Max-Planck-Institut für Mathematik
Bonn

Noncommutative 'Spaces' and 'Stacks'

by

Alexander Rosenberg

Max-Planck-Institut für Mathematik
Preprint Series 2011 (70)

Noncommutative ’Spaces’ and ’Stacks’

Alexander Rosenberg

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Kansas State University
Manhatten, KS 66506
USA

MPIM 11-70

Noncommutative 'Spaces' and Stacks.

Alexander Rosenberg

Preface

This work grew around a series of examples of smooth noncommutative 'spaces' constructed together with Maxim Kontsevich more than a decade ago. The first example was the noncommutative projective 'space'. Then we moved to noncommutative Grassmannians and flag varieties and their generalizations, which include the noncommutative version of Quot schemes. Finally, we found a general combinatorial construction producing noncommutative 'spaces'. These 'spaces' have remarkable properties and a "classical" flavor, although many of them do not even exist in commutative algebraic geometry. Each of the examples, and the general construction, started with a functor from the category of associative algebras to *Sets* interpreted as a presheaf of sets on the category of noncommutative affine schemes. Every such presheaf gives rise to a category of *quasi-coherent modules* on it. The latter represents the corresponding 'space'.

A self-contained exposition of these examples and the construction and the formalism required for (and triggered by) their elementary analysis turned out to be a sketch of a considerable part of basics of noncommutative algebraic geometry, which includes notions and properties of smooth and étale morphisms and noncommutative stacks. In order to give a better overview and include these examples into landscape of noncommutative algebraic geometry, the sketch is complemented with some other, previously known, important examples of noncommutative 'spaces' and several relevant digressions. All together formed the present manuscript.

Acknowledgements.

The essential part of this monograph is a result of collaboration with Maxim Kontsevich. Many details were shaped in the course of lectures given by the author at the Kansas State University. Thus, the facts of Chapter I appeared as a background material in graduate courses on Noncommutative algebraic geometry. The constructions and basic properties of noncommutative Grassmannians and flag varieties were a part of an algebra course attended by undergraduate and graduate students. Their questions influenced presentation here and their lecture notes were helpful for the recovery of some forgotten arguments while writing Chapters II and III of this manuscript. Special thanks to Bryan Bischof for indicating a number of typos in the preliminary version of the book.

This work was helped very much by the support and hospitality of the Max Planck Institut für Mathematik in Bonn and the Institute des Hautes Études Scientifiques, Bures-sur-Yvette, France. The National Science Foundation supported my research from 2000 until 2006.

Contents

Chapter IV

Quasi-Coherent Sheaves on Fibred Categories and Noncommutative Spaces

Chapter V

Noncommutative Grassmannians and Related Constructions

Introduction

The background material consists of some basic facts of "categorical" noncommutative algebraic geometry – 'spaces' represented by categories and morphisms of 'spaces' represented by their inverse image functors. Then follows a sketch, in the same spirit, of different finiteness and (formal) smoothness conditions, which are used, in particular, to single out (relative) noncommutative schemes and more general locally affine 'spaces', as well as some important special classes of the them (like semi-separated and separated noncommutative schemes, noncommutative algebraic 'spaces' etc.). Then we temporarily abandon categorical philosophy and plunge in the direction of (pre)sheaves of sets representable by noncommutative 'spaces' and schemes. The main thrust here is on introducing and studying examples which can be qualified as noncommutative "flag varieties" and "generic flag varieties". The first class includes, as a special case, noncommutative Grassmannians and the second one – generic Grassmannians, which are noncommutative versions of *Quot schemes* (they are not schemes, however). Two points of view meet again when we introduce quasi-coherent (sheaves of) modules on fibred categories. This leads to noncommutative stacks. Applying the so called "local constructions", we extend the examples of Grassmannians and flag varieties to non-affine base. Then we make a move, which might be interpreted as another way of "globalization" – defining geometry whose initial data is an action of a monoidal category on a category. The main immediate reason is that some key examples of non-commutative schemes – like quantum base affine space and flag variety and, especially, quantum D-schemes over them, require this point of view, as well as many (actually, most of) other examples. It is also useful for a better understanding the nature of (formal) smoothness and some other finiteness conditions. We introduce, in the framework of this setting, a general construction of noncommutative spaces starting from a combinatorial data and certain "initial conditions". The combinatorial data with initial conditions produce a presheaf of sets on the category of affine schemes in the monoidal category, and the action of this monoidal category gives rise (via a canonically associated fibred category) to the category of quasi-coherent modules.

The text is organized as follows.

Chapter I contains a refined exposition of well known facts on 'spaces' represented by categories based mostly on [R, Ch. VII], [R3], [R4]. It gives a necessary background on continuous and affine morphisms and flat descent. Examples which appear here do not go beyond cones of non-unital algebras and Proj of non-unital graded algebras. Their main particular cases are quantum base affine 'spaces', quantum flag varieties of semi-simple Lie algebras and associated with them quantum D-schemes (first introduced in [LR]).

Chapter II outlines a simple general formalism of finiteness conditions, representability and smoothness, which allows to single out schemes, algebraic spaces and more general locally affine 'spaces' in different settings of noncommutative algebraic geometry.

In Chapter III, we introduce geometry in which the role of 'spaces' is played by functors from the category of associative unital algebras to *Sets*, or, what is the same, presheaves of sets on the category of noncommutative affine schemes. Applying the general nonsense formalism of Chapter II, we obtain, in this setting, the notions of Zariski open immersions, locally affine 'spaces' and schemes, formally smooth and smooth 'spaces' and morphisms of 'spaces', Zariski, smooth and étale pretopologies etc.. We sketch properties of closed immersions and separated morphisms of (pre)sheaves of sets. We illustrate all these notions with examples which might be regarded as *toy* models of noncommutative Grassmannians, and flag varieties, and their generalized versions. The characteristic "toy" is due to the fact that these presheaves are convenient approximations of the noncommutative Grassmannians and flag varieties. The transition from "toy" to "real" consists of taking the associated sheaf of the "toy" model for an *appropriate* pretopology on the category of noncommutative affine schemes. A natural appropriate pretopology for all the examples of this chapter (as well as the examples of Chapter V) is the *smooth* pretopology.

It is worth to mention that the restrictions of the sheaves of sets representing our varieties to the category of commutative schemes are commutative schemes with the same name, whenever the latter exist (there is no commutative prototype of the Grassmannian of hundred-dimensional subspaces of the one-dimensional vector space).

Chapter IV is dedicated to quasi-coherent sheaves. We start with introducing the category of *sheaves* and the category of *quasi-coherent sheaves* on fibred categories and study their general properties. For every presheaf of sets $\mathfrak{X}$ on the category of noncommutative affine schemes, we define the category of quasi-coherent sheaves on $\mathfrak{X}$ as the category of quasi-coherent sheaves on the fibred category naturally associated with $\mathfrak{X}$. An important observation is that, given a pretopology τ on the category of affine noncommutative schemes, the category of quasi-coherent presheaves on a presheaf of sets $\mathfrak{X}$ coincides with the category of quasi-coherent sheaves on the associated sheaf of sets, provided that the pretopology τ is of effective descent. The smooth pretopology of Chapter III is of effective descent. This implies that the categories of quasi-coherent presheaves on the toy Grassmannians and toy flag varieties introduced in Chapter III are equivalent to the categories of quasi-coherent sheaves on "real" noncommutative 'spaces' with the corresponding names.

In Chapter V, we approach geometries which "live" in different monoidal categories. That is we outline an appropriate formalism and construct important (classes of) examples of noncommutative varieties in this setting. The necessity to do this follows already from [LR]: the quantum flag variety of a semi-simple Lie algebra $\mathfrak{g}$ and the corresponding D-scheme live in the monoidal category of $\mathbb{Z}^n$-graded vector spaces, where n is the rank of the lattice of integral weights, endowed with a braiding determined by the Cartan matrix of $\mathfrak{g}$. It is worth to mention that, thanks to an adequate generality, the classically looking constructions introduced and studied in Chapter III – generalized Grassmannians and flag varieties, are better understood. We conclude the chapter with a combinatorial construction, which can be regarded as a machine creating big classes of (not only smooth) noncommutative varieties. These classes include all examples considered earlier in the text and much more.

Each chapter has a fairly detailed summary, which complements this introduction.

There are two appendices. The first appendix gives a short exposition of fibred categories and cartesian functors; the second appendix contains elementary facts on actions of monoidal categories needed for Chapter V (and a little bit for Chapter I).

A remarkable feature of this whole story is that everything is derived from several notions carrying, with a good reason, familiar names, which are of surprising generality. Thus, affine morphisms – one of the first key notions, appear as morphisms between 'spaces' represented by arbitrary categories. The finiteness conditions, which are seen everywhere and are used here, in particular, to single out locally affine 'spaces' and schemes, are defined even in a more general framework. Same holds for (formally) smooth and étale morphisms, open immersions, closed immersions and related to the latter separated morphisms. Even classical examples, like Grassmannians and flag varieties are special cases of very general constructions, which make sense for non-additive categories. Note by passing that same holds for geometric picture – spectral theory, which is the subject of [R15].

Commutative algebraic geometry turns out to be a special case of geometric phenomena existing in a much greater generality. Even the geometrization of noncommutative algebras and abelian categories is a very small part of actual possibilities.

Chapter I
'Spaces' Represented by Categories. Flat Descent.

According to Grothendieck's philosophy (in Manin's interpretation [M1, p.83]), *to do geometry you really don't need a space, all you need is a category of sheaves on this space.*

Definition of the **Proj** of an associative $\mathbb{Z}_+$-graded algebra is one of the applications of this thesis. The affine case fits naturally into this viewpoint: for any associative unital k-algebra R, the category of quasi-coherent sheaves on the corresponding affine scheme is identified with the category $R-mod$ of left R-modules.

So that the correspondence [spaces $\longrightarrow$ categories] includes the correspondence [spaces $\longrightarrow$ algebras] of affine geometry.

The universal category of 'spaces'.

Any incarnation of the "space $\longmapsto$ category" philosophy (where the word "category" is understood in its straightforward sense) is expressed as a contra-variant pseudo-functor from some category $\mathfrak{B}$ of "spaces" to Cat. This pseudo-functor can be interpreted as a choice of an *inverse image* functor f^* for each morphism f of $\mathfrak{B}$ and an isomorphism, $\alpha_{f,g}$, from f^*g^* to $(gf)^*$ for any composable pair of morphisms f, g. Isomorphic pseudo-functors give equivalent theories. All these equivalent theories are encoded in the fibred category associated with the pseudo-functor in question.

There is a natural fibred category

$$\mathfrak{Cat}^o \xrightarrow{\pi} |Cat|^o \tag{1}$$

defined as follows. The category $|Cat|^o$ has the same objects as Cat^{op} and morphisms from X to Y are isomorphism classes of functors from the category C_Y corresponding Y to the category C_X. We call the objects of the category $|Cat|^o$ 'spaces', or *'spaces' represented by categories.* Objects of the category $\mathfrak{Cat}^o$ are pairs (X, M), where X is a 'space' and M is an object of the category C_X. Morphisms from (X, M) to (Y, N) are pairs (f^*, ξ), where f^* is a functor $C_Y \longrightarrow C_X$ and ξ a morphism $f^*(N) \longrightarrow M$. The functor π maps every object (X, M) of $\mathfrak{Cat}^o$ to the object X of $|Cat|^o$ and every morphism $(X, M) \xrightarrow{(f^*,\xi)} (Y, N)$ to the morphism $X \xrightarrow{f} Y$ whose inverse image functor is f^*.

The fibred category (1) is universal in the sense that every fibred category $\mathfrak{F} \xrightarrow{\pi_\beta} \mathfrak{B}$ corresponding to a pseudo-functor $\mathfrak{B}^{op} \xrightarrow{\beta} Cat$ is the pull-back of (1) along the composition $\mathfrak{B} \longrightarrow |Cat|^o$ of $\mathfrak{B} \xrightarrow{\beta^{op}} Cat^{op}$ with the projection $Cat^{op} \xrightarrow{\pi} |Cat|^o$.

Thus, $|Cat|^o$ can be regarded as the universal category of spaces. We call its objects *'spaces' represented by categories.* The whole "space $\longmapsto$ category" philosophy is encoded in the fibred category of 'spaces' $\mathfrak{Cat}^o \xrightarrow{\pi} |Cat|^o$.

This observation is the starting point of the chapter. The immediate purpose is to define and study geometry (or geometries) inside of the category $|Cat|^o$, like commutative scheme theory is defined and studied inside of the category of locally ringed spaces.

What we really study and use is not so much 'spaces', but certain classes of morphisms of 'spaces'. The most important among them are *continuous', flat*, and *affine* morphisms introduced in [R]. A morphism is continuous if its inverse image functor has a right adjoint (called a direct image functor), and flat if, in addition, the inverse image functor is left exact (i.e. preserves finite limits). A continuous morphism is called *affine* if its direct image functor is conservative (i.e. it reflects isomorphisms) and has a right adjoint.

Affine and locally affine 'spaces'. Given an object S of the category $|Cat|^o$, we define the category $\mathbf{Aff}_S$ of affine S-spaces as the full subcategory of $|Cat|^o/S$ whose objects are affine morphisms to S. The functor from $\mathbf{Aff}_S$ to $|Cat|^o$ is the composition of the inclusion functor $\mathbf{Aff}_S \hookrightarrow |Cat|^o/S$ and the canonical functor $|Cat|^o/S \longrightarrow |Cat|^o$. The choice of the object S influences drastically the rest of the story. Thus, if $S = \mathbf{Sp}\mathbb{Z}$ (i.e. C_S is the category of abelian groups), then $\mathbf{Aff}_S$ is naturally equivalent to the category opposite to the category $\mathfrak{Ass}$ which is defined as follows: objects of $\mathfrak{Ass}$ are associative unital rings and morphisms are conjugation classes of unital ring morphisms. If C_S is the category **Sets**, then the category $\mathbf{Aff}_S$ is equivalent to the category $\mathfrak{Ass}_{\mathcal{E}}^{op}$, where objects of $\mathfrak{Ass}_{\mathcal{E}}$ are monoids and morphisms are conjugation classes of monoid homomorphisms.

Locally affine objects are defined in an obvious way, once a notion of a cover (a *quasi-pretopology*) is fixed. We introduce several canonical quasi-pretopologies on the category $|Cat|^o$. Their common feature is the following: if a set of morphisms to X is a cover, then the set of their inverse image functors is conservative and all inverse image functors are exact in a certain mild way. If, in addition, morphisms of covers are continuous, X has a finite affine cover, and the category C_S has finite limits, this requirement suffices to recover the object X from the covering data uniquely up to isomorphism (i.e. the category C_X is recovered uniquely up to equivalence) via 'flat descent'.

In Section 1, we remind first notions of *categoric geometry* ('spaces' represented by categories, morphisms represented by their inverse image functors, continuous, flat and affine morphisms) and sketch several examples of noncommutative spaces which are among illustrations and/or motivations of constructions of this work.

In Section 2, we study general properties of the category of $|Cat|^o$ of 'spaces' starting from the existence (and a description) of arbitrary limits and colimits.

In Section 3, we describe continuous morphisms from an arbitrary "space" X to the *categoric spectrum* $\mathbf{Sp}(R)$ of a ring R. We argue that continuous morphisms $X \longrightarrow \mathbf{Sp}(R)$

are in bijective correspondence with isomorphism classes of *right R-modules* $\mathcal{O}$ in the category C_X (i.e. R-modules in the opposite category C_X^{op}). In the case $X = \mathbf{Sp}(T)$ for some ring T, this correspondence expresses a classical fact (a theorem by Eilenberg and Moore) which in the our pseudo-geometric language sounds as follows: inverse image functors of continuous morphisms $\mathbf{Sp}(T) \longrightarrow \mathbf{Sp}(R)$ are given by (T, R)-bimodules.

In Section 4, we start to study continuous morphisms via monads and comonads associated with them, using as a main tool the Beck's theorem characterizing the so called *monadic* and *comonadic* morphisms. For a monad $\mathcal{F}$ on a "space" X (i.e. on the category C_X), we define the *categoric spectrum* of $\mathcal{F}$ as the "space" $\mathbf{Sp}(\mathcal{F}/X)$ corresponding to the category of $\mathcal{F}$-modules. The categoric spectrum of a monad is a natural generalization of the categoric spectrum of a ring. Dually, for any comonad $\mathcal{G}$ on X, we define its *cospectrum*, $\mathbf{Sp}^o(X\backslash\mathcal{G})$ as the space corresponding to the category $\mathcal{G} - Comod$ of $\mathcal{G}$-comodules.

In Section 5, we exploit the fact that an affine morphism to X is isomorphic to the canonical morphism $\mathbf{Sp}(\mathcal{F}/X) \longrightarrow X$ for a *continuous* monad $\mathcal{F} = (F, \mu)$. Here 'continuous' means that the functor F has a right adjoint. A consequence of this fact is that any affine morphism $Y \longrightarrow \mathbf{Sp}(R)$ is equivalent to the morphism $\mathbf{Sp}(T) \longrightarrow \mathbf{Sp}(R)$ corresponding to a ring morphism $R \longrightarrow T$. In particular, a direct image functor of any affine morphism $\mathbf{Sp}(S) \longrightarrow \mathbf{Sp}(R)$ is a composition of a Morita equivalence and the "restriction of scalars" (pull-back) functor corresponding to a ring morphism.

In Section 6, we study *affine flat descent*. If $U \xrightarrow{f} X$ is a flat conservative affine morphism ('conservative' means that f^* reflects isomorphisms), then it follows from Beck's theorem that X is isomorphic to $\mathbf{Sp}^o(U\backslash\mathcal{G}_f)$, where $\mathcal{G}_f = (G_f, \delta_f)$ is a *continuous* comonad. 'Continuous' means that the functor G_f has a right adjoint. In the case $U = \mathbf{Sp}(R)$ for some ring R, continuous comonads are given by coalgebras in the category of R-bimodules.

The main commutative example is an arbitrary semi-separated quasi-compact scheme. Recall that a scheme X is *semi-separated* if it has an affine cover $\{U_i \hookrightarrow X \mid i \in J\}$ such that all finite intersections of the open subschemes U_i are affine.

In Section 7, we introduce the Cone of a non-unital monad and the Proj of a non-unital graded monad. Motivated by important constructions of representation theory of classical and quantum groups and enveloping algebras, we consider Hopf actions on non-unital rings and induced actions on the corresponding quasi-affine 'spaces'. Applying general facts to the action of the enveloping algebra of a semi-simple (or reductive) Lie algebra, we realize the category of D-modules on the base affine space and the flag variety as categories of quasi-coherent sheaves on resp. the cone and the Proj of a graded ring naturally associated with the Lie algebra. This setting is extended to actions of the quantized enveloping algebra of a semi-simple Lie algebra on the quantum base affine 'space' and the quantum flag variety.

1. 'Spaces' represented by categories. Examples.

1.0. The category of 'spaces'. Continuous, flat, and affine morphisms. We denote by $|Cat|^o$ the category whose objects are *svelte* (i.e. equivalent to small) categories with respect to some fixed universum. We call objects of $|Cat|^o$ *'spaces'*, or *'spaces' represented by categories*. For any 'space' X, we denote by C_X the corresponding category (regarded as an *appropriate* category of 'sheaves' on X), and for any svelte category $\mathcal{A}$, we denote by $|\mathcal{A}|$ the *underlying 'space'* of $\mathcal{A}$ defined by $C_{|\mathcal{A}|} = \mathcal{A}$. Morphisms from X to Y are isomorphism classes of functors $C_Y \longrightarrow C_X$. A functor $C_Y \longrightarrow C_X$ representing a morphism $X \xrightarrow{f} Y$ is, usually, denoted by f^* and called an *inverse image functor* of f. We shall write $f = [F]$ to indicate that f is a morphism having an inverse image functor F. The composition of morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ is defined by $g \circ f = [f^* \circ g^*]$.

1.0.1. Definitions. (a) A morphism of 'spaces' $X \xrightarrow{f} Y$ is *continuous* if its inverse image functor has a right adjoint, $C_X \xrightarrow{f_*} C_Y$ (called a *direct image* functor of f).

(b) A continuous morphism is called *flat* if its inverse image functor is *left exact*, i.e. it preserves finite limits.

(c) A continuous morphism $X \xrightarrow{f} Y$ is called *affine* if its direct image functor is conservative (i.e. it reflects isomorphisms) and has a right adjoint, $C_Y \xrightarrow{f^!} C_X$.

1.1. The categoric spectrum of a unital ring. For an associative unital ring R, we define the *categoric spectrum* of R as the object $\mathbf{Sp}(R)$ of $|Cat|^o$ such that $C_{\mathbf{Sp}(R)} = R-mod$. Let $R \xrightarrow{\phi} S$ be a unital ring morphism and $R-mod \xrightarrow{\bar{\phi}^*} S-mod$ the functor $S \otimes_R -$. The canonical right adjoint to $\bar{\phi}^*$ is the *restriction of scalars* functor – the *pull-back along* the ring morphism ϕ. A right adjoint to $\bar{\phi}_*$ is given by

$$R-mod \xrightarrow{\bar{\phi}^!} S-mod, \quad L \longmapsto Hom_R(\phi_*(S), L). \tag{1}$$

The map

$$\Big(R \xrightarrow{\phi} S\Big) \longmapsto \Big(\mathbf{Sp}(S) \xrightarrow{\bar{\phi}} \mathbf{Sp}(R)\Big)$$

is a functor

$$Rings^{op} \xrightarrow{\mathbf{Sp}} |Cat|^o$$

which takes values in the subcategory formed by affine morphisms.

The image $\mathbf{Sp}(R) \xrightarrow{\bar{\phi}} \mathbf{Sp}(T)$ of a ring morphism $T \xrightarrow{\phi} R$ is flat (resp. faithful) morphism of 'spaces' iff $R \xrightarrow{\phi} S$ is a flat (resp. faithful) ring morphism, i.e. it turns R into a flat (resp. faithful) right T-module.

1.1.1. Continuous, flat, and affine morphisms from $\mathbf{Sp}(S)$ to $\mathbf{Sp}(R)$. Let R and S be associative unital rings. A morphism $\mathbf{Sp}(S) \xrightarrow{f} \mathbf{Sp}(R)$ with an inverse image functor f^* is continuous iff

$$f^* \simeq \mathcal{M}\otimes_R \ : L \longmapsto \mathcal{M} \otimes_R L \tag{2}$$

for an (S, R)-bimodule $\mathcal{M}$ defined uniquely up to isomorphism. The functor

$$f_* = Hom_S(\mathcal{M}, -) : N \longmapsto Hom_S(\mathcal{M}, N) \tag{3}$$

is a direct image of f. It follows that the morphism f is conservative iff $\mathcal{M}$ is *faithful* as a right R-module, i.e. the functor $\mathcal{M} \otimes_R -$ is faithful. The direct image functor (3) is conservative iff $\mathcal{M}$ is a cogenerator in the category of left S-modules, i.e. for any nonzero S-module N, there exists a nonzero S-module morphism $\mathcal{M} \longrightarrow N$.

The morphism f is flat iff $\mathcal{M}$ is flat as a right R-module.

The functor (3) has a right adjoint, $f^!$, iff f_* is isomorphic to the tensoring (over S) by a bimodule. This happens iff $\mathcal{M}$ is a projective S-module of finite type. The latter is equivalent to the condition: the natural functor morphism

$$\mathcal{M}_S^* \otimes_S \ \longrightarrow Hom_S(\mathcal{M}, -)$$

is an isomorphism. Here $\mathcal{M}_S^* = Hom_S(M, S)$. In this case, $f^! \simeq Hom_R(\mathcal{M}_S^*, -)$. In particular, taking $M = \phi_*(S)$, we recover (1) above.

1.2. The graded version. Let $\mathcal{G}$ be a monoid and R a $\mathcal{G}$-graded unital ring. We define the 'space' $\mathbf{Sp}_{\mathcal{G}}(R)$ by taking as $C_{\mathbf{Sp}_{\mathcal{G}}(R)}$ the category $gr_{\mathcal{G}}R - mod$ of left $\mathcal{G}$-graded R-modules. There is a natural functor

$$gr_{\mathcal{G}}R - mod \xrightarrow{\phi_*} R_0 - mod$$

which assigns to each graded R-module its zero component ('zero' is the unit element of the monoid $\mathcal{G}$). The functor ϕ_* has a left adjoint, ϕ^*, which maps every R_0-module M to the graded R-module $R \otimes_{R_0} M$. The adjunction arrow $Id_{R_0 - mod} \longrightarrow \phi_*\phi^*$ is an isomorphism. This means that the functor ϕ^* is fully faithful, or, equivalently, the functor ϕ_* is (equivalent to) a localization.

The functors ϕ_* and ϕ^* are regarded as respectively a direct and an inverse image functor of a morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$. It follows from the above that the morphism ϕ is affine iff ϕ is an isomorphism (i.e. ϕ^* is an equivalence of categories).

In fact, if ϕ is affine, the functor ϕ_* should be conservative. Since ϕ_* is a localization, this means, precisely, that ϕ_* is an equivalence of categories.

1.3. The cone of a non-unital ring. Let R_0 be a unital associative ring, and let R_+ be an associative ring, non-unital in general, in the category of R_0-bimodules; i.e. R_+ is endowed with an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. Let R denote the augmented ring described by this data ($R = R_0 \oplus R_+$ as an R_0-bimodule) and $\mathcal{T}_{R_+}$ the full subcategory of the category $R-mod$ whose objects are R-modules annihilated by R_+. Let $\mathcal{T}^-_{R_+}$ be the *Serre* subcategory (that is a full subcategory closed by taking subquotients, extensions, and arbitrary direct sums) of the category $R-mod$ spanned by $\mathcal{T}_{R_+}$. One can see that objects of $\mathcal{T}^-_{R_+}$ are all R-modules whose elements are annihilated by a (depending on the element) power of R_+.

We denote the quotient category $R - mod/\mathcal{T}^-_{R_+}$ by $C_{\mathbf{Cone}(R_+)}$ defining this way the 'space' *cone of* R_+. The localization functor $R-mod \xrightarrow{u^*} R-mod/\mathcal{T}^-_{R_+}$ is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. If R_+ is a unital ring, then u is an isomorphism (see 7.2.2.1). The composition of the morphism u with the canonical affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor*.

1.4. The graded version: $\mathbf{Proj}_{\mathcal{G}}$. Let $\mathcal{G}$ be a monoid and $R = R_0 \oplus R_+$ a $\mathcal{G}$-graded ring with zero component R_0. Then we have the category $gr_{\mathcal{G}}R - mod$ of $\mathcal{G}$-graded R-modules and its full subcategory $gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R - mod$ whose objects are graded modules annihilated by the ideal R_+. We define the 'space' $\mathbf{Proj}_{\mathcal{G}}(R)$ by setting

$$C_{\mathbf{Proj}_{\mathcal{G}}(R)} = gr_{\mathcal{G}}R - mod/gr_{\mathcal{G}}\mathcal{T}^-_{R_+}.$$

Here $gr_{\mathcal{G}}\mathcal{T}^-_{R_+}$ is the Serre subcategory of the category $gr_{\mathcal{G}}R - mod$ spanned by $gr_{\mathcal{G}}\mathcal{T}_{R_+}$. One can show that $gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = gr_{\mathcal{G}}R - mod \cap \mathcal{T}^-_{R_+}$. Therefore, we have a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R).$$

The localization functor $gr_{\mathcal{G}}R - mod \longrightarrow C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is an inverse image functor of a continuous morphism $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}_{\mathcal{G}}(R)$. The composition $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\mathfrak{v}} \mathbf{Sp}(R_0)$ of the morphism $\mathfrak{v}$ with the canonical morphism $\mathbf{Sp}_{\mathcal{G}}(R) \xrightarrow{\phi} \mathbf{Sp}(R_0)$ defines $\mathbf{Proj}_{\mathcal{G}}(R)$ as a 'space' over $\mathbf{Sp}(R_0)$. Its direct image functor is called the *global sections functor*.

1.4.1. Cone and Proj of a $\mathbb{Z}_+$-graded ring. Let $R = \bigoplus_{n \geq 0} R_n$ be a $\mathbb{Z}_+$-graded ring, $R_+ = \bigoplus_{n \geq 1} R_n$ its 'irrelevant' ideal. Thus, we have the *cone* of R_+, $\mathbf{Cone}(R_+)$, and $\mathbf{Proj}(R) = \mathbf{Proj}_{\mathbb{Z}}(R)$, and a canonical morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Proj}(R)$.

1.5. Skew cones and skew projective 'spaces'. Let A be an arbitrary associative k-algebra and $\mathbf{q}$ a matrix $[q_{ij}]_{i,j\in J}$ with entrees in k such that $q_{ij}q_{ji} = 1$ for all $i,j \in J$ and $q_{ii} = 1$ for all $i \in J$. Let $R = A_{\mathbf{q}}[\mathbf{x}]$ denote a skew polynomial algebra corresponding to this data. Here $\mathbf{x} = (x_i \mid i \in J)$ is a set of indeterminates satisfying the relations

$$x_i x_j = q_{ij} x_j x_i \quad \text{for all } i,j \in J, \tag{1}$$

$$x_i r = r x_i \quad \text{for all } i \in J \text{ and } r \in k \tag{2}$$

For any $i \in J$, set $S_i = \{x_i^n \mid n \geq 1\}$. Each of S_i is a left and right Ore set in R, and the Serre subcategory $\mathcal{T}^-_{R_+}$ is generated by all R-modules whose elements are annihilated by some elements of $\bigcup\limits_{i\in J} S_i$. This implies that the localization functors

$$R - mod \longrightarrow S_i^{-1}R - mod$$

factor through the localization functor $R - mod \longrightarrow C_{\mathbf{Cone(R_+)}}$, and the induced localization $C_{\mathbf{Cone(R_+)}} \xrightarrow{u_i^*} S_i^{-1}R - mod$ form a conservative family. The corresponding family of morphisms of 'spaces' $\{\mathbf{Sp}(S_i^{-1}R) \xrightarrow{u_i} \mathbf{Cone}(R_+) \mid i \in J\}$ is an *affine cover* of the cone $\mathbf{Cone}(R_+)$. It follows that the algebra $S_i^{-1}R$ is isomorphic to $A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}]$.

Let $\mathcal{G} = \mathbb{Z}^J$; and let γ_i, $i \in J$, denote the canonical generators of the group $\mathcal{G}$. Assigning to each x_i the parity γ_i, we turn the skew polynomial algebra $R = A_{\mathbf{q}}[\mathbf{x}]$ into a $\mathcal{G}$-graded algebra with $R_0 = A$. Each of the localizations $R - mod \longrightarrow S_i^{-1}R - mod$ induces a localization $gr_{\mathcal{G}}R - mod \longrightarrow gr_{\mathcal{G}}S_i^{-1}R - mod$ which maps the kernel of the localization $gr_{\mathcal{G}}R - mod \longrightarrow C_{\mathbf{Proj}(R)}$ to zero. Therefore, it is the composition of $gr_{\mathcal{G}}R - mod \longrightarrow C_{\mathbf{Proj}(R)}$ and a localization

$$C_{\mathbf{Proj}(R)} \xrightarrow{\mathfrak{v}_i^*} gr_{\mathcal{G}}S_i^{-1}R - mod = gr_{\mathcal{G}}A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}] - mod. \tag{3}$$

Let $\mathcal{G}_i$ denote the quotient group $\mathcal{G}/\mathbb{Z}\gamma_i$. The category $gr_{\mathcal{G}}A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}] - mod$ in (3) is naturally equivalent to the category $gr_{\mathcal{G}_i}A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod$ of left $\mathcal{G}_i$-graded modules over the skew polynomial algebra $A_{\mathbf{q}_i}[\mathbf{x}/x_i]$. Here $\mathbf{x}/x_i$ denotes $\{x_j/x_i | j \in J, j \neq i\}$, and $\mathbf{q}_i$ denotes the matrix $[q_{ni}q_{nm}q_{mi}^{-1}]_{n,m\in J-\{i\}}$ (cf. [R, I.7.2.2.4]). Note that $A_{\mathbf{q}_i}[\mathbf{x}/x_i]$ is the $\mathcal{G}_i$-component of the algebra $A_{\mathbf{q}}[\mathbf{x}, x_i^{-1}]$ of the 'functions' on $\mathbf{Cone}(R)/|\mathbb{S}'_i|$.

