On normal Darboux coordinates

Boris Fedosov

Dedicated to Professor M.I. Vishik on the occasion of his eightieth birthday

Abstract. For a given symplectic connection we try to find Darboux local coordinates for which connection coefficients take the simplest form, the so-called normal Darboux coordinates. To this end we introduce a nonlinear Cayley transform which gives a convenient parametrization of local symplectomorphisms and prove the existence and uniqueness of such coordinates on the level of jets. At the same time we construct a counterexample showing non-correctness of the problem of finding normal Darboux coordinates.

1. Introduction

Recall the notion of normal local coordinates on a manifold with an affine connection. Let M be a smooth manifold with an affine torsion-free connection ∂ and $x^1, x^2, \ldots, x^n$ be local coordinates with the origin $x_0 \in M$. We call them *normal* or *geodesic coordinates* with respect to the connection ∂ if the following identity holds

$$\Gamma^i_{jk}(x)x^j x^k \equiv 0 \tag{1.1}$$

where Γ^i_{jk} are connection coefficients (Christoffel symbols) of ∂ in a given coordinate chart. This identity has an invariant meaning: the rays emanating from the origin are geodesics, that is

$$\partial_{\dot{x}(t)}\dot{x}(t) = \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0. \tag{1.2}$$

Here $x^i(t) = x^i t,\ t \in [0, \infty)$ is a ray through the point $x = \{x^1, x^2, \ldots, x^n\} \neq x_0$. Thus, for the rays we have $\ddot{x}^i = 0,\ \dot{x}^i = x^i$, so that (1.2) reduces to (1.1).

The theory of ordinary differential equations applied to (1.2) gives the following existence theorem.

Theorem 1.1. *For any $x_0 \in M$ there exists a neighborhood O and a normal local coordinate system on O with the origin x_0. This system is unique up to linear change of variables.*

Thus, at any point $x_0 \in M$ we have essentially unique local coordinate system for which connection coefficients take the simplest form (1.1). Normal coordinates play an important role in local differential-geometrical computations.

2000 *Mathematics Subject Classification.* 53D05, 53D22.

The object of our interest in this paper is a symplectic connection ∂ on a symplectic manifold (M, ω). A detailed treatment of symplectic connections may be found in [**GRS**]. We consider Darboux local coordinates $x^1, x^2, \dots, x^{2n}$ on M for which the coefficients of the symplectic form $\omega = 1/2\omega_{ij}dx^i \wedge dx^j$ are constant. It is known (see, e.g. [**GRS**]) that the connection coefficients

$$\Gamma_{ijk} = \omega_{im}\Gamma^m_{jk}$$

are completely symmetric in i, j, k. Indeed, the symmetry in j, k follows from the vanishing of the torsion and the symmetry in i, k reflects the fact that ∂ preserves the tensor ω_{ij}. We are looking for the local Darboux coordinates for which the coefficients $\Gamma_{ijk}(x)$ have the simplest form similar to (1.1).

Definition 1.2. Local Darboux coordinates are called *normal* if the following identity holds

$$\Gamma_{ijk}(x)x^i x^j x^k \equiv 0. \tag{1.3}$$

To avoid confusion, we use the name *geodesic* coordinates for those satisfying (1.1) reserving the name *normal* Darboux coordinates for those satisfying (1.3). Note, that geodesic coordinates for a given symplectic connection are not Darboux ones in general, so they are not normal coordinates in the sense of Definition 1.2, although they satisfy (1.3).

By analogy with Theorem 1.1 we formulate the following conjecture.

Conjecture 1.3. *For a given symplectic connection ∂ and any $x_0 \in M$ there exists a local normal Darboux coordinate system with the origin at x_0. It is unique up to linear symplectic change of variables.*

In [**F**, Chapter 2] a weaker version was proved.

Theorem 1.4. *For any $x_0 \in M$ there exists a local Darboux coordinate system with the origin at x_0 such that the function (1.3) has a zero of infinite order at the origin, that is*

$$\Gamma_{ijk}(x)x^i x^j x^k = O(|x|^\infty). \tag{1.4}$$

Any two such systems differ by a linear symplectic change of variables up to infinite order. More precisely, if $x_1 = \{x_1^1, x_1^2, \dots, x_1^{2n}\}$ and $x_2 = \{x_2^1, x_2^2, \dots, x_2^{2n}\}$ are such two systems, then there exists a matrix $U \in Sp(2n)$ such that

$$x_2 - Ux_1 = O(|x_1|^\infty).$$

We will use the term *week normal Darboux coordinates* if they satisfy (1.4). Theorem 1.4 means that the conjecture is true on the level of formal Taylor series. For real analytic data this theorem gives a real analytic symplectic transformation to normal Darboux coordinates, provided the formal series are convergent. So, our conjecture resembles the Newlander-Nirenberg theorem on integrable almost complex structures [**NN**].

The notion of normal Darboux coordinates was introduced in the book [**F**] for the purposes of deformation quantization. For these applications Theorem 1.1 was sufficient. But the conjecture remains a challenging problem which I tried to solve many times in the affirmative but in vain. Each time it became more evident that the conjecture should be false.

