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SEPARATION OF PERIODS OF QUARTIC SURFACES

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Abstract. We give an effective lower bound on the distance between two distinct periods of a given quartic surface defined over the algebraic numbers. The main ingredient is the determination of height bounds on components of the Noether–Lefschetz loci. This makes it possible to study the Diophantine properties of periods of quartic surfaces and to certify a part of the numerical computation of their Picard groups.

1. Introduction

Periods are a countable set of complex numbers containing all the algebraic numbers as well as many of the transcendental constants of nature. In light of the ubiquity of periods in mathematics and the sciences, Kontsevich and Zagier (2001) ask for the development of an algorithm to check for the equality of two given periods. We solve this problem for periods coming from quartic surfaces by giving a computable separation bound, that is, a lower bound on the minimum distance between distinct periods.

Let $f \in \mathbb{C}[w, x, y, z]_4$ be a homogeneous quartic polynomial defining a smooth quartic X_f in $\mathbb{P}^3(\mathbb{C})$. The periods of X_f are the integrals of a nowhere vanishing holomorphic 2-form on X_f over integral 2-cycles in X_f. The periods can also be given in the form of integrals of a rational function

$$\frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{f(1, x, y, z)}, \tag{1}$$

where γ is a 3-cycle in $\mathbb{C}^3 \backslash X_f$. The integral (1) depends only on the homology class of γ. These periods form a group under addition. The geometry of quartic surfaces dictates that there are only 21 independent 3-cycles in $\mathbb{C}^3 \backslash X_f$. These give 21 periods $\alpha_1, \dots, \alpha_{21} \in \mathbb{C}$ such that the integral over any other 3-cycle is an integer linear combination of these periods.

It is possible to compute the periods to high precision (Sertöz 2019), typically to thousands of decimal digits, and to deduce from them interesting algebraic invariants such as the Picard group of X_f (Lairez and Sertöz 2019). This point of view has been fruitful for computing algebraic invariants for algebraic curves from their periods (Booker et al. 2016; Bruin et al. 2019; Costa et al. 2019; van Wamelen 1999).

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For quartic surfaces, the computation of the Picard group reduces to computing the lattice in $\mathbb{Z}^{21}$ of integer relations $x_1\alpha_1 + \cdots + x_{21}\alpha_{21} = 0$, $x_i \in \mathbb{Z}$. A basis for this lattice can be guessed from approximate α_i's using lattice reduction algorithms. But is it possible to *prove* that all guessed relations are true relations? Previous work related to this question (Simpson 2008) require explicit construction of algebraic curves on X_f, which becomes challenging very quickly. Instead, we give a method of proving relations by checking them at a predetermined finite precision. At the moment, this is equally challenging, but we conjecture that the numerical approach can be made asymptotically faster, see §4.5 for details.

The Lefschetz theorem on $(1,1)$-classes (§2.2) associates a divisor on X_f to any integer relation between the periods of X_f. In turn, the presence of a divisor imposes algebraic conditions on the coefficients of f. Such algebraic conditions define the *Noether–Lefschetz loci* on the space of quartic polynomials (§3). In addition to the degree computations of Maulik and Pandharipande (2013), we give height bounds on the polynomial equations defining the Noether–Lefschetz loci (Theorem 14). These lead to our main result (Theorem 17): Assume f has integer coefficients, then for $x_i \in \mathbb{Z}$,

$$x_1\alpha_1 + \cdots + x_{21}\alpha_{21} = 0 \text{ or } |x_1\alpha_1 + \cdots + x_{21}\alpha_{21}| > 2^{-c^{\max_i |x_i|^9}} \tag{2}$$

for some constant $c > 0$ depending only on f . The constant c is computable in rather simple terms and without prior knowledge of the Picard group of X_f. The result generalizes to f with algebraic coefficients (Theorem 19).

As a consequence of this separation bound, we apply a construction in the manner of Liouville (1851) and prove, for instance, that the number

$$\sum_{n\geqslant 0} (2 \uparrow\uparrow 3n)^{-1} \tag{3}$$

is not a quotient of two periods of a single quartic surface defined over $\mathbb{Q}$, where $2 \uparrow\uparrow 3n$ denotes an exponentiation tower with $3n$ twos (Theorem 20, with $\theta_{i+1} = 2^{2^{2^{\theta_i}}}$).

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2. Periods and deformations

2.1. **Construction of the period map.** For any non-zero homogeneous polynomial f in $\mathbb{C}[w,x,y,z]$, let X_f denote the surface in $\mathbb{P}^3$ defined as the zero locus of f. Let $R \doteq \mathbb{C}[w,x,y,z]$ and let $R_4 \subset R$ be the subspace of degree 4 elements. Let $U_4 \subset R_4$ denote the dense open subset of all homogeneous polynomials f of degree 4 such that X_f is smooth. For our purposes, it will be useful to consider not only the periods of a single quartic surface X_f but also the *period map* to study the dependence of periods on f.

The topology of X_f does not depend on f as long as X_f is smooth: given two polynomials f and $g \in U_4$, we can connect them by a continuous path in U_4 and the surface X_f deforms continuously along this path, giving a homeomorphism $X_f \simeq X_g$, which is uniquely defined up to isotopy. In particular, if we fix a base point $b \in U_4$, then for every $f \in \widetilde{U}_4$, where $\widetilde{U}_4$ is a universal covering of U_4, we have a uniquely determined isomorphism of cohomology groups $H^2(X_b,\mathbb{Z}) \simeq H^2(X_f,\mathbb{Z})$. Let $H_\mathbb{Z}$ denote the second cohomology group of X_b, which is isomorphic to $\mathbb{Z}^{22}$ (e.g. Huybrechts 2016, §1.3.3).

An element of $\widetilde{U}_4$ can be viewed as a polynomial $f \in U_4$ together with an identification of $H^2(X_f,\mathbb{Z})$ with $H_\mathbb{Z}$. We often work locally around a given polynomial f and, in that case, we do not actively distinguish between U_4 and its universal covering.

The group $H_\mathbb{Z}$ is endowed with an even unimodular pairing

$$(x,y) \in H_\mathbb{Z} \times H_\mathbb{Z} \to x \cdot y \in \mathbb{Z}, \tag{4}$$

given by the intersection form on cohomology. Through this pairing, the second homology and cohomology groups are canonically identified with one another. For K3 surfaces, such as smooth quartic surfaces in $\mathbb{P}^3$, the structure of the lattice $H_\mathbb{Z}$ with its intersection form is explicitly known (ibid., Proposition **1**.3.5). The fundamental class of a generic hyperplane section of X_f gives an element of $H_\mathbb{Z}$ denoted by h.

Furthermore, the complex cohomology group $H^2(X_f,\mathbb{C})$, which is just $H_\mathbb{C} \doteq H_\mathbb{Z} \otimes \mathbb{C}$, is isomorphic to the corresponding de Rham cohomology $H^2_{\mathrm{dR}}(X_f,\mathbb{C})$ group as follows. Elements of $H^2_{\mathrm{dR}}(X_f,\mathbb{C})$ are represented by differential 2-forms. To a form Ω one associates the element $\Theta(\Omega)$ of $H^2(X_f,\mathbb{C})$ given by the map

$$\Theta(\Omega)\colon [\gamma] \in H_2(X_f,\mathbb{C}) \mapsto \int_\gamma \Omega \in \mathbb{C}. \tag{5}$$

The group $H^2_{\mathrm{dR}}(X_f,\mathbb{C})$ has a distinguished element Ω_f, a nowhere vanishing holomorphic 2-form, described below. Every other holomorphic 2-form on X_f is a scalar multiple of Ω_f (ibid., Example **1**.1.3). Mapping Ω_f to $H_\mathbb{C}$ gives rise to the *period map*

$$\mathcal{P}\colon f \in \widetilde{U}_4 \mapsto \omega_f \doteq \Theta(\Omega_f) \in H_\mathbb{C}. \tag{6}$$

The coordinates of the *period vector* ω_f, in some fixed basis of $H_\mathbb{Z}$, generates the group of periods of X_f.

To make the connection clear, we first consider the tube map

$$T\colon H_2(X_f,\mathbb{Z}) \to H_3(\mathbb{P}^3 \backslash X_f,\mathbb{Z}), \tag{7}$$

constructed as follows (Griffiths 1969, §3). Let $\varepsilon > 0$ be small enough. For $x \in X$, the normal ε-circle over x is the set of all points $y \in \mathbb{P}^3$ such that $d(y, X) = \varepsilon$ and x is the closest point to y in X (which is unique if ε is small enough), that is $d(x, y) = \varepsilon$. The union of all normal ε-circles over the points of an effective 2-cycle $\gamma \in H_2(X, \mathbb{Z})$ is a 3-cycle in $\mathbb{P}^3 \backslash X$, denoted by $T(\gamma)$. The map T is a surjective morphism and its kernel is generated by the class of a hyperplane section of X_f. We choose Ω_f so that the following identity holds

$$\int_\gamma \Omega_f = \frac{1}{2\pi i} \int_{T(\gamma)} \frac{\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z}{f(1, x, y, z)}. \tag{8}$$

Therefore, in view of (5), the coefficients of ω_f in a basis of $H_\mathbb{Z}$ coincides with periods as defined in (1).

The image $\mathcal{D}$ of the period map $\mathcal{P}$ is called the *period domain*. It admits a simple description:

$$\mathcal{D} \doteq \mathcal{P}(\widetilde{U}_4) = \{w \in H_\mathbb{C} \backslash \{0\} \mid w \cdot h = 0, w \cdot w = 0, w \cdot \overline{w} > 0\}, \tag{9}$$

where "$\cdot$" denotes the intersection form on $H_\mathbb{Z}$ and h the fundamental class of a hyperplane section, as introduced above (Huybrechts 2016, Chapter 6). Moreover, by the local Torelli theorem for K3 surfaces (ibid., Proposition **6**.2.8), the map $\mathcal{P}$ is a submersion; its derivative at any point of $\widetilde{U}_4$ is surjective.

2.2. **The Lefschetz (1,1)-theorem.** The linear integer relations between the periods of a quartic surface X_f are in correspondence with formal linear combinations of algebraic curves in X_f. Let $C \subset X_f$ be an algebraic curve. Its fundamental class is the element $[C]$ of $H_\mathbb{Z}$ obtained as the Poincaré dual of the homology class of C. The Picard group $\operatorname{Pic}(X_f)$ of X_f is the sublattice of $H_\mathbb{Z}$ spanned by the fundamental classes of algebraic curves.

It follows from the definition that for any class $[\Omega] \in H^2_{\mathrm{dR}}(X_f)$ of a differential 2-form on X_f,

$$[C] \cdot \Theta(\Omega) = \int_C \Omega. \tag{10}$$

Moreover, if Ω is a holomorphic 2-form, then $\int_C \Omega = 0$ because the restriction of Ω to the complex 1-dimensional subvariety C vanishes. In particular $[C] \cdot \omega_f = 0$. It turns out that this condition characterizes the elements of $\operatorname{Pic}(X_f)$.

More precisely, let $H^{1,1}(X_f) \subset H_\mathbb{C}$ denote the space orthogonal to ω_f and $\overline{\omega}_f$, the conjugate of ω_f, with respect to the intersection form. This space is a direct summand in the Hodge decomposition of $H^2(X_f, \mathbb{C})$.