Let 'spaces' U_i are defined by $C_{U_i} = gr_{\mathcal{G}_i}A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod$. Note that if the cardinality of J is greater than one, then the natural morphisms $U_i \xrightarrow{u_i} \mathbf{Proj}_{\mathcal{G}}(R)$ do not form an affine cover of $\mathbf{Proj}_{\mathcal{G}}(R)$ over $\mathbf{Sp}(A)$, because the composition of $\mathfrak{v}_{i*}$ with the direct image of the projection $\mathbf{Proj}_{\mathcal{G}}(R) \xrightarrow{\pi} \mathbf{Sp}(A)$ is isomorphic to the functor

$$gr_{\mathcal{G}_i}A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod \longrightarrow A - mod$$

which assigns to each $\mathcal{G}_i$-graded module (resp. $\mathcal{G}_i$-graded module morphism) its zero component. If the group $\mathcal{G}_i$ is non-trivial, this functor is not faithful, hence the morphism $\pi \circ u_i$ is not affine.

1.5.1. The projective q-'space' $\mathbf{P}^r_{\mathbf{q}}$. Let again $R = A_{\mathbf{q}}[\mathbf{x}]$, $\mathbf{x} = (x_0, x_1, \ldots, x_r)$. But, we take $\mathcal{G} = \mathbb{Z}$ with the natural order; and set the parity of each x_i equal to 1. One can repeat with $\mathbf{Cone}_{\mathbb{Z}}(R)$ and $\mathbf{P}^r_{\mathbf{q}} = \mathbf{Proj}_{\mathbb{Z}}(R)$ the same pattern as with $\mathbf{Cone}(R)$ and $\mathbf{P}^r_{\mathcal{G}} = \mathbf{Proj}_{\mathcal{G}}(R)$. Only this time the quotient groups $\mathcal{G}_i$ will be trivial, and we obtain a picture very similar to the classical one: $\mathbf{P}^r_{\mathbf{q}}$ is a $\mathbb{Z}$-scheme covered by $r+1$ affine spaces $A_{\mathbf{q}_i}[\mathbf{x}/x_i] - mod$, $i = 0, 1, \ldots, r$.

1.6. The base affine 'space' and the flag variety of a reductive Lie algebra. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{G}$ be the group of integral weights of $\mathfrak{g}$ and $\mathcal{G}_+$ the semigroup of nonnegative integral weights. Let $R = \oplus_{\lambda \in \mathcal{G}_+} R_\lambda$, where R_λ is the vector space of the (canonical) irreducible finite dimensional representation with the highest weight λ. The module R is a $\mathcal{G}$-graded algebra with the multiplication determined by the projections $R_\lambda \otimes R_\nu \longrightarrow R_{\lambda+\nu}$ for all $\lambda, \nu \in \mathcal{G}_+$. It is well known that the algebra R is isomorphic to the algebra of regular functions on the *base affine space* of $\mathfrak{g}$. Recall that the *base affine space* of $\mathfrak{g}$ (which is not affine, but a quasi-affine scheme) is the quotient space G/U, where G is a connected simply connected algebraic group with the Lie algebra $\mathfrak{g}$, and U is its maximal unipotent subgroup.

The category $C_{\mathbf{Cone}(R)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space Y of the Lie algebra $\mathfrak{g}$. The category $Proj_{\mathcal{G}}(R)$ is equivalent to the category of quasi-coherent sheaves on the flag variety of $\mathfrak{g}$.

1.7. The quantized base affine 'space' and quantized flag variety of a semi-simple Lie algebra. Let now $\mathfrak{g}$ be a semi-simple Lie algebra over a field k of zero characteristic, and let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of $\mathfrak{g}$. Define the $\mathcal{G}$-graded algebra $R = \oplus_{\lambda \in \mathcal{G}_+} R_\lambda$ the same way as above. This time, however, the algebra R is not commutative. Following the classical example, we call $\mathbf{Cone}(R)$ *the quantum base affine 'space'* and $\mathbf{Proj}_{\mathcal{G}}(R)$ the *quantum flag variety* of $\mathfrak{g}$.

1.7.1. Canonical affine covers of the base affine 'space' and the flag variety. Let W be the Weyl group of the Lie algebra $\mathfrak{g}$. Fix a $w \in W$. For any $\lambda \in \mathcal{G}_+$, choose a nonzero w-extremal vector $e^\lambda_{w\lambda}$ generating the one dimensional vector subspace of R_λ formed by the vectors of the weight $w\lambda$. Set $S_w = \{k^* e^\lambda_{w\lambda} \mid \lambda \in \mathcal{G}_+\}$. It follows from the Weyl character formula that $e^\lambda_{w\lambda} e^\mu_{w\mu} \in k^* e^{\lambda+\mu}_{w(\lambda+\mu)}$. Hence S_w is a multiplicative set. It was proved by Joseph [Jo] that S_w is a left and right Ore subset in R. The Ore sets $\{S_w \mid w \in W\}$ determine a conservative family of affine localizations

$$\mathbf{Sp}(S_w^{-1}R) \longrightarrow \mathbf{Cone}(R), \quad w \in W, \tag{4}$$

of the quantum base affine 'space' and a conservative family of affine localizations

$$\mathbf{Sp}_{\mathcal{G}}(S_w^{-1}R) \longrightarrow \mathbf{Proj}_{\mathcal{G}}(R), \quad w \in W,$$

of the quantum flag variety. We claim that the category $gr_{\mathcal{G}} S_w^{-1}R - mod$ is naturally equivalent to $(S_w^{-1}R)_0 - mod$. By 1.2, it suffices to verify that the canonical functor

$$gr_{\mathcal{G}} S_w^{-1}R - mod \longrightarrow (S_w^{-1}R)_0 - mod,$$

which assigns to every graded $S_w^{-1}R$-module its zero component is faithful; i.e. the zero component of every nonzero $\mathcal{G}$-graded $S_w^{-1}R$-module is nonzero. This is, indeed, the case, because if z is a nonzero element of the λ-component of a $\mathcal{G}$-graded $S_w^{-1}R$-module, then $(e_{w\lambda}^{\lambda})^{-1}z$ is a nonzero element of the zero component of this module.

2. Basic properties of the category of 'spaces'.

2.1. Initial objects of $|Cat|^o$. The category $\bullet$ with one (identical) morphism (in particular with one object) is an initial object of $|Cat|^o$. A morphism $A \xrightarrow{f} B$ in $|Cat|^o$ with an inverse image functor f^* is an isomorphism iff f^* is a category equivalence. In particular, $X \in Ob|Cat|^o$ is an initial object of $|Cat|^o$ iff the category C_X is a connected groupoid; i.e. all arrows of C_X are invertible and there are arrows between any two objects.

Notice that, for any 'space' X, the set $|Cat|^o(X, \bullet)$ of morphisms $X \to \bullet$ is isomorphic to the set $|X|$ of isomorphism classes of objects of the category C_X.

The category $|Cat|^o$ has no "real" final objects: its unique final object is the 'space' represented by the empty category.

2.2. Proposition. *The category $|Cat|^o$ has small limits and colimits.*

Proof. (a) Let $\{X_i \mid i \in J\}$ be a set of objects of $|Cat|^o$. Then $X^J = \prod_{i \in J} X_i$ and $X_J = \coprod_{i \in J} X_i$ are defined by

$$C_{X^J} = \coprod_{i \in J} C_{X_i} \quad \text{and} \quad C_{X_J} = \prod_{i \in J} C_{X_i}.$$

(b) Every pair of arrows, $X \overset{f}{\underset{g}{\rightrightarrows}} Y$, in $|Cat|^o$ has a cokernel.

Let $C_Y \overset{f^*}{\underset{g^*}{\rightrightarrows}} C_X$ be inverse image functors of respectively f and g. Let C_Z denote the category whose objects are pairs (x, ϕ), where $x \in ObC_Y$ and ϕ is an isomorphism

$f^*(x) \xrightarrow{\sim} g^*(x)$. A morphism from (x, ϕ) to (y, ψ) is a morphism $\xi : x \to y$ such that the diagram

$$\begin{array}{ccc} f^*(x) & \xrightarrow{f^*(\xi)} & f^*(y) \\ \phi \downarrow & & \downarrow \psi \\ f^*(x) & \xrightarrow{f^*(\xi)} & f^*(y) \end{array}$$

commutes. Denote by $\mathfrak{h}^*$ the forgetful functor $C_Z \longrightarrow C_Y$, $(x, \phi) \longmapsto x$. Let $Y \xrightarrow{w} W$ be a morphism in $|Cat|^o$ with an inverse image functor w^* such that $w \circ f = w \circ g$. This means that there exists an isomorphism $f^* \circ w^* \xrightarrow{\psi} g^* \circ w^*$. The pair (w^*, ψ) defines a functor $\gamma^*_{w^*,\psi} : C_W \longrightarrow C_Z$, $b \longmapsto (w^*(b), \psi(b))$. A different choice, w_1^*, of the inverse image functor of w and an isomorphism $\psi_1 : w_1^* \circ f^* \xrightarrow{\sim} w_1^* \circ g^*$ produces a functor $\gamma^*_{w_1^*,\psi_1}$ isomorphic to $\gamma^*_{w^*,\psi}$. This shows that the morphism $Y \longrightarrow Z$ having the inverse image $\mathfrak{h}^*$ is the cokernel of the pair (f, g). The existence of cokernels and (small) coproducts is equivalent to the existence of arbitrary (small) colimits.

(c) Every pair of arrows, $X \underset{g}{\overset{f}{\rightrightarrows}} Y$, in $|Cat|^o$ has a kernel.

Let $C_Y \underset{g^*}{\overset{f^*}{\rightrightarrows}} C_X$ be inverse image functors of resp. f and g. Denote by $\mathfrak{D}_{f^*,g^*}$ the diagram scheme defined as follows:

$$Ob\mathfrak{D}_{f^*,g^*} = ObC_Y \coprod ObC_X \quad \text{and} \quad Hom\mathfrak{D}_{f^*,g^*} = HomC_X \coprod \Sigma_{f^*,g^*},$$

where

$$\Sigma_{f^*,g^*} = \{f^*(x) \overset{s_x}{\to} x,\ x \overset{t_x}{\to} g^*(x) \mid x \in ObC_Y\}.$$

Consider the category $\mathcal{P}a\mathfrak{D}_{f^*,g^*}$ of paths of the diagram $\mathfrak{D}_{f^*,g^*}$ together with the natural embeddings $HomC_X \xrightarrow{\tau} Hom\mathcal{P}a\mathfrak{D}_{f^*,g^*} \longleftarrow \Sigma_{f^*,g^*}$ which define the corresponding diagrams. We denote by $\mathcal{P}\mathfrak{D}_{f^*,g^*}$ the quotient of the category $\mathcal{P}a\mathfrak{D}_{f^*,g^*}$ by the minimal equivalence relation such that

$$\tau(\alpha \circ \beta) \sim \tau(\alpha) \circ \tau(\beta) \quad \text{and} \quad \tau(id_x) \sim id_{\tau(x)}$$

for all composable arrows α, β and for all $x \in ObC_X$.

Finally, we denote by C_W the quotient category $\Sigma^{-1}_{f^*,g^*}\mathcal{P}\mathfrak{D}_{f^*,g^*}$. It follows from the construction that the object W of the category $|Cat|^o$ defined this way is the kernel of the pair (f, g). Details are left to the reader. ■

3. Continuous morphisms to the categoric spectrum of a ring and 'structure sheaves'. Let R be an associative unital ring. For a morphism $X \xrightarrow{f} \mathbf{Sp}(R)$

with an inverse image functor f^*, we denote by $\mathcal{O}$ the object $f^*(R)$. It follows that the object $\mathcal{O}$ is determined by the pair (X, f) uniquely up to isomorphism. The functor f^* defines a monoid morphism $End_R(R) \longrightarrow C_X(\mathcal{O}, \mathcal{O})$ whose composition with the canonical ring isomorphism $R^o \xrightarrow{\sim} End_R(R)$ gives a monoid morphism $R \xrightarrow{\phi_f} C_X(\mathcal{O}, \mathcal{O})^o$. Here $C_X(\mathcal{O}, \mathcal{O})^o$ denotes the monoid opposite to $C_X(\mathcal{O}, \mathcal{O})$. If the category C_X is preadditive and the functor f^* is additive, the morphism ϕ_f is a unital ring morphism.

In general, the object $\mathcal{O}$ does not determine the morphism f. It does, however, if f is continuous:

3.1. Proposition. *Let $X \xrightarrow{f} \mathbf{Sp}(R)$ be a continuous morphism. Then*
(a) The morphism f is determined by $\mathcal{O} = f^*(R)$ uniquely up to isomorphism.
(b) There exists a coproduct of any small set of copies of $\mathcal{O}$.
(c) The object $\mathcal{O}$ has a structure of an R-module in the category C_X^{op}. In particular, $\mathcal{O}$ is an abelian cogroup in the category C_X (i.e. an abelian group in C_X^{op}) and the canonical map $R \xrightarrow{\phi_f} C_X(\mathcal{O}, \mathcal{O})^o$ is a ring morphism.

Proof. (a) Let f_* be a direct image functor of f (i.e. a right adjoint to f^*). we have functorial isomorphisms $C_X(f^*(R), M) \simeq Hom_R(R, f_*(M)) \simeq f_*(M)$ which shows that the direct image functor f_* of the morphism f is naturally isomorphic to the functor $M \longmapsto C_X(f^*(R), M)$, where the object $C_X(f^*(R), M)$ is endowed a natural R-module structure determined by the composition of the isomorphism $R \xrightarrow{\sim} Hom_R(R, R)^o$ and $Hom_R(R, R)^o \longrightarrow C_X(f^*(R), f^*(R))^o$ and the $C_X(f^*(R), f^*(R))^o$-module structure on $C_X(f^*(R), M)$ given by the composition of arrows

$$C_X(f^*(R), f^*(R))^o \otimes C_X(f^*(R), M) \longrightarrow C_X(f^*(R), M).$$

Therefore the inverse image functorf^* of f is defined uniqely up to isomorphism (being a left adjoint to f_*) by the object $f^*(R)$.

(b) Since the functor f^* preserves colimits, there exists a coproduct of any set of copies of the object $\mathcal{O} = f^*(R)$.

(c) The assertion follows from the isomorphism $f_* \simeq C_X(\mathcal{O}, -)$ and the fact that f_* takes values in the category of R-modules. ∎

3.2. Global sections functor. We call R-modules in the category C_X^{op} *right R-modules in C_X*, or *right R-modules on the 'space' X*. The right R-module $\mathcal{O}$ in 3.1 is viewed as the 'structure sheaf' on the 'space' X over $\mathbf{Sp}(R)$.

We denote $C_X(\mathcal{O}, \mathcal{O})^o$ by $\Gamma_X \mathcal{O}$. The functor

$$C_X \xrightarrow{f_{\mathcal{O}^*}} \Gamma_X \mathcal{O} - mod, \quad M \longmapsto C_X(\mathcal{O}, M)$$

will be called the *global sections functor* on $(X, \mathcal{O})$. In particular, $\Gamma_X\mathcal{O} = f_{\mathcal{O}*}(\mathcal{O})$ is the ring of global sections of the 'structure sheaf' $\mathcal{O}$. It follows from 3.1(c) that the functor $f_* = C_X(\mathcal{O}, -)$ is naturally decomposed into

$$\begin{array}{ccc} C_X & \xrightarrow{f_{\mathcal{O}*}} & \Gamma_X\mathcal{O} - mod \\ f_* \searrow & & \swarrow \bar{\phi}_{f*} \\ & R - mod & \end{array} \qquad (1)$$

where $\bar{\phi}_{f*}$ is the pull-back by the ring morphism $R \xrightarrow{\phi_f} \Gamma_X\mathcal{O}$ defining a right R-module structure on $\mathcal{O}$.

3.3. $\mathbb{Z}$-'spaces'. Let $X \xrightarrow{f} \mathbf{Sp}(R)$ be a continuous morphism with an inverse image functor f^*, and let $\mathcal{O} = f^*(R)$.

3.3.1. Lemma. *The global sections functor $f_{\mathcal{O}*}$ is a direct image functor of a continuous morphism, $X \xrightarrow{f_{\mathcal{O}}} \mathbf{Sp}(\Gamma_X\mathcal{O})$, iff any pair of arrows $\mathcal{O}^{\oplus I} \rightrightarrows \mathcal{O}^{\oplus J}$ between coproducts of copies of $\mathcal{O}$ has a cokernel in C_X.*

Proof. The inverse image functor $f^*_{\mathcal{O}}$ assigns to a free $\Gamma_X\mathcal{O}$-module $\Gamma_X\mathcal{O}^{\oplus J}$ the coproduct $\mathcal{O}^{\oplus J}$ of J copies of the object $\mathcal{O}$. ■

3.3.2. The category of $\mathbb{Z}$-spaces. Denote by $|Cat|^o_{\mathbb{Z}}$ the category whose objects are all pairs $(X, \mathcal{O})$, where X is an object of $|Cat|^o$ and $\mathcal{O}$ is an abelian group in C_X^{op} such that there exist coproducts of small sets of copies of $\mathcal{O}$ and any pair of arrows $\mathcal{O}^{\oplus I} \rightrightarrows \mathcal{O}^{\oplus J}$ between coproducts of copies of $\mathcal{O}$ has a cokernel in C_X. Morphisms from $(X, \mathcal{O})$ to $(X', \mathcal{O}')$ are morphisms $X \xrightarrow{f} X'$ such that there exists an isomorphism $f^*(\mathcal{O}') \xrightarrow{\sim} \mathcal{O}$. Composition is defined in an obvious way. Objects of the category $|Cat|^o_{\mathbb{Z}}$ will be called $\mathbb{Z}$-'spaces'.

3.3.2.1. A reformulation. By 3.3.1, $\mathbb{Z}$-spaces are pairs $(X, \mathcal{O})$ such that $\mathcal{O}$ is an abelian group in the category C_X^{op} and the canonical functor

$$C_X \xrightarrow{f_{\mathcal{O}*}} \Gamma_X\mathcal{O} - mod, \quad M \longmapsto C_X(\mathcal{O}, M), \qquad (6)$$

has a left adjoint; or, equivalently, $f_{\mathcal{O}*}$ is a direct image functor of a continuous morphism.

3.3.2.2. Example. If C_X is an additive category with small coproducts and cokernels, then $(X, \mathcal{O})$ is a $\mathbb{Z}$-space for any $\mathcal{O} \in ObC_X$.

3.3.3. Affine $\mathbb{Z}$-spaces. We call a $\mathbb{Z}$-space $(X, \mathcal{O})$ *affine* if the canonical morphism $X \xrightarrow{f_{\mathcal{O}}} \mathbf{Sp}(\Gamma_X\mathcal{O})$ is an isomorphism; i.e. the functor $f_{\mathcal{O}*}$ (see (6)) is a category equivalence.

By a Mitchel's theorem, affine $\mathbb{Z}$-spaces are pairs $(X, \mathcal{O})$, where C_X is an abelian category with small coproducts, and $\mathcal{O}$ is a projective cogenerator of finite type. We denote by $\mathbf{Aff}_{\mathbb{Z}}$ the full subcategory of the category $|Cat|^o_{\mathbb{Z}}$ formed by affine $\mathbb{Z}$-spaces.

The functor

$$Rings^{op} \xrightarrow{\mathbf{Sp}} |Cat|^o, \quad R \longmapsto \mathbf{Sp}(R),$$

gives rise to the functor

$$Rings^{op} \xrightarrow{\mathbf{Sp}_{\mathbb{Z}}} |Cat|^o_{\mathbb{Z}}, \quad R \longmapsto (\mathbf{Sp}(R), R)$$

which takes values in the subcategory $\mathbf{Aff}_{\mathbb{Z}}$. We denote the image of the functor $\mathbf{Sp}_{\mathbb{Z}}$ by $\mathfrak{Aff}_{\mathbb{Z}}$. Thus, objects of the category $\mathfrak{Aff}_{\mathbb{Z}}$ are pairs $(\mathbf{Sp}(R), R)$ and morphisms from $(\mathbf{Sp}(R), R) \longrightarrow (\mathbf{Sp}(T), T)$ are morphisms $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(T)$ corresponding to unital ring morphisms $T \longrightarrow R$. The functor $\mathbf{Sp}_{\mathbb{Z}}$ induces an inclusion functor $\mathfrak{Aff}_{\mathbb{Z}} \xrightarrow{\gamma_*} |Cat|^o_{\mathbb{Z}}$ which takes values in the subcategory of affine $\mathbb{Z}$-spaces.

3.3.4. Proposition. *The functor $\mathfrak{Aff}_{\mathbb{Z}} \xrightarrow{\gamma_*} |Cat|^o_{\mathbb{Z}}$ is fully faithful and has a left adjoint. In particular, the functor γ_* induces an equivalence of $\mathfrak{Aff}_{\mathbb{Z}}$ and the category of affine $\mathbb{Z}$-spaces.*

Proof. Let f be a morphism $(X, \mathcal{O}) \longrightarrow (X', \mathcal{O}')$. A choice of an inverse image functor, $C_{X'} \xrightarrow{f^*} C_X$, of f and an isomorphism $f^*(\mathcal{O}') \xrightarrow{\lambda} \mathcal{O}$ determines a ring morphism $\Gamma_{X'}\mathcal{O}' \xrightarrow{\psi_{f^*,\lambda}} \Gamma_X\mathcal{O}$. One can check that the corresponding morphism of categoric spectra, $\mathbf{Sp}(\Gamma_X\mathcal{O}) \longrightarrow \mathbf{Sp}(\Gamma_{X'}\mathcal{O}')$, does not depend on choices. Thus we have a functor $|Cat|^o_{\mathbb{Z}} \xrightarrow{\gamma^*} \mathfrak{Aff}_{\mathbb{Z}}$. By 3.3.1, we have a natural morphism $Id_{|Cat|^o_{\mathbb{Z}}} \xrightarrow{\eta_\gamma} \gamma_*\gamma^*$. And there is an isomorphism $\gamma^*\gamma_* \xrightarrow{\epsilon_\gamma} Id_{\mathfrak{Aff}_{\mathbb{Z}}}$. These are adjunction morphisms. Since ϵ_γ is an isomorphism, the functor γ_* is fully faithful. ∎

3.3.4.1. Proposition. *Let $X \in Ob|Cat|^o$ be such that the category C_X has cokernels of pairs of morphisms. Then $(X, \mathcal{O})$ is a $\mathbb{Z}$-space for any object $\mathcal{O}$ of C_X such that there exists a coproduct of any small set of copies of $\mathcal{O}$. Continuous morphisms $X \longrightarrow \mathbf{Sp}(R)$ are in one-to-one correspondence with isomorphism classes of right R-modules $(\mathcal{O}, \phi)$ in C_X such that $(X, \mathcal{O})$ is a $\mathbb{Z}$-space. In particular, if C_X is an abelian category with small coproducts, then morphisms $X \longrightarrow \mathbf{Sp}(R)$ are in bijective correspondence with isomorphism classes of right R-modules in C_X.*

Proof. The assertion is a corollary of 3.3.4. ∎

3.3.4.2. Example. Let $X = \mathbf{Sp}(S)$ for some associative unital ring S. By 3.3.4.1, continuous morphisms from $\mathbf{Sp}(S) \longrightarrow \mathbf{Sp}(R)$ are in bijective correspondence with isomorphism classes of right R-modules in the category $S-mod$. Notice that the category of right R-modules in $S - mod$ is isomorphic to the category of (S,R)-bimodules. If $\mathcal{O}$ is a (S,R)-bimodule corresponding to a morphism $\mathbf{Sp}(S) \xrightarrow{f} \mathbf{Sp}(R)$, then $L \longmapsto Hom_S(\mathcal{O}, L)$ is a direct image functor of f. Therefore $N \longmapsto \mathcal{O} \otimes_R N$ is an inverse image functor of f.

3.3.5. The category $\mathfrak{Ass}$. Let $\mathfrak{Ass}$ denote the category whose objects are associative rings; and morphisms from a ring R to a ring S are equivalence classes of ring morphisms $R \longrightarrow S$ by the following equivalence relation: two ring morphisms $R \overset{\phi}{\underset{\psi}{\rightrightarrows}} S$ are equivalent if they are conjugated, i.e. $\phi(-) = t\psi(-)t^{-1}$ for an invertible element t of S.

3.3.5.1. Proposition. *Two ring morphisms, $R \overset{\phi}{\underset{\psi}{\rightrightarrows}} S$, are conjugated iff the corresponding inverse image functors, $R - mod \overset{\phi^*}{\underset{\psi^*}{\rightrightarrows}} S - mod$, are isomorphic.*

Proof. The assertion is a consequence of a much more general fact 5.7.1. For the reader convenience, we give below a direct argument.

(a) Suppose that ψ and ϕ are conjugate, i.e. there exists an invertible element, t, of S such that $\psi(r) = t\phi(r)t^{-1}$ for all $r \in R$. For any R-module $\mathcal{M} = (M, m)$, we have a commutative diagram

$$\begin{array}{ccc} S \otimes M & \xrightarrow{\cdot t} & S \otimes M \\ \gamma_\psi \downarrow & & \downarrow \gamma_\phi \\ S \otimes_{R,\psi} M & \xrightarrow{\lambda_t} & S \otimes_{R,\phi} M \end{array} \qquad (1)$$

Here $\cdot t$ denotes the S-module morphism $s \otimes z \longmapsto st \otimes z$ for all $s \in S,\ z \in M$; $\gamma_\psi,\ \gamma_\phi$ are canonical epimorphisms.

In fact, for any $s \in S,\ r \in R,\ z \in M$, $\gamma_\psi(s\psi(r) \otimes z) = \gamma_\psi(s \otimes r \cdot z)$, and $\cdot t(s \otimes r \cdot z) = st \otimes r \cdot z$. On the other hand, $\cdot t(s\psi(r) \otimes z) = s\psi(r)t \otimes z = st\phi(r) \otimes z$, and $\gamma_\phi(st\phi(r) \otimes z) = \gamma_\phi(st \otimes r \cdot z)$. Since γ_ψ is by definition the cokernel of two maps

$$S \otimes_k R \otimes_k M \overset{\psi_l}{\underset{\psi_r}{\rightrightarrows}} S \otimes_k M, \quad s \otimes r \otimes z \overset{\psi_l}{\longmapsto} s\psi(r) \otimes z, \text{ and } s \otimes r \otimes z \overset{\psi_r}{\longmapsto} s \otimes r \cdot z,$$

it follows the existence of a (necessarily unique) morphism $S \otimes_{R,\psi} M \xrightarrow{\lambda_t} S \otimes_{R,\phi} M$ such that the diagram (1) commutes; i.e. λ_t is given by $\gamma_\psi(s \otimes z) \longmapsto \gamma_\phi(st \otimes z)$.

(b) Conversely, suppose $\phi,\ \psi$ are unital ring morphisms such that there is a functorial isomorphism $\psi^* \xrightarrow{u} \phi^*$. Identifying both $\phi^*(R)$ and $\psi^*(R)$ with the left S-module S, we obtain, in particular, an S-module morphism $S \xrightarrow{u(R)} S$. Since S is a ring with unit, $u(R)$ equals to $\cdot t : s \longmapsto st$ for some $t \in S$. Since u is a functor morphism, for any $r \in R$,

$u(R) \circ \psi^*(\cdot r) = \phi^*(\cdot r) \circ u(R)$. This means that for any $s \in S$, $s\psi(r)t = st\phi(r)$, hence $\psi(r) = t\phi(r)t^{-1}$. ∎

3.3.5.2. Corollary. *The functor*

$$Rings^{op} \xrightarrow{\mathbf{Sp}_{\mathbb{Z}}} |Cat|^o_{\mathbb{Z}}, \quad R \longmapsto (\mathbf{Sp}(R), R)$$

induces an isomorphism of categories $\mathfrak{Ass}^{op} \xrightarrow{\sim} \mathfrak{Aff}_{\mathbb{Z}}$, *hence an equivalence of categories* $\mathfrak{Ass}^{op} \longrightarrow \mathbf{Aff}_{\mathbb{Z}}$.

Proof. This follows from 3.3.4 and 3.3.5.1. ∎

3.3.6. Remark. If S is a commutative ring, then for any ring R, the surjection $Rings(R,S) \longrightarrow \mathfrak{Ass}(R,S)$ is a bijective map. In particular, the full subcategory of $\mathfrak{Ass}$ formed by commutative rings is isomorphic to the category $CRings$ of commutative rings. Thus, the equivalence of categories $\mathfrak{Ass}^{op} \longrightarrow \mathbf{Aff}_{\mathbb{Z}}$ induces an equivalence between the category $CRings^{op}$ opposite to the category of commutative unital rings and the full subcategory $\mathbf{CAff}_{\mathbb{Z}}$ formed by affine $\mathbb{Z}$-spaces $(X, \mathcal{O})$ such that the global sections ring $\Gamma_X\mathcal{O}$ is commutative. This shows, by passing, that the category $\mathbf{CAff}_{\mathbb{Z}}$ of commutative affine $\mathbb{Z}$-spaces is equivalent to the category of commutative affine schemes in the usual sense.

4. Monads, comonads, and continuous morphisms.

4.1. Monads and their categoric spectrum. Let Y be an object of $|Cat|^o$. A *monad on the 'space'* Y is by definition a monad on the category C_Y, i.e. a pair (F, μ), where F is a functor $C_Y \longrightarrow C_Y$ and μ a morphism $F^2 \longrightarrow F$ (multiplication) such that $\mu \circ F\mu = \mu \circ \mu F$ and there exists a morphism $Id_{C_Y} \xrightarrow{\eta} F$ (called the *unit* element of the monad (F, μ)), which is uniquely determined by the equalities $\mu \circ F\eta = id_F = \mu \circ \eta F$.

The latter follows from the following simple consideration: if $Id_{C_Y} \xrightarrow{\eta} F \xleftarrow{\eta'} Id_{C_Y}$ are morphisms such that $\mu \circ F\eta' = id_F = \mu \circ \eta F$, then $\eta = \mu \circ F\eta' \circ \eta = \mu \circ F\eta \circ \eta' = \eta'$.

A morphism from a monad $\mathcal{F} = (F, \mu)$ to a monad $\mathcal{F}' = (F', \mu')$ is given by a functor morphism $F \longrightarrow F'$ such that $\varphi \circ \mu = \mu' \circ \varphi \odot \varphi$ and $\varphi \circ \eta = \eta'$. Here $\varphi \odot \varphi = F'\varphi \circ \varphi F$, and η, η' are units of the monads resp. $\mathcal{F}$ and $\mathcal{F}'$. The composition of morphisms is defined naturally, so that the map forgetting monad structures, i.e. sending a monad morphism $(F, \mu) \xrightarrow{\varphi} (F', \mu')$ to the natural transformation $F \xrightarrow{\varphi} F'$, is a functor.

For a 'space' Y, we denote by $\mathfrak{Mon}_Y$ the category of monads on Y.

Given a monad $\mathcal{F} = (F, \mu)$ on Y, we denote by $(\mathcal{F}/Y) - mod$, or simply by $\mathcal{F} - mod$, the category of $(\mathcal{F}/Y)$-modules. Its objects are pairs (M, ξ), where $M \in ObC_Y$ and ξ is a morphism $F(M) \longrightarrow M$ such that $\xi \circ F\xi = \xi \circ \mu(M)$ and $\xi \circ \eta(M) = id_M$. Morphisms from (M, ξ) to (M', ξ') are given by morphisms $M \xrightarrow{g} M'$ such that $\xi' \circ Fg = g \circ \xi$.

We denote by $\mathbf{Sp}(\mathcal{F}/Y)$ the 'space' represented by the category $(\mathcal{F}/Y) - mod$ of $(\mathcal{F}/Y)$-modules. The object $\mathbf{Sp}(\mathcal{F}/Y)$ is regarded as the *categoric spectrum of* $\mathcal{F}$. The forgetful functor

$$(\mathcal{F}/Y) - mod \xrightarrow{f_*} C_Y, \quad (M, \xi) \longmapsto M,$$

is a right adjoint to the functor

$$C_Y \xrightarrow{f^*} (\mathcal{F}/Y) - mod, \quad L \longmapsto (F(L), \mu(L)), \; (L \xrightarrow{g} N) \longmapsto (f^*(L) \xrightarrow{F(g)} f^*(L')).$$

In other words, we have a canonical continuous morphism $\mathbf{Sp}(\mathcal{F}/Y) \xrightarrow{f} Y$.

