shown to be equivariant with respect to the $(j+1)$-cubes action.

Section 5 covers, in a rather terse survey manner, many of the basic properties the spaces $\mathcal{K}_{n,j}$ which have not already been mentioned. A curiosity is put forward: two seemingly distinct null homotopies of the inclusion $\mathcal{K}_{n,1} \to \mathcal{K}_{n+1,1}$ are described, giving a mysterious map $\Sigma\mathcal{K}_{n,1} \to \mathcal{K}_{n+1,1}$. This leads to a question about the existence of a 'Freudenthal suspension' $\Sigma^2\mathcal{K}_{n,1} \to \mathcal{K}_{n+1,1}$. Basic properties of other natural maps such as $\mathcal{K}_{n,j} \to \Omega\mathcal{K}_{n,j-1}$ and the Smale-Hirsch map $SH : \mathcal{K}_{n,j} \to \Omega^j V_{n,j}$ are described.

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2 Basic relations between embedding spaces

This section describes some basic relationships between the spaces: $\mathcal{K}_{n,j}$, $\mathrm{EC}(j,M)$, $\mathrm{Emb}(S^j,S^n)$, $\mathrm{Emb}(S^j,\mathbb{R}^n)$, $\mathrm{Emb}(D^j,D^n)$, $\mathcal{P}_{n,j}$ and $\mathrm{PEC}(j,M)$. The essential spirit of the results is that most homotopy questions about these spaces reduce to studying $\mathcal{K}_{n,j}$ and $\mathcal{P}_{n,j}$.

Given a neat embedding $f : D^j \to D^n$, the restriction to the boundary is an embedding $f_{|\partial D^j} : S^{j-1} \to S^{n-1}$. On a global level, restriction defines a function

$$\mathrm{Emb}(D^j, D^n) \to \mathrm{Emb}(S^{j-1}, S^{n-1})$$

which is a fibration [15, 59]. In this paper 'fibration' means Serre fibration. The above map is known to be more than a fibration, it is a locally trivial fibre-bundle [59]. Fibrations need not be onto. In this example, the fibration is onto the isotopy classes of 'slice' knots (and not all knots are slice, see [37] for examples). Thus, the homotopy-type of the fibre can change as one changes base-space components, and fibres are allowed to be empty.

Consider $\mathrm{Emb}(S^{j-1}, S^{n-1})$ to be a based space, with base-point the standard inclusion $S^{j-1} \subset S^{n-1}$. The fibre of $\mathrm{Emb}(D^j, D^n) \to \mathrm{Emb}(S^{j-1}, S^{n-1})$ over the base-point has the homotopy-type of $\mathcal{K}_{n,j}$. There is a similar fibration $\mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$ defined by restriction to the 'free face.' The next theorem shows that this fibration induces the fibration $\mathrm{Emb}(D^j, D^n) \to \mathrm{Emb}(S^{j-1}, S^{n-1})$.

Theorem 2.1 *For $n - j > 0$ there are homotopy-equivalences:*

$$\mathrm{Emb}(D^j, D^n) \simeq \mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{P}_{n,j}$$

$$\mathrm{Emb}(S^{j-1}, S^{n-1}) \simeq \mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n-1,j-1}$$

Moreover, the homotopy fibre sequence $\mathcal{K}_{n,j} \to \mathrm{Emb}(D^j, D^n) \to \mathrm{Emb}(S^{j-1}, S^{n-1})$ fits into a commutative diagram of 6 homotopy fibre sequences:

$$\begin{tikzcd}
\mathcal{K}_{n,j} \arrow[r] \arrow[d] & \mathcal{P}_{n,j} \arrow[r] \arrow[d] & \mathcal{K}_{n-1,j-1} \arrow[d] \\
\mathcal{K}_{n,j} \arrow[r] \arrow[d] & \mathrm{Emb}(D^j, D^n) \arrow[r] \arrow[d] & \mathrm{Emb}(S^{j-1}, S^{n-1}) \arrow[d] \\
* \arrow[r] & V_{n,j} \arrow[r] & V_{n,j}
\end{tikzcd}$$

Proof In [10] a homotopy-equivalence $\mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n-1,j-1} \to \mathrm{Emb}(S^{j-1}, S^{n-1})$ was constructed. The basic idea is to consider S^{n-1} to be the one-point compactification of $\mathbb{R}^{n-1}$, this gives an inclusion $\mathcal{K}_{n-1,j-1} \to \mathrm{Emb}(S^{j-1}, S^{n-1})$. The action of SO_n on S^{n-1} gives an extension $\mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n-1,j-1} \to \mathrm{Emb}(S^{j-1}, S^{n-1})$. $\mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n-1,j-1}$ fibres over $V_{n,j} = \mathrm{SO}_n/\mathrm{SO}_{n-j}$ by projection onto the first coordinate. $\mathrm{Emb}(S^{j-1}, S^{n-1})$ fibres over a space homotopy-equivalent to $V_{n,j}$ by restriction to a fixed hemi-sphere $B \subset S^{j-1}$, $\mathrm{Emb}(S^{j-1}, S^{n-1}) \to \mathrm{Emb}(B, S^{n-1}) \simeq V_{n,j}$ [15]. This makes $\mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n-1,j-1} \to \mathrm{Emb}(S^{j-1}, S^{n-1})$ a map of fibrations.

The same idea can be applied to $\mathrm{Emb}(D^j, D^n)$. Let $B \subset \partial D^j = S^{j-1}$ be as above. Let $\mathrm{Emb}(D^j \text{ rel } B, D^n)$ denote the subspace of $\mathrm{Emb}(D^j, D^n)$ which is fixed point-wise on B. There is a fibre bundle $\mathrm{Emb}(D^j \text{ rel } B, D^n) \to \mathrm{Emb}(D^j, D^n) \to \mathrm{Emb}(B, S^{n-1})$ given by restriction to B. The base-space has the homotopy-type of $V_{n,j} \simeq \mathrm{SO}_n/\mathrm{SO}_{n-j}$ and as in the previous paragraph, there is a map of fibrations

$$\mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} \mathrm{Emb}(D^j \text{ rel } B, D^n) \to \mathrm{Emb}(D^j, D^n).$$

That $\mathrm{Emb}(D^j \text{ rel } B, D^n)$ has the same homotopy-type as $\mathcal{P}_{n,j}$ is a fairly standard argument, see for example the 2nd half of Corollary 6 of [7]. $\square$

When $n=j$, the above argument proves that $\mathrm{Emb}(D^n,D^n)$ has the homotopy-type of $O_n\times\mathcal{P}_{n,j}$. Similarly, $\mathrm{Emb}(S^{n-1},S^{n-1})=\mathrm{Diff}(S^{n-1})$ has the homotopy-type of $O_n\times\mathcal{K}_{n-1,n-1}$. This case appears in [29].

There is a similar relationship between $\mathrm{Emb}(S^j,\mathbb{R}^n)$ and $\mathcal{K}_{n,j}$. For this proposition, identify $\overline{\mathbb{R}^n}$ (the one-point compactification of $\mathbb{R}^n$) with S^n via stereographic projection. This makes SO_n the stabiliser of ∞ under the SO_{n+1} action on S^n. Denote the projection map $\mathrm{SO}_{n+1}\to S^n$ by π. Given $f\in\mathcal{K}_{n,j}$ let $\overline{f}\in\mathrm{Emb}(S^j,S^n)$ be the one-point compactification of f. Notice that the space

$$\{(A,f): A\in\mathrm{SO}_{n+1}, \pi(a)\in S^n\setminus img(\overline{f}), f\in\mathcal{K}_{n,j}\}$$

fibres over $C\rtimes\mathcal{K}_{n,j}$ with fibre SO_n, for

$$C\rtimes\mathcal{K}_{n,j}=\{(p,f): p\in S^n\setminus img(\overline{f}), f\in\mathcal{K}_{n,j}\}.$$

Denote $\{(A,f): A\in\mathrm{SO}_{n+1}, \pi(a)\in S^n\setminus img(\overline{f}), f\in\mathcal{K}_{n,j}\}$ by $(C\rtimes\mathcal{K}_{n,j})^*(\pi)$. Consider $(C\rtimes\mathcal{K}_{n,j})^*(\pi)$ to be the pull-back of π over $\mathbb{R}^n$. Since π is trivial over $\mathbb{R}^n$, the pull-back must be as well.

$$\mathrm{SO}_n\times(C\rtimes\mathcal{K}_{n,j})\simeq(C\rtimes\mathcal{K}_{n,j})^*(\pi).$$

Notice that SO_{n-j} acts on $(C\rtimes\mathcal{K}_{n,j})^*(\pi)$ from the left, by considering $\mathrm{SO}_{n-j}\subset\mathrm{SO}_{n+1}$ to be the group that leaves $S^j=\overline{\mathbb{R}^j}$ in S^n fixed point-wise.

Proposition 2.2 *Provided $n-j>0$ there is a homotopy-equivalence*

$$\mathrm{SO}_{n-j}\backslash(C\rtimes\mathcal{K}_{n,j})^*(\pi)\to\mathrm{Emb}(S^j,\mathbb{R}^n).$$

Induced by the map $(A,f)\longmapsto A^{-1}\circ\overline{f}$. Moreover, there is a homotopy-equivalence

$$\mathrm{SO}_{n-j}\backslash(C\rtimes\mathcal{K}_{n,j})^*(\pi)\to\mathrm{SO}_n\times_{\mathrm{SO}_{n-j}}(C\rtimes\mathcal{K}_{n,j})$$

where the action of SO_{n-j} on SO_n is by left multiplication.

Proof Observe that $\mathrm{Emb}(S^j,\mathbb{R}^n)$ fibres over $V_{n,j}$. The fibre can be identified with $\{f\in\mathcal{K}_{n,j}: 0\notin f(\mathbb{R}^j)\}$. $C\rtimes\mathcal{K}_{n,j}$ fibres over a ball with fibre $\{f\in\mathcal{K}_{n,j}: 0\notin f(\mathbb{R}^j)\}$, thus there is a homotopy-fibre sequence

$$C\rtimes\mathcal{K}_{n,j}\to\mathrm{Emb}(S^j,\mathbb{R}^n)\to V_{n,j}$$

$(C\rtimes\mathcal{K}_{n,j})^*(\pi)$ similarly fibres over $V_{n,j}$ giving a commutative ladder of homotopy fibre sequences

$$\begin{tikzcd}
C\rtimes\mathcal{K}_{n,j} \arrow[r] & \mathrm{Emb}(S^j,\mathbb{R}^n) \arrow[r] & V_{n,j} \\
C\rtimes\mathcal{K}_{n,j} \arrow[r] \arrow[u] & (C\rtimes\mathcal{K}_{n,j})^*(\pi) \arrow[r] \arrow[u] & V_{n,j} \arrow[u]
\end{tikzcd}$$

Let $(A,f)\in(C\rtimes\mathcal{K}_{n,j})^*(\pi)$, then A is a matrix whose first column vector is $\pi(A)$, the remaining vectors are in the tangent space to $\mathbb{R}^n$ at $\pi(A)$. Let $[A]_{\pi(A)}$ denote the representation of A with respect to the standard framing of $\mathbb{R}^n$ at $\pi(A)$. Consider the map $(C\rtimes\mathcal{K}_{n,j})^*(\pi)\to SO_n\times(C\rtimes\mathcal{K}_{n,j})$ given by sending the pair (A,f) to $\big([A]_{\pi(A)},(\pi(A),f)\big)$. This map is equivariant with respect to the action of SO_{n-j} since if $B\in\mathrm{SO}_{n-j}$ then $B.(A,f)=(BA,Bf)$, which is sent to $\big([BA]_{\pi(BA)},(\pi(BA),Bf)\big)=\big([BA]_{B\pi(A)},B.(A,f)\big)$, but $[BA]_{B\pi(A)}=B[A]_{\pi(A)}$ by a change of variables argument, giving the result. □

A basic fact and conventions about homotopy-fibres is given for future reference.

Lemma 2.3 *Let $p : E \to B$ be a fibration. Let $e \in E$ and $b \in B$ be the base-points of E and B respectively, with $p(e) = b$. Let $i : F \to E$ be the fibre inclusion. Let $R(F) = \{(a, h) : a \in F, h : [0,1] \to E, h(0) = p(a)\}$ then the map $R(i) : R(F) \to E$ given by evaluation $h(1)$ is a fibration, and $\pi_F : R(F) \to F$ given by projection onto F is a homotopy-equivalence. The fibre of the map $R(i) : R(F) \to E$ is the space $HF(i) = \{h : [0,1] \to E, h(0) \in F, h(1) = e\}$, and the map $p_* : HF(i) \to \Omega B$ given by post-composition with p is a weak homotopy-equivalence, giving a fibration:*

$$\Omega E \to HF(i) \to F$$

and a homotopy-commutative diagram

$$\begin{tikzcd}
 & \Omega B & F \arrow[r, "i"] & E \arrow[r, "p"] & B \\
\Omega E \arrow[r] & HF(i) \arrow[u, "p_*"] \arrow[r] \arrow[ur] & R(F) \arrow[u, "\simeq"', "\pi_F"] \arrow[ur, "R(i)"'] & &
\end{tikzcd}$$

The map $HF(i) \to F$ is sometimes called the 'connecting map' or the 'boundary map' as it induces the same map as the connecting map in the homotopy long exact sequence of the fibration p.

The next two results are a modest generalisation of observations due to Goodwillie (unpublished), Sinha [73], Turchin [79] and Salvatore [67], concerning the monodromy of the fibration $\mathrm{EC}(j, D^{n-j}) \to \mathcal{K}_{n,j}$ and the Smale-Hirsch map $\mathcal{K}_{n,j} \to \Omega^j V_{n,j}$.

Theorem 2.4 *The homotopy fibre sequence*

$$\Omega^j \mathrm{SO}_{n-j} \to \mathrm{EC}(j, D^{n-j}) \to \mathcal{K}_{n,j}$$

is trivial for $j = 1$, and also for $n - j \leq 2$. There is a pull-back diagram of homotopy fibre sequences:

$$\begin{tikzcd}
\Omega^j \mathrm{SO}_{n-j} \arrow[r] \arrow[d] & \Omega^j \mathrm{SO}_{n-j} \arrow[d] \\
\mathrm{EC}(j, D^{n-j}) \arrow[r] \arrow[d] & P\Omega^{j-1}\mathrm{SO}_{n-j} \arrow[d] \\
\mathcal{K}_{n,j} \arrow[r, "cl"] & \Omega^{j-1}\mathrm{SO}_{n-j}
\end{tikzcd}$$

Where $\Omega^j \mathrm{SO}_{n-j} \to P\Omega^{j-1}\mathrm{SO}_{n-j} \to \Omega^{j-1}\mathrm{SO}_{n-j}$ is the path-loop fibration of the space $\Omega^{j-1}\mathrm{SO}_{n-j}$. The classifying map $cl : \mathcal{K}_{n,j} \to \Omega^{j-1}\mathrm{SO}_{n-j}$ fits into a commutative diagram

$$\begin{tikzcd}
 & & \Omega^j V_{n,n-j} \arrow[d] \\
\Omega^j \mathrm{SO}_j \arrow[r] & \Omega^j V_{n,j} \arrow[r] & \Omega^j G_{n,j} \equiv \Omega^j G_{n,n-j} \arrow[d, "mono"] \\
 & \mathcal{K}_{n,j} \arrow[u, "SH"] \arrow[r, "cl"] & \Omega^{j-1}\mathrm{SO}_{n-j}
\end{tikzcd}$$

where 'SH' is the Smale-Hirsch map, $V_{n,j}$ is the Stiefel manifold of j linearly independent vectors in $\mathbb{R}^n$, $\mathrm{SO}_j \to V_{n,j} \to G_{n,j}$ is the canonical fibration for the Grassmanian of oriented j-dimensional subspaces of $\mathbb{R}^n$. 'mono' is the j-fold looping of the classifying map $G_{n,n-j} \to B\mathrm{SO}_{n-j}$ for the bundle $\mathrm{SO}_{n-j} \to V_{n,n-j} \to G_{n,n-j}$. Identify $G_{n,j}$ with $G_{n,n-j}$ via the oriented orthogonal complement.

