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Abstract

Tropical differential equations are introduced and an algorithm is designed which tests solvability of a system of tropical linear differential equations within the complexity polynomial in the size of the system and in its coefficients. Moreover, we show that there exists a minimal solution, and the algorithm constructs it (in case of solvability). This extends a similar complexity bound established for tropical linear systems. In case of tropical linear differential systems in one variable a polynomial complexity algorithm for testing its solvability is designed.

We prove also that the problem of solvability of a system of tropical non-linear differential equations in one variable is NP-hard, and this problem for arbitrary number of variables belongs to NP. Similar to tropical algebraic equations, a tropical differential equation expresses the (necessary) condition on the dominant term in the issue of solvability of a differential equation in power series.

Introduction

Tropical algebra deals with the tropical semi-rings $\mathbb{Z}_+$ of non-negative integers or $\mathbb{Z}_+ \cup \{\infty\}$ endowed with the operations $\{\min, +\}$, or with the tropical semi-fields $\mathbb{Z}$ or $\mathbb{Z} \cup \{\infty\}$ endowed with the operations $\{\min, +, -\}$ (see e. g. [6], [7], [8]).

A *tropical linear differential equation* is a tropical linear polynomial of the form

$$\min_{i,j}\{a_i^{(j)} + x_i^{(j)},\, a\} \qquad (1)$$

where the coefficients a, $a_i^{(j)} \in \mathbb{Z}_+ \cup \{\infty\}$, and a variable $x_i^{(j)}$ is treated as "j-th derivative of $x_i := x_i^{(0)}$".

For a subset $S_i \subset \mathbb{Z}_+$ we define the *valuation*

$$Val_{S_i}(\{j \geq 0\}) := Val_{S_i}(\{x_i^{(j)}\}_{j\geq 0}) : \mathbb{Z}_+ \to \mathbb{Z}_+ \cup \{\infty\}$$

of variable x_i as follows. For each $j \geq 0$ take the minimal $s \in S_i$ (provided that it does exist) such that $s \geq j$ and put $Val_{S_i}(j) := s - j$: in case when such s does not exist put $Val_{S_i}(j) := \infty$. We use a shorthand

$$Val_{S_1,\dots,S_n} := Val_{S_1} \times \cdots \times Val_{S_n} : \mathbb{Z}_+^n \to (\mathbb{Z}_+ \cup \{\infty\})^n.$$

Observe that if X_i is a power series in t with the support $\{t^s,\ s \in S_i\}$ then $Val_{S_i}(j)$ is the order $ord_t(X_i^{(j)})$ at zero of the j-th derivative $X_i^{(j)}$.

We say that $S_1, \dots, S_n$ is a *solution of the tropical linear differential equation (1)* if the minimum $\min_{i,j}\{a_i^{(j)} + Val_{S_i}(j),\ a\}$ is attained at least twice or is infinite (as it is accustomed in tropical mathematics [6], [7]). The latter is a necessary condition of solvability in power series in t of a linear differential equation $\sum_{i,j} A_{i,j} \cdot X_i^{(j)} = A$ in several indeterminates $X_1, \dots, X_n$. Namely, the orders of power series coefficients equal $ord_t(A_{i,j}) = a_{i,j}$, $ord_t(A) = a$ and the support of X_i is S_i. More precisely, (1) expresses that at least two lowest terms of the expansion in power series of the differential equation have the same exponents, which is similar to that the tropical equations concern the lowest terms of the expansions in Puiseux series of algebraic equations.

We study solvability of a system of tropical linear differential equations

$$\min_{i,j}\{a_{i,l}^{(j)} + x_i^{(j)},\ a_l\},\ 1 \leq l \leq k \tag{2}$$

where $1 \leq i \leq n$, $0 \leq j \leq r$ and for all finite coefficients $a_{i,l}^{(j)}$, $a_l \in \mathbb{Z}$ we have $0 \leq a_{i,l}^{(j)}$, $a_l \leq M$. Thus, the bit-size of (2) is bounded by $knr \log_2(M+2)$.

We say that a solution $T_1, \dots, T_n$ of (2) is *minimal* if the inequality $Val_{T_1,\dots,T_n} \leq Val_{S_1,\dots,S_n}$ holds pointwise for any solution $S_1, \dots, S_n$ of (2).