4.1.1. Example. Let R, S be unital associative rings. Any unital ring morphism $S \xrightarrow{\varphi} R$ defines a monad, $R_\varphi^\sim = (R_\varphi, \mu_\varphi)$, on the affine 'space' $Y = \mathbf{Sp}(S)$: the endofunctor R_φ is $M \longmapsto R \otimes_S M$, and the multiplication μ_φ is induced by the multiplication on the ring R. Notice that the category $(R_\varphi^\sim/\mathbf{Sp}(S))$-modules is isomorphic to the category $R - mod$ of left R-modules; in particular, $\mathbf{Sp}(R_\varphi^\sim/\mathbf{Sp}(S)) \simeq \mathbf{Sp}(R)$. The canonical morphism $\mathbf{Sp}(R_\varphi^\sim/\mathbf{Sp}(S)) \longrightarrow \mathbf{Sp}(S)$ has the restriction of scalars $R - mod \xrightarrow{\phi_*} S - mod$ as a direct image functor. Consistently with our previous notations, we write $\mathbf{Sp}(R)$ instead of $\mathbf{Sp}(R/\mathbf{Sp}\mathbb{Z})$.

4.1.2. Example. Any monoid morphism $\mathcal{M} \xrightarrow{\phi} \mathcal{N}$ defines a monad, $\mathcal{F} = (F, \mu)$, on $Y = \mathbf{Sp}(\mathcal{M}/\mathcal{E})$, where the functor F is $\mathcal{M} \boxtimes_{\mathcal{N}} -$. It maps any left $\mathcal{M}$-set (L, ξ) to the cokernel of the pair of morphisms $\mathcal{N} \times \mathcal{M} \times L \rightrightarrows \mathcal{N} \times L$, where one arrow is $\mathcal{N} \times \xi$ and another is the composition of the maps

$$\mathcal{N} \times \mathcal{M} \times L \xrightarrow{\mathcal{N} \times \phi \times L} \mathcal{N} \times \mathcal{N} \times L \xrightarrow{\nu \times L} \mathcal{N} \times L.$$

Here $\mathcal{N} \times \mathcal{N} \xrightarrow{\nu} \mathcal{N}$ is the multiplication on $\mathcal{N}$. The multiplication $F^2 \xrightarrow{\mu} F$ is induced by the multiplication on $\mathcal{N}$. The canonical morphism $\mathbf{Sp}(\mathcal{F}/\mathbf{Sp}(\mathcal{M}/\mathcal{E})) \longrightarrow \mathbf{Sp}(\mathcal{M}/\mathcal{E})$ has the restriction of scalars functor $\mathcal{N} - sets \xrightarrow{\phi_*} \mathcal{M} - sets$ as a direct image functor.

4.1.3. Example: localizations of modules. Let R be an associative unital ring and $\mathfrak{F}$ a set of left ideals in R. Denote by $R - mod_{\mathfrak{F}}$ the full subcategory of $R - mod$ whose objects are R-modules M such that the canonical morphism

$$M \longrightarrow Hom_R(m, M), \quad z \longmapsto (r \mapsto r \cdot z) \text{ for all } r \in m \text{ and } z \in M, \tag{1}$$

is an isomorphism for all $m \in \mathfrak{F}$. The inclusion functor

$$R - mod_{\mathfrak{F}} \xrightarrow{\mathfrak{j}_{\mathfrak{F}*}} R - mod \qquad (2)$$

preserves limits, hence it has a left adjoint, $\mathfrak{j}_{\mathfrak{F}}^*$. Since $\mathfrak{j}_{\mathfrak{F}*}$ is fully faithful, $\mathfrak{j}_{\mathfrak{F}}^*$ is a localization functor. The R-module $R_{\mathfrak{F}} = \mathfrak{j}_{\mathfrak{F}*}\mathfrak{j}_{\mathfrak{F}}^*(R)$ has a structure of a ring uniquely determined by the fact that the adjunction arrow, $R \xrightarrow{\eta_{\mathfrak{F}}} R_{\mathfrak{F}}$ is a ring morphism. There is a canonical functor morphism

$$R_{\mathfrak{F}} \otimes_R - \xrightarrow{\tau_{\mathfrak{F}}} \mathfrak{j}_{\mathfrak{F}*}\mathfrak{j}_{\mathfrak{F}}^*. \qquad (3)$$

4.1.3.1. Suppose all ideals in $\mathfrak{F}$ are projective modules. Then the inclusion functor (2) is exact. This implies that the morphism

$$R_{\mathfrak{F}} \otimes_R M \xrightarrow{\tau_{\mathfrak{F}}(M)} \mathfrak{j}_{\mathfrak{F}*}\mathfrak{j}_{\mathfrak{F}}^*(M)$$

is an isomorphism for any R-module M of finite type.

4.1.3.2. If $\mathfrak{F}$ consists of projective ideals of finite type, then (2) is strictly exact, i.e. it has a right adjoint. In this case, the localization $R - mod \longrightarrow R - mod_{\mathfrak{F}}$ is an affine morphism, or, equivalently, the functor morphism (3) is an isomorphism. Therefore, in this case, the category $R - mod_{\mathfrak{F}}$ is equivalent to the category $R_{\mathfrak{F}} - mod$ of left $R_{\mathfrak{F}}$-modules.

4.1.3.3. Note. In general, the localization $\mathfrak{j}_{\mathfrak{F}}$ is not flat, i.e. the functor $\mathfrak{j}_{\mathfrak{F}}^*$ is not exact. Denote by $\mathfrak{F}^-$ the set of all left ideals of the ring R such that the canonical morphism $M \longrightarrow Hom_R(m, M)$ is an isomorphism for all $M \in ObR - mod_{\mathfrak{F}}$. Clearly $R - mod_{\mathfrak{F}^-} = R - mod_{\mathfrak{F}}$. It follows from results of Gabriel (cf. [Gab], or [BD, Ch. 6]) that the localization $\mathfrak{j}_{\mathfrak{F}}$ is flat iff $\mathfrak{F}^-$ is a *radical filter*; i.e. with any left ideal m, the set $\mathfrak{F}^-$ contains left ideals $(m : r) = \{a \in R \mid ar \in m\}$ for all $r \in R$ and all left ideals n in R such that $(n : r) \in \mathfrak{F}^-$ for all $r \in m$. These conditions are equivalent to that the full subcategory $T_{\mathfrak{F}^-}$ of $R - mod$ whose objects are all R-modules M such that every element of M is annihilated by some ideal $m \in \mathfrak{F}^-$, is a Serre subcategory.

4.1.4. Curves. Let R be a ring of the homological dimension one, or, equivalently, every left ideal in R is projective. Then for any set of left ideals $\mathfrak{F}$, the inclusion functor (2) is exact. If, in addition, R is left noetherian, then the functor (2) is strictly exact.

4.2. Morphisms of monads and morphisms of their categoric spectra. Let Y be an object of $|Cat|^o$ and $\mathcal{F}$, $\mathcal{F}'$ monads on Y. Any monad morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{F}'$ induces the 'pull-back' functor

$$(\mathcal{F}'/Y) - mod \xrightarrow{\varphi_*} (\mathcal{F}/Y) - mod, \quad (M, \xi) \longmapsto (M, \xi \circ \varphi(M)).$$

This correspondence defines a functor $\mathfrak{Mon}_Y^{op} \longrightarrow Cat/C_Y$ which takes values in the full subcategory of Cat/C_Y whose objects are functors $C_Z \longrightarrow C_Y$ having a left adjoint.

4.2.1. Reflexive pairs of arrows, weakly continuous functors and monads. Recall that a pair of arrows $M \underset{g_2}{\overset{g_1}{\rightrightarrows}} L$ in C_Y is called *reflexive*, if there exists a morphism $L \xrightarrow{h} M$ such that $g_1 \circ h = id_M = g_2 \circ h$.

We call a functor *weakly continuous* if it preserves cokernels of reflexive pairs of arrows.

We call a monad $\mathcal{F} = (F, \mu)$ on Y *weakly continuous* if the functor $C_Y \xrightarrow{F} C_Y$ is weakly continuous. We denote by $\mathfrak{Mon}_Y^w$ the full subcategory of the category $\mathfrak{Mon}_Y$ whose objects are weakly continuous monads on Y.

4.2.2. Lemma. *Suppose that the category C_Y has cokernels of reflexive pairs of morphisms. Let $\mathcal{F}'$ be a weakly continuous monad on Y. Then, for any monad morphism $\mathcal{F} \xrightarrow{\varphi} \mathcal{F}'$, the restriction of scalars functor*

$$(\mathcal{F}'/Y) - mod \xrightarrow{\varphi_*} (\mathcal{F}/Y) - mod$$

has a left adjoint. In particular, the restriction of the map

$$(\mathcal{F}/Y) \longmapsto \mathbf{Sp}(\mathcal{F}/Y), \quad \varphi \longmapsto [\varphi^*]$$

to the category of weakly continuous monads is a functor,

$$(\mathfrak{Mon}_Y^w)^{op} \xrightarrow{\mathbf{Sp}_Y} |Cat|^o, \tag{1}$$

which takes values in the subcategory $|Cat|^o_{\mathfrak{c}}$ of $|Cat|^o$ formed by continuous morphisms.

Proof. The left adjoint, $(\mathcal{F}/Y) - mod \xrightarrow{\varphi^*} (\mathcal{F}'/Y) - mod$ assigns to each $(\mathcal{F}/Y)$-module $(M, F(M) \xrightarrow{\xi} M)$ the cokernel of the pair of arrows

$$F'F(M) \underset{F'\xi}{\overset{\mu' \circ F'\varphi}{\rightrightarrows}} F'(M). \tag{1}$$

Since, by hypothesis, F' preserves cokernels of reflexive pairs and both arrows (1) are $\mathcal{F}'$-module morphisms, there exists a unique $\mathcal{F}'$-module structure on the cokernel of (1). Details are left to the reader. ■

4.2.3. Note. Suppose that the category C_X has colimits of certain type $\mathfrak{D}$, and let $\mathcal{F} = (F, \mu)$ be a monad on X such that the functor F preserves colimits of this type. Then the category $(\mathcal{F}/X) - mod$ has colimits of this type.

In fact, for a diagram $\mathfrak{D} \xrightarrow{\mathcal{D}} (\mathcal{F}/X)-mod$, the colimit of the composition $f_*\circ\mathcal{D}$ (where f_* is the forgetful functor $(\mathcal{F}/X) - mod \longrightarrow C_X$) has a unique $\mathcal{F}$-module structure, $\xi_{\mathcal{D}}$. The $\mathcal{F}$-module $(colim(f_* \circ \mathcal{D}), \xi_{\mathcal{D}})$ is a colimit of the diagram $\mathcal{D}$.

In particular, if $\mathcal{F} = (F, \mu)$ is a weakly continuous monad on X, and the category C_X has cokernels of reflexive pairs of arrows, then the category $(\mathcal{F}/X) - mod$ has cokernels of reflexive pairs of arrows.

4.3. Comonads and their cospectrum. A *comonad* on a 'space' Y is the same as a monad on the dual object ('space') Y^o defined by $C_{Y^o} = C_Y^{op}$. In other words, a comonad on Y is a pair (G, δ), where G is a functor $C_Y \longrightarrow C_Y$ and δ a functor morphism $G \longrightarrow G^2$ (a comultiplication) such that $G\delta \circ \delta = \delta G \circ \delta$ and $G\epsilon \circ \delta = id_G = \epsilon G \circ \delta$ for a uniquely determined morphism $G \xrightarrow{\epsilon} Id_{C_Y}$ (a counit).

We denote the category of comonads on Y by $\mathfrak{CMon}_Y$. Duality provides a natural category isomorphism $\mathfrak{CMon}_Y \simeq \mathfrak{Mon}_{Y^o}$.

Comodules over a comonad $\mathcal{G} = (G, \delta)$ correspond to modules over the dual monad on Y^o. In terms of Y, a $\mathcal{G}$-comodule is a pair (M, ξ), where $M \in ObC_Y$ and ξ a morphism $M \longrightarrow G(M)$ such that $\delta(M) \circ \xi = G\xi \circ \xi$ and $\epsilon(M) \circ \xi = id_M$. We denote the category of comodules over $\mathcal{G}$ by $(Y\backslash\mathcal{G}) - Comod$, or simply by $\mathcal{G} - Comod$.

We denote by $\mathbf{Sp^o}(Y\backslash\mathcal{G})$ the object of $|Cat|^o$ (or Cat^{op}) such that the corresponding category is $(Y\backslash\mathcal{G}) - Comod$. This definition can be rephrased as follows:

$$\mathbf{Sp^o}(Y\backslash\mathcal{G}) = \mathbf{Sp}(\mathcal{G}^o/Y^o)^o. \tag{1}$$

Here $\mathcal{G}^o$ is the monad (G^o, δ^o) on Y^o dual to the comonad $\mathcal{G}$.

We call $\mathbf{Sp^o}(Y\backslash\mathcal{G})$ the *cospectrum of the comonad* $\mathcal{G}$ *in* $|Cat|^o$.

By duality, there is a canonical continuous morphism $Y \xrightarrow{g} \mathbf{Sp^o}(Y\backslash\mathcal{G})$ with an inverse image functor

$$(Y\backslash\mathcal{G}) - Comod \xrightarrow{g^*} C_Y, \quad (M, \xi) \longmapsto M, \tag{2}$$

and having a direct image functor

$$C_Y \xrightarrow{g_*} (Y\backslash\mathcal{G}) - Comod, \quad L \longmapsto (G(L), \delta(L)). \tag{3}$$

4.3.1. Example. Let R be an associative unital ring and $\mathcal{H} = (H, \delta)$ a coalgebra in the monoidal category of R-bimodules. This means that H is an R-bimodule, δ an R-bimodule morphism $H \longrightarrow H \otimes_R H$ such that $\delta \otimes_R id_H \circ \delta = id_H \otimes_R \delta \circ \delta$, and there exists a (necessarily unique) R-bimodule morphism $H \otimes_R H \xrightarrow{\epsilon} R$ such that

$$\lambda_r(H) \circ \epsilon \otimes_R id_H \circ \delta = id_H = \lambda_l(H) \circ id_H \otimes_R \epsilon \circ \delta.$$

Here $R \otimes_R H \xrightarrow{\lambda_l(H)} H$ and $H \otimes_R R \xrightarrow{\lambda_r(H)} H$ are canonical isomorphisms. The coalgebra $\mathcal{H}$ induces a comonad on the category $R-mod$ of left R-modules tensoring by H over R, $L \longmapsto H \otimes_R L$, as a functor and with comultiplication $H \otimes_R - \longrightarrow H \otimes_R H \otimes_R -$ induced by the comultiplication δ. The canonical morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp^o}(\mathbf{Sp}(R)\backslash\mathcal{H})$ has the forgetful functor $(\mathbf{Sp}(R)\backslash\mathcal{H}) - Comod \longrightarrow R - mod$ as an inverse image functor.

4.3.2. Functoriality of the cospectrum. Let $\mathcal{G} = (G, \delta)$ and $\mathcal{G}' = (G', \delta')$ be comonads on Y and ψ a comonad morphism $\mathcal{G} \longrightarrow \mathcal{G}'$; i..e. ψ is a morphism of functors $G \longrightarrow G'$ such that $\delta' \circ \psi = \psi \odot \psi \circ \delta$ and $\epsilon' \circ \psi = \epsilon$. Here $\psi \odot \psi = G'\psi \circ \psi G$, and ϵ, ϵ' are counits of the comonads resp. $\mathcal{G}$ and $\mathcal{G}'$. The morphism ψ induces the 'pull-back' functor

$$(Y\backslash\mathcal{G}) - Comod \xrightarrow{\psi^*} (Y\backslash\mathcal{G}') - Comod, \quad (M, \xi) \longmapsto (M, \psi(M) \circ \xi), \tag{4}$$

which is regarded as an inverse image functor of a morphism

$$\mathbf{Sp^o}(Y\backslash\mathcal{G}') \xrightarrow{\mathbf{Sp^o_Y}(\psi)} \mathbf{Sp^o}(Y\backslash\mathcal{G}) \tag{5}$$

The map (4) defines a functor

$$\mathfrak{CMon}_Y^{op} \xrightarrow{\widetilde{\mathbf{Sp}}_Y^o} (C_Y\backslash Cat^{op})_{\mathfrak{c}}, \tag{6}$$

where $(C_Y\backslash Cat^{op})_{\mathfrak{c}}$ denotes the full subcategory of the category $C_Y\backslash Cat^{op}$ whose objects are continuous morphisms.

4.3.2.1. Proposition. *The functor (6) is fully faithful and has a right adjoint.*

Proof. Let

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ f \nwarrow & & \nearrow g \\ & Y & \end{array}$$

be a morphism in $(C_Y\backslash Cat^{op})_{\mathfrak{c}}$ given by the commutative diagram

$$\begin{array}{ccc} C_Z & \xrightarrow{h^*} & C_X \\ g^* \searrow & & \swarrow f^* \\ & C_Y & \end{array}$$

of functors. Fix direct image functors of f and g and the corresponding adjunction arrows. Set

$$\varphi_h = f^*(f_*\epsilon_g \circ \eta_f h^* g_*) : g^* g_* \longrightarrow f^* f_*. \tag{7}$$

One can check that ϕ_h is a comonad morphism $\mathcal{G}_g \longrightarrow \mathcal{G}_f$ and the map $\tilde{\Gamma}_Y : h \longmapsto \varphi_h$ is functorial. The composition $\tilde{\Gamma}_Y \circ \widetilde{\mathbf{Sp}}_Y^o$ is the identical functor, which provides one of the adjunction arrows and shows that the functor $\widetilde{\mathbf{Sp}}_Y^o$ is fully faithful. We leave the other adjunction arrow to the reader (it is defined in 4.4). ■

Recall that a pair of arrows $M \overset{g_1}{\underset{g_2}{\rightrightarrows}} L$ is called *coreflexive*, if there exists a morphism $L \xrightarrow{h} M$ such that $h \circ g_1 = id_M = h \circ g_2$.

4.3.2.2. Lemma. *Let $\mathcal{G} = (G, \delta)$ and $\mathcal{G}' = (G', \delta')$ be comonads on Y and ψ a comonad morphism $\mathcal{G} \longrightarrow \mathcal{G}'$. Suppose that the category C_Y has kernels of coreflexive pairs of morphisms and the functor G preserves these kernels. Then the functor ψ^* in (4) has a right adjoint, i.e. the morphism $\mathbf{Sp}_Y^o(\psi)$ (see (5)) is continuous.*

Proof. The assertion is the dual version of 4.2.2. ■

4.4. Beck's theorem. Let $X \xrightarrow{f} Y$ be a continuous morphism in $|Cat|^o$ with inverse image functor f^*, direct image functor f_*, and adjunction morphisms

$$Id_{C_Y} \xrightarrow{\eta_f} f_*f^* \quad \text{and} \quad f^*f_* \xrightarrow{\epsilon_f} Id_{C_X}.$$

Let $\mathcal{G}_f$ denote the comonad (G_f, δ_f), where $G_f = f^*f_*$ and $\delta_f = f^*\eta_f f_*$. There is a commutative diagram

$$\begin{array}{ccc} C_Y & \xrightarrow{\tilde{f}^*} & (X\backslash\mathcal{G}_f) - Comod \\ {\scriptstyle f^*} \searrow & & \swarrow {\scriptstyle \check{f}^*} \\ & C_X & \end{array} \qquad (1^o)$$

Here $\tilde{f}^*$ is the canonical functor

$$C_Y \longrightarrow (X\backslash\mathcal{G}_f) - Comod, \quad M \longmapsto (f^*(M), f^*\eta_f(M)),$$

and $\check{f}^*$ is the forgetful functor $(X\backslash\mathcal{G}_f) - Comod \longrightarrow C_X$. The diagram (1^o) is regarded as the diagram of inverse image functors of the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\check{f}} & \mathbf{Sp^o}(X\backslash\mathcal{G}_f) \\ {\scriptstyle f} \searrow & & \swarrow {\scriptstyle \tilde{f}} \\ & Y & \end{array} \qquad (2^o)$$

in $|Cat|^o$. The following statement is one of the versions of the Beck's theorem.

4.4.1. Theorem. *Let $X \xrightarrow{f} Y$ be a continuous morphism.*

(a) If the category C_X has kernels of coreflexive pairs of arrows, then the functor $\tilde{f}^*$ has a right adjoint, $\tilde{f}_*$, i.e. $\mathbf{Sp^o}(X\backslash\mathcal{G}_f) \xrightarrow{\tilde{f}} Y$ *is a continuous morphism.*

(b) If, in addition, f is weakly flat, i.e. the functor f^* preserves kernels of coreflexive pairs, then the adjunction arrow $\tilde{f}^*\tilde{f}_* \xrightarrow{\epsilon_{\tilde{f}}} Id_{(X\backslash\mathcal{G}_f)-Comod}$ is an isomorphism, i.e. $\hat{f}_*$ is a fully faithful functor, or, equivalently, $\tilde{f}^*$ is a localization.

(c) If, in addition to (a) and (b), f^* reflects isomorphisms, then the adjunction arrow $Id_{C_Y} \xrightarrow{\eta_{\tilde{f}}} \tilde{f}_*\tilde{f}^*$ is an isomorphism too, i.e. $\tilde{f}$ is an isomorphism.

Proof. See [MLM], IV.4.2, or [ML], VI.7. ∎

We need also the dual version of the theorem 4.4.1. Let $\mathcal{F}_f$ denote the monad (F_f, μ_f), where $F_f = f_*f^*$ and $\mu_f = f_*\epsilon_f f^*$. There is a commutative diagram

$$\begin{array}{ccc} C_X & \xrightarrow{\bar{f}_*} & (\mathcal{F}_f/Y) - mod \\ & f_* \searrow \quad \swarrow \hat{f}_* & \\ & C_Y & \end{array} \qquad (1)$$

Here $\bar{f}_*$ is the canonical functor

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod, \quad M \longmapsto (f_*(M), f_*\epsilon_f(M)),$$

$\hat{f}_*$ the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$.

4.4.2. Theorem. *Let $X \xrightarrow{f} Y$ be a continuous morphism.*

(a) If the category C_Y has cokernels of reflexive pairs of arrows, then the functor $\bar{f}_*$ has a left adjoint, $\bar{f}^*$; hence $\bar{f}_*$ is a direct image functor of a continuous morphism $\bar{X} \xrightarrow{f} \mathbf{Sp}(\mathcal{F}_f/Y)$.

(b) If, in addition, the functor f_* preserves cokernels of reflexive pairs, then the adjunction arrow $\bar{f}^*\bar{f}_* \longrightarrow Id_{C_X}$ is an isomorphism, i.e. $\bar{f}_*$ is a localization.

(c) If, in addition to (a) and (b), the functor f_* is conservative, then $\bar{f}_*$ is a category equivalence.

If the condition (a) in 4.4.2 holds, then to the diagram (1), there corresponds a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\bar{f}} & \mathbf{Sp}(\mathcal{F}_f/Y) \\ & {}_f \searrow \quad \swarrow {}_{\hat{f}} & \\ & Y & \end{array} \qquad (2)$$

in $|Cat|^o$. If the condition (c) in 4.4 holds, the morphism $\bar{f}$ in (2) is an isomorphism.

Thus, given a continuous morphism $X \xrightarrow{f} Y$ such that the category C_Y has cokernels of reflexive pairs of arrows, we have a commutative diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{\bar{f}} & \mathbf{Sp}(\mathcal{F}_f/Y) \\ & \check{f} \swarrow & & \searrow^{f} \quad \swarrow \hat{f} & \\ \mathbf{Sp^o}(X\backslash\mathcal{G}_f) & \xrightarrow{\tilde{f}} & Y & & \end{array} \qquad (3)$$

Notice that the diagrams (1) and (1^o) are uniquely defined by the data $(f^*, f_*, \epsilon_f, \eta_f)$, since the monad $\mathcal{F}_f$, the comonad $\mathcal{G}_f$, and the functors $\bar{f}_* : C_X \longrightarrow (F_f, Y) - mod$ in (1) and $\tilde{f}^* : C_Y \longrightarrow (X\backslash\mathcal{G}_f) - Comod$ are defined in terms of this data. Given the functor f^* (resp. f_*), the rest of the data, f_*, ϵ_f, η_f (resp. f^*, ϵ_f, η_f), is determined uniquely up to isomorphism. Thus, the monad $\mathcal{F}_f$ and the comonad $\mathcal{G}_f$ in the diagrams (1) and (1^o) are determined by f^* uniquely up to isomorphism.

4.5. Weakly flat and weakly affine morphisms. We call a functor *weakly continuous* (resp. *weakly flat*) if it preserves cokernels of reflexive pairs of arrows (resp. kernels of coreflexive pairs of arrows). We call a monad (F, μ) *weakly continuous.* if the functor F is weakly continuous, and a comonad (G, δ) *weakly flat*, if the functor G is weakly flat.

Let $\mathfrak{Mon}_X^w$ denote the full subcategory of the category $\mathfrak{Mon}_X$ of monads on X generated by weakly continuous monads; and let $\mathfrak{CMon}_X^w$ be the full subcategory of the category $\mathfrak{CMon}_X$ of comonads on X generated by weakly flat comonads on X.

We call a continuous morphism $X \xrightarrow{f} Y$ *weakly affine* if its direct image functor is conservative and weakly continuous and the category C_X has cokernels of reflexive pairs of arrows. We denote by $\mathbf{Aff}_Y^w$ the full subcategory of the category $|Cat|^o/Y$ whose objects are weakly affine morphisms to the 'space' Y.

Dually, we call a continuous morphism $X \xrightarrow{f} Y$ *weakly flat* (resp. *weakly fflat*), if its inverse image functor is weakly flat (resp. weakly flat and conservative) and the category C_X has kernels of coreflexive pairs of arrows. We denote by $Fflat_X^w$ the full subcategory of the category $X\backslash|Cat|^o$ whose objects are pairs (X, γ), where γ is a weakly fflat morphisms from X.

4.5.1. Proposition. *Let*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ f \searrow & & \swarrow g \\ & Z & \end{array}$$

be a commutative diagram in $|Cat|^o$. Suppose C_Z has cokernels of reflexive pairs of arrows. If f and g are weakly affine, then h is weakly affine.

Dually, if

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ f\nwarrow & & \nearrow g \\ & Z & \end{array}$$

is a commutative diagram in $|Cat|^o$ such that C_Z has kernels of coreflexive pairs of arrows and the morphisms f and g are weakly fflat, then h is weakly fflat.

Proof. Fix inverse and direct image functors of the morphisms f and g together with adjunction morphisms. By hypothesis, the canonical functors $C_X \longrightarrow (\mathcal{F}_f/Z) - mod$ and $C_Y \longrightarrow (\mathcal{F}_g/Z) - mod$ are category equivalences. Here $\mathcal{F}_f = (f_* f^*, \mu_f)$ and $\mathcal{F}_g = (g_* g^*, \mu_g)$ are the monads associated with (the choice of inverse and direct image functors and adjunction morphisms of) respectively f and g. It follows from the dual version of 4.3.2.1 that a choice of an inverse image functor h^* of the morphism h determines a monad morphism $\mathcal{F}_g \xrightarrow{\phi_h} \mathcal{F}_f$ such that the diagram

$$\begin{array}{ccc} C_Y & \xrightarrow{\sim} & (\mathcal{F}_g/Z) - mod \\ h^* \downarrow & & \downarrow \phi_h^* \\ C_X & \xrightarrow{\sim} & (\mathcal{F}_f/Z) - mod \end{array}$$

quasi-commutes. Here ϕ_h^* is the inverse image functor associated with the monad morphism ϕ_h (a left adjoint to the pull-back functor). The pull-back functor, ϕ_{h_*}, is, evidently, conservative and weakly continuous. The latter follows from the fact that the monads $\mathcal{F}_f$ and $\mathcal{F}_g$ are weakly continuous. ■

4.5.2. Proposition. *(a) Suppose that the category C_Y has cokernels of reflexive pairs of arrows. The map $(\mathcal{F}/Y) \longmapsto (\mathbf{Sp}(\mathcal{F}/Y) \to Y)$ defines a full functor*

$$(\mathfrak{Mon}_Y^w)^{op} \xrightarrow{\mathbf{Sp}_Y^w} \mathbf{Aff}_Y^w.$$

(b) Dually, if the category C_Y has kernels of coreflexive pairs of arrows, then the map $(X\backslash\mathcal{G}) \longmapsto (X \to \mathbf{Sp^o}(X\backslash\mathcal{G}))$ defines a full functor

$$(\mathfrak{CMon}_X^w)^{op} \xrightarrow{\mathbf{Sp}_X^{o\,w}} Fflat_X^w.$$

Proof. These facts follow from Beck's Theorem, Proposition 4.5.1, and the following

4.5.3. Lemma. *Let $X \xrightarrow{f} Y$ be a continuous morphism with a direct image functor f_* and an inverse image functor f^*.*

(a) Suppose the morphism f is monadic and the category C_Y has colimits of a type $\mathfrak{S}$. Then f_* preserves colimits of the type $\mathfrak{S}$ iff the functor $F_f = f_*f^*$ has this property.

(b) Dually, if the morphism f is comonadic and the category C_X has limits of a certain type, then f^* preserves these limits iff the functor $G_f = f^*f_*$ does the same.

Recall that a continuous morphism $X \xrightarrow{f} Y$ is called *comonadic* if the induced morphism $\mathbf{Sp^o}(X\backslash\mathcal{G}_f) \xrightarrow{\tilde{f}} Y$ is an isomorphism.

Dually, a continuous morphisms $X \xrightarrow{f} Y$ (in $|Cat|^o$, or in Cat^{op}) is called *monadic* if the associated morphism $\bar{X} \xrightarrow{f} \mathbf{Sp}(\mathcal{F}_f/Y)$ is an isomorphism.

The proof of the lemma and details of the proof of 4.5.2 are left to the reader. ■

4.6. Weakly continuous monads and weakly affine 'spaces'.

4.6.1. Elements of monads. Conjugated morphisms. Let $\widetilde{\mathcal{C}} = (\mathcal{C}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ be a monoidal category (cf. A2.4). Morphisms $\mathbb{I} \longrightarrow V$ are called *elements of the object* V. We denote the set of the elements of V by $|V|$. If $\mathcal{A} = (A, \mu)$ is a unital algebra in $\widetilde{\mathcal{C}}$, then the set $|A|$ of elements of A has a structure of a monoid with the multiplication defined by

$$|A| \times |A| \longrightarrow |A|, \quad (x, y) \longmapsto \mu \circ x \odot y \circ \phi,$$

where $\phi = \mathfrak{l}_{\mathbb{I}}(\mathbb{I}) = \mathfrak{r}_{\mathbb{I}}(\mathbb{I})$ is the isomorphism $\mathbb{I} \xrightarrow{\sim} \mathbb{I} \odot \mathbb{I}$. We denote this monoid by $|\mathcal{A}|$. It follows that the unit element of $|\mathcal{A}|$ is the unit $\mathbb{I} \xrightarrow{e} A$ of the algebra $\mathcal{A}$. We denote by $\mathcal{A}^\star$ the group $|\mathcal{A}|^\star$ of invertible elements of the monoid $|\mathcal{A}|$.

Fix an algebra $\mathcal{B} = (B, \mu)$ in $\widetilde{\mathcal{C}}$. Elements of $\mathcal{B}$ act on B by left and right multiplications defined by the commutative diagram

$$\begin{array}{ccccc}
\mathbb{I} \odot B & \xrightarrow{t \odot B} & B \odot B & \xleftarrow{B \odot s} & B \odot \mathbb{I} \\
\wr\downarrow & & \mu\downarrow & & \downarrow\wr \\
B & \xrightarrow{t\cdot} & B & \xleftarrow{\cdot s} & B
\end{array}$$

for any elements, t, s of B. One can check that

(a) $t \cdot \circ s \cdot = (ts)\cdot$ and $\cdot t \circ \cdot s = \cdot(st)$.