Framed and unframed pseudoisotopy embedding spaces are more directly related, as the forgetful map $\mathrm{PEC}(j, D^{n-j}) \to \mathcal{P}_{n,j}$ is a homotopy-equivalence.

Proof The observation of the existence of the above pull-back diagram first appears in Turchin's work [79] for $j = 1$. The idea is to divide $\mathbf{I}^j$ into $\mathbf{I} \times \mathbf{I}^{j-1}$. Given a knot $f \in \mathcal{K}_{n,j}$, let νf be its normal bundle, and consider parallel transport (using the connection inherited as a submanifold of Euclidean space $\mathbb{R}^n$) from $\nu f_{|\{-1\}\times \mathbf{I}^{j-1}}$ to $\nu f_{\{1\}\times \mathbf{I}^{j-1}}$, this is an element of $\Omega^{j-1}\mathrm{SO}_{n-j}$. The map $\mathrm{EC}(j, D^{n-j}) \to P\Omega^{j-1}\mathrm{SO}_{n-j}$ is defined similarly, only along the paths $\mathbf{I} \times \{x\} \subset \mathbf{I} \times \mathbf{I}^{j-1}$ $f \in \mathrm{EC}(j, D^{n-j})$ one has a pre-defined framing of $\nu f_{|\mathbb{R}^j \times \{0\}^{n-j}}$ which can be compared to the parallel transport framing, giving the bundle map.

Observe that the way $\mathcal{K}_{n,j} \to \Omega^{j-1}\mathrm{SO}_{n-j}$ is defined, it factors as a composite $\mathcal{K}_{n,j} \to \Omega^j G_{n,j} \equiv \Omega^j G_{n,n-j} \to \Omega^{j-1}\mathrm{SO}_{n-j}$. $\mathcal{K}_{n,j} \to \Omega^j G_{n,j}$ is the 'tangent space map.' $G_{n,j}$ is the Grassmanian of j-dimensional subspaces of $\mathbb{R}^n$. $\mathrm{mono} : \Omega^j G_{n,n-j} \to \Omega^{j-1}\mathrm{SO}_{n-j}$ is the j-fold looping of the classifying map of the bundle $\mathrm{SO}_{n-j} \to V_{n,n-j} \to G_{n,n-j}$.

For the fibration $\mathrm{PEC}(j, D^{n-j}) \to \mathcal{P}_{n,j}$ observe the fibre has the homotopy-type of the path-space $P\Omega^{j-1}\mathrm{SO}_{n-j}$. □

The homotopy-class of the Smale-Hirsch map $SH : \mathcal{K}_{n,j} \to \Omega^j V_{n,j}$ is not so well understood. There are results concerning the induced map $SH : \pi_0\mathcal{K}_{n,j} \to \pi_j V_{n,j}$ in two cases: Kervaire proved it to be trivial provided $2n - 3j \geq 2$ [40]. In the co-dimension 2 case $n - j = 2$, Hughes and Melvin showed that $SH : \pi_0\mathcal{K}_{n,j} \to \pi_j V_{n,j}$ has non-trivial image if and only if $j \equiv 3 \bmod 4$ [35], moreover they gave a rather appealing description of the immersions that can be realised as embeddings. Eckholm and Szücs [18, 19] have recently given more geometric interpretations of the obstruction to an immersion having a representative that is an embedding.

Theorem 2.5 *The Smale-Hirsch map $SH : \mathcal{K}_{n,j} \to \Omega^j V_{n,j}$ fits into a homotopy-commutative diagram*

$$\begin{tikzcd} \mathcal{K}_{n,j} \arrow[rr, "SH"] \arrow[dr] & & \Omega^j V_{n,j} \\ & \Omega^j V_{n-1,j-1} \arrow[ur, "\Omega^j(i)"'] & \end{tikzcd}$$

where $i : V_{n-1,j-1} \to V_{n,j}$ is the fibre-inclusion of the fibration $V_{n-1,j-1} \to V_{n,j} \to S^{n-1}$.

Proof Consider the commutative diagram of spaces and maps:

$$\begin{tikzcd} \mathcal{K}_{n,j} \arrow[r] \arrow[d, "SH"] & \mathcal{P}_{n,j} \arrow[r] \arrow[d, "SH"] & \mathcal{K}_{n-1,j-1} \arrow[d, "SH"] \\ \Omega^j V_{n,j} \arrow[r] & \Omega^{j-1} HF(i) \arrow[r] & \Omega^{j-1} V_{n-1,j-1} \end{tikzcd}$$

$HF(i)$ is the homotopy-fibre of i. By Proposition 2.3, there is a homotopy-equivalence $HF(i) \simeq \Omega S^{n-1}$.

The Smale-Hirsch map $SH : \mathcal{P}_{n,j} \to \Omega^j S^{n-1}$ is given by differentiation in the vertical 'pseudo-isotopy' direction. The map $h : [0,3] \times \mathbb{R}^j \times \mathcal{P}_{n,j} \to S^{n-1}$ given by:

$$h(t, x_1, \cdots, x_j, f) = \begin{cases} n(\frac{\partial f}{\partial x_1}(x_1, \cdots, x_j)) & t = 0 \\ n(f(x_1 + t, x_2, \cdots, x_j) - f(x_1, \cdots, x_j)) & 0 < t \leq 2 \\ p_{t-2}(n(f(x_1 + 2, x_2, \cdots, x_j) - f(x_1, \cdots, x_j))) & 2 \leq t \leq 3 \end{cases}$$

is a null-homotopy of the Smale-Hirsch map, provided $p : [0,1] \times S^{n-1} \setminus \{-1\} \to S^{n-1} \setminus \{-1\}$ is a deformation-retraction of $S^{n-1} \setminus \{-1\}$ to $\{1\} \subset S^{n-1}$, and $n : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is the function $n(v) = \frac{v}{|v|}$. $\square$

Theorems 2.4 and 2.5 combine to give a commutative diagram involving the maps $cl : \mathcal{K}_{n,j} \to \Omega^{j-1}\mathrm{SO}_{n-j}$ and $SH : \mathcal{K}_{n,j} \to \Omega^j V_{n,j}$.

$$\begin{tikzcd}
 & & \Omega^j V_{n-1,n-j} \arrow[d] \\
\Omega^j V_{n,j} & \Omega^j V_{n-1,j-1} \arrow[l] \arrow[r, "\Omega^j \perp"] & \Omega^j G_{n-1,j-1} \equiv \Omega^j G_{n-1,n-j} \arrow[d] \\
 & \mathcal{K}_{n,j} \arrow[ul, "SH"] \arrow[u] \arrow[r, "cl"] & \Omega^{j-1}\mathrm{SO}_{n-j}
\end{tikzcd}$$

3 Spinning and graphing in high co-dimensions

This section is devoted to the concepts surrounding a new proof that $\pi_0\mathcal{K}_{n,j}$ is a group, provided $n - j > 2$. The proof is quite simple: show that the total-space of the fibration $\mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$ is connected. This forces the boundary map $\pi_1\mathcal{K}_{n-1,j-1} \to \pi_0\mathcal{K}_{n,j}$ from the homotopy long exact sequence to be an epi-morphism. Showing that $\mathcal{P}_{n,j}$ is connected reduces to showing that every neat embedding of D^j in D^n is isotopic (through neat embeddings) to a linear inclusion. The remainder of the section elaborates on ingredients used in the proof and its consequences. The boundary map $\Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ is shown to be homotopic an explicitly-defined graphing map $\mathrm{gr}_1 : \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ in Proposition 3.2. Propositions 3.4 and 3.6 demonstrate that gr_1 is a variant of Litherland's deform-spinning construction [47]. Goodwillie's dissertation is invoked, showing that gr_1 is a surprisingly highly-connected map. This allows the computation of the first non-trivial homotopy groups of $\mathcal{K}_{n,j}$ provided $2n - 3j - 3 \geq 0$. Using some computations of Victor Turchin and a quadrisecants argument, an explicit generator is constructed for $\pi_{2n-6}\mathcal{K}_{n,1}$. Via spinning, this gives new explicit constructions of Haefliger's spheres $\pi_0\mathcal{K}_{n,j}$ for $2n - 3j - 3 = 0$.

The next proposition is an old result which is known to hold in far greater generality [34, 22]. Goodwillie's generalisation will later be used in this paper. So strictly speaking, this proposition is redundant. The proof is included as several later developments in this section build on it, making it the natural starting point.

Proposition 3.1 *Provided $n-j>2$, the map $\pi_1\mathcal{K}_{n-1,j-1} \to \pi_0\mathcal{K}_{n,j}$ is an epi-morphism. The spaces $\mathrm{Emb}(D^j,D^n)$ and $\mathcal{P}_{n,j}$ are connected.*

Proof Once $\mathrm{Emb}(D^j,D^n)$ is shown to be connected, the remaining results follow from the homotopy long exact sequences of the fibrations $\mathcal{K}_{n,j} \to \mathcal{P}_{n,j} \to \mathcal{K}_{n-1,j-1}$ and $\mathcal{P}_{n,j} \to \mathrm{Emb}(D^j,D^n) \to V_{n,j}$ from Theorem 2.1.

- Consider $n=4$. The path-connectivity of $\mathrm{Emb}(D^1,D^4)$ is well-known and appears in many places. Let $f \in \mathrm{Emb}(D^1,D^4)$, and isotope it to be standard on the boundary: $f(-1)=(-1,0,0,0)$ and $f(1)=(1,0,0,0)$. Let $v \in S^3$. By Sard's theorem, the projection of f into the orthogonal complement of v is generically an embedding. Choose one such value for v such that $c=\langle v,(1,0,0,0)\rangle > 0$. Then the formula $f(t)-a\langle f(t),v\rangle v + act\cdot v$ describes a path (parametrised by $a\in[0,1]$) in $\mathrm{Emb}(D^1,D^4)$, starting at f and ending at a function which is monotone increasing in the direction of v, thus isotopic to $t \longmapsto (t,0,0,0)$ by the straight-line homotopy.
- Consider $n=5$. As in the previous case, isotope $f\in \mathrm{Emb}(D^2,D^5)$ to be standard on the boundary, and let $f_a : D^2 \to D^5$ for $a\in[0,1]$ be the straight-line homotopy from f to the standard inclusion. By the weak Whitney immersion theorem, one can assume f_a is generically an embedding, with only finitely many times a for which it has an isolated, regular double point. Wu [85] developed a 1-parameter 'Whitney trick' for this situation, to remove the double points from the family.
- Consider the case $n\geq 6$ and let $e: D^j \to D^n$ be a proper embedding. Let $B\subset D^j$ be the open ball of radius $\frac{1}{2}$, centred about the origin. Consider $D^j = D^j\times\{0\}^{n-j}\subset D^n$. By a local linearisation, isotope e so that it agrees with inclusion on B, $e(x)=x$ for all $x\in B$. Let U be the open ball of radius $\frac{1}{2}$ centred about 0 in D^n, and isotope e so that $e(D^j)\cap U = e(B)$. Let $W = D^n\setminus U$, $M_1=\partial U$ and $M_2=\partial D^n$. $\partial W = M_1 \sqcup M_2$. $M_i \to W$ is a homotopy-equivalence for $i\in\{1,2\}$, since W is a product. Let $V = e(D^j\setminus B)$ with $V_1 = W_1\cap V$, $V_2 = W_2\cap V$, and let $f: V_1\times[\frac{1}{2},1]\to W$ be the map defined by $f(v,t)=e(2tv)$. f maps $V_1\times[\frac{1}{2},1]$ diffeomorphically to V. Corollary 3.2 of [76] states that f extends to a diffeomorphism of pairs $f:(W_1,V_1)\times[\frac{1}{2},1]\to(W,V)$. Therefore it further extends to a diffeomorphism of pairs $f:(D^n,D^j)\to(D^n,img(e))$. So $e=f\circ h$ where h is the standard inclusion $h: D^j\to D^n$. Given an orientation-preserving diffeomorphism f of D^n it acts on $\mathrm{Emb}(D^j,D^n)$, but the action is trivial on $\pi_0\mathrm{Emb}(D^j,D^n)$ – the idea is that one can linearise f on the complement of a neighbourhood of a point in the boundary of D^n.

□

The earliest claim in the literature that $\mathrm{Emb}(D^j,D^n)$ is connected for $n-j>2$ seems to be made by Haefliger. It appears in his AMS math review [28] of Zeeman's paper [87]. Perhaps the above proof is similar to what Haefliger had in mind, as he states the result follows from Smale's paper [76]. It would be interesting to know if there are any more elementary proofs.

The fibre-sequence $\mathcal{K}_{n,j}\to\mathcal{P}_{n,j}\to\mathcal{K}_{n-1,j-1}$ 'backs-up' to a fibre-sequence $\Omega\mathcal{K}_{n-1,j-1}\to\mathcal{K}_{n,j}\to\mathcal{P}_{n,j}$ by Lemma 2.3. The remainder of this section is devoted to the properties of the 'connecting map' $\Omega\mathcal{K}_{n-1,j-1}\to\mathcal{K}_{n,j}$ and its relatives.

Proposition 3.2 *The connecting-map $\Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ is homotopic to*

$$\begin{array}{ccc} \Omega\mathcal{K}_{n-1,j-1} & \xrightarrow{\quad \mathrm{gr}_1 \quad} & \mathcal{K}_{n,j} \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ f & \longmapsto & [(t_0,t_1,\cdots,t_{j-1}) \longmapsto (t_0, f(t_0)(t_1,\cdots,t_{j-1}))] \end{array}$$

and the connecting map $\Omega\mathrm{EC}(j-1,M) \to \mathrm{EC}(j,M)$ is homotopic to

$$\begin{array}{ccc} \Omega\mathrm{EC}(j-1,M) & \xrightarrow{\quad \mathrm{gr}_1 \quad} & \mathrm{EC}(j,M) \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ f & \longmapsto & [(t_0,t_1,\cdots,t_{j-1},m) \longmapsto (t_0, f(t_0)(t_1,\cdots,t_{j-1},m))] \end{array} .$$

Proof The two cases are essentially the same, so restrict attention to the fibration

$$\mathrm{EC}(j,M) \xrightarrow{i} \mathrm{PEC}(j,M) \xrightarrow{p} \mathrm{EC}(j-1,M) .$$

By Lemma 2.3

$$HF(i) = \{f : [0,1] \to \mathrm{PEC}(j,M), f(0) = Id_{\mathbb{R}^j \times M}, f(1) \in \mathrm{EC}(j,M)\}.$$

The map $HF(i) \to \Omega\mathrm{EC}(j-1,M)$ defined in Lemma 2.3 is a weak homotopy equivalence. Palais has proved that every embedding space has the homotopy-type of a CW-complex [60]. Strictly speaking, he proves embedding spaces are dominated by CW-complexes, but at that time it was a well-known theorem of Whitehead's that a space dominated by a CW-complex has the homotopy-type of a (perhaps different) CW-complex [84]. The further fact that the various loop space and homotopy-fibre constructions send spaces with the homotopy-type of CW-complexes to spaces having the homotopy-type of CW-complexes is due to Milnor [54]. Thus, $HF(i) \to \Omega\mathrm{EC}(j-1,M)$ is a homotopy-equivalence.