The Lefschetz (1,1)-theorem (Griffiths and Harris 1978, p. 163) asserts that the lattice of integer relations coincide with the Picard group:

$$\operatorname{Pic}(X_f) = H_\mathbb{Z} \cap H^{1,1}(X_f). \tag{11}$$

Noting that for any $\gamma \in H_{\mathbb{Z}}$, $\overline{\gamma} = \gamma$, where $\overline{\gamma}$ denotes the complex conjugate, we have $\overline{\omega_f \cdot \gamma} = \overline{\omega}_f \cdot \gamma$, so that (11) becomes

$$\operatorname{Pic}(X_f) = \{\gamma \in H_{\mathbb{Z}} \mid \gamma \cdot \omega_f = 0\}\,. \tag{12}$$

2.3. **A deformation argument.** Let $\gamma_1, \dots, \gamma_{22}$ be a basis of $H_{\mathbb{Z}}$. The space $H_{\mathbb{R}}$ (resp. $H_{\mathbb{C}}$) is endowed with the coefficient wise Euclidean (resp. Hermitian) norm

$$\left\| \sum_{i=1}^{22} x_i \gamma_i \right\|^2 \doteq \sum_{i=1}^{22} |x_i|^2\,. \tag{13}$$

For $\gamma \in H_{\mathbb{Z}}$, if $|\gamma \cdot \omega_f|$ is small enough, then γ is close to being an integer relation between the periods of X_f. We want to argue that, in this case, γ is a genuine integer relation between the periods of X_g for some polynomial $g \in U_4$ close to f.

Recall $f, g \in \widetilde{U}_4$ means f and g are smooth quartics with second cohomology identified with $H_{\mathbb{Z}}$. The space $\widetilde{U}_4$ inherits a metric from U_4 so that $\widetilde{U}_4 \to U_4$ is locally isometric. The metric on $U_4 \subset R_4 \simeq \mathbb{C}^{35}$ is induced by an inner product. The choice of an inner product will change the distances but this is absorbed into the constants in the statements below.

Let $f \in \widetilde{U}_4$ be fixed. For any $g \in R_4$ and $t \in \mathbb{C}$ small enough, the polynomials $f + tg \in R_4$ lift canonically to $\widetilde{U}_4$. For any $\gamma \in H_{\mathbb{C}}$ we consider the map

$$\phi_{\gamma,g}(t) \doteq \gamma \cdot \mathcal{P}(f + tg) \tag{14}$$

which is well-defined and analytic in a neighbourhood of 0 in $\mathbb{C}$.

Lemma 1. *There is a constant $C > 0$, depending only on f, such that for any $\gamma \in H_{\mathbb{C}}$ satisfying $\gamma \cdot h = 0$ and $|\gamma \cdot \overline{\omega}_f|\,\|\omega_f\| \leqslant \frac{1}{2}\|\gamma\|(\omega_f \cdot \overline{\omega}_f)$, there is a monomial $m \in R_4$ for which $\left|\phi'_{\gamma,m}(0)\right| \geqslant C\|\gamma\|$.*

Proof. Observe that $\phi'_{\gamma,m}(0) = \gamma \cdot \mathrm{d}_f\mathcal{P}(m)$. It follows that any constant C satisfying the following inequality would work, provided the infimum is not zero,

$$C < \inf_{\|\gamma\|=1} \max_m |\gamma \cdot \mathrm{d}_f\mathcal{P}(m)|\,, \tag{15}$$

with the infimum taken over γ satisfying $h \cdot \gamma = 0$ and $|\gamma \cdot \overline{\omega}_f|\,\|\omega_f\| \leqslant \frac{1}{2}(\omega_f \cdot \overline{\omega}_f)$. If the infimum is zero, it is realized by some γ of norm one that annihilates $\mathrm{d}_f\mathcal{P}(m)$ for each monomial m. It follows that γ is orthogonal (with respect to the intersection product) to the tangent space $T_{\omega_f}\mathcal{D}$ of $\mathcal{D}$ at ω_f. By (9),

$$T_{\omega_f}\mathcal{D} = \{w \in H_{\mathbb{C}} \mid w \cdot h = w \cdot \omega_f = 0\}\,. \tag{16}$$

It follows that $\gamma = ah + b\omega_f$ for some $a, b \in \mathbb{C}$. The condition $\gamma \cdot h = 0$ implies $a = 0$ (note that $\omega_f \cdot h = 0$ because $\omega_f \in \mathcal{D}$). Since $\|\gamma\| = 1$ we have $|\gamma \cdot \overline{\omega}_f| = \|\omega_f\|^{-1}(\omega_f \cdot \overline{\omega}_f)$ which is a contradiction. □

The next statement is proved using the following result of Smale (1986). Let ϕ be an analytic function on a maximal open disc around 0 in $\mathbb{C}$ with $\phi'(0) \neq 0$.

We define

$$\gamma_{\mathrm{Smale}}(\phi) \doteq \sup_{k\geqslant 2}\left|\frac{1}{k!}\frac{\phi^{(k)}(0)}{\phi'(0)}\right|^{\frac{1}{k-1}} \text{ and } \beta_{\mathrm{Smale}}(\phi) \doteq \left|\frac{\phi(0)}{\phi'(0)}\right|. \tag{17}$$

If $\beta_{\mathrm{Smale}}(\phi)\gamma_{\mathrm{Smale}}(\phi) \leqslant \frac{1}{34}$, then there is a $t \in \mathbb{C}$ such that $|t| \leqslant 2\beta_{\mathrm{Smale}}(\phi)$ and $\phi(t) = 0$ (Smale 1986; see also Blum et al. 1998, Chapter 8, Theorem 2).

Proposition 2. *For any $f \in \widetilde{U}_4$, there exists C_f and $\varepsilon_f > 0$ such that for all $\varepsilon < \varepsilon_f$ the following holds. For any $\gamma \in H_{\mathbb{R}}$, if $\gamma \cdot h = 0$ and $|\gamma\cdot\omega_f| \leqslant \varepsilon\|\gamma\|$ then there is a monomial $m \in R_4$ and $t \in \mathbb{C}$ such that $|t| \leqslant C_f\varepsilon$ and $\gamma\cdot\omega_{f+tm} = 0$.*

Proof. Let $\gamma \in H_{\mathbb{R}}$ such that $\gamma \cdot h = 0$ and

$$|\gamma\cdot\omega_f| \leqslant \left(\frac{\omega_f\cdot\overline{\omega}_f}{2\|\omega_f\|}\right)\|\gamma\|. \tag{18}$$

Since γ has real coefficients, we have $|\gamma\cdot\omega_f| = |\gamma\cdot\overline{\omega}_f|$ and we may apply Lemma 1 to obtain a monomial m and a constant C such that

$$\left|\phi'_{\gamma,m}(0)\right| \geqslant C\|\gamma\|. \tag{19}$$

It follows in particular that

$$\beta_{\mathrm{Smale}}(\phi_{\gamma,m}) \leqslant \frac{|\gamma\cdot\omega_f|}{C\|\gamma\|}. \tag{20}$$

Moreover, for any $k \geqslant 2$, and using $C \leqslant 1$,

$$\left|\frac{1}{k!}\frac{\phi^{(k)}_{\gamma,m}(0)}{\phi'_{\gamma,m}(0)}\right|^{\frac{1}{k-1}} \leqslant C^{-1}\left|\frac{\phi^{(k)}_{\gamma,m}(0)}{\|\gamma\|}\right|^{\frac{1}{k-1}} = C^{-1}\left|\frac{\gamma}{\|\gamma\|}\cdot \mathrm{d}^k_f\mathcal{P}(m,\dots,m)\right|^{\frac{1}{k-1}} \tag{21}$$

$$\leqslant C^{-1}\left|\!\left|\!\left|\tfrac{1}{k!}\mathrm{d}^k_f\mathcal{P}\right|\!\right|\!\right|^{\frac{1}{k-1}}, \tag{22}$$

where $|\!|\!|\cdot|\!|\!|$ is the operator norm defined as

$$\left|\!\left|\!\left|\tfrac{1}{k!}\mathrm{d}^k_f\mathcal{P}\right|\!\right|\!\right| \doteq \sup_{\gamma\in H_{\mathbb{C}}}\sup_{h_1,\dots,h_k}\frac{\left|\gamma\cdot\frac{1}{k!}\mathrm{d}^k_f\mathcal{P}(h_1,\dots,h_n)\right|}{\|\gamma\|\|h_1\|\cdots\|h_n\|}, \tag{23}$$

with supremum taken over $h_1,\dots,h_n \in \mathbb{C}[w,x,y,z]_4$. It follows that

$$\gamma_{\mathrm{Smale}}(\phi_{\gamma,m}) \leqslant C^{-1}\sup_{k\geqslant 2}\left|\!\left|\!\left|\tfrac{1}{k!}\mathrm{d}^k_f\mathcal{P}\right|\!\right|\!\right|^{\frac{1}{k-1}}. \tag{24}$$

Let Γ denote the right-hand side of (24). By Smale's theorem, together with (20) and (24), if $|\gamma\cdot\omega_f| \leqslant \frac{1}{34}C^2\Gamma^{-1}\|\gamma\|$, then there is a $t \in \mathbb{C}$ such that $|t| \leqslant 2C^{-1}|\gamma\cdot\omega_f|$ and $\gamma\cdot\mathcal{P}(f+tm) = 0$. The claim follows with $C_f \doteq 2C^{-1}$ and

$$\varepsilon_f \doteq \min\left(\tfrac{1}{34}C^2\Gamma^{-1}, \frac{\omega_f\cdot\overline{\omega}_f}{2\|\omega_f\|}\right). \qquad \square \tag{25}$$

The constants C_f and ε_f are actually computable with simple algorithms. The constant from Lemma 1 is not hard to get with elementary linear algebra. It only remains to compute an upper bound for Γ. We address this issue in §2.4.

Corollary 3. *For any $f \in \widetilde{U}_4$, any $\varepsilon < \varepsilon_f$, and any $\gamma \in H_{\mathbb{Z}}$, if $|\gamma \cdot \omega_f| \leqslant \frac{1}{4}\varepsilon$ then there exists a monomial $m \in R_4$ and $t \in \mathbb{C}$ such that $|t| \leqslant C_f \varepsilon$ and $\gamma \in \operatorname{Pic}(X_{f+tm})$.*

Proof. We may assume that $\gamma \cdot \omega_f \neq 0$ (otherwise choose any m and $t = 0$). Let $\gamma' = \gamma - \frac{1}{4}(\gamma \cdot h)h$. Since $h \cdot h = 4$, we have $\gamma' \cdot h = 0$. Moreover $\gamma' \cdot \omega_f = \gamma \cdot \omega_f \neq 0$. In particular, $\gamma' \neq 0$ and since $\gamma' \in \frac{1}{4}H_{\mathbb{Z}}$, we have $\|\gamma'\| \geqslant \frac{1}{4}$ and then

$$|\gamma' \cdot \omega_f| \leqslant 4\|\gamma'\| \, |\gamma \cdot \omega_f| \leqslant \varepsilon \|\gamma'\|, \tag{26}$$

and Proposition 2 applies. □

2.4. **Effective bounds for the higher derivatives of the period map.** In the proof of Proposition 2, only the quantity Γ is not clearly computable. We show in this section how to compute an upper bound for Γ using the Griffiths–Dwork reduction. We follow here Griffiths (1969).

