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ABELIAN IDEALS OF A BOREL SUBALGEBRA AND ROOT SYSTEMS, II

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ABSTRACT. Let $\mathfrak g$ be a simple Lie algebra with a Borel subalgebra $\mathfrak b$ and $\mathfrak{Ab}$ the set of abelian ideals of $\mathfrak b$. Let Δ^+ be the corresponding set of positive roots. We continue our study of combinatorial properties of the partition of $\mathfrak{Ab}$ parameterised by the long positive roots. In particular, the union of an arbitrary set of maximal abelian ideals is described, if $\mathfrak g \ne \mathfrak{sl}_n$. We also characterise the greatest lower bound of two positive roots, when it exists, and point out interesting subposets of Δ^+ that are modular lattices.

INTRODUCTION

Let $\mathfrak g$ be a simple Lie algebra over $\mathbb C$, with a triangular decomposition $\mathfrak g = \mathfrak u \oplus \mathfrak t \oplus \mathfrak u^-$. Here $\mathfrak t$ is a Cartan and $\mathfrak b = \mathfrak u \oplus \mathfrak t$ is a fixed Borel subalgebra. The theory of abelian ideals of $\mathfrak b$ is based on their relationship, due to D. Peterson, with the *minuscule elements* of the affine Weyl group $\widehat W$ (see Kostant's account in [6]; another approach is presented in [2]). In this note, we elaborate on some topics related to the combinatorial theory of abelian ideals, which can be regarded as a sequel to [11]. We mostly work in the combinatorial setting, i.e., the abelian ideals of $\mathfrak b$, which are sums of root spaces of $\mathfrak u$, are identified with the corresponding sets of positive roots.

Let Δ be the root system of $(\mathfrak g, \mathfrak t)$ in the vector space $V = \mathfrak t^*_{\mathbb R}$, Δ^+ the set of positive roots in Δ corresponding to $\mathfrak u$, Π the set of simple roots in Δ^+, and θ the highest root in Δ^+. Then W is the Weyl group and $(\ ,\ )$ is a W-invariant scalar product on V. We equip Δ^+ with the usual partial ordering '$\succcurlyeq$'. An *upper ideal* (or just an *ideal*) of $(\Delta^+, \succcurlyeq)$ is a subset $I \subset \Delta^+$ such that if $\gamma \in I, \nu \in \Delta^+$, and $\nu + \gamma \in \Delta^+$, then $\nu + \gamma \in I$. An upper ideal I is *abelian*, if $\gamma' + \gamma'' \notin \Delta^+$ for all $\gamma', \gamma'' \in I$. The set of minimal elements of I is denoted by $\min(I)$. It also makes sense to consider the maximal elements of the complement of I, denoted $\max(\Delta^+ \setminus I)$.

Write $\mathfrak{Ab}$ (resp. $\mathfrak{Ad}$) for the set of all abelian (resp. all upper) ideals of Δ^+ and think of them as posets with respect to inclusion. The upper ideal *generated by* γ is $I\langle \succcurlyeq \gamma \rangle = \{\nu \in \Delta^+ \mid \nu \succcurlyeq \gamma\}$. Then $\min(I\langle \succcurlyeq \gamma \rangle) = \{\gamma\}$. A root $\gamma \in \Delta^+$ is said to be *commutative*, if $I\langle \succcurlyeq \gamma \rangle \in \mathfrak{Ab}$. Write Δ^+_{com} for the set of all commutative roots. This notion was introduced

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in [9], and the subset $\Delta^+_{\sf com}$ for each Δ is explicitly described in [9, Theorem 4.4]. Note that $\Delta^+_{\sf com} \in \mathfrak{Ad}$.

Let $\mathfrak{Ab}^o$ denote the set of nonempty abelian ideals and Δ^+_l the set of long positive roots. In [8, Sect. 2], we defined a mapping $\tau : \mathfrak{Ab}^o \to \Delta^+_l$, which is onto. Letting $\mathfrak{Ab}_\mu = \tau^{-1}(\mu)$, we get a partition of $\mathfrak{Ab}^o$ parameterised by Δ^+_l. Each $\mathfrak{Ab}_\mu$ is a subposet of $\mathfrak{Ab}$ and, moreover, $\mathfrak{Ab}_\mu$ has a unique minimal and unique maximal element (ideal) [8, Sect. 3]. These extreme abelian ideals in $\mathfrak{Ab}_\mu$ are denoted by $I(\mu)_{\sf min}$ and $I(\mu)_{\sf max}$. Then $\{I(\alpha)_{\sf max} \mid \alpha \in \Pi_l\}$ are exactly the maximal abelian ideals of $\mathfrak b$.

In this article, we first establish a property of $(\Delta^+, \succcurlyeq)$, which seems to be new. It was proved in [11, Appendix] that, for any $\eta_1, \eta_2 \in \Delta^+$, there exists the *least upper bound*, denoted $\eta_1 \vee \eta_2$. Moreover, an explicit formula for $\eta_1 \vee \eta_2$ is also given. Here we prove that the *greatest lower bound*, $\eta_1 \wedge \eta_2$, exists if and only if ${\sf supp}(\gamma_1) \cap {\sf supp}(\gamma_2) \ne \varnothing$. Furthermore, if $\eta_i = \sum_{\alpha\in\Pi} c_{i\alpha}\alpha$, then $\eta_1 \wedge \eta_2 = \sum_{\alpha\in\Pi} \min\{c_{1\alpha}, c_{2\alpha}\}\alpha$. This also implies that $I\langle \succcurlyeq\eta\rangle$ is a modular lattice for any $\eta \in \Delta^+$, see Theorem 2.4. Another example a modular lattice inside Δ^+ is the subposet $\Delta_\alpha(i) = \{\gamma \in \Delta^+ \mid {\sf ht}_\alpha(\gamma) = i\}$, where $\alpha \in \Pi$ and ${\sf ht}_\alpha(\gamma)$ is the coefficient of α in the expression of γ via Π.

Using properties of '$\vee$' and '$\wedge$' and $\mathbb Z$-gradings of $\mathfrak g$, we prove uniformly that if Δ is not of type $\mathbf{A}_n$, then $\Delta^+_{\sf nc} := \Delta^+ \setminus \Delta^+_{\sf com}$ has the unique maximal element, which is $\lfloor\theta/2\rfloor := \sum_{\alpha\in\Pi}\lfloor {\sf ht}_\alpha(\theta)/2\rfloor\alpha$, see Section 3. In particular, $\lfloor\theta/2\rfloor$ is a root. (Note that if Δ is of type $\mathbf{A}_n$, then $\lfloor\theta/2\rfloor = 0$ and $\Delta^+_{\sf nc} = \varnothing$.) We also describe the maximal abelian ideals $I(\alpha)_{\sf max}$ if ${\sf ht}_\alpha(\theta)$ is odd.

In Section 4, we study the sets of maximal and minimal elements related to abelian ideals of the form $I(\alpha)_{\sf min}$ and $I(\alpha)_{\sf max}$, with $\alpha \in \Pi_l := \Pi \cap \Delta^+_l$.