Note that (2) extends tropical linear systems when for all the occurring derivatives $x_i^{(j)}$ we have $j = 0$. Thus, the complexity bound of testing solvability of (2) in the next theorem generalizes the similar complexity bound of solvability of tropical linear systems from [1], [2], [4].

Theorem 0.1 *If a system (2) of tropical linear differential equations is solvable then it has the (unique) minimal solution. There is an algorithm which tests solvability of (2) and in case of solvability yields its minimal solution within the complexity polynomial in $knrM$.*

Note that $S \subset T \subset \mathbb{Z}_+$ iff the inequality $Val_S \geq Val_T$ holds pointwise. For $S_1, \dots, S_n, T_1, \dots, T_n \subset \mathbb{Z}_+$ we have for the pointwise minimum

$$Val_{(S_1,\dots,S_n)\vee(T_1,\dots,T_n)} := Val_{S_1\cup T_1,\dots S_n\cup T_n} = \min\{Val_{S_1,\dots,S_n},\, Val_{T_1,\dots,T_n}\}.$$

Assume now that (2) has a solution $S_1, \dots, S_n$. If $s \in S_i$ such that $s \geq r$ then one can replace S_i by adding to it all the integers greater than s (while keeping $S_1, \dots, S_n$ still to be a solution of (2)). Therefore, one can suppose w.l.o.g. that for every $1 \leq i \leq n$ either S_i is finite and moreover $S_i \subset \{0, \dots, r-1\}$ or the complement $\mathbb{Z}_+ \setminus S_i$ is finite. Thus, if we define $V := Val_{\bigvee(S_1,\dots,S_n)}$ where $\bigvee$ ranges over all the solutions $S_1, \dots, S_n$ of (2) then $\bigvee$ can be taken over a finite number of solutions, hence it can be reduced to a single solution $T_1, \dots, T_n$, thereby $V = Val_{T_1,\dots,T_n}$ and $T_1, \dots, T_n$ is the minimal solution of (2) which proves the first statement of Theorem 0.1.

Also we design a polynomial complexity algorithm for solving systems of the type (1) in case of one variable ($n = 1$).

Theorem 0.2 *There is an algorithm which tests solvability of a system (1) of tropical linear differential equations in one variable x and yields its minimal solution in case of solvability within the polynomial complexity. More precisely, the complexity is bounded by $O(kr\log(rM))$.*

1 Bound on the minimal solution of a tropical linear differential equation

Our next goal is to bound the (finite) complements $\mathbb{Z} \setminus S_i$. For each $1 \leq i \leq n$ with a finite $\mathbb{Z} \setminus S_i$ denote by $m_i \in S_i$ the minimal element of S_i such that $m_i \geq r$. If for some $1 \leq l \leq k$ the inequality $\min_{i,j}\{a_{i,l}^{(j)} + Val_{S_i}(j),\, a_l\} > M + r$ holds then for every $1 \leq i_0 \leq n$, $0 \leq j_0 \leq r$ for which this minimum is attained: $a_{i_0,l}^{(j_0)} + Val_{S_{i_0}}(j_0) = \min_{i,j}\{a_{i,l}^{(j)} + Val_{S_i}(j),\, a_l\}$ we have $Val_{S_{i_0}}(j_0) = m_{i_0} - j_0$.

Consider a graph G which for each finite $S_i, 1 \leq i \leq n$ contains a vertex w_i and for each S_i with a finite complement $\mathbb{Z}_+ \setminus S_i$ contains two vertices w_i, w_i^∞. A derivative $x_i^{(j)}$ corresponds to a vertex w_i^∞ iff $Val_{S_i}(j) = m_i - j$ (provided that $\mathbb{Z}_+ \setminus S_i$ is finite), else $x_i^{(j)}$ corresponds to w_i. Also G contains a vertex w_0 to which corresponds every free term $a_l, 1 \leq l \leq k$.