Firstly, as a variant of (8) avoiding dehomogeneization, we write

$$\mathcal{P}(f) = \left(\frac{1}{2\pi i} \int_{T(\gamma_i)} \frac{\mathrm{Vol}}{f} \right)_{1 \leqslant i \leqslant 22} \tag{27}$$

where Vol is the projective volume form

$$\mathrm{Vol} \doteq w\mathrm{d}x\mathrm{d}y\mathrm{d}z - x\mathrm{d}w\mathrm{d}y\mathrm{d}z + y\mathrm{d}w\mathrm{d}x\mathrm{d}z - z\mathrm{d}w\mathrm{d}x\mathrm{d}y. \tag{28}$$

For any $k > 0$ and $a \in R_{4k-4}$, we denote

$$\int \frac{a \,\mathrm{Vol}}{f^k} \doteq \left(\frac{1}{2\pi i} \int_{T(\gamma_i)} \frac{a \,\mathrm{Vol}}{f^k} \right)_{1 \leqslant i \leqslant 22} \in H_{\mathbb{C}}. \tag{29}$$

For any $h \in R_4$ close enough to 0, we have the power series expansion

$$\int \frac{\mathrm{Vol}}{f+h} = \sum_{k \geqslant 1} (-1)^{k-1} \int \frac{h^{k-1} \,\mathrm{Vol}}{f^k}. \tag{30}$$

Proposition 4. *For any $k \geqslant 3$, there is a linear map $G_k \colon R_{4k-4} \to R_8$ such that*

$$\int \frac{a}{f^k} \mathrm{Vol} = \int \frac{G_k(a)}{f^3} \mathrm{Vol}\,.$$

Moreover, there is a computable constant C, which depends only on f, such that for any $k \geqslant 3$, $|||G_k||| \leqslant C^{k-3}$, where R is endowed with the 1-norm (55).

Before we begin the proof of proposition, let us show that this is enough to bound Γ. Let $A \colon a \in R_8 \mapsto \int \frac{a}{f^3} \mathrm{Vol} \in H_{\mathbb{C}}$, then, using (30) we obtain

$$\int \frac{\mathrm{Vol}}{f+h} = \sum_{k \geqslant 1} (-1)^{k-1} A(G_k(h^{k-1})), \tag{31}$$

and it follows that

$$\tfrac{1}{k!} \mathrm{d}_f^k \mathcal{P}(h_1, \ldots, h_k) = (-1)^k A(G_{k+1}(h_1 \cdots h_k)). \tag{32}$$

In particular,

$$\left\| \tfrac{1}{k!} \mathrm{d}_f^k \mathcal{P}(h_1, \dots, h_k) \right\| \leqslant \left\| A \right\| \left\| G_{k+1} \right\| \left\| h_1 \cdots h_n \right\|_1 \tag{33}$$

$$\leqslant \left\| A \right\| \left\| G_{k+1} \right\| \left\| h_1 \right\|_1 \cdots \left\| h_n \right\|_1, \tag{34}$$

and therefore $\left\| \tfrac{1}{k!} \mathrm{d}_f^k \mathcal{P} \right\| \leqslant \left\| A \right\| C^{k+1}$, and it follows

$$\Gamma \leqslant C \max\left(\left\| A \right\| C^2, 1 \right) \tag{35}$$

2.4.1. *Proof of Proposition 4.* Let $R = \mathbb{C}[w, x, y, z]$. We define two families of maps for this proof. First, for $d \geqslant 12$, a multivariate division map $Q_d \colon R_d \to R^4_{d-3}$, such that for any $a \in R_d$,

$$a = \sum_{i=0}^{3} Q_d(a)_i \, \partial_i f. \tag{36}$$

Note that such a map exists as soon as $d \geqslant 12$ by a theorem due to Macaulay (see Lazard 1977, Corollaire, p. 169). The choice of Q_d is not unique. We fix Q_{12} arbitrarily and define $Q_d(a)$, for $d > 12$ and $a \in R_{12}$, as follows. Write $a = \sum_{i=0}^{3} x_i a_i$, in such a way that the terms of the sum have disjoint monomial support, and define

$$Q_d(a) = \sum_{i=0}^{3} x_i Q_{d-1}(a_i). \tag{37}$$

It is easy to check that this definition satisfies (36).

Second, for $k \geqslant 3$, we define $G_k \colon R_{4k-4} \to R_8$ as follows. Begin with $G_3 = \mathrm{id}$ and then define G_k for $k \geqslant 4$ inductively as follows. For $a \in R_{4k-4}$ we write $(b_0, \dots, b_3) = Q_{4k-4}(a)$ and define

$$G_k(a) \doteq G_{k-1}\left(\frac{1}{k-1} (\partial_0 b_0 + \cdots + \partial_3 b_3) \right). \tag{38}$$

This map is the Griffiths–Dwork reduction, and it satisfies

$$\int_\gamma \frac{a\Omega}{f^k} = \int_\gamma \frac{G_k(a)\Omega}{f^3}. \tag{39}$$

Lemma 5. *For any $d \geqslant 12$, $\left\| Q_d \right\| \leqslant \left\| Q_{12} \right\|$, where R is endowed with the 1-norm and R^4 with the norm $\|(f_0, \dots, f_3)\|_1 \doteq \|f_0\|_1 + \cdots + \|f_3\|_1$.*

Proof. For any $a \in R_d$,

$$\|Q_d(a)\|_1 = \sum_{i=0}^{3} \|Q_d(a)_i\|_1 \leqslant \sum_{i=0}^{3} \sum_{j=0}^{3} \|x_j Q_{d-1}(a_j)_i\|_1 \tag{40}$$

$$= \sum_{i=0}^{3} \sum_{j=0}^{3} \|Q_{d-1}(a_j)_i\|_1 = \sum_j \|Q_{d-1}(a_j)\|_1 \tag{41}$$

$$\leqslant \left\| Q_{d-1} \right\| \sum_j \|a_j\|_1 = \left\| Q_{d-1} \right\| \|a\|_1, \tag{42}$$

using, for the last equality, that the terms a_j have disjoint monomial support. □

Lemma 6. *For any $k \geqslant 3$, $|||G_k||| \leqslant (4\,|||Q_{12}|||)^{k-3}$, where R is endowed with the 1-norm.*

Proof. We proceed by induction on k (the base case $k = 3$ is trivial since $G_3 = \mathrm{id}$). Let $a \in R_{4k-4}$ and $(b_0, \dots, b_3) = Q_{4k-4}(a)$. By (38), we have

$$\|G_k(a)\|_1 \leqslant \frac{|||G_{k-1}|||}{k-1} \left(\|\partial_0 b_0\|_1 + \cdots + \|\partial_3 b_3\|_1\right). \tag{43}$$

By induction hypothesis, $|||G_{k-1}||| \leqslant (4\,|||Q_{12}|||)^{k-4}$ and moreover $\|\partial_i b_i\|_1 \leqslant (4k - 7)\|b_i\|_1$, since each b_i has degree $4k - 7$. If follows that

$$\|G_k(a)\|_1 \leqslant (4\,|||Q_{12}|||)^{k-4} \frac{4k-7}{k-1} \left(\|b_0\|_1 + \cdots + \|b_3\|_1\right). \tag{44}$$

Next, we note that $\|b_0\|_1 + \cdots + \|b_3\|_1 = \|Q_{4k-4}(a)\|_1$ and, by Lemma 5, $|||Q_{4k-4}(a)||| \leqslant |||Q_{12}|||$. Therefore

$$\|G_k(a)\|_1 \leqslant (4\,|||Q_{12}|||)^{k-3} \|a\|_1, \tag{45}$$

and the claim follows. □

3. The Noether–Lefschetz locus

3.1. **Basic properties.** We define the Noether–Lefschetz locus for quartic surfaces and review a few classical properties, especially algebraicity, with a view towards Theorem 14 about the degree and the height of the equations defining the components of the Noether–Lefschetz locus.

3.1.1. *Definition.* The Noether–Lefschetz locus of quartics $\mathcal{NL}$ is the set of all $f \in U_4$ such that the rank of $\mathrm{Pic}(X_f)$ is at least 2. Equivalently, in view of (12), $\mathcal{NL}$ is the set of quartic polynomials f whose primitive periods (1) are $\mathbb{Z}$-linearly dependent.

The set $\mathcal{NL}$ is locally the union of smooth analytic hypersurfaces in U_4. To see this, let $\widetilde{\mathcal{NL}}$ be the lift of $\mathcal{NL}$ in the universal covering $\widetilde{U}_4$ of U_4. Recall $\mathcal{P} \colon \widetilde{U}_4 \to \mathcal{D}$ is the period map. The Lefschetz (1,1)-theorem implies

$$\widetilde{\mathcal{NL}} = \bigcup_{\gamma \in H_{\mathbb{Z}} \setminus \mathbb{Z}h} \mathcal{P}^{-1} \left\{ w \in \mathcal{D} \mid w \cdot \gamma = 0 \right\}. \tag{46}$$

That is, $\widetilde{\mathcal{NL}}$ is the pullback of smooth hyperplane sections of $\mathcal{D}$. Since $\mathcal{P}$ is a submersion, $\widetilde{\mathcal{NL}}$ is the union of smooth analytic hypersurfaces. It follows that $\mathcal{NL}$ is locally the union of smooth analytic hypersurfaces.

We break $\mathcal{NL}$ into algebraic pieces as follows. For any integers d and g, let $\mathcal{NL}_{d,g}$ be the set

$$\mathcal{NL}_{d,g} = \left\{ f \in U_4 \mid \exists \gamma \in \mathrm{Pic}(X_f) \setminus \mathbb{Z}h : \gamma \cdot h = d \text{ and } \gamma \cdot \gamma = 2g - 2 \right\}, \tag{47}$$

By replacing γ by $\gamma + h$ or $-\gamma$, we observe that

$$\mathcal{NL}_{d,g} = \mathcal{NL}_{d+4, g+d+2} = \mathcal{NL}_{-d,g}. \tag{48}$$

In particular, $\mathcal{NL}_{d,g}$ is equal to some $\mathcal{NL}_{d',g'}$ with $d' > 0$ and $g' \geqslant 0$, so that

$$\mathcal{NL} = \bigcup_{d>0} \bigcup_{g\geqslant 0} \mathcal{NL}_{d,g}. \tag{49}$$

For $\gamma \in H_\mathbb{Z}$, let $\Delta(\gamma) = (h\cdot\gamma)^2 - 4\gamma\cdot\gamma$. It is the opposite of the discriminant of the lattice generated by h and γ in $H_\mathbb{Z}$, with respect to the intersection product (and it is zero if $\gamma \in \mathbb{Z}h$). It follows from the Hodge index theorem (see Hartshorne 1977, Theorem V.1.9) that for any $f \in U_4$ and any $\gamma \in \operatorname{Pic}(X_f)$, $\Delta(\gamma) \geqslant 0$, with equality if and only if $\gamma \in \mathbb{Z}h$. If $\gamma\cdot h = d$ and $\gamma\cdot\gamma = 2g-2$, then $\Delta(\gamma) = d^2 - 8g + 8$. We obtain therefore that for any $d > 0$ and $g \geqslant 0$,

$$\mathcal{NL}_{d,g} = \begin{cases} \{f \in U_4 \mid \exists \gamma \in \operatorname{Pic}(X_f) : \gamma\cdot h = d \text{ and } \gamma\cdot\gamma = 2g-2\} & \text{if } d^2 > 8g-8 \\ \varnothing & \text{otherwise.} \end{cases} \tag{50}$$