Theorem 0.1. *If $S \subset \Pi_l$ is arbitrary and Δ is not of type $\mathbf{A}_n$, then there is the bijection*

$$\eta \in \min\bigl(\bigcap_{\alpha\in S} I(\alpha)_{\sf min}\bigr) \stackrel{1:1}{\longmapsto} \eta' = \theta - \eta \in \max\bigl(\Delta^+ \setminus \bigcup_{\alpha\in S} I(\alpha)_{\sf max}\bigr).$$

Our proof is conceptual and relies on the fact θ is a multiple of a fundamental weight if Δ is not of type $\mathbf{A}_n$. For $\mathbf{A}_n$, the same bijection holds if S is a **connected** subset on the Dynkin diagram. The case in which $\#S = 1$ was considered earlier in [11, Theorem 4.7]. This has some interesting consequences if $S = \Pi_l$ and hence $\bigcup_{\alpha\in\Pi_l} I(\alpha)_{\max} = \Delta^+_{\sf com}$, see Proposition 4.6.

In Section 5, we describe the interval $[\lfloor\theta/2\rfloor, \theta - \lfloor\theta/2\rfloor]$ inside the poset Δ^+.

1. Preliminaries

We have $\Pi = \{\alpha_1, \dots, \alpha_n\}$, the vector space $V = \oplus_{i=1}^n \mathbb R\alpha_i$, the Weyl group W generated by simple reflections s_α ($\alpha \in \Pi$), and a W-invariant inner product $(\ ,\ )$ on V. Set $\rho =$

$\frac{1}{2}\sum_{\nu\in\Delta^+}\nu$. The partial ordering '$\preccurlyeq$' in Δ^+ is defined by the rule that $\mu \preccurlyeq \nu$ if $\nu-\mu$ is a non-negative integral linear combination of simple roots. Write $\mu\prec\nu$, if $\mu\preccurlyeq\nu$ and $\mu\ne\nu$. If $\mu=\sum_{i=1}^n c_i\alpha_i\in\Delta$, then $\mathsf{ht}_{\alpha_i}(\mu):=c_i$, $\mathsf{ht}(\mu):=\sum_{i=1}^n c_i$ and $\mathsf{supp}(\mu)=\{\alpha_i\in\Pi\mid c_i\ne 0\}$.

The Heisenberg ideal $\mathcal{H}:=\{\gamma\in\Delta^+\mid(\gamma,\theta)\ne 0\}=\{\gamma\in\Delta^+\mid(\gamma,\theta)>0\}\in\mathfrak{Ad}$ plays a prominent role in the theory of abelian ideals and posets $\mathfrak{Ab}_\mu=\tau^{-1}(\mu)$.

Let us collect some known results that are frequently used below.

- If $I\in\mathfrak{Ad}$ is not abelian, then there exist $\eta,\eta'\in I$ such that $\eta+\eta'=\theta$, see [8, p. 1897]. Therefore, $I\notin\mathfrak{Ab}$ if and only if $I\cap\mathcal{H}\notin\mathfrak{Ab}$.
- $I=I(\mu)_{\mathsf{min}}$ for some $\mu\in\Delta^+_l$ if and only if $I\subset\mathcal{H}$ [8, Theorem 4.3];
- $\#I(\mu)_{\mathsf{min}}=(\rho,\theta^\vee-\mu^\vee)+1$ [8, Theorem 4.2(4)];
- For $I\in\mathfrak{Ab}^o$, we have $I\in\mathfrak{Ab}_\mu$ if and only if $I\cap\mathcal{H}=I(\mu)_{\mathsf{min}}$ [11, Prop. 3.2];
- The set of (globally) maximal abelian ideals is $\{I(\alpha)_{\mathsf{max}}\mid\alpha\in\Pi_l\}$ [8, Corollary 3.8].
- For any $\mu\in\Delta^+_l$, there is a unique element of minimal length in W that takes θ to μ [8, Theorem 4.1]. Writing w_μ for this element, one has $\ell(w_\mu)=(\rho,\theta^\vee-\mu^\vee)$ [8, Theorem 4.1].
- Let $\mathcal{N}(w)$ be the inversion set of $w\in W$. By [12, Lemma 1.1],
$$I(\mu)_{\mathsf{min}}=\{\theta\}\cup\{\theta-\gamma\mid\gamma\in\mathcal{N}(w_\mu)\}.$$

For each $\eta\in\mathcal{H}\setminus\{\theta\}$ there is a unique $\eta'\in\mathcal{H}\setminus\{\theta\}$ such that $\eta+\eta'$ is a root, and this root is θ. It is well known that $\#\mathcal{H}=2(\rho,\theta^\vee)-1=2h^*-3$, where h^* is the *dual Coxeter number* of Δ. Since $\#I(\alpha)_{\mathsf{min}}=(\rho,\theta^\vee)=h^*-1$ for $\alpha\in\Pi_l$, the ideal $I(\alpha)_{\mathsf{min}}$ contains θ and exactly a half of elements of $\mathcal{H}\setminus\{\theta\}$, cf. also [11, Lemma 3.3].

Although the affine Weyl group and minuscule elements are not explicitly used in this paper, their use is hidden in properties of the posets $\mathfrak{Ab}_\mu$, $\mu\in\Delta^+_l$, and ideals $I(\mu)_{\mathsf{min}}$, $I(\mu)_{\mathsf{max}}$. Important properties of the maximal abelian ideals are also obtained in [3, 16].

We refer to [1], [4, § 3.1] for standard results on root systems and Weyl groups and to [15, Chapter 3] for posets.

2. The greatest lower bound in Δ^+

It is proved in [8, Appendix] that the poset $(\Delta^+,\succcurlyeq)$ is a join-semilattice. i.e., for any pair $\eta,\eta'\in\Delta^+$, there is the least upper bound (= *join*), denoted $\eta\vee\eta'$. Furthermore, there is a simple explicit formula for '$\vee$', see [8, Theorem A.1]. However, Δ^+ is not a meet-semilattice. We prove below that under a natural constraint the greatest lower bound (= *meet*) exists and can explicitly be described. Afterwards, we provide some applications of this property in the theory of abelian ideals.

Definition 1. Let $\eta,\eta'\in\Delta^+$. The root ν is the greatest lower bound (or *meet*) of η and η' if

- $\eta\succcurlyeq\nu$, $\eta'\succcurlyeq\nu$;

- if $\eta \succcurlyeq \kappa$ and $\eta' \succcurlyeq \kappa$, then $\nu \succcurlyeq \kappa$.

The meet of η and η', if it exists, is denoted by $\eta \wedge \eta'$.

Obviously, if $\alpha, \alpha' \in \Pi$, then their meet does not exist. But as we see below, the only reason for such a failure is that their supports are disjoint.