If there are $1 \leq l \leq k$, $1 \leq i_0, i_1 \leq n$, $0 \leq j_0, j_1 \leq r$ such that

$$a_{i_0,l}^{(j_0)} + Val_{S_{i_0}}(j_0) = a_{i_1,l}^{(j_1)} + Val_{S_{i_1}}(j_1) = \min_{i,j}\{a_{i,l}^{(j)} + Val_{S_i}(j),\, a_l\} < \infty \qquad (3)$$

then we connect in G by an edge vertices which correspond to the derivatives $x_{i_0}^{(j_0)}$ and $x_{i_1}^{(j_1)}$. Instead of $a_{i_0,l}^{(j_0)} + Val_{S_{i_0}}(j_0)$ could be a_l, then we consider the vertex w_0.

If a connected component of G contains only vertices of the form w_i^∞ with $m_i > r$ then for each w_i^∞ from this component we replace m_i by $m_i - 1$, or in other terms, augment S_i by $m_i - 1$, preserving so modified $S_1, \dots, S_n$ (for which we keep the same notation) to be still a solution of (2). After that G can be modified (we use the same notation for the modified G), and we continue this process. Eventually, we arrive to a solution $S_1, \dots, S_n$ whose graph G has no connected component satisfying the described property.

Therefore, each connected component of G contains a vertex of the form w_{i_0} (or perhaps, $w_{i_0}^\infty$ with $m_{i_0} = r$) fulfilling (3) for suitable j_0, l. Then $Val_{S_{i_0}}(j_0) \leq r$ (there is a possibility that instead of $a_{i_0,l}^{(j_0)} + Val_{S_{i_0}}(j_0)$ we consider the free term a_l), hence $Val_{S_1}(j_1) \leq M + r$ follows from (3). Thus, for every vertex w_i^∞ from this connected component there is a path in G of a length at most $n-1$ connecting it with a vertex of the form w_{i_2} (or perhaps, $w_{i_2}^\infty$ with $m_{i_2} = r$). Therefore, there is $0 \leq j \leq r$ such that $m_i - j = Val_{S_i}(j) \leq (n-1)(M+r)$ (one can show the latter inequality following along the path and applying the above argument). Thus, we conclude with the following lemma.

Lemma 1.1 *For each $1 \leq i \leq n$ for which $\mathbb{Z}_+ \setminus S_i$ is finite the bound $m_i \leq N := (n-1)(M+r) + r$ holds.*

This lemma extends Lemmas 1.2, 2.2 [4] established for tropical linear systems.

2 Algorithm testing solvability and producing the minimal solution of a system of tropical linear differential equations

Now we proceed to design an algorithm which tests whether a system (2) is solvable and if yes then yields its minimal solution $T_1, \dots, T_n$. The algorithm starts with the setting $T_1 = \dots = T_n = \{0, \dots, N\}$ (see Lemma 1.1), perhaps, being not a solution of (2), and then modifies $T_1, \dots, T_n$ recursively while a current $T_1, \dots, T_n$ is not a solution. If eventually a current $T_1, \dots, T_n$ becomes a solution then it is the minimal solution. We show by recursion that for any solution $S_1, \dots, S_n$ of (2) the pointwise inequality $Val_{S_1,\dots,S_n} \geq Val_{T_1,\dots,T_n}$ holds for a current $T_1, \dots, T_n$.

If a current $T_1, \dots, T_n$ is not a solution of (2) then two cases can emerge. In the first case there exist $1 \leq i_0 \leq n$, $0 \leq j_0 \leq r$, $1 \leq l \leq k$ such that a finite minimum $\min_{i,j}\{a_{i,l}^{(j)} + Val_{T_i}(j)\} < a_l$ is attained at a single pair i_0, j_0. Let $Val_{T_{i_0}}(j_0) = s - j_0$ where $s \in T_{i_0}$ is the minimal element of T_{i_0} such that $s \geq j_0$. The algorithm modifies T_{i_0} discarding s from it. The inequality $Val_{S_1,\dots,S_n} \geq Val_{T_1,\dots,T_n}$ still holds for any solution $S_1, \dots, S_n$ of (2) since $S_i \subset T_i$, $1 \leq i \leq n$. Note that if $s = m_{i_0} < N$ (see Lemma 1.1) then m_{i_0} increases by one. If $s = N(= m_{i_0})$ then S_{i_0} is finite due to Lemma 1.1 and the algorithm discards from the current T_{i_0} all integers $p > N$. In the second case a_l is the unique minimum in $\min_{i,j}\{a_{i,l}^{(j)} + Val_{T_i}(j),\, a_l\}$, then system (2) has no solution.