The main purpose of this paper is to present a counterexample. More precisely, we show that the linearized problem has no smooth solution in any neighborhood of a given point x_0. Strictly speaking, this is not a direct counterexample for the Conjecture 1.3. Our example means that the solution (if exists) has no differential with respect to parameters. In other words, the problem of finding normal Darboux coordinates is non-correct.

Our considerations are purely local, so we deal with the symplectic space $\mathbb{R}^{2n}$ with a standard symplectic form $\omega = 1/2\omega_{ij}dx^i \wedge dx^j$, $\omega_{ij} = const$. The main technical tool is a special parametrization of local symplectomorphisms which we call *non-linear Cayley transform.* Its description is given is Section 2. In Section 3 we use it to prove Theorem 1.4 and in Section 4 give a standard application of this theorem. All this is a necessary background to the last Section 5 where the counterexample is constructed.

I would like to thank A.Weinstein for a stimulating discussion. In particular, he suggested to apply the so-called Moser trick to prove the conjecture. Having thought this application over, I came to a guess that the conjecture might be false.

2. Non-linear Cayley transform

The Cayley transform

$$U = (1 - A)(1 + A)^{-1}; \quad A = (1 - U)(1 + U)^{-1} \tag{2.1}$$

gives a convenient rational parametrization of symplectic matrices $U \in Sp(2n)$ by means of infinitesimally symplectic matrices $A \in sp(2n)$. It is defined, in particular, for A sufficiently close to the identity matrix 1 and gives a one-to-one correspondence between a neighborhood of 0 in $sp(2n)$ and a neighborhood of 1 in $Sp(2n)$. More detail on Cayley transform can be found in the book [**W**]. Since infinitesimally symplectic matrix $A = (a^k_j)$ corresponds to a symmetric matrix

$$S = (s_{ij}) = (\omega_{ik}a^k_j),$$

the Cayley transform may be viewed as a correspondence between quadratic forms $S = 1/2s_{ij}x^ix^j$ (we denote the quadratic form and the corresponding symmetric matrix by the same letters) and linear symplectic transformations $U \in Sp(2n)$.

Consider now non-linear symplectomorphisms $y = f(x)$ of a neighborhood of $0 \in \mathbb{R}^{2n}$ which are sufficiently close to the identity map. We always assume that $f(0) = 0$. Introduce new variables

$$z = \frac{x + y}{2} = \frac{x + f(x)}{2} \tag{2.2}$$

and consider a one-form

$$\omega\left(\frac{y - x}{2}, dz\right) = \frac{1}{4}\omega_{ij}(y - x)^i d(x + y)^j \tag{2.3}$$

Lemma 2.1. *The form (2.3) is closed.*

Proof. The differential is proportional to

$$\omega(dy - dx, dx + dy) = \omega(dy, dy) - \omega(dx, dx) + \omega(dy, dx) - \omega(dx, dy).$$

The first two summands on the right give 0 since ω is preserved by symplectomorphisms. The last two summands also give 0 because $\omega(dy, dx) = \omega(dx, dy)$. Indeed, transposing dy and dx, we have to change the sign twice: because ω is skew-symmetric and because exterior differentials anticommute. ∎

We introduce a generating function $S(z)$ of the symplectomorphism f by the identity

$$dS(z) = \omega\left(\frac{y-x}{2}, dz\right) \tag{2.4}$$

or

$$\frac{x-y}{2} = \nabla S(z) \tag{2.5}$$

where ∇S means the symplectic gradient of S, that is, the vector with coordinates

$$\nabla^i S = \omega^{ij}\frac{\partial S}{\partial z^j}.$$

From (2.2) and (2.5) we obtain

$$x = z + \nabla S(z), \quad y = z - \nabla S(z) \tag{2.6}$$

which gives a parametrization of symplectomorphisms f by generating functions $S(z)$. The correspondence is one-to-one if we set $S(0) = 0$. Of course, it is defined if x can be expressed from (2.2) as an implicit function of z (respectively, z can be expressed from the first equation (2.6) as an implicit function of x). In this sense f must be sufficiently close to the identity map or $S(z)$ must be sufficiently small. Note that the condition $f(0) = 0$ implies that $\nabla S = 0$, so S has a second order zero at the origin. It is easy to see that for quadratic forms $S = 1/2 s_{ij} z^i z^j$ we obtain the linear Cayley transform (2.1). Obviously, we also have a linear Cayley transform between the differential df of a symplectomorphism and the second differential $d^2 S$ of the generating function. For these reasons we call the correspondence given by (2.6) the *non-linear Cayley transform* and the function $S(z)$ the *Cayley generating function.*

Generating functions of symplectomorphisms have a wide field of applications, e.g., in the theory of Fourier integral operators (FIO). Yet our version differs from those used in FIO. Indeed, the latter require a separation of variables into coordinates and momenta, while in our version all the variables are treated on equal footing. This drastically simplifies the formulas making them more manageable.

