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Abstract

Centralizers of rank one in the first Weyl algebra have genus zero

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Take $a \in A_1$ (the first Weyl algebra). Rank of the centralizer $C(a)$ is the greatest common divisor of the orders of elements in $C(a)$ (orders as differential operators).

This note contains a proof of the following.

Theorem. If the centralizer $C(a)$ of $a \in A_1$ (defined over the field $\mathbb{C}$ of complex numbers) has rank 1 then $C(a)$ can be embedded into a polynomial ring $\mathbb{C}[z]$.

The classical works of Burchnall and Chaundy where the systematic research of commuting differential operators was initiated are also devoted primarily to the case of rank 1 but the coefficients of the operators considered by them are analytic functions. Burchnall and Chaundy treated only monic differential operators which doesn't restrict generality if the coefficients are analytic functions. Situation is completely different if the coefficients are polynomial.

Before we proceed with a proof, here is a short refresher on the first Weyl algebra.

Definition. The first Weyl algebra A_1 is an algebra over a field K generated by two elements (denoted here by x and ∂) which satisfy a relation

$\partial x - x\partial = 1$. When characteristic of K is zero it has a natural representation over the ring of polynomials $K[x]$ by operators of multiplication by x and the derivative ∂ relative to x. Hence the elements of the Weyl algebra can be thought of as differential operators with polynomial coefficients. They can be written as ordinary polynomials: $a = \sum c_{i,j} x^i \partial^j$ with ordinary addition but a more complicated multiplication. If characteristic of K is zero then the centralizer of any element $a \in A_1 \setminus K$ is a commutative subalgebra of A_1. This theorem which was first proved by Issai Schur in 1904 (see [S]) has somewhat entertaining history which is described in [ML].

Given $\rho,\ \sigma \in \mathbb{Z}$ it is possible to define a weight function on A_1 by $w(x) = \rho,\ w(\partial) = \sigma,\ w(x^i\partial^j) = \rho i + \sigma j,\ w(a) = \max w(x^i\partial^j) | c_{i,j} \neq 0$ for $a = \sum c_{i,j} x^i \partial^j$ and the leading form $\bar{a}$ of a by $\bar{a} = \sum c_{i,j} x^i \partial^j | w(x^i\partial^j) = w(a)$. One of the nice properties of A_1 which was used by Dixmier in his seminal research of the first Weyl algebra (see [D]) is the following property of the leading forms of elements of A_1: if $\rho + \sigma > 0$ then $\overline{[a,b]} = \{\bar{a}, \bar{b}\}$ for $a, b \in A_1$ where $[a,b] = ab - ba$ and $\{\bar{a}, \bar{b}\} = \bar{a}_\partial \bar{b}_x - \bar{a}_x \bar{b}_\partial$ is the standard Poisson bracket of $\bar{a},\ \bar{b}$ as commutative polynomials ($\bar{a}_\partial$ etc. are the corresponding partial derivatives) provided $\{\bar{a}, \bar{b}\} \neq 0$.

The main ingredient of the considerations below is this property of the leading forms.

To make considerations clearer the reader may use the Newton polygons of elements of A_1. The Newton polygon of $a \in A_1$ is the convex hull of those points (i,j) on the plane for which $c_{i,j} \neq 0$. The Newton polygons of elements of A_1 are less sensible than the Newton polygons of polynomials in two variables because they depend on the way one chooses to record elements of A_1 but only those edges which are independent of the choice will be used.

First case $a = \partial^n + \sum_{i=1}^n a_i \partial^{n-i}$.

Consider the leading form α of a which contains ∂^n, is not a monomial, and has non-zero weight. This is possible if $a \notin \mathbb{C}[\partial]$. (If $a \in \mathbb{C}[\partial]$ then $C(a) = \mathbb{C}[\partial]$.)

Since the leading forms of the elements from $C(a)$ are Poisson commutative with α they are proportional to the fractional powers of α (as a commutative polynomial). Because the rank of $C(a)$ is 1 we should have $\alpha = c(\partial + c_1 x^k)^n$. Hence there exists an automorphism which makes the

Newton polygon of a smaller (say, area wise), and after an automorphism $\psi: \ x \to x, \ \partial \to \partial + p(x)$ we have $\psi(a) \in \mathbb{C}[\partial]$; therefore $C(a) = \mathbb{C}[\psi^{-1}(\partial)]$.

