A COMPUTATION OF THE BASIC INVARIANT J^+ FOR CLOSED 2-VERTEX CURVES

MASAAKI UMEHARA

Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26
D-53225 Bonn

GERMANY

Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-16, Toyonaka
Osaka 560
JAPAN

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Max-Planck-Institut für Mathematik, Gottfried-Claren-Str.26, 53225 Bonn, Germany, umehara@mpim-bonn.mpg.de and Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-16, Toyonaka, Osaka 560, Japan, umehara@math.wani.osaka-u.ac.jp

ABSTRACT. In this paper, we will give an explicit and simple formula for J^+-invariant of closed 2-vertex curves. As an application, we give an affirmative answer to a problem conjectured by Arnold [A2], related to Legendrian knots in $\mathbf{R}^2 \times S^1$.

Introduction.

An immersed plane curve is called *normal* if all crossings are transversal and only double points. Recently, Arnold [A1] introduced new invariants J^+, J^- and St for normal closed plane curves, which are additive under connected sum. (See also [A3].) By taking fields of co-oriented contact elements of immersed plane curves, they induce Legendrian knots in the unit sphere bundle $ST^*\mathbf{R}^2 = \mathbf{R}^2 \times S^1$. The invariant J^+ has extra importance, because it can be extended for Legendrian knots in $ST^*\mathbf{R}^2$ which coincides with that of its projection into the plane whenever the projection has no singularity.

Vertices of the plane curves are points which attain local maxima or minima of their curvature functions. We shall prove the following theorem, which was conjectured by Arnold [A2;§30].

Theorem. *Let γ be an immersed plane curve. If the induced Legendrian embedding $L_\gamma : S^1 \to ST^*\mathbf{R}^2$ belongs to the component of that of unit circle in $\mathbf{R}^2$, then γ has no fewer than four vertices.*

In the previous work [KU], the intersection sequences of closed normal 2-vertex curves are classified by combinatorial method. In this paper, we will give an explicit and simple formula for the invariant J^+ of closed normal 2-vertex curves using the classification, and apply it for a proof of Theorem. Here we give an outline of it:(The detail are discussed in latter sections)

We assume an oriented closed normal curve $\gamma(t) : S^1 \to \mathbf{R}^2$ has only two vertices, say $t = a$ is maximal and $t = b$ is minimal. The curve is divided by two arcs $I^+ \cup I^- := S^1 \setminus \{a, b\}$ such that $\gamma^- := \gamma|_{I^-}$ (resp. $\gamma^+ := \gamma|_{I^+}$) is a *negative scroll* (resp. *positive scroll*), namely it is a curvature decreasing (resp increasing) arc. We explain the main result in [KU] for the following 2-vertex curve γ.

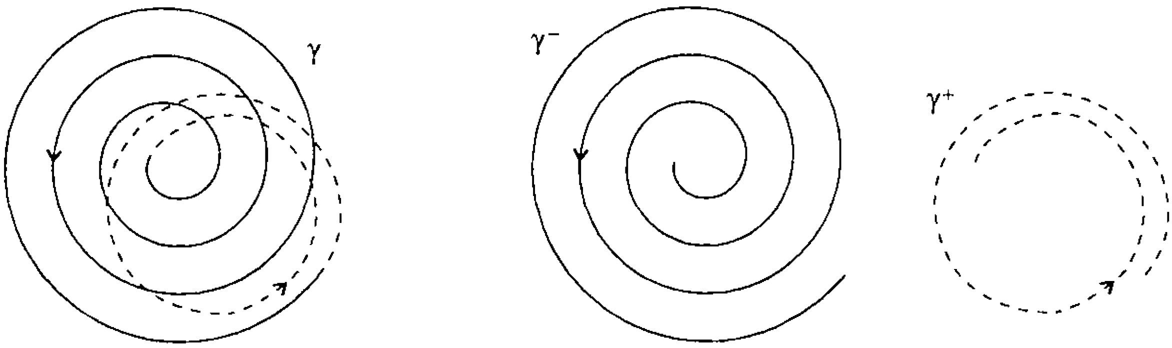


Figure 1.