It is in fact more natural to introduce, for $\Delta > 0$, the following locus

$$\mathcal{NL}_\Delta \doteq \{f \in U_4 \mid \exists\gamma \in \operatorname{Pic}(X_f) : \Delta(\gamma) = \Delta\} \tag{51}$$

$$= \bigcup_{\substack{d>0 \\ d^2 \equiv \Delta \bmod 8}} \mathcal{NL}_{d, \frac{d^2-\Delta}{8}+1}. \tag{52}$$

Due to (48), $\mathcal{NL}_\Delta$ reduces to a single $\mathcal{NL}_{d,g}$. Namely,

$$\mathcal{NL}_\Delta = \begin{cases} \mathcal{NL}_{4t, 2t^2 + \frac{8-\Delta}{8}}, & \text{if } \Delta \equiv 0 \bmod 8, \\ \mathcal{NL}_{4t+1, 2t^2+t+\frac{9-\Delta}{8}} & \text{if } \Delta \equiv 1 \bmod 8, \\ \mathcal{NL}_{4t+2, 2t^2+2t+\frac{12-\Delta}{8}}, & \text{if } \Delta \equiv 4 \bmod 8, \\ \varnothing & \text{otherwise,} \end{cases} \tag{53}$$

where $t = \lceil \frac{1}{4}\sqrt{\Delta} \rceil$. Conversely, each $\mathcal{NL}_{d,g} = \mathcal{NL}_{d^2-8g+8}$.

3.1.2. *Algebraicity.* For any $d > 0$ and $g \geqslant 0$, the set $\mathcal{NL}_{d,g}$ is either empty or an algebraic hypersurface in U_4. This is a classical result (e.g. Voisin 2003, Theorem 3.32) which we recall here to obtain an explicit algebraic description of $\mathcal{NL}_{d,g}$.

Lemma 7. *For any $f \in U_4$, $d > 0$ and $g \geqslant 0$ we have: $f \in \mathcal{NL}_{d,g}$ if and only if X_f contains an effective divisor with Hilbert polynomial $t \mapsto dt + 1 - g$.*

Proof. Assume that X_f contains an effective divisor C with Hilbert polynomial $t \mapsto td + 1 - g$. Since X_f is smooth, C is a locally principal divisor and gives an element γ of $\operatorname{Pic} X_f$. The integer d is the degree of C, so it is the number of points in the intersection with a generic hyperplane, that is $d = \gamma\cdot h$. Moreover, g is the arithmetic genus of C, which is determined by $2g - 2 = \gamma\cdot\gamma$ (Hartshorne 1977, Ex. III.5.3(b) and V.1.3(a)). So $f \in \mathcal{NL}_{d,g}$.

Conversely, let $f \in \mathcal{NL}_{d,g}$. By definition, there is a divisor C on X_f such that its class γ in $\operatorname{Pic} X_f$ satisfies $\gamma\cdot h = d$ and $\gamma\cdot\gamma = 2g-2$. From the Riemann–Roch theorem for surfaces (ibid., p. V.1.6) we get:

$$\dim H^0(X, \mathcal{O}_X(C)) + \dim H^0(X, \mathcal{O}_X(-C)) \geqslant \tfrac{1}{2}\gamma\cdot\gamma + 2 = g + 1 > 0$$

so that either C or $-C$ must be linearly equivalent to an effective divisor. Since $\gamma \cdot h > 0$, $-C$ can not be effective and therefore C must be. As above, the Hilbert polynomial of C is given by $t \mapsto dt + 1 - g$. □

In light of Lemma 7, the algebraicity of $\mathcal{NL}_{d,g}$ is proved by using the Hilbert scheme $\mathcal{H}_{d,g}$. The Hilbert scheme $\mathcal{H}_{d,g}$ of degree d and genus g curves in $\mathbb{P}^3$ is a projective scheme that parametrizes all the subschemes of $\mathbb{P}^3$ whose Hilbert polynomial is $t \mapsto dt + 1 - g$.

The Hilbert scheme $\mathcal{H}_{d,g}$ may contain components that are not desirable for our purposes. For example $\mathcal{H}_{3,0}$, which contains twisted cubics in $\mathbb{P}^3$, contains two irreducible components (Piene and Schlessinger 1985): a 12-dimensional component that is the closure of the space of all smooth cubic rational curves in $\mathbb{P}^3$; and a 15-dimensional component parametrizing the union of a plane cubic curve with a point in $\mathbb{P}^3$. We would be only interested in the first, not in the second component. So we introduce $\mathcal{H}'_{d,g}$, the union of components of $\mathcal{H}_{d,g}$ obtained by removing the components that does not correspond to locally-complete-intersection pure-dimensional subschemes of $\mathbb{P}^3$.

When $d^2 > 8g - 8$, Lemma 7 can be rephrased as

$$\mathcal{NL}_{d,g} = \operatorname{proj}_1 \left\{ (f, C) \in U_4 \times \mathcal{H}'_{d,g} \;\middle|\; C \subset X_f \right\}, \tag{54}$$

where proj_1 denotes the projection $U_4 \times \mathcal{H}'_{d,g} \to U_4$. Since $\mathcal{H}'_{d,g}$ is a projective variety, and the condition $C \subset X_f$ is algebraic, this shows that $\mathcal{NL}_{d,g}$ is a closed subvariety of U_4 (for more details about this construction, see Voisin 2003, §3.3).

We note furthermore that $\mathcal{NL}_{d,g}$ is clearly invariant under the action of the Galois group of algebraic numbers. Therefore, it can be defined over the rational numbers.

As a consequence, for any nonnegative integers d and g, there is a squarefree primitive homogeneous polynomial $\mathrm{NL}_{d,g} \in \mathbb{Z}[u_1, \dots, u_{35}]$ in the 35 coefficients of the general quartic polynomial that is unique up to sign and whose zero locus is $\mathcal{NL}_{d,g}$ in U_4. Similarly, we define NL_Δ upto sign as the unique squarefree primitive polynomial vanishing exactly on $\mathcal{NL}_\Delta$.

3.2. **Height of multiprojective varieties.** The mainstay of our results is a bound on the degree and size of the coefficients of the polynomials $\mathrm{NL}_{d,g}$. The determination of these bounds is based on (54) and involves the theory of heights of multiprojective varieties as developped by D'Andrea et al. (2013), and, before them, Bost et al. (1991), Krick et al. (2001), Philippon (1995), and Rémond (2001a,b), among others. We recall here the results that we need, following D'Andrea et al. (2013).

3.2.1. *Heights of polynomials.* Let $f = \sum_\alpha c_\alpha \mathbf{x}^\alpha \in \mathbb{C}[x_1, \dots, x_n]$. We recall the following different measures of height of f:

$$\|f\|_1 \doteq \sum_\alpha |c_\alpha|, \tag{55}$$

$$\|f\|_{\sup} \doteq \sup_{|x_1|=\cdots=|x_n|=1} |f(\mathbf{x})|, \tag{56}$$

$$m(f) \doteq \int_{[0,1]^n} \log\left|f\left(e^{2\pi i t_1}, \dots, e^{2\pi i t_n}\right)\right| \mathrm{d}t_1 \cdots \mathrm{d}t_n. \tag{57}$$

Lemma 8 (D'Andrea et al. 2013, Lemma 2.30)**.** *For any homogeneous polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$,*

$$\exp(m(f)) \leqslant \|f\|_{\sup} \leqslant \|f\|_1 \leqslant \exp(m(f))(n+1)^{\deg f}.$$

3.2.2. *The extended Chow ring.* The extended Chow ring (ibid., Definition 2.50) is a tool to track a measure of height of multiprojective varieties when performing intersections and projections. We present here a very brief summary. Bold letters refer to multi-indices and all varieties are considered over $\mathbb{Q}$. Let $\mathbf{n} \in \mathbb{N}^r$ and let $\mathbb{P}^{\mathbf{n}}$ be the multiprojective space $\mathbb{P}^{\mathbf{n}} = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_m}$.

An *algebraic cycle* is a finite $\mathbb{Z}$-linear combination $\sum_V n_V V$ of irreducible subvarieties of $\mathbb{P}^{\mathbf{n}}$. The *irreducible components* of an algebraic cycle as above are the irreducible varieties V such that $n_V \neq 0$. An algebraic cycle is *equidimensional* if all its irreducible components have the same dimension. An algebraic cycle is *effective* if $n_V \geqslant 0$ for all V. The *support* of X, denoted by $\operatorname{supp} X$, is the union of the irreducible components of X.

Let $A^*(\mathbb{P}^{\mathbf{n}}; \mathbb{Z})$ be the extended Chow ring, namely

$$A^*(\mathbb{P}^{\mathbf{n}}; \mathbb{Z}) \doteq \mathbb{R}[\eta, \theta_1, \dots, \theta_m]/(\eta^2, \theta_1^{n_1+1}, \dots, \theta_m^{n_m+1}), \tag{58}$$

where θ_i is the class of the pullback of a hyperplane from $\mathbb{P}^{n_i}$ and η is used to keep track of heights of varieties. For two elements a and b of this ring, we write $a \leqslant b$ when the coefficients of $b - a$ in the monomial basis are nonnegative.

To an algebraic cycle X of $\mathbb{P}^{\mathbf{n}}$ we associate an element $[X]_{\mathbb{Z}}$ of $A^*(\mathbb{P}^{\mathbf{n}}; \mathbb{Z})$ (ibid., Definition 2.50). If X is effective, then $[X]_{\mathbb{Z}} \geqslant 0$. The coefficients of the terms in $[X]_{\mathbb{Z}}$ for monomials not involving η record the usual multi-degrees of X. The terms involving η record mixed canonical heights of X. The definition of these heights is based on the heights of various Chow forms associated to X (ibid., §2.3). For the computations in this paper, we only need the following results.

Let $f \in \mathbb{Z}[\mathbf{x_1}, \dots, \mathbf{x_r}]$ be a nonzero multihomogeneous polynomial with respect to the group of variables $\mathbf{x}_1, \dots, \mathbf{x}_n$. We assume that f is *primitive*, that is, the g.c.d. of the coefficients of f is 1. The element associated in $A^*(\mathbb{P}^{\mathbf{n}}; \mathbb{Z})$ to the hypersurface $V(f) \subset \mathbb{P}^{\mathbf{n}}$ is (ibid., Proposition 2.53(2))

$$[V(f)]_{\mathbb{Z}} = m(f)\eta + \deg_{\mathbf{x}_1}(f)\theta_1 + \cdots + \deg_{\mathbf{x}_r}(f)\theta_r. \tag{59}$$

To such a polynomial f, we also associate (ibid., Eq. (2.57))

$$[f]_{\sup} \doteq \log(\|f\|_{\sup})\eta + \deg_{\mathbf{x}_1}(f)\theta_1 + \cdots + \deg_{\mathbf{x}_r}(f)\theta_r. \tag{60}$$

3.2.3. *Arithmetic Bézout theorem.* Let X be an effective cycle and H a hypersurface in $\mathbb{P}^{\mathbf{n}}$. They *intersect properly* if no irreducible component of X is in H. When X and H intersect properly, ones defines an intersection product $X \cdot H$, that is an effective cycle supported on $X \cap H$. If X is equidimensional of dimension r, then $X \cdot H$ is equidimensional of dimension $r-1$.