Second case $a = x^m \partial^n + \sum_{i=1}^n a_i \partial^{n-i}, \ m > 0.$

As above, the leading forms of elements of $C(a)$ are proportional to the fractional powers of α (as a commutative polynomial) as long as α is the leading form of a relative to weights $\rho, \ \sigma$ of x and ∂ provided $\rho + \sigma > 0$ and the weight of α is not zero. Because of that and since the rank is assumed to be one n divides m. We have now two possibilities for α of a non-zero weight: it is $c(x^d \partial + c_1 x^k)^n$ and either $k \geq d$ or $k < d - 1$. If we picture the Newton polygon of a on a plane where the x axis is horizontal and the ∂ axis is vertical then a may have both leading forms with one corresponding to the right edge containing the vertex (m, n) and another corresponding to the left edge containing this vertex. In any case a has a non-trivial zero weight form (which can degenerate to a monomial $x^m \partial^n$).

Lemma 1. If a has the leading form of weight zero then $C(a)$ is a subring of a ring of polynomials in one variable.
Proof. Assume that the weights for which a has the leading form of weight zero are ρ for x and σ for ∂ where $\rho, \ \sigma \in \mathbb{Z}$ and relatively prime (and $\rho + \sigma \geq 0$). Then any $b \in C(a)$ has a non-zero leading form $\bar{b}$ of weight zero (relative to $\rho, \ \sigma$) because a zero weight form and a non-zero weight form cannot commute i.e. if $\rho + \sigma > 0$ then the Poisson bracket of these forms is not zero and if $\rho + \sigma = 0$ then $\bar{a} \in \mathbb{C}[x\partial]$ and only elements of $\mathbb{C}[x\partial]$ commute with it. Hence the restriction map $b \to \bar{b}$ is an isomorphism. An algebra generated by all $\bar{b}$ is a subalgebra of $x^{-\sigma}\partial^{\rho}$ if $\rho > 0$ (of $x^{\sigma}\partial^{-\rho}$ if $\rho < 0$). $\square$

General case. $a = a_0(x)\partial^n + \sum_{i=1}^n a_i \partial^{n-i}$.

In this case $a_0 = \alpha^n, \ \alpha \in \mathbb{C}[x]$. We may assume that $\alpha \notin \mathbb{C}$ and that $\alpha(0) = 0$ (applying an automorphism $x \to x + c, \ \partial \to \partial$ if necessary). The left leading edge of the Newton polygon containing a point (m', n) (where

$m' = \mathrm{ord}(a_0)$) corresponds to either a form of weight zero or $c(x^d\partial + c_1x^k)^n$ where $0 < k < d-1$ in which case we still have a weigh zero form $x^{m'}\partial^n$ and the Lemma 1 shows that $C(a)$ is isomorphic to a subring of $\mathbb{C}[z]$.

Since the proof of the Theorem turned out to be too simple and too short we can complement it by an attempt to describe the rank one centralizers more precisely. In the first case it is already done, the centralizer is isomorphic to $\mathbb{C}[z]$. Actually if $z = \partial + p(x)$ for an appropriate $p(x) \in \mathbb{C}[x]$ then $C(a) = \mathbb{C}[z]$.

It would be interesting to describe a for which $C(a) \subsetneq \mathbb{C}[z]$ and the second case will provide us with such examples. Recall that in this case we have two possibilities for the leading form of a non-zero weight α: it is $c(x^d\partial + c_1x^k)^n$ and either $k \geq d$ or $k < d-1$ and that the Newton polygon of a on a plane where the x axis is horizontal and ∂ axis is vertical may have the right edge containing the vertex (m, n) which corresponds to the leading form with $k \geq d$ and the left edge containing this vertex which corresponds to the leading form with $k < d-1$. Let us assume that the zero weight leading form is $x^m\partial^n$.