The crossing of γ is called *positive* (resp. *negative*) if γ^+ crosses to γ^- from the left (resp. right). We use small capitals for positive crossings. Let a be a positive crossing. If a negative crossing is the first crossing at which the future part of γ^- from a meets the past part from γ^+ to a. Then this negative crossing is expressed by a^*. (It was shown in [KU] that a^* coincides with the first crossing at which the past part from γ^+ from a meets the future part from γ^- from a. Moreover $*$-pairing is unique.)

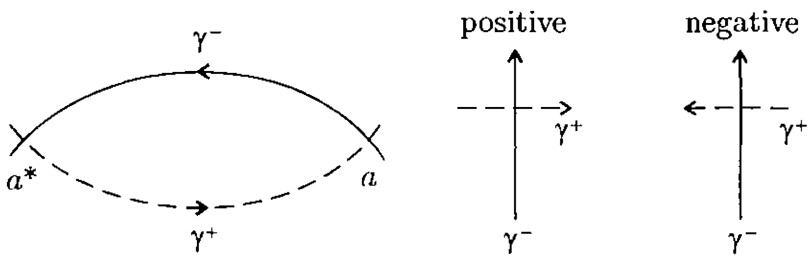


Figure 2.

If a negative crossing does not make a $*$-pairing, it is called *solitary negative crossing* and denoted by large capital. The star pairing is easy to find: one only seeks a leaf figure as in Fig.2 according to this rule, we can label the crossings as follows:

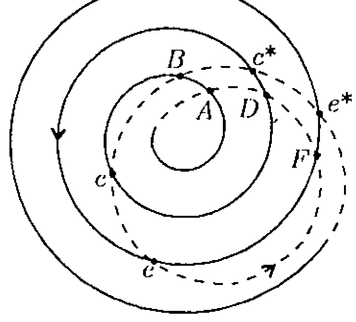


Figure 3.

The intersection sequences of γ^+ and γ^- are written as follows

$$\gamma^- : ABcDc^*eFe^* = AB[c : D][e : F],$$
$$\gamma^+ : e^*c^*BceFDA = [e^*c^* : B]FDA,$$

where we use the following convention for reducing the intersection sequences

$$\text{Type T}: \quad A_1A_2\ldots A_n,$$
$$\text{Type S}: \quad [a_1a_2\ldots a_n : B] := a_1\ldots a_nBa_n^*\ldots a_1^*.$$

We define the length of the each type of words by

$$|A_1A_2\ldots A_n| := n, \qquad |[a_1a_2\ldots a_n : B]| := n+1.$$

The two intersection sequences are mutually translated by the following head-picking rule:(The rule is to pick up heads of groups made by brackets until a capital letter.)

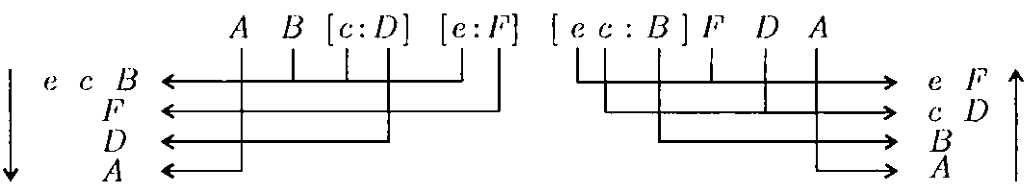


Figure 4.

Moreover, the grammars of a negative intersection sequence are characterized as follows.

(**a**) The intersection sequence consists of the word of type T and type S and written by the form $T_0S_1T_1S_2T_2\cdots S_kT_k$. Each T_j may possible to be empty.

(**b**) $|T_0| > 0$, $|T_0| \geq |S_1|$ and $|S_j| + |T_j| \geq |S_{j+1}|$ $(j = 1, \ldots, k)$.

Conversely if such a sequence of words, say $AB[c : D][e : F]$ is given abstractly, we can restore the curve by the following way: (The matrix in below is defined in [KU;§4])