An explicit homotopy-inverse of $\Omega\mathrm{EC}(j-1,M) \to HF(i)$ is exhibited. Given $f \in \Omega\mathrm{EC}(j-1,M)$, consider the object

$$(t,t_1,\cdots,t_j,m) \longmapsto \begin{cases} (t_1, f(t_1)(t_2,\cdots,t_j,m)) & \text{for } 2t-1 \leq t_1 \\ (t_1, f(2t-1)(t_2,\cdots,t_j,m)) & \text{for } t_1 \leq 2t-1 \end{cases}$$

This would be the 'right' map $\Omega\mathrm{EC}(j-1,M) \to HF(i)$ (with loop-space parameter t) if it was a smooth function in the variable t_1. Consider a smooth 'wet blanket' function $b : \mathbb{R} \to \mathbb{R}$ with the properties:

- $b(x) = x$ for all $x \leq 0$
- $b(x) = 1/2$ for all $x \geq 1$
- $b'(x) \geq 0$ for all $x \in \mathbb{R}$.

Such a function can be obtained in closed-form as

$$b(x) = \int_0^x \left(1 - \int_0^x B(x)dx\right) dx$$

where $B : \mathbb{R} \to \mathbb{R}$ is any smooth function such that $B(\frac{1}{2}+x) = B(\frac{1}{2}-x)$ and $B(x) \geq 0$ for all $x \in \mathbb{R}$, with $B(x) = 0$ for all $|x - \frac{1}{2}| \geq \frac{1}{2}$ and $\int_{-\infty}^{\infty} B(x)dx = 1$.

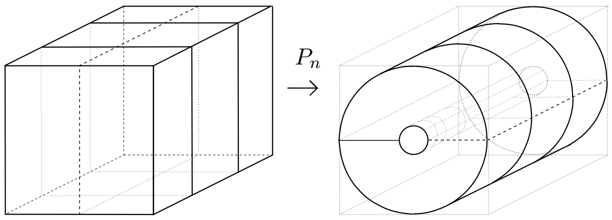
For $t \in \mathbb{R}$ define $b_t : \mathbb{R} \to \mathbb{R}$ as $b_t(x) = b(x-t)+t$. Consider the function $\Omega\mathrm{EC}(j-1,M) \to HF(i)$ defined by sending $f \in \Omega\mathrm{EC}(j-1,M)$ to $\tilde{f} \in HF(i)$ by the formula

$$\tilde{f}(t)(t_1,\cdots,t_j,m) = \left(t_1, f(b_{\frac{-3+5t}{2}}(t_1))(t_2,t_3,\cdots,t_j,m)\right) \qquad (*)$$

The composite $\Omega\mathrm{EC}(j-1,M) \to HF(i) \to \Omega\mathrm{EC}(j-1,M)$ is obtained by setting $t_1 = 1$ in $(*)$, thus f is sent to the map $\Big[(t,t_2,\cdots,t_j,m) \longmapsto f(b_{\frac{-3+5t}{2}}(1))(t_2,\cdots,t_j,m)\Big] \in \Omega\mathrm{EC}(j-1,M)$ which is just a reparametrisation of f by $b_{\frac{-3+5t}{2}}(1)$ (thought of as a function of t). Since $b_{\frac{-3+5t}{2}}(1)$ is an increasing function of t it is homotopic to the identity. □

Zeeman proved that the complements of certain co-dimension two 'twist-spun' knots fibre over S^1 [88]. Litherland later went on to formulate a more general notion of spinning, at the time called 'deform-spinning,' further generalising Zeeman's theorem to this context [47]. The Zeeman-Litherland results are important for a number of reasons – one being that they are an excellent source of embeddings of 3-manifolds in S^4, as the Seifert-surfaces of embeddings of S^2 in S^4. The next proposition points out that the connecting map $\mathrm{gr}_1 : \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ is a mild variation of Litherland's spinning construction.

Given a topological space X, denote the space of continuous functions $f : S^1 \equiv \mathbb{R}/2\mathbb{Z} \to X$ by LX called the 'free loop space of X.' Define $P_2 : \mathbf{I}^2 \to \mathbf{I}^2$ by $P_2(t_1,t_2) = (\frac{t_2+2}{3}\cos(\pi t_1), \frac{t_2+2}{3}\sin(\pi t_1))$ and $P_n : \mathbf{I}^n \to \mathbf{I}^n$ as $P_n = P_2 \times Id_{\mathbf{I}^{n-2}}$. Notice P_n is an embedding on the interior of $\mathbf{I}^n$, and is globally one-to-one except for the equality $P_n(-1,t_2,t_3,\cdots,t_n) = P_n(1,t_2,\cdots,t_n)$.



Definition 3.3 Given $f \in L\mathcal{K}_{n-1,j-1}$, let $h : \mathbb{R}^j \to \mathbb{R}^n$ be the function $h(t_0,t_1,\cdots,t_{j-1}) = (t_0, f(t_0)(t_1,\cdots,t_{j-1}))$, and consider the composite $P_n \circ h \circ P_j^{-1}$. It is well-defined on the image of P_j. On $\partial P_j(\mathbf{I}^j)$ it agrees with the standard inclusion $\mathbb{R}^j \to \mathbb{R}^n$. Define $\mathrm{gr}_1(f) \in \mathcal{K}_{n,j}$ to be the unique extension of $P_n \circ h \circ P_j^{-1}$ such that $\mathrm{gr}_1(f)_{|\mathbb{R}^j \setminus P_j(\mathbf{I}^j)}$ agrees with the standard inclusion.

Proposition 3.4 *The diagram*

$$\begin{tikzcd}
L\mathcal{K}_{n-1,j-1} \arrow[r, "\mathrm{gr}_1"] & \mathcal{K}_{n,j} \\
\Omega\mathcal{K}_{n-1,j-1} \arrow[u] \arrow[ur, "\mathrm{gr}_1"'] &
\end{tikzcd}$$

is homotopy-commutative.

Proof There exists a 1-parameter family $P_n(t) : \mathbf{I}^n \to \mathbf{I}^n$ for $t \in [0,1]$ satisfying $P_n(0) = P_n$, $P_n(1) = Id_{\mathbf{I}^n}$, such that for all $t \in (0,1]$ the function $P_n(t) : \mathbf{I}^n \to \mathbf{I}^n$ is an embedding. Substituting $P_n(t)$ for P_n in the definition of $\mathrm{gr}_1 : L\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ gives the desired homotopy. □

In the literature, Litherland spinning is not defined as the map $\mathrm{gr}_1 : L\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$, but what Litherland defined in [47], when appropriately adapted to the smooth category, turns out to be precisely gr_1. This is the content of Proposition 3.6.

$\mathrm{EC}(n,*)$ is the group of diffeomorphisms of $\mathbb{R}^n$ whose support is contained in $\mathbf{I}^n$, thus it acts (by composition on the left) on $\mathcal{K}_{n,j}$. Notice that if $n-j>0$, $f \in \mathcal{K}_{n,j}$ and $g \in \mathrm{EC}(n,*)$ then $g \circ f$ is in the the same path-component of $\mathcal{K}_{n,j}$ as f. In fact, much more is true. Let $\mathcal{K}_{n,j}(f)$ denote the path-component of f in $\mathcal{K}_{n,j}$.

Lemma 3.5 *Provided $n-j>0$ and $f \in \mathcal{K}_{n,j}$, the map $\mathrm{EC}(n,*) \to \mathcal{K}_{n,j}$ given by sending $g \in \mathrm{EC}(n,*)$ to $g \circ f$ is a null-homotopic fibration whose image is $\mathcal{K}_{n,j}(f)$. The fibre of this fibration is denoted* $\mathrm{Diff}(\mathbf{I}^n, f)$.

Proof That the map is a fibration is classical [15]. That the image contains $\mathcal{K}_{n,j}(f)$ follows from the isotopy extension theorem. Consider an orientation-preserving affine-linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ such that $L(\mathbf{I}^n) \subset \mathbf{I}^n$. Given $g \in \mathrm{EC}(n,*)$ notice that $L \circ g \circ L^{-1} \in \mathrm{EC}(n,*)$, moreover the support of $L \circ g \circ L^{-1}$ is contained in $L(\mathbf{I}^n)$. The space of orientation-preserving affine linear transformations of $\mathbb{R}^n$ which preserves $\mathbf{I}^n$ is connected, thus there is a path L_t in this space such that $L_0 = Id_{\mathbb{R}^n}$ and $L_1 = L$. The function

$$\begin{array}{ccc} [0,1] \times \mathrm{EC}(n,*) & \longrightarrow & \mathcal{K}_{n,j} \\ \Psi & & \Psi \\ (t,g) & \longmapsto & L_t \circ g \circ L_t^{-1} \circ f \end{array}$$

is a null-homotopy of the map $\mathrm{EC}(n,*) \to \mathcal{K}_{n,j}$ provided $L(\mathbf{I}^n) \cap f(\mathbb{R}^j) = \phi$, which can always be arranged provided $n-j>0$, by Sard's theorem. $\square$

The map $\pi_1\mathcal{K}_{n,j}(f) \to \pi_0\mathrm{Diff}(\mathbf{I}^n, f)$ is therefore a bijection onto the subgroup of $\pi_0\mathrm{Diff}(\mathbf{I}^n, f)$ which is the kernel of the forgetful map $\pi_0\mathrm{Diff}(\mathbf{I}^n, f) \to \pi_0\mathrm{EC}(n,*)$. Given an element $g \in \pi_1\mathcal{K}_{n,j}(f)$, let $\tilde{g} \in \pi_0\mathrm{Diff}(\mathbf{I}^n, f)$ be its image. Given $g \in \pi_1\mathcal{K}_{n,j}(f)$ and $\mathrm{gr}_1 g \in \mathcal{K}_{n+1,j+1}$ denote the one-point compactification by $\overline{\mathrm{gr}_1 g} \in \mathrm{Emb}(S^{j+1}, S^{n+1})$.

Starting from an element $h \in \mathrm{Diff}(\mathbf{I}^n, f)$ which is in the kernel of the forgetful map $\mathrm{Diff}(\mathbf{I}^n, f) \to \pi_0\mathrm{EC}(n,*)$, Litherland gave a 'surgery' description [47] of an embedding $S^{j+1} \to S^{n+1}$. Consider $\mathbf{I}^{n+2}$ to be the product $\mathbf{I}^{n+2} = \mathbf{I}^n \times \mathbf{I}^2$, so $\partial\mathbf{I}^{n+2} = \mathbf{I}^n \times (\partial\mathbf{I}^2) \cup (\partial\mathbf{I}^n) \times \mathbf{I}^2$. Think of $\mathbf{I}^n \times (\partial\mathbf{I}^2)$ as a trivial $\mathbf{I}^n$-bundle over $\partial\mathbf{I}^2$, therefore it is diffeomorphic to the bundle over $\partial\mathbf{I}^2$ with fibre $\mathbf{I}^n$ and monodromy given by h. Call this space $\mathbf{I}^n \times_h \partial I^2$. Since h acts as the identity on $\partial\mathbf{I}^n$, the boundary of $\mathbf{I}^n \times_h \partial\mathbf{I}^2$ is canonically identified with $\partial\mathbf{I}^n \times \partial\mathbf{I}^2$. Thus the union

$$\left((\mathbf{I}^n, f) \times_h \partial\mathbf{I}^2\right) \cup (\partial\mathbf{I}^n, \partial\mathbf{I}^j) \times \mathbf{I}^2$$

makes sense as a manifold pair. Identify $\partial\mathbf{I}^{n+2}$ with $S^{n+1} \subset \mathbb{R}^{n+2}$ by radial projection from the origin. Thus, $\left((\mathbf{I}^n, f) \times_h \partial\mathbf{I}^2\right) \cup (\partial\mathbf{I}^n, \partial\mathbf{I}^j) \times \mathbf{I}^2$ describes an embedding of S^{j+1} in S^{n+1}. This is Litherland's deform-spun knot construction [47].

Proposition 3.6 *Given $g \in \pi_1\mathcal{K}_{n,j}(f)$, the 'Litherland spun' knot $\left((\mathbf{I}^n, f) \times_{\tilde{g}} \partial\mathbf{I}^2\right) \cup (\partial\mathbf{I}^n, \partial\mathbf{I}^j) \times \mathbf{I}^2$ and $\overline{\mathrm{gr}_1 g} \in \mathrm{Emb}(S^{j+1}, S^{n+1})$ are isotopic, once S^{n+1} is identified with $\partial\mathbf{I}^{n+2}$ via radial projection.*

Proof The key step is to remember that the identification of $\mathbf{I}^n \times (\partial\mathbf{I}^2)$ with $\mathbf{I}^n \times_{\tilde{g}} \partial I^2$ is made via the null-isotopy of $\tilde{g}$ when considered as an element of $\mathrm{EC}(n,*)$. Under this identification, the two definitions are identical. $\square$

Given $f \in \mathcal{K}_{n,j}$ and $g \in \Omega\mathcal{K}_{n,j}(f)$, let C_f be the complement of an open tubular neighbourhood of $\overline{f}$ in S^n. By the above argument, the complement of $\overline{\mathrm{gr}_1(g)}$ in S^{n+1} is diffeomorphic to $C_f \rtimes_{\tilde{g}} S^1$ union a 2-handle and an $(n-j+1)$-handle. Here $C_f \rtimes_{\tilde{g}} S^1$ indicates the C_f bundle over S^1 with monodromy induced by $\tilde{g}$. This gives a presentation

$$\pi_1 C_{\mathrm{gr}_1(g)} \simeq \pi_1 C_f / \langle \tilde{g}.x = x \ \forall x \in \pi_1 C_f \rangle$$

where $\langle \tilde{g}.x = x \ \forall x \in \pi_1 C_f \rangle$ is the normal subgroup of $\pi_1 C_f$ generated by the relations $\tilde{g}.x = x$ for all $x \in \pi_1 C_f$.