If $k \geq d$ an automorphism $x \to x,\ \partial \to \partial - c_1x^{k-d}$ makes the Newton polygon of a smaller and after several similar steps we will obtain the Newton polygon with the right edge parallel to the bisectrix of the first quadrant. The leading form which corresponds to this edge cannot be treated as a commutative polynomial. The leading forms of elements of $C(a)$ are fractional powers of this form but as an element of A_1 (not as a commutative polynomial).

If the left edge of this Newton polygon is not parallel to the bisectrix we can consider the centralizer of a in $\mathbb{C}[\partial, x, x^{-1}]$ and proceed with automorphisms $x \to x,\ \partial \to \partial - c_1x^{k-d}$ where $k - d < -1$. Hence there exists an automorphism $x \to x,\ \partial \to \partial + q(x)$ of $\mathbb{C}[\partial, x, x^{-1}]$ such that $\psi(a) = x^{m-n}p(t)$ where $t = x\partial$ (here $q(x)$ is a Laurent polynomial while $p(t)$ is a polynomial).

Of course $C(a) = \psi^{-1}(C(x^{m-n}p(t))$ and the rank of $C(x^{m-n}p(t))$ is one. Since the rank is 1 we can find an element $b \in D_1$ (the skew field of fractions of A_1) commuting with $a_1 = x^{m-n}p(t)$ of the form $x^{d-1}r(t)$ where $r(t) \in \mathbb{C}(t)$ and $\deg(r) = 1$.

If r is a polynomial then $C(a_1) = \mathbb{C}[x^{d-1}r]$; if $d = 1$ then $C(a_1) = \mathbb{C}[t]$; if r is not a polynomial and $d > 1$ then some powers of $x^{d-1}r$ are polynomials. In the last case $r(t) \in \mathbb{C}(t)$ but $r(t)r(t+d-1)\ldots r(t+(k-1)(d-1)) \in \mathbb{C}[t]$

since $tx = x(t+1)$. We can reduce this to $r(t)r(t+1)\dots r(t+k-1) \in \mathbb{C}[t]$ by rescaling t and r.

By shifting t if necessary we may assume that one of the roots of $r(t)$ is 0 and represent r as a product $r_0 r_1$ where all roots and poles of r_0 are in $\mathbb{Z}$ and all roots and poles of r_1 are not in $\mathbb{Z}$. It is clear that $r_0(t)r_0(t+1)\dots r_0(t+k-1) \in \mathbb{C}[t]$ and $r_1(t)r_1(t+1)\dots r_1(t+k-1) \in \mathbb{C}[t]$. Since $\deg(r) = 1$ and $\deg(r_i) \geq 0$ (because $k\deg(r_i) \geq 0$) degree of one of the r_i is equal to zero and $r_i(t)r_i(t+1)\dots r_i(t+k-1) \in \mathbb{C}$ for this r_i. But then $r_i(t) = r_i(t+k)$ which is impossible for a non-constant rational function. Since $r_0(0) = 0$ and $r \neq 0$ we see that r_1 is a constant and all roots and poles of r are in $\mathbb{Z}$.

We can assume now that 0 is the largest root of r and write $r = ts(t)$ where $s(t) = \frac{\prod_{i=1}^{p}(t+\lambda_i)}{\prod_{i=1}^{p}(t+\mu_i)} \in \mathbb{C}(t) \setminus \mathbb{C},\ \lambda_i \in \mathbb{Z},\ \mu_i \in \mathbb{Z}$, and $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_p,\ \mu_1 \leq \mu_2 \leq \dots \leq \mu_p$. If $\mu_1 < 0$ then $r(t)r(t+1)\dots r(t+k-1)$ would have a pole at $t = -\mu_1$. Hence $\mu_1 > 0$ and all poles of $s(t)$ are negative integers while all zeros of $s(t)$ are non-positive integers.