The following statement is an arithmetic Bézout bound that not only bounds the degree, as with the classical Bézout bound, but also the height of an intersection.

Theorem 9 (ibid., Theorem 2.58)**.** *Let X be an effective equidimensional cycle on $\mathbb{P}^{\mathbf{n}}$ and $f \in \mathbb{Z}[\mathbf{x}_1, \dots, \mathbf{x}_m]$. If X and $V(f)$ intersect properly, then $[X \cdot V(f)]_{\mathbb{Z}} \leqslant [X]_{\mathbb{Z}} \cdot [f]_{\sup}$.*

This theorem can be applied (as in ibid., Corollary 2.61) to bound the height of the irreducible components of a variety in terms of its defining equations.

Proposition 10. *Let $Z \subset \mathbb{P}^{\mathbf{n}}$ be an equidimensional variety and let X be $V(f_1, \dots, f_s) \cap Z$, where f_i is a multihomogeneous polynomial of multidegree at most $\mathbf{d}$ and sup-norm at most L. Let X_r be the union of all the irreducible components of X of codimension r in Z. Then*

$$[X_r]_{\mathbb{Z}} \leqslant [Z]_{\mathbb{Z}} \left(\log(sL)\eta + \sum_{i=1}^m d_i \theta_i \right)^r .$$

Proof. Let (y_{ij}) be a new group of variables, with $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$. Let $g_i \doteq \sum_{j=1}^s y_{ij} f_j$ and $X' \doteq V(g_1, \dots, g_r)$ in $\mathbb{P}^k \times Z$, with $k = rs - 1$ We first claim that $\mathbb{P}^k \times X_r$ is a union of components of X'. Indeed, let ξ_0 be the generic point of $\mathbb{P}^k$ and ξ_1 be the generic point of a component Y of X_r, so that $\xi = (\xi_0, \xi_1)$ is the generic point of the component $\mathbb{P}^k \times Y$ of $\mathbb{P}^k \times X_r$. Since X has codimension r at ξ_1, the generic linear combinations $g_1, \dots, g_r$ form a regular sequence at ξ (in other words, they form a regular sequence at ξ_1 for generic values of the v_{ij}). Therefore, X' has codimension r at ξ. Since $\mathbb{P}^k \times Y \subseteq X'$, it follows that $\mathbb{P}^k \times Y$ is a component of X'.

Let X'_r be the union of the components of codimension r of X'. The argument above shows that $[\mathbb{P}^k \times X_r]_{\mathbb{Z}} \leqslant [X'_r]_{\mathbb{Z}}$. Besides, by repeated application of (ibid., Corollary 2.61),

$$[X'_r]_{\mathbb{Z}} \leqslant [\mathbb{P}^k \times Z]_{\mathbb{Z}} \prod_{i=1}^r [g_i]_{\sup}. \tag{61}$$

We compute, using (59) that

$$[g_i]_{\sup} \leqslant \log(sL)\eta + \theta_0 + \sum_{i=1}^s d_i \theta_i. \tag{62}$$

Finally, we note that $[\mathbb{P}^k \times X_r]_{\mathbb{Z}} = [X_r]_{\mathbb{Z}}$ and $[\mathbb{P}^k \times Z]_{\mathbb{Z}} = [Z]_{\mathbb{Z}}$ (ibid., Proposition 2.51.3 and 2.66). ∎

Proposition 11. *Let X be an equidimensional closed subvariety of $\mathbb{P}^k \times \mathbb{P}^{\mathbf{n}}$ and let $Y \subset \mathbb{P}^{\mathbf{n}}$ be the projection of X. If Y is equidimensional, then*

$$\theta_0^k [Y]_{\mathbb{Z}} \leqslant \theta_0^{\dim X - \dim Y} [X]_{\mathbb{Z}} \quad \in \quad A^*(\mathbb{P}^k \times \mathbb{P}^{\mathbf{n}}; \mathbb{Z}),$$

where θ_0 is the variable attached to $\mathbb{P}^k$ in the extended Chow ring of $\mathbb{P}^k \times \mathbb{P}^{\mathbf{n}}$.

Proof. We will argue by induction on $r \doteq \dim X - \dim Y$. When $r = 0$, this is (D'Andrea et al. 2013, Proposition 2.64).

Suppose now that $r > 0$ and X is irreducible. Let $\mathbb{Q}[\mathbf{y}, \mathbf{x}_1, \dots, \mathbf{x}_\mathbf{m}]$ denote the multihomogeneous coordinate ring of $\mathbb{P}^k \times \mathbb{P}^{\mathbf{n}}$. There is an i, $0 \leqslant i \leqslant k$, such that $H \doteq V(y_i) \subset \mathbb{P}^k \times \mathbb{P}^{\mathbf{n}}$ intersects X properly (otherwise X would be included in all $V(y_i)$ and would be empty). Since the fibers of $X \to Y$ are positive dimensional, H intersects each fiber. In particular, the set-theoretical projections of X and $X \cap H$ coincide. As X is irreducible, so is Y. In particular, there is an irreducible component $X' \subset X \cap H$ that maps to Y. By induction hypothesis applied to X', $\theta_0^k [Y]_\mathbb{Z} \leqslant \theta_0^{\dim X' - \dim Y} [X']_\mathbb{Z}$. Moreover, $[X']_\mathbb{Z} \leqslant [X]_\mathbb{Z} [y_i]_{\sup}$, and, in view of (60), $[y_i]_{\sup} = \theta_0$. The claim follows.

If X is reducible, then we apply the inequality above to each of the irreducible components of Y together with an irreducible component of X mapping onto that component. ☐

3.3. **Explicit equations for the Noether–Lefschetz loci.** Following Gotzmann (1978), Bayer (1982), and the exposition of Lella (2012), we describe the equations defining the Hilbert schemes of curves in $\mathbb{P}^3$. An explicit description of the Noether–Lefschetz loci $\mathcal{NL}_{d,g}$ follows.

3.3.1. *The Hilbert schemes of curves.* For $d > 0$ and $g \geqslant 0$ let $\mathcal{H}_{d,g}$ be the Hilbert scheme of curves of degree d and genus g in $\mathbb{P}^3$. It parametrizes subschemes of $\mathbb{P}^3$ with Hilbert polynomial $p(m) \doteq dm + 1 - g$. Smooth curves in $\mathbb{P}^3$ of degree d and genus g, in particular, have Hilbert polynomial $p(m)$. Let $R = \mathbb{C}[w, x, y, z]$ be the homogeneous coordinate ring of $\mathbb{P}^3$. For $m \geqslant 0$, let R_m denote the mth homogeneous part of R and let $q(m) = \dim R_m - p(m)$.

The Hilbert scheme $\mathcal{H}_{d,g}$ can be realized in a Grassmannian variety as follows. A subscheme X of $\mathbb{P}^3$ is uniquely defined by a saturated homogeneous ideal I of R. If the Hilbert polynomial of X is p, then I is the saturation of the ideal generated by the degree r slice $I_r \doteq I \cap R_r$ (Gotzmann 1978; Bayer 1982, §II.10), where

$$r = \binom{d}{2} + 1 - g, \tag{63}$$

is the Gotzmann number of p (Bayer 1982, §II.1.17). For practical reasons, we need $r \geqslant 4$, so we define instead

$$r = \max\left(\binom{d}{2} + 1 - g, 4\right). \tag{64}$$

So X is entirely determined by I_r, which is a $q(r)$-dimensional subspace of R_r.

Let $\mathbb{G}$ be the Grassmannian variety of $q(r)$-dimensional subspaces of R_r. As a set, one can construct $\mathcal{H}_{d,g}$ as the subset of all $\Xi \in \mathbb{G}$ such that the ideal generated by Ξ in R defines a subscheme of $\mathbb{P}^3$ with Hilbert polynomial p. In fact, $\mathcal{H}_{d,g}$ is a subvariety that is defined by the following condition (ibid., §VI.1):

(65) $$\mathcal{H}_{d,g} = \{\Xi \in \mathbb{G} \mid \dim(R_1\Xi) \leqslant q(r+1)\},$$

where R_1 is the space of linear forms in w, x, y, z, so that $R_1\Xi$ is a subspace of R_{r+1}.

Several authors gave explicit equations for $\mathcal{H}_{d,g}$ in the Plücker coordinates (Bayer 1982; Brachat et al. 2016; Gotzmann 1978; Grothendieck 1961). We will prefer here a more direct path that avoids the Plücker embedding.

3.4. **Equations for the relative Hilbert scheme.** Define the relative Hilbert scheme of curves inside quartic surfaces

(66) $$\mathcal{H}_{d,g}(4) \doteq \{(f, C) \in \mathbb{P}(R_4) \times \mathcal{H}_{d,g} \mid C \subset V(f)\},$$

for each $d > 0$, $g \geqslant 0$.

We define the following auxiliary spaces to better describe (66). First, define the following ambient space

(67) $$\mathcal{A} \doteq \mathbb{P}(R_4) \times \mathbb{P}\left(\operatorname{End}(\mathbb{C}^{q(r)-N_{r-4}}, R_r)\right) \times \mathbb{P}\left(\operatorname{End}(R_{r+1}, \mathbb{C}^{p(r+1)})\right).$$

Second, let $\mathcal{B} = \{(f, \phi, \psi) \in \mathcal{A}\}$ be the set of all triples satisfying the conditions

(i) $R_{r-3}f \subseteq \ker\psi$,
(ii) $R_1 \operatorname{im}(\phi) \subseteq \ker\psi$,
(iii) $\operatorname{im}\phi \cap R_{r-4}f = 0$,
(iv) ϕ and ψ are full rank.

Finally, we denote by $\overline{\mathcal{B}}$ the Zariski closure of $\mathcal{B}$.

Lemma 12. *The map $\mathcal{B} \to \mathcal{H}_{d,g}(4)$ defined by $(f, \phi, \psi) \mapsto (f, R_{r-4}f + \operatorname{im}\phi)$ is well defined and surjective.*

Proof. Let $(f, \phi, \psi) \in \mathcal{B}$ and let $\Xi = R_{r-4}f + \operatorname{im}\phi$. Constraint (iv) implies that $\operatorname{im}\phi$ has dimension $q(r) - N_{r-4}$. Together with Constraint (iii), we have $\dim\Xi = q(r)$. Moreover, Constraint (iv) implies that $\ker\psi$ has dimension $q(r+1)$. In particular Since $R_1\Xi = R_{r-3}f + R_1 \operatorname{im}\phi$, Constraints (i) and (ii) implies that $R_1\Xi$ has dimension at most $q(r+1)$. So, $\Xi \in \mathcal{H}_{d,g}(4)$. Since $R_{r-4}f \subseteq \Xi$, the polynomial f is in the saturation of the ideal generated by Ξ. Hence, $(f, \Xi) \in \mathcal{H}_{d,g}(4)$.