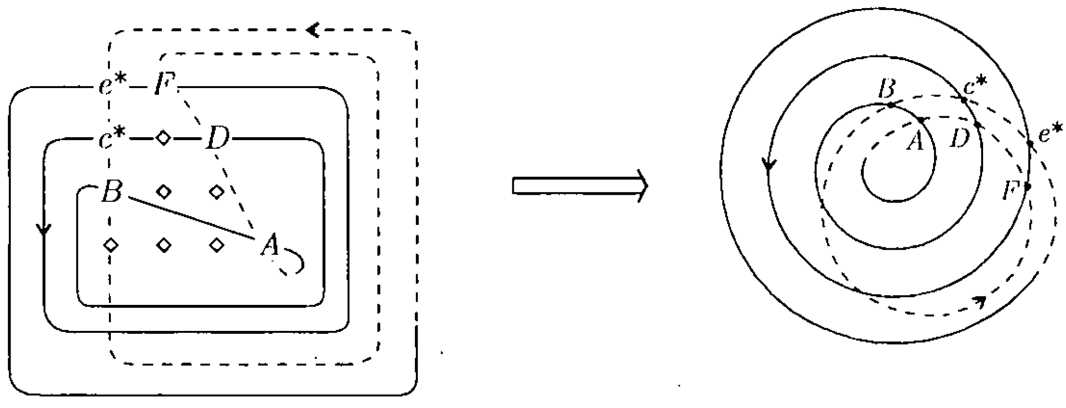


Figure 5.

These rule and grammars will play important role to prove the following(See §1)

Theorem A. *Let γ be a normal closed 2-vertex curve with n-crossings and p is the number of positive crossing on the curvature decreasing arc γ^-. Then*

(i) *The rotation number (i.e. index) i_γ can take any values among the sequence*

$$-(m+1),\ -(m-1),\ -(m-3), ...,\ m-3,\ m-1,\ m+1,$$

where $m = n - 2p$ is the number of solitary negative crossings.

(ii) *The J^+ invariant of the curve γ is given by*

$$J^+(\gamma) = 2p - \frac{1}{2}\{i_\gamma^2 + m^2 - 1\}. \tag{1}$$

Since vertices on curves are invariant under Möbious transformations, 2-vertex curves are considered on a sphere $S^2 = \mathbf{R}^2 \cup \{\infty\}$. An invariant SJ^+ for a plane curve γ is defined by

$$SJ^+(\gamma) := J^+(\gamma) + \frac{i_\gamma^2}{2}, \tag{2}$$

which is invariant under the diffeomorphisms on the sphere S^2 (See [A2:§27]). This invariant is more convenient for 2-vertex closed curves. By (1) and (2), we have

$$SJ^+(\gamma) = 2p - \frac{1}{2}(m^2 - 1). \tag{3}$$

By applying the above grammars (a) and (b) carefully, we will prove the following estimate (See §2).

Theorem B. *For any normal closed 2-vertex curve, the spherical invariant SJ^+ is nonpositive.*

As a corollary of the estimate, we get the following

Corollary C. *Let γ is a closed normal curve such that $i_\gamma = 1$ and $J^+(\gamma) = 0$. Then γ has at least four vertices.*

(Proof of Theorem.)

Theorem follows from the above corollary for normal curves. In fact, the induced Legendrian embedding L_γ is isotopic to that of unit circle L_{S^1} then $i_\gamma = 1$ and $J^+(\gamma) = 0$ (See [A3]). So we suppose that γ is not a normal curve. Since any 2-vertex curve is divided by 2-embedded arcs γ^+ and γ^-, all crossings are at most double points. Moreover, by the assumption of Theorem, the curve is a projection of a Legendrian embedding, any crossings are not dangerous self-tangency.(See [A3:§18]) So any small deformation of the curve does not change the isotopy class of induced Legendrian embeddings. Then by Theorem 2.5 in [KU], such a small deformation can be taken to preserve the number of vertices, and the curve is deformed to a normal curve. Thus the original curve has also four vertices. $\square$

§1. Computations of J^+ and SJ^+

The followings are the table of the convex 2-vertex curves which have only negative crossings.

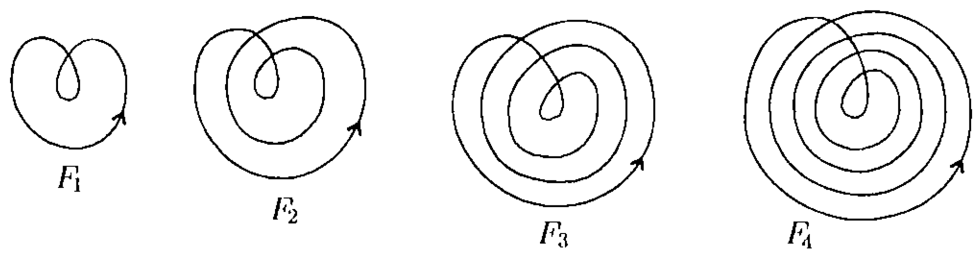


Figure 6.