Example 3.7 *If $g \in \Omega\mathcal{K}_{3,1}(f)$ is the Gramain element (rotation by 2π about the long axis), its action on $\pi_1 C_f$ is conjugation by the meridian. Thus $\pi_1 C_{\mathrm{gr}_1(g)}$ is trivial, as all knot groups are 'normally generated' by a meridian. This observation anticipates the Zeeman-Litherland theorem, which states that $\mathrm{gr}_1(g)$ is the unknot [88, 47] whenever g is the Gramain element. The Zeeman-Litherland theorem is stated in full generality in Section 5.*

The spaces $\mathcal{K}_{n,n} = \mathrm{EC}(n,*)$ are the groups of diffeomorphisms of a cube, and have the homotopy-type of $\mathrm{Diff}(D^n)$, the group of diffeomorphisms of a disc which are the identity on the boundary. The maps $\mathrm{gr}_1 : \Omega\mathcal{K}_{n,n} \to \mathcal{K}_{n+1,n+1}$ have been studied in this context. Define $\mathrm{gr}_2 : \Omega^2\mathcal{K}_{n,j} \to \mathcal{K}_{n+2,j+2}$ to be the composite $\mathrm{gr}_1 \circ \Omega\mathrm{gr}_1$ where $\Omega\mathrm{gr}_1 : \Omega^2\mathcal{K}_{n,j} \to \Omega\mathcal{K}_{n+1,j+1}$ is the induced map of gr_1. Similarly define $\mathrm{gr}_i : \Omega^i\mathcal{K}_{n,j} \to \mathcal{K}_{n+i,j+i}$. In the literature [4, 83, 24] elements of $\pi_0\mathcal{K}_{n,n}$ which are in the image of $\mathrm{gr}_i : \pi_i\mathcal{K}_{n-i,n-i} \to \pi_0\mathcal{K}_{n,n}$ but which are not in the image of gr_{i+1} are typically said to have Gromoll degree i.

Definition 3.8 An element $f \in \pi_0\mathcal{K}_{n,j}$ has (Gromoll) degree i if it is in the image of the i-th graphing map $\mathrm{gr}_i : \pi_i\mathcal{K}_{n-i,j-i} \to \pi_0\mathcal{K}_{n,j}$ but not in the image of the $(i+1)$-st graphing map gr_{i+1}.

Proposition 3.9 (1) *The Gromoll degree of the elements of $\pi_0\mathcal{K}_{n,j}$ is at least $2n-2j-4$ for all $n \geq j > 0$.*

(2) *$\mathcal{K}_{n,j}$ is $(2n-3j-4)$-connected for all $n \geq j \geq 1$. Provided $2n-3j-3 \geq 0$ and $n-j > 2$ the first non-trivial homotopy group of $\mathcal{K}_{n,j}$ is*

$$\pi_{2n-3j-3}\mathcal{K}_{n,j} \simeq \begin{cases} \mathbb{Z} & j = 1 \text{ or } n-j \text{ is odd} \\ \mathbb{Z}_2 & j > 1 \text{ and } n-j \text{ is even} \end{cases}$$

The elements of $\pi_0\mathcal{K}_{n,j}$ for $2n-3j-3 = 0$ have Gromoll degree $(j-1)$, ie: $\mathrm{gr}_{j-1} : \pi_{j-1}\mathcal{K}_{n-j+1,1} \to \pi_0\mathcal{K}_{n,j}$ is onto.

(3) *$\mathrm{Emb}(S^j, S^n)$ is $\min\{(2n-3j-4),(n-j-2)\}$-connected for all $n \geq j \geq 1$. Let $m = \min\{2n-3j-3, n-j-1\}$. Provided $2n-3j-3 \geq 0$ and $n-j > 2$ the first non-trivial homotopy-group of $\mathrm{Emb}(S^j, S^n)$ is*

$$\pi_m\mathrm{Emb}(S^j,S^n) \simeq \begin{cases} \mathbb{Z} & 2n-3j-3 < n-j-1, (j=1 \text{ or } n-j \text{ odd}) \\ \mathbb{Z} & 2n-3j-3 > n-j-1, n-j \text{ even} \\ \mathbb{Z}_2 & 2n-3j-3 < n-j-1, j > 1 \text{ and } n-j \text{ even} \\ \mathbb{Z}_2 & 2n-3j-3 > n-j-1, n-j \text{ odd} \\ \mathbb{Z}\oplus\mathbb{Z}_2 & 2n-3j-3 = n-j-1 \end{cases}$$

(4) $\mathrm{Emb}(S^j,\mathbb{R}^n)$ *is* $\min\{2n-3j-4, n-j-2\}$ *connected for all* $n \geq j+2 \geq 3$. *Let* $m = \min\{2n-3j-3, n-j-1\}$. *Provided* $2n-3j-3 \geq 0$ *and* $n-j>2$ *the first non-trivial homotopy group of* $\mathrm{Emb}(S^j,\mathbb{R}^n)$ *is*

$$\pi_m \mathrm{Emb}(S^j,\mathbb{R}^n) \simeq \begin{cases} \mathbb{Z} & 2n-3j-3 < n-j-1, (j=1 \text{ or } n-j \text{ odd}) \\ \mathbb{Z}_2 & 2n-3j-3 < n-j-1, j>1 \text{ and } n-j \text{ even} \\ \mathbb{Z} & 2n-3j-3 > n-j-1 \\ \mathbb{Z}^2 & 2n-3j-3 = n-j-1, (j=1 \text{ or } n-j \text{ odd}) \\ \mathbb{Z}\oplus\mathbb{Z}_2 & 2n-3j-3 = n-j-1, j>1 \text{ and } j \text{ even} \end{cases}$$

(5) $\mathcal{P}_{n,j}$ *is* $(2n-2j-5)$*-connected for all* $n-j>2$.

(6) $\mathrm{Emb}(D^j,D^n)$ *is* $(n-j-2)$*-connected for all* $n-j>2$.

Proof (5) That $\mathcal{P}_{n,j}$ is $2n-3j-5$ connected follows directly from Goodwillie's dissertation [22] (see Theorem C on page 9, and the comments immediately afterwards).

(6) This result follows from (5) and Theorem 2.1.

(1) Consider the homotopy fibre-sequence $\Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j} \to \mathcal{P}_{n,j}$ from Proposition 3.2. Since $\mathcal{P}_{n,j}$ is $(2n-2j-5)$-connected, $\pi_1\mathcal{K}_{n-1,j-1} \to \pi_0\mathcal{K}_{n,j}$ is epic for $n-j>2$. Moreover, $\pi_2\mathcal{K}_{n-2,j-2} \to \pi_1\mathcal{K}_{n-1,j-1}$ is also epic, as $\pi_1\mathcal{P}_{n-1,j-1}$ is trivial. The result follows by induction.

(2) There is a computation of the 3rd stage of the Goodwillie tower for $\mathcal{K}_{n,1}$ in [9]. This is a $(2n-6)$-connected map $\mathcal{K}_{n,1} \to AM_3$. AM_3 is known to have the homotopy-type of the 3-fold loop-space on the homotopy fibre of the inclusion $S^{n-1}\vee S^{n-1} \to S^{n-1}\times S^{n-1}$, thus $\mathcal{K}_{n,1}$ is $(2n-7)$-connected. The first non-trivial integral homology group of $\mathcal{K}_{n,1}$ is computed by Victor Turchin [78] (see the computations for the homology of the complexes CT_0D^{even} and CT_0D^{odd} for $j=4$, $i=2$). Turchin's result is that $H_{2n-6}(\mathcal{K}_{n,1};\mathbb{Z}) \simeq \mathbb{Z}$, so by the Hurewicz Theorem, $\pi_{2n-6}\mathcal{K}_{n,1} \simeq \mathbb{Z}$. That verifies the result for $\mathcal{K}_{n,1}$.

Consider the space $\mathcal{K}_{n+j,j+1}$ for $j \geq 1$. The fibre-sequence

$$\Omega\mathcal{K}_{n+j-1,j} \to \mathcal{K}_{n+j,j+1} \to \mathcal{P}_{n+j,j+1}$$

has a $(2n-7)$-connected base-space. In the special case of $j=1$ the fibre has first non-trivial homotopy group in dimension $2n-7$. But $\pi_{2n-7}\mathcal{P}_{n+1,2}$ is trivial, thus $\pi_{2n-6}\mathcal{K}_{n,1} \to \pi_{2n-7}\mathcal{K}_{n+1,2}$ is epic with kernel generated by the image of $\pi_{2n-6}\mathcal{P}_{n+1,2}$, giving the isomorphism

$$\pi_{2n-7}\mathcal{K}_{n+1,2} \simeq \pi_{2n-6}\mathcal{K}_{n,1}/img\,(\pi_{2n-6}\mathcal{P}_{n+1,2}).$$

Repeat the argument for $j>1$, inductively assuming that the first non-trivial homotopy group of $\Omega\mathcal{K}_{n+j-1,j}$ is $\pi_{2n-j-6}\Omega\mathcal{K}_{n+j-1,j}$ and isomorphic to $\pi_{2n-6}\mathcal{K}_{n,1}/img\,(\pi_{2n-6}\mathcal{P}_{n+1,2})$. Since $\mathcal{P}_{n+j,j+1}$ is $(2n-7)$-connected, the map $\pi_{2n-j-6}\Omega\mathcal{K}_{n+j-1,j} \to \pi_{2n-j-6}\mathcal{K}_{n+j,j+1}$ is an isomorphism of first non-trivial homotopy-groups, thus for all $j \geq 1$ there is an isomorphism $\pi_{2n-j-6}\mathcal{K}_{n+j,j+1} \simeq \pi_{2n-6}\mathcal{K}_{n,1}/img\,(\pi_{2n-6}\mathcal{P}_{n+1,2})$.

Setting j equal to $2n-6$ gives the isomorphism

$$\pi_0\mathcal{K}_{3n-6,2n-5} \simeq \pi_{2n-6}\mathcal{K}_{n,1}/img\,(\pi_{2n-6}\mathcal{P}_{n+1,2}).$$

Haefliger's computations [27] completes the proof:

$$\pi_0\mathcal{K}_{3n-6,2n-5} \simeq \begin{cases} \mathbb{Z}_2 & \text{for } n \geq 4 \text{ odd} \\ \mathbb{Z} & \text{for } n \geq 4 \text{ even.} \end{cases}$$

(3) Theorem 2.1 gives us a homotopy equivalence $\mathrm{Emb}(S^j, S^n) \simeq \mathrm{SO}_{n+1} \times_{\mathrm{SO}_{n-j}} \mathcal{K}_{n,j}$. Since $\mathrm{SO}_{n+1}/\mathrm{SO}_{n-j} \equiv V_{n+1,j+1}$ is $(n-j-1)$-connected, the homotopy long exact sequence of the fibration $\mathcal{K}_{n,j} \to \mathrm{Emb}(S^j, S^n) \to V_{n+1,j+1}$ tells us that $\mathrm{Emb}(S^j, S^n)$ is $min\{n-j-1, 2n-3j-4\}$-connected. Since the bundle $\mathrm{Emb}(S^j, S^n) \to V_{n+1,j+1}$ is split, the first non-trivial homotopy group of $\mathrm{Emb}(S^j, S^n)$ can be computed directly.

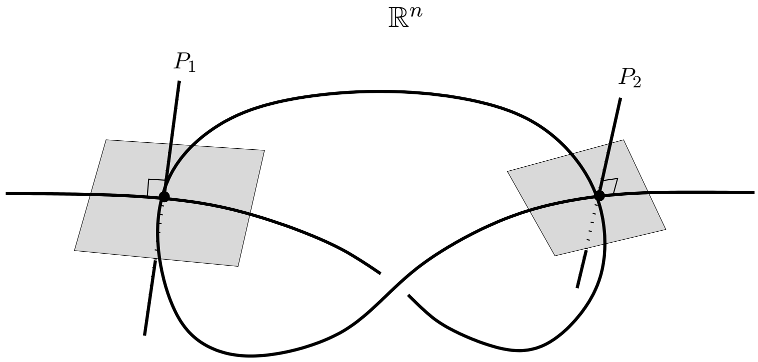
(4) For $\mathrm{Emb}(S^j, \mathbb{R}^n)$ use the homotopy equivalence $\mathrm{Emb}(S^j, \mathbb{R}^n) \simeq \mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} (C \rtimes \mathcal{K}_{n,j})$ from Proposition 2.2. The bundles $C \rtimes \mathcal{K}_{n,j} \to \mathcal{K}_{n,j}$ and $\mathrm{SO}_n \times_{\mathrm{SO}_{n-j}} (C \rtimes \mathcal{K}_{n,j}) \to V_{n,j}$ are split, so the computation follows directly. □

An interesting corollary is that there are 'exotic families' of smooth 2-discs in the 6-disc.

Corollary 3.10 *$\pi_{2n-6}\mathcal{P}_{n+1,2}$ has rank at least 1 provided $n \geq 5$ is odd.*

Brian Munson gave a lower bound of $\min\{2n-3j-4, n-j-2\}$ on the connectivity of $\mathrm{Emb}(S^j, \mathbb{R}^n)$. Proposition 3.9 proves that Munson's lower bound is sharp.

The rest of this section is devoted to a geometric construction of the generators of $\pi_{2n-6}\mathcal{K}_{n,1}$ for $n \geq 4$. Take a 'long' immersion $f : \mathbb{R} \to \mathbb{R}^3 \subset \mathbb{R}^n$ having two regular double points $f(t_1) = f(t_3)$, $f(t_2) = f(t_4)$ with $t_1 < t_2 < t_3 < t_4 \in \mathbb{R}$ such that one of the four resolutions of f in $\mathbb{R}^3$ is a trefoil knot. Let Tf_i be the tangent space to $f(\mathbb{R})$ at t_i. Let P_1 be the orthogonal complement to $Tf_1 \oplus Tf_3$ in $\mathbb{R}^n$, and P_2 the orthogonal complement of $Tf_2 \oplus Tf_4$ in $\mathbb{R}^n$. P_1 and P_2 are $(n-2)$-dimensional, so if S_1 and S_2 are the unit sphere of P_1 and P_2 respectively they are both $(n-3)$-dimensional. There is a 'resolution function' $r : S_1 \times S_2 \to \mathcal{K}_{n,1}$ given by perturbing f near the double points via bump-functions whose directions are prescribed by the pair $(v_1, v_2) \in S_1 \times S_2$. The claim is that r is a generator of $H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z}) \simeq \mathbb{Z}$.



One could potentially trace through the computations of Turchin and Vassiliev [78, 81] to verify that r generates $H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$. The following approach is perhaps more direct. It is inspired by the quadrisecant description of the type-2 Vassiliev invariant for knots $\mathbb{R}^3$ [9]. The idea is to construct an integral co-homology class $\nu_2 \in H^{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$ such that if $x \in H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$ is represented as an oriented $(2n-6)$-dimensional manifold mapping into $\mathcal{K}_{n,1}$ then $\nu_2(x)$ can be computed as a signed count of the number of alternating quadrisecants along the family of long knots represented by x. Every class in $H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$ is realisable as a map from an oriented $(2n-6)$-dimensional manifold M to $\mathcal{K}_{n,1}$ since $\mathcal{K}_{n,1}$ is $(2n-7)$-connected (Proposition 3.9). Moreover, by the Hurewicz theorem, M can be assumed to be S^{2n-6}, as $\pi_{2n-6}\mathcal{K}_{n,1} \simeq H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$.