The algorithm terminates when either $T_1, \dots, T_n$ is a solution of (2), in this case $T_1, \dots, T_n$ is the minimal solution, or the algorithm detects that (2) has no solution.

When (2) is homogeneous, i. e. $a_l = \infty$, $1 \leq l \leq k$, system (2) has a solution with all infinite functions Val_{S_i}, $1 \leq i \leq n$. It can happen that the algorithm terminates with all $T_1, \dots, T_n$ being void, that means that the infinite solution of (2) is its unique one.

To bound the complexity of the algorithm observe that it runs at most $n(N+1)$ steps because at each step at least one of the current sets $T_1, \dots, T_n \subset \{0, \dots, N\}$ decreases. The cost of each step is polynomial in $knr \log M$ (the algorithm for every $1 \leq i \leq n$ stores m_i, provided that $\mathbb{Z}_+ \setminus T_i$ is finite, and also stores $T_i \cap \{0, \dots, r-1\}$). This completes the proof of Theorem 0.1.

3 Polynomial complexity solving systems of tropical linear differential equations in one variable

We design a polynomial complexity algorithm for solving a system of tropical linear differential equations (2) in one variable x. The algorithm basically follows the algorithm from Theorem 0.1 with a few modifications. In fact, the algorithm designed in this Section is a version of the algorithm from Theorem 0.1, the modification consists in that its steps are ordered in a special way (observe that at each step of the algorithm from Theorem 0.1 there could be several choices of an element to be discarded from the current set T). We use the notations from Section 2.

First, if there exists $s < r$ from T such that a finite minimum $\min_j\{a_l^{(j)} + Val_T(j)\}$ is attained at a unique j_0 for some $1 \leq l \leq k$ and it holds $Val_T(j_0) = s - j_0$ then the algorithm discards s from T. This is also a step of the algorithm from Theorem 0.1, and we refer to it as a *step of the finite type*. In other words, the algorithm designed in this Section has a preference in discarding elements s which are less than r. Denote by $s_0 \geq r$ the minimal element of $T \cap [r, \infty)$.

Second, let otherwise s_0 be the only candidate to be discarded from T for all the equations from (2) which are not satisfied by T. Then the algorithm from Theorem 0.1 would just discard s_0, while the algorithm under description discards from T possibly more elements at one step.

For each $1 \le l \le k$ consider a unique $0 \le j_0 \le r$ (provided that it does exist, i. e. the l-th equation is not satisfied by T) such that $Val_T(j_0) = s_0 - j_0$ and $a_l^{(j_0)} + s_0 - j_0 = \min_j\{a_l^{(j)} + Val_T(j)\}$. Take the maximal p_l (or the infinity when it is not defined) such that

$$a_l^{(j_0)} + s_0 - j_0 + p_l \le a_l^{(j)} + Val_T(j) \tag{4}$$

for any $0 \le j < r$ for which $Val_T(j) = s_1 - j$ for suitable $T \ni s_1 < r$. Observe that $p_l \ge 1$.

Denote by p the maximum of all such p_l. Let $p = p_{l_0}$ for an appropriate $1 \le l_0 \le k$. If $p = \infty$ then the algorithm discards from T all the elements $s \ge r$. Else, if $p < \infty$ then the algorithm discards all the elements s from T such that $s_0 \le s < s_0 + p$. In other words, the algorithm replaces the minimal element s_0 of $T \cap [r, \infty)$ by $s_0 + p$, we call this step of the algorithm a *jump*. Clearly, the jump replaces p steps of the algorithm from Theorem 0.1 each consisting in discarding just one element from T (so, discarding consecutively $s_0, s_0 + 1, \dots, s_0 + p - 1$) due to the unique monomial $a_{l_0}^{(j_0)} + x^{(j_0)}$ at which the minimum in (2) is attained for the l_0-th equation.