A fraction $\frac{t+\lambda_i}{t+\mu_i}$ can be presented as $\frac{f_i(t)}{f_i(t+1)}$ if $\lambda_i < \mu_i$ or as $\frac{f_i(t+1)}{f_i(t)}$ if $\lambda_i > \mu_i$ (recall that $\lambda_i \neq \mu_i$; indeed, $\frac{t+d}{t} = \frac{(t+1)(t+2)\dots(t+d)}{t(t+1)\dots(t+d-1)}$ if $d > 0$, take the reciprocal fraction if $d < 0$). Because of that $s(t)$ can be written as $\frac{s_1(t)s_2(t+1)}{s_1(t+1)s_2(t)},\ s_i(t) \in \mathbb{C}[t]$. Write $s_1(t) = s_3(t)s_4(t),\ s_2(t) = s_4(t)s_5(t)$ where $s_4(t)$ is the greatest common divisor of $s_1(t),\ s_2(t)$. Then $s(t) = \frac{s_3(t)s_4(t)s_4(t+1)s_5(t+1)}{s_3(t+1)s_4(t+1)s_4(t)s_5(t)} = \frac{s_3(t)s_5(t+1)}{s_3(t+1)s_5(t)}$. A polynomial $s_3(t)$ cannot have positive roots. Indeed, the largest such root couldn't be canceled by a root of $s_5(t)$ or $s_3(t+1)$ and this root would be a root of $s(t)$.

Now, $q(t) = r(t)r(t+1)\dots r(t+k-1) = t(t+1)\dots(t+k-1)\frac{s_3(t)s_5(t+k)}{s_3(t+k)s_5(t)}$ is a polynomial. If $s_3(t) \notin \mathbb{C}$ then all roots of $s_3(t+k)$ are less then $1-k$ and the smallest root of $s_3(t+k)$ would be a pole of $q(t)$. Hence $s_3(t) \in \mathbb{C},\ \ s(t) = \frac{s_5(t+1)}{s_5(t)}$, and $q(t) = t(t+1)\dots(t+k-1)\frac{s_5(t+k)}{s_5(t)}$.

We can uniquely write $s_5(t) = \prod_{i\in I}\phi_{k,p_i}(t+i)$, where $\phi_{k,p}(t) = \prod_{j=0}^{p}(t+jk)$, and all p_i are maximal possible. Then $\frac{s_5(t+k)}{s_5(t)} = \prod_{i\in I}\frac{t+i+p_ik+k}{t+i}$ and $t+i_1 \neq t+i_2+p_{i_2}k+k$ for all $i_1, i_2 \in I$ because of the maximality of p_i. Hence $I \subset \{1,\dots,k-1\}$ and each i is used only once.

As we have seen, all roots of $s_5(t)$ are of multiplicity 1 and since $(xr)^N = x^N\prod_{i=0}^{N-1}(t+i)\frac{s_5(t+N)}{s_5(t)}$ the elements $(xr)^N \in \mathbb{C}[x]$ for sufficiently large N. Therefore the rank of $C(xr)$ is one. Observe that the rank is not stable under automorphisms: the rank of $C(\phi(xr))$ where $\phi(x) = x + t^M,\ \phi(t) = t$

is $M+1$.

We understood the structure of $C(a)$ when $a = x^{m-n}p(t)$. Are there substantially different examples of centralizers of rank one for which $C(a) \neq \mathbb{C}[a]$? Consider the case of an order 2 element commuting with an order 3 element which was completely researched in the work [BC] of Burchnall and Chaundry for analytic coefficients. They showed (and for this case it is a straightforward computation) that monic commuting operators of orders 2 and 3 can be reduced to $A = \partial^2 - 2\psi(x),\ B = \partial^3 - 3\psi(x)\partial - \frac{3}{2}\psi'(x)$ where $\psi''' = 12\psi\psi'$ i.e. $\psi'' = 6\psi^2 + c_1$ and $(\psi')^2 = 4\psi^3 + c_1\psi + c_2$ (a Weierstrass function). The only rational (even algebraic) solution in this case is (up to a substitution) $\psi = x^{-2}$ when $c_1 = c_2 = 0$. (If ψ is a rational function then the curve parameterized by $\psi,\ \psi'$ has genus zero, so $4\psi^3 + c_1\psi + c_2 = 4(\psi-\lambda)^2(\psi-\mu)$ and ψ is not an algebraic function of x if $\lambda \neq \mu$.) The corresponding operator $A = x^{-2}(t-2)(t+1) = (x^{-1}\frac{t^2-1}{t})^2$ is homogeneous.