(b) for any $t \in \mathcal{B}^\star$, the composition $Ad(t) = t \cdot \circ \cdot t^{-1}$ is an algebra homomorphism $\mathcal{B} \longrightarrow \mathcal{B}$. It follows from (a) that $Ad(t)$ is an automorphism and $Ad(t) \circ Ad(s) = Ad(ts)$ for any two invertible elements of the algebra $\mathcal{B}$. In particular, $Ad(t^{-1}) = Ad(t)^{-1}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras in $\widetilde{\mathcal{C}}$ and $\mathcal{A} \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \mathcal{B}$ algebra morphisms. We say that the morphisms φ and ψ are *conjugated* if $\psi = t \cdot \varphi \cdot t^{-1} = Ad(t) \circ \varphi$ for some $t \in \mathcal{B}^\star$.

We denote by $Alg(\widetilde{\mathcal{C}})$ the category of unital algebras in the monoidal category $\widetilde{\mathcal{C}}$ and by $\mathfrak{Ass}(\widetilde{\mathcal{C}})$ the category whose objects are unital algebras and morphisms are conjugation classes of algebra morphisms. The composition of morphisms in $\mathfrak{Ass}(\widetilde{\mathcal{C}})$ is induced by the composition in the category of algebras, i.e. it is uniquely determined by the condition that the projection $Alg(\widetilde{\mathcal{C}}) \longrightarrow \mathfrak{Ass}(\widetilde{\mathcal{C}})$ which is identical on objects and assigns to every algebra morphism its conjugation class is a functor.

In what follows, we need this formalism in the case of the (strict) monoidal category $\widetilde{End}(C_X) = (End(C_X), \circ, Id_{C_X})$ of endofunctors of a category C_X and some of its monoidal subcategories. Namely, we shall consider the subcategory $End_w(C_X)$ of *weakly continuous* functors, i.e. functors preserving the cokernels of *reflexive* pairs of arrows (see 4.2.1). The category of algebras in the monoidal category $\widetilde{End_w}(C_X) = (End_w(C_X), \circ, Id_{C_X})$ coincides with the category $\mathfrak{Mon}^w_X$ of weakly continuous monads on the category C_X.

4.6.2. Proposition. *Let $\mathcal{F}$ and $\mathcal{G}$ be monads on the category C_X, and let $\mathcal{F} \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \mathcal{G}$ be monad morphisms. The direct image functors ψ_*, φ_* are isomorphic one to another iff the morphisms ψ and φ are conjugated.*

Proof. (a) Suppose that ψ and φ are conjugated, i.e. $\psi = t \cdot \varphi \cdot t^{-1}$ for some $t \in \mathcal{G}^\star$. For any $\mathcal{G}$-module $\mathcal{M} = (M, \xi)$, we define morphism $M \xrightarrow{t_\mathcal{M}} M$ as the composition $\xi \circ t(M)$. The claim is that the morphism $t_\mathcal{M}$ is a morphism $\psi_*(\mathcal{M}) \longrightarrow \varphi_*(\mathcal{M})$, i.e. the diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{F(t_\mathcal{M})} & F(M) \\ \xi_\psi \downarrow & & \downarrow \xi_\varphi \\ M & \xrightarrow{t_\mathcal{M}} & M \end{array} \qquad (1)$$

commutes. Here $\xi_\psi = \xi \circ \psi(M)$ and $\xi_\varphi = \xi \circ \varphi(M)$. Consider the diagram

$$\begin{array}{ccccc} F^2(M) & \xrightarrow{\mu_\mathcal{F}(M)} & F(M) & \xrightarrow{F(\xi \circ t(M))} & F(M) \\ \psi \odot \psi(M) \downarrow & & \psi(M) \downarrow & & \downarrow \varphi(M) \\ G^2(M) & \xrightarrow{\mu(M)} & G(M) & \xrightarrow{G(\xi \circ t(M))} & G(M) \\ \mu(M) \downarrow & & \xi \downarrow & & \downarrow \xi \\ G(M) & \xrightarrow{\xi} & M & \xrightarrow{\xi \circ t(M)} & M \end{array} \qquad (2)$$

whose right half is a detailed version of the diagram (1). The morphism $\psi \odot \psi$ is by definition $G\psi \circ \psi F$. Notice that the left half of the diagram (2) is commutative which implies the equalities

$$t_{\mathcal{M}} \circ \xi_\psi \circ \mu_{\mathcal{F}}(M) = \xi \circ t(M) \circ \xi \circ \psi(M) \circ \mu_{\mathcal{F}}(M) =$$
$$\xi \circ G\xi \circ tG(M) \circ \mu(M) \circ \psi \odot \psi(M) =$$
$$\xi \circ \mu(M) \circ tG(M) \circ \mu(M) \circ \psi \odot \psi(M) =$$
$$\xi \circ \mu(M) \circ tG(M) \circ \psi(M) \circ \mu_{\mathcal{F}}(M).$$

Since $F^2 \xrightarrow{\mu_{\mathcal{F}}} F$ is an epimorphism, in particular, $\mu_{\mathcal{F}}(M)$ is an epimorphism, it follows from the equality

$$t_{\mathcal{M}} \circ \xi_\psi \circ \mu_{\mathcal{F}}(M) = \xi \circ \mu(M) \circ tG(M) \circ \psi(M) \circ \mu_{\mathcal{F}}(M)$$

that

$$\begin{aligned} t_{\mathcal{M}} \circ \xi_\psi &= \xi \circ (\mu(M) \circ tG(M)) \circ \psi(M) = \\ &\xi \circ (\mu \circ tG \circ \psi)(M) = \xi \circ (t \cdot \circ \psi)(M). \end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned} &\xi_\varphi \circ F(t_{\mathcal{M}}) \stackrel{\text{def}}{=} \xi \circ \varphi(M) \circ F\xi \circ F(t_{\mathcal{M}}) = \xi \circ G\xi \circ Gt(M) \circ \varphi(M) = \\ &\xi \circ (\mu(M) \circ Gt(M)) \circ \varphi(M) = \xi \circ (\mu \circ Gt \circ \varphi)(M) = \xi \circ (\cdot t \circ \varphi)(M). \end{aligned} \tag{4}$$

By hypothesis, $\psi = t \cdot \varphi \cdot t^{-1}$, which means precisely that $t \cdot \circ \psi = \cdot t \circ \varphi$. In particular, $\xi \circ (t \cdot \circ \psi)(M) = \xi \circ (\cdot t \circ \varphi)(M)$, which implies, as follows from (3) and (4), the claimed commutativity of the diagram (1): $t_{\mathcal{M}} \circ \xi_\psi = \xi_\varphi \circ F(t_{\mathcal{M}})$.

The map, which assigns to every $\mathcal{G}$-module $\mathcal{M} = (M, \xi)$ the morphism

$$\psi_*(\mathcal{M}) = (M, \xi_\psi) \xrightarrow{t_{\mathcal{M}}} \varphi_*(\mathcal{M}) = (M, \xi_\varphi)$$

is a functor isomorphism $\psi_* \xrightarrow{\sim} \varphi_*$.

(b) Conversely, let $\psi_* \xrightarrow{u} \varphi_*$ be a functor isomorphism. In particular, we have an isomorphism $\psi_* g^* \xrightarrow{ug^*} \varphi_* g^*$, where g^* is the functor $C_X \longrightarrow (\mathcal{G}/X) - mod$ (– the canonical left adjoint to the forgetful functor $(\mathcal{G}/X) - mod \xrightarrow{g_*} C_X$), which assigns to every object V of C_X the $\mathcal{G}$-module $(G(V), \mu(V))$.

Set, for convenience, $\lambda_u = ug^*$. The diagram

$$\begin{array}{ccc} FG & \xrightarrow{F\lambda_u} & FG \\ {\scriptstyle \varphi G}\downarrow & & \downarrow{\scriptstyle \psi G} \\ G^2 & \xrightarrow{G\lambda_u} & G^2 \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ G & \xrightarrow{\lambda_u} & G \end{array} \tag{4}$$

commutes due to the fact that $\lambda_u(V)$ is an $\mathcal{F}$-module morphism for every $V \in ObC_X$. Let $\mathfrak{t}$ denote the composition $\lambda_u \circ e$, where $Id_{C_X} \xrightarrow{e} G$ is the unit element of the monad $\mathcal{G} = (G, \mu)$. Since $\mu \circ Ge = id_G$, it follows from the commutativity of the lower square of the diagram (4) that

$$\lambda_u = \lambda_u \circ \mu \circ Ge = \mu \circ G\lambda_u \circ Ge = \mu \circ G\mathfrak{t} \stackrel{\text{def}}{=} \cdot\mathfrak{t}.$$

For every $\mathcal{G}$-module $\mathcal{M} = (M, \xi)$, there is a commutative diagram

$$\begin{array}{ccc} G(M) & \xrightarrow{\lambda_u(M)} & G(M) \\ \xi \downarrow & & \downarrow \xi \\ M & \xrightarrow{u(\mathcal{M})} & M \end{array} \qquad (6)$$

which is due to the fact that $G(M) \xrightarrow{\xi} M$ is a morphism of $\mathcal{G}$-modules $g^*g_*(\mathcal{M}) \longrightarrow \mathcal{M}$, hence it can be regarded as the image of this morphism by resp. ψ_* and φ_*. It follows from the commutativity of (6) and the equality $\xi \circ e(M) = id_M$ that

$$u(\mathcal{M}) = u(\mathcal{M}) \circ \xi \circ e(M) = \xi \circ \lambda_u(M) \circ e(M) = \mathfrak{t}_{\mathcal{M}}.$$

Since λ_u is an isomorphism, the element $\mathfrak{t}$ is invertible. ■

4.6.2.1. Corollary. *Let $\mathcal{F} = (F, \mu_{\mathcal{F}})$ and $\mathcal{G} = (G, \mu_{\mathcal{G}})$ be monads on a category C_X, and let $\mathcal{F} \overset{\varphi}{\underset{\psi}{\rightrightarrows}} \mathcal{G}$ be monad morphisms having inverse image functors*

$$(\mathcal{F}/X) - mod \overset{\varphi^*}{\underset{\psi^*}{\rightrightarrows}} (\mathcal{G}/X) - mod.$$

The inverse image functors of the morphisms φ and ψ are isomorphic iff the morphisms are conjugated.

Proof. The inverse image functors of φ and ψ are isomorphic iff their direct image functors are isomorphic. So that the assertion follows from 4.6.2. ■

4.6.2.2. The category $\mathfrak{Ass}^w_X$. Fix a category C_X. The category $\mathfrak{Ass}^w_X$ is defined as follows. Its objects are weakly continuous monads on C_X, i.e. $Ob\mathfrak{Ass}^w_X = Ob\mathfrak{Mon}^w_X$; morphisms are conjugation classes of monad morphisms (see 4.6.1).

4.6.3. Proposition. *Let C_X be a category with cokernels of reflexive pairs of arrows. Let $\mathbf{Aff}^w_X$ denote the full subcategory of the category $|Cat|^o_X$ generated by weakly* *affine morphisms. The functor* $\mathbf{Sp}_X^w : \mathfrak{Mon}_X^w \longrightarrow \mathbf{Aff}_X^w$ *which assigns to every weakly continuous monad* $\mathcal{F}$ *on* C_X *the object* $(\mathbf{Sp}(\mathcal{F}), \mathbf{Sp}(\mathcal{F}) \to X)$ *factors through the category* $\mathfrak{Ass}_X^w$. *The corresponding functor* $\mathfrak{Ass}_X^w \longrightarrow \mathbf{Aff}_X^w$ *is a category equivalence.*

Proof. The fact that the functor $\mathbf{Sp}_X^w : \mathfrak{Mon}_X^w \longrightarrow \mathbf{Aff}_X^w$ factors through $\mathfrak{Ass}_X^w$ and that the functor $\mathfrak{Ass}_X^w \longrightarrow \mathbf{Aff}_X^w$ is faithful is a consequence of 4.6.2. It follows from (the argument of) 4.5.3 that all morphisms of the category $\mathbf{Aff}_X^w$ are weakly affine and the functor $\mathfrak{Ass}_X^w \longrightarrow \mathbf{Aff}_X^w$ is full. Its quasi-inverse is given by assigning to each weakly affine morphism $Y \xrightarrow{f} X$ the weakly continuous monad $\mathcal{F}_f = (f_* f^*, \mu_f)$. ∎

5. Continuous monads and affine morphisms. Duality. A functor is called *continuous* if it has a right adjoint. A monad $\mathcal{F} = (F, \mu)$ on a 'space' Y (i.e. on the category C_Y) is called *continuous*, if the functor F is continuous. Since continuous functors preserve colimits, a continuous monad is weakly continuous.

Dually, a comonad $\mathcal{G} = (G, \delta)$ on Y is called *cocontinuous* if the functor G has a left adjoint. In other words, a cocontinuous comonad on Y is the same as a continuous monad on Y^o. A cocontinuous monad is weakly flat.

5.1. Duality. Let $\mathcal{F} = (F, \mu)$ be a continuous monad on Y; i.e. the functor F has a right adjoint, $F^\wedge$. The multiplication $F^2 \xrightarrow{\mu} F$ induces a morphism $F^\wedge \xrightarrow{\delta} (F^\wedge)^2$ which is a comonad structure on $F^\wedge$ with the counit $F^\wedge \xrightarrow{\epsilon} Id_{C_Y}$ induced by the unit $Id_{C_Y} \xrightarrow{\eta} F$ of the monad $\mathcal{F}$. Thus, we have a comonad, $\mathcal{F}^\wedge = (F^\wedge, \mu^\wedge)$ dual to the monad $\mathcal{F}$. The map which assigns to any morphism $F(L) \longrightarrow L$, $L \in ObC_Y$, the dual morphism $L \longrightarrow F^\wedge(L)$ induces an isomorphism of categories

$$\Phi : (\mathcal{F}/Y) - mod \xrightarrow{\ \sim\ } (Y\backslash\mathcal{F}^\wedge) - Comod \qquad (1)$$

such that the diagram

$$\begin{array}{ccc} (\mathcal{F}/Y) - mod & \xrightarrow{\ \Phi\ } & (Y\backslash\mathcal{F}^\wedge) - Comod \\ \hat{f}_* \searrow & & \swarrow \check{f}^* \\ & C_Y & \end{array} \qquad (2)$$

commutes. Here $\check{f}^*$ denotes the functor forgetting $\mathcal{F}^\wedge$-comodule structure.

It follows from the construction that $\mathcal{F}^\wedge$ is a cocontinuous comonad on Y determined by the monad $\mathcal{F}$ uniquely up to isomorphism.

Conversely, to any cocontinuous comonad, $\mathcal{G} = (G, \delta)$, on Y, there corresponds a continuous monad $\mathcal{G}^\vee = (G^\vee, \delta^\vee)$, where $G^\vee$ is a left adjoint to G. The monad $\mathcal{G}^\vee$ is determined by $\mathcal{G}$ uniquely up to isomorphism, and we have a comonad and monad isomorphisms, respectively

$$\mathcal{G} \xrightarrow{\sim} (\mathcal{G}^\vee)^\wedge \quad \text{and} \quad \mathcal{F} \xrightarrow{\sim} (\mathcal{F}^\wedge)^\vee.$$

5.2. Proposition. *A monad $\mathcal{F} = (F, \mu)$ on Y is continuous iff the canonical morphism* $\mathbf{Sp}(\mathcal{F}/Y) \xrightarrow{\hat{f}} Y$ *is affine. Dually, a comonad $\mathcal{G} = (G, \delta)$ on Y is cocontinuous iff the canonical morphism $Y \longrightarrow \mathbf{Sp^o}(Y\backslash\mathcal{G})$ is coaffine.*

Proof. A canonical direct image functor of $\hat{f}$ is the forgetful functor

$$(\mathcal{F}/Y) - mod \xrightarrow{\hat{f}_*} C_Y, \ (M, \xi) \longmapsto M.$$

Since the functor $\hat{f}_*$ is conservative, the morphism $\hat{f}$ is affine iff $\hat{f}_*$ has a right adjoint.

(a) If $\hat{f}^!$ is a right adjoint to $\hat{f}_*$, then the functor $F^\wedge = \hat{f}_*\hat{f}^!$ is a right adjoint to $F = \hat{f}_*\hat{f}^*$. Here $\hat{f}^*$ denotes the functor $L \longmapsto (F(L), \mu(L))$.

(b) Conversely, suppose $\mathcal{F} = (F, \mu)$ is a continuous monad on Y; i.e. the functor F has a right adjoint, $F^\wedge$. The functor $\breve{f}^*$ in the diagram (2) has a right adjoint, $\breve{f}_*$, which maps every object M of C_Y to the $(Y\backslash\mathcal{F}^\wedge)$-comodule $(\mathcal{F}^\wedge(M), M \stackrel{\delta(M)}{\longrightarrow} (F^\wedge)^2(M))$. It follows from the commutativity of (2) that the functor

$$\hat{f}^! = \Phi^{-1} \circ \breve{f}_* : C_Y \longrightarrow \mathcal{F} - mod$$

is a right adjoint to the forgetful functor $\mathcal{F} - mod \xrightarrow{\hat{f}_*} C_Y$. Since $\hat{f}_*$ is, obviously, conservative, it is a direct image functor of an affine morphism $\mathbf{Sp}(\mathcal{F}/Y) \longrightarrow Y$. ∎

5.2.1. Corollary. *Suppose that the category C_Y has cokernels of reflexive pairs of arrows. A continuous morphism $X \xrightarrow{f} Y$ in $|Cat|^o$ is affine iff its direct image functor $C_X \xrightarrow{f_*} C_Y$ is the composition of a category equivalence*

$$C_X \longrightarrow (\mathcal{F}_f/Y) - mod$$

for a continuous monad $\mathcal{F}_f$ in C_Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$. The monad $\mathcal{F}_f$ is determined by f uniquely up to isomorphism.

Proof. The conditions of the Beck's theorem are fulfilled if f is affine, hence f_* is the composition of an equivalence $C_X \longrightarrow (\mathcal{F}_f/Y) - mod$ for a monad $\mathcal{F}_f = (f_*f^*, \mu_f)$ in C_Y and the forgetful functor $(\mathcal{F}_f/Y) - mod \longrightarrow C_Y$ (see (1)). The functor $F_f = f_*f^*$ has a right adjoint $f_*f^!$, where $f^!$ is a right adjoint to f_*. The rest follows from 5.2. ∎

5.3. Proposition. *Suppose X is an object of $|Cat|^o$ such that the category C_X has kernels of reflexive pairs of arrows. Let $\mathcal{F}_f = (F_f, \mu_f)$ and $\mathcal{F}_g = (F_g, \mu_g)$ be continuous monads on X. Then, for any monad morphism $\mathcal{F}_f \xrightarrow{\varphi} \mathcal{F}_g$, the corresponding morphism*

$$\mathbf{Sp}(\mathcal{F}_f/X) \xrightarrow{\mathbf{Sp}(\varphi)} \mathbf{Sp}(\mathcal{F}_g/X)$$

is affine.

Proof. The morphism φ induces a dual comonad morphism $\mathcal{F}_g^\wedge \xrightarrow{\hat{\varphi}} \mathcal{F}_f^\wedge$ such that the diagram

$$\begin{array}{ccc} (\mathcal{F}_g/X) - mod & \xrightarrow{\varphi_*} & (\mathcal{F}_f/X) - mod \\ \Phi_{\mathcal{F}_g} \downarrow & & \downarrow \Phi_{\mathcal{F}_f} \\ (X\backslash\mathcal{F}_g^\wedge) - Comod & \xrightarrow{\hat{\varphi}^*} & (X\backslash\mathcal{F}_f^\wedge) - Comod \end{array} \qquad (3)$$

commutes. Here $\Phi_{\mathcal{F}_f}$ and $\Phi_{\mathcal{F}_g}$ are the canonical category isomorphisms (cf. 5.1). Since the category C_X has kernels of coreflexive pairs of arrows and the functor $F_g^\wedge$ preserves limits (in particular, it preserves kernels of pairs of arrows), the functor $\hat{\varphi}^*$ has a right adjoint (cf. 4.3.2.2), hence φ_* has a right adjoint. Since the functor φ_* is conservative, the morphism $\mathbf{Sp}(\varphi)$ is affine. ■

5.4. Proposition. *Let*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ f \searrow & & \swarrow g \\ & Z & \end{array}$$

be a commutative diagram in $|Cat|^o$. Suppose that the category C_Z has cokernels of reflexive pairs of arrows. If the morphisms f and g are affine, then the morphism h is affine too.

Proof. Fix inverse and direct image functors of f and g together with adjunction morphisms. By the Beck's theorem, the canonical functors

$$C_X \longrightarrow (\mathcal{F}_f/Z) - mod \quad \text{and} \quad C_Y \longrightarrow (\mathcal{F}_g/Z) - mod$$

are category equivalences. Here $\mathcal{F}_f = (f_*f^*, \mu_f)$ and $\mathcal{F}_g = (g_*g^*, \mu_g)$ are monads associated with resp. f and g. By 4.5.3, a choice of an inverse image functor h^* of the morphism h determines a monad morphism $\mathcal{F}_g \xrightarrow{\phi_h} \mathcal{F}_f$ such that the diagram

$$\begin{array}{ccc} C_Y & \xrightarrow{\sim} & (\mathcal{F}_g/Z) - mod \\ h^* \downarrow & & \downarrow \phi_h^* \\ C_X & \xrightarrow{\sim} & (\mathcal{F}_f/Z) - mod \end{array}$$

quasi-commutes. By 5.3, since the monads $\mathcal{F}_g$ and $\mathcal{F}_f$ are continuous, the direct image functor ϕ_{h*} (the pull-back by the morphism ϕ_h) has a right adjoint, $\phi_h^!$. ■

For $Z \in Ob|Cat|^o$, denote by $\mathbf{Aff}_Z$ the full subcategory of $|Cat|^o/Z$ whose objects are affine morphisms. Let $|Cat|^o_{aff}$ be the subcategory of $|Cat|^o$ formed by affine morphisms.

5.4.1. Proposition. *Suppose that the category C_Z has cokernels of reflexive pairs of arrows. Then the natural embedding*

$$|Cat|^o_{aff}/Z \longrightarrow \mathbf{Aff}_Z$$

is an isomorphism of categories.

Proof. The assertion is a corollary of 5.4. ∎

5.5. Proposition. *Let $X \xrightarrow{f} Y$ be an affine morphism in $|Cat|^o$. If the category C_Y is additive (resp. abelian, resp. abelian with small coproducts, resp. a Grothendieck category), then the category C_X has the same property.*

Proof. By 5.2, the category C_X is equivalent to the category $(\mathcal{F}_f/Y)-mod$ of (F_f/Y)-modules for a continuous monad $\mathcal{F} = (F_f, \mu_f)$ on Y. Since the functor F_f has a right adjoint and the category C_Y is additive, F_f is additive and preserves colimits of arbitrary small diagrams. This implies that for any diagram $D \xrightarrow{\mathfrak{D}} \mathcal{F}-mod$, the object $\mathrm{colim}(f_* \circ \mathfrak{D})$ (where f_* is the forgetful functor $(\mathcal{F}/Y)-mod \longrightarrow C_Y$) has a unique $(\mathcal{F}_f/Y)$-module structure $\xi_{\mathfrak{D}}$ such that all morphisms $f_*\mathfrak{D}(x) \longrightarrow \mathrm{colim}(f_* \circ \mathfrak{D})$ are $(\mathcal{F}/Y)$-module morphisms $\mathfrak{D}(x) \longrightarrow (\mathrm{colim}(f_* \circ \mathfrak{D}), \xi_{\mathfrak{D}})$. This implies the assertion. Details are left to the reader. ∎

5.6. Affine morphisms to $\mathbf{Sp}(R)$.

5.6.1. Proposition. *Let R be an associative unital ring. A continuous morphism $X \xrightarrow{f} \mathbf{Sp}(R)$ in $|Cat|^o$ is affine iff its direct image functor, $C_X \xrightarrow{f_*} R - mod$, is the composition of an equivalence of categories $C_X \longrightarrow R_f - mod$ for an associative unital ring R_f and the restriction of scalars functor $R_f - mod \xrightarrow{\phi_*} R - mod = C_Y$ for a ring morphism $R \xrightarrow{\phi} R_f$ determined by f uniquely up to isomorphism.*

Proof. (i) The morphism $\mathbf{Sp}(S) \longrightarrow \mathbf{Sp}(R)$ corresponding to a ring morphism $R \longrightarrow S$ is affine by 1.1. Isomorphisms are affine and the composition of affine morphisms is affine.

(ii) Conversely, suppose that $X \xrightarrow{f} \mathbf{Sp}(R)$ is an affine morphism. Then the functor $f_*f^* : R - mod \longrightarrow R - mod$ has a right adjoint, hence it is isomorphic to the functor $R_f \otimes_R - : L \longmapsto R_f \otimes_R L$ for some R-bimodule R_f. The monad structure on f_*f^* induces an associative ring structure, $R_f \otimes_R R_f \xrightarrow{m_f} R_f$, on R_f (in the monoidal category of R-bimodules); and the adjunction morphism $Id_{R-mod} \xrightarrow{\eta_f} f_*f^*$ corresponds to a ring morphism $R \xrightarrow{\phi} R_f$ so that the diagrams of functor morphisms

$$\begin{array}{ccc} Id_{R-mod} & \xrightarrow{\sim} & R \otimes_R - \\ \eta_f \downarrow & & \downarrow \phi \otimes_R \\ f_*f^* & \xrightarrow{\sim} & R_f \otimes_R - \end{array} \quad \text{and} \quad \begin{array}{ccc} (f_*f^*)^2 & \xrightarrow{\sim} & R_f \otimes_R R_f \otimes_R - \\ \mu_f \downarrow & & \downarrow m_f \\ f_*f^* & \xrightarrow{\sim} & R_f \otimes_R - \end{array} \qquad (2)$$

commute. Thus we have a commutative diagram

$$\begin{array}{ccc} (\mathcal{F}_f/\mathbf{Sp}(R)) - mod & \stackrel{\sim}{\longrightarrow} & R_f - mod \\ \hat{f}_* \searrow & & \swarrow \phi_* \\ & R - mod & \end{array} \qquad (3)$$

whose horizontal arrow is an isomorphism of categories. Combining with the commutative diagram (1), we obtain a commutative diagram

$$\begin{array}{ccc} C_X & \stackrel{\sim}{\longrightarrow} & R_f - mod \\ f_* \searrow & & \swarrow \phi_* \\ & R - mod & \end{array} \qquad (4)$$

where the horizontal arrow is an equivalence of categories.

Notice that $R_f = f_*f^*(R)$. Therefore, the ring morphism $R \stackrel{\phi}{\longrightarrow} R_f$ is defined uniquely up to isomorphism by a choice of an inverse image functor f^*. ■

5.6.2. A comparison of two descriptions. Let $X \stackrel{f}{\longrightarrow} \mathbf{Sp}(R)$ be an affine morphism. Being continuous, the morphism f is determined uniquely up to isomorphism by the object $\mathcal{O} = f^*(R)$, and a right R-module structure $R \longrightarrow \Gamma_X\mathcal{O} = C_X(\mathcal{O},\mathcal{O})^o$ (cf. 3.1). By 3.3, we have a commutative diagram of direct image functors of continuous morphisms

$$\begin{array}{ccc} C_X & \stackrel{f_{\mathcal{O}*}}{\longrightarrow} & \Gamma_X\mathcal{O} - mod \\ f_* \searrow & & \swarrow \bar{\phi}_{f*} \\ & R - mod & \end{array} \qquad (1)$$

Here $\bar{\phi}_{f*}$ is the pull-back by the ring morphism $R \stackrel{\phi_f}{\longrightarrow} \Gamma_X\mathcal{O}$ defining a right R-module structure on $\mathcal{O}$. The morphism $f_{\mathcal{O}*}$ has an inverse image functor $f^*_{\mathcal{O}}$ which maps the left module $\Gamma_X\mathcal{O}$ to $\mathcal{O}$. The adjunction morphism $\Gamma_X\mathcal{O} \longrightarrow f_{\mathcal{O}*}f^*_{\mathcal{O}}(\Gamma_X\mathcal{O})$ is an isomorphism. Since morphisms f and $\mathbf{Sp}\Gamma_X\mathcal{O} \stackrel{\bar{\phi}_f}{\longrightarrow} \mathbf{Sp}(R)$ are affine, the morphism $f_{\mathcal{O}}$ is affine too (cf. 5.3). In particular, $f_{\mathcal{O}*}f^*_{\mathcal{O}}$ has a right adjoint, hence it preserves colimits. Since $\Gamma_X\mathcal{O}$ is a generator of the category $\Gamma_X\mathcal{O} - mod$, the isomorphness of the adjunction arrow $\Gamma_X\mathcal{O} \longrightarrow f_{\mathcal{O}*}f^*_{\mathcal{O}}(\Gamma_X\mathcal{O})$ implies that $M \longrightarrow f_{\mathcal{O}*}f^*_{\mathcal{O}}(M)$ is an isomorphism for any $\Gamma_X\mathcal{O}$-module M. This means that the functor $f^*_{\mathcal{O}}$ is fully faithful, hence $f_{\mathcal{O}*}$ is a localization. Since by condition $f_{\mathcal{O}*}$ is conservative, it is a category equivalence.

This shows that C_X is naturally equivalent to the category of $\Gamma_X\mathcal{O}$-modules. Thus, the ring morphism $R \longrightarrow R_f$ in 5.6.1 is isomorphic to the ring morphism $R \longrightarrow \Gamma_X\mathcal{O}$ defining a right R-module structure on the object $\mathcal{O}$.

This observation was made before (in [R], Ch.7) using a slightly different argument.

5.7. Continuous monads and relative affine schemes. Let $\mathfrak{Mon}^{\mathfrak{c}}_X$ denote the category of continuous monads on X and let $\mathfrak{Ass}_X$ denote the quotient of the category $\mathfrak{Mon}^{\mathfrak{c}}_X$ by the relation of conjugation. That is objects of $\mathfrak{Ass}_X$ are continuous monads on X and morphisms are conjugation classes of monad morphisms.

5.7.1. Proposition. *Let C_X be a category with cokernels of coreflexive pairs of arrows. Let $\mathbf{Aff}_X$ denote the full subcategory of the category $|Cat|^o_X$ generated by affine morphisms. The functor*

$$(\mathfrak{Mon}^{\mathfrak{c}}_X)^{op} \xrightarrow{\mathbf{Sp}_X} \mathbf{Aff}_X$$

which assigns to every continuous monad $\mathcal{F}$ on C_X the object $(\mathbf{Sp}(\mathcal{F}), \mathbf{Sp}(\mathcal{F}) \to X)$ factors through the category $\mathfrak{Ass}_X$. The corresponding functor $\mathfrak{Ass}_X \xrightarrow{\mathfrak{Sp}_X} \mathbf{Aff}_X$ is a category equivalence.

Proof. By 5.4, morphisms of the category $\mathbf{Aff}_X$ are affine: if

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & f \searrow \quad \swarrow g & \\ & X & \end{array}$$

is a commutative diagram in $|Cat|^o$ with affine morphisms f and g, then the morphism h is affine too. By 5.2.1, a weakly affine morphism $Y \xrightarrow{f} X$ is affine iff the corresponding monad $\mathcal{F}_f$ is continuous. The assertion follows now from 4.6.3. ■

5.7.2. Proposition. *Let C_X be a category with cokernels of reflexive pairs of arrows. Let $End_c(C_X)$ denote the category of continuous endofunctors of C_X. Suppose that the inclusion functor $End_c(C_X) \longrightarrow End(C_X)$ has a left adjoint. Then*

$$\mathfrak{Ass}_X \xrightarrow{\mathfrak{Sp}_X} |Cat|^o_X, \quad \mathcal{F} \longmapsto \mathbf{Sp}(\mathcal{F}/X),$$

is a fully faithful functor which has a left adjoint.