Observe that after a jump either the (new current) T provides a solution of (2) or the algorithm can execute a step of the finite type because of the choice of p, see (4), so discards from T some element $s < r$.

As in Section 2 the algorithm terminates when either a current T provides a solution of (2) (being the minimal solution as it was proved in Section 2) or the algorithm exhausts T (which means that $T \cap [0, N] = \emptyset$, see Sections 1, 2). In the latter case if system (2) is homogeneous then it has the (unique) infinite solution, otherwise a non-homogeneous system has no solutions (again similar to Section 2).

To estimate the complexity of the algorithm note that after a jump the algorithm executes a step of the finite type, i. e. discards from T an element $s < r$. Therefore, the number of steps of the algorithm does not exceed $2r$ taking into the account that the number of steps of the finite type is less or equal than r. To bound the jump p observe that $a_{l_0}^{(j_0)} + s_0 - j_0 + p = a_{l_0}^{(j_1)} + s_1 - j_1$ for appropriate $j_1, s_1 < r$ (cf. (4)). Since $j_0 \le r$ we deduce that $s_0 + p < 2r + M$, hence $p < r + M$. Thus, one can estimate the complexity by $O(kr\log(rm))$, and we complete the proof of Theorem 0.2.

4 NP-hardness of solvability of tropical non-linear differential equations in one variable

Now generalizing tropical linear differential equations (see the Introduction) we consider systems of tropical *non-linear* differential equations of the form

$$\min_{\{P\}}\{a_P + \sum_{(i,j)\in P} x_i^{(j)}\} \tag{5}$$

where the coefficients $a_P \in \mathbb{Z}_+$ and the minimum ranges over a certain (finite) family of finite multisets P of pairs (i,j). We view $|P|$ as the degree of the monomial $a_P + \sum_{(i,j)\in P} x_i^{(j)}$.

Similar to the case of tropical linear differential equations (see the Introduction), we observe that the solvability of (5) is necessary for the solvability in power series in t of a non-linear differential equation $\sum_{\{P\}} A_P \cdot \prod_{(i,j)\in P} X_i^{(j)} = 0$ where $ord_t(A_P) = a_P$.

We prove that the problem of solvability (with a set $S \subset \mathbb{Z}_+$ similar to tropical linear differential equations, see the Introduction) of a system of equations of the form (5) is NP-hard already in the case of a single variable x. Mention that in [8] NP-completeness of the solvability of tropical non-linear systems (in several variables) is established.

We prove NP-hardness by means of reducing a 3-SAT boolean formula Φ in n variables $y_0, \dots, y_{n-1}$ (see e. g. [3]) to a system E_Φ of equations of the form (5) in a single variable x, preserving the property of solvability.

The system E_Φ contains (linear) equations

$$\min\{x^{(2j+1)}, 0\},\ 0 \le j \le 2n-1 \tag{6}$$

These equations mean that the valuation of each even derivative $x^{(2j)}$, $0 \le j \le 2n-1$ equals either 0 or 1. Also E_Φ contains (quadratic) equations

$$\min\{x^{(2j)} + x^{(2j+2n)}, 1\},\ 0 \le j \le n-1 \tag{7}$$

They mean that either $Val(x^{(2j)}) = 0$, $Val(x^{(2j+2n)}) = 1$ (which corresponds to the value "true" of the variable y_j, $0 \le j \le n-1$ of Φ) or $Val(x^{(2j)}) = 1$, $Val(x^{(2j+2n)}) = 0$ (which corresponds to the value "false" of y_j, respectively). Finally, for each 3-clause of Φ, say of the form $\neg y_{j_1} \vee y_{j_2} \vee \neg y_{j_3}$ we add to E_Φ a (linear) equation

$$\min\{x^{(2j_1+2n)}, x^{(2j_2)}, x^{(2j_3+2n)}, 0\} \tag{8}$$

Clearly, Φ is equivalent to the solvability of the system obtained by uniting (6), (7) and (8) for all 3-clauses of Φ. Thus, we have proved

Proposition 4.1 *The problem of solvability of systems of tropical non-linear differential equations in a single variable is NP-hard.*

5 Solvability of systems of tropical non-linear differential equations is in NP

Next we prove that the problem of solvability of systems of k tropical non-linear differential equations of the form (5) of degrees $|P| \leq d$ fulfilling the bounds: $0 \leq a_P \leq M$, $0 \leq j \leq r$, $1 \leq i \leq n$ (in an arbitrary number n of variables) belongs to NP. First, similar to Lemma 1.1 and using the notations from Section 1, we show that if a system has a solution (with some $S_1, \dots, S_n \subset \mathbb{Z}_+$) then it possesses a sufficiently small solution.