In our case we have $A = f(x)^2\partial^2 + f_1(x)\partial + f_2(x)$. (The leading form for $w(x) = 0,\ w(\partial) = 1$ must be the square of a polynomial.) Here are computations for this case. $A = (f\partial)^2 - ff'\partial + f_1(x)\partial + f_2(x) = [f\partial + \frac{1}{2}(\frac{f_1}{f} - f')]^2 - \frac{1}{2}(\frac{f_1}{f} - f')'f - \frac{1}{4}(\frac{f_1}{f} - f')^2 + f_2$. Denote $f\partial + \frac{1}{2}(\frac{f_1}{f} - f')$ by D. Then $A = D^2 - 2\phi(x)$ where $\phi \in \mathbb{C}(x)$. Analogously to Burchnall and Chaundry case if there is an operator of order 3 commuting with A then it can be written as $B = D^3 - 3\phi D - \frac{3}{2}\phi' f$ (this follows from [BC] but will be clear from the condition $[A,B] = 0$ as well). In order to find an equation for ϕ we should compute $[A,B]$. Observe that $[D, g(x)] = g'f,\ [D^2, g] = 2g'fD + (g'f)'f,\ [D^3, g] = 3g'fD^2 + 3(g'f)'fD + ((g'f)'f)'f$. Hence $[A,B] = -3[D^2, \phi D + \frac{1}{2}\phi' f] + 2[D^3 - 3\phi D, \phi] = -3[(2\phi' fD + (\phi' f)'f)D + (\phi' f)'fD + \frac{1}{2}((\phi' f)'f)'f] + 2[3\phi' fD^2 + 3(\phi' f)'fD + ((\phi' f)'f)'f] - 6\phi\phi' f = (-6\phi' f + 6\phi' f)D^2 + (-6(\phi' f)'f + 6(\phi' f)'f)D - \frac{3}{2}((\phi' f)'f)'f + 2((\phi' f)'f)'f - 6\phi\phi' f = \frac{1}{2}((\phi' f)'f)'f - 6\phi\phi' f$. Therefore $((\phi' f)'f)' = 12\phi\phi',\ (\phi' f)'f = 6\phi^2 + c_1,\ (\phi' f)'\phi' f = 6\phi^2\phi' + c_1\phi',\ (\phi' f)^2 = 4\phi^3 + 2c_1\phi + c_2$ and we have a parameterization of an elliptic curve. Since $f, \phi \in \mathbb{C}(x)$ this curve must have genus 0, i.e. $4\phi^3 + 2c_1\phi + c_2 = 4(\phi-\lambda)^2(\phi-\mu)$. Take $z = \frac{\phi' f}{2(\phi-\lambda)}$. Then $\phi - \mu = z^2,\ \phi' f = 2z(z^2-\delta^2)$ where $\delta^2 = \lambda - \mu,\ 2zz'f = 2z(z^2-\delta^2)$ and $z'f = z^2 - \delta^2$.

Assume that $\delta \neq 0$. Since we can re-scale f and z by $f \to 2\delta f,\ z \to \delta z$, without loss of generality $\delta^2 = 1$. Then $z'f = z^2 - 1$ and $\int \frac{dz}{z^2-1} = \int \frac{dx}{f}$. Recall that $f \in \mathbb{C}[x]$. Since $2\int \frac{dz}{z^2-1} = \ln \frac{z-1}{z+1}$ all zeros of f have multiplicity

1 and $\int \frac{dx}{f} = \ln(\prod_i (x-\nu_i)^{c_i})$ where $\{\nu_i\}$ are the roots of f and $c_i = f'(\nu_i)$. Therefore $\frac{z-1}{z+1} = c\prod_i (x-\nu_i)^{2c_i}$ and $z = \frac{1+c\prod_i (x-\nu_i)^{2c_i}}{1-c\prod_i (x-\nu_i)^{2c_i}}$. Now it is time to recall that $2\phi = \frac{1}{2}(\frac{f_1}{f} - f')'f + \frac{1}{4}(\frac{f_1}{f} - f')^2 - f_2$ and $f^2\phi = f^2(z^2+\mu) \in \mathbb{C}[x]$ accordingly.