Definition 3.11 Given two points $x, y \in \mathbb{R}^n$ let $[x, y]$ denote the oriented line segment in $\mathbb{R}^n$, starting at x and ending at y. An alternating quadrisecant in $C_4(\mathbb{R}^n)$ is a point $(x_1, x_2, x_3, x_4) \in C_4(\mathbb{R}^n)$ such that $[x_1, x_4] \subset [x_3, x_2]$ as an oriented subinterval. $C_k M$ denotes the configuration space of distinct k-tuples of points in M, $C_k M = \{x \in M^k : x_i \neq x_j \ \forall\ i \neq j\}$. Provided M is a manifold, let $C_k[M]$ denotes the (real oriented) Fulton-Macpherson compactification of $C_k M$, as in [9]. $C_k[M]$ is a compact manifold, provided M is compact. The 'real oriented' Fulton-Macpherson compactification has the property that the inclusion $C_k M \to C_k[M]$ is a homotopy-equivalence.

Let $AQ_n \subset C_4[\mathbb{R}^n]$ denote the closure of the set of all alternating quadrisecants in $C_4(\mathbb{R}^n)$. Let $C_4'[\mathbb{R}] = \overline{\{t \in C_4(\mathbb{R}) : t_1 < t_2 < t_3 < t_4\}}$. Given $f \in \mathcal{K}_{n,1}$ let $AQ_n(f) \subset C_4'[\mathbb{R}]$ denote the pull-back of AQ_n. More generally, if $f : M \to \mathcal{K}_{n,1}$ is smooth, define $AQ_n(f) \subset M \times C_4'[\mathbb{R}]$ as the pull-back of AQ_n.

Given a closed, oriented $(2n-6)$-dimensional manifold M and a map $f : M \to \mathcal{K}_{n,1}$ such that $f_* : M \times C_4'[\mathbb{R}] \to C_4[\mathbb{R}^n]$ is transverse to AQ_n, $AQ_n(f) \subset M \times C_4'[\mathbb{R}]$ is a 0-dimensional submanifold whose normal bundle is oriented by the map. A well-defined integer invariant $\nu_2(f) \in \mathbb{Z}$ is defined as the signed count (of the relative orientations) of the points in $AQ_n(f)$. The sign of each point of $AQ_n(f)$ could be computed by a formula analogous to the one in Proposition 6.2 of [9]. Lemma 3.12 is the key technical lemma needed to show that $\nu_2(f)$ is an invariant of the homology class of f.

Given $f \in \mathcal{K}_{n,1}$ let $\Gamma(f) \in (0, \infty]$ be the 'cut radius' of f in $\mathbb{R}^n$, defined as the supremum over all R such that the exponential map from f's radius-R normal disc bundle to $\mathbb{R}^n$ is an embedding. $\Gamma : \mathcal{K}_{n,1} \to (0, \infty]$ can be shown to be a continuous function, as $\Gamma(f)$ is the minimum of two continuous quantities 1) the focal radius of f (which can be computed in terms of the 2nd fundamental form of f) and 2) the minimum of the distances L such that there exists two geodesics segments, each of length L, emanating from a point in $\mathbb{R}^n$ and terminating in $f(\mathbb{R})$, orthogonal to the tangent space of $f(\mathbb{R})$. This kind of continuity argument is standard in differential geometry, see Proposition 4.1 in §III of [66] for example.

Lemma 3.12 *Every $x \in H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$ represented by a manifold $f : M \to \mathcal{K}_{n,1}$ can be perturbed so that f_* is transverse to AQ_n.*

Proof Let R be the cut radius of f, $R = \min\{\Gamma(f(x)) : x \in M\}$. Let $b : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth function satisfying:

- $b(x) = 0$ for all $|x| \geq 1$
- $b(x) = b(-x)$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{\infty} b(x)dx = 1$
- $b'(x) > 0$ for all $-1 < x < 0$.

For $\epsilon > 0$ and $t \in \mathbb{R}$ let $b_{\epsilon,t} : \mathbb{R} \to \mathbb{R}$ be defined as $b_{\epsilon,t}(x) = \frac{1}{\epsilon} b(\frac{x-t}{\epsilon})$. By a compactness argument, there exists an $m \in \mathbb{Z}$ (perhaps very large) such that if $I_1, \cdots, I_m$ is the partition of $\mathbf{I}$ into m equal-length sub-intervals, then for all $x \in M$ and $j \in \{1, 2, \cdots, m\}$, $f(x)(I_j)$ is contained in the radius $R/2$ tubular neighbourhood of $f(x)$.

Consider the function $\tilde{f}$ defined as

$$\begin{array}{ccc} M \times (\mathbb{R}^n)^m \times \mathbb{R} & \xrightarrow{\tilde{f}} & \mathbb{R}^n \\ \cup & & \cup \\ (x, v_1, \cdots, v_m, t) & \longmapsto & f(t) + \sum_{j=1}^m b_{\frac{3}{2m}, p_j}(t) v_j \end{array}$$

where $p_j \in I_j$ is the mid-point of the interval I_j. Since embeddings are an open subset of the space of all 'long' smooth maps from $\mathbb{R}$ to $\mathbb{R}^n$ [33], in some neighbourhood U of 0 in $(\mathbb{R}^n)^m$, a restriction of $\tilde{f}$ can be thought of as a map $\overline{f} : M \times U \to \mathcal{K}_{n,1}$. Consider a point $(x, y, t_1, t_2, t_3, t_4)$ of $AQ_n(\overline{f}) \subset M \times U \times C'_4[\mathbb{R}]$. For each i, t_i and t_{i+1} cannot both be elements of some common I_j since $(f(t_1), f(t_2), f(t_3), f(t_4))$ is an alternating quadrisecant. Thus $\overline{f}_* : M \times U \times C'_4[\mathbb{R}] \to C_4[\mathbb{R}^n]$ is transverse to AQ_n. By the Transversality Theorem [25], f can be approximated by a map $M \to \mathcal{K}_{n,1}$ such that the induced map $M \times C'_4[\mathbb{R}] \to C_4[\mathbb{R}^n]$ is transverse to AQ_n. □

Theorem 3.13 $\nu_2 \in H^{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z})$ *is a well-defined cohomology class. Moreover,* $\nu_2(r) = \pm 1$, *forcing* r *to be a generator of* $H_{2n-6}(\mathcal{K}_{n,1}; \mathbb{Z}) \simeq \mathbb{Z}$.

Proof An alternating quadrisecant can never appear on $\partial(M \times C'_4[\mathbb{R}])$ nor can a 1-parameter family of alternating quadrisecants run off to infinity, thus, by the Transversality Extension Theorem (see for example Chapter 2 of [25]) $\nu_2(f)$ is well-defined integer invariant of the homology class of f.

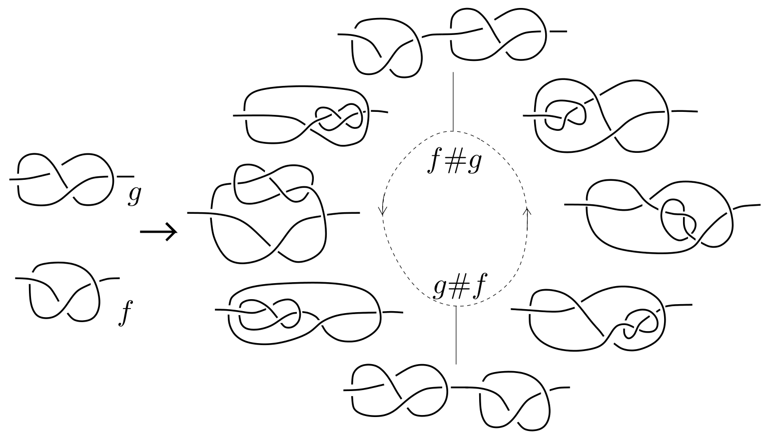
In the picture of the 'immersed trefoil' $f : \mathbb{R} \to \mathbb{R}^3 \subset \mathbb{R}^n$ there are no quadrisecants, except the 'degenerate' quadrisecant that consisting of the secant between the two pairs of double-points. Consider all the possible resolutions r of this immersed trefoil. r only has 4 resolutions in $\mathbb{R}^3 \subset \mathbb{R}^n$, so these are the only 4 resolutions that could possibly have quadrisecants. Moreover, only the resolution which is a trefoil in $\mathbb{R}^3$ has a quadrisecant. □

Since $\mathcal{K}_{n,1}$ is $(2n-7)$-connected, by the Hurewicz Theorem $\pi_{2n-6}\mathcal{K}_{n,1} \simeq \mathbb{Z}$ is generated by any map $\tilde{r} : S^{2n-6} \to \mathcal{K}_{n,1}$ homologous to r. One can explicitly construct such a map – attachment of an $(n-3)$-handle to $S_1 \times S_2 \times [0,1]$ along $S_1 \times \{*\} \times \{1\}$ gives a cobordism between $S_1 \times S_2$ and S^{2n-6}. $r_{|S_1 \times \{*\}}$ is null so r extends over the cobordism. $\tilde{r}$ can be chosen to be the restriction of this cobordism to S^{2n-6}.

4 Actions of operads of little cubes on embedding spaces

This section is devoted to the study of the iterated loop-space structures on the embedding spaces $\mathcal{K}_{n,j}$ and $\mathrm{EC}(j, D^n)$, especially focusing on the compatibility of these structures with Litherland spinning gr_1. The context of these results comes from the work of Boardman, Vogt and May [5, 49, 50]. They give a very simple criterion for recognising if a space X has the homotopy-type of an n-fold loop-space, being that X admits an action of the operad of little n-cubes, and that the induced monoid structure on $\pi_0 X$ is that of a group. A useful reference for operads relevant to topology, including operads of cubes, is the book of Markl, Shnider and Stasheff [48].

There is an action of the operad of j-cubes on the spaces $\mathrm{EC}(j, M)$ and $\mathcal{K}_{n,j}$ given by concatenation (see Definition 4.2). The first instance of an action of the operad of $(j+1)$-cubes on any space of the form $\mathrm{EC}(j, M)$ was given by Morlet [56]. The Cerf-Morlet 'Comparison Theorem' states that $\mathrm{EC}(j, *) \simeq \Omega^{j+1}(PL_j/O_j)$ (see [12] or [42] for a proof). Here PL_j is the group of PL-automorphisms of $\mathbb{R}^j$ (given a suitable topology) and O_j is the group of linear isometries of $\mathbb{R}^j$.



The first 'hint' of a higher cubes action on the spaces $\mathrm{EC}(j, M)$ for M non-trivial would perhaps be the work of Schubert [68]. Schubert demonstrated that the connect-sum pairing turns $\pi_0\mathcal{K}_{3,1}$ into a free commutative monoid on the isotopy-classes of prime long knots, where the demonstration of commutativity involved 'pulling one knot through another' as in the figure above.

In 'Little cubes and long knots' [7] this idea was extended to construct a $(j+1)$-cubes action on the spaces $\mathrm{EC}(j, M)$ for an arbitrary compact manifold M. By some elementary considerations, this also gives an action of the operad of $(j+1)$-cubes on $\mathcal{K}_{n,j}$ for all $n - j \leq 2$. Schubert's theorem that $\pi_0\mathcal{K}_{3,1}$ is a free commutative monoid over the isotopy classes of prime long knots generalises in this context to say that $\mathcal{K}_{3,1}$ is a free 2-cubes object over the based space $\mathcal{P} \sqcup \{*\}$ where $\mathcal{P} \subset \mathcal{K}_{3,1}$ is the subspace of prime long knots. This can be thought of as a precise 'space level' non-uniqueness result for the connect-sum decomposition of knots, whereas Schubert's result states uniqueness on the level of isotopy classes of knots.

There is a major conceptual gap between the Cerf-Morlet 'Comparison Theorem' and the freeness of $\mathcal{K}_{3,1}$ as a 2-cubes object. Getting a better understanding of this defect was one of the primary motivations behind this paper.

Definition 4.1

- A (single) little n-cube is a function $L : \mathbf{I}^n \to \mathbf{I}^n$ such that $L = l_1 \times \cdots \times l_n$ where each $l_i : \mathbf{I} \to \mathbf{I}$ is affine-linear and increasing ie: $l_i(t) = a_i t + b_i$ for some $0 \leq a_i < 1$ and $b_i \in \mathbb{R}$.
- Let CAut_n denote the monoid of affine-linear automorphisms of $\mathbb{R}^n$ of the form $L = l_1 \times \cdots \times l_n$ where $l_i : \mathbb{R} \to \mathbb{R}$ affine linear and increasing, and $L(\mathbf{I}^n) \subset \mathbf{I}^n$.
- Given a little n-cube L a mild abuse of notation is to consider $L \in \mathrm{CAut}_n$ by taking the unique affine-linear extension of L to $\mathbb{R}^n$.
- The space of j little n-cubes $\mathcal{C}_n(j)$ is the space of maps $L : \sqcup_{i=1}^j \mathbf{I}^n \to \mathbf{I}^n$ such that the restriction of L to the interior of its domain is an embedding, and the restriction of L to any connected component of its domain is a little n-cube. Given $L \in \mathcal{C}_n(j)$ let L_i denote the restriction of L to the i-th copy of $\mathbf{I}^n$. By convention $\mathcal{C}_n(0)$ is taken to be a point. This makes the union $\sqcup_{j=0}^{\infty}\mathcal{C}_n(j)$ into an operad, called the operad of little n-cubes $\mathcal{C}_n$ [49].

- There is an action of CAut_n on $\mathrm{EC}(n,M)$ given by

$$\mu : \mathrm{CAut}_n \times \mathrm{Emb}(\mathbb{R}^n \times M, \mathbb{R}^n \times M) \to \mathrm{Emb}(\mathbb{R}^n \times M, \mathbb{R}^n \times M)$$

$$\mu(L,f) = (L \times Id_M) \circ f \circ (L^{-1} \times Id_M)$$

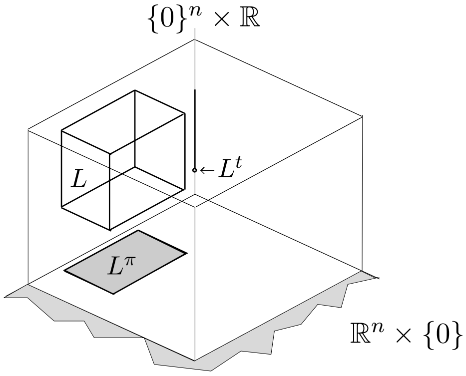
  In the above formula, L^{-1} is the inverse of L in the group of affine-linear isomorphisms of $\mathbb{R}^n$. The above action is denoted $\mu(L,f) = L.f$. There is an action of CAut_j on $\mathcal{K}_{n,j}$ defined essentially the same way.

An action of the operad of j-cubes on both $\mathcal{K}_{n,j}$ and $\mathrm{EC}(j,M)$ where the associated multiplication on $\pi_0\mathcal{K}_{n,j}$ is the connect-sum operation, is given next.