Conversely, let $(f, \Xi) \in \mathcal{H}_{d,g}(4)$, then $R_{r-4}f \subset \Xi$ and there is a full rank map $\phi\colon \mathbb{C}^{q(r)-N_{r-4}} \to R_r$ such that $\operatorname{im}\phi$ complements $R_{r-4}f$ in Ξ. Furthermore, $\dim R_1\Xi \leqslant q(r+1)$, because $\Xi \in \mathcal{H}_{d,g}$, so there is a full rank map $\psi\colon R_{r+1} \to \mathbb{C}^{p(r+1)}$ such that $R_1\Xi \subseteq \ker\psi$. So (f, Ξ) is the image of $(f, \phi, \psi) \in \mathcal{B}$. ☐

Lemma 13. *For any $a \geqslant 0$, let $\overline{\mathcal{B}}_a$ be the union of the codimension a components of $\overline{\mathcal{B}}$. Then*

$$\left[\overline{\mathcal{B}}_a\right]_{\mathbb{Z}} \leqslant (15\log(d+2)\,\eta + \theta_1 + \theta_2 + \theta_3)^a$$

Proof. Let $\mathcal{B}'$ be the closed set defined by the constraints (i) and (ii). The constraints (iii) and (iv) are open, so any component of $\overline{\mathcal{B}}$ is a component of $\mathcal{B}'$. In particular $[\overline{\mathcal{B}}_a]_{\mathbb{Z}} \leqslant [\mathcal{B}'_a]_{\mathbb{Z}}$.

Constraint (i) is expressed with $p(r+1)N_{r-3}$ polynomial equations of multidegree $(1,0,1)$ (w.r.t. f, ϕ and ψ respectively). Namely, $\psi(mf) = 0$ for every monomial m in R_{r-3}. Each $p(r+1)$ components of the equation $\psi(mf) = 0$ involves a sum of 35 terms (since f, as a quartic polynomial, contains only 35 terms) with coefficients 1. So the 1-norm of these constraints is at most 35 (which is also at most N_r, since $r \geqslant 4$).

Constraint (ii) is expressed with $4p(r+1)(q(r) - N_{r-4})$ polynomial equations of multidegree $(0,1,1)$. Namely, $\psi(v\phi(e)) = 0$ for any basis vector e and any variable $v \in \{w,x,y,z\}$. Each $p(r+1)$ component of the equation $\psi(v\phi(e)) = 0$ involves a sum of N_r terms with coefficients 1. So the 1-norm of these constraints is at most N_r.

The claim is then a consequence of Proposition 10, with $s = p(r+1)N_{r-3} + 4p(r+1)(q(r) - N_{r-4})$ and $L = N_r$. We check routinely, with Mathematica, that $sL \leqslant (d+2)^{15}$. □

Theorem 14. *There is an absolute constant $A > 0$ such that for any $d > 0$ and $g \geqslant 0$ we have*

$$\deg(\mathrm{NL}_{d,g}) \leqslant A^{d^9} \text{ and } \|\mathrm{NL}_{d,g}\|_1 \leqslant 2^{A^{d^9}}.$$

Proof. We assume $\mathcal{NL}_{d,g}$ is non-emtpy, since these inequalities are trivially satisfied if $\mathcal{NL}_{d,g} = \varnothing$ with $\mathrm{NL}_{d,g} = 1$. Let $P_2 \doteq \mathbb{P}\left(\mathrm{End}(\mathbb{C}^{q(r)-N_{r-4}}, R_r)\right)$ and $P_3 \doteq \mathbb{P}\left(\mathrm{End}(R_{r+1}, \mathbb{C}^{p(r+1)})\right)$ denote the second and third factors of $\mathcal{A}$. Let $\alpha \doteq (q(r) - N_{r-4})N_r - 1$ and $\beta \doteq p(r+1)N_{r+1} - 1$ denote the dimensions of P_2 and P_3 respectively. Let $\mathcal{E}$ be the projection of $\overline{\mathcal{B}}$ on $\mathbb{P}(R_4) \times P_2$. The fibers of the map $\overline{\mathcal{B}} \to \mathcal{E}$ are projective subspaces of P_3 since Constraints (i) and (ii) are linear in ψ. The dimension of these fibers are $\beta' \doteq p(r+1)^2 - 1$. So, by Proposition 11,

$$\theta_3^{\beta}[\mathcal{E}]_{\mathbb{Z}} \leqslant \theta_3^{\beta'}\left[\overline{\mathcal{B}}\right]_{\mathbb{Z}}. \tag{68}$$

Next, the map $\mathcal{B} \to \mathcal{H}_{d,g}(4)$ factors through $\mathcal{E}$ and the fibers of the corresponding map $\mathcal{E} \to \mathcal{H}_{d,g}(4)$ have dimension $\alpha' \doteq (q(r) - N_{r-4})q(r) - 1$. Finally, let e be the dimension of the fibers of the map $\mathcal{H}_{d,g}(4) \to \mathcal{NL}_{d,g}$. (If this dimension is not generically constant, we work one component at a time.) Once again, by Proposition 11, we obtain

$$\theta_2^{\alpha}[\mathcal{NL}_{d,g}]_{\mathbb{Z}} \leqslant \theta_2^{\alpha'+e}[\mathcal{E}]_{\mathbb{Z}}. \tag{69}$$

Since $[\mathcal{NL}_{d,g}]_{\mathbb{Z}} = m(\mathrm{NL}_{d,g})\eta + \deg(\mathrm{NL}_{d,g})\theta_1$, taking $L = 15\log(d+2)$, we get

$$\deg \mathrm{NL}_{d,g} \leqslant \text{coeff of } \theta_1\theta_2^{\alpha-\alpha'-e}\theta_3^{\beta-\beta'} \text{ in } (L\eta + \theta_1 + \theta_2 + \theta_3)^{\alpha+\beta-\alpha'-\beta'-e+1} \tag{70}$$

$$\leqslant 3^{\alpha+\beta-\alpha'-\beta'-e+1}. \tag{71}$$

The exponent is a polynomial in d and g. Unless $d^2 \geqslant 8g-8$, $\mathcal{NL}_{d,g}$ is empty. So, we may bound the exponent with a polynomial only in d, which turns out to be of degree 9. Therefore, $\deg \mathrm{NL}_{d,g} \leqslant A^{d^9}$ for some constant $A > 0$.

Similarly,

$$m(\mathrm{NL}_{d,g}) \leqslant \text{coeff of } \eta\theta_2^{\alpha-\alpha'-e}\theta_3^{\beta-\beta'} \text{ in } (L\eta+\theta_1+\theta_2+\theta_3)^{\alpha+\beta-\alpha'-\beta'-e+1} \tag{72}$$

$$\leqslant (\alpha+\beta-\alpha'-\beta'-e+1)L3^{\alpha+\beta-\alpha'-\beta'-e} \tag{73}$$

$$\leqslant 2^{O(d^9)}. \tag{74}$$

By D'Andrea et al. (2013, Lemma 2.30.3),

$$\|\mathrm{NL}_{d,g}\|_1 \leqslant \exp(m(\mathrm{NL}_{d,g}))36^{\deg \mathrm{NL}_\Delta}, \tag{75}$$

and this implies the claim, for some other constant $A > 0$. □

For the following, we write $a \uparrow b$ for a^b. This is a right-associative operation.

Corollary 15. *There is an absolute constant $A > 0$ such that for any $\Delta > 0$,*

$$\deg(\mathrm{NL}_\Delta) \leqslant A \uparrow \Delta \uparrow \tfrac{9}{2} \text{ and } \|\mathrm{NL}_\Delta\|_1 \leqslant 2 \uparrow A \uparrow \Delta \uparrow \tfrac{9}{2}.$$

In fact, one can obtain the following explicit bounds

$$\deg(\mathrm{NL}_\Delta) \leqslant 3^{(\Delta+20)^{9/2}} \text{ and } \log_2 \|\mathrm{NL}_\Delta\|_1 \leqslant (\Delta+60)^5 3^{(\Delta+20)^{9/2}}.$$

Proof. The first statement follows directly from (53) and Theorem 14 using a different A. The second statement is found by carrying out the arguments in the proof of Theorem 14 with the help of a computer algebra system. □

3.5. **How good are these bounds?** We can compare our degree bounds for NL_Δ to the exact degrees computed by Maulik and Pandharipande (2013), from which it actually follows that

$$\deg \mathrm{NL}_\Delta = O(\Delta^{\frac{19}{2}}). \tag{76}$$

This sharper bound does not directly imply a sharper bound on the height of NL_Δ but it suggests the following conjecture. This would improve subsequently Theorems 17 and 20. In particular, Equation (2) would be exponential in the size of the coefficients, as opposed to being doubly exponential.

Conjecture 16. *There is a constant $c > 0$ such that for any $\Delta > 0$,*

$$\|\mathrm{NL}_\Delta\|_1 \leqslant c \uparrow \Delta \uparrow \tfrac{19}{2}.$$

Now we turn to the details of (76). Following Maulik and Pandharipande (2013) (but replacing q by q^8), consider the following power series

$$A \doteq \sum_{n\in\mathbb{Z}} q^{n^2}, \quad B \doteq \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2}, \quad \Psi = 108 \sum_{n>0} q^{8n^2}, \tag{77}$$

and Θ defined by

$$2^{22}\Theta \doteq 3A^{21} - 81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 - 20007A^{16}B^5 - 169092A^{15}B^6 - 120636A^{14}B^7 - 621558A^{13}B^8 - 292796A^{12}B^9 - 1038366A^{11}B^{10} - 346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} - 361908A^7B^{14} - 56364A^6B^{15} - 60021A^5B^{16} - 4812A^4B^{17} - 1881A^3B^{18} - 27A^2B^{19} + B^{21}. \tag{78}$$

From (Maulik and Pandharipande 2013, Corollary 2), we have, for any $\Delta > 0$,

$$\deg \mathrm{NL}_\Delta \leqslant \text{coefficient of } q^\Delta \text{ in } \Theta - \Psi. \tag{79}$$

In fact, this is an equality when the components of $\mathcal{NL}_\Delta$ are given appropriate multiplicities. Let $\Theta[k]$ denote the coefficient of q^k in Θ. By (79), we only need to bound $\Theta[\Delta]$ in order to bound $\deg \mathrm{NL}_\Delta$. To do so, replace every negative sign in the definition of Θ by a positive sign, including those in B, to obtain the *coefficientwise* inequality

$$\Theta \leqslant 6\Big(\sum_{n\in\mathbb{Z}} q^{n^2}\Big)^{21}. \tag{80}$$

The coefficient of q^k in $\left(\sum_{n\in\mathbb{Z}} q^{n^2}\right)^{21}$ is

$$r_{21}(k) \doteq \#\left\{(a_1,\dots,a_{21}) \in \mathbb{Z}^{21} \;\middle|\; \sum_i a_i^2 = k\right\}. \tag{81}$$

The asymptotic bound $r_d(k) = O(x^{\frac{d}{2}-1})$, for $d > 4$, is well known (e.g. Krätzel 2000, Satz 5.8).

4. Separation bound

We now state and prove the main results. Recall that $a \uparrow b = a^b$ is right associative and for $\gamma \in H_{\mathbb{Z}}$ we defined the discriminant $\Delta(\gamma)$ as $(\gamma\cdot h)^2 - 4\gamma\cdot\gamma$.