Lemma 2.1 (see [14, Lemma 3.1]). *Suppose that $\gamma \in \Delta^+$ and $\alpha, \beta \in \Pi$. If $\gamma - \alpha, \gamma - \beta \in \Delta^+$, then either $\gamma - \alpha - \beta \in \Delta^+$ or $\gamma = \alpha + \beta$ and hence α, β are adjacent in the Dynkin diagram.*

Lemma 2.2 (see [13, Lemma 3.2]). *Suppose that $\gamma \in \Delta^+$ and $\alpha, \beta \in \Pi$. If $\gamma + \alpha, \gamma + \beta \in \Delta^+$, then $\gamma + \alpha + \beta \in \Delta^+$.*

Let us provide a reformulation of these lemmata in terms of '$\vee$' and '$\wedge$'. To this end, we note that in the previous lemma, $(\gamma + \alpha) \wedge (\gamma + \beta) = \gamma$.

Proposition 2.3. *Let $\eta_1, \eta_2 \in \Delta^+$.*

(i) *If $\eta_1 \vee \eta_2$ covers both η_1 and η_2, then either $\eta_1 \vee \eta_2 = \alpha + \beta = \eta_1 + \eta_2$ for some adjacent $\alpha, \beta \in \Pi$, or η_1 and η_2 both cover $\eta_1 \wedge \eta_2$;*
(ii) *If $\eta_1 \wedge \eta_2$ exists and η_1 and η_2 both cover $\eta_1 \wedge \eta_2$, then $\eta_1 \vee \eta_2$ covers both η_1 and η_2.*

For any two roots $\eta = \sum_{\alpha\in\Pi} c_\alpha \alpha$ and $\eta' = \sum_{\alpha\in\Pi} c'_\alpha \alpha$, one defines two elements of the root lattice, $\min(\eta, \eta') = \sum_{\alpha\in\Pi} \min\{c_\alpha, c'_\alpha\}\alpha$ and $\max(\eta, \eta') = \sum_{\alpha\in\Pi} \max\{c_\alpha, c'_\alpha\}\alpha$. Recall that the poset $(\Delta^+, \succcurlyeq)$ is graded and the rank function is the usual *height* of a root, i.e., $\mathsf{ht}(\eta) = \sum_{\alpha\in\Pi} c_\alpha$. We also set $\mathsf{ht}_\alpha(\eta) := c_\alpha$.

Theorem 2.4.

1) *For any $\gamma \in \Delta^+$, the upper ideal $I\langle\succcurlyeq\gamma\rangle$ is a modular lattice;*
2) *the meet $\gamma_1 \wedge \gamma_2$ exists if and only if $\mathsf{supp}(\gamma_1) \cap \mathsf{supp}(\gamma_2) \neq \varnothing$. In this case, one has $\gamma_1 \wedge \gamma_2 = \min(\gamma_1, \gamma_2)$.*

Proof. 1) By [8, Theorem A.1(i)], the join always exists in Δ^+ and formulae for '$\vee$' show that $\gamma_1 \vee \gamma_2 \in I\langle\succcurlyeq\gamma\rangle$ whenever $\gamma_1, \gamma_2 \in I\langle\succcurlyeq\gamma\rangle$. Therefore, $I\langle\succcurlyeq\gamma\rangle$ is a join-semilattice with a unique minimal element. Hence the meet also exists for any $\gamma_1, \gamma_2 \in I\langle\succcurlyeq\gamma\rangle$, see [15, Prop. 3.3.1]. That is, $I\langle\succcurlyeq\gamma\rangle$ is a lattice. Note that, for $\gamma_1, \gamma_2 \in I\langle\succcurlyeq\gamma\rangle$, the first possibility in Proposition 2.3(i) does not realise. Therefore, using Proposition 2.3 with $I\langle\succcurlyeq\gamma\rangle$ in place of Δ^+ and [15, Prop. 3.3.2], we conclude that $I\langle\succcurlyeq\gamma\rangle$ is a modular lattice.

Yet, this does not provide a formula for the meet and leaves a theoretical possibility that $\gamma_1 \wedge \gamma_2$ depends on γ.

2) If $\mathsf{supp}(\gamma_1) \cap \mathsf{supp}(\gamma_2) = \varnothing$, then there are no roots ν such that $\gamma_1 \succcurlyeq \nu$ and $\gamma_2 \succcurlyeq \nu$. Conversely, if $\mathsf{supp}(\gamma_1) \cap \mathsf{supp}(\gamma_2) \neq \varnothing$, then $\gamma_1, \gamma_2 \in I\langle\succcurlyeq\gamma\rangle$ for some γ. Using again [15, Prop. 3.3.2], the modularity of the lattice $I\langle\succcurlyeq\gamma\rangle$ implies that $\mathsf{ht}(\gamma_1 \vee \gamma_2) + \mathsf{ht}(\gamma_1 \wedge \gamma_2) =$

$\mathsf{ht}(\gamma_1)+\mathsf{ht}(\gamma_2)$, where $\gamma_1\wedge\gamma_2$ is taken inside $I\langle\succcurlyeq\gamma\rangle$. It is clear that $\gamma_1\wedge\gamma_2 \preccurlyeq \min(\gamma_1,\gamma_2)$. Moreover, in this situation, the formulae of [8, Theorem A.1(i)] imply that $\gamma_1\vee\gamma_2=\max(\gamma_1,\gamma_2)$. Therefore, $\mathsf{ht}(\min(\gamma_1,\gamma_2))=\mathsf{ht}(\gamma_1\wedge\gamma_2)$ and thereby $\min(\gamma_1,\gamma_2)=\gamma_1\wedge\gamma_2$. □

Remark 2.5. A special class of modular lattices inside Δ^+ occurs in connection with $\mathbb{Z}$-gradings of $\mathfrak{g}$. For $\alpha\in\Pi$, set $\Delta_\alpha(i)=\{\gamma\in\Delta \mid \mathsf{ht}_\alpha(\gamma)=i\}$. It is known that $\Delta_\alpha(i)$ has a unique minimal and a unique maximal element, see Section 3. It is also clear that $\gamma_1\wedge\gamma_2$ and $\gamma_1\vee\gamma_2\in\Delta_\alpha(i)$ for all $\gamma_1,\gamma_2\in\Delta_\alpha(i)$. Hence $\Delta_\alpha(i)$ is a **modular** lattice. (It was already noticed in [8, Appendix] that $\Delta_\alpha(i)$ is a lattice.)

Remark 2.6. In what follows, we have to distinguish the $\mathbf{A}_n$-case from the other types. One the reasons is that θ is not a multiple of a fundamental weight only for $\mathbf{A}_n$. In all other types, there is a unique $\alpha_\theta\in\Pi$ such that $(\theta,\alpha_\theta)\ne 0$. For the $\mathbb{Z}$-grading associated with α_θ, one then has $\Delta_{\alpha_\theta}(1)=\mathcal{H}\setminus\{\theta\}$ and $\Delta_{\alpha_\theta}(2)=\{\theta\}$. That is, $\mathcal{H}\setminus\{\theta\}$ (or just $\mathcal{H}$) has a unique minimal element, which is α_θ, if and only if Δ is not of type $\mathbf{A}_n$. This provides the following consequence of Theorem 2.4:

If Δ is not of type $\mathbf{A}_n$, then for all $\eta_1,\eta_2\in\mathcal{H}\setminus\{\theta\}$, the meet $\eta_1\wedge\eta_2$ exists and lies in $\mathcal{H}\setminus\{\theta\}$.
This is going to be used several times in Section 4.