Definition 4.2 $k_i : \mathcal{C}_j(i) \times (\mathcal{K}_{n,j})^i \to \mathcal{K}_{n,j}$, $k_i : \mathcal{C}_j(i) \times \mathrm{EC}(j,M)^i \to \mathrm{EC}(j,M)$ is defined by the rule $k_i(L_1,\cdots,L_i,f_1,\cdots,f_i) = L_1.f_1 \circ \cdots \circ L_i.f_i$. In the case of the space $\mathcal{K}_{n,j}$, given $f,g \in \mathcal{K}_{n,j}$ with disjoint support, $f \circ g$ is defined so that $f \circ g(x) = \begin{cases} f(x) & \text{if } f(x) \neq x \\ g(x) & \text{if otherwise.} \end{cases}$

Definition 4.3 extends the j-cubes action on $\mathrm{EC}(j,M)$ to a $(j+1)$-cubes action.

Definition 4.3
- Given j little $(n+1)$-cubes, $L = (L_1,\cdots,L_j) \in \mathcal{C}_{n+1}(j)$ define the j-tuple of (non-disjoint) little n-cubes $L^\pi = (L_1^\pi,\cdots,L_j^\pi)$ by the rule $L_i^\pi = l_{i,1} \times \cdots \times l_{i,n}$ where $L_i = l_{i,1} \times \cdots \times l_{i,n+1}$. Similarly define $L^t \in \mathbf{I}^j$ by $L^t = (L_1^t,\cdots,L_j^t)$ where $L_i^t = l_{i,n+1}(-1)$.



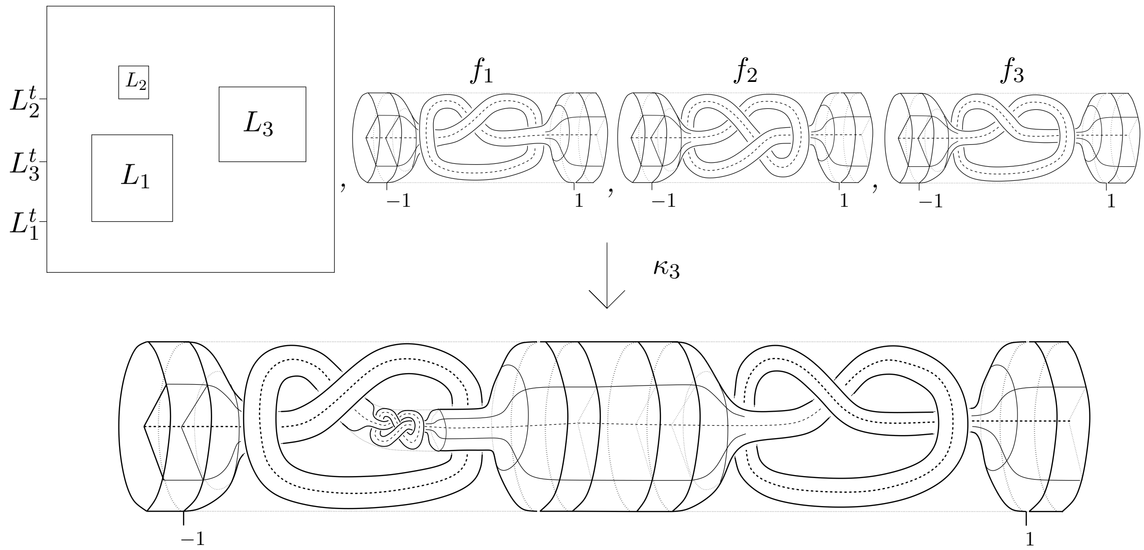
- The action of the operad of little $(n+1)$-cubes on the space $\mathrm{EC}(n,M)$ is given by the maps $\kappa_j : \mathcal{C}_{n+1}(j) \times \mathrm{EC}(n,M)^j \to \mathrm{EC}(n,M)$ for $j \in \{1,2,\cdots\}$ defined by

$$\kappa_j(L_1,\cdots,L_j,f_1,\cdots,f_j) = L^\pi_{\sigma(1)}.f_{\sigma(1)} \circ L^\pi_{\sigma(2)}.f_{\sigma(2)} \circ \cdots \circ L^\pi_{\sigma(j)}.f_{\sigma(j)}$$

  where $\sigma : \{1,\cdots,j\} \to \{1,\cdots,j\}$ is any permutation such that $L^t_{\sigma(1)} \leq L^t_{\sigma(2)} \leq \cdots \leq L^t_{\sigma(j)}$. The map $\kappa_0 : \mathcal{C}_{n+1}(0) \times \mathrm{EC}(n,M)^0 \to \mathrm{EC}(n,M)$ is the inclusion of a point $*$ in $\mathrm{EC}(n,M)$, defined so that $\kappa_0(*) = Id_{\mathbb{R}^n \times M}$.

Theorem 4.4 *[7] For any compact manifold M and any integer $n \geq 0$ the maps κ_j for $j \in \{0,1,2,\cdots\}$ define an action of the operad of little $(n+1)$-cubes on $\mathrm{EC}(n,M)$.*

Example 4.5



$L_1^t < L_3^t < L_2^t$ so $\sigma = (23)$ and $\kappa_3(L_1, L_2, L_3, f_1, f_2, f_3) = L_1^\pi.f_1 \circ L_3^\pi.f_3 \circ L_2^\pi.f_2$, which explains the figure-8 knot being 'inside' of the trefoil on the left hand side of the picture.

In the definition of $\mathrm{EC}(n, M)$, if one replaces the condition that the support of f is contained in $\mathbf{I}^n \times M$ with it being contained in $D^n \times M$ one obtains a homotopy-equivalent space $\mathrm{ED}(n, M)$. By a similar construction to Definition 4.3, one also obtains an action of the operad of unframed little $(n+1)$-discs on $\mathrm{ED}(n, M)$. Since $\pi_0\mathcal{K}_{n,j}$ is a group for $n-j > 2$, $\mathrm{EC}(j, D^{n-j})$ an $(n+1)$-fold loop space. Next is a construction of analogous operad actions on the spaces $\mathrm{PEC}(n, M)$.

Definition 4.6 $\kappa_j : \mathcal{C}_n(j) \times \mathrm{PEC}(n, M)^j \to \mathrm{PEC}(n, M)$ for $j \in \{1, 2, \cdots\}$ is defined by

$$\kappa_j(L_1, \cdots, L_j, f_1, \cdots, f_j) = L_{\sigma(1)}.f_{\sigma(1)} \circ L_{\sigma(2)}.f_{\sigma(2)} \circ \cdots \circ L_{\sigma(j)}.f_{\sigma(j)}$$

where $\sigma : \{1, \cdots, j\} \to \{1, \cdots, j\}$ is any permutation such that $L^t_{\sigma(1)} \leq L^t_{\sigma(2)} \leq \cdots \leq L^t_{\sigma(j)}$.

Proposition 4.7 *The maps κ_* define an action of the operad of little n-cubes on* $\mathrm{PEC}(n, M)$.

Proof There are three axioms to verify.

(1) Identity. Let $Id_{\mathbf{I}^n}$ be the identity n-cube, then $\kappa_1(Id_{\mathbf{I}^n}, f) = Id_{\mathbf{I}^n}.f = f$ by design.

(2) Symmetry. We need to verify that $\kappa_n(L.\alpha, f.\alpha) = \kappa_n(L, f)$, for $\alpha \in \Sigma_n$.
Let
$$\kappa_j(L, f) = L_{\sigma(1)}.f_{\sigma(1)} \circ L_{\sigma(2)}.f_{\sigma(2)} \circ \cdots \circ L_{\sigma(j)}.f_{\sigma(j)}$$
and
$$\kappa_j(L.\alpha, f.\alpha) = L_{\alpha\sigma'(1)}.f_{\alpha\sigma'(1)} \circ L_{\alpha\sigma'(2)}.f_{\alpha\sigma'(2)} \circ \cdots \circ L_{\alpha\sigma'(j)}.f_{\alpha\sigma'(j)}$$
where $\sigma, \sigma' \in S_n$ satisfy $L^t_{\sigma(1)} \leq \cdots \leq L^t_{\sigma(n)}$ and $L^t_{\alpha\sigma'(1)} \leq \cdots \leq L^t_{\alpha\sigma'(n)}$. Up to the ambiguity in our choice of σ and σ' one can assume $\sigma' = \alpha^{-1}\sigma$, giving the result.

(3) Associativity. We need to verify the diagram below commutes:

$$\begin{tikzcd}
\mathcal{C}_n(m) \times \left(\mathcal{C}_n(j_1) \times \mathrm{PEC}(n,M)^{j_1} \times \cdots \times \mathcal{C}_n(j_m) \times \mathrm{PEC}(n,M)^{j_m}\right) \arrow[r] \arrow[d] & \mathcal{C}_n(m) \times \mathrm{PEC}(n,M)^m \arrow[d] \\
\mathcal{C}_n(j_1+\cdots+j_m) \times \mathrm{PEC}(n,M)^{j_1+\cdots+j_m} \arrow[r] & \mathrm{PEC}(n,M)
\end{tikzcd}$$

Given something in the top-left corner, consider what it maps to in the bottom-right corner, going around both ways. Either way around the diagram, one gets a composite of functions of the form $L_i.L_{i,p}.f_{i,p}$, in some order. The difference in the order of composition is irrelevant as our definition only allows functions to appear in different relative orders if they have disjoint supports.

□

Proposition 4.8 *Both the fibre-inclusion and projection maps in the fibration*

$$\mathrm{EC}(n,M) \to \mathrm{PEC}(n,M) \to \mathrm{EC}(n-1,M)$$

are maps of little n-cubes objects. The graphing map $\mathrm{gr}_1 : \Omega\mathrm{EC}(n-1,M) \to \mathrm{EC}(n,M)$ is a map of $(n+1)$-cubes object.

Proof The map $\mathrm{PEC}(n,M) \to \mathrm{EC}(n-1,M)$ is of course restriction to the $\{1\} \times \mathbb{R}^{n-1} \times M$ 'face', followed by the natural identification with $\mathbb{R}^{n-1} \times M$.

$$\kappa_j(L_1,\cdots,L_j,f_1,\cdots,f_j) = L_{\sigma(1)}.f_{\sigma(1)} \circ L_{\sigma(2)}.f_{\sigma(2)} \circ \cdots \circ L_{\sigma(j)}.f_{\sigma(j)}$$

Once restricted to $\{1\} \times \mathbb{R}^{n-1} \times M$ it becomes the composite

$$L^\pi_{\sigma(1)}.f_{\sigma(1)|\{1\}\times\mathbb{R}^{n-1}\times M} \circ L^\pi_{\sigma(2)}.f_{\sigma(2)|\{1\}\times\mathbb{R}^{n-1}\times M} \circ \cdots \circ L^\pi_{\sigma(j)}.f_{\sigma(j)|\{1\}\times\mathbb{R}^{n-1}\times M}$$

which is precisely

$$\kappa_j(L_1,\cdots,L_j,f_{1|\{1\}\times\mathbb{R}^{n-1}\times M},\cdots,f_{j|\{1\}\times\mathbb{R}^{n-1}\times M}).$$

Consider the $(n+1)$-cubes action on $\Omega\mathrm{EC}(n-1,M)$. Given i little $(n+1)$-cubes $L = (L_1,\cdots,L_i)$ let $L^\alpha = (L^\alpha_1,\cdots,L^\alpha_i) \in \mathcal{C}_1(1)^i$ be the projection on the 1st coordinate, and let $L^\beta = (L^\beta_1,\cdots,L^\beta_i) \in \mathcal{C}_j(1)^i$ be their projections on the remaining n coordinates. The $(n+1)$-cubes action on $\Omega\mathrm{EC}(n-1,M)$ is given by κ' defined below:

$$\kappa'_i(L_1,\cdots,L_i,f_1,\cdots,f_i) := \kappa_i(L^\beta_1,\cdots,L^\beta_i,L^\alpha_1.f_1,\cdots,L^\alpha_i.f_i) \quad (1)$$

$$= L^{\beta\pi}_{\sigma(1)}.L^\alpha_{\sigma(1)}.f_{\sigma(1)} \circ L^{\beta\pi}_{\sigma(2)}.L^\alpha_{\sigma(2)}.f_{\sigma(2)} \circ \cdots \circ L^{\beta\pi}_{\sigma(i)}.L^\alpha_{\sigma(i)}.f_{\sigma(i)} \quad (2)$$

$L^\alpha_i.f_i$ is the $\mathcal{C}_1$-action on $\Omega\mathrm{EC}(n-1,M)$ (reparametrisation in the loop-space coordinate) and L^β_i acts on this via the $\mathcal{C}_n$-action on $\mathrm{EC}(n-1,M)$. $\sigma \in \Sigma_i$ is any permutation such that $L^{\beta t}_{\sigma(1)} \leq L^{\beta t}_{\sigma(2)} \leq \cdots \leq L^{\beta t}_{\sigma(i)}$.

Consider applying the map gr_1:

$$\mathrm{gr}_1 : \Omega\mathrm{EC}(n-1,M) \ni F \longmapsto ((t_0,t,v) \longmapsto (t_0,F(t_0)(t,v))) \in \mathrm{EC}(n,M)$$

Observe that $\mathrm{gr}_1(L^{\beta\pi}_{\sigma(p)}.L^\alpha_{\sigma(p)}.f_{\sigma(p)}) = L^\pi_{\sigma(p)}.\mathrm{gr}_1(f_{\sigma(p)})$ thus

$$\mathrm{gr}_1(\kappa'_i(L_1,\cdots,L_i,f_1,\cdots,f_i)) = L^\pi_{\sigma(1)}.\mathrm{gr}_1(f_{\sigma(1)}) \circ L^\pi_{\sigma(2)}.\mathrm{gr}_1(f_{\sigma(2)}) \circ \cdots \circ L^\pi_{\sigma(i)}.\mathrm{gr}_1(f_{\sigma(i)}) \quad (3)$$

$$= \kappa_i(L_1,\cdots,L_i,\mathrm{gr}_1(f_1),\cdots,\mathrm{gr}_1(f_i)) \quad (4)$$

since gr_1 commutes with $\circ$. □

5 Survey

Much of this paper has been devoted to studying the map $\mathrm{gr}_1 : \Omega\mathcal{K}_{n-1,j-1} \to \mathcal{K}_{n,j}$ and the pseudoisotopy formalism for embedding spaces. This section is more survey in nature, mentioning what is known on the homotopy-type of the embedding spaces $\mathcal{K}_{n,j}$ and the properties of natural maps into and out of these spaces, focusing largely on the issues most closely related to iterated loop-space structures on these spaces and $\mathrm{EC}(j, D^{n-j})$.

Proposition 5.1 is a generalisation of the classical theorem that an embedding of S^1 in S^3 unknots in S^4. It is based loosely on the argument in Rolfsen's textbook [63]. The argument itself is likely much older.

Proposition 5.1 *The natural inclusion $\mathbb{R}^n \to \mathbb{R}^{n+1}$ induces an inclusion $i : \mathcal{K}_{n,1} \to \mathcal{K}_{n+1,1}$ which is null-homotopic.*

Proof Two null-homotopies of i will be constructed, giving a map $\mathcal{K}_{n,1} \to \Omega\mathcal{K}_{n+1,1}$.