We call F_m the fundamental 2-vertex curve of order m. Since $SJ^+(F_m)$ is Möbious invariant, we have

$$SJ^+(F_m) = SJ^+(F_m^{\#}), \tag{4}$$

where $F_m^{\#}$ is the 2-vertex curve of rotation number $i_{F_m} - 1$ obtained by a suitable Möbious transformation $T = (az+b)/(cz+d) : \mathbf{C} \cup \{\infty\} \to \mathbf{C} \cup \{\infty\}$ $(ad - bc \neq 0)$.

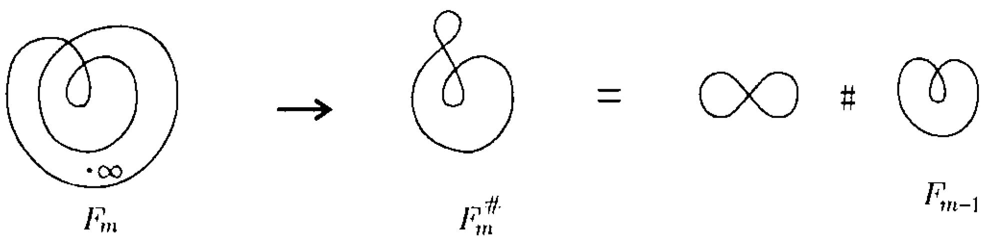


Figure 7.

By using the additivity of J^+-invariant under connected sum, (2) and (4), one can easily get the following two formulas

$$J^+(F_m) = -m(m+1), \tag{5}$$

$$SJ^+(F_m) = -\frac{1}{2}(m^2 - 1). \tag{6}$$

First we prove the following lemma, which contains a special case of Theorem A:

Lemma 1. *Let γ be a convex normal closed curve with n-crossings. Assume that γ is divided 2-embedded arcs*[1] *γ^- and γ^+ such that the intersection sequences γ^- and γ^+ satisfy the same translation rule and the grammars as that of 2-vertex curves. Then*

(i) *The the rotation number i_γ coincides with the number $m = n - 2p$ of solitary negative crossings, where p is the number of positive crossings.*

[1] If a closed curve is divided into two embedded arcs, it is called semisimple. The semisimplicity plays an important roles to prove the 6-vertex theorem in [U].

(ii) *The J^+ invariant of the curve γ is given by:*

$$J^+(\gamma) = 2p + J^+(F_m) = 2p - m(m+1). \tag{7}$$

Proof. First we explain the outline of the proof by using the example $AB[c:D][e:F]$ in Introduction. The main idea is very simple, one applies J^+-operation for this curve as follows:(For the definition of J^+-operation, see [A1] or [A3].)

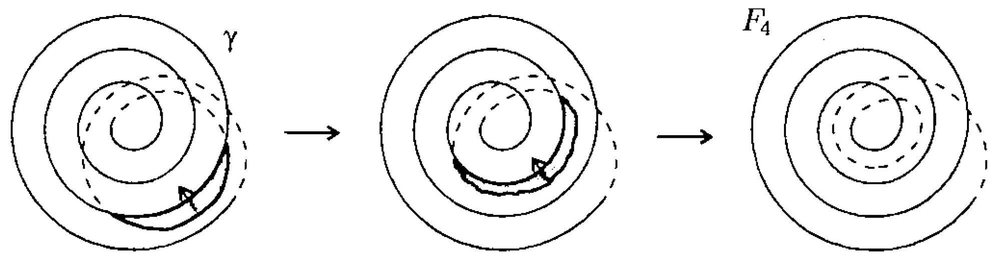


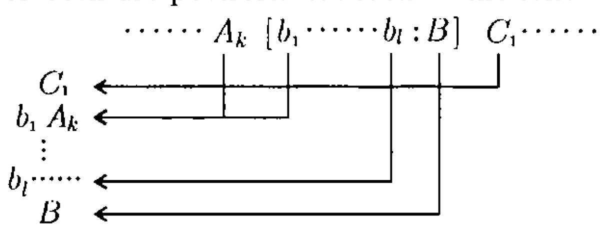
Figure 8.