3. $\mathbb{Z}$-GRADINGS AND NON-COMMUTATIVE ROOTS

If $\gamma\in\Delta^+_{\mathsf{com}}$, then γ belongs to a maximal abelian ideal. Since $I(\alpha)_{\mathsf{max}}$, $\alpha\in\Pi_l$, are all the maximal abelian ideals in Δ^+, we have

$$\Delta^+_{\mathsf{com}}=\bigcup_{\alpha\in\Pi_l} I(\alpha)_{\mathsf{max}}.$$

Set $\Delta^+_{\mathsf{nc}}=\Delta^+\setminus\Delta^+_{\mathsf{com}}$. In this section, we obtain an *a priori* description of Δ^+_{nc}. Let us introduce special elements of the root lattice

$$\lfloor\theta/2\rfloor=\sum_{\alpha\in\Pi}\lfloor\mathsf{ht}_\alpha(\theta)/2\rfloor\alpha \ \text{ and } \ \lceil\theta/2\rceil=\sum_{\alpha\in\Pi}\lceil\mathsf{ht}_\alpha(\theta)/2\rceil\alpha. \tag{3·1}$$

Hence $\lfloor\theta/2\rfloor+\lceil\theta/2\rceil=\theta$. Note that $\lfloor\theta/2\rfloor=0$ if and only if $\theta=\sum_{\alpha\in\Pi}\alpha$, i.e., Δ is of type $\mathbf{A}_n$.

Lemma 3.1. *Suppose that Δ is not of type $\mathbf{A}_n$, so that $\lfloor\theta/2\rfloor\ne 0$.*

1. *If $\gamma\in\Delta^+_{\mathsf{nc}}$, then $\mathsf{ht}_\alpha\gamma\leqslant\lfloor\mathsf{ht}_\alpha(\theta)/2\rfloor$ for all $\alpha\in\Pi$, i.e., $\gamma\preccurlyeq\lfloor\theta/2\rfloor$.*
2. *If $\gamma_1,\gamma_2\preccurlyeq\lfloor\theta/2\rfloor$, then $\gamma_1\vee\gamma_2\preccurlyeq\lfloor\theta/2\rfloor$.*

Proof. (1) Obvious.

(2) By [8, Theorem A.1], if $\mathsf{supp}(\gamma_1)\cup\mathsf{supp}(\gamma_2)$ is connected, then $\gamma_1\vee\gamma_2=\max(\gamma_1,\gamma_2)$ and the assertion is clear. Otherwise, $\gamma_1\vee\gamma_2=\gamma_1+(\text{connecting root})+\gamma_2$. Recall that if the union of supports is not connected, then there is a (unique) chain of simple roots

that connects them. If this chain consists of $\alpha_{i_1},\dots,\alpha_{i_s}$, then the "connecting root" is $\alpha_{i_1}+\dots+\alpha_{i_s}$. Here we only need the condition that $\mathsf{ht}_\alpha(\theta)\geqslant 2$ for any α in the connecting chain. Indeed, the roots in this chain are not extreme in the Dynkin diagram, and outside type $\mathbf{A}_n$ the coefficients of non-extreme simple roots are always $\geqslant 2$. □

Remark. For $\mathbf{A}_n$, $\lfloor\theta/2\rfloor=0$ and hence $\Delta^+_{\mathsf{nc}}=\varnothing$.

Set $\mathcal{A}=\{\gamma\in\Delta^+ \mid \gamma\preccurlyeq\lfloor\theta/2\rfloor\}$. Then $\mathcal{A}\ne\varnothing$ if and only if Δ is not of type $\mathbf{A}_n$. It follows from Lemma 3.1 that

- $\Delta^+_{\mathsf{nc}}\subset\mathcal{A}$;
- $\mathcal{A}$ has a unique maximal element.

Our goal is to prove that $\Delta^+_{\mathsf{nc}}=\mathcal{A}$ and $\max(\mathcal{A})=\{\lfloor\theta/2\rfloor\}$. The latter essentially boils down to the assertion that $\lfloor\theta/2\rfloor$ is a root.

For an arbitrary $\alpha\in\Pi$, consider the $\mathbb{Z}$-grading $\mathfrak{g}=\bigoplus_{i\in\mathbb{Z}}\mathfrak{g}_\alpha(i)$ corresponding to α. That is, the set of roots of $\mathfrak{g}_\alpha(i)$ is $\Delta_\alpha(i)$, see Remark 2.5. In particular, $\alpha\in\Delta_\alpha(1)$ and $\Pi\setminus\{\alpha\}\subset\Delta_\alpha(0)$. Here $\mathfrak{l}:=\mathfrak{g}_\alpha(0)$ is reductive and contains the Cartan subalgebra $\mathfrak{t}$. By an old result of Kostant (see [7] and Joseph's exposition in [5, 2.1]), each $\mathfrak{g}_\alpha(i)$, $i\ne 0$, is a simple $\mathfrak{l}$-module. Therefore, $\Delta_\alpha(i)$ has a unique minimal and a unique maximal element. The following is a particular case of Theorem 2.3 in [7].

Proposition 3.2. *If $i+j\leqslant\mathsf{ht}_\alpha(\theta)$, then $0\ne[\mathfrak{g}_\alpha(i),\mathfrak{g}_\alpha(j)]=\mathfrak{g}_\alpha(i+j)$.*

Once one has proved that $[\mathfrak{g}_\alpha(i),\mathfrak{g}_\alpha(j)]\ne 0$, the equality $[\mathfrak{g}_\alpha(i),\mathfrak{g}_\alpha(j)]=\mathfrak{g}_\alpha(i+j)$ stems from the fact that $\mathfrak{g}_\alpha(i+j)$ is a simple $\mathfrak{l}$-module. We derive from this result two corollaries.

Corollary 3.3. *For any $\mu\in\Delta_\alpha(i)$, there is $\nu\in\Delta_\alpha(j)$ such that $\mu+\nu\in\Delta_\alpha(i+j)$.*

Proof. Let $e_\mu\in\mathfrak{g}_\alpha(i)$ be a root vector for μ. Assume that the property in question does not hold. Then $[e_\mu,\mathfrak{g}_\alpha(j)]=0$. Hence $[L{\cdot}e_\mu,\mathfrak{g}_\alpha(j)]=0$, where $L\subset G$ is the connected reductive group with Lie algebra $\mathfrak{l}$. Since the linear span of an L-orbit in a simple L-module is the whole space, this implies that $[\mathfrak{g}_\alpha(i),\mathfrak{g}_\alpha(j)]=0$, which contradicts the proposition. □

Set $d_\alpha=\lfloor\mathsf{ht}_\alpha(\theta)/2\rfloor$, and let μ_{d_α} be the lowest weight in $\Delta_\alpha(d_\alpha)$.

Corollary 3.4. $\mu_{d_\alpha}\in\Delta^+_{\mathsf{nc}}$.