Substitute the solution into each equation of the form (5), then the valuations of some derivatives $x_i^{(j)}$ can equal $m_i - j$, the valuations of all the other derivatives consider as being fixed. We treat the system (after this substitution) as an input of the linear programming problem (expressing that the minimum in (5) is attained at least at two terms) with respect to the indeterminates m_i (for all i for which they are defined), and the fixed valuations consider as the coefficients of the input. Therefore, this input possesses a solution with m_i bounded (due to Hadamard's inequality on determinants) by $N_1 := n! \cdot (M + rd) \cdot d^n$. Note that this bound is worse than the bound on N established in Lemma 1.1 for systems of tropical *linear* differential equations.

Since in order to give a solution $S_1, \dots, S_n$ it suffices just to specify $S_i \cap [0, r]$, $1 \leq i \leq n$ and $m_i \leq N_1$ (for i for which it does exist), we get the following

Proposition 5.1 *The problem of solvability of systems of tropical non-linear differential equations belongs to NP.*

6 Further research

Similar to tropical linear systems (cf. [1], [2], [4]) it is an open problem, whether one can solve system (2) of tropical linear differential equations within the complexity polynomial in $knr \log M$ (in other words, within the proper polynomial complexity)? In Section 3 a polynomial complexity algorithm is designed for testing solvability of systems of tropical linear differential equations in one variable ($n = 1$). Is there a polynomial complexity algorithm for similar systems in, say a constant number $n \geq 2$ of variables?

It is known (see [1], [2], [4]) that the problem of solvability of systems of tropical linear equations is in the complexity class $NP \cap coNP$. Does the problem of solvability of systems of the type (2) of tropical linear differential equations belong to $coNP$? Proposition 5.1 implies that even a more general problem of solvability of systems of the type (5) of tropical non-linear differential equations lies in NP.

We say that $S_1, \dots, S_n$ is a *Laurent solution* of (2) if for every $1 \leq i \leq n$ either $S_i \subset \mathbb{Z}_+$ is as we considered above or $S_i = \{b\}$ is a singleton for some negative integer $0 > b \in \mathbb{Z}$. In the latter case $Val_b(j) = b - j$. This corresponds to the order of the j-th derivative of a Laurent polynomial of the form $ct^b(1 + O(t))$ for a (complex) coefficient c. If all sets among $S_1, \dots, S_n$ are negative singletons then the solvability of (2) reduces to the solvability of a tropical linear system. The question is, what is the complexity of testing whether (2) has a Laurent solution? Actually, one can extend this setting from Laurent solutions to solutions of the form $S_i = \{b\}$ where $b \in \mathbb{R} \setminus \mathbb{Z}_+$. This corresponds to a necessary condition of solvability of a system of linear differential equations in Puiseux series (when $b \in \mathbb{Q}$) or in Hahn series (when $b \in \mathbb{R}$, see e. g. [5]).

For a tropical linear differential monomial $a + x^{(j)}$, $a, j \in \mathbb{Z}_+$ define its derivative as $\min\{a - 1 + x^{(j)}, a + x^{(j+1)}\}$ when $a \geq 1$ or as $x^{(j+1)}$ when $a = 0$ (which mimics the usual derivation law). We spread this definition of the derivative to all tropical linear differential equations of the form (2) by the tropical linearity. The tropical ideal generated by the derivatives of all the orders of tropical linear differential equations is called the *tropical linear differential ideal* generated by these equations. Is it possible to test solvability of a tropical linear differential ideal? Lest there would be a misunderstanding, we note that a solution of a tropical linear differential equation is not necessary a solution of the tropical ideal generated by this equation.

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