Proof. It follows from 5.7.1 that the functor $\mathfrak{Sp}_X$ is fully faithful. Let Φ_X denote a left adjoint to the inclusion functor $End_c(C_X) \longrightarrow End(C_X)$. The map which assigns to each object $(Y, Y \xrightarrow{f} X)$ of the category $|Cat|^o_X$ the continuous monad $\Phi_X(\mathcal{F}_f)$ extends naturally to a functor $|Cat|^o_X \longrightarrow \mathfrak{Ass}_X$. This functor is a left adjoint to the functor $\mathfrak{Sp}_X$. Details are left to the reader. ■

5.7.2.1. Example. Let $X = \mathbf{Sp}(R)$ for some associative unital ring R. Then for every functor $C_X \xrightarrow{F} C_X$, the left R-module $F(R)$ has a natural structure of a right R-module which together turn $F(R)$ into an R-bimodule. The map which assigns to each endofunctor F on C_X the continuous functor $\mathcal{M} \longmapsto F(R) \otimes_R \mathcal{M}$ extends to a functor which is a left adjoint to the inclusion functor $End_c(C_X) \longrightarrow End(C_X)$.

5.7.2.2. Note. If the category C_X is additive, then the category $End(C_X)$ of endofunctors might be replaced in 5.7.2 by the category $End_a(C_X)$ of additive endofunctors. Notice that the subcategory $End_{\mathfrak{c}}(C_X)$ of continuous endofunctors is contained in $End_a(C_X)$, because any continuous functor between additive categories preserves coproducts, hence it is additive.

As a corollary of 5.7.2, we obtain

5.7.3. Theorem. *(a) Every morphism*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & \mathbf{Sp}(\mathbb{Z}) & \end{array} \qquad (1)$$

of $\mathbb{Z}$-affine schemes is affine (i.e. the morphism $X \xrightarrow{f} Y$ is affine).

(b) The functor

$$\mathbf{Rings}^{op} \xrightarrow{\mathbf{Sp}} |Cat^o|_{\mathbb{Z}}, \quad R \longmapsto \mathbf{Sp}(R),$$

is full and factors through an equivalence of categories $\mathfrak{Ass}^{op} \xrightarrow{\approx} \mathbf{Aff}_{\mathbb{Z}}$.

(c) The fully faithful functor

$$\mathfrak{Ass}^{op} \longrightarrow |Cat^o|_{\mathbb{Z}}$$

(– the composition of the equivalence $\mathfrak{Ass}^{op} \xrightarrow{\approx} \mathbf{Aff}_{\mathbb{Z}}$ and the full embedding of $\mathbf{Aff}_{\mathbb{Z}}$ into $|Cat|^o$) is right adjoint to the 'global sections functor' $X \longmapsto \Gamma\mathcal{O}_X \stackrel{\mathrm{def}}{=} C_X(\mathcal{O}_X, \mathcal{O}_X)^o$.

6. Flat descent.

6.1. Continuous, flat comonads. A comonad $\mathcal{G} = (G, \delta)$ on a 'space' X (i.e. the category C_X) is called

— *continuous*, if the functor G has a right adjoint;

— *flat*, if the functor G preserves finite limits;

— *weakly flat*, if the functor G preserves kernels of coreflexive pairs of arrows;

— *conservative*, if the functor G is conservative.

6.2. Proposition. *Let $X \xrightarrow{f} Y$ be a continuous morphism, and let C_X have kernels of coreflexive pairs of morphisms. The morphism $X \xrightarrow{f} Y$ in $|Cat|^o$ is affine, flat (resp. weakly flat), and conservative iff its inverse image functor $C_Y \xrightarrow{f^*} C_X$ is the composition of an equivalence of categories*

$$C_Y \longrightarrow (X\backslash\mathcal{G}_f) - Comod$$

for a continuous flat (resp. weakly flat) conservative comonad $\mathcal{G}_f$ on X and the forgetful functor

$$(X\backslash\mathcal{G}_f) - Comod \longrightarrow C_X.$$

The comonad $\mathcal{G}_f$ is determined by f uniquely up to isomorphism.

Proof. The conditions of the Beck's theorem are fullfiled, if f is weakly flat and conservative, hence f^* is the composition of an equivalence

$$C_Y \longrightarrow (X\backslash\mathcal{G}_f) - Comod$$

for a comonad $\mathcal{G}_f = (f^* f_*, \delta_f)$ on X and the forgetful functor

$$(X\backslash\mathcal{G}_f) - Comod \longrightarrow C_X.$$

If the morphism f is affine, then the functor $G_f = f^* f_*$ has a right adjoint $f^! f_*$, where $f^!$ is a right adjoint to f_*.

Let now $\mathcal{G} = (G, \delta)$ be a continuous comonad on X and f^* the forgetful functor $(X\backslash\mathcal{G}) - Comod \longrightarrow C_X$. The functor f_* which assigns to each object M of C_X the $\mathcal{G}$-comodule $(G(M), \delta(M))$ is right adjoint to f^*: the canonical adjunction arrows are

$$f^* f_* = G \xrightarrow{\epsilon_f} Id_{C_X} \quad \text{and} \quad Id_{C_Y} \xrightarrow{\eta_f} f_* f^*,$$

where $C_Y = \mathcal{G} - Comod$, ϵ_f is the counit of the monad $\mathcal{G}$ and

$$\eta_f(M, \xi) = \xi : (M, \xi) \longrightarrow f_* f^*(M, \xi) = (G(M), \delta(M))$$

for any $\mathcal{G}$-comodule $(M, M \xrightarrow{\xi} G(M))$.

Let $G^!$ be a right adjoint to G, and let $GG^! \xrightarrow{\epsilon'} Id_{C_X}$ and $Id_{C_X} \xrightarrow{\eta'} G^! G$ be adjunction arrows. Let $f^!$ denote the functor

$$C_Y = (X\backslash\mathcal{G}) - Comod \longrightarrow C_X, \quad (M, \xi) \longmapsto G^!(M).$$

Since $f^! f_* = G^! G$, the adjunction arrow η' is a morphism $Id_{C_X} \longrightarrow f^! f_*$. The composition $f_* f^!$ assigns to each $\mathcal{G}$-comodule (M, ξ) the $\mathcal{G}$-comodule $f_* G^!(M) = (GG^!(M), \delta G^!(M))$. One can check that the adjunction arrow $GG^!(M) \xrightarrow{\epsilon'(M)} M$ is a $\mathcal{G}$-comodule morphism $f_* G^!(M) \longrightarrow (M, \xi)$, i.e. the diagram

$$\begin{array}{ccc} GG^!(M) & \xrightarrow{\epsilon'(M)} & M \\ {\scriptstyle \delta G^!(M)} \downarrow & & \downarrow {\scriptstyle \xi} \\ G^2 G^!(M) & \xrightarrow{G\epsilon'(M)} & G(M) \end{array} \qquad (1)$$

commutes. This implies that $\epsilon' f^*$ and η' are adjunction morphisms, hence the assertion. ∎

6.2.1. Corollary. *Let a morphism $X \xrightarrow{f} Y$ be affine, weakly flat, and conservative. If the category C_X is additive (resp. abelian, resp. abelian with small coproducts, resp. a Grothendieck category), then the category C_Y has the same property, and the morphism f is flat.*

Proof. Under the hypothesis, the category C_Y is equivalent to the category of $(X\backslash\mathcal{G}_f)$-comodules for a continuous comonad $\mathcal{G}_f = (G_f, \delta_f)$ on X. Since the category C_X is additive and the functor $C_X \xrightarrow{G_f} C_X$ has a right adjoint, it preserves small colimits; in particular, G_f is additive. Since $G_f = f^* f_*$, the functor f_* preserves all small limits, and the functor f^* preserves kernels of coreflexive pairs of arrows, the functor G_f preserves kernels of coreflexive pairs of arrows too. For additive categories (more generally, for categories with coproducts and a zero object) functors which preserve kernels of coreflexive pairs of arrows preserve kernels of any pairs of arrows. Thus, G_f preserves kernels of any pairs of arrows and, being additive, finite products (which coincide with finite coproducts), hence G_f reserves limits of arbitrary finite diagrams. Therefore, the category $(X\backslash\mathcal{G}_f) - Comod$ has limits of finite diagrams which are preserved (and reflected) by the forgetful functor

$$(X\backslash\mathcal{G}_f) - Comod \longrightarrow C_X.$$

This implies the additivity of $(X\backslash\mathcal{G}_f) - Comod$. The rest follows from the compatibility of G_f with small colimits (cf. the argument of 5.5). ∎

6.3. Affine, flat morphisms from $\mathbf{Sp}(R)$. If R is an associative ring and $\mathcal{G}$ a comonad on $\mathbf{Sp}(R)$, we shall write for convenience $(R\backslash\mathcal{G})$ instead of $(\mathbf{Sp}(R)\backslash\mathcal{G})$.

6.3.1. Proposition. *A continuous morphism $\mathbf{Sp}(R) \xrightarrow{f} X$ in $|Cat|^o$ is flat, conservative, and affine iff its inverse image functor, $C_X \xrightarrow{f^*} R - mod$, is the composition of an equivalence of categories*

$$C_X \longrightarrow (R\backslash\mathcal{H}_f) - Comod$$

for a coalgebra $\mathcal{H}_f = (H_f, \delta_f)$ in the category of R-bimodules such that H_f is a flat right R-module, and the forgetful functor $(R\backslash\mathcal{H}_f) - mod \longrightarrow R - mod$.

Proof. Let $\mathbf{Sp}(R) \xrightarrow{f} X$ be a flat, conservative, and affine morphism with an inverse image functor f^*. By 6.2, the functor $C_X \xrightarrow{f^*} R - mod$ is the composition of a category equivalence $C_X \longrightarrow (R\backslash\mathcal{G}_f) - Comod$ for a comonad $\mathcal{G}_f = (G_f, \delta_f)$ on $\mathbf{Sp}(R)$ and the forgetful functor $(R\backslash\mathcal{G}_f) - mod \longrightarrow R - mod$. Since the comonad $\mathcal{G}_f$ is continuous, the functor G_f is isomorphic to the functor $H_f \otimes_R -$ for an R-bimodule H_f (equal to $G_f(R)$). The comultiplication $G_f \xrightarrow{\delta_f} G_f^2$ induces a comultiplication $H_f \longrightarrow H_f \otimes_R G_f$ (see the argument of 5.6.1).

Conversely, let $\mathcal{H} = (H, \delta)$ be a coalgebra in the category of R-bimodules, and let f^* denote the forgetful functor

$$(R\backslash\mathcal{H}) - Comod \longrightarrow R - mod, \quad (M, M \to H(M)) \longmapsto M. \tag{1}$$

The functor f^* has a right adjoint,

$$L \longmapsto \mathcal{H} \otimes_R L = (H \otimes_R L, \delta \otimes_R L) \tag{2}$$

(see the argument of 6.2). The comonad $\mathcal{H}\otimes_R$ is continuous, since the functor $H \otimes_R -$ has a right adjoint, $Hom_R(H, -)$. The functor f^* being flat is equivalent to the flatness of H as a right R-module. The assertion follows now from 6.2. ■

6.3.2. Example: semi-separated schemes and algebraic spaces. Let $\mathcal{X}$ be a scheme, or an algebraic space. Recall that an affine cover $\{U_i \xrightarrow{u_i} \mathcal{X} \mid i \in J\}$ of $\mathcal{X}$ is called *semi-separated* if all finite intersections of $U_i \xrightarrow{u_i} \mathcal{X}$ are affine. A scheme (or an algebraic space) is called *semi-separated* if it has a semi-separated cover. Evidently, every separated algebraic space (or scheme) is semi-separated.

If $\{U_i \to \mathcal{X} \mid i \in J\}$ is a semi-separated finite cover of $\mathcal{X}$, then the corresponding morphism

$$\mathcal{U} = \coprod_{i\in J} U_i \xrightarrow{\ \pi\ } \mathcal{X}$$

is affine, which implies that the space of relations $\mathcal{R} = \coprod_{i,j\in J} U_i \times_{\mathcal{X}} U_j \simeq \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ is affine too. Since morphisms u_i are étale, their inverse image functors, u_i^* are flat and the family $\{u_i^* |\ i \in J\}$ is conservative. The latter means exactly that an inverse image functor π^* of the morphism π is flat and conservative. It follows by construction, that the inverse images of projections $\mathcal{R} \rightrightarrows \mathcal{U}$ are flat and conservative (equivalently, faithfully flat). And they are affine, since both $\mathcal{R}$ and $\mathcal{U}$ are affine.

7. The Cone and the Proj.

7.1. Continuous morphisms, monads, and localizations. Let $X \xrightarrow{q} Y$ be a localization with an inverse image functor $C_Y \xrightarrow{q^*} C_X$. Let Σ_q denote the class of all morphisms s in C_Y such that $q^*(s)$ is invertible. A functor $C_Y \xrightarrow{F} C_Y$ is compatible with the localization q iff $F(\Sigma_q) \subseteq \Sigma_q$. In this case, there exists a unique functor $C_X \xrightarrow{\bar{F}} C_X$ such that $q^* \circ F = \bar{F} \circ q^*$.

7.1.1. Proposition. *Let $X \xrightarrow{q} Y$ be a continuous localization.*

(a) A functor $C_Y \xrightarrow{F} C_Y$ is compatible with q iff the canonical morphism

$$q^* \circ F \longrightarrow q^* \circ F \circ q_* q^* \qquad (1)$$

is an isomorphism.

(b) Suppose $C_Y \xrightarrow{F} C_Y$ is compatible with the localization q, and let $\bar{F}$ be the functor $C_X \longrightarrow C_X$ such that $q^* \circ F = \bar{F} \circ q^*$.

(i) If C_Y has colimits of certain type, then C_X has colimits of this type. If F preserves colimits of this type, then the functor $\bar{F}$ has the same property.

(ii) If C_Y has limits of certain type, then C_X has limits of this type. If F and q^* preserve limits of this type (e.g. finite limits), then the functor $\bar{F}$ has the same property.

(iii) If F has a right adjoint, then $\bar{F}$ has a right adjoint.

Proof. (a) The assertion follows from 7.1.4.1.1 applied to $f^* = q^* \circ F$.

(b) (i) Let $\mathcal{D} : D \longrightarrow C_X$ be a small diagram such that there exists $colim(q_*\mathcal{D})$. Then there exists the colimit of $\mathcal{D}$ and $colim(\mathcal{D}) = q^* colim(q_*\mathcal{D})$. Suppose the functor F preserves the colimit of $q_*\mathcal{D}$. Since $\bar{F} \simeq q^* F q_*$, we have:

$$\bar{F}(colim(\mathcal{D})) \simeq q^* F q_* q^*(colim(q_*\mathcal{D})) \simeq q^* F(colim(q_*\mathcal{D})) \simeq q^*(colim(F q_*\mathcal{D})$$

(the second isomorphism here is due to the isomorphism $q^* F \xrightarrow{\sim} q^* F q_* q^*$). Since q^* has a right adjoint, it preserves colimits. Therefore

$$q^*(colim(F q_*\mathcal{D}) \simeq colim(q^* F q_*\mathcal{D}) \simeq colim(\bar{F}\mathcal{D})$$

whence the assertion.

(ii) Let $\mathcal{D} : D \longrightarrow C_X$ be a small diagram such that there exists $\lim(q_*\mathcal{D})$. Then, by [GZ, I.1.4], there exists the limit of $\mathcal{D}$ and $\lim(\mathcal{D}) = q^*(\lim(q_*\mathcal{D}))$. Let the functor F preserve the colimit of $q_*\mathcal{D}$. As in (i), we have:

$$\bar{F}(\lim(\mathcal{D})) \simeq q^* F q_* q^*(\lim(q_*\mathcal{D})) \simeq q^* F(\lim(q_*\mathcal{D})) \simeq q^*(\lim(F q_*\mathcal{D})$$

If q^* preserves limit of $Fq_*\mathcal{D}$, we continue as follows:

$$q^*(\lim(Fq_*\mathcal{D}) \simeq (\lim(q^*Fq_*\mathcal{D}) \simeq \lim(\bar{F}\mathcal{D}).$$

(iii) Let $F^!$ be a right adjoint to F. Set $\bar{F}^! = q^*F^!q_*$. By (a), $\bar{F} \simeq q^*Fq_*$. Thus we have morphisms

$$\bar{F}\bar{F}^! \xrightarrow{\sim} (q^*Fq_*q^*)F^!q_* \xrightarrow{\sim} q^*FF^!q_* \xrightarrow{q^*\epsilon_F q_*} q^*q_* \xrightarrow{\epsilon_q} Id_{C_X}. \qquad (3)$$

and

$$Id_{C_X} \xrightarrow{\sim} q^*q_* \xrightarrow{q^*\eta_F q_*} q^*F^!Fq_* \xrightarrow{q^*F^!\eta_q Fq_*} q^*F^!q_*q^*Fq_* \xrightarrow{\sim} \bar{F}^!\bar{F}. \qquad (4)$$

The compositions of the sequence of morphisms resp. (3) and (4) are adjunction arrows. ∎

7.1.2. Proposition. *Let $X \xrightarrow{q} Y$ be a localization and $\mathcal{F} = (F, \mu)$ a monad on Y such that the endofunctor F is compatible with q. Then the monad $\mathcal{F}$ induces a monad, $\bar{\mathcal{F}} = (\bar{F}, \bar{\mu})$, on X defined uniquely up to isomorphism.*

(i) If $\mathcal{F}$ is continuous (i.e. F has a right adjoint), then the monad $\bar{\mathcal{F}}$ is continuous.

(ii) If C_Y has colimits of certain type, then C_X has colimits of this type. If F preserves colimits of this type, then $\bar{F}$ has the same property.

(iii) If C_Y has limits of certain type, then C_X has limits of this type. If F and q^* preserve limits of this type, then $\bar{F}$ has the same property.

Proof. Fix an inverse image, q^*, of the localization q. Let $\bar{F}$ be a unique endofunctor $\bar{F} : C_X \longrightarrow C_X$ such that $q^* \circ F = \bar{F} \circ q^*$. Then $q^* \circ F^2 = \bar{F}^2 \circ q^*$, and, by the universal property of localizations, there exists a unique morphism $\bar{\mu} : \bar{F}^2 \longrightarrow \bar{F}$ such that $q^*\mu = \bar{\mu}q^*$. We leave to the reader verifying that $\bar{\mu}$ is a monad structure on $\bar{F}$.

The assertions (i), (ii), (iii) follow from the corresponding assertions of 7.1.1. ∎

7.1.2.1. Remark. The same assertion holds for comonads. In fact, the first part is obtained by dualization. The parts (i) and (ii) are statements about endofunctors.

7.2. Cones of non-unital monads and rings.

7.2.1. Non-unital monads. Let X be a 'space' such that C_X is an additive category, and let $\mathbb{F}_+ = (F_+, \mu)$ be a non-unital additive monad on X; i.e. F_+ is an additive functor $C_X \longrightarrow C_X$ and μ is a functor morphism $F_+^2 \longrightarrow F_+$ such that $\mu \circ F_+\mu = \mu \circ \mu F_+$. Let $\mathbb{F}_+ - mod_1$ denote the category of non-unital $\mathbb{F}_+$-modules. Its objects are pairs (M, ξ), where $M \in ObC_X$ and ξ a morphism $F_+(M) \longrightarrow M$ such that $\xi \circ \mu(M) = \xi \circ F_+\xi$. A morphism $(M, \xi) \longrightarrow (M', \xi')$ is given by a morphism $M \xrightarrow{f} M'$ such that $\xi' \circ F_+(f) = f \circ \xi$. Composition is defined naturally, so that the map which assigns to each $\mathbb{F}_+$-module

(M, ξ) the object M and to every $\mathbb{F}_+$-module morphism $(M, \xi) \xrightarrow{f} (M', \xi')$ the morphism $M \xrightarrow{f} M'$ is a functor, $\mathbb{F}_+ - mod_1 \xrightarrow{\mathfrak{f}_*} C_X$. This functor has a canonical left adjoint, $\mathfrak{f}^*$, which maps every object N of C_X to the $\mathbb{F}_+$-module $(N \oplus F(N), \xi_N)$, where the action $F_+(N \oplus F_+(N)) = F_+(N) \oplus F_+^2(N) \xrightarrow{\xi_N} N \oplus F_+(N)$ is the composition of the morphism

$$F_+(N) \oplus F_+^2(N) \xrightarrow{(id_{F_+(N)}, \mu(N))} F_+(N)$$

and the embedding $F_+(N) \longrightarrow N \oplus F_+(N)$.

Thus, $\mathfrak{f}_*\mathfrak{f}^* = Id_{C_X} \oplus F_+$. We denote $Id_{C_X} \oplus F_+$ by F and the monad corresponding to the pair of adjoint functors $\mathfrak{f}_*, \mathfrak{f}^*$ by $\mathbb{F} = (F, \mu_1)$. It is easy to see that the category $\mathbb{F}_+ - mod_1$ of non-unital $\mathbb{F}_+$-modules is isomorphic to the category $\mathbb{F} - mod$ of unital $\mathbb{F}$-modules. There is a natural embedding $C_X \longrightarrow \mathbb{F}_+ - mod$ which assigns to each object M of C_X the $\mathbb{F}_+$-module $(M, 0)$. We denote the image of C_X in $\mathbb{F}_+ - mod$ (i.e. the full subcategory generated by trivial modules) by $\mathcal{T}_{\mathbb{F}_+}$.

7.2.2. A reminder on Serre subcategories. Let $C_{\mathfrak{Z}}$ be an abelian category and $\mathcal{T}$ its subcategory. We denote by $\mathcal{T}^-$ the full subcategory of the category $C_{\mathfrak{Z}}$ generated by all objects of $C_{\mathfrak{Z}}$ whose nonzero subquotients have nonzero subobjects from $\mathcal{T}$. One can show that the subcategory $\mathcal{T}^-$ is *thick*, i.e. it is closed under taking arbitrary subquotients and extensions. A subcategory $\mathcal{T}$ is called a *Serre* subcategory if $\mathcal{T} = \mathcal{T}^-$. If $C_{\mathfrak{Z}}$ is a so-called (AB5) category (in particular, it has small coproducts), then Serre subcategories are precisely thick subcategories closed under small coproducts. If $C_{\mathfrak{Z}}$ is a Grothendieck category (i.e. an (AB5) category with generators), then Serre subcategories are those thick subcategories $\mathcal{T}$ for which the localization functor $C_{\mathfrak{Z}} \longrightarrow C_{\mathfrak{Z}}/\mathcal{T}$ has a right adjoint.

7.2.3. The cone of a non-unital monad. Suppose that C_X is an abelian category. We denote by $C_{\mathbf{Cone}(\mathbb{F}_+/X)}$ the quotient, $\mathbb{F}_+ - mod_1/\mathcal{T}_{\mathbb{F}_+}^-$, of the category $\mathbb{F}_+ - mod_1$ by the smallest Serre subcategory containing $\mathcal{T}_{\mathbb{F}_+}$. This defines a 'space' $\mathbf{Cone}(\mathbb{F}_+/X)$.

7.2.3.1. Proposition. *If $\mathbb{F}_+$ is a unital monad, then $C_{\mathbf{Cone}(\mathbb{F}_+/X)}$ is naturally equivalent to the category $\mathbb{F}_+ - mod$ of unital $\mathbb{F}_+$-modules, i.e. the 'space' $\mathbf{Cone}(\mathbb{F}_+/X)$ is isomorphic to $\mathbf{Sp}(\mathbb{F}_+/X)$.*

Proof. If the monad $\mathbb{F}_+ = (F_+, \mu)$ is unital with the unit element $Id \xrightarrow{e} F_+$, then there is a monad epimorphism $F = Id \oplus F_+ \xrightarrow{\gamma} F_+$ defined by (e, id_{F_+}). The corresponding pull-back functor, γ_*, is the inclusion functor $\mathbb{F}_+ - mod$ into $\mathbb{F}_+ - mod_1$. Its left adjoint, γ^* assigns to each object (M, ξ) of $\mathbb{F}_+ - mod_1$ the cokernel of the pair of morphisms $M \underset{\xi e(M)}{\overset{id_M}{\rightrightarrows}} M$. Since the functor γ_* is fully faithful, its left adjoint γ^* is an exact localization, and the kernel of γ^* coincides with the subcategory $\mathcal{T}_{\mathbb{F}_+}$. Therefore, $\mathcal{T}_{\mathbb{F}_+}$ is, in this case, a Serre subcategory, i.e. $\mathcal{T}_{\mathbb{F}_+} = \mathcal{T}_{\mathbb{F}_+}^-$, whence the assertion. ∎

7.2.3.2. Corollary. *If* $\mathbb{F}_+$ *is a unital monad, then* $\mathbf{Sp}(\mathbb{F}/X) \simeq \mathbf{Sp}(\mathbb{F}_+/X) \coprod X$.

Proof. If $\mathbb{F}_+$ is a unital monad, then (by the argument of 7.2.3.1) $\mathbb{F} \simeq \mathbb{F}_+ \prod Id_X$, where Id_X denotes the identical monad (Id_{C_X}, id). This implies that the category $\mathbb{F} - mod$ of $\mathbb{F}$-modules is equivalent to the product $(\mathbb{F}_+ - mod) \prod C_X$, hence the assertion. ∎

7.3. The cone of an associative ring. Let $X = \mathbf{Sp}(R)$, where R_0 is a unital associative ring, and let R_+ be an R_0*-ring.* The latter means that R_0 is an associative ring, not unital in general, in the category of R_0-bimodules; i.e. the mutiplication in R_+ is given by an R_0-bimodule morphism $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$ satisfying the associativity condition. The R_0-ring R_+ defines a non-unital monad $\mathbb{F}_+ = (F_+, \mu)$ on X, where F_+ is the endofunctor $R_+ \otimes_{R_0} -$ on $C_X = R_0 - mod$ and $F_+^2 \xrightarrow{\mu} F_+$ is induced by the multiplication $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$. The category $\mathbb{F}_+ - mod_1$ of non-unital $\mathbb{F}_+$-modules is the category of unital R_0-modules endowed with a non-unital R_+-module structure compatible with the action of R_0 on the module and on R_+. We write $R_+ - mod_1$ instead of $\mathbb{F}_+ - mod_1$ and $\mathcal{T}_{R_+}$ instead of $\mathcal{T}_{\mathbb{F}_+}$. By definition $\mathcal{T}_{\mathbb{F}_+}$ is the full subcategory of $R_+ - mod_1$ spanned by modules with zero action.

The associated *augmented* monad $\mathbb{F}$ (cf. 7.2.1) is isomorphic to the monad associated with the unital R_0-ring $R = R_0 \oplus R_+$ which we call the *augmented* R_0*-ring* corresponding to R_+. The category $R_+ - mod_1$ is isomorphic to the category $R - mod$ of unital R-modules.

We shall write $\mathbf{Cone}(R_+/R_0)$, or simply $\mathbf{Cone}(R_+)$, instead of $\mathbf{Cone}(\mathbb{F}_+/\mathbf{Sp}(R_0))$.

The category $R_+ - mod_1$ will be identified with $R - mod$ whenever it is convenient. Thus, $\mathcal{T}_{R_+}$ is viewed as the full subcategory of $R - mod$ whose objects are modules annihilated by the *irrelevant* ideal R_+; and we write $C_{\mathbf{Cone}(R_+)} = R - mod/\mathcal{T}_{R_+}^-$, where $\mathcal{T}_{R_+}^-$ is the smallest Serre subcategory of the category $R - mod$ containing $\mathcal{T}_{R_+}$ (getting back the definition of a cone in 1.3). The localization functor

$$R - mod \xrightarrow{u^*} R - mod/\mathcal{T}_{R_+}^-$$

is an inverse image functor of a morphism of 'spaces' $\mathbf{Cone}(R_+) \xrightarrow{u} \mathbf{Sp}(R)$. The functor u^* has a (necessarily fully faithful) right adjoint, i.e. the morphism u is continuous. The composition of the morphism u with the natural affine morphism $\mathbf{Sp}(R) \longrightarrow \mathbf{Sp}(R_0)$ is a continuous morphism $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R_0)$. Its direct image functor is (regarded as) the *global sections functor.*

7.3.1. Proposition. *If* R_+ *is a unital ring, then* $\mathbf{Cone}(R_+/R_0) \simeq \mathbf{Sp}(R_+)$ *and* $\mathbf{Sp}(R) \simeq \mathbf{Sp}(R_+) \coprod \mathbf{Sp}(R_0)$.

Proof. The assertion follows from 7.2.3.1 and 7.2.3.2. ∎

7.3.2. Lemma. *Let $\mathcal{J}$ be a two-sided ideal in the ring R contained in R_+ (i.e. a two-sided ideal in R_+ which is an R_0-bimodule). Let $T_{R_+|\mathcal{J}}$ denote the full subcategory of $R-mod$ whose objects are R-modules annihilated by $\mathcal{J}$; and let $T^-_{R|\mathcal{J}}$ be the Serre subcategory spanned by $T_{R|\mathcal{J}}$. The quotient category $R-mod/T^-_{R|\mathcal{J}}$ is equivalent to $C_{\mathbf{Cone}(\mathcal{J})}$.*

Proof. The embedding $\mathcal{J} \hookrightarrow R$ induces a unital ring morphism $\widetilde{\mathcal{J}} \stackrel{\iota}{\longrightarrow} R$, where $\widetilde{\mathcal{J}}$ is the ring $R_0 \oplus \mathcal{J}$ with natural multiplication. The pull-back functor $R-mod \stackrel{\iota_*}{\longrightarrow} \widetilde{\mathcal{J}}-mod$ induces a functor from the subcategory $T_{R|\mathcal{J}}$ to the subcategory $T_{\mathcal{J}}$. Since the functor ι_* is exact (in a strong sense, that is it preserves small limits and colimits), it maps the Serre subcategory $T^-_{R|\mathcal{J}}$ to the Serre subcategory $T^-_{\mathcal{J}}$. Thus we have a commutative diagram

$$\begin{array}{ccc} R-mod & \stackrel{\iota_*}{\longrightarrow} & \widetilde{\mathcal{J}}-mod \\ \uparrow & & \uparrow \\ T^-_{R|\mathcal{J}} & \longrightarrow & T^-_{\mathcal{J}} \end{array} \qquad (1)$$

of exact functors. Therefore the functor ι_* induces a functor

$$R-mod/T^-_{R|\mathcal{J}} \longrightarrow \widetilde{\mathcal{J}}-mod/T^-_{\mathcal{J}}. \qquad (2)$$

The functor (2) is a category equivalence. In fact, let $\widetilde{\mathcal{J}}-mod \stackrel{\Psi}{\longrightarrow} R-mod$ be the functor which assigns to every $\widetilde{\mathcal{J}}$-module M the R-module $\mathcal{J}M$. The cokernel of the embedding $\mathcal{J}M \hookrightarrow M$ belongs to $T_{\mathcal{J}}$, hence the localization $\widetilde{\mathcal{J}}-mod \longrightarrow \widetilde{\mathcal{J}}-mod/T^-_{\mathcal{J}}$ maps this embedding to an isomorphism. We assign to each object M of $\widetilde{\mathcal{J}}-mod$ the composition of the functor Ψ and the localization $R-mod \longrightarrow R-mod/T^-_{R|\mathcal{J}}$. It follows that this functor factors through the localization $\widetilde{\mathcal{J}}-mod \longrightarrow \widetilde{\mathcal{J}}-mod/T^-_{\mathcal{J}}$, i.e. it defines (uniquely) a functor $\widetilde{\mathcal{J}}-mod/T^-_{\mathcal{J}} \stackrel{\Phi}{\longrightarrow} R-mod/T^-_{R|\mathcal{J}}$. The functor Φ is a quasi-inverse to the functor (2). ∎

7.3.3. Example: quasi-affine schemes. Quasi-affine schemes are defined (in [EGA II, 5.1.1]) as open *quasi-compact* subschemes of affine schemes. Open subschemes of $\mathbf{Spec}A$ are in bijective correspondence with the radical ideals in A. Quasi-compactness of an open set defined by an ideal $\mathcal{J}$ means that $\mathcal{J}$ is the radical of its finitely generated subideal (this holds in noncommutative case too, see [R, I.5.6]). One can show that the category of quasi-coherent sheaves on the open subscheme of $\mathbf{Spec}A$ defined by the ideal $\mathcal{J}$ is equivalent to the quotient category $A-mod/T^-_{A|\mathcal{J}}$. By 7.3.2, the latter category is equivalent to the category $C_{\mathbf{Cone}(\mathcal{J})} = \widetilde{\mathcal{J}} - mod/T^-_{\mathcal{J}}$ of modules on the cone $\mathbf{Cone}(\mathcal{J})$ of the (non-unital) R_0-ring $\mathcal{J}$.