We finish this section by indicating the relations between Hamiltonian flows and Cayley generating functions. Let $f_t(x)$ be a Hamiltonian flow on $\mathbb{R}^{2n}$ generated by a Hamiltonian vector field

$$X^i(x,t) = \omega^{ij}\frac{\partial H(x,t)}{\partial x^j} = \nabla^i H.$$

LEMMA 2.2. *The generating function $S_T(z)$ of the symplectomorphism $f_T(x)$ at a point*

$$z = \frac{x + f_T(x)}{2}$$

is equal to the action along the trajectory $f_t(x)$, $t \in [0,T]$, more precisely,

$$S_T(z) = \frac{1}{2}\int_0^T \left\{\frac{1}{2}\omega(f_t(x) - x, \dot f_t(x)) - H(f_t(x), t)\right\} dt. \tag{2.7}$$

PROOF. Fixing T and differentiating (2.7), we obtain

$$
\begin{aligned}
dS_T &= \frac{1}{2}\int_0^T \left\{ \frac{1}{2}\omega(f_t(x)-x, d\dot{f}_t,) + \frac{1}{2}\omega(df_t - dx, \dot{f}_t(x)) - \frac{\partial H}{\partial x^i} df_t^i \right\} dt \\
&= \frac{1}{4}\omega\ (f_t - x, df_t)|_0^T + \frac{1}{2}\int_0^T \left\{ -\frac{1}{2}\omega(dx, \dot{f}_t,) + \omega(df_t, \dot{f}_t) - \frac{\partial H}{\partial x^i} df_t^i \right\} dt \\
&= \frac{1}{4}\omega(f_T - x, df_T) + \frac{1}{4}\omega(f_T - x, dx).
\end{aligned}
$$

The last two terms under the integral sign vanish since

$$\dot{f}^i = \omega^{ij}\frac{\partial H}{\partial x^j},$$

so that

$$dS_T(z) = \frac{1}{4}\omega(f_T(x) - x, d(f_T(x) + x))$$

which coincides with (2.4). □

Finally, the Hamilton-Jacobi equation for the action $S_T(z)$ takes the following form.

LEMMA 2.3. *As the function of z and T, $S_T(z)$ satisfies the equation*

$$\frac{\partial S_T}{\partial T} + \frac{1}{2}H(z - \nabla S_T(z), T) = 0. \tag{2.8}$$

PROOF. Taking x, T as independent variables, we have

$$z = \frac{x + f_T(x)}{2}.$$

Thus,

$$\frac{dS_T}{dT} = \frac{\partial S_T}{\partial T} + \frac{\partial S_T}{\partial z^i}\frac{\dot{f}_T}{2},$$

and, substituting $\partial S_T/\partial z^i$ from (2.4), we can write further

$$\frac{dS_T}{dT} = \frac{\partial S_T}{\partial T} + \frac{1}{4}\omega(f_T - x, \dot{f}_T).$$

On the other hand, it follows from (2.7)

$$\frac{dS_T}{dT} = \frac{1}{4}\omega(f_T - x, \dot{f}_T) - \frac{1}{2}H(f_T, T).$$

Equating these two expressions for dS_T/dT, we obtain

$$\frac{\partial S_T}{\partial T} + \frac{1}{2}H(f_T, T) = 0,$$

and Equation (2.8) follows since

$$f_T = z - \nabla S_T(z).$$

□

3. Cayley generating function for normal Darboux coordinates

Consider two Darboux charts $x_1 = \{x_1^1, x_1^2, \dots, x_1^{2n}\}$ and $x_2 = \{x_2^1, x_2^2, \dots, x_2^{2n}\}$ with the same origin. They are connected by a symplectomorphism $x_2 = f(x_1)$ such that $f(0) = 0$. We will look for symplectomorphisms from a given Darboux coordinate system x_1 to a normal Darboux coordinate system x_2. First of all observe that the Equation (1.3) is invariant under linear symplectomorphisms. This means that we can normalize f by the condition $df(0) = 1$. Under this normalization f may be defined in a neighborhood of the origin by means of a generating function $S(z)$ such that

$$S(0) = 0, \quad dS(0) = 0, \quad d^2S(0) = 0,$$

that is, $S(z)$ has the third-order zero at the origin. We are going to derive a differential equation for $S(z)$.

Let $\Gamma^1_{ijk}(x_1)$, $\Gamma^2_{ijk}(x_2)$ be connection coefficients of a symplectic connection ∂ in Darboux coordinates x_1 and x_2. Set

$$z = \frac{x_1 + x_2}{2} = \frac{x_1 + f(x_1)}{2}.$$

Since $df(0) = 1$, this equation defines x_1 as an implicit function $x_1 = x_1(z)$ and $x_2 = x_2(z) = 2z - x_1(z)$.