Let $j_t : \mathcal{K}_{n,1} \to \mathcal{K}_{n,1}$ for $t \in \mathbf{I} = [-1, 1]$ be defined as $j_t(f)(x) = \frac{f((1+t^2)x - t^3) + (t^3, 0, \cdots, 0)}{1+t^2}$. j_0 is the identity, yet j_1 consists of knots which are standard outside of $[0, 1]$, and j_{-1} consists of knots which are standard outside of $[-1, 0]$.

Let $b : \mathbb{R} \to \mathbb{R}$ be a C^∞-smooth function with the properties that:

- $b(x) = 0$ for all $|x| \geq 1$.
- $b(x) = b(-x)$ for all $x \in \mathbb{R}$.
- $b'(x) > 0$ for all $-1 < x < 0$.

Let $B : \mathbb{R} \to \mathbb{R}^{n+1}$ satisfy $B(x) = (x, 0, \cdots, 0, b(x))$. Let $C : \mathbb{R} \to \mathbb{R}^{n+1}$ satisfy $C(x) = (x, 0, \cdots, 0, 0)$.

Given $f \in \mathcal{K}_{n,1}$, consider the function $F : \mathbf{I} \times \mathbb{R} \to \mathbb{R}^{n+1}$ defined as

$$F_t(x) = \begin{cases} i(j_{3t}(f))(x) & \text{for } |t| \in [0, \frac{1}{3}], x \in \mathbb{R} \\ (2 - 3|t|)\, i\left(j_{\frac{t}{|t|}}(f)\right)(x) + (3|t| - 1)B(x) & \text{for } |t| \in [\frac{1}{3}, \frac{2}{3}], x \in \mathbb{R} \\ (3 - 3|t|)B(x) + (3|t| - 2)C(x) & \text{for } |t| \in [\frac{2}{3}, 1], x \in \mathbb{R} \end{cases}$$

F, restricted to either $[0, 1] \times \mathbb{R}$ or $[-1, 0] \times \mathbb{R}$ is a null-homotopy of i. □

It is not known whether or not $F : \mathcal{K}_{n,1} \to \Omega\mathcal{K}_{n+1,1}$ is null-homotopic. The adjoint of F, $\Sigma\mathcal{K}_{n,1} \to \mathcal{K}_{n+1,1}$ is the direct-analogue of the 'Freudenthal suspension map for configuration spaces' [17] $\Sigma C_k\mathbb{R}^n \to C_k\mathbb{R}^{n+1}$ which is known to induce an isomorphism on the 1st non-trivial homology groups of the spaces provided $n > 1$. But in this case, first non-trivial homology group of $\Sigma\mathcal{K}_{n,1}$ is in dimension $2n - 5$, while for $\mathcal{K}_{n+1,1}$ it is in dimension $2n - 4$.

Using the same constructions, one can construct null-homotopies of the inclusions $\mathcal{K}_{n,j} \to \mathcal{K}_{n+j,j}$ for all $j > 0$.

Question 5.2
- *For each n and j, what is the smallest i such that inclusion $\mathcal{K}_{n,j} \to \mathcal{K}_{n+i,j}$ is null-homotopic?*
- *Is $F : \Sigma\mathcal{K}_{n,1} \to \mathcal{K}_{n+1,1}$ defined in Proposition 5.1 null-homotopic?*

- *If the answer to the previous question is positive, then does F have two distinct null-homotopies? Is there a 'Freudenthal suspension map' $\Sigma^2\mathcal{K}_{n,1} \to \mathcal{K}_{n+1,1}$ inducing an isomorphism of $H_{2n-4}\Sigma^2\mathcal{K}_{n,1}$ and $H_{2n-4}\mathcal{K}_{n+1,1}$?*

There is a 'fibrewise restriction' map $R : \mathcal{K}_{n,j} \to \Omega\mathcal{K}_{n,j-1}$, thinking of $\mathbb{R}^j$ as $\mathbb{R}\times\mathbb{R}^{j-1}$. If $2n-3j-3 \geq 0$ this map is exactly $(2n-3j-3)$-connected, as the first non-trivial homotopy groups of the two spaces are in different dimensions. These maps have been studied in some detail by Morlet and Goodwillie. The 'Morlet Disjunction Lemma' (see for example [22], page 9) is a theorem on the connectivity of this map in the context of arbitrary pseudoisotopy embedding spaces.

Proposition 5.3 *There is a homotopy-equivalence $\mathcal{K}_{n,n} \to \Omega\mathcal{K}_{n,n-1}$.*

Proof There are homotopy-equivalences $\mathcal{K}_{n,n} \simeq \mathrm{EC}(n,*)$ and $\mathcal{K}_{n,n-1} \simeq \mathrm{EC}(n-1,\mathbf{I})$ given by the fibrations in Proposition 2.4. Restriction to $\mathbb{R}^{n-1}\times\mathbf{I}$ gives a map $\mathrm{EC}(n,*) \to \mathrm{EC}(n-1,\mathbf{I})$ which is homotopic to a fibration, whose fibre has the homotopy-type of $\mathrm{EC}(n,*)^2$. The fibre-inclusion map $\mathrm{EC}(n,*)^2 \to \mathrm{EC}(n,*)$ is homotopic to multiplication in the group $\mathrm{EC}(n,*)$ (the homotopy is constructed via the $(n+1)$-cubes action on $\mathrm{EC}(n,*)$). Thus, the homotopy fibre of the map $\mathrm{EC}(n,*)^2 \to \mathrm{EC}(n,*)$ is $\mathrm{EC}(n,*)$. By Lemma 2.3, this homotopy-fibre has the homotopy-type of $\Omega\mathrm{EC}(n-1,\mathbf{I})$. □

The above argument is a mild variant of Hatcher's arguments where he gives various equivalent statements of the Smale conjecture [29]. A way to look at the above proposition is that studying the homotopy-type of the spaces $\mathrm{Emb}(S^{n-1},S^n)$ and $\mathrm{Diff}(S^n)$ ultimately reduces to studying the homotopy-types of the spaces $\mathcal{K}_{n,n-1}$ and $\mathcal{K}_{n,n}$. Since $\Omega\mathcal{K}_{n,n-1} \simeq \mathcal{K}_{n,n}$, the study of the homotopy-properties of these spaces is essentially identical modulo $\pi_0\mathcal{K}_{n,n-1} \simeq \pi_0\mathrm{Emb}(S^{n-1},S^n)$. The next result compiles the major theorems on $\pi_0\mathcal{K}_{n,n-1}$.

Theorem 5.4

- *[51, 6] If $f : S^{n-1} \to S^n$ is a smooth embedding, then $f(S^{n-1})$ bounds a topological disc.*
- *[76] The disc D^n has a unique smooth structure for $n \geq 6$.*
- *(Corollary of the above two results) If $f : S^{n-1} \to S^n$ is a smooth embedding, then $f(S^{n-1})$ bounds a smooth disc provided $n \geq 5$. Thus, $\mathrm{Emb}(S^{n-1},S^n)/\mathrm{Diff}(S^{n-1})$ is connected. See [43] for a modern account of the results in Smale's paper [76].*
- *For $n \in \{2,3\}$, $\mathrm{Emb}(S^{n-1},S^n)$ is known to be connected. For $n=2$ this is the Schoenflies theorem. See [71] for a historical account. For $n=3$ it is the combination of Alexander's theorem [2], and Smale's theorem [75].*
- *Whether or not $\mathrm{Emb}(S^3,S^4)$ is connected is called the smooth Schoenflies problem in dimension 4. Scharlemann [70] and Poenaru [61] have some partial results on this problem.*

Observe that an element of $\mathrm{Emb}(S^{n-1},S^n)$ is isotopic to the standard inclusion if and only if it extends to an embedding of D^n in S^n. The above observation that $\pi_0\mathrm{Emb}(S^{n-1},S^n)/\mathrm{Diff}(S^{n-1})$ is connected for $n \geq 5$ allows the extension of the long exact sequence from Theorem 2.1.

$$\cdots \to \pi_1\mathcal{K}_{n-1,n-1} \to \pi_0\mathcal{K}_{n,n} \to \pi_0\mathcal{P}_{n,n} \to \pi_0\mathcal{K}_{n-1,n-1} \to \pi_0\mathcal{K}_{n,n-1} \to 0$$

Thus, for $n \geq 5$ $\pi_0\mathcal{K}_{n,n-1}$ is isomorphic to the groups of homotopy n-spheres θ^n [43]. θ^n is known to be finite, and many of these groups have been computed, for example $\theta^5 = 0$, $\theta^6 = 0$, $\theta^7 \simeq \mathbb{Z}_{28}$, $\theta^8 \simeq \mathbb{Z}_2$, θ^9 is known to have 8 elements, θ^{10} is known to have 6 elements, $\theta^{11} \simeq \mathbb{Z}_{992}$.

Theorem 5.5 *[16] $\mathcal{P}_{n,n}$ is connected for $n \geq 6$. So there is an isomorphism of groups $\pi_0\mathrm{Diff}(D^{n-1}) \simeq \pi_0\mathrm{Emb}(S^{n-1}, S^n)$ and an epimorphism $\pi_1\mathrm{Diff}(D^{n-1}) \to \pi_0\mathrm{Diff}(D^n)$.*

A metric g on S^n is said to be round if for any points $x, y \in S^n$ there is an isometry of g carrying x to y which can also be chosen to send an orthonormal basis in T_xS^n to any orthonormal basis in T_yS^n. Let $\mathbb{M}^n$ denote the space of round Riemann metrics on S^n.

Proposition 5.6 *[29] $\mathbb{M}^n$ has the same homotopy-type as $\mathcal{K}_{n,n} \simeq \mathrm{Diff}(D^n)$.*

Proof There is a fibration $\mathbb{M}^n \to (0, \infty)$ given by taking the volume of the metric. The fibre of this map is a $\mathrm{Diff}^+(S^n)$-homogeneous space, with isotropy group SO_{n+1}. Theorem 2.1 tells us that $\mathcal{K}_{n,n} \simeq \mathrm{Diff}(D^n)$ is also the base-space of such a homotopy-fibre sequence $\mathrm{SO}_{n+1} \to \mathrm{Diff}^+(S^n) \to \mathrm{Diff}(D^n)$. □

Smale [75] and Hatcher [29] have proved that $\mathrm{Diff}(D^n)$ is contractible for $n = 2$ and $n = 3$ respectively. That $\mathrm{Diff}(D^1)$ is contractible follows from an averaging argument, or equivalently from the 'length' classification of connected closed 1-dimensional Riemann manifolds via Proposition 5.6. The space of Riemann metrics on S^n is contractible since it is an affine space, making the homotopy-type of $\mathrm{Diff}(D^n)$ the complete obstruction to $\mathbb{M}^n$ being a deformation-retract of the space of all Riemann metrics on S^n.

$\mathrm{Diff}(D^n)$ is an $(n+1)$-fold loop space [7, 56, 12] whose $(n+1)$-fold delooping is $PL(n)/O_n$ [12, 56]. As of yet, their does not appear to be any direct methods of studying the homotopy-type of $PL(n)$. In particular, essentially nothing is known about the homotopy-type of $\mathrm{Diff}(D^4)$. Farrell and Hsiang computed the rational homotopy of $\mathrm{Diff}(D^n)$ in a range.

Theorem 5.7 *[20] Provided $0 \leq i < \min\{\frac{n-4}{3}, \frac{n-7}{2}\}$*

$$\pi_i\mathrm{Diff}(D^n) \otimes \mathbb{Q} \simeq \begin{cases} \mathbb{Q} & \textit{provided } 4|(i+1) \\ 0 & \textit{otherwise} \end{cases}$$

The bound $i < \min\{\frac{n-4}{3}, \frac{n-7}{2}\}$ is known as Igusa's stable range [36]. Roughly this the range where $\pi_i\mathcal{P}_{n,n}$ can be related to K-theory. Antonelli, Burghelea and Khan had shown earlier that $H_*\mathrm{Diff}(D^n)$ is not finitely-generated for $n \geq 7$ [4].

The spaces $\mathcal{K}_{n+2,n}$ are in the realm of 'traditional' co-dimension 2 knot theory, on which there is a plethora of literature. The majority of the literature focuses on issues related to isotopy classification, ie: $\pi_0\mathcal{K}_{n+2,n}$. Some good general references are Kawauchi [37], Hillman [32], Ranicki [62] and Kervaire-Weber [41].

The homotopy-type of $\mathcal{K}_{3,1}$ is described, component-by-component, as an iterated fibre bundle.

Theorem 5.8 *[30, 31, 8, 11, 7] Given a long knot $f \in \mathcal{K}_{3,1}$, let $\mathcal{K}_{3,1}(f)$ denote the path component in $\mathcal{K}_{3,1}$ containing f. Then $\mathcal{K}_{3,1}(f)$ has the homotopy-type of:*

(1) $\{*\}$ *if* f *is the unknot.*

(2) $S^1 \times \mathcal{K}_{3,1}(g)$ *if* f *is a cable of* g.

(3) $C_n(\mathbb{R}^2) \times_{\Sigma_f} \prod_{i=1}^n \mathcal{K}_{3,1}(f_i)$ *if* $f = f_1 \# \cdots \# f_n$ *is the prime decomposition of* f, *with* $n \geq 2$. Σ_f *is the subgroup of* Σ_n *corresponding to the partition of* $\{1, 2, \cdots, n\}$ *defined by the equivalence relation* $i \sim j$ *if and only if* $\mathcal{K}_{3,1}(f_i) = \mathcal{K}_{3,1}(f_j)$.

(4) $S^1 \times \left(SO_2 \times_{A_f} \prod_{i=1}^n \mathcal{K}_{3,1}(f_i)\right)$ *if* $f = (f_1, \cdots, f_n) \bowtie L$ *is hyperbolically-spliced. Here* L *is some hyperbolic link* $L = (L_0, L_1, \cdots, L_n)$ *in* S^3 *with the* L_0 *component 'long'. Define* B_L *to be the group of orientation-preserving hyperbolic isometries of* $S^3 \setminus L$ *which extend to* L, *preserving* L_0 *and its orientation.* $B_L \to \mathrm{Diff}(S^3, L_0)$ *is a faithful representation, giving an embedding of* B_L *in* $\mathrm{Diff}(L_0)$ *(thus conjugate to a subgroup of* SO_2*). Similarly, there is a homomorphism* $B_L \to \pi_0 \mathrm{Diff}(L_1 \cup \cdots \cup L_n) \equiv \Sigma_n^+$ *the signed symmetric group of* $\{1, 2, \cdots, n\}$. Σ_n^+ *acts on* $\mathcal{K}_{3,1}^n$ *by permutation of the factors and knot inversion. Let* A_f *be the subgroup of* $B_L \subset \Sigma_n^+$ *that preserves* $\prod_{i=1}^n \mathcal{K}_{3,1}(f_i)$.