7.3.4. Functoriality. Let $R_0 - Rings$ denote the category of (not necessarily unital) R_0-rings. A morphism of such rings, $R_+ \xrightarrow{\phi} S_+$, is an R_0-bimodule morphism compatible with multiplication. The morphism ϕ induces the pull-back functor

$$S_+ - mod_1 \xrightarrow{\phi_*} R_+ - mod_1$$

which maps the subcategory T_{S_+} of trivial S_+-modules to the category T_{R_+} of trivial R_+-modules. Since ϕ_* is exact and preserves small colimits, it maps the Serre subcategory $T^-_{S_+}$ spanned by T_{S_+} to the Serre subcategory $T^-_{R_+}$ spanned by T_{R_+}. Therefore, ϕ_* induces a unique functor

$$S_+ - mod_1/T^-_{S_+} \xrightarrow{\bar{\phi}_*} R_+ - mod_1/T^-_{R_+} \tag{3}$$

such that the diagram

$$\begin{array}{ccc} S_+ - mod_1/T^-_{S_+} & \xrightarrow{\bar{\phi}_*} & R_+ - mod_1/T^-_{R_+} \\ q^*_S \uparrow & & \uparrow q^*_R \\ S_+ - mod_1 & \xrightarrow{\phi_*} & R_+ - mod_1 \end{array} \tag{4}$$

commutes. In general, the functor $\bar{\phi}_*$ does not have a left adjoint, hence it cannot be interpreted as a direct image functor of a continuous morphism.

7.3.4.1. The category $R_0 - Rings_1$. We denote by $R_0 - Rings_1$ the subcategory of $R_0 - Rings$ formed by R_0-ring morphisms $R_+ \xrightarrow{\phi} S_+$ whose inverse image functor, $R_+ - mod_1 \xrightarrow{\phi^*} S_+ - mod_1$, is *compatible* with the localizations at resp. $T^-_{R_+}$ and $T^-_{S_+}$. The compatibility means that there exists a functor $C_{\mathbf{Cone}(R_+)} \xrightarrow{\bar{\phi}^*} C_{\mathbf{Cone}(S_+)}$ such that the diagram

$$\begin{array}{ccc} C_{\mathbf{Cone}(R_+)} = R_+ - mod_1/T^-_{R_+} & \xrightarrow{\bar{\phi}^*} & S_+ - mod_1/T^-_{S_+} = C_{\mathbf{Cone}(S_+)} \\ q^*_R \uparrow & & \uparrow q^*_S \\ R_+ - mod_1 & \xrightarrow{\phi^*} & S_+ - mod_+ \end{array} \tag{5}$$

commutes. Thanks to the universal property of localizations, the functor $\bar{\phi}$ is uniquely determined by the commutativity of (5).

Evidently, all ring isomorphisms belong to $R_0 - Rings_1$. It follows from the universal property of localizations that the composition of morphisms of $R_0 - Rings_1$ belongs to $R_0 - Rings_1$; i.e. $R_0 - Rings_1$ is, indeed, a subcategory of the category $R_0 - Rings$. The map $R_+ \longmapsto \mathbf{Cone}(R_+)$ extends to a functor $R_0 - Rings_1^{op} \longrightarrow |Cat|^o$ which we denote by **Cone**.

7.3.4.2. Remarks. (a) For any morphism $R_+ \xrightarrow{\varphi} S_+$ of $R_0 - Rings$, the functor

$$\bar{\varphi}^* = q_S^* \varphi_1^* q_{R*} : C_{\mathbf{Cone}(R_+)} \longrightarrow C_{\mathbf{Cone}(S_+)} \qquad (6)$$

might be regarded as an inverse image functor of a morphism $\mathbf{Cone}(S_+) \xrightarrow{\bar{\varphi}} \mathbf{Cone}(R_+)$.

Notice, however, that the map $\varphi \longmapsto \bar{\varphi}$ is not functorial, unless morphisms are picked from the subcategory $R_0 - Rings_1$.

(b) For any morphism $R_+ \xrightarrow{\varphi} S_+$ of $R_0 - Rings_1$, the corresponding morphism $\mathbf{Cone}(S_+) \xrightarrow{\bar{\varphi}} \mathbf{Cone}(R_+)$ is continuous.

This follows from the fact that the functor $C_{\mathbf{Cone}(R_+)} \xrightarrow{q_{R*}} R_+ - mod_1$ has a right adjoint and from the formula (6).

7.3.4.3. Proposition. *Let S_+ be an R_0-ring, e a central idempotent element in S_+ (i.e. $e^2 = e$). Then $R_+ = \{r \in S \mid re = er = r\}$ is an R_0-subring in S_+, and the inclusion $R \hookrightarrow S$ is a morphism of $R_0 - Rings_1$.*

Proof is left to the reader. ■

7.3.5. Remark. For any R_0-ring R_+, we have a canonical morphism (Zariski open immersion) $\mathbf{Cone}(R_+) \longrightarrow \mathbf{Sp}(R)$, $R = R_0 \oplus R_+$, which depends functorially on R_+ (a functor from $R_0 - Rings_1^{op}$). This morphism can be regarded as a noncommutative analogue of the Stone compactification of a locally compact space. If the ring R_+ is unital, then $\mathbf{Sp}(R)$ is the disjoint union of $\mathbf{Sp}(R_+)$ and $\mathbf{Sp}(R_0)$ (see 7.3).

7.3.6. Hopf actions and cross-products. Let R_0 be an associative unital k-algebra. We call an R_0-ring R_+ an $(R_0|k)$*-ring* if the R_0-ring structure makes R_+ a k-algebra, i.e. $\lambda r = r\lambda$ for all $r \in R_+$ and $\lambda \in k$. Let $\mathcal{H} = (\delta, H, \mu)$ be a k-bialgebra. Here $H \xrightarrow{\delta} H \otimes_k H \xrightarrow{\mu} H$ are resp. comultiplication and multiplication. Recall that a Hopf action of $\mathcal{H}$ on a k-algebra R_+ is a unital $\mathcal{H}$-module structure on R such that the multiplication $R_+ \otimes_k R_+ \longrightarrow R_+$ is an $\mathcal{H}$-module morphism. We assume that $\mathcal{H}$ acts trivially on R_0. Then the cross-product $R_+ \# \mathcal{H}$ is an $(R_0|k)$-ring.

The Hopf action of $\mathcal{H}$ on R_+ induces an endofunctor, $\widetilde{H}$, on the category $R_+ - mod_1$. This endofunctor assigns to any (non-unital) R_+-module $\mathcal{M} = (M, R_+ \otimes_k M \xrightarrow{\xi} M)$ the R_+-module $\mathcal{H} \otimes_k \mathcal{M} = (H \otimes_k M, \xi_{\mathcal{H}})$, where the action $\xi_{\mathcal{H}}$ is the composition of

$$R_+ \otimes_k H \otimes_k M \xrightarrow{\sim} H \otimes_k R_+ \otimes_k M \longrightarrow H \otimes_k H \otimes_k R_+ \otimes_k M \longrightarrow H \otimes_k R_+ \otimes_k M \longrightarrow H \otimes_k M.$$

Here the second arrow is induced by the comultiplication δ, the third arrow by the action τ, and the fourth arrow by the R-module structure ξ. The multiplication $H \otimes_k H \xrightarrow{\mu} H$ induces a monad structure, $\widetilde{H}^2 \xrightarrow{\widetilde{\mu}} \widetilde{H}$, on $\widetilde{H}$. One can see that the category $R_+\#\mathcal{H}-mod_1$ is isomorphic to the category $\widetilde{\mathcal{H}} - mod$, where $\widetilde{\mathcal{H}}$ denotes the monad $(\widetilde{H}, \widetilde{\mu})$. This follows from the observation that the functor $\widetilde{H}$ is isomorphic to $R\#\mathcal{H} \otimes_R -$, where $R = R_0 \oplus R_+$ is the augmented R_0-ring. This observation implies, on the other hand, that the functor $\widetilde{H}$ is continuous (i.e. it has a right adjoint) and that there is a natural isomorphism between the category $R_+\#\mathcal{H}-mod_1$ and the category $(\widetilde{\mathcal{H}}/R)-mod$ of modules over the monad $\widetilde{\mathcal{H}}$. Here we write $(\widetilde{\mathcal{H}}/R)$ instead of $(\widetilde{\mathcal{H}}/\mathbf{Sp}(R))$ and identify $R_+ - mod$ with $R - mod$. Thus, we have a natural isomorphism $\mathbf{Sp}(\widetilde{\mathcal{H}}/R) \xrightarrow{\sim} \mathbf{Sp}(R\#\mathcal{H})$ such that the diagram

$$\begin{array}{ccc} \mathbf{Sp}(\widetilde{\mathcal{H}}/R) & \xrightarrow{\sim} & \mathbf{Sp}(R\#\mathcal{H}) \\ & \searrow \quad \swarrow & \\ & \mathbf{Sp}(R) & \end{array} \tag{7}$$

commutes.

The following assertion provides another family of morphisms of $R_0 - Rings_1$.

7.3.6.1. Proposition. *Let $H \otimes_k R_+ \xrightarrow{\tau} R_+$ be a Hopf action of an k-bialgebra $\mathcal{H} = (\delta, H, \mu)$ on a $(R_0|k)$-ring R_+. Suppose the functor $H \otimes_k -$ is flat. Then the monad $\widetilde{\mathcal{H}}$ on $\mathbf{Sp}(R)$ induces a monad $\bar{\mathcal{H}}$ on $\mathbf{Cone}(R_+)$ such that there is a canonical commutative diagram*

$$\begin{array}{ccc} \mathbf{Sp}(\bar{\mathcal{H}}/\mathbf{Cone}(R_+)) & \xrightarrow{\sim} & \mathbf{Cone}(R_+\#\mathcal{H}) \\ & \searrow \quad \swarrow & \\ & \mathbf{Cone}(R_+) & \end{array} \tag{8}$$

of affine morphisms.

In particular, the canonical morphism $R_+ \longrightarrow R_+\#\mathcal{H}$ belongs to $R_0 - Rings_1$.

Proof. It follows that the functor $\widetilde{H}$ maps the subcategory T_{R_+} to itself. Since the functor $H \otimes_k - : k - mod \longrightarrow k - mod$ is flat (i.e. it is exact and preserves colimits of small diagrams), the functor $R_+ - mod_1 \xrightarrow{\widetilde{H}} R_+ - mod_1$ is flat too. Therefore, the Serre subcategory $T^-_{R_+}$ is stable under $\widetilde{H}$, and the functor $\widetilde{H}$ induces a continuous functor $R_+ - mod_1/T^-_{R_+} = C_{\mathbf{Cone}(R_+)} \xrightarrow{\bar{H}} C_{\mathbf{Cone}(R_+)}$. By 7.1.2, the multiplication $\widetilde{H}^2 \xrightarrow{\widetilde{\mu}} \widetilde{H}$ induces a multiplication $\bar{H}^2 \xrightarrow{\bar{\mu}} \bar{H}$. The isomorphism of categories

$$R_+\#\mathcal{H} - mod_1 \xrightarrow{\sim} (\widetilde{\mathcal{H}}/R) - mod_1$$

mentioned above induces an isomorphism $C_{\mathbf{Cone}(R\#\mathcal{H})} \xrightarrow{\sim} (\bar{\mathcal{H}}/\mathbf{Cone}(R)){-}mod$, regarded as an inverse image functor of an isomorphism $\mathbf{Sp}(\bar{\mathcal{H}}/\mathbf{Cone}(R)) \xrightarrow{\sim} \mathbf{Cone}(R\#\mathcal{H})$ such that the diagram (8) commutes. The monad $(\widetilde{\mathcal{H}}/R)$ is continuous (i.e. the functor $\widetilde{H}$ has a right adjoint), because $\widetilde{H}$ is isomorphic to the (obviously) continuous functor $R\#\mathcal{H}\otimes_R -$. By 7.1.2(i), this implies that the monad $\bar{\mathcal{H}}$ on $\mathbf{Cone}(R)$ is continuous. By 5.2, the latter means precisely that the natural morphism $\mathbf{Sp}(\bar{\mathcal{H}}/\mathbf{Cone}(R)) \longrightarrow \mathbf{Cone}(R)$ is affine. Therefore, by the commutativity of (8), the morphism $\mathbf{Cone}(R\#\mathcal{H}) \longrightarrow \mathbf{Cone}(R)$ is affine. This shows, in particular, that the canonical morphism $R_+ \longrightarrow R_+\#\mathcal{H}$ belongs to $R_0 - Rings_1$. ■

7.4. Noncommutative projective spectra.

7.4.1. Proj$_{\mathcal{G}}$. Fix a monoid $\mathcal{G}$. Let $\mathbb{F}_+ = (F_+, \mu)$ be a $\mathcal{G}$-graded (non-unital in general) monad on X. Let $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$ denote the category of $\mathcal{G}$-graded non-unital $\mathbb{F}_+$-modules and preserving gradings morphisms. Let

$$gr_{\mathcal{G}}\mathbb{F}_+ - mod_1 \xrightarrow{\pi^*} \mathbb{F}_+ - mod_1 \qquad (1)$$

be the functor forgetting the grading. We denote by $gr_{\mathcal{G}}\mathcal{T}_{\mathbb{F}_+}$ the preimage of the subcategory $\mathcal{T}_{\mathbb{F}_+}$ in $gr_{\mathcal{G}}\mathbb{F}_+{-}mod_1$. Let $C_{\mathbf{Proj}_{\mathcal{G}}}(\mathbb{F}_+)$ be the quotient category $gr_{\mathcal{G}}\mathbb{F}_+{-}mod_1/gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+}$. This defines a 'space' $\mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+) = \mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+/X)$.

7.4.2. Actions. Let $\mathcal{G}$ be a monoid. An action of $\mathcal{G}$ on a 'space' X is a monoidal functor $\mathcal{G} \xrightarrow{\mathfrak{L}} \widetilde{End}(C_X)$. Here $\mathcal{G}$ is viewed as a discrete monoidal category and $\widetilde{End}(C_X)$ denote the (strict) monoidal category of endofunctors $C_X \longrightarrow C_X$; i.e. $\widetilde{End}(C_X) = (End(C_X), \circ)$.

7.4.2.1. Examples. (a) Let $\mathbb{F}_+$ be a (non-unital in general) $\mathcal{G}$-graded monad on a 'space' X; and let $C_Y = gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$. For any $\mathcal{G}$-graded $\mathbb{F}_+$-module $N = \oplus_{\nu\in\mathcal{G}}N_\nu$ and any $\gamma \in \mathcal{G}$, we denote by $N[\gamma]$ the $\mathcal{G}$-graded $\mathbb{F}_+$-module defined by $N[\gamma]_\sigma = N_{\sigma\gamma}$. This defines a *strict* action of $\mathcal{G}$ on the 'space' Y. Here *strict* means that the monoidal functor $\mathcal{G} \xrightarrow{\mathfrak{L}} \widetilde{End}(C_X)$ is strict, that is $N[\gamma_1\gamma_2] = (N[\gamma_2])[\gamma_1]$ for all N.

(b) The action of $\mathcal{G}$ on the 'space' Y in (a) (i.e. on the category $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$), induces an action of $\mathcal{G}$ on $\mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+)$.

Recall that a full subcategory $\mathbb{T}$ of an abelian category is called *topologizing* if it is closed under taking subquotients and finite coproducts.

7.4.3. Proposition. *Let $\mathbb{T}$ be a $\mathcal{G}$-stable, topologizing subcategory of $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$, and let $\widetilde{\mathbb{T}}$ denote the image of $\mathbb{T}$ in $\mathbb{F}_+ - mod_1$. Then ${\pi^*}^{-1}(\widetilde{\mathbb{T}}^-) \subseteq \mathbb{T}^-$.*

If the 'space' X has the property (sup) and the functor F_+ preserves supremums of subobjects, then $\pi^{*^{-1}}(\widetilde{\mathbb{T}}^-) = \mathbb{T}^-$.

Proof. (a) Since the functor (1) is exact, the preimage, $\pi^{*^{-1}}(\widetilde{\mathbb{T}}^-)$, of the Serre subcategory $\widetilde{\mathbb{T}}^-$ is a Serre subcategory of the category $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$. The inclusion $\pi^{*^{-1}}(\widetilde{\mathbb{T}}^-) \subseteq \mathbb{T}^-$ is equivalent to that every nonzero object of $\pi^{*^{-1}}(\widetilde{\mathbb{T}}^-)$ has a nonzero subobject which belongs to $\mathbb{T}$; or, what is the same, for any nonzero object, N, of $\pi^{*^{-1}}(\widetilde{\mathbb{T}}^-)$, there exists a nonzero morphism $L \xrightarrow{g} N$, with $L \in Ob\mathbb{T}$. We can and will assume that L is generated by one of its homogeneous components. Then $\mathbb{F}_+ - mod_1(L, N)$ is a $\mathcal{G}$-graded $\mathbb{Z}$-module, and some of homogeneous components of the morphism g are nonzero. Replacing the module L by the module $L[\gamma]$ for an appropriate $\gamma \in \mathcal{G}$, we can assume that the homogeneous component of g of zero degree is nonzero. Thus, there exists a nonzero morphism $L[\gamma] \longrightarrow N$ of graded $\mathbb{F}_+$-modules. Since the subcategory $\mathbb{T}$ is stable under the action of $\mathcal{G}$, the object $L[\gamma]$ belongs to $\mathbb{T}$.

(b) Since X has the property (sup) and F preserves supremums, both categories, $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1$ and $\mathbb{F}_+ - mod_1$ possess this property too. Therefore, every object, M, of $\mathbb{T}^-$ has a filtration, $\{M_i \mid i \geq 0\}$ such that $M_i = sup(M_j | j < i)$, if i is a limit ordinal, and M_{i+1}/M_i belongs to $\mathbb{T}$. But, this implies that M is an object of $\widetilde{\mathbb{T}}^-$; i.e. we have the inverse inclusion, $\mathbb{T}^- \subseteq \pi^{*^{-1}}(\widetilde{\mathbb{T}}^-)$. ■

7.4.3.1. Corollary. *(a) $\pi^{*^{-1}}(\mathcal{T}^-_{\mathbb{F}_+}) \subseteq gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+}$.*

(b) If the 'space' X has the property (sup) and the functor F_+ preserves supremums of subobjects, then $gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+} = \pi^{*^{-1}}(\mathcal{T}^-_{\mathbb{F}_+})$.

Proof. Set $\mathbb{T} = gr_{\mathcal{G}}\mathcal{T}_{\mathbb{F}_+}$. Then $\widetilde{\mathbb{T}}^-$ coincides with $\mathcal{T}^-_{\mathbb{F}_+}$, hence the assertion. ■

7.4.3.2. Corollary. *Suppose that X has the property (sup) and F_+ preserves supremums of subobjects. Then the forgetful functor (1) induces a faithful exact functor*

$$C_{\mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+)} \xrightarrow{\mathfrak{p}^*} C_{\mathbf{Cone}(\mathbb{F}_+)} \qquad (2)$$

Proof. By 7.4.3.1(b), $gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+} = \pi^{*^{-1}}(\mathcal{T}^-_{\mathbb{F}_+})$, where $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1 \xrightarrow{\pi^*} \mathbb{F}_+ - mod_1$ is the forgetful functor. The functor $\pi^{*^{-1}}$ induces a faithful functor between quotient categories

$$gr_{\mathcal{G}}\mathbb{F}_+ - mod_1/gr_{\mathcal{G}}\mathcal{T}^-_{\mathbb{F}_+} \longrightarrow \mathbb{F}_+ - mod_1/\mathcal{T}^-_{\mathbb{F}_+}.$$

This functor is exact because the inclusion functor $gr_{\mathcal{G}}\mathbb{F}_+ - mod_1 \longrightarrow \mathbb{F}_+ - mod_1$ is exact. Hence the assertion. ■

The functor (2) is regarded as an inverse image functor of a morphism ('projection') $\mathbf{Cone}(\mathbb{F}_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+)$.

7.4.3.3. The Proj of an associative ring. Let R_0 be an associative unital ring and $\mathcal{G}$ a monoid. Let R_+ be a $\mathcal{G}$-graded R_0-ring, which is, by definition, a $\mathcal{G}$-graded ring in the category of R_0-bimodules. Then we have the category $gr_{\mathcal{G}}R_+ - mod_1$ of $\mathcal{G}$-graded R_+-modules and its subcategory $gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R_+ - mod_1$. We obtain the 'space' $\mathbf{Proj}_{\mathcal{G}}(R_+) = \mathbf{Proj}_{\mathcal{G}}(R)$ defined by

$$C_{\mathbf{Proj}_{\mathcal{G}}(R_+)} = gr_{\mathcal{G}}R_+ - mod_1/gr_{\mathcal{G}}\mathcal{T}^-_{R_+}.$$

Since the conditions of 7.4.3.1(b) hold, $gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = gr_{\mathcal{G}}R_+ - mod_1 \cap \mathcal{T}^-_{R_+}$, and, therefore, we have a canonical projection

$$\mathbf{Cone}(R_+) \xrightarrow{\mathfrak{p}} \mathbf{Proj}_{\mathcal{G}}(R_+).$$

Taking $X = \mathbf{Sp}(R_0)$ (i.e. $C_X = R_0 - mod$), we can identify R_+ with the monad $\mathbb{F}_+ = (F_+, \mu)$, where $F_+ = R_+ \otimes_{R_0} -$ and $F_+^2 \xrightarrow{\mu} F_+$ is determined by the multiplication $R_+ \otimes_{R_0} R_+ \xrightarrow{m} R_+$. If R_+ is $\mathcal{G}$-graded, then the monad $\mathbb{F}_+$ is $\mathcal{G}$-graded. There are natural isomorphisms $\mathbf{Cone}(R_+) \simeq \mathbf{Cone}(\mathbb{F}_+)$ and $\mathbf{Proj}_{\mathcal{G}}(R_+) \simeq \mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+)$.

We have recovered the construction 1.4 illustrated by examples 1.5, 1.6, and 1.7. One can approach to these examples from a different side, via Hopf actions.

7.5. Hopf actions. Let $\mathcal{G}$ be a monoid and R_0 an associative unital k-algebra. For a $\mathcal{G}$-graded $(R_0|k)$-ring R_+, we denote by $gr_{\mathcal{G}}R_+ - mod_1$ the category of non-unital $\mathcal{G}$-graded R_+-modules. Let $\mathcal{H} = (\delta, H, \mu)$ be a $\mathcal{G}$-graded k-bialgebra with comultiplication δ and multiplication μ; and let $\mathcal{H} \otimes_k R_+ \xrightarrow{\tau} R_+$ is a Hopf action compatible with grading. Recall that a Hopf action of $\mathcal{H}$ on a k-algebra R_+ is a unital $\mathcal{H}$-module structure on R such that the multiplication $R_+ \otimes_k R_+ \longrightarrow R_+$ is an $\mathcal{H}$-module morphism. We assume that $\mathcal{H}$ acts trivially on R_0. Then the cross-product $R_+ \# \mathcal{H}$ is a $\mathcal{G}$-graded $(R_0|k)$-ring.

The Hopf action of $\mathcal{H}$ on R_+ induces an endofunctor, $H_{\mathcal{G}}$, on the category $gr_{\mathcal{G}}R_+ - mod_1$ which assigns to any (non-unital) $\mathcal{G}$-graded R_+-module $\mathcal{M} = (M, R_+ \otimes_k M \xrightarrow{\xi} M)$ the $\mathcal{G}$-graded R_+-module $\mathcal{H} \otimes_k \mathcal{M} = (H \otimes_k M, \xi_{\mathcal{H}})$, where the action $\xi_{\mathcal{H}}$ is same as in the non-graded case (cf. 7.3.6). The multiplication $H \otimes_k H \xrightarrow{\mu} H$ gives rise to a monad $\mathcal{H}_{\mathcal{G}} = (H_{\mathcal{G}}, \mu_{\mathcal{G}})$ (like in 7.3.6); and the category $gr_{\mathcal{G}}R_+ \# \mathcal{H} - mod_1$ is isomorphic to the category $\mathcal{H}_{\mathcal{G}} - mod$. By an argument similar to that of 7.3.6, the monad $\mathcal{H}_{\mathcal{G}}$ is continuous (i.e. the functor $H_{\mathcal{G}}$ has a right adjoint) which is equivalent to that the forgetful functor $\mathcal{H}_{\mathcal{G}} - mod \longrightarrow gr_{\mathcal{G}}R_+ - mod_1$ is a direct image functor of an affine morphism.

7.5.1. Proposition. *Let $H \otimes_k R_+ \xrightarrow{\tau} R_+$ be a Hopf action of an k-bialgebra $\mathcal{H} = (\delta, H, \mu)$ on a $\mathcal{G}$-graded $(R_0|k)$-ring R_+. Suppose the functor $H \otimes_k -$ is flat. Then the monad $\widetilde{\mathcal{H}}$ on $\mathbf{Sp}(R)$ induces a monad $\bar{\mathcal{H}}$ on $\mathbf{Proj}_{\mathcal{G}}(R_+)$ such that there is a canonical commutative diagram*

$$\begin{array}{ccc} \mathbf{Sp}(\bar{\mathcal{H}}/\mathbf{Proj}_{\mathcal{G}}(R_+)) & \xrightarrow{\sim} & \mathbf{Proj}_{\mathcal{G}}(R_+\#\mathcal{H}) \\ & \searrow \quad \swarrow & \\ & \mathbf{Proj}_{\mathcal{G}}(R_+) & \end{array} \tag{8}$$

of affine morphisms.

Proof. The argument is similar to that of 7.3.6.1. Details are left to the reader. ■

7.5.2. Example. Let G be a connected reductive algebraic group over an algebraically closed field k of zero characteristic. Fix a Borel subgroup B, a maximal unipotent subgroup U, and a maximal torus H chosen in a compatible way: H and U are subgroups of B, and $B = HU$). Let R be the algebra of regular functions on the homogeneous space G/U (called after I. M. Gelfand the 'base affine space'). The algebra R is the direct sum of all simple finite dimensional modules, each appears once; i.e. $R = \oplus_{\lambda \geq 0} R_\lambda$, where λ runs through nonnegative integral weights. Then $R_0 = k$, and $R_+ = \oplus_{\lambda > 0} R_\lambda$ is a $\mathcal{G}$-graded k-algebra. Here $\mathcal{G}$ is the group of integral weights of the group G.

The category $C_{\mathbf{Cone}(R_+)}$ is equivalent to the category of quasi-coherent sheaves on the base affine space G/U. The category $C_{\mathbf{Proj}_{\mathcal{G}}(R_+)}$ is equivalent to the category of quasi-coherent sheaves on the flag variety G/B. We refer for details to [LR4].

7.5.2.1. Note. If the group G is simply connected, this construction can be given in terms of the Lie algebra $\mathfrak{g}$ of G and its Cartan subalgebra $\mathfrak{h}$, as it is done in 1.6.

7.5.2.2. D-modules. By construction, there is a Hopf action on R of the universal enveloping (Hopf) algebra $U(\mathfrak{g})$. Consider instead of R the crossed product $R_+\#U(\mathfrak{g})$.

The universal enveloping algebra, $\mathcal{H}$, of the Cartan subalgebra, $\mathfrak{h}$, acts on the algebra R according the decomposition $R = \oplus_{\lambda \geq 0} R_\lambda$: each R_λ is a one-dimensional representation of $\mathcal{H}$ with the weight λ tensored by the vector space R_λ. This is a Hopf action commuting with the action of $U(\mathfrak{g})$, hence it determines to a Hopf action of $\widetilde{U}(\mathfrak{g}) = U(\mathfrak{g}) \otimes_k \mathcal{H}$ on R_+.

The category $C_{\mathbf{Cone}(R_+\#\widetilde{U}(\mathfrak{g}))}$ is equivalent to the category of D-modules on the base affine space G/U.

The category $C_{\mathbf{Proj}_{\mathcal{G}}(R_+\#\widetilde{U}(\mathfrak{g}))}$ is equivalent to the category of D-modules on the flag variety G/B.

We can express these facts saying that the category of D-modules on the base affine space G/U is the category of quasi-coherent sheaves on the noncommutative quasi-affine 'space' $\mathbf{Cone}(R_+\#\widetilde{U}(\mathfrak{g}))$ and the category of D-modules on the flag variety G/B is the category of quasi-coherent sheaves on the noncommutative 'space' $\mathbf{Proj}_{\mathcal{G}}(R_+\#\widetilde{U}(\mathfrak{g}))$. Both are semi-separated (actually, separated) noncommutative schemes.

7.5.3. Example: quantum affine base space and quantum flag variety. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of a semi-simple Lie algebra, $\mathfrak{g}$, and let $\mathcal{H}$ be its maximal torus (this time canonical). We define R and R_+ as in 7.5.2; i.e. $R = \bigoplus_{\lambda\geq 0} R_\lambda$ and $R_+ = \bigoplus_{\lambda>0} R_\lambda$, where R_λ is the simple $U_q(\mathfrak{g})$-module with the highest weight λ. The multiplication is given by choosing projections $R_\lambda \otimes R_\mu \longrightarrow R_{\lambda+\mu}$ for different λ and μ in an appropriate way (see [LR4] for details).

We define the *quantum base affine space of* $\mathfrak{g}$ as the 'space' $\mathbf{Cone}(R_+)$ and the *quantum flag variety of* $\mathfrak{g}$ as the 'space' $\mathbf{Proj}_{\mathcal{G}}(R_+)$.

7.5.3.1. D-modules on the quantum base affine space and the quantum flag variety. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra of a semi-simple Lie algebra, $\mathfrak{g}$, and let $\mathcal{H}$ be its maximal a torus. Let $R = \oplus_{\lambda\geq 0} R_\lambda$ be the algebra of functions on the quantum base affine space, and $R_+ = \oplus_{\lambda\geq 0} R_\lambda$ the *quantum base affine space of* $\mathfrak{g}$ which is by definition the spectrum $\mathbf{Cone}(R_+)$ of the algebra R_+ (see 7.5.2).

The maximal torus $\mathcal{H}$ acts on R_+, and this action commutes with the action of $U_q(\mathfrak{g})$. Thus R_+ has a structure of a $\widetilde{U}_q(\mathfrak{g})$-module, where $\widetilde{U}_q(\mathfrak{g}) = U_q(\mathfrak{g}) \otimes_k \mathcal{H}$. By 7.1.2, $\widetilde{U}_q(\mathfrak{g})$ induces a continuous monad, $U_q^\sim(\mathfrak{g})$, on $\mathbf{Cone}(R_+)$. And we have the commutative diagram

$$\begin{array}{ccc} \mathbf{Sp}(U_q^\sim(\mathfrak{g})/\mathbf{Cone}(R_+)) & \xrightarrow{\ \sim\ } & \mathbf{Cone}(R_+\#\widetilde{U}_q(\mathfrak{g})) \\ & \searrow \quad \swarrow & \\ & \mathbf{Cone}(R_+) & \end{array}$$

The action of $\widetilde{U}_q(\mathfrak{g})$ on R_+ respects $\mathcal{G}$-grading, hence it induces a continuous monad, $\bar{U}_q(\mathfrak{g})$, on $\mathbf{Proj}_{\mathcal{G}}(R_+)$; and we have a commutative diagram

$$\begin{array}{ccc} \mathbf{Sp}(\bar{U}_q(\mathfrak{g})/\mathbf{Proj}_{\mathcal{G}}(R_+)) & \xrightarrow{\ \sim\ } & \mathbf{Proj}_{\mathcal{G}}(R_+\#\widetilde{U}_q(\mathfrak{g})) \\ & \searrow \quad \swarrow & \\ & \mathbf{Proj}_{\mathcal{G}}(R_+) & \end{array}$$

of projective 'spaces' whose diagonal arrows are affine morphisms.

Chapter II
Locally Affine 'Spaces' and Schemes. Smoothness

The first section is dedicated to generalities on finiteness conditions – locally finitely presentable morphisms and objects, and representable morphisms in a simple-minded setting. Namely, our initial (and, with the exception of examples, the only) data here is a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ whose domain $\mathfrak{A}$ is thought as the category of 'local' objects (like affine schemes) and range, $\mathfrak{B}$, as the category of "spaces". We introduce morphisms of the category $\mathfrak{B}$ representable by a given class of morphisms of the category $\mathfrak{A}$. Morphisms representable by the class of all morphisms of $\mathfrak{A}$ are called simply "representable", or "affine". As a non-trivial application of this formalism, we define closed immersions and, as their derivatives, separated objects and morphisms of the category of "spaces" $\mathfrak{B}$.