Consequently $zf = c_1 \frac{1+c\prod_i (x-\nu_i)^{c_i}}{1-c\prod_i (x-\nu_i)^{c_i}} \prod (x-\nu_i) \in \mathbb{C}[x]$ which is possible only if the rational function $1 - c\prod_i (x-\nu_i)^{c_i}$ doesn't have zeros. We can write $\prod_i (x-\nu_i)^{c_i}$ as $\frac{\prod_i (x-\nu_i)^{c_i^+}}{\prod_i (x-\nu_i)^{c_i^-}}$ where $c_i^\pm \in \mathbb{Z}^+$. Then $\prod_i (x-\nu_i)^{c_i^-} - c\prod_i (x-\nu_i)^{c_i^+} \in \mathbb{C}$ which is possible only if $c = 1$.

Since $z = \frac{\prod_i (x-\nu_i)^{c_i^-} + \prod_i (x-\nu_i)^{c_i^+}}{\prod_i (x-\nu_i)^{c_i^-} - \prod_i (x-\nu_i)^{c_i^+}}$ we see that $z \in \mathbb{C}[x]$. So to produce a $2, 3$ commuting pair we should find a polynomial solution to $f = \frac{z^2-1}{z'}$. If $f,\ z$ are given then $A = (f\partial + \psi)^2 - 2(z^2-\mu),\ B = (f\partial+\psi)^3 - 3(z^2-\mu)(f\partial+\psi) - 3zz'f$ is a commuting pair for any $\psi \in \mathbb{C}[x]$ (indeed, $f\psi \in \mathbb{C}[x]$ and $\psi^2 + f\psi' \in \mathbb{C}[x]$, hence $\psi \in \mathbb{C}[x]$). Constant $\mu = -\frac{2}{3}$ since we assumed that $\lambda - \mu = 1$ and $2\lambda + \mu = 0$ because the equation is $(\phi' f)^2 = 4\phi^3 + 2c_1\phi + c_2$.

Here is a series of examples: $z = 1 + x^n,\ f = \frac{x}{n}(2+x^n),\ \phi = (1+x^n)^2 + \frac{2}{3} = x^n(2+x^n) + \frac{1}{3}$ which correspond to $A = [\frac{x}{n}(2+x^n)\partial + \psi]^2 - 2(x^n(2+x^n) + \frac{1}{3})$ Even the simplest one, $A = [x(2+x)\partial]^2 - 2[x(2+x) + \frac{1}{3}]$ cannot be made homogeneous.

It seems that a complete classification of $(2,3)$ pairs of rank one is a daunting task. Our condition on z is that z assumes values ± 1 when $z' = 0$. Let us call such a polynomial admissible. We can look only at reduced monic polynomials $z(x) = x^n + a_2x^{n-2} + \ldots$ because a substitution $x \to ax + b$ preserves admissibility. Also $\lambda^n z(\lambda^{-1}x)$ preserves admissibility if $\deg(z) = n$ and $\lambda^n = 1$.

Examples above are just one value case. Say, an admissible cubic polynomial is $x^3 - 3 \cdot 2^{\frac{-2}{3}}x$. If $z = (x-\nu)^i(x+\nu)^j + 1$ then it is admissible when $\nu^{i+j} = (-1)^{i-1}2^{1-i-j}\frac{(i+j)^{i+j}}{i^i j^j}$. If a composition $h(g(x))$ is admissible then $g' = 0$ and $h' = 0$ should imply that $h(g(x)) = \pm 1$. Hence $h(x)$ should be an admissible function. As far as g is concerned $g' = 0$ should imply that the value of g belongs to the preimage of ± 1 for h which is less restrictive if this preimage is large. Because of that it is hard to imagine a reasonable classification of all admissible polynomials. On the other hand $z^2 \equiv 1 \pmod{z'}$ for $z = x^n + a_2x^{n-2} + \cdots + a_n$ leads to $n-1$ equations on $n-1$ variables with apparently finite number of solutions for each n. Say, for $n = 4$ all admissible polynomials are $x^4 \pm\ 1;\ x^4 + ax^2 + \frac{1}{8}a^2,\ a^4 =$