Case (2) above is considered to apply to torus knots – think of a torus knot as a cable of the unknot, thus the component of a torus knot has the homotopy-type of S^1. A hyperbolic knot is thought of as a hyperbolically-spliced knot where L is a 1-component hyperbolic link, thus such a component has the homotopy-type of $S^1 \times S^1$. Since every knot can be obtained from the unknot by iterated cabling, connect-sum and hyperbolic splicing operations [11], the above result describes the homotopy-type of $\mathcal{K}_{3,1}(f)$ for any $f \in \mathcal{K}_{3,1}$. To be clear, if the knot f has j tori in the JSJ-decomposition of its complement, to obtain an answer for the homotopy-type of $\mathcal{K}_{3,1}(f)$, one would have to apply Theorem 5.8 $j+1$ times. A detailed justification for the above theorem is given in the reference [8]. The homotopy-equivalence in part (3) of Theorem 5.8 is induced by the action of the operad of 2-cubes on $\mathcal{K}_{3,1}$. Another way to state (3) is that $\mathcal{K}_{3,1}$ is a free 2-cubes object, with generating space $\mathcal{P} \sqcup \{*\}$, $\mathcal{P} \subset \mathcal{K}_{3,1}$ the space of prime long knots. By the work of May [49], the group-completion $\Omega B\mathcal{K}_{3,1}$ of the knot space has a particularly simple structure, $\Omega B\mathcal{K}_{3,1} \simeq \Omega^2\Sigma^2(\mathcal{P} \sqcup \{*\})$. Fred Cohen and the author have used these results to compute the homology of many components of $\mathcal{K}_{3,1}$ [10]. In the process it became clear that the homotopy-type and homology of $\mathcal{K}_{3,1}$ would likely have a more elegant description if one could prove that $\mathcal{K}_{3,1}$ had an action of the operad of framed little 2-discs.

Question 5.9 *Can one define an action of the operad of framed* $(n+1)$*-discs on the spaces* $\mathrm{ED}(n, D^k)$, *in a 'natural geometric manner' similar to Definition 4.3?* $\mathrm{ED}(n, D^k)$ *refers to the comments preceding Definition 4.6.*

The topic of $\pi_0\mathcal{K}_{4,2}$ has a few new references. Carter and Saito have constructed an analogue of Reidermeister theory [14]. Kamada has constructed an analogue of the Alexander-Markov theorem from dimension 3 [38]. It is possible that there are other types of Alexander-Markov theorems in dimension four. For example, at present not known if every element of $\pi_0\mathcal{K}_{4,2}$ is Litherland spun. As an additional advertisement for Litherland spinning, a statement of the Zeeman-Litherland theorem is given.

Theorem 5.10 *[88, 47] (Zeeman-Litherland Theorem) Let* $g \in \Omega\mathcal{K}_{n+2,n}(f)$ *be such that* $\tilde{g} \in \pi_0\mathrm{Diff}(\mathbf{I}^{n+2}, f)$ *preserves a Seifert surface for* f. *Let* $G \in \pi_0\mathrm{Diff}(\mathbf{I}^{n+2}, f)$ *denote the Gramain element (a meridional Dehn twist). If* $k \in \mathbb{Z} \setminus \{0\}$ *then the complement of* $\mathrm{gr}_1(G^k g) \in \mathcal{K}_{n+3,n+1}$ *fibres over* S^1.

For $n = 1$ Litherland went on to identify the fibre in several cases. From a practical point of view, the Zeeman-Litherland theorem is a useful tool for constructing embeddings of 3-manifolds into S^4, as fibres of fibred knot complements [65]. The possible types of Litherland-spun knots is parametrised by $\pi_0 L\mathcal{K}_{3,1}$. By the results in [7, 8], the group $\pi_1\mathcal{K}_{3,1}(f)$ can be computed directly from the JSJ-decomposition of C_f, perhaps allowing one to answer the questions:

Question 5.11 • *Does every 2-knot in S^4 have the Alexander polynomial of a Litherland-spun knot?*

- *Is $\mathrm{gr}_1 : \pi_0 L\mathcal{K}_{3,1} \to \pi_0\mathcal{K}_{4,2}$ onto?*

Up to a homotopy-equivalence, the spaces $\mathrm{ED}(j, D^{n-j})$ and $\mathrm{EC}(j, D^{n-j})$ admit an action of the operad of framed little $(j+1)$-discs, provided $n - j > 2$. This is because they are $(j+1)$-fold loop spaces. This argument does not apply when $n - j = 2$ since $\pi_0\mathrm{EC}(n, D^2)$ is never a group. This will be explained in the next proposition.

Proposition 5.12 • *$\pi_0\mathcal{K}_{n+2,n}$ is not a group for all $n \geq 1$.*

- *The map $\pi_0\mathcal{K}_{n+1,n} \to \pi_0\mathcal{K}_{n+2,n}$ induced by inclusion $\mathbb{R}^{n+1} \to \mathbb{R}^{n+2}$ is injective and maps onto the maximal subgroup of $\pi_0\mathcal{K}_{n+2,n}$ provided $n \geq 4$.*

Proof To prove the first point, non-invertible elements are constructed. Start with $f_1 \in \mathcal{K}_{3,1}$ a trefoil knot. Then $\pi_1 C_f$ is the braid group on 3 strands. Let $g_1 = 0 \in \pi_1\mathcal{K}_{3,1}(f_1)$ be the constant loop, and observe that the complement of $f_2 = \mathrm{gr}_1(g_1) \in \mathcal{K}_{4,2}$ also has the braid group on 3 strands as its fundamental group. Continuing, this constructs for all $n \geq 1$ a knot $f_n \in \mathcal{K}_{n+2,n}$ whose complement has the braid group on 3 strands as its fundamental group. f_n is non-invertible in the monoid $\pi_0\mathcal{K}_{n+2,n}$ by Proposition 2.3.4 of [86]. This is because if $h \in \mathcal{K}_{n+2,n}$ then the complement of the connect-sum $f_n\#h$, $C_{f_n\#h}$ has the homotopy-type of the union of C_{f_n} and C_h where C_{f_n} and C_h intersect along a meridional circle, so by the canonical form for amalgamated free products, $\pi_1 C_{f_n\#h}$ contains $\pi_1 C_{f_n}$.

By the above argument, if $f \in \pi_0\mathcal{K}_{n+2,n}$ is invertible, $\pi_1 C_f \simeq \mathbb{Z}$. By a Mayer-Vietoris sequence argument, $H_i C_f = 0$ for all $i > 1$. Thus, C_f has the homotopy-type of a circle. By Levigne's unknotting theorem [46] (provided $n \geq 4$) or Wall's unknotting theorem [82] (for $n = 3$), f is in the image of $\pi_0\mathcal{K}_{n+1,n}$.

The last item to prove is that the map $\pi_0\mathcal{K}_{n+1,n} \to \pi_0\mathcal{K}_{n+2,n}$ is injective. Consider $S^n \subset S^{n+1} \subset S^{n+2}$. Let $f : S^n \to S^{n+2}$ be an embedding with $f(S^n) = S^n$. By Theorem 2.1 we could equivalently prove that if f extends to an embedding $F : D^{n+1} \to S^{n+2}$, then there is another extension of f, $F' : D^{n+1} \to S^{n+1}$. Identify the complement of an open tubular neighbourhood of S^n in S^{n+2} with $S^1 \times D^{n+1}$. Thus, F, if it exists, is an embedding $F : D^{n+1} \to S^1 \times D^{n+1}$ such that $F(\partial D^{n+1}) = \{1\} \times \partial D^{n+1}$. By Farrell's proof of the relative Browder-Livesay-Leving-Farrell fibration theorem [21], there is a diffeomorphism $G : S^1 \times D^{n+1} \to S^1 \times D^{n+1}$ such that $G(F(D^{n+1})) = \{1\} \times D^{n+1}$ and $G_{|S^1\times\partial D^{n+1}}$ is the identity on $S^1 \times \partial D^{n+1}$. Farrell's theorem requires $n \geq 4$. □

I would like to thank Larry Siebenmann for suggesting the Browder-Livesay-Leving-Farrell fibration theorem.

The above proposition implies that $\mathrm{EC}(n, D^2)$ is not a free $(n+1)$-cubes object provided there exists exotic $(n+1)$-spheres, so no direct analogue of [7] is true in high dimensions. Of course, $\mathrm{EC}(1, D^2)$ is not a free object, either, as it splits as a product of $\mathbb{Z}$ with the free object $\mathcal{K}_{3,1}$. One might hope that for $n > 1$, $\mathrm{EC}(n, D^2) \simeq \mathcal{K}_{n+2,n}$ is closely related to a free $(n+1)$-cubes object, but there are yet further obstructions. Provided $n \geq 3$, $\pi_0\mathcal{K}_{n+2,n}/\pi_0\mathcal{K}_{n+1,n}$ (this is the isotopy classes of the images of the elements of $\mathcal{K}_{n+2,n}$) is not a free commutative monoid. Kearton proved this in the $n = 3$ case, which has since been generalised to all $n \geq 3$. Bayer-Fluckiger went on to prove the non-existence of a 'cancellation law' ie: one can satisfy $a + b = a + c$ with $b \neq c$. See Kearton's survey [39] for details.

Question 5.13

- *What is the group-completion of the monoid $\pi_0\mathcal{K}_{n+2,n}$?*
- *Can one characterise the monoid structure on $\pi_0\mathcal{K}_{n+2,n}$ for $n \geq 2$?*
- *If $f \in \mathcal{K}_{n+2,n}$ is a connect-sum of two non-trivial knots, the action of the operad of $(n+1)$-cubes on $\mathcal{K}_{n+2,n}$ gives a map $S^n \to \mathcal{K}_{n+2,n}(f)$. Is this map a non-trivial element of $\pi_n\mathcal{K}_{n+2,n}(f)$?*

The remainder of the survey will focus on the high co-dimension case: $\mathcal{K}_{n,j}$ for $n - j > 2$. For references, Adachi's survey has been around for a few years [1]. It focuses on topics such as the Whitney trick, and the Smale-Hirsch immersion theorem. Skopenkov has a recent survey article [74] which is concerned with $\pi_0\mathcal{K}_{n,j}$. Goodwillie, Klein and Weiss have recently put put together a survey of what is known about embedding spaces from the point of view of disjunction [23].

There have been computations of some of the groups $\pi_0\mathcal{K}_{n,j}$. From Proposition 3.9, the first non-trivial homotopy-group of $\mathcal{K}_{n,j}$ is in dimension $2n - 3j - 3$ (provided $2n - 3j - 3 \geq 0$). Along the $2n - 3j - 3 = 0$ line there is $\pi_0\mathcal{K}_{3,1}$ which is the free commutative monoid on $\pi_0\mathcal{P}$, the isotopy-classes of prime long knots [68]. Provided $j > 1$ and $2n - 3j - 3 = 0$, there are Haefliger's computations [27]:

$$\pi_0\mathcal{K}_{n,j} \simeq \begin{cases} \mathbb{Z} & j \equiv 3 (\mathrm{mod}\ 4) \\ \mathbb{Z}_2 & j \equiv 1 (\mathrm{mod}\ 4) \end{cases}$$

The generator being Haefliger's Borromean rings construction [26], also sometimes called the 'trefoil' [74]. The generator has also been described (Theorem 3.13) as an iterated graphing construction applied to r, the resolution of an immersion of $\mathbb{R}$ in Euclidean space, corresponding to the $\bigotimes$ chord-diagram. More recently, another spinning construction involving r has recently been developed by Roseman and Takase [64].

The work of Haefliger [27], Milgram [53], Kreck and Skopenkov [44] gives $\pi_0\mathcal{K}_{n,j}$ along the $n - j > 2$ part of the $2n - 3j - 3 = -1$ line. Their computations are:

$$\pi_0\mathcal{K}_{n,j} \simeq \begin{cases} 0 & j \equiv 2 \text{ or } 6 (\mathrm{mod}\ 4) \\ \mathbb{Z}_{12} & (n,j) = (7,4) \\ \mathbb{Z}_4 & j \equiv 4 (\mathrm{mod}\ 8), j \geq 12 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & j \equiv 0 (\mathrm{mod}\ 8) \end{cases}$$

The above results give the next corollary as a direct analogue to Theorem 3.9.

Corollary 5.14

- *$\pi_{6n}\mathcal{K}_{3n+4,2}$ is non-trivial and has $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ as a quotient for all $n \geq 1$.*
- *$\pi_{6n+2}\mathcal{K}_{3n+5,2}$ is non-trivial and has $\mathbb{Z}_4$ as a quotient for all $n \geq 0$ ($\mathbb{Z}_{12}$ for $n = 0$).*

Question 5.15 *What is the structure of the groups $\pi_2\mathcal{K}_{5,2}$ and $\pi_6\mathcal{K}_{7,2}$. Further, find explicit geometric representatives for the embeddings, in analogy to Theorem 3.13.*

The technique of Haefliger [27] involves two main steps. The first step is the construction of an isomorphism $\pi_0\mathcal{K}_{n,j} \simeq C_j^{n-j}$ where C_j^{n-j} is the group of concordance classes of embeddings of S^j in S^n. This step is formally analogous to Proposition 3.1. Using a Thom-type construction, Haefliger constructs an isomorphism between C_j^{n-j} and a multi-relative homotopy group $C_j^n \simeq \pi_{j+1}(G; SO, G_{n-j})$ where $SO = \varinjlim(\mathrm{SO}_1 \to \mathrm{SO}_2 \to \mathrm{SO}_3 \to \cdots)$ is the stable special-orthogonal group, G_n is the space of degree 1 self-maps of S^{n-1}, with G the analogous stable object, defined via suspensions $G = \varinjlim(G_1 \to G_2 \to G_3 \to \cdots)$. This reduces the computation of $\pi_0\mathcal{K}_{n,j}$ to rather traditional difficult problems common to surgery theory [62]: homotopy groups of spheres and orthogonal groups.

Takase [77] has recently proved that any embedding of $S^{4k-1} \to S^{6k}$ can be extended to an embedding of $(S^{2k} \times S^{2k}) \setminus D^{4k} \to S^{6k}$. Takase gives a rather explicit formula for determining the isotopy class of an element of $\mathrm{Emb}(S^{4k-1}, S^{6k})$ that simplifies Haefliger's characteristic class computations [26].

The work of Volic, Lambrechts and Turchin [45] gives the homology $H_*(\mathcal{K}_{n,1}; \mathbb{Q})$ for $n \geq 4$ as the homology of a differential graded algebra, by showing the collapse of the rational Vassiliev spectral sequence. Turchin has found a Poisson algebra structure for this DGA [79, 78], which motivated the author's construction of the 2-cubes action on $\mathcal{K}_{3,1}$. Salvatore [67], building on the work of Sinha [73] has recently constructed a 2-cubes action on $\mathcal{K}_{n,1}$ for $n \geq 4$. The structure of $\mathcal{K}_{n,1}$ and $\mathrm{EC}(1, D^{n-1})$ as 2-cubes objects for $n \geq 4$ remains mysterious. One would hope that constructions having the flavour of Mostovoy's [57] 'short rope' spaces, or Anderson and Hsiang's 'bounded embedding spaces' [3] could give useful geometric models that one could use to get homotopy-theoretic information on $B^j\mathcal{K}_{n,j}$, $B^2\mathcal{K}_{n,1}$, $B^{j+1}\mathrm{EC}(j, M)$. Not only is there a lack of proofs that these spaces are the appropriate iterated classifying spaces, but, even if they were, its not clear how one could use such results to study the spaces $\mathcal{K}_{n,j}$.

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