In Section 2, we enrich our initial data with a class $\mathfrak{M}$ of morphisms of the category $\mathfrak{B}$ (playing the role of "infinitesimal" morphisms) and introduce formally $\mathfrak{M}$-smooth and formally $\mathfrak{M}$-étale morphisms. In classical examples, morphisms of $\mathfrak{M}$ come from the category $\mathfrak{A}$ of 'affine' objects; but, for a general formalism, there is no need in this assumption. Actually, there is no need in the category $\mathfrak{A}$ as long as local finiteness conditions are not involved. The local finiteness conditions (hence the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$) reappear in Section 3, where we define $\mathfrak{M}$-smooth and $\mathfrak{M}$-étale morphisms, as well as $\mathfrak{M}$-open immersions.

In Section 4, we adjoin to the data one more parameter – a pretopology on the category $\mathfrak{B}$. The purpose is to pass from a given class $\mathfrak{P}$ of morphisms of $\mathfrak{B}$ to the class $\mathfrak{P}^{\tau}$ of those morphisms which 'locally' belong to $\mathfrak{P}$. This extension of the class $\mathfrak{P}$ is natural (for instance, smoothness is expected to be a local property) and it is much more flexible; in particular, it is stable, under certain conditions, with respect to sheafification functor (the advantages of the latter property become clear in Chapter 3).

In Section 5, we define smooth, étale, and Zariski pretopologies on the category $\mathfrak{B}$ (all three depending on a class of "infinitesimal" morphisms $\mathfrak{M}$) and the notions of a scheme, an algebraic space and a locally affine space with respect to smooth pretopology.

Unfortunately, *flat covers* of noncommutative spaces, which, thanks to flat descent, are at least of the same importance for noncommutative algebraic geometry as for commutative one, do not form a pretopology – base change invariance fails. By this reason, we outline in Section 6 the first steps of a slightly weaker gluing formalism, in which pretopologies are replaced by 'quasi-pretopologies'. In Section 7, we apply this formalism to define locally affine 'spaces' and schemes in the category $|Cat|^o$ (– *absolute* case) and *relative* locally affine 'spaces' and schemes via **fppf** quasi-pretopology. It is also applied to introduce locally affine presheaves of sets (with respect to **fppf**, or a *flat Zariski* quasi-pretopology) on the category of affine noncommutative schemes.

1. Generalities on finiteness conditions.

1.0. The local data and some of its special cases. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ which will be usually referred to as a 'local data'. Such reference means that $\mathfrak{A}$ is regarded as the category of 'local', or 'affine', objects, $\mathfrak{B}$ as the category of "spaces", and the functor $\mathfrak{F}$ associates with each 'affine' object a space.

Examples one might keep in mind for this work are as follows.

1.0.1. Commutative affine schemes and geometric spaces. In a standard commutative prototype, $\mathfrak{B}$ is the category of locally ringed topological spaces, otherwise called *geometric spaces*, $\mathfrak{A}$ is the category opposite to the category $CAlg_k$ of commutative unital k-algebras, and the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ assigns to every commutative k-algebra the corresponding affine scheme.

1.0.2. Subcanonical presites and (pre)sheaves of sets. Let τ be a *subcanonical* pretopology on a category $\mathfrak{A}$. Recall that 'subcanonical' means that all representable presheaves are sheaves. So that the Yoneda embedding

$$\mathfrak{A} \longrightarrow \mathfrak{A}^\wedge, \quad \mathcal{M} \longmapsto \widehat{\mathcal{M}} = \mathfrak{A}(-, \mathcal{M}),$$

induces a full embedding of the category $\mathfrak{A}$ into the category $(\mathfrak{A}, \tau)^\wedge$ of sheaves of sets on the presite $(\mathfrak{A}, \tau)$. The latter is our functor $\mathfrak{F}$ in this situation.

1.0.3. Presheaves and sheaves of sets on noncommutative affine schemes. Let $\mathfrak{A}$ be the category $\mathbf{Aff}_k$ of *affine* noncommutative k-schemes, which is, by definition, the category opposite to the category Alg_k of associative unital algebras over a commutative ring k. We take as $\mathfrak{B}$ the category $\mathbf{Aff}_k^\wedge$ of presheaves of sets on $\mathbf{Aff}_k$, or, what is the same, functors $Alg_k \longrightarrow Sets$. The functor $\mathfrak{F}$ is the Yoneda embedding, which maps an affine scheme corresponding to a k-algebra $\mathcal{R}$ to the presheaf $\mathcal{R}^\vee = Alg_k(\mathcal{R}, -)$.

1.0.3.1. Sheaves of sets. The category $\mathfrak{A}$ is $\mathbf{Aff}_k$ as in 1.0.3, and $\mathfrak{B}$ is the category $(\mathfrak{A}, \tau)^\wedge$ of sheaves of sets with respect to a subcanonical pretopology τ. Our predominant choice is the *smooth* pretopology defined later in the text.

1.0.4. Affine 'spaces' and 'spaces' over a 'space'. Fix a 'space' $\mathcal{S}$ – an object of the category $|Cat|^o$. We take as $\mathfrak{A}$ the category $\mathbf{Aff}_\mathcal{S}$ of affine $\mathcal{S}$-'spaces', as $\mathfrak{B}$ the category $|Cat|^o/\mathcal{S}$ of 'spaces' over $\mathcal{S}$ and as $\mathfrak{F}$ the natural embedding of $\mathbf{Aff}_\mathcal{S}$ into $|Cat|^o/\mathcal{S}$. It follows from I.5.4 that $\mathfrak{F}$ is a full embedding.

1.0.4.1. Monads and 'spaces'. Let $C_{\mathcal{S}}$ be a svelte category with cokernels of reflexive pairs of arrows. Let $\mathfrak{Mon}^{\mathfrak{c}}_{\mathcal{S}}$ be the category of continuous monads on the category $C_{\mathcal{S}}$. We take as $\mathfrak{A}$ the category opposite to $\mathfrak{Mon}^{\mathfrak{c}}_{\mathcal{S}}$, as $\mathfrak{B}$ the category $|Cat|^o/\mathcal{S}$ of 'spaces' over $\mathcal{S}$ and as $\mathfrak{F}$ the functor which assigns to every continuous monad $\mathcal{F}$ on the category $C_{\mathcal{S}}$ the pair $(\mathbf{Sp}(\mathcal{F}/\mathcal{S}), \mathbf{Sp}(\mathcal{F}/\mathcal{S}) \longrightarrow \mathcal{S})$, where $\mathbf{Sp}(\mathcal{F}/\mathcal{S})$ is the *categoric spectrum* of the monad $\mathcal{F}$ – the 'space' represented by the category $(\mathcal{F}/\mathcal{S}) - mod$ of $\mathcal{F}$-modules, and $\mathbf{Sp}(\mathcal{F}/\mathcal{S}) \longrightarrow \mathcal{S}$ is a natural morphism, whose direct image functor is the forgetful functor $(\mathcal{F}/\mathcal{S}) - mod \longrightarrow C_{\mathcal{S}}$. It follows from I.4.6.3 and I.5.2 that the functor $\mathfrak{F}$ is full.

1.1. Locally cofinite and finitely copresentable objects. Fix a local data – a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$. We call an object X of $\mathfrak{B}$ *locally cofinite*, or $(\mathfrak{A}, \mathfrak{F})$*-cofinite* (resp. *locally finitely copresentable*, or $(\mathfrak{A}, \mathfrak{F})$*-finitely copresentable*), if, for any small diagram (resp. for any filtered projective system) $\delta\mathfrak{D} \xrightarrow{\mathfrak{D}} \mathfrak{A}$ such that there exists $\lim(\mathfrak{F} \circ \mathfrak{D})$, the canonical map

$$\mathrm{colim}\ \mathfrak{B}(\mathfrak{F} \circ \mathfrak{D}, X) \longrightarrow \mathfrak{B}(\lim(\mathfrak{F} \circ \mathfrak{D}), X) \qquad (1)$$

is an isomorphism.

We denote by $\mathfrak{B}^{\mathfrak{f}}_{\mathfrak{A},\mathfrak{F}}$ (resp. by $\mathfrak{B}^{\mathfrak{fp}}_{\mathfrak{A},\mathfrak{F}}$) the full subcategory of the category $\mathfrak{B}$ generated by locally cofinite (resp. by locally finitely copresentable) objects.

1.1.1. Cofinite and finitely copresentable objects. If $\mathfrak{F}$ is the identical functor $\mathfrak{B} \longrightarrow \mathfrak{B}$, then we omit "locally" and denote the full subcategory of $\mathfrak{B}$ generated by cofinite (resp. locally *finitely copresentable*) objects by $\mathfrak{B}^{\mathfrak{f}}$ (resp. by $\mathfrak{B}^{\mathfrak{fp}}$).

1.2. Locally finite and locally finitely presentable objects. They are defined dually: an object $\mathfrak{X}$ of the category $\mathfrak{B}$ is called *locally finite* (resp. *locally finitely presentable*) if the canonical morphism

$$\mathrm{colim}\ \mathfrak{B}(\mathfrak{X}, \mathfrak{F} \circ \mathfrak{D}) \longrightarrow \mathfrak{B}(\mathfrak{X}, \mathrm{colim}(\mathfrak{F} \circ \mathfrak{D})) \qquad (2)$$

is an isomorphism for any small diagram (resp. filtered inductive system) $\delta\mathfrak{D} \xrightarrow{\mathfrak{D}} \mathfrak{A}$, provided that $\mathrm{colim}(\mathfrak{F} \circ \mathfrak{D})$ exists.

The notations are dual to those of 1.1. Namely, we denote by $\mathfrak{B}^{\mathfrak{A},\mathfrak{F}}_{\mathfrak{f}}$ (resp. by $\mathfrak{B}^{\mathfrak{A},\mathfrak{F}}_{\mathfrak{fp}}$) the full subcategory of the category $\mathfrak{B}$ generated by locally finite (resp. by locally finitely presentable) objects.

1.2.1. Finite and finitely presentable objects. If $\mathfrak{F}$ is the identical functor $\mathfrak{B} \longrightarrow \mathfrak{B}$, then we drop "locally" and denote the full subcategory of $\mathfrak{B}$ generated by finite objects by $\mathfrak{B}_{\mathfrak{f}}$ and the full subcategory generated by *finitely presentable* objects by $\mathfrak{B}_{\mathfrak{fp}}$.

1.3. Proposition. *Let* $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ *be a fully faithful functor.*

(a) If the functor $\mathfrak{F}$ preserves colimits of small diagrams, then an object X of the category $\mathfrak{A}$ is finite iff the object $\mathfrak{F}(X)$ is locally finite.

(b) Suppose that $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ preserves colimits of inductive systems. Then an object X of the category $\mathfrak{A}$ is finitely presentable iff the object $\mathfrak{F}(X)$ is locally finitely presentable.

Proof. (a) Suppose that X is a finite object of the category $\mathfrak{A}$. Then, for any small diagram $\delta\mathfrak{D} \xrightarrow{\mathfrak{D}} \mathfrak{A}$ such that the colimits of of $\mathfrak{D}$ and of $\mathfrak{F}\circ\mathfrak{D}$ exists, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{colim}\mathfrak{A}(X, \mathfrak{D}(-)) & \longrightarrow & \mathfrak{A}(X, \mathrm{colim}\mathfrak{D}) \\ \downarrow & & \downarrow \\ \mathrm{colim}\mathfrak{B}(\mathfrak{F}(X), \mathfrak{F}\circ\mathfrak{D}(-)) & \longrightarrow & \mathfrak{B}(\mathfrak{F}(X), \mathfrak{F}(\mathrm{colim}\mathfrak{D})) \\ & \searrow \quad \mathfrak{B}(\mathfrak{F}(X), \mathrm{colim}(\mathfrak{F}\circ\mathfrak{D})) \quad \nearrow & \end{array} \qquad (1)$$

Since $\mathfrak{F}$ is, by hypothesis, a fully faithful functor, the vertical arrows of (1) are isomorphisms. Since $\mathfrak{F}$ preserves colimits, the right diagonal arrow is an isomorphism too. So that, under the conditions, the upper horizontal arrow of (1) is an isomorphism iff the left diagonal arrow is an isomorphism.

(b) Same argument applied to inductive systems. ∎

1.4. Proposition. *Fix a local data $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$.*

(a) If $\delta\Psi \xrightarrow{\Psi} \mathfrak{B}$ is a finite diagram which maps objects of $\delta\Psi$ to finitely presentable objects, then the colimit of Ψ (if any) is a finitely presentable object.

(b) Every retract of a locally finite object is locally finite.

(c) Any retract of a locally finitely presentable object is locally finitely presentable.

Proof. (a) The assertion follows from the natural isomorphism $\mathfrak{B}(\mathrm{colim}\Psi, -) \simeq \lim \mathfrak{B}(\Psi, -)$ and the fact that filtered colimits commute with finite limits.

(b) Let X be a locally finite object, and let Y be a retract of X; i.e. there exist morphisms $Y \xrightarrow{\phi} X$ and $X \xrightarrow{\psi} Y$ such that $\psi \circ \phi = id_X$. Let $\mathfrak{p}$ denote the corresponding projector: $\mathfrak{p} = \phi \circ \psi$. For any diagram $\delta\mathcal{D} \xrightarrow{\mathcal{D}} \mathfrak{B}$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{colim}\mathfrak{B}(X, \mathfrak{F}\circ\mathcal{D}) & \xrightarrow{\sim} & \mathfrak{B}(X, \mathfrak{F}(\mathrm{colim}\mathcal{D})) \\ id \downarrow\downarrow \mathfrak{B}(\mathfrak{p}, -) & & id \downarrow\downarrow \mathfrak{B}(\mathfrak{p}, -) \\ \mathrm{colim}\mathfrak{B}(X, \mathfrak{F}\circ\mathcal{D}) & \xrightarrow{\sim} & \mathfrak{B}(X, \mathfrak{F}(\mathrm{colim}\mathcal{D})) \\ \phi_\bullet \downarrow & & \downarrow \phi_\bullet \\ \mathrm{colim}\mathfrak{B}(Y, \mathfrak{F}\circ\mathcal{D}) & \longrightarrow & \mathfrak{B}(Y, \mathfrak{F}(\mathrm{colim}\mathcal{D})) \end{array}$$

whose vertical diagrams $\cdot \rightrightarrows \cdot \to \cdot$ are exact and two upper horizontal arrows are isomorphisms. Therefore, the canonical morphism $\mathrm{colim}\mathfrak{B}(Y,\mathcal{D}) \longrightarrow \mathfrak{B}(Y,\mathrm{colim}\mathcal{D})$ is bijective, which shows that the object Y is locally finite.

(c) The argument of (b) applied to filtered diagrams proves the assertion. ∎

1.5. Proposition. *(a) Finite objects of a category $\mathfrak{B}$ are projective with respect to the class $\mathfrak{E}^{\mathfrak{s}}_{\mathfrak{B}}$ of strict epimorphisms of $\mathfrak{B}$. In particular, if $\mathcal{X}$ is a finite object, then every strict epimorphism to $\mathcal{X}$ splits.*

(b) Suppose that the $\mathfrak{B}$ is an additive category with small colimits. Then finite objects of $\mathfrak{B}$ are precisely finitely presentable objects which are projective with respect to strict epimorphisms.

Proof. (a) Let $\mathcal{X}$ be a finite object, $\mathfrak{X} \xrightarrow{\mathfrak{t}} \mathfrak{Y}$ a strict epimorphism and $\mathcal{X} \xrightarrow{\xi} \mathfrak{Y}$ an arbitrary morphism. By definition, $\mathfrak{X} \xrightarrow{\mathfrak{t}} \mathfrak{Y}$ is a strict epimorphism if $\mathfrak{Y}$ is colimit of a diagram $\mathcal{D}$ formed by a set of pairs of arrows $\mathcal{Z} \rightrightarrows \mathfrak{X}$ equalizing $\mathfrak{X} \xrightarrow{\mathfrak{t}} \mathfrak{Y}$. Since

$$\mathrm{colim}\mathfrak{B}(\mathcal{X},\mathcal{D}) \longrightarrow \mathfrak{B}(\mathcal{X},\mathrm{colim}\mathcal{D}) = \mathfrak{B}(\mathcal{X},\mathfrak{Y})$$

is an isomorphism, there exists a cone $\mathcal{X} \longrightarrow (\mathcal{Z} \rightrightarrows \mathfrak{X})$ whose image in $\mathfrak{B}(\mathcal{X},\mathfrak{Y})$ is the morphism ξ. In particular, we obtain a morphism $\mathcal{X} \xrightarrow{\widetilde{\xi}} \mathfrak{X}$ such that $\xi = \mathfrak{t} \circ \widetilde{\xi}$.

(b1) If $\mathfrak{B}$ is an additive category with infinite coproducts, then, for every finitely presentable object $\mathcal{V}$ of the category $\mathfrak{B}$, the functor $\mathfrak{B}(\mathcal{V},-)$ preserves infinite coproducts.

This is due to the fact that $\mathfrak{B}(\mathcal{V},-)$ preserves finite coproducts thanks to the additivity of $\mathfrak{B}$, and infinite coproducts are colimits of the filtered set of their finite subcoproducts.

(b2) The projectivity of $\mathcal{V}$ with respect to strict epimorphisms implies that $\mathfrak{B}(\mathcal{V},-)$ preserves cokernels of pairs of arrows. So that if, in addition, the object $\mathcal{V}$ is finitely presentable and the category $\mathfrak{B}$ has colimits of small diagrams, then the functor $\mathfrak{B}(\mathcal{V},-)$ preserves arbitrary colimits. ∎

1.6. Examples: modules and tensor algebras.

1.6.1. Modules. If $\mathfrak{B}$ is the category of modules over an associative unital algebra, then finitely presentable objects are finitely presentable modules in the usual sense, and it follows from 1.5(b) that finite objects are finitely generated projective modules.

1.6.2. Tensor algebras. Let $\mathfrak{B}$ be the category Alg_k of associative unital k-algebras. For any k-module $\mathcal{M}$, we denote by $T_k(\mathcal{M})$ the tensor algebra of the k-module $\mathcal{M}$.

1.6.2.1. Proposition. *(a) The tensor algebra $T_k(\mathcal{M})$ of a k-module $\mathcal{M}$ is a finitely presentable object of Alg_k iff $\mathcal{M}$ is a finitely presentable k-module.*

(b) The tensor algebra $T_k(\mathcal{M})$ is a projective object with respect to strict epimorphisms of Alg_k iff $\mathcal{M}$ is a projective k-module.

Proof. (a1) The functor

$$k-mod \longrightarrow Alg_k, \quad \mathcal{M} \longmapsto T_k(\mathcal{M}),$$

is a left adjoint to the forgetful functor $Alg_k \xrightarrow{\mathfrak{f}_*} k-mod$; and the latter preserves colimits of inductive systems. So that, for any inductive system $\delta\mathfrak{D} \xrightarrow{\mathfrak{D}} Alg_k$, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{colim} Alg_k(T_k(\mathcal{M}),\mathfrak{D}) & \longrightarrow & Alg_k(T_k(\mathcal{M}),\mathrm{colim}\mathfrak{D}) \\
\wr\downarrow & & \downarrow\wr \\
\mathrm{colim} Hom_k(\mathcal{M},\mathfrak{f}_*\circ\mathfrak{D}) & \longrightarrow & Hom_k(\mathcal{M},\mathfrak{f}_*(\mathrm{colim}(\mathfrak{D})) \\
& \searrow \quad\quad \nearrow & \\
& Hom_k(\mathcal{M},\mathrm{colim}(\mathfrak{f}_*\circ\mathfrak{D})) &
\end{array} \qquad (1)$$

whose vertical arrows are isomorphisms by adjunction. The right diagonal arrow is an isomorphism, because the functor $\mathfrak{f}_*$ preserves colimits of filtered diagrams. So that the upper horizontal arrow is an isomorphism iff the left diagonal arrow is an isomorphism. This shows that the tensor algebra $T_k(\mathcal{M})$ is a finitely presentable object of the category Alg_k, if $\mathcal{M}$ is a finitely presentable k-module.

(a2) Let ϕ_k denote the functor $k-mod \longrightarrow Alg_k$ which maps every k-module $\mathcal{N}$ to $k\oplus\mathcal{N}$ with zero multiplication on $\mathcal{N}$. Let $\delta\mathcal{D} \xrightarrow{\mathcal{D}} k-mod$ be an inductive system. Then there are natural isomorphisms

$$\mathrm{colim}(\phi_k\circ\mathcal{D}) \simeq \phi_k(\mathrm{colim}\mathcal{D}) \quad \text{and} \quad \mathrm{colim}(\mathfrak{f}_*\circ\phi_k\circ\mathfrak{D}) \simeq k\oplus\mathrm{colim}\mathcal{D}.$$

Therefore, replacing in (1) the diagram $\mathfrak{D}$ by the composition $\phi_k\circ\mathcal{D}$, we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{colim} Alg_k(T_k(\mathcal{M}),\phi_k\circ\mathcal{D}) & \longrightarrow & Alg_k(T_k(\mathcal{M}),\mathrm{colim}(\phi_k\circ\mathcal{D})) \\
\wr\downarrow & & \downarrow\wr \\
\mathrm{colim} Hom_k(\mathcal{M},\mathfrak{f}_*\phi_k\circ\mathcal{D}) & \longrightarrow & Hom_k(\mathcal{M},\mathfrak{f}_*(\mathrm{colim}(\phi_k\circ\mathcal{D})) \\
\wr\downarrow & & \downarrow\wr \\
\mathrm{colim} Hom_k(\mathcal{M},k\oplus\mathcal{D}) & \longrightarrow & Hom_k(\mathcal{M},k\oplus\mathrm{colim}(\mathcal{D})) \\
\wr\downarrow & & \downarrow\wr \\
\mathcal{M}^*\oplus\mathrm{colim} Hom_k(\mathcal{M},\mathcal{D}) & \longrightarrow & \mathcal{M}^*\oplus Hom_k(\mathcal{M},\mathrm{colim}(\mathcal{D}))
\end{array} \qquad (2)$$

whose vertical arrows are isomorphisms and $\mathcal{M}^* = Hom_k(\mathcal{M},k)$. It follows that all arrows of (2) are isomorphisms iff any of horizontal arrows is an isomorphism. The lowest horizontal arrow is an isomorphism iff the canonical map

$$\mathrm{colim} Hom_k(\mathcal{M},\mathcal{D}) \longrightarrow Hom_k(\mathcal{M},\mathrm{colim}(\mathcal{D})) \qquad (3)$$

is an isomorphism. So that if $T_k(\mathcal{M})$ is a finitely presentable object of the category Alg_k, then the upper horizontal arrow of the diagram (2) is an isomorphism for any inductive system $\delta\mathcal{D} \xrightarrow{\mathcal{D}} k-mod$, which implies (is equivalent to) that (3) is an isomorphism, i.e. $\mathcal{M}$ is a finitely presentable k-module.

(b1) Strict epimorphisms of algebras are precisely those morphisms which the forgetful functor $Alg_k \xrightarrow{\mathfrak{f}_*} k-mod$; maps to module epimorphisms. Therefore, a left adjoint to $\mathfrak{f}_*$ maps projective modules to to the objects, which are projective with respect to strict epimorphisms, because $Alg_k(T_k(\mathcal{M}),\lambda) \simeq Hom_k(\mathcal{M},\mathfrak{f}_*(\lambda))$ for any algebra morphism λ.

(b2) The functor $k-mod \xrightarrow{\phi_k} Alg_k$ (defined in (a2) above) maps epimorphisms of modules to strict epimorphisms of algebras, and it follows from the way ϕ_k acts on arrows that, for any module morphism $\mathcal{L} \xrightarrow{\gamma} \mathcal{N}$, the morphism $Alg_k(T_k(\mathcal{M}),\phi_k(\gamma)) \simeq Hom_k(\mathcal{M},\mathfrak{f}_*\phi_k(\gamma))$ is surjective iff $Hom_k(\mathcal{M},\gamma)$ is surjective. This shows that the module $\mathcal{M}$ is projective if its tensor algebra is projective with respect to strict epimorphisms of algebras. ■

1.6.2.2. Corollary. *The following conditions on a k-module $\mathcal{M}$ are equivalent:*

(a) $\mathcal{M}$ is a finitely generated projective;

(b) $T_k(\mathcal{M})$ is a finitely presentable object of Alg_k which is projective with respect to strict epimorphisms.

Proof. By 1.6.2.1, a k-module $\mathcal{M}$ is finitely presentable iff the k-algebra $T_k(\mathcal{M})$ is a finitely presentable object of Alg_k and $\mathcal{M}$ is projective iff $T_k(\mathcal{M})$ is projective with respect to strict epimorphisms of algebras. Therefore, (b) is equivalent to $\mathcal{M}$ being a finitely presentable projective module, which is the same as a finitely generated projective. ■

1.6.3. Proposition. *A k-algebra R is a projective object of Alg_k with respect to strict epimorphisms iff it is a retract of a tensor algebra $T_k(\mathcal{V})$ for a projective module $\mathcal{V}$; that is there exists a split algebra epimorphism $T_k(\mathcal{V}) \longrightarrow R$.*

Proof. Since, for any associative unital k-algebra R, the adjunction morphism

$$\mathfrak{f}^*\mathfrak{f}_*(\mathcal{R}) = T_k(\mathfrak{f}_*(R)) \longrightarrow R$$

is a strict epimorphism, there exists a strict epimorphism $T_k(\mathcal{V}) \longrightarrow R$ with $\mathcal{V}$ a projective (or free) k-module. If R is projective with respect to strict epimorphisms, this strict epimorphism splits. Conversely, if $T_k(\mathcal{V}) \longrightarrow R$ splits, then R is a projective object of Alg_k, because, by 1.6.2.1, the tensor algebra $T_k(\mathcal{V})$ is a projective object. ■

1.6.4. Note. Even if $\mathcal{M}$ is a free k-module of finite rank, its tensor algebra $T_k(\mathcal{M})$ is not a finite object of the category Alg_k. This follows from the fact that the functor

$Alg_k(T_k(\mathcal{M}), -)$ is isomorphic to $Hom_k(\mathcal{M}, \mathfrak{f}_*(-))$ and the latter functor does not preserve coproducts in the category Alg_k (which are usually called "star-products").

In fact, if $\mathcal{L}$ and $\mathcal{N}$ are non-trivial (say, free) k-modules, then

$$Alg_k(T_k(\mathcal{M}), T_k(\mathcal{L}) \coprod T_k(\mathcal{N})) \simeq Alg_k(T_k(\mathcal{M}), T_k(\mathcal{L} \oplus \mathcal{N})) \simeq$$
$$Hom_k(\mathcal{M}, \mathfrak{f}_*(T_k(\mathcal{L} \oplus \mathcal{N}))) \not\simeq Hom_k(\mathcal{M}, \mathfrak{f}_*(T_k(\mathcal{L}))) \coprod Hom_k(\mathcal{M}, \mathfrak{f}_*(T_k(\mathcal{N}))).$$

Here the first isomorphism is due to the fact that the functor $T_k(-)$ preserves colimits (because it has a right adjoint); in particular it preserves coproducts.

1.7. Weakly finite and locally weakly finite objects. It follows from 1.6 that the notion of a finite object becomes too restrictive in non-additive categories, and it makes sense to single out a weaker class of objects suggested by 1.5 and 1.6.4.

1.7.1. Definitions. (a) We call an object $\mathcal{X}$ of a category $\mathfrak{B}$ *weakly finite* if it is finitely presentable and the functor $\mathfrak{B}(\mathcal{X}, -)$ preserves cokernels of reflexive pairs of arrows.

Recall that a pair of arrows $M \underset{g_2}{\overset{g_1}{\rightrightarrows}} L$ in C_Y is called *reflexive*, if there exists a morphism $L \xrightarrow{h} M$ such that $g_1 \circ h = id_M = g_2 \circ h$.

(b) Given a functor (– a local data) $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$, we call an object $\mathcal{X}$ of the category $\mathfrak{B}$ *locally weakly finite* if it is locally finitely presentable and the functor $\mathfrak{B}(\mathcal{X}, \mathfrak{F}(-))$ preserves cokernels of reflexive pairs of arrows.

(c) *Weakly cofinite* and *locally weakly cofinite* objects of $\mathfrak{B}$ are defined dually.

1.7.2. Note. Finite, weakly finite and finitely presentable objects appear in linear algebra, while locally weakly cofinite and locally finitely copresentable objects play crucial role in algebraic geometry. The following examples (and the rest of the manuscript) illustrate this observation.

1.8. Noncommutative vector fibers. By definition, the category $\mathbf{Aff}_k$ of *noncommutative affine schemes* over k is the category opposite to the category Alg_k of associative unital k-algebras. We take as $\mathfrak{B}$ the category $\mathbf{Aff}_k^\wedge$ of presheaves of sets on $\mathbf{Aff}_k$, which we identify with the category $Alg_k^\vee$ of functors $Alg_k \longrightarrow Sets$, and as local data the Yoneda embedding $\mathbf{Aff}_k \longrightarrow \mathbf{Aff}_k^\wedge$. The Yoneda embedding identifies $\mathbf{Aff}_k$ with the full subcategory of $Alg_k^\vee$ generated by the presheaves of sets $R^\vee = Alg_k(R, -)$.

1.8.1. Vector fiber of a module. The affine scheme

$$T_k(\mathcal{M})^\vee \stackrel{\text{def}}{=} Alg_k(T_k(\mathcal{M}), -) \simeq Hom_k(\mathcal{M}, \mathfrak{f}_*(-))$$

corresponding to the tensor algebra $T_k(\mathcal{M})$ of a k-module $\mathcal{M}$ is denoted by $\mathbb{V}_k(\mathcal{M})$ and called the *vector fiber* of the k-module $\mathcal{M}$.

1.8.2. Proposition. *(a) The vector fiber $\mathbb{V}_k(\mathcal{M})$ is a finitely copresentable object of the category $\mathbf{Aff}_k$ iff the module $\mathcal{M}$ is finitely presentable.*

(b) The vector fiber $\mathbb{V}_k(\mathcal{M})$ is an injective object of the category $\mathbf{Aff}_k$ with respect to strict monomorphisms iff the k-module $\mathcal{M}$ is projective.

(c) The vector fiber $\mathbb{V}_k(\mathcal{M})$ is a weakly cofinite object of $\mathbf{Aff}_k$ iff $\mathcal{M}$ is a finitely generated projective k-module.

Proof. The assertion follows from 1.6.2.1 and 1.6.2.2. ∎

1.9. Objects of locally (co)finite type. A standard notion (– an imitation of Grothendieck's definition) is as follows. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$. An object X of $\mathfrak{B}$ is *of locally finite type*, or *of* $(\mathfrak{A}, \mathfrak{F})$*-finite type*, if, for any inductive system $\delta\mathfrak{D} \xrightarrow{\mathfrak{D}} \mathfrak{A}$ such that there exists $\operatorname{colim}(\mathfrak{F} \circ \mathfrak{D})$, the canonical map

$$\operatorname{colim} \mathfrak{B}(X, \mathfrak{F} \circ \mathfrak{D}) \longrightarrow \mathfrak{B}(X, \operatorname{colim}(\mathfrak{F} \circ \mathfrak{D})) \tag{1}$$

is injective. We denote by $\mathfrak{B}^{\mathfrak{A},\mathfrak{F}}_{\mathfrak{ft}}$ the full subcategory of the category $\mathfrak{B}$ generated by all objects of locally finite type.

Objects of locally *cofinite* type are defined dually, and the full subcategory of the category $\mathfrak{B}$ they generate is denoted by $\mathfrak{B}^{\mathfrak{ft}}_{\mathfrak{A},\mathfrak{F}}$.