PROPOSITION 3.1. *The following transition formula holds*

$$\Gamma^2_{ijk}(x_2)\frac{\partial x_2^i}{\partial z^p}\frac{\partial x_2^j}{\partial z^q}\frac{\partial x_2^k}{\partial z^r} = \Gamma^1_{ijk}(x_1)\frac{\partial x_1^i}{\partial z^p}\frac{\partial x_1^j}{\partial z^q}\frac{\partial x_1^k}{\partial z^r} + 2\frac{\partial^3 S}{\partial z^p \partial z^q \partial z^r}. \tag{3.1}$$

PROOF. We start with the usual transition formula for connection coefficients from coordinates x_1 or x_2 (denoted simply x for short) to coordinates z. Let $\Gamma^i_{jk}(x)$ and $\widetilde{\Gamma}^i_{jk}(z)$ be corresponding connection coefficients. Then

$$\frac{\partial x^l}{\partial z^m}\widetilde{\Gamma}^m_{jk}(z) = \Gamma^l_{qr}(x)\frac{\partial x^q}{\partial z^j}\frac{\partial x^r}{\partial z^k} + \frac{\partial^2 x^l}{\partial z^j \partial z^k}.$$

Multiplying by $\partial x^p/\partial z^i \omega_{pl}$ and using the relation

$$\frac{\partial x^p}{\partial z^i}\omega_{pl}\frac{\partial x^l}{\partial z^m} = \omega_{im}(z)$$

for coefficients of the symplectic form, we obtain

$$\widetilde{\Gamma}_{ijk}(z) = \Gamma_{pqr}(x)\frac{\partial x^p}{\partial z^i}\frac{\partial x^q}{\partial z^j}\frac{\partial x^r}{\partial z^k} + \omega_{pl}\frac{\partial x^p}{\partial z^i}\frac{\partial^2 x^l}{\partial z^j \partial z^k}.$$

Let us write this formula taking $x = x_1$ or $x = x_2$. Then by (2.6) $x_{1,2} = z \pm \nabla S$, so

$$\frac{\partial^2 x^p_{1,2}}{\partial z^j \partial z^k} = \pm \omega^{lm}\frac{\partial^3 S}{\partial z^m \partial z^j \partial z^k}.$$

Equating the right-hand sides, we get

$$\Gamma^2_{pqr}(x_2)\frac{\partial x_2^p}{\partial z^i}\frac{\partial x_2^q}{\partial z^j}\frac{\partial x_2^r}{\partial z^k} - \frac{\partial x_2^m}{\partial z^i}\frac{\partial^3 S}{\partial z^m \partial z^j \partial z^k}$$
$$= \Gamma^1_{pqr}(x_1)\frac{\partial x_1^p}{\partial z^i}\frac{\partial x_1^q}{\partial z^j}\frac{\partial x_1^r}{\partial z^k} + \frac{\partial x_1^m}{\partial z^i}\frac{\partial^3 S}{\partial z^m \partial z^j \partial z^k}.$$

Taking into account that $x_1^m + x_2^m = 2z^m$, we come to (3.1). □

Thus, we can express Γ^2 as

$$\begin{aligned}\Gamma^2_{ijk}(x_2) \\ = \Gamma^1_{pqr}(x_1)\frac{\partial x_1^p}{\partial x_2^i}\frac{\partial x_1^q}{\partial x_2^j}\frac{\partial x_1^r}{\partial x_2^k} + 2\frac{\partial^3 S}{\partial z^p \partial z^q \partial z^r}\frac{\partial z^p}{\partial x_2^i}\frac{\partial z^q}{\partial x_2^j}\frac{\partial z^r}{\partial x_2^k}\end{aligned} \tag{3.2}$$

and further

$$\begin{aligned}\Gamma^2_{ijk}(x_2)x_2^i x_2^j x_2^k \\ = \Gamma^1_{pqr}(x_1)\frac{\partial x_1^p}{\partial x_2^i}\frac{\partial x_1^q}{\partial x_2^j}\frac{\partial x_1^r}{\partial x_2^k}x_2^i x_2^j x_2^k + 2\frac{\partial^3 S}{\partial z^p \partial z^q \partial z^r}\frac{\partial z^p}{\partial x_2^i}\frac{\partial z^q}{\partial x_2^j}\frac{\partial z^r}{\partial x_2^k}x_2^i x_2^j x_2^k,\end{aligned} \tag{3.3}$$

where

$$x_1 = z + \nabla S(z); \quad x_2 = z - \nabla S(z). \tag{3.4}$$

Equating (3.3) to 0, we come to a differential equation for the generating function of the symplectic transformation to normal Darboux coordinates

$$2\frac{\partial^3 S}{\partial z^i \partial z^j \partial z^k} b^i b^j b^k + \Gamma^1_{ijk}(z + \nabla S)a^i a^j a^k = 0 \tag{3.5}$$

where a and b are vector fields given by the formulas

$$a = \frac{\partial(z + \nabla S)}{\partial z}\left(\frac{\partial(z - \nabla S)}{\partial z}\right)^{-1}(z - \nabla S) \tag{3.6}$$

$$b = \left(\frac{\partial(z - \nabla S)}{\partial z}\right)^{-1}(z - \nabla S). \tag{3.7}$$

Thus, we come to a non-linear third order differential equation having degeneracy at the origin. The solution S gives a Cayley generating function for a symplectomorphism f to normal Darboux coordinates x_2. Unfortunately, we can not solve this equation, so we replace it by a more simple one

$$2\frac{\partial^3 S}{\partial z^i \partial z^j \partial z^k} z^i z^j z^k + \Gamma^1_{ijk}(z) z^i z^j z^k = 0 \tag{3.8}$$

obtained from (3.5), (3.6) and (3.7) by dropping ∇S everywhere.