Proof. By Corollary 3.3, there is $\lambda\in\Delta_\alpha(d_\alpha)$ such that $\mu_{d_\alpha}+\lambda$ is a root in $\Delta_\alpha(2d_\alpha)$. Since $\mu_{d_\alpha}\preccurlyeq\gamma$, the upper ideal in Δ^+ generated by μ_{d_α} is not abelian. □

This allows us to obtain the promised characterisation of Δ^+_{nc}.

Theorem 3.5. *If $\lfloor\theta/2\rfloor\ne 0$, i.e., Δ is not of type $\mathbf{A}_n$, then $\lfloor\theta/2\rfloor$ is the unique maximal element of Δ^+_{nc}. Furthermore, $\lfloor\theta/2\rfloor\in\mathcal{H}$.*

Proof. It was noticed above that $\Delta^+_{\mathsf{nc}} \subset \mathcal{A}$, $\mathcal{A}$ has a unique maximal element, say $\hat\nu$, and $\hat\nu \preccurlyeq \lfloor\theta/2\rfloor$. By Corollary 3.4, for any $\alpha\in\Pi$, there is $\mu_\alpha \in \Delta^+_{\mathsf{nc}}$ such that $\mathsf{ht}_\alpha(\mu_\alpha) = d_\alpha$. Therefore $\bigvee\limits_{\alpha\in\Pi} \mu_\alpha \succcurlyeq \lfloor\theta/2\rfloor$. On the other hand, $\mu_\alpha \preccurlyeq \hat\nu$ for each α and hence $\bigvee\limits_{\alpha\in\Pi} \mu_\alpha \preccurlyeq \hat\nu \preccurlyeq \lfloor\theta/2\rfloor$. Thus, $\lfloor\theta/2\rfloor = \hat\nu$ is a root. If α_θ is the unique simple root such that $(\theta,\alpha_\theta)\ne 0$, then $\mathsf{ht}_{\alpha_\theta}(\theta) = 2$. Therefore $\lfloor\theta/2\rfloor \in \mathcal{H}$ whenever Δ is not $\mathbf{A}_n$. □

The fact that $\lfloor\theta/2\rfloor$ is the unique maximal non-commutative root has been observed in [9, Sect. 4] via a case-by-case analysis.

Example 3.6. If Δ is of type $\mathbf{E}_8$, then $\theta = \begin{smallmatrix}2345642\\3\end{smallmatrix}$ and $\lfloor\theta/2\rfloor = \begin{smallmatrix}1122321\\1\end{smallmatrix}$.

Remark 3.7. In the proof of Corollary 3.4 and then Theorem 3.5, we only need the property, which follows from Proposition 3.2, that $[\mathfrak{g}_\alpha(d_\alpha),\mathfrak{g}_\alpha(d_\alpha)] = \mathfrak{g}_\alpha(2d_\alpha)$.

For $\alpha\in\Pi$ with $\mathsf{ht}_\alpha(\theta) = 2$ or 3, this means that $[\mathfrak{g}_\alpha(1),\mathfrak{g}_\alpha(1)] = \mathfrak{g}_\alpha(2)$, which is obvious. This covers all classical simple Lie algebras, $\mathbf{E}_6$, and $\mathbf{G}_2$. For $\mathbf{E}_7$, $\mathbf{E}_8$, and $\mathbf{F}_4$, there are $\alpha\in\Pi$ such that $\mathsf{ht}_\alpha(\theta)\in\{4,5,6\}$. Then the required relation is $[\mathfrak{g}_\alpha(2),\mathfrak{g}_\alpha(2)] = \mathfrak{g}_\alpha(4)$ or $[\mathfrak{g}_\alpha(3),\mathfrak{g}_\alpha(3)] = \mathfrak{g}_\alpha(6)$. This can easily be verified case-by-case. However, our intention is to provide a case-free treatment of this property.

Another consequence of Kostant's theory [7] is that one obtains an explicit presentation of some maximal abelian ideals.

Proposition 3.8. *Suppose that* $\mathsf{ht}_\alpha(\theta) = 2d_\alpha + 1$ *is odd. Then* $\mathfrak{a} := \bigoplus_{j\geqslant d_\alpha+1}\mathfrak{g}_\alpha(j)$ *(i.e.,* $\Delta_{\mathfrak{a}} := \bigcup_{j\geqslant d_\alpha+1}\Delta_\alpha(j)$ *in the combinatorial set up) is a maximal abelian ideal of* $\mathfrak{b}$.

Proof. Obviously, $\mathfrak{a}$ is abelian. Let $\lambda\in\Delta_\alpha(d_\alpha)$ be the highest weight. It follows from the simplicity of all $\mathfrak{l}$-modules $\mathfrak{g}_\alpha(i)$ that λ is the only maximal element of $\Delta^+\setminus\Delta_{\mathfrak{a}}$. Therefore, it suffices to prove that the upper ideal $\Delta_{\mathfrak{a}}\cup\{\lambda\}$ is not abelian. Indeed, there is $\nu\in\Delta_\alpha(d_\alpha+1)$ such that $\nu+\lambda$ is a root (apply Corollary 3.3 with $i = d_\alpha$ and $j = d_\alpha+1$.) □

This prompts the following question. Suppose that $\mathsf{ht}_\alpha(\theta) = 2d_\alpha+1$. Then $\mathfrak{a} = I(\beta)_{\mathsf{max}}$ for some $\beta\in\Pi_l$. What is the relationship between α and β? We say below that $\alpha\in\Pi$ is *odd*, if $\mathsf{ht}_\alpha(\theta)$ is odd.

Example 3.9. 1) If $\mathsf{ht}_\alpha(\theta) = 1$, i.e., $d_\alpha = 0$, then $\mathfrak{a}$ is the (abelian) nilradical of the corresponding maximal parabolic subalgebra. Then $\beta = \alpha$. This covers all simple roots and all maximal abelian ideals in type $\mathbf{A}_n$.

2) For Δ of type $\mathbf{D}_n$ or $\mathbf{E}_n$, there are exactly three odd simple roots α.

– For $\mathbf{D}_n$, these are the endpoints of the Dynkin diagram and $d_\alpha = 0$. That is, again $\alpha = \beta$ in these cases.

– For $\mathbf{E}_n$, there are also odd simple roots with $d_\alpha\geqslant 1$ and then $\beta\ne\alpha$.

Nevertheless, the related maximal abelian ideals always correspond to the extreme nodes of the Dynkin diagram! Moreover, one always has $\mathsf{ht}_\beta(\theta)=d_\alpha+1$. (Similar things happen for $\mathbf{F}_4$ and $\mathbf{G}_2$.) It might be interesting to find a reason behind it.

Below is the table of all exceptional cases with $d_\alpha\geqslant 1$. The numbering of simple roots follows [4, Tables]. In particular, the numbering for $\mathbf{E}_8$ is $\genfrac{}{}{0pt}{}{1\,2\,3\,4\,5\,6\,7}{8}$ and the extreme nodes correspond to $\alpha_1,\alpha_7,\alpha_8$.

	$\mathbf{E}_6$	$\mathbf{E}_7$		$\mathbf{E}_8$			$\mathbf{F}_4$	$\mathbf{G}_2$
α	α_3	α_3	α_5	α_2	α_4	α_8	α_3	α_1
d_α	1	1	1	1	2	1	1	1
β	α_6	α_7	α_6	α_1	α_8	α_7	α_4	α_2
$\mathsf{ht}_\beta(\theta)$	2	2	2	2	3	2	2	2

4. Bijections related to the maximal abelian ideals

In this section, we consider abelian ideals of the form $I(\alpha)_{\mathsf{min}}$ and $I(\alpha)_{\mathsf{max}}$ for $\alpha\in\Pi_l$, and their derivatives (intersections and unions).

The following is Theorem 4.7 in [11].

Theorem 4.1. *For any $\alpha\in\Pi_l$, there is a one-to-one correspondence between $\min\bigl(I(\alpha)_{\min}\bigr)$ and $\max\bigl(\Delta^+\setminus I(\alpha)_{\max}\bigr)$. Namely, if $\eta\in\max\bigl(\Delta^+\setminus I(\alpha)_{\max}\bigr)$, then $\eta':=\theta-\eta\in\min\bigl(I(\alpha)_{\min}\bigr)$, and vice versa.*

It formally follows from this theorem that $\min\bigl(I(\alpha)_{\min}\bigr)$ and $\max\bigl(\Delta^+\setminus I(\alpha)_{\max}\bigr)$ both belong to $\mathcal{H}$. This is clear for the former, since $I(\alpha)_{\mathsf{min}}\subset\mathcal{H}$. And the key point in the proof of Theorem 4.1 was to demonstrate *a priori* that $\max\bigl(\Delta^+\setminus I(\alpha)_{\max}\bigr)\subset\mathcal{H}$.

Below, we provide a generalisation of Theorem 4.1, which is even more general than [11, Theorem 4.9], i.e., we will **not** assume that $S\subset\Pi_l$ be connected. Another improvement is that we give a conceptual proof of that generalisation, while Theorem 4.9 in [11] was proved case-by-case and no details has been given there.

The following is a key step for our generalisation of Theorem 4.1.

Theorem 4.2. *Suppose that $S\subset\Pi_l$ and $\gamma\in\max\bigl(\Delta^+\setminus\bigcup_{\alpha\in S}I(\alpha)_{\max}\bigr)$. If Δ is not of type $\mathbf{A}_n$, then $\gamma\in\mathcal{H}$.*

Proof. Here we have to distinguish two possibilities: either $\gamma\in\Delta^+_{\mathsf{nc}}$ or $\gamma\in\Delta^+_{\mathsf{com}}$.

(1) Suppose that $\gamma\in\Delta^+_{\mathsf{nc}}$ and assume that $\gamma\not\in\mathcal{H}$. Then there are $\eta,\eta'\succ\gamma$ such that $\eta+\eta'=\theta$, see [8, p. 1897]. Here both η and η' belong to $\mathcal{H}\cap\bigl(\bigcup_{\alpha\in S}I(\alpha)_{\max}\bigr)=\bigcup_{\alpha\in S}I(\alpha)_{\min}$. Since Δ is not of type $\mathbf{A}_n$, $\mathcal{H}$ has a unique minimal element (= the unique simple root that is not orthogonal to θ). Therefore, $\mu:=\eta\wedge\eta'$ exists and belongs to $\mathcal{H}$. (The existence of $\eta\wedge\eta'$ also follows from Theorem 2.4(1).) Since $\eta,\eta'\succcurlyeq\mu$, we have $\mu\in\Delta^+_{\mathsf{nc}}$. This implies

that $\mu \notin \bigcup_{\alpha\in S} I(\alpha)_{\min}$ and hence $\mu \notin \bigcup_{\alpha\in S} I(\alpha)_{\max}$. By the definition of meet, $\gamma \preccurlyeq \mu$. Furthermore, $\gamma \notin \mathcal{H}$ and $\mu \in \mathcal{H}$. Hence $\gamma \prec \mu$ and γ is not maximal in $\Delta^+ \setminus \bigcup_{\alpha\in S} I(\alpha)_{\max}$. A contradiction!

(2) Suppose that $\gamma \in \Delta^+_{\mathsf{com}}$. Consider the abelian ideal $J = I\langle \succcurlyeq\gamma\rangle$. By the assumption, $J \setminus \{\gamma\} \subset \bigcup_{\alpha\in S} I(\alpha)_{\max}$. On the other hand, since $J \not\subset I(\alpha)_{\mathsf{max}}$ for each $\alpha \in S$, we conclude that

$$J \cap \mathcal{H} \not\subset I(\alpha)_{\mathsf{max}} \cap \mathcal{H} = I(\alpha)_{\min},$$

see [11, Prop. 3.2]. For each $\alpha \in S$, we pick $\eta_\alpha \in (J \cap \mathcal{H}) \setminus I(\alpha)_{\mathsf{min}}$. Then $\eta_\alpha \succcurlyeq \gamma$. Since Δ is not of type $\mathbf{A}_n$, the meet $\eta := \underset{\alpha\in S}{\wedge} \eta_\alpha$ exists and belong to $\mathcal{H}$ (Remark 2.6) and also $\eta \succcurlyeq \gamma$. Note also that $\eta \notin I(\alpha)_{\min}$ for each $\alpha \in S$. (Otherwise, if $\eta \in I(\alpha_0)_{\min}$, then $\eta_{\alpha_0} \in I(\alpha_0)_{\min}$ as well.) Therefore, $\eta \notin \bigcup_{\alpha\in S} I(\alpha)_{\min}$ and hence $\eta \notin \bigcup_{\alpha\in S} I(\alpha)_{\max}$ (because $\eta \in \mathcal{H}$). As γ is assumed to be maximal in $\Delta^+ \setminus \bigcup_{\alpha\in S} I(\alpha)_{\max}$, we must have $\gamma = \eta \in \mathcal{H}$. □

Remark. For $\mathbf{A}_n$, this theorem remains true if we add the hypothesis that $S \subset \Pi_l$ is a *connected* subset in the Dynkin diagram, see also Example 4.4.

Theorem 4.3. *If $S \subset \Pi_l$ is arbitrary and Δ is not of type $\mathbf{A}_n$, then there is the bijection*

$$\eta \in \min\Big(\bigcap_{\alpha\in S} I(\alpha)_{\min}\Big) \xrightarrow{1:1} \eta' = \theta - \eta \in \max\Big(\Delta^+ \setminus \bigcup_{\alpha\in S} I(\alpha)_{\max}\Big).$$

Proof. (1) Suppose that $\eta \in \min\big(\bigcap_{\alpha\in S} I(\alpha)_{\min}\big)$. As Δ is not of type $\mathbf{A}_n$, there is a unique $\alpha_\theta \in \Pi$ such that $(\theta, \alpha_\theta) \neq 0$. Then $\theta - \alpha_\theta \in \mathcal{H}$ is the only root covered by θ. Therefore, $\theta - \alpha_\theta \in I(\alpha)_{\min}$ for all $\alpha \in \Pi_l$. Hence $\eta \neq \theta$ and hence $\eta' = \theta - \eta$ is a root (in $\mathcal{H}$). Since $\eta \in I(\alpha)_{\mathsf{min}}$, we have $\eta' \notin I(\alpha)_{\mathsf{min}}$, see [11, Lemma 3.3]. And this holds for each $\alpha \in S$. Hence $\eta' \notin \bigcup_{\alpha\in S} I(\alpha)_{\min}$ and thereby $\eta' \notin \bigcup_{\alpha\in S} I(\alpha)_{\max}$.

Assume that η' is not maximal in $\Delta^+ \setminus \bigcup_{\alpha\in S} I(\alpha)_{\max}$, i.e., $\eta' + \beta \notin \bigcup_{\alpha\in S} I(\alpha)_{\max}$ for some $\beta \in \Pi$. Again, $\eta' \prec \theta - \alpha_\theta$, hence $\eta' + \beta \in \mathcal{H} \setminus \{\theta\}$. Then $\theta - (\eta' + \beta) = \eta - \beta \in \mathcal{H}$ and arguing "backwards" we obtain that $\eta - \beta \in \bigcap_{\alpha\in S} I(\alpha)_{\min}$, which contradicts the fact that η is minimal.

(2) By Theorem 4.2, if $\eta' \in \max\big(\Delta^+ \setminus \bigcup_{\alpha\in S} I(\alpha)_{\max}\big)$, then $\eta' \in \mathcal{H}$. Under these circumstances, the previous part of the proof can be reversed. □

Example 4.4. Suppose that Δ is of type $\mathbf{A}_n$, with the usual numbering of simple roots. Then $I(\alpha_i)_{\mathsf{max}} = I\langle \succcurlyeq\alpha_i\rangle$ for all i and $\mathcal{H} = I(\alpha_1)_{\mathsf{max}} \cup I(\alpha_n)_{\mathsf{max}}$, where

$$I(\alpha_1)_{\mathsf{min}} = I(\alpha_1)_{\mathsf{max}} = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_1 - \varepsilon_n, \varepsilon_1 - \varepsilon_{n+1} = \theta\},$$
$$I(\alpha_n)_{\mathsf{min}} = I(\alpha_n)_{\mathsf{max}} = \{\varepsilon_n - \varepsilon_{n+1}, \dots, \varepsilon_2 - \varepsilon_{n+1}, \varepsilon_1 - \varepsilon_{n+1}\}.$$

If $S = \{\alpha_1, \alpha_n\}$, then S is not connected for $n \geqslant 3$, $I(\alpha_1)_{\mathsf{min}} \cap I(\alpha_n)_{\mathsf{min}} = \{\theta\}$, and $\max\big(\Delta^+ \setminus (I(\alpha_1)_{\mathsf{max}} \cup I(\alpha_n)_{\mathsf{max}})\big) = \{\varepsilon_2 - \varepsilon_n\}$. That is, Theorems 4.2 and 4.3 do not apply here. However, both remain true if S is assumed to be connected and $S \neq \Pi$. For instance, suppose that $S = \{\alpha_i, \alpha_{i+1}, \dots, \alpha_j\}$ with $1 < i < j < n$. Then $\min\bigl(\bigcap_{\alpha\in S} I(\alpha)_{\min}\bigr) = \{\varepsilon_1 - \varepsilon_{j+1}, \varepsilon_i - \varepsilon_{n+1}\}$ and $\max\bigl(\Delta^+ \setminus \bigcup_{\alpha\in S} I(\alpha)_{\max}\bigr) = \{\varepsilon_1 - \varepsilon_i, \varepsilon_{j+1} - \varepsilon_{n+1}\}$.

If $S = \Pi$, then $\bigcap_{\alpha\in\Pi} I(\alpha)_{\min} = \{\theta\}$ and $\Delta^+ = \bigcup_{\alpha\in\Pi} I(\alpha)_{\max}$.

As a by-product of Theorem 4.3, we derive a property of maximal abelian ideals outside type **A**. Given $S \subset \Pi_l$, let $\langle S\rangle$ be the smallest connected subset of Π_l containing S.

Theorem 4.5. *Let $S \subset \Pi_l$. Then*

(i) $\bigcap_{\alpha\in S} I(\alpha)_{\mathsf{min}} = \bigcap_{\alpha\in\langle S\rangle} I(\alpha)_{\mathsf{min}}$;

(ii) *if $\Delta \ne \mathbf{A}_n$, then* $\bigcup_{\alpha\in S} I(\alpha)_{\mathsf{max}} = \bigcup_{\alpha\in\langle S\rangle} I(\alpha)_{\mathsf{max}}$.

Proof. (i) By [11, Theorem 2.1], $\bigcap\limits_{\alpha\in S} I(\alpha)_{\mathsf{min}} = I(\gamma)_{\mathsf{min}}$, where $\gamma = \bigvee\limits_{\alpha\in S} \alpha$. It remains to notice that $\bigvee\limits_{\alpha\in S} \alpha = \sum_{\alpha\in\langle S\rangle} \alpha = \bigvee\limits_{\alpha\in\langle S\rangle} \alpha$.

(ii) This follows from (i) and Theorem 4.3. Namely, if Δ is not of type $\mathbf{A}_n$, then

$$\max\bigl(\Delta^+ \setminus \bigcup_{\alpha\in S} I(\alpha)_{\max}\bigr) = \max\bigl(\Delta^+ \setminus \bigcup_{\alpha\in\langle S\rangle} I(\alpha)_{\max}\bigr).$$

Hence both unions also coincide. □

The equality $\bigcap\limits_{\alpha\in S} I(\alpha)_{\mathsf{min}} = I(\bigvee\limits_{\alpha\in S} \alpha)_{\mathsf{min}}$ has interesting consequences. By [8, Prop. 4.6], the minimal elements of the abelian ideal $I(\gamma)_{\mathsf{min}}$ have the following description:

Let $w_\gamma \in W$ be a unique element of minimal length such that $w_\gamma(\theta) = \gamma$. If $\beta \in \Pi$ and $(\beta, \gamma^\vee) = -1$, then $w_\gamma^{-1}(\beta + \gamma) = w_\gamma^{-1}(\beta) + \theta \in \min(I(\gamma)_{\mathsf{min}})$. Conversely, any element of $\min(I(\gamma)_{\mathsf{min}})$ is obtained in this way.

For any γ of the form $\bigvee\limits_{\alpha\in S} \alpha$, the required simple roots β are easily determined, which yields the maximal elements of $\Delta^+ \setminus \bigcup_{\alpha\in S} I(\alpha)_{\max}$. We consider below the particular case in which $S = \Pi_l$.

Proposition 4.6. *Set $|\Pi_l| = \sum_{\alpha\in\Pi_l} \alpha$. If $|\Pi_l| \ne \theta$, i.e., Δ is not of type $\mathbf{A}_n$, then there is a unique $\hat{\boldsymbol\alpha} \in \Pi$ such that $|\Pi_l| + \hat{\boldsymbol\alpha}$ is a root. More precisely,*

– *if $\Delta \in \{\mathbf{D}\text{-}\mathbf{E}\}$, then $\hat{\boldsymbol\alpha}$ is the branching point in the Dynkin diagram;*

– *if $\Delta \in \{\mathbf{B}\text{-}\mathbf{C}\text{-}\mathbf{F}\text{-}\mathbf{G}\}$, then $\hat{\boldsymbol\alpha}$ is the unique short root that is adjacent to a long root in the Dynkin diagram.*

In all these cases, $w_{|\Pi_l|}^{-1}(\hat{\boldsymbol\alpha}) = -\lfloor\theta/2\rfloor$.

Proof. If $S = \Pi_l$, then $\bigcup_{\alpha\in\Pi_l} I(\alpha)_{\max} = \Delta^+_{\mathsf{com}}$. Hence $\max\bigl(\Delta^+ \setminus \bigcup_{\alpha\in\Pi_l} I(\alpha)_{\max}\bigr) = \{\lfloor\theta/2\rfloor\}$, see Theorem 3.5. Therefore, by Theorem 4.3, the unique minimal element of $I(|\Pi_l|)_{\mathsf{min}} = \bigcap_{\alpha\in\Pi_l} I(\alpha)_{\min}$ is $\theta - \lfloor\theta/2\rfloor =: \lceil\theta/2\rceil$. This means that there is a unique simple root $\hat{\boldsymbol\alpha}$ such that $(|\Pi_l|^\vee, \hat{\boldsymbol\alpha}) = -1$, i.e., $|\Pi_l| + \hat{\boldsymbol\alpha}$ is a root. Since $w_{|\Pi_l|}^{-1}(|\Pi_l| + \hat{\boldsymbol\alpha}) = \theta + w_{|\Pi_l|}^{-1}(\hat{\boldsymbol\alpha}) = \theta - \lfloor\theta/2\rfloor$, the last assertion follows.

Clearly, $\hat{\boldsymbol{\alpha}}$ specified in the statement satisfies the condition that $(|\Pi_l|, \hat{\boldsymbol{\alpha}}) < 0$. □

The A_n-case can partially be included in the DE-picture, if we formally assume that $\hat{\boldsymbol{\alpha}} = 0$ (because there is no branching point).

5. On the interval $[\lfloor\theta/2\rfloor, \lceil\theta/2\rceil]$

In this section, we first assume that Δ is not of type A_n. Since $\lfloor\theta/2\rfloor \in \mathcal{H}$, we have $\lceil\theta/2\rceil = \theta - \lfloor\theta/2\rfloor \in \mathcal{H}$ and also $\lfloor\theta/2\rfloor \preccurlyeq \lceil\theta/2\rceil$. We consider the interval between $\lfloor\theta/2\rfloor$ and $\lceil\theta/2\rceil$ in Δ^+. Let h be the Coxeter number of Δ.

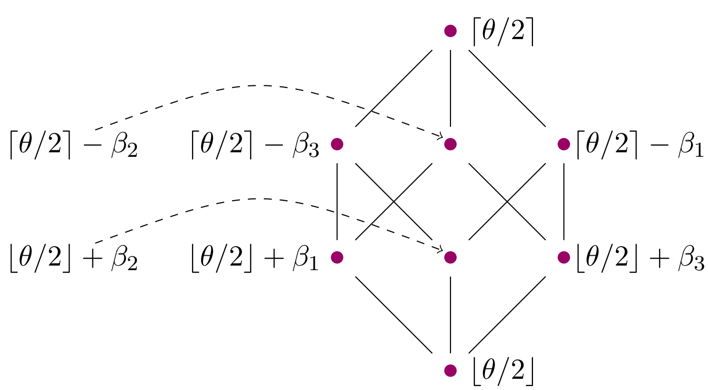
Proposition 5.1. *Set* $\mathfrak{J} = \{\gamma \in \Delta^+ \mid \lfloor\theta/2\rfloor \preccurlyeq \gamma \preccurlyeq \lceil\theta/2\rceil\}$.
- *if* $\Delta \in \{\mathsf{D}\text{-}\mathsf{E}\}$, *then* $\mathfrak{J} \simeq \mathbb{B}^3$ *and* $\mathsf{ht}(\lceil\theta/2\rceil) = (h/2) + 1$;
- *if* $\Delta \in \{\mathsf{B}\text{-}\mathsf{C}\text{-}\mathsf{F}\text{-}\mathsf{G}\}$, *then* $\mathfrak{J}$ *is a segment and* $\mathsf{ht}(\lceil\theta/2\rceil) = h/2$.

Proof. This can be verified case-by-case, but we also provide some *a priori* hints.
It follows from the definition of $\lfloor\theta/2\rfloor$, see Eq. (3·1), that

$$\lceil\theta/2\rceil - \lfloor\theta/2\rfloor = \theta - 2\lfloor\theta/2\rfloor = 2\lceil\theta/2\rceil - \theta = \sum_{\alpha:\ \mathsf{ht}_\alpha(\theta)\ \text{odd}} \alpha,$$

the sum of all odd simple roots. Let $\mathcal{O} \subset \Pi$ denote the set of odd roots. Then $\mathsf{ht}(\lceil\theta/2\rceil) - \mathsf{ht}(\lfloor\theta/2\rfloor) = \#\mathcal{O}$.

• In the simply-laced case, $(\theta - 2\lfloor\theta/2\rfloor, \lfloor\theta/2\rfloor^\vee) = 1 - 4 = -3$. Therefore, there are at least three $\alpha \in \mathcal{O}$ such that $(\alpha, \lfloor\theta/2\rfloor^\vee) = -1$, i.e., $\lfloor\theta/2\rfloor + \alpha \in \Delta^+$. On the other hand, for any $\gamma \in \Delta^+$, there are at most three $\alpha \in \Pi$ such that $\gamma + \alpha \in \Delta^+$ [10, Theorem 3.1(i)]. Thus, there are exactly three odd roots α_i such that $\lfloor\theta/2\rfloor + \alpha_i \in \Delta^+$. Actually, there are only three odd roots in the $\{\mathsf{D}\text{-}\mathsf{E}\}$-case. Hence every odd root can be added to $\lfloor\theta/2\rfloor$. Likewise, $(2\lceil\theta/2\rceil - \theta, \lceil\theta/2\rceil^\vee) = 3$ and the same three roots can be subtracted from $\lceil\theta/2\rceil$. This yields all six roots strictly between $\lfloor\theta/2\rfloor$ and $\lceil\theta/2\rceil$. If $\mathcal{O} = \{\beta_1, \beta_2, \beta_3\}$, then $\mathfrak{J}$ is as follows:



• In the non-simply laced cases, there is always a unique odd root and hence $\mathfrak{J} = \{\lfloor\theta/2\rfloor, \lceil\theta/2\rceil\}$. □

Remark 5.2. If Δ is of type $\mathbf{A}_n$, then $\lfloor\theta/2\rfloor = 0$ and $\lceil\theta/2\rceil = \theta$. Then $\mathfrak{J} = \Delta^+ \cup \{0\}$. However, this poset is not a modular lattice.

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