After two times of J^+-operations, this curve is deformed to F_4. The number 4 is the number of solitary negative crossings A, B, D, F. These J^+-operations do not change the rotation number of the curve, the original 2-vertex curve has also rotation number 4 and the J^+-invariant should be $4 + J^+(F_4)$.

Now we prove the general case by induction for the number of positive crossings p. If γ has only negative crossings, it is equivalent to F_m as a spherical curve, since γ has the same signed Gauss word. (Two spherical curves are equivalent iff their signed Gauss words coincide. See [S] for details.) Moreover γ is equivalent to F_m as a plane curve since γ is convex. Thus by (5), the assertions are true for $p = 0$. Suppose that the assertions are true when the number of positive crossing is less than p. Assume γ has p times of positive crossings, which is of type $T_0 S_1 T_1 \cdots S_k T_k$ $(k \geq 1)$. We pay attention to the final three words of this intersection sequence. We set

$$\begin{aligned} T_{k-1} :=& A_1 A_2, \ldots A_k, \\ S_k :=& [b_1 \ldots b_\ell : B], \\ T_k :=& C_1 C_2 \ldots C_r, \end{aligned}$$

where $k = 0$, $r = 0$ or both are possible. We look at the following translation rule:



γ^- : $\cdots$ A_k $\cdots$ b_1 $\cdots$ $\boxed{b_l B}$ b_l^* $\cdots$ $b_l^* C_1$ $\cdots\cdots$

γ^+ : $\cdots$ b_l^* $\cdots$ $\boxed{b_l B}$ $\cdots$

Figure 9.

By this head picking rule mentioned in Introduction, γ^+ and γ^- have both no crossings between b_ℓ and B. Now we get the following two leaf figured domains D_1 and D_2 edged by (b_ℓ, b_ℓ^*) and (b_ℓ, B) respectively.

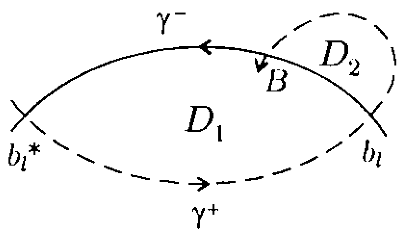


Figure 10.

In fact, since b_ℓ is positive and b_ℓ^* is negative, the boundary ∂D_1 is left-turning at b_ℓ and b_ℓ^*. So the convexity of γ yields that of ∂D_1. Moreover boundedness of D_2 follows also by the convexity of γ. (Consider the intersection point P between the tangent line at B and ∂D_1, then the simple close arc at P should be convex and the following figure is impossible.)

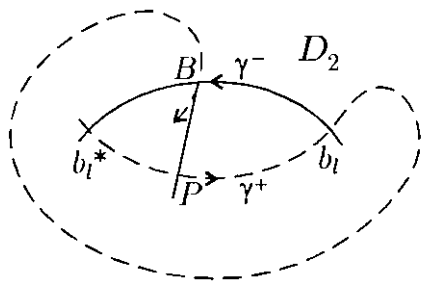


Figure 11.

Thus we can apply J^+-operation at the leaf figured domain D_2 such that new curve $\hat{\gamma}$ after the operation keeps its convexity. Then it can be easily verified that the intersection sequence of $\hat{\gamma}^-$ should be

$$\cdots A_1, A_2, ..., A_k[b_1, ..., b_{\ell-1} : b_\ell^*]C_1, ..., C_r,$$

which satisfies the grammars (a) and (b). Moreover the intersection sequence of $\hat{\gamma}^+$ is obtained by the head-picking rule from the above new sequence. (Here b_ℓ^* turns to be the solitary negative crossing of $\hat{\gamma}$.) By the construction, we have $J^+(\gamma) = J^+(\hat{\gamma}) + 2$ and $i_\gamma = i_{\hat{\gamma}}$. The number of positive crossings of $\hat{\gamma}$ is $p-1$. So, by the assumption of induction, we proved the lemma. □

Figure 12.