Lemma 3.2. *Equation (3.8) has a unique solution with the third order zero at the origin*

$$S(z) = -\frac{1}{4}\int_0^1 (1-t)^2 \Gamma^1_{ijk}(tz) z^i z^j z^k dt. \tag{3.9}$$

Proof. Consider equality (3.8) on a ray $zt, \quad t \geq 0$. It may be written as

$$2\frac{d^3 S(tz)}{dt^3} + \Gamma^1_{ijk}(tz) z^i z^j z^k = 0.$$

Integrating three times with respect to t and using that $S(tz)$ has a third order zero at $t = 0$, we come to (3.9). □

We are able now to prove Theorem 1.4.

Proof. Let us start with a given Darboux coordinate system $x_1 = \{x_1^1, x_1^2, \dots, x_1^{2n}\}$ and corresponding Christoffel symbols $\Gamma^1_{ijk}(x_1)$. Using (3.9), define the Cayley generating function $S_1(z)$ and the corresponding symplectomorphism f_1 by the non-linear Cayley transform (3.4). We obtain new Darboux coordinates $x_2 = f_1(x_1) = \{x_2^1, x_2^2, \dots, x_2^{2n}\}$ and new Christoffel symbols $\Gamma^2_{ijk}(x)$ given by (3.2). Next, repeat this procedure taking x_2 and $\Gamma^2_{ijk}(x_2)$ as starting data and find $S_2(z), f_2$ and so on. At the N-th step we get

$$x_N,\ \Gamma^N_{ijk}(x_N),\ S_N(z),\ f_N,$$

where $x_N = \{x_N^1, x_N^2, \dots, x_N^{2n}\}$ is a Darboux coordinate system, Γ^N are Christoffel symbols of the symplectic connection ∂ for these coordinates, S_N the Cayley generating function and f_N is a symplectomorphism to the next Darboux coordinates $x_{N+1} = f_N(x_N)$. Finally, the composition

$$F_N = f_N \circ f_{N-1}, \circ \cdots \circ f_1$$

gives a symplectomorphism from the original coordinates x_1 to x_{N+1}.

Using induction, let us show that

$$\Gamma^N_{ijk}(x_N) x_N^i x_N^j x_N^k = O(|x_1|^{N+2}). \tag{3.10}$$

It is evident for $N = 1$. Supposing that (3.10) is true and using (3.9) for $S_N(z)$ with Γ^1 replaced by Γ^N, we find

$$S_N(z) = O(|z|^{N+2}), \quad \nabla S_N(z) = O(|z|^{N+1}).$$

To estimate $\Gamma^{N+1}_{ijk}(x_{N+1}) x^i_{N+1} x^j_{N+1} x^k_{N+1}$, we use (3.3) with x_1 replaced by x_N, x_2 by x_{N+1}, S by S_N and Γ^1_{ijk} by Γ^N_{ijk}. Thus,

$$\begin{aligned} &\Gamma^{N+1}_{ijk}(x_{N+1}) x^i_{N+1} x^j_{N+1} x^k_{N+1} \\ &= \Gamma^N_{pqr}(z + \nabla S_N) a^p_N a^q_N a^r_N + 2 \frac{\partial^3 S_N(z)}{\partial z^p \partial z^q \partial z^r} b^p_N b^q_N b^r_N \end{aligned} \tag{3.11}$$

where, similarly to (3.6), (3.7),

$$a_N = \frac{\partial(z + \nabla S_N)}{\partial z} \left(\frac{\partial (z - \nabla S_N)}{\partial z} \right)^{-1} (z - \nabla S_N) \tag{3.12}$$

$$b_N = \frac{\partial(z - \nabla S_N)^{-1}}{\partial z} (z - \nabla S_N) \tag{3.13}$$

and $x_{N+1} = z - \nabla S_N(z) = z + O(|z|^{N+1})$. Introduce a function

$$\varphi(\tau) = \Gamma^N_{pqr}(z + \tau \nabla S) a^p_N(\tau) a^q_N(\tau) a^r_N(\tau) + 2 \frac{\partial^3 S_N(z)}{\partial z^p \partial z^q \partial z^r} b^p_N(\tau) b^q_N(\tau) b^r_N(\tau)$$

where $a_N(\tau), b_N(\tau)$ are defined by (3.12), (3.13) with ∇S_N replaced by $\tau \nabla S_N$. Then the right-hand side of (3.11) is

$$\varphi(1) = \varphi(0) + (\varphi(1) - \varphi(0)) = (\varphi(1) - \varphi(0))$$

since

$$\varphi(0) = \Gamma^N_{pqr}(z) z^p z^q z^r + 2 \frac{\partial^3 S_N}{\partial z^p z^q z^r} z^p z^q z^r = 0$$

by construction. Thus,

$$\varphi(1) = \varphi(1) - \varphi(0) = O(\nabla S_N) = O(|z|^{N+1}),$$

proving the desired estimate for the left-hand side of (3.11).