64; $x^4 - 3a^2x^2 + 2\sqrt{2}a^3x + \frac{21}{8}a^4,\ 337a^8 = 64.$

Remark. The number of admissible polynomials of a given degree is finite. Indeed consider first $n-2$ homogeneous equations on the coefficients $a_2, \ldots, a_n$. They are satisfied if $z^2 \equiv c \pmod{z'}$ where $c \in \mathbb{C}$. If one of the components of the variety defined by these equations is more than one-dimensional then (by Affine Dimension Theorem) its intersection with the hypersurface given by the last homogeneous equation will be at least one-dimensional while condition $z^2 \equiv 0 \pmod{z'}$ is satisfied only by $z = x^n$ (recall that we are considering only reduced monic polynomials).

If $\delta = 0$ then $(\phi' f)^2 = 4(\phi - \lambda)^3$ and $(\phi - \lambda)^{-1/2} = -\int \frac{dx}{f}$.

Lemma 2. If $f \in \mathbb{C}[x]$ and $\int \frac{dx}{f}$ is a rational function then f is a monomial, i.e. $f = a(x-b)^d$. [1]

Proof. If $g' = \frac{1}{f}$ for $g \in \mathbb{C}(x)$ then $g = \frac{h}{f},\ h \in \mathbb{C}[x]$ since the poles of g are the zeros of f and if the multiplicity of a zero of f is d then the corresponding pole of g has the multiplicity $d-1$. An equality $g' = \frac{1}{f}$ can be rewritten as $h'f - hf' = f$. If $\deg(h) > 1$ then $\deg(h'f) > \deg(f)$. Hence the leading coefficients of polynomials $h'f$ and hf' are the same which is possible only when $\deg(h) = \deg(f)$. Therefore there exists a $c \in \mathbb{C}$ for which $\deg(h - cf) < \deg(f)$. Since $(h-cf)'f - (h-cf)f' = f$ we can conclude that $\deg(h_1) = 1$ for $h_1 = h - cf$. Changing the variable we may assume that $h_1 = c_1x$ and then $c_1(f - xf') = f$ which is possible only if $f = ax^d$. $\square$

Hence when $\delta = 0$ we may assume that $f = x^d$. Then $(\phi - \lambda)^{-1/2} = -\int \frac{dx}{x^d} = \frac{1}{(1-d)x^{d-1}} + b$ where $b = 0$ since $f^2\phi \in \mathbb{C}[x]$ and $\phi - \lambda = c(x-\nu)^{2(d-1)}$. Therefore $A = (x^{d-1}t + \psi(x))^2 - 2cx^{2(d-1)} - 2c\lambda$. Finally an automorphism $x \to x,\ t \to t - x^{2-d}\psi$ makes $A + 2c\lambda$ homogeneous for the weight $w(x) = 1,\ w(t) = 0$.

These computations show that a description of the structure of centralizers of rank one in A_1 is sufficiently challenging. Can the ring of regular functions of a genus zero curve with one place at infinity be realised as a centralizer of an element of A_1? Here is a more approachable relevant ques-

[1] I was unable to find a published proof for this observation. This proof is a result of discussions with J. Bernstein and A. Volberg

tion: is there an element of $A \in D_1 \setminus A_1$ for which $p(A) \in A_1$ for a given a polynomial $p(x) \in \mathbb{C}[x]$?

Remarks. It seems that the definition of rank of a centralizer as the greatest common divisor of the orders of its elements appeared in a work of Wilson [W].

There are many papers discussing commuting differential operators, providing examples of such pairs, and applications of such pairs to PDE. An interested reader can find a rather exhaustive reference list in the recently published work [BZ] of Burban and Zheglow.

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