1.9.1. Proposition. *If an object X of the category $\mathfrak{B}$ is of locally finite type and $X \longrightarrow Y$ is an epimorphism, then Y is an object of locally finite type too.*

Proof. This follows from the fact that the vertical arrows of the commutative diagram

$$\begin{array}{ccc} \operatorname{colim} \mathfrak{B}(Y, \mathfrak{F} \circ \mathfrak{D}) & \longrightarrow & \mathfrak{B}(Y, \operatorname{colim}(\mathfrak{F} \circ \mathfrak{D})) \\ \downarrow & & \downarrow \\ \operatorname{colim} \mathfrak{B}(X, \mathfrak{F} \circ \mathfrak{D}) & \longrightarrow & \mathfrak{B}(X, \operatorname{colim}(\mathfrak{F} \circ \mathfrak{D})) \end{array}$$

are monomorphisms. So that if the lower horizontal arrow is a monomorphism, then the upper horizontal arrow is a monomorphism too. ∎

1.10. Objects of locally strictly finite type. Proposition 1.9.1 shows that there might be (depending on the local data, more precisely, on the category $\mathfrak{B}$) too many locally finite objects in the classical sense. The following notion gives what we actually expect.

1.10.1. Definition. We call an object $\mathcal{X} \in Ob\mathfrak{B}$ an object of *locally strictly finite type* if there is a strict epimorphism $\mathfrak{X} \longrightarrow \mathcal{X}$ with $\mathfrak{X}$ a locally finitely presentable object.

1.10.2. Proposition. *(a) The tensor algebra $T_k(\mathcal{M})$ of a k-module $\mathcal{M}$ is of strictly finite type iff $\mathcal{M}$ is a finitely generated k-module.*

(b) An associative unital k-algebra $\mathcal{R}$ is an object of strictly finite type in Alg_k iff there exists a strict epimorphism $T_k(\mathcal{P}) \longrightarrow \mathcal{R}$, where $\mathcal{P}$ is a finitely generated projective (or a free of finite rank) k-module.

Proof. (a) This follows from 1.6.2.2 and the fact that $T_k(-)$ maps module epimorphisms to strict epimorphisms of (tensor) algebras.

(b1) It follows from 1.6.2.2 that if there exists an epimorphism $T_k(\mathcal{P}) \longrightarrow \mathcal{R}$ with $\mathcal{P}$ a finitely generated projective, then $\mathcal{R}$ is an algebra of strictly finite type.

(b2) As to the inverse implication, it suffices to prove the existence of a strict epimorphism $T_k(\mathcal{P}) \longrightarrow \mathcal{R}$ with $\mathcal{P}$ a free k-module of finite rank in the case when $\mathcal{R}$ is a finitely presentable object of the category Alg_k.

The k-module $\mathfrak{f}_*(\mathcal{R})$ is the colimit of its finitely generated submodules which form an inductive system, $\mathfrak{S}$. By adjunction, we have an inductive system of algebra morphisms

$$T_k(\mathcal{M}) \xrightarrow{\gamma_{\mathcal{M}}} \mathcal{R}, \quad \mathcal{M} \in \mathfrak{S},$$

whose colimit is the canonical strict epimorphism

$$\mathfrak{f}^*\mathfrak{f}_*(\mathcal{R}) = T_k(\mathfrak{f}_*(\mathcal{R})) \longrightarrow \mathcal{R}.$$

(because the functor $\mathfrak{f}^* = T_k(-)$ preserves colimits). Let $\mathcal{R}_{\mathcal{M}}$ denote the image of the algebra $T_k(\mathcal{M})$ in $\mathcal{R}$. Since, by hypothesis, the algebra $\mathcal{R}$ is finitely presentable and $\mathrm{colim}(\mathcal{R}_{\mathcal{M}} \mid \mathcal{M} \in \mathfrak{S}) = \mathcal{R}$, there exists $\mathcal{M} \in \mathfrak{S}$ such that $\mathcal{R}_{\mathcal{M}} = \mathcal{R}$. Taking an epimorphism $\mathcal{P} \xrightarrow{\mathfrak{e}} \mathcal{M}$, where $\mathcal{P}$ is a free module of finite rank, we obtain the claimed strict algebra epimorphism $T_k(\mathcal{P}) \longrightarrow \mathcal{R}$ as the composition of the strict epimorphism $T_k(\mathcal{M}) \longrightarrow \mathcal{R}$ and $T_k(\mathcal{P}) \xrightarrow{T_k(\mathfrak{e})} T_k(\mathcal{M})$. ■

1.10.2.1. Corollary. *Let a k-algebra $\mathcal{R}$ be a projective object of Alg_k with respect to strict epimorphisms. Then $\mathcal{R}$ is of strictly finite type iff it is finitely presentable.*

Proof. Let $\mathcal{R}$ be of strictly finite type. Then, by 1.10.2(b), there is a strict epimorphism $T_k(\mathcal{P}) \xrightarrow{\mathfrak{t}} \mathcal{R}$ with $\mathcal{P}$ a finitely generated projective k-module. If the algebra $\mathcal{R}$ is a projective object of Alg_k with respect to strict epimorphisms, then $T_k(\mathcal{P}) \xrightarrow{\mathfrak{t}} \mathcal{R}$ splits; that is $\mathcal{R}$ is a retract of the algebra $T_k(\mathcal{P})$. Since, by 1.6.2.1, $T_k(\mathcal{P})$ is a finitely presentable object of Alg_k, it follows from 1.4(c) that $\mathcal{R}$ is finitely presentable too. ■

1.10.3. Proposition. *Let $\mathcal{S} \xrightarrow{\varphi} \mathcal{R}$ be a strict epimorphism k-algebras and $\mathcal{S}$ is finitely presentable. Then $\mathcal{R}$ is finitely presentable iff $Ker(\varphi)$ is finitely generated as a two-sided ideal.*

Proof. (a) Let $\delta\mathfrak{D} \xrightarrow{\mathfrak{D}} Alg_k$ be a filtered diagram of k-algebras and $\mathcal{R} \xrightarrow{\mathfrak{f}} \mathrm{colim}\mathfrak{D}$ a k-algebra morphism. Since the algebra $\mathcal{S}$ is finitely presentable, the canonical morphism

$$\mathrm{colim} Alg_k(\mathcal{S}, \mathfrak{D}) \longrightarrow Alg_k(\mathcal{S}, \mathrm{colim}\mathfrak{D})$$

is an isomorphism. In particular, the composition $\mathcal{S} \xrightarrow{\mathfrak{f}\circ\varphi} \mathrm{colim}\mathfrak{D}$ factors through a morphism $\mathcal{S} \xrightarrow{\mathfrak{f}_x} \mathfrak{D}(x)$ for some $x \in Ob\delta\mathfrak{D}$. For every object $(y, x \xrightarrow{\nu} y)$ of $x\backslash\mathfrak{D}$, we denote by $\mathfrak{J}_\nu$ the kernel of the composition of $\mathcal{S} \xrightarrow{\mathfrak{f}_x} \mathfrak{D}(x)$ and $\mathfrak{D}(x) \xrightarrow{\mathfrak{D}(\nu)} \mathfrak{D}(y)$. It follows that the supremum of the family $\{\mathfrak{J}_\nu \mid (y,\nu) \in Ob(x\backslash\mathfrak{D})\}$ contains $Ker(\varphi)$. Since the latter is finitely generated (as a two-sided ideal), $Ker(\varphi) \subseteq \mathfrak{J}_\nu$ for some $(y,\nu) \in Ob(x\backslash\mathfrak{D})$; so that the composition $\mathcal{S} \xrightarrow{\mathfrak{D}(\nu)\mathfrak{f}_x} \mathfrak{D}(y)$ factors through the strict epimorphism $\mathcal{S} \xrightarrow{\varphi} \mathcal{R}$. This shows that $\mathcal{R}$ is finitely presentable if ($\mathcal{S}$ is finitely presentable and) the two-sided ideal $Ker(\varphi)$ is finitely generated.

(b) Conversely, let $(\mathfrak{J}_\alpha)$ be a filtered system of two-sided finitely generated ideals of the algebra $\mathcal{S}$ such that $\bigcup \mathfrak{J}_\alpha = Ker(\varphi)$. Then there is an isomorphism $\mathcal{R} \xrightarrow{\sim} \mathrm{colim}(\mathcal{S}/\mathfrak{J}_\alpha)$. If $\mathcal{R}$ is finitely presentable, then this isomorphism factors through some $\mathcal{S}/\mathfrak{J}_\nu$. In other words, the canonical epimorphism $\mathcal{S}/\mathfrak{J}_\nu \xrightarrow{\varphi_\nu} \mathcal{R}$ splits. The latter implies that $Ker(\varphi_\nu)$ is a finitely generated two-sided ideal. Therefore, the two-sided ideal $Ker(\varphi)$ is finitely generated. ■

1.10.3.1. Corollary. *An associative unital k-algebra $\mathcal{R}$ is finitely presentable iff there exists a strict epimorphism $T_k(\mathcal{P}) \xrightarrow{\varphi} \mathcal{R}$, where $\mathcal{P}$ is a finitely generated projective (or a free of finite rank) k-module such that $Ker(\varphi)$ is a finitely generated two-sided ideal of the tensor algebra $T_k(\mathcal{P})$.*

Proof. The assertion follows from 1.10.3 and 1.6.2.1 (or 1.6.2.2). ■

1.11. Locally finitely presentable morphisms and morphisms locally of finite type. Fix a functor (– a local data) $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$.

We call a morphism $X \xrightarrow{f} Y$ of $\mathfrak{B}$ *locally finitely presentable*, or $(\mathfrak{A},\mathfrak{F})$-*finitely presentable*, if for any filtered inductive system $\delta\mathfrak{D} \xrightarrow{\mathfrak{D}} \mathfrak{F}/Y$, the canonical morphism

$$\mathrm{colim}\ \mathfrak{B}/Y((X,f), \mathfrak{F}_Y \circ \mathfrak{D}) \longrightarrow \mathfrak{B}/Y((X,f), \mathrm{colim}(\mathfrak{F}_Y \circ \mathfrak{D})) \qquad (1)$$

is an isomorphism, provided that $\mathrm{colim}(\mathfrak{F}_Y \circ \mathfrak{D})$ exists. Here $\mathfrak{F}_Y$ denotes the functor induced by $\mathfrak{F}$:

$$\mathfrak{F}/Y \longrightarrow \mathfrak{B}/Y, \quad (\mathcal{V}, \mathfrak{F}(\mathcal{V}) \to Y) \longmapsto (\mathfrak{F}(\mathcal{V}), \mathfrak{F}(\mathcal{V}) \to Y).$$

We say that a morphism $X \xrightarrow{f} Y$ of $\mathfrak{B}$ is *locally of finite type*, or *of* $(\mathfrak{A}, \mathfrak{F})$*-finite type*, if, for any filtered inductive system $\mathcal{D} \xrightarrow{\mathfrak{D}} \mathfrak{F}/Y$, the canonical morphism (1) is a monomorphism, whenever $\mathrm{colim}(\mathfrak{F}_Y \circ \mathfrak{D})$ exists.

It follows from these definitions that if the category $\mathfrak{B}$ has a final object, $\bullet$, then an object X of $\mathfrak{B}$ is of $(\mathfrak{A}, \mathfrak{F})$-finite type (resp. $(\mathfrak{A}, \mathfrak{F})$-finitely presentable) iff the unique morphism $X \longrightarrow \bullet$ is of $(\mathfrak{A}, \mathfrak{F})$-finite type (resp. $(\mathfrak{A}, \mathfrak{F})$-finitely presentable).

1.11.1. Dual notions. These are the notions of a locally finitely copresentable morphism and a morphism of locally cofinite type.

1.11.2. Proposition. *Let* $\Sigma^1_{\mathfrak{A}}$ *(resp.* $\Sigma^0_{\mathfrak{A}}$*) denote the class of all* $(\mathfrak{A}, \mathfrak{F})$*-finitely copresentable morphisms (resp. morphisms of* $(\mathfrak{A}, \mathfrak{F})$*-cofinite type) of the category* $\mathfrak{B}$*.*

(a) Both $\Sigma^0_{\mathfrak{A}}$ *and* $\Sigma^1_{\mathfrak{A}}$ *are closed under compositions and contain all isomorphisms.*

(b) If the morphism f *in the cartesian square*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\widetilde{g}} & X \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ \mathfrak{Y} & \xrightarrow{g} & Y \end{array}$$

belongs to $\Sigma^i_{\mathfrak{A}}$*, then* f' *belongs to* $\Sigma^i_{\mathfrak{A}}$*,* $i = 0, 1$*.*

(c) Suppose that $X \xrightarrow{f} Y$ *and* $Z \xrightarrow{h} W$ *are morphisms over an object* S *which belong to* $\Sigma^i_{\mathfrak{A}}$*. If* $X \times_S Z$ *and* $Y \times_S W$ *exist, then the morphism*

$$X \times_S Z \xrightarrow{f \times_S h} Y \times_S W$$

belongs to the same class $\Sigma^i_{\mathfrak{A}}$*,* $i = 0, 1$*.*

(d) If the composition $g \circ f$ *of two morphisms is* $(\mathfrak{A}, \mathfrak{F})$*-finitely presentable and the morphism* g *is of* $(\mathfrak{A}, \mathfrak{F})$*-finite type, then* f *is* $(\mathfrak{A}, \mathfrak{F})$*-finitely presentable.*

Proof. (a1) The class $\Sigma^1_{\mathfrak{A}}$ of $(\mathfrak{A}, \mathfrak{F})$-finitely presentable morphisms contains all isomorphisms and is contained in the class $\Sigma^0_{\mathfrak{A}}$ of morphisms of $(\mathfrak{A}, \mathfrak{F})$-finite type.

(a2) Let morphisms $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ belong to the class $\Sigma^i_{\mathfrak{A}}$, $i = 0, 1$.

The claim is that their composition, $X \xrightarrow{g \circ f} Z$, belongs to the same class; i.e. for any filtered projective system $D \xrightarrow{\mathfrak{D}} \mathfrak{F}/Z$ such that $\lim(\mathfrak{F}_Z \circ \mathfrak{D})$ exists, the canonical map

$$\mathrm{colim}\; \mathfrak{B}/Z(\mathfrak{F}_Z \circ \mathfrak{D}, (X, gf)) \longrightarrow \mathfrak{B}/Z(\lim(\mathfrak{F}_Z \circ \mathfrak{D}), (X, gf))$$

is injective if $i = 0$ and bijective if $i = 1$.

(i) First, we consider the case $i = 0$.

Let (u_ν) and (u'_ν) be two inductive systems of arrows $\mathfrak{F}_Z \circ \mathfrak{D}(\nu) \longrightarrow (X, gf)$, $\nu \in ObD$, (i.e. $gfu_\nu = gfu'_\nu$ for all ν) such that the compositions of u_ν and u'_ν with the canonical morphism $\lim(\mathfrak{F}_Z \circ \mathfrak{D}) \xrightarrow{p_\nu} \mathfrak{F}_Z \circ \mathfrak{D}(\nu)$ are equal. With more reason, $(fu_\nu)p_\nu = (fu'_\nu)p_\nu$. Since $Y \xrightarrow{g} Z$ is of $(\mathfrak{A}, \mathfrak{F})$-finite type, $fu_\mu = fu'_\mu$ for an appropriate μ. Replacing $\mathfrak{D}$ by the composition, $\mathfrak{D}_\mu$, of $\mathfrak{D}$ with the canonical functor $\mu\backslash D \longrightarrow D$, we can regard (u_ν) and (u'_ν) as inductive systems of arrows $\mathfrak{D}_\mu(\nu) \longrightarrow (X, f)$, $\nu \in Ob\mu\backslash D$, which equalize the canonical morphism

$$\lim(\mathfrak{F}_Z \circ \mathfrak{D}) = \lim(\mathfrak{F}_Z \circ \mathfrak{D}_\mu) \xrightarrow{\ p_\nu\ } \mathfrak{F}_Z \circ \mathfrak{D}_\mu(\nu).$$

Since $X \xrightarrow{f} Y$ belongs to $\Sigma^0_{\mathfrak{A}}$, there exists λ such that $u_\lambda = u'_\lambda$; i.e. the systems (u_ν) and (u'_ν) define the same element of $\operatorname{colim} \mathfrak{B}/Z(\mathfrak{F}_Z \circ \mathfrak{D}, (X, gf))$.

(ii) Suppose now that the morphisms f and g belong to the class $\Sigma^1_{\mathfrak{A}}$.

Let $D \xrightarrow{\mathfrak{D}} \mathfrak{A}/Z$ be a filtered projective system, and let $\lim(\mathfrak{F}_Z \circ \mathfrak{D}) \xrightarrow{h} (X, gf)$ be an arbitrary morphism. Consider the morphism $\lim(\mathfrak{F}_Z \circ \mathfrak{D}) \xrightarrow{\ fh\ } (Y, g)$.

Since $Y \xrightarrow{g} Z$ is from $\Sigma^1_{\mathfrak{A}}$, there exists a unique element $\mathfrak{u}$ of $\operatorname{colim} \mathfrak{B}/Z(\mathfrak{F}_Z \circ \mathfrak{D}, (Y, g))$ whose image in $\mathfrak{B}/Z(\lim(\mathfrak{F}_Z \circ \mathfrak{D}), (Y, g))$ coincides with $f \circ h$. Let (u_ν) be an inductive system of arrows $\{\mathfrak{F}_Z \circ \mathfrak{D}_\mu(\nu) \longrightarrow (Y, g)\}$ representing the element $\mathfrak{u}$; that is for some $\mu \in ObD$, there are commutative diagrams

$$\begin{array}{ccc} \lim(\mathfrak{F}_Z \circ \mathfrak{D}_\mu) & \xrightarrow{\ h\ } & (X, g \circ f) \\ {\scriptstyle p_\nu}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{F}_Z \circ \mathfrak{D}_\mu(\nu) & \xrightarrow{\ u_\nu\ } & (Y, g) \end{array} \tag{1}$$

for $\nu \in Ob\mu\backslash D$. Here $\mathfrak{D}_\mu$ is the composition of $\mu\backslash D \longrightarrow D \xrightarrow{\mathfrak{D}} \mathfrak{F}/Z$.

The commutative diagrams (1) mean that there is a diagram $\mu\backslash D \xrightarrow{\widetilde{\mathfrak{D}}_\mu} \mathfrak{F}/Y$ such that $\mathfrak{D}_\mu$ is the composition of $\widetilde{\mathfrak{D}}_\mu$ and the natural functor

$$\mathfrak{F}/Y \xrightarrow{g_*} \mathfrak{F}/Z, \quad (V, \xi) \longmapsto (V, g \circ \xi).$$

Since the arrow $X \xrightarrow{f} Y$ belongs to the class $\Sigma^1_{\mathfrak{A}}$ of $(\mathfrak{A}, \mathfrak{F})$-finitely presentable morphisms, there is a unique element of $\operatorname{colim} \mathfrak{B}/Y(\mathfrak{F}_Y \circ \widetilde{\mathfrak{D}}_\mu, (X, f))$ whose image in $\mathfrak{B}/Y(\lim(\mathfrak{F}_Z \circ \mathfrak{D}_\mu), (X, f))$ is given by h. Here we use that $\lim \mathfrak{D}_\mu = \lim \mathfrak{D}$.

(b) Let

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\widetilde{g}} & X \\ {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ \mathfrak{Y} & \xrightarrow{g} & Y \end{array}$$

be a cartesian square. Suppose that the morphism $X \xrightarrow{f} Y$ belongs to $\Sigma^1_{\mathfrak{A}}$. The claim is that the morphism $\mathfrak{X} \xrightarrow{f'} \mathfrak{Y}$ belongs to $\Sigma^1_{\mathfrak{A}}$.

Let $D \xrightarrow{\mathfrak{D}} \mathfrak{F}/\mathfrak{Y}$ be a filtered projective system, and let $\lim(\mathfrak{F}_{\mathfrak{Y}} \circ \mathfrak{D}) = (V, V \xrightarrow{v} \mathfrak{Y})$. Fix a morphism $(V, v) \xrightarrow{h} (\mathfrak{X}, f')$ in $\mathfrak{B}/\mathfrak{Y}$. Since f belongs to $\Sigma^1_{\mathfrak{A}}$, the morphism $(V, gv) \xrightarrow{\widetilde{g}h} (X, f)$ is the image of a unique element, $\mathfrak{u}$, of $\operatorname{colim} \mathfrak{B}/Y(g_*(\mathfrak{F}_{\mathfrak{Y}} \circ \mathfrak{D}), (X, f))$.

Here $g_*(\mathfrak{F}_{\mathfrak{Y}} \circ \mathfrak{D})$ is the diagram $D \longrightarrow \mathfrak{B}/Y$ obtained by composing $\mathfrak{F}_{\mathfrak{Y}} \circ \mathfrak{D}$ with g.

Let (u_ν) be an inductive system of arrows $\{g_*(\mathfrak{F}_{\mathfrak{Y}} \circ \mathfrak{D})(\nu) \longrightarrow (X, f)\}$ representing the element $\mathfrak{u}$. Then the diagrams

$$\begin{array}{ccc} (V, gv) & \xrightarrow{h} & (\mathfrak{X}, f\widetilde{g}) \\ {\scriptstyle p_\nu}\downarrow & & \downarrow{\scriptstyle \widetilde{g}} \\ g_*\mathfrak{D}(\nu) & \xrightarrow{u_\nu} & (X, f) \end{array}$$

commute. By the universal property of cartesian squares, there exists a unique morphism $\mathfrak{F}_{\mathfrak{Y}} \circ \mathfrak{D}(\nu) \xrightarrow{u'_\nu} (\mathfrak{X}, f')$ such that $u_\nu = \widetilde{g}u'_\nu$.

(c)&(d) The proof of the remaining assertions follows a similar routine. We leave detailed arguments to the reader. ■

1.11.3. Morphisms of locally strictly finite type. We call a morphism $X \xrightarrow{\mathfrak{f}} Y$ of locally strictly finite type, if there exists a commutative diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mathfrak{t}} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle \mathfrak{f}} & \\ & Y & \end{array}$$

in which $\mathfrak{X} \xrightarrow{g} Y$ is a finitely presentable morphism and $\mathfrak{X} \xrightarrow{\mathfrak{t}} X$ a strict epimorphism.

1.11.4. Proposition. *Suppose that the class of strict epimorphisms of the category $\mathfrak{B}$ is stable under pull-backs along arbitrary morphisms. Then pull-backs of morphisms of locally strictly finite type are morphisms of locally strictly finite type.*

Proof. Let

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\mathfrak{f}}} & \widetilde{Y} \\ \downarrow & cart & \downarrow \\ X & \xrightarrow{\mathfrak{f}} & Y \end{array}$$

be a cartesian with $X \xrightarrow{\mathfrak{f}} Y$ a morphism of locally strictly finite type. The claim is that its pull-back, $\widetilde{X} \xrightarrow{\widetilde{\mathfrak{f}}} \widetilde{Y}$ is of locally strictly finite type.

In fact, let $\mathfrak{X} \xrightarrow{\mathfrak{t}} X$ be a strict epimorphism such that the composition $\mathfrak{f} \circ \mathfrak{t}$ is a locally finitely presentable morphism. By hypothesis, arbitrary pull-backs of $\mathfrak{X} \xrightarrow{\mathfrak{t}} X$ exist and are strict epimorphisms. In particular, there exist the diagram

$$\begin{array}{ccccc} \widetilde{\mathfrak{X}} & \xrightarrow{\widetilde{\mathfrak{t}}} & \widetilde{X} & \xrightarrow{\widetilde{\mathfrak{f}}} & \widetilde{Y} \\ \downarrow & cart & \downarrow & cart & \downarrow \\ \mathfrak{X} & \xrightarrow{\mathfrak{t}} & X & \xrightarrow{\mathfrak{f}} & Y \end{array}$$

whose both squares are cartesian. Therefore, the square

$$\begin{array}{ccc} \widetilde{\mathfrak{X}} & \xrightarrow{\widetilde{\mathfrak{f}} \circ \widetilde{\mathfrak{t}}} & \widetilde{Y} \\ \downarrow & cart & \downarrow \\ \mathfrak{X} & \xrightarrow{\mathfrak{f} \circ \mathfrak{t}} & Y \end{array}$$

is cartesian. By 1.11.2, any pull-back of a locally finitely presentable morphism is a finitely presentable morphism. Therefore, since the lower horizontal arrow, $\mathfrak{f} \circ \mathfrak{t}$, is finitely presentable, the upper horizontal arrow, $\widetilde{\mathfrak{X}} \xrightarrow{\widetilde{\mathfrak{f}} \circ \widetilde{\mathfrak{t}}} \widetilde{Y}$ is finitely presentable too, hence the assertion. ■

2. Representability. Semi-separated and separated objects and morphisms. Closed immersions.

2.0. Definition. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$.

Let $\mathcal{P}$ be a class of morphisms of the category $\mathfrak{A}$. We call a morphism $X \xrightarrow{f} Y$ of the category $\mathfrak{B}$ *representable* by morphisms of $\mathcal{P}$, or $(\mathcal{P}, \mathfrak{F})$*-representable*, if, for any morphism $\mathfrak{F}(V) \xrightarrow{g} Y$, there exist a morphism $\mathfrak{F}(W) \xrightarrow{\tilde{g}} X$ and an arrow $W \xrightarrow{v} V$ from $\mathcal{P}$ such that

$$\begin{array}{ccc} \mathfrak{F}(W) & \xrightarrow{\ \tilde{g}\ } & X \\ {\scriptstyle \mathfrak{F}(v)}\downarrow & cart & \downarrow{\scriptstyle f} \\ \mathfrak{F}(V) & \xrightarrow{\ g\ } & Y \end{array}$$

is a cartesian square; in particular, it commutes.

We denote by $\widehat{\mathcal{P}}_{\mathfrak{F}}$ the class of all morphisms of the category $\mathfrak{B}$ representable by morphisms of $\mathcal{P}$.

2.1. Proposition. *(a) If the class $\mathcal{P}$ contains all identical morphisms of the category $\mathfrak{A}$, then every isomorphism in the category $\mathfrak{B}$ belongs to the class $\widehat{\mathcal{P}}_{\mathfrak{F}}$.*

(b) The class $\widehat{\mathcal{P}}_{\mathfrak{F}}$ of $(\mathcal{P}, \mathfrak{F})$-representable morphisms is closed under pull-backs.

(c) If the class $\mathcal{P}$ is closed under composition, then $\widehat{\mathcal{P}}_{\mathfrak{F}}$ has the same property.

(d) Suppose that the class $\mathcal{P}$ is stable under pull-backs and the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is full and preserves pull-backs. Then the image $\mathfrak{F}(\mathcal{P})$ of the class $\mathcal{P}$ is contained in $\widehat{\mathcal{P}}_{\mathfrak{F}}$.

Proof. The assertions (a) and (d) are obvious.

The assertions (b) and (c) follow from the general nonsense fact that the composition of cartesian squares is a cartesian square: if in the commutative diagram

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ {\scriptstyle f''}\downarrow & cart & {\scriptstyle f'}\downarrow & cart & \downarrow{\scriptstyle f} \\ Y'' & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

both squares are cartesian, then the square

$$\begin{array}{ccc} X'' & \longrightarrow & X \\ {\scriptstyle f''}\downarrow & cart & \downarrow{\scriptstyle f} \\ Y'' & \longrightarrow & Y \end{array}$$

is cartesian. Details are left to the reader. ∎

2.2. Note. Let $\mathcal{P}$ be a class of morphisms of the category $\mathfrak{A}$ and $\mathcal{P}^\infty$ the smallest class of morphisms of $\mathfrak{A}$ containing $\mathcal{P}$ and closed under composition. In other words, $\mathcal{P}^\infty$ consists of all possible compositions of arrows from the class $\mathcal{P}$.

For any functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$, we have the equality $\widehat{(\mathcal{P}^\infty)}_{\mathfrak{F}} = (\widehat{\mathcal{P}}_{\mathfrak{F}})^\infty$.

This follows from the fact that the composition of cartesian squares is a cartesian square (see the argument of 2.1).

2.3. Proposition. *Let $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ be a full functor preserving pull-backs and $\mathcal{P}$ a class of morphisms of the category $\mathfrak{A}$ closed under pull-backs. Then $\mathfrak{F}(\mathcal{P}) \subseteq \widehat{\mathcal{P}}_{\mathfrak{F}}$.*

Proof. The argument is left to the reader. ■

2.4. Note. All functors $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ in the main body of the work (and examples 1.0.1–1.0.4) to which formalism of this chapter is intended satisfy the conditions of 2.3.

2.5. Representable morphisms. If the class $\mathcal{P}$ coincides with the class of all morphisms of the category $\mathfrak{A}$, then we call $(\mathcal{P}, \mathfrak{F})$-representable morphisms *representable*, or, sometimes, *affine*, and denote this class by $\widehat{\mathfrak{A}}_{\mathfrak{F}}$.

2.6. Semi-separated objects and morphisms.

2.6.1. Definition. An object $\mathcal{G}$ of the category $\mathfrak{B}$ is *semi-separated*, if there exists a product $\mathcal{G} \times \mathcal{G}$ and the diagonal morphism $\mathcal{G} \xrightarrow{\Delta_{\mathcal{G}}} \mathcal{G} \times \mathcal{G}$ is representable.

2.6.2. Proposition. *Let $\mathfrak{A}$ be a category with finite limits; and let the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ be full and preserve finite products and pull-backs. Let G be an object of $\mathfrak{B}$ such that there is a product $G \times G$. The following conditions are equivalent:*
(a) For every object X of $\mathfrak{A}$, any morphism $\mathfrak{F}(X) \longrightarrow G$ is representable.
(b) The object G is semi-separated.

Proof. (a)⇒(b). Let X be an object of $\mathfrak{A}$ and $\mathfrak{F}(X) \xrightarrow{f} G \times G$ an arbitrary morphism. Taking compositions of f with the projections $G \times G \rightrightarrows G$, we obtain a pair of morphisms $\mathfrak{F}(X) \overset{f_1}{\underset{f_2}{\rightrightarrows}} G$. Their fiber product, $\mathfrak{F}(X) \times_G \mathfrak{F}(X)$, is a part of the cartesian square

$$\begin{array}{ccc} G & \xrightarrow{\Delta_G} & G \times G \\ \uparrow & cart & \uparrow f_1 \times f_2 \\ \mathfrak{F}(X) \times_G \mathfrak{F}(X) & \longrightarrow & \mathfrak{F}(X) \times \mathfrak{F}(X) \end{array}$$

The morphism $\mathfrak{F}(X) \xrightarrow{f} G \times G$ is the composition of the diagonal morphism

$$\mathfrak{F}(X) \xrightarrow{\Delta_{\mathfrak{F}(X)}} \mathfrak{F}(X) \times \mathfrak{F}(X)$$

and the morphism

$$\mathfrak{F}(X) \times \mathfrak{F}(X) \xrightarrow{f_1 \times f_2} G \times G.$$

Therefore, the cartesian square

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & G \\ \downarrow & cart & \downarrow \Delta_G \\ \mathfrak{F}(X) & \xrightarrow{f} & G \times G \end{array}$$

is the composition of two cartesian squares:

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathfrak{F}(X)\times_G\mathfrak{F}(X) & \longrightarrow & G \\ \big\downarrow & cart & \big\downarrow & cart & \big\downarrow \Delta_G \\ \mathfrak{F}(X) & \xrightarrow{\Delta_{\mathfrak{F}(X)}} & \mathfrak{F}(X)\times\mathfrak{F}(X) & \xrightarrow{f_1\times f_2} & G\times G \end{array} \qquad (1)$$

Since the category $\mathfrak{A}$ has finite limits, in particular products, the presheaf $\mathfrak{F}(X)\times\mathfrak{F}(X)$ is representable: $\mathfrak{F}(X)\times\mathfrak{F}(X)\simeq\mathfrak{F}(X\times X)$. The condition (a) implies that $\mathfrak{F}(X)\times_G\mathfrak{F}(X)$ is representable. It follows from the left cartesian square in (1) (and the fact that $\mathfrak{A}$ has fibred products) that the presheaf $\mathcal{F}$ is representable. Therefore, the diagonal morphism $G \xrightarrow{\Delta_G} G\times G$ is representable.

(b)⇒(a). Let X, Y be objects of $\mathfrak{A}$ and $\mathfrak{F}(X)\longrightarrow G\longleftarrow\mathfrak{F}(Y)$ arbitrary morphisms. Consider the cartesian square

$$\begin{array}{ccc} G & \xrightarrow{\Delta_G} & G\times G \\ \big\uparrow & cart & \big\uparrow \\ \mathfrak{F}(X)\times_G\mathfrak{F}(Y) & \longrightarrow & \mathfrak{F}