The sequence $x_{N+1} = f_N(x_N) = F_N(x_1)$ of symplectomorphisms has the property that for any fixed k the sequence of differentials

$$d^k x_{N+1}\big|_{x_1=0} = d^k F_N(0) \tag{3.14}$$

stabilizes for N large enough. Indeed, we see by induction that $x_N = O(|x_1|)$ and further

$$d^k x_{N+1} = d^k(x_N + O(|x_N|^{N+1})) = d^k x_N + d^k O(|x_1|^{N+1})$$

where the second term vanishes at the origin for $N+1$ larger than k. To complete the proof, it remains to show that there exists a symplectomorphism $F(x_1)$ whose k-th order differential $d^k F(0)$ at the origin coincides with the stabilized value of (3.14). Passing from symplectomorphisms $F_N(x_1)$ to their generating functions, we see that this problem is equivalent to the well-known construction of a function with the given Taylor expansion at the origin. □

We would like to emphasize that this proof does not mean the convergence of the sequence $F_N(x_1)$ in a fixed neighborhood of the origin in some norm.

4. Application

Here we describe a standard application of Theorem 1.4 resembling the application of normal coordinates in Riemannian geometry.

THEOREM 4.1. *Let*

$$P(\Gamma_{ijk}(x), \frac{\partial}{\partial x^l}\Gamma_{ijk}(x), \frac{\partial^2}{\partial x^{l_1}\partial x^{l_2}}\Gamma_{ijk}(x), \dots)$$

be a polynomial with constant coefficients in Christoffel symbols and their partial derivatives for some Darboux local coordinates. If this polynomial does not depend on the choice of Darboux coordinates, then there exists a polynomial

$$Q(R_{ijkl}, \partial_m R_{ijkl}, \partial_{m_1}\partial_{m_2} R_{ijkl})$$

in the curvature tensor and its covariant derivatives such that $P \equiv Q$.

PROOF. We use the fact that the value of P is independent of the choice of Darboux local coordinates. So, to calculate the value of P at a point $x_0 \in M$, introduce week normal Darboux coordinates of Theorem 1.4 with the origin x_0. Thus, it is sufficient to prove that the partial derivatives of the Christoffel symbols at the origin are polynomials in the curvature tensor and its covariant derivatives. Let us denote

$$\Gamma_{ijk,l_1 l_2 \dots l_m} = \frac{\partial^m}{\partial x^{l_1} \dots \partial x^{l_m}}\Gamma_{ijk}(0). \tag{4.1}$$

The property (1.3) means that the completely symmetric part $\Gamma_{(ijk,l_1 l_2 \dots l_m)}$ of the tensor (4.1) vanishes. On the other hand, there is a separate symmetry in i, j, k and $l_1, l_2, \dots, l_m$. Hence we may write (4.1) in the form

$$\frac{1}{(m+3)!}\sum_{s\in S_{m+3}} (\Gamma_{ijk,l_1 l_2 \dots l_m} - \Gamma_{s(ijk,l_1 l_2 \dots l_m)})$$

where S_{m+3} means the permutation group and $s(ijk, l_1 l_2 \dots l_m)$ means the result of a permutation s applied to the whole set of indices. Since any permutation is a sequence of transpositions, each summand may be represented as a sum of the following terms

$$\begin{aligned}
&\Gamma_{ijk,l_1l_2\dots l_m} - \Gamma_{ijl,kl_2\dots l_m}\\
&= \frac{\partial^{m-1}}{\partial x^{l_2}\dots\partial x^{l_m}}\left(\frac{\partial\Gamma_{ijk}}{\partial x^{l_1}} - \frac{\partial\Gamma_{ijl_1}}{\partial x_k}\right)(0)\\
&= -\frac{\partial^{m-1}}{\partial x^{l_2}\dots\partial x^{l_m}}R_{ijkl_1}(0) + \frac{\partial^{m-1}}{\partial x^{l_2}\dots\partial x^{l_m}}(\Gamma_{ikp}\Gamma_{jl_1q} - \Gamma_{il_1p}\Gamma_{jkq})(0)\omega^{pq}\\
&= -\partial_{l_2}\dots\partial_{l_m}R_{ijkl_1}(0) + P_{m-1},
\end{aligned}$$

where P_{m-1} denotes a polynomial in partial derivatives of Γ_{ijk} at the origin containing the derivatives of the order at most $m-1$. Thus, the theorem follows by induction. $\square$

5. Counterexample

The aim of this section is to show that the problem of finding Darboux coordinates is a non-correct one. The term correctness means as usual the following three properties: existence, uniqueness and smooth dependence on parameters. We construct a counterexample showing that at least the third property is violated. Of course, it would be more interesting to prove directly that Equation (3.5) has no smooth solutions with a third-order zero at the origin, but we managed to prove it only for variational equation. The reason is that the linearized equation is easier to treat than the original non-linear one.

To derive the linearization of (3.5), consider Christoffel symbols in the form

$$\Gamma_{ijk}(x) + \tau\delta\Gamma_{ijk}(x) \tag{5.1}$$

where Γ_{ijk} satisfies (1.3), $\delta\Gamma_{ijk}$ is an arbitrary symmetric tensor and τ is a small parameter. At $\tau = 0$ we have a trivial solution $S = 0$ of (3.5) and we look for a variation of the trivial solution. Thus, we set for $\tau \neq 0$

$$S = \tau\delta S(z) + o(\tau). \tag{5.2}$$

Substituting (5.1) and (5.2) into (3.5) and extracting linear terms in τ, we come to the equation

$$\begin{aligned}
&2\frac{\partial^3\delta S(z)}{\partial z^i\partial z^j\partial z^k}z^iz^jz^k + 6\Gamma_{ijk}(z)z^jz^k\nabla^i\left(z^l\frac{\partial\delta S}{\partial z^l} - 2\delta S\right)\\
&= -\delta\Gamma_{ijk}(z)z^iz^jz^k,
\end{aligned} \tag{5.3}$$

the variation of (3.5). Introducing new functions

$$u = z^l\frac{\partial\delta S}{\partial z^l} - 2\delta S,$$

$$f = -\frac{1}{2}\delta\Gamma_{ijk}(z)z^iz^jz^k,$$

rewrite (5.3) in the form

$$x^ix^j\frac{\partial^2 u}{\partial x^i\partial x^j} + 3\Gamma_{ijk}(z)z^iz^k\nabla^i u = f.$$

Further, differentiating the identity $\Gamma_{jkl}(z)z^jz^kz^l \equiv 0$, we get

$$q_i(z) := 3\Gamma_{ijk}(z)z^jz^k = -\frac{\partial\Gamma_{jkl}(z)}{\partial z^i}z^jz^kz^l,$$

hence our equation becomes

$$z^i z^j \frac{\partial^2 u}{\partial z^i \partial z^j} + q_i(z)\nabla^i u = f(z) \tag{5.4}$$

where $f(z)$ may be an arbitrary function having a third-order zero at the origin. The coefficients $q_j(z)$ have also a third-order zero and satisfy the relation

$$z^i q_i(z) = z^i \frac{\partial}{\partial z^i}(\Gamma_{jkl}(z) z^j z^k z^l) - 3\Gamma_{jkl}(z) z^j z^k z^l \equiv 0. \tag{5.5}$$

Equation (5.4) exhibits quite different properties depending on $\dim M$. First we consider briefly the case $\dim M = 2$. The relation (5.5) takes the form

$$z^1 q_1(z) + z^2 q_2(z) = 0,$$

so that

$$q_1 = a(z) z^2; \quad q_2 = -a(z) z^1$$

where

$$a(z) = \int_0^1 \frac{\partial q_1}{\partial z^2}(z^1, tz^2) dt = -\int_0^1 \frac{\partial q_2}{\partial z^1}(tz^1, z^2) dt.$$

Thus, $a(z)$ is a smooth function having at least a second-order zero at the origin. Further, taking $\omega^{12} = -\omega^{21} = 1$, we get

$$\begin{aligned} q_i \nabla^i u &= q_1 \omega^{12} \frac{\partial u}{\partial z^2} + q_2 \omega^{21} \frac{\partial u}{\partial z^1} \\ &= a(z) \left(z^1 \frac{\partial u}{\partial z^1} + z^2 \frac{\partial u}{\partial z^2} \right), \end{aligned}$$

so that (5.4) may be rewritten in the form

$$z^i z^j \frac{\partial^2 u}{\partial z^i \partial z^j} + a(z) z^i \frac{\partial u}{\partial z^i} = f(z). \tag{5.6}$$

Restricting this equation to the rays, we come to an ordinary differential equation

$$\frac{d^2 u(tz)}{dt^2} + \frac{a(tz)}{t} \frac{du(tz)}{dt} = \frac{f(tz)}{t^2}.$$

It may be solved explicitly giving a unique solution of (5.6) with a third-order zero

$$u(z) = \int_0^1 dt \int_0^t \frac{f(\tau z)}{\tau^2} \exp\left(-\int_t^t au \frac{a(sz)}{s} ds \right) d\tau.$$

In contrast, for $\dim M > 2$, Equation (5.4) can not be reduced to an ordinary differential equation along the rays and this is a reason why the existence theorem is not true. We construct a counterexample for $\dim M = 4$ taking non-vanishing components ω^{ij} in the form $\omega^{12} = -\omega^{21} = 1$; $\omega^{34} = -\omega^{43} = 1$ and defining non-vanishing components of $\Gamma_{ijk}(z)$ for $i \le j \le k$ by

$$\Gamma_{113} = \Gamma_{223} = \frac{1}{3} z^3, \ \Gamma_{114} = \Gamma_{224} = \frac{1}{3} z^4,$$

$$\Gamma_{113} = \Gamma_{144} = -\frac{1}{3} z^1, \ \Gamma_{233} = \Gamma_{244} = -\frac{1}{3} z^2.$$

The coefficients $q_i(z)$ are then equal to

$$q_1(z) = z_1(z_3^2 + z_4^2), \quad q_2(z) = z_2(z_3^2 + z_4^2)$$

$$q_3(z) = -z_3(z_1^2 + z_2^2), \quad q_4(z) = -z_4(z_1^2 + z_2^2),$$

so that (5.5) is obviously fulfilled (here we have used subscripts instead of superscripts not to confuse them with squares). Thus,

$$\begin{aligned} q_i\nabla^i u &= q_1\frac{\partial u}{\partial z_2} - q_2\frac{\partial u}{\partial z_1} + q_3\frac{\partial u}{\partial z_4} - q_4\frac{\partial u}{\partial z_3} \\ &= (z_3^2+z_4^2)\left(z_1\frac{\partial u}{\partial z_2} - z_2\frac{\partial u}{\partial z_1}\right) \\ &-(z_1^2+z_2^2)\left(z_3\frac{\partial u}{\partial z_4} - z_4\frac{\partial u}{\partial z_3}\right) \\ &= r_2^2\frac{\partial u}{\partial \varphi_1} - r_1^2\frac{\partial u}{\partial \varphi_2} \end{aligned}$$

where in the last line we have introduced polar coordinates r_1, φ_1 in the plane z_1, z_2 and r_2, φ_2 in the plane z_3, z_4. In these polar coordinates Equation (5.4) takes the form

$$r_i r_j \frac{\partial^2 u}{\partial r_i \partial r_j} + r^2\frac{\partial u}{\partial \varphi_1} - r_1^2\frac{\partial u}{\partial \varphi_2} = f \tag{5.7}$$

(summation over $i, j = 1, 2$ is understood).

Suppose that f is a sufficiently smooth function and u is a classical solution of (5.7) satisfying $u = O(|z|^3)$. Denoting $u_{m,n}(r_1, r_2)$, $f_{m,n}(r_1, r_2)$ Fourier coefficients, we obtain from (5.7)

$$r_i r_j \frac{\partial^2 u_{mn}}{\partial r_i \partial r_j} + (imr_2^2 - inr_1^2)u_{mn} = f_{mn}.$$

Now the restriction to the ray is possible giving for the function $u_{mn}(tr_1, tr_2)$ an ordinary differential equation

$$\begin{aligned} &\frac{d^2 u_{mn}(tr_1, tr_2)}{dt^2} + (imr_2^2 - inr_1^2)u_{mn}(tr_1, tr_2) \\ &= \frac{1}{t^2} f_{mn}(tr_1, tr_2) \end{aligned}$$

which can be solved explicitly. Here $t \in [0, \infty)$ and the right-hand side satisfies an estimate $O(t)$. Taking into account that $u_{mn} = O(t^3)$, we obtain that u_{mn} is uniquely defined by the following formula

$$u_{mn} = \int_0^t \frac{\sinh\sqrt{inr_1^2 - imr_2^2}(t-\tau)}{\sqrt{inr_1^2 - imr_2^2}} \frac{1}{\tau^2} f_{mn}(\tau r_1, \tau r_2) dt. \tag{5.8}$$

Let us take, in particular,

$$f = r_1^2 r_2^{2k} \sum_{n=-\infty}^{\infty} \frac{e^{in\varphi_2}}{n^{2k}} \tag{5.9}$$

where $k \in \mathbb{N}$. Clearly, this function belongs to $C^{2k-2}(\mathbb{R}^4)$. From (5.8) it follows that only u_{0n} do not vanish. Denoting them simply u_n and taking $t = 1$ in (5.8), we obtain

$$\begin{aligned} u_n(r_1, r_2) &= \frac{r_1^2 r_2^{2k}}{n^{2k}} \int_0^1 \frac{\sinh\sqrt{in}r_1(1-\tau)}{\sqrt{in}r_1} \tau^{2k} d\tau \\ &= \frac{(2k)!}{(inr_1)^{k+1}} \left(\cosh\sqrt{in}r_1 - \sum_{l=0}^{k} \frac{(inr_1^2)^l}{(2l)!}\right) \frac{r_1^2 r_2^{2k}}{n^{2k}}. \end{aligned}$$

Thus, if there exists a classical solution of (5.7), it should be represented by the Fourier series

$$(2k)!r_1^2 r_2^{2k}\left(\sum_{n=-\infty}^{\infty}\frac{\cosh\sqrt{in}r_1}{(inr_1)^{k+1}n^{2k}}e^{in\varphi_2}\right.$$
$$\left.-\sum_{n=-\infty}^{\infty}\sum_{l=0}^{k}\frac{e^{in\varphi_2}}{(inr_1)^{k-l+1}(2l)!n^{2k}}\right), \tag{5.10}$$

and this series must be convergent since the classical solution belongs at least to C^2. But the series (5.10) is divergent for any $r_1 \neq 0$. Indeed, the second series in (5.10) converges, while the first one has coefficients proportional to

$$\frac{\cosh\sqrt{in}r_1}{n^{3k+1}},$$

so they do not tend to 0 for $n \to \pm\infty$.

So, the classical solution of (5.7) does not exist in any neighborhood of the origin although the right-hand sides (5.9) may be of arbitrary high smoothness.

References

[GRS] I. Gelfand, V. Retakh and M. Shubin, *Fedosov Manifolds* Advances in Mathematics, **136** (1998), 104–140.

[F] B. Fedosov, *Deformation Quantization and Index Theory*, Akademie Verlag, Berlin, 1996.

[NN] A. Newlander and L. Nirenberg, *Complex Analytic Coordinates in Almost Complex Manifolds*, Annals of Math., **65** 1957, 391–404.

[W] H. Weyl *The Classical Groups*, Princeton University Press, Princeton, New Jersey, 1939.

Mathématique Physique, Université de Bourgogne, BP 47870, F-21078, Dijon Cedex, France

E-mail address: fedosov@u-bourgogne.fr