Remark. In the proof of Lemma 1, the convexity of γ is not essential. In fact, if γ is considered as a spherical curve, we need not distinguish the two shapes in Figure 12 for J^+ operations. Without the convexity of γ, the above proof still leads the following formula

$$SJ^+(\gamma) = 2p + SJ^+(F_m).$$

(Proof of Theorem A.) Let γ be a normal closed 2-vertex curve. Consider the osculating circle C at the minimal vertex and take a point $P \in C$ which is not any crossing of γ. By a Möbious transformation which maps P to the infinity, the image $T \circ \gamma$ is a convex 2-vertex curve. The number of the crossings n and positive crossings p are equal to those of γ. By Lemma 1, the rotation number of $T \circ \gamma$ is $m = n - 2p$. By a suitable Möbious transformation, the rotation number of $T \circ \gamma$ can take the any values between

$$-(m+1),\ -(m-1), \ldots, m-1,\ m+1.$$

So we get the first assertion. Since $SJ^+(T \circ \gamma) = SJ^+(\gamma)$, the second assertion follows from (2) and (7). □

§2. An estimate for SJ^+

In this section, we prove Theorem B. The key is the following

Lemma 2. *Let γ is a closed normal 2-vertex curve. Then the total number n of crossings and the positive crossings p satisfy the following inequality*

$$p \leq \frac{(m-1)^2}{4}, \tag{8}$$

where $m = n - 2p$ is the number of solitary negative crossings.

Proof. We fix the number m of solitary negative crossings, and seek the maximal number of positive crossings among all such closed normal 2-vertex curves. Assume curvature decreasing arc γ^- of a 2-vertex closed curve γ is the form $T_0S_1T_1 \cdots S_kT_k$ $(k \geq 0)$. We assume that γ attains the maximum number of positive crossings. (Such γ exists, because the possibility of 2-vertex curves with given m is finite.) Then obviously, the inequality in (b) of Introduction must be equality, that is $|T_0| = |S_1|$, $|S_j|+|T_j| = |S_{j+1}|$. (For example, if $|T_0| > |S_1|$ and $S_1 = [a_1a_2...a_n; B]$, then one can replace S_1 by $S_1' = [a_1a_2...a_nc_1..c_r; B]$, where $c_1, ..., c_r$ $(r = |T_0| - n)$ are new small letters. Then the new sequence $T_0S_1'T_1 \cdots S_kT_k$ also satisfies the grammars (a) and (b) in Introduction, but this contradicts the maximalities of γ. The other equalities $|S_j| + |T_j| = |S_{j+1}|$ are also obtained by the same method.) Now suppose that T_j $(j \geq 1)$ is a non-empty word. We set

$$\begin{aligned} T_{j-1} &:= A_1A_2...A_s, \\ S_j &:= [b_1...b_\ell : B], \\ T_j &:= C_1C_2...C_r. \end{aligned}$$

Define new words $\hat{T}_j$ and $\hat{S}_j$ by

$$\hat{T}_{j-1} := A_1A_2...A_sC_1C_2...C_r,$$
$$\hat{S}_j := [b_1...b_\ell e_1...e_r : B],$$

where $e_1, ..., e_r$ are new small capitals. Then the new sequence

$$T_0S_1T_1 \cdots S_{j-1}\hat{T}_{j-1}\hat{S}_jS_{j+1} \cdots S_kT_k$$

satisfies $|S_{j-1}| + |\hat{T}_{j-1}| = |\hat{S}_j|$, $|\hat{S}_j| = |S_{j+1}|$. Thus, there is a 2-vertex curve corresponding to the new word and it has the same number of solitary negative crossings but the number of positive crossings is $p + r$. This contradicts the maximality of γ. Hence T_j $(j \geq 1)$ should be all empty and γ has the expression of the form $T_0S_1S_2 \cdots S_k$ such that $|T_0| = |S_1| = \cdots = |S_k|(= r)$. In the above expression, the number m of solitary negative crossings is $m = |T_0| + k(= r + k)$, and the number p of positive crossings is $(r-1)k$. Thus we have

$$p = (r-1)k = (r-1)(m-r).$$

Now we still have the freedom of the choice of number $r = 1, ..., m$. But for any choice of r, it does not exceed the maximum of the function $f(x) = (x-1)(m-x)$. Hence we have the inequality $p \leq (m-1)^2/4$. □

(Proof of Theorem B.) By (3) and Lemma 2, we get the following estimates

$$SJ^+(\gamma) = 2p - \frac{1}{2}(m^2 - 1) \leq 1 - m.$$

Since $|T_0| > 0$, any 2-vertex curve has at least one solitary negative crossing, so $m \geq 1$. Thus $SJ^+(\gamma)$ is non-positive. □

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