Theorem 17. *For any $f \in \mathbb{Z}[w,x,y,z] \cap U_4$ there is a computable constant $c > 1$ such that for any $\gamma \in H^2(X_f,\mathbb{Z})$, if $\gamma\cdot\omega_f \neq 0$, then*

$$|\gamma\cdot\omega_f| > \left(2 \uparrow c \uparrow \Delta(\gamma) \uparrow \tfrac{9}{2}\right)^{-1}.$$

4.1. Multiplicity of Noether–Lefschetz loci.

The multiplicity of some nonzero polynomial $F \in \mathbb{C}[x_1,\dots,x_s]$ at a point $p \in \mathbb{C}^s$ is the unique integer k such that all partial derivatives of F of order $< k$ vanish at p and some partial derivative of order k does not. It is denoted by $\mathrm{mult}_p F$.

The multiplicity of NL_Δ at some $f \in U_4$ is related to the elements of $\mathrm{Pic}(X_f)$ with discriminant Δ. For $\Delta > 0$, let E_Δ be a set of representatives of the equivalence classes of the relation $\sim$ on $H_{\mathbb{Z}}$ defined by

$$\gamma \sim \gamma' \text{ if } \exists a \in \mathbb{Q}^*, b \in \mathbb{Q} : \gamma' = a\gamma + bh. \tag{82}$$

Lemma 18. *For any $f \in U_4$ and any $\Delta > 0$,*

$$\operatorname{mult}_f \mathrm{NL}_\Delta = \#\left(\operatorname{Pic} X_f \cap E_\Delta\right).$$

Proof. Let $\widetilde{\mathcal{NL}}_\Delta$ be the lift of $\mathcal{NL}_\Delta$ in $\widetilde{U}_4$. Arguing as in §3.1.1, $\widetilde{\mathcal{NL}}_\Delta$ is the union of smooth analytic hypersurfaces:

$$\widetilde{\mathcal{NL}}_\Delta = \bigcup_{\eta \in E_\Delta} \mathcal{P}^{-1}\left\{w \in \mathcal{D} \mid w \cdot \eta = 0\right\}. \tag{83}$$

Then the same holds locally for $\mathcal{NL}_\Delta$.

For any $f \in U_4$ it follows from the smoothness of branches of $\mathcal{NL}_\Delta$ that $\operatorname{mult}_f \mathrm{NL}_\Delta$ is exactly the number of branches meeting at f. The branches meeting at f are described by the elements of $\operatorname{Pic} X_f$ with discriminant Δ. Two elements γ and γ' describe the same branch (that is the same hyperplane section of $\mathcal{D}$) if and only if $\gamma' \sim \gamma$. So $\operatorname{mult}_f \mathrm{NL}_\Delta$ is exactly the number of equivalence classes in $\{\gamma \in \operatorname{Pic} X_f \mid \Delta(\gamma) = \Delta\}$ for this relation. □

4.2. **Proof of Theorem 17.** We first apply Corollary 3. Let $\varepsilon = 4\,|\gamma \cdot \omega_f|$. The corollary gives constants $C_f > 0$ and $\varepsilon_f > 0$ (depending only on f) such that if $\varepsilon < \varepsilon_f$ then there exists a monomial $m \in R_4$ and $t \in \mathbb{C}$ such that

$$|t| \leqslant C_f \varepsilon \tag{84}$$

and

$$\gamma \in \operatorname{Pic} X_{f+tm}. \tag{85}$$

Assume that $\varepsilon < \varepsilon_f$ and let t and m be as above. As u varies, the number $\#\left(\operatorname{Pic}(X_{f+um}) \cap E_\Delta\right)$ has a strict local maximum at $u = t$. By Lemma 18, so does $\operatorname{mult}_{f+um} \mathrm{NL}_{\Delta(\gamma)}$. In particular, there is some higher-order partial derivative of NL_Δ which vanishes at $f + tm$ but not at $f + um$, for u close to but not equal to t. Let $\alpha \in \mathbb{N}^{35}$ be the multi-index for which

$$P \doteq \frac{1}{\alpha_1! \cdots \alpha_{35}!} \frac{\partial^{|\alpha|} \mathrm{NL}_\Delta}{\partial u^\alpha} \in \mathbb{Z}[u_1, \ldots, u_{35}] \tag{86}$$

is this derivative. For a monomial $u^\beta \doteq u_1^{\beta_1} \cdots u_{35}^{\beta_{35}}$ we have

$$\frac{1}{\alpha_1! \cdots \alpha_{35}!} \frac{\partial^{|\alpha|} u^\beta}{\partial u^\alpha} = \prod_{i=1}^{35} \binom{\beta_i}{\alpha_i} u^{\beta - \alpha}. \tag{87}$$

Since $\binom{\beta_i}{\alpha_i} \leqslant 2^{\beta_i}$, it follows that

$$\left\| \frac{1}{\alpha_1! \cdots \alpha_{35}!} \frac{\partial^{|\alpha|} \mathrm{NL}_\Delta}{\partial u^\alpha} \right\|_1 \leqslant 2^{\deg \mathrm{NL}_\Delta} \|\mathrm{NL}_\Delta\|_1. \tag{88}$$

Let $Q \in \mathbb{Z}[x]$ be the integer polynomial $Q(x) \doteq P(f + xm)$. By construction $Q \neq 0$ and $Q(t) = 0$. Clearly $\deg Q \leqslant \deg \mathrm{NL}_\Delta$, and we check that

$$\|Q\|_1 \leqslant \|P\|_1 \left(\|f\|_1 + 1\right)^{\deg P}. \tag{89}$$

and then

$$\|Q\|_1 \leqslant 2^{\deg \mathrm{NL}_\Delta} \|\mathrm{NL}_\Delta\|_1 \left(\|f\|_1 + 1\right)^{\deg \mathrm{NL}_\Delta}. \tag{90}$$

From Corollary 15, we find a constant c depending only on f such that

$$\deg Q \leqslant c \uparrow \Delta \uparrow \tfrac{9}{2} \text{ and } \|Q\|_1 \leqslant 2 \uparrow c \uparrow \Delta \uparrow \tfrac{9}{2}. \tag{91}$$

We write $Q = \sum_{i=0}^{\deg Q} q_i x^i$. Let k be the smallest integer such that $q_k \neq 0$. Since $Q(t) = 0$, it follows that

$$\left|q_k t^k\right| \leqslant \sum_{i=k+1}^{\deg Q} \left|q_i t^i\right| \tag{92}$$

If $\varepsilon < C_f^{-1}$, we have $|t| < 1$, by (84), and it follows that $\left|q_k t^k\right| \leqslant \left|t^{k+1}\right| \|Q\|_1$. Since $q_k \in \mathbb{Z}$ and nonzero, it follows that

$$|t| \geqslant \frac{1}{\|Q\|_1}. \tag{93}$$

By (84), this leads to

$$\varepsilon \geqslant \left(2 \uparrow c \uparrow \Delta \uparrow \tfrac{9}{2}\right)^{-1}, \tag{94}$$

for some other constant c which depends only on f. Recall that (94) holds with the assumption that $\varepsilon \leqslant \varepsilon_f$ and $\varepsilon < C_f^{-1}$. However, we can choose c large enough so that the right-hand side of (94) is smaller that ε_f and C_f^{-1}. Then (94) holds unconditionally, concluding the proof of Theorem 17. □

4.3. **Quartics with algebraic coefficients.** Let $K \subset \mathbb{C}$ be a number field of degree $D = [K : \mathbb{Q}]$. Suppose that $f \in K[w, x, y, z]_4 \cap U_4$ has coefficients that are algebraic integers in K. Let $H > 0$ be an upper bound for the absolute logarithmic Weil height for the coefficient vector of f (Waldschmidt 2000, p.77).

Theorem 19. *Let f and $H, D > 0$ be as above. Then there is a computable constant $c > 1$ such that for any $\gamma \in H^2(X_f, \mathbb{Z})$, if $\gamma \cdot \omega_f \neq 0$, we have*

$$|\gamma \cdot \omega_f| > \left(2 \uparrow c \uparrow \Delta(\gamma) \uparrow \tfrac{9}{2}\right)^{-D(1+H)}.$$

Proof. The proof of Theorem 17 carries through, with the sole exception that $Q(x)$ no longer has integer coefficients. If q_k is the first non-zero coefficient of $Q(x)$, then q_k is an algebraic integer defined by a polynomial expression $\widetilde{q}_k(f)$ in coefficients of f with $\widetilde{q}_k$ having integer coefficients. Therefore, the number "1" in (93) must be replaced by a suitable lower bound on the norm of q_k. For this, we use Liouville's inequality (ibid., Proposition 3.14):

$$|q_k| \geqslant \|\widetilde{q}_k\|_1^{-D+1} e^{-DH \deg \widetilde{q}_k}. \tag{95}$$

It is easy to see that $\deg \widetilde{q}_k \leqslant \deg \mathrm{NL}_\Delta$ and $\|\widetilde{q}_k\|_1 \leqslant 2^{\deg \mathrm{NL}_\Delta} \|\mathrm{NL}_\Delta\|_1$, the latter can be bounded by $\|Q\|_1$. The result follows for some other c. □

4.4. **Numbers *à la* Liouville.** Let $(\theta_i)_{i \geqslant 0}$ be a sequence of positive integers such that θ_i is a strict divisor of θ_{i+1} for all $i \geqslant 0$ (in particular $\theta_i \geqslant 2^i$.) Consider the number

$$L_\theta \doteq \sum_{i=0}^{\infty} \theta_i^{-1}.$$

As a corollary to the separation bound obtained in Theorem 17, the following result states that L_θ is not a ratio of periods of quartic surfaces when θ grows too fast.

Theorem 20. *If $|\theta_{i+1}| \geqslant 2 \uparrow 2 \uparrow \theta_i \uparrow 10$, for all i large enough, then L_θ is not equal to $\frac{\gamma_1 \cdot \omega_f}{\gamma_2 \cdot \omega_f}$ for any $\gamma_1, \gamma_2 \in H_{\mathbb{Z}}$ and any $f \in U_4$ with rational coefficients.*

Proof. Let $l_k = \sum_{i=0}^{k} \theta_i^{-1}$. Since θ_i divides θ_{i+1}, we can write $l_k = \frac{u_k}{\theta_k}$ for some integer u_k. And since the divisibility is strict, $\theta_i \geqslant 2^i$ and $u_k \leqslant 2\theta_k$. Moreover

$$0 < L_\theta - l_k \leqslant 2\theta_{k+1}^{-1}, \tag{96}$$

using $\theta_{k+i+1} \geqslant 2^i \theta_{k+1}$, for any $i \geqslant 0$. Assume now that $L_\theta = \frac{\gamma_1 \cdot \omega_f}{\gamma_2 \cdot \omega_f}$ for some $\gamma_1, \gamma_2 \in H_{\mathbb{Z}}$ and some $f \in U_4$ with rational coefficients. Then, with

$$\gamma_k \doteq \theta_k \gamma_1 - u_k \gamma_2, \tag{97}$$

we check that $\Delta(\gamma_k) = O(\theta_k^2)$ and that

$$0 < |\gamma_k \cdot \omega_f| = |\theta_k| \, |\gamma_2 \cdot \omega_f| \, |L_\theta - l_k| \leqslant C \frac{\theta_k}{\theta_{k+1}}, \tag{98}$$

for some constant C. By Theorem 17, we obtain therefore

$$(2 \uparrow c \uparrow \theta_k \uparrow 9)^{-1} \leqslant C \frac{\theta_k}{\theta_{k+1}}, \tag{99}$$

for some constant $c > 0$ which depends only on f. This contradicts the assumption on the growth of θ. □

4.5. **Computational complexity.** Given a polynomial $f \in \mathbb{Z}[w, x, y, z] \cap U_4$ and a cohomology class $\gamma \in H^2(X_f, \mathbb{Z})$, we can decide if $\gamma \in \operatorname{Pic}(X_f)$ (that is $\gamma \cdot \omega_f = 0$) as follows:

(a) Compute the constant c in Theorem 17;
(b) Let $\varepsilon = \left(2 \uparrow c \uparrow \Delta(\gamma) \uparrow \frac{9}{2}\right)^{-1}$ and compute an approximation $s \in \mathbb{C}$ of the period $\gamma \cdot \omega_f$ such that $|s - \gamma \cdot \omega_f| < \frac{1}{2}\varepsilon$.

Then γ is in $\operatorname{Pic}(X_f)$ if and only if $|s| < \frac{1}{2}\varepsilon$.

Computing the Picard group itself is an interesting application of this procedure. Algorithms for computing the Picard group of X_f, or even just the rank of it, break the problem into two: a part gives larger and larger lattices inside $\operatorname{Pic}(X_f)$ while the other part gets finer and finer upper bounds on the rank of $\operatorname{Pic}(X_f)$ (Charles 2014; Hassett et al. 2013; Poonen et al. 2015). The computation stops when the two parts meet. Approximations from the inside are based on finding sufficiently many elements of $\operatorname{Pic}(X_f)$. So while deciding the membership of γ in $\operatorname{Pic}(X_f)$ can be solved by computing $\operatorname{Pic}(X_f)$ first, it makes sense not to assume prior knowledge of the Picard group and to study the complexity of deciding membership as $\Delta(\gamma) \to \infty$, with f fixed.

Step (a) does not depend on γ, so only the complexity of Step (b) matters, that is the numerical approximation of $\gamma \cdot \omega_f$. This approximation amounts to numerically solving a Picard–Fuchs differential equation (Sertöz 2019) and the complexity is $(\log \frac{1}{\varepsilon})^{1+o(1)}$ (Beeler et al. 1972; Hoeven 2001; Mezzarobba 2010, 2016). With the value of ε in Step (b), we have a complexity bound of $\exp(\Delta(\gamma)^{O(1)})$ for deciding membership.

For the sake of comparison, we may speculate about an approach that would decide the membership of γ in $\operatorname{Pic}(X_f)$ by trying to construct an explicit algebraic divisor on X_f whose cohomology class is equal to γ. It would certainly need to decide the existence of a point satisfying some algebraic conditions in some Hilbert scheme $\mathcal{H}_{d,g}$, with $d = O(\Delta^{\frac{1}{2}})$ and $g = O(\Delta)$ (see §3.1.1). Embedding $\mathcal{H}_{d,g}$ (or some fibration over it, as we did in §3.4) in some affine chart of a projective space of dimension $d^{O(1)}$ will lead to a complexity of $\exp(\Delta(\gamma)^{O(1)})$ for deciding membership in this way.

However, if Conjecture 16 holds true, then the complexity of the numerical approach for deciding membership would reduce to $\Delta(\gamma)^{O(1)}$.

References

D. A. Bayer (1982). "The Division Algorithm and the Hilbert Scheme". Harvard University.

M. Beeler, R. W. Gosper, and R. Schroeppel (1972). *Hakmem.* AI memo 239. MIT Artificial Intelligence Laboratory.

L. Blum, F. Cucker, M. Shub, and S. Smale (1998). *Complexity and Real Computation.* Springer. DOI: 10/fb9984.

A. R. Booker, J. Sijsling, A. V. Sutherland, J. Voight, and D. Yasaki (2016). "A Database of Genus-2 Curves over the Rational Numbers". In: *LMS J. Comput. Math.* 19.A, pp. 235–254. DOI: 10/ggck8h.

J.-B. Bost, H. Gillet, and C. Soulé (1991). "Un analogue arithmétique du théorème de Bézout". In: *C. R. Acad. Sci. Paris Sér. I Math.* 312.11, pp. 845–848.

J. Brachat, P. Lella, B. Mourrain, and M. Roggero (2016). "Extensors and the Hilbert Scheme". In: *Ann. Scuola Norm. Sci.* 1, pp. 65–96. DOI: 10/ggjcwf.

N. Bruin, J. Sijsling, and A. Zotine (2019). "Numerical Computation of Endomorphism Rings of Jacobians". In: *Open Book Ser.* 2.1, pp. 155–171. DOI: 10/ggck8d.

F. Charles (2014). "On the Picard Number of K3 Surfaces over Number Fields". In: *Algebra Number Theory* 8.1, pp. 1–17. DOI: 10/ggck85.

E. Costa, N. Mascot, J. Sijsling, and J. Voight (2019). "Rigorous Computation of the Endomorphism Ring of a Jacobian". In: *Math. Comp.* 88.317, pp. 1303–1339. DOI: 10/ggck8g.

C. D'Andrea, T. Krick, and M. Sombra (2013). "Heights of varieties in multiprojective spaces and arithmetic Nullstellensätze". In: *Ann. Sci. Éc. Norm. Supér.* 46.4, pp. 549–627. DOI: 10/gbdc5k.

G. Gotzmann (1978). "Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes". In: *Math. Z.* 158.1, pp. 61–70. DOI: 10/d4vszd.

P. A. Griffiths (1969). "On the Periods of Certain Rational Integrals". In: *Ann. of Math.* 2nd ser. 90, pp. 460–541. DOI: 10/dq3zc6.

P. A. Griffiths and J. Harris (1978). *Principles of Algebraic Geometry*. Wiley-Interscience.

A. Grothendieck (1961). "Techniques de construction et théorèmes d'existence en géométrie algébrique IV : les schémas de Hilbert". In: *Séminaire Bourbaki : années 1960/61, exposés 205-222*. Séminaire Bourbaki 6. Société mathématique de France, pp. 249–276.

R. Hartshorne (1977). *Algebraic Geometry*. Springer-Verlag.

B. Hassett, A. Kresch, and Y. Tschinkel (2013). "Effective Computation of Picard Groups and Brauer-Manin Obstructions of Degree Two K3 Surfaces over Number Fields". In: *Rendiconti Circolo Mat. Palermo* 62.1, pp. 137–151. DOI: 10/ggck83.

J. van der Hoeven (2001). "Fast Evaluation of Holonomic Functions near and in Regular Singularities". In: *J. Symb. Comput.* 31.6, pp. 717–743. DOI: 10/dtv7b5.

D. Huybrechts (2016). *Lectures on K3 Surfaces*. Cambridge University Press. DOI: 10/fh6x.

M. Kontsevich and D. Zagier (2001). "Periods". In: *Mathematics Unlimited*. Springer, pp. 771–808. DOI: 10/fh6w.

T. Krick, L. M. Pardo, and M. Sombra (2001). "Sharp Estimates for the Arithmetic Nullstellensatz". In: *Duke Math. J.* 109.3, pp. 521–598. DOI: 10/cxzmw8.

E. Krätzel (2000). *Analytische Funktionen in der Zahlentheorie*. Teubner-Texte zur Mathematik. Vieweg+Teubner Verlag. DOI: 10/c6pqkk.

P. Lairez and E. C. Sertöz (2019). "A Numerical Transcendental Method in Algebraic Geometry: Computation of Picard Groups and Related Invariants". In: *SIAM J. Appl. Algebra Geometry*, pp. 559–584. DOI: 10/ggck6n.

D. Lazard (1977). "Algèbre linéaire sur $K[X_1, \dots, X_n]$, et élimination". In: *Bull. Soc. Math. France* 105.2, 165–190.

P. Lella (2012). "Computable Hilbert Schemes". Università degli Studi di Torino. arXiv: 1202.4384.

J. Liouville (1851). "Sur des classes très-étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationnelles algébriques". In: *J. Mathématiques Pures Appliquées*. 1st ser. 16, pp. 133–142.

D. Maulik and R. Pandharipande (2013). "Gromov-Witten Theory and Noether-Lefschetz Theory". In: *A Celebration of Algebraic Geometry*. Vol. 18. Clay Math. Proc. Amer. Math. Soc., Providence, RI, pp. 469–507.

M. Mezzarobba (2010). "NumGFun: A Package for Numerical and Analytic Computation with D-Finite Functions". In: *Proceedings of the 35th International Symposium on Symbolic and Algebraic Computation*. Ed. by S. M. Watt. ACM, pp. 139–146. DOI: 10/cg7w72.

– (2016). *Rigorous Multiple-Precision Evaluation of D-Finite Functions in Sagemath*. arXiv: 1607.01967.

P. Philippon (1995). "Sur Des Hauteurs Alternatives. III". In: *J. Math. Pures Appl. (9)* 74.4, pp. 345–365.

R. Piene and M. Schlessinger (1985). "On the Hilbert Scheme Compactification of the Space of Twisted Cubics". In: *Am. J. Math.* 107.4, pp. 761–774. DOI: 10/fvp9zp.

B. Poonen, D. Testa, and R. van Luijk (2015). "Computing Néron–Severi Groups and Cycle Class Groups". In: *Compos. Math.* 151.4, pp. 713–734. DOI: 10/ggck84.

G. Rémond (2001a). "Géométrie Diophantienne Multiprojective". In: *Introduction to Algebraic Independence Theory*. Vol. 1752. Lecture Notes in Math. Springer, pp. 95–131. DOI: 10/dk6pjk.

– (2001b). "Élimination Multihomogène". In: *Introduction to Algebraic Independence Theory*. Vol. 1752. Lecture Notes in Math. Springer, pp. 53–81. DOI: 10/dp99xs.

E. C. Sertöz (2019). "Computing Periods of Hypersurfaces". In: *Math. Comp.* 88.320, pp. 2987–3022. DOI: 10/ggck7t.

C. Simpson (2008). "Algebraic Cycles from a Computational Point of View". In: *Theoretical Computer Science*. Computational Algebraic Geometry and Applications 392.1, pp. 128–140. DOI: 10/c86jfm.

S. Smale (1986). "Newton's Method Estimates from Data at One Point". In: *The Merging of Disciplines: New Directions in Pure, Applied, and Computational Mathematics (Laramie, Wyo., 1985)*. Springer, pp. 185–196.

P. van Wamelen (1999). "Examples of Genus Two CM Curves Defined over the Rationals". In: *Math. Comput.* 68.225, pp. 307–321. DOI: 10/fxg626.

C. Voisin (2003). *Hodge Theory and Complex Algebraic Geometry II*. Trans. by L. Schneps. 1st ed. Cambridge University Press. DOI: 10/cxbxj9.

M. Waldschmidt (2000). *Diophantine approximation on linear algebraic groups*. Vol. 326. Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, pp. xxiv+633. DOI: 10/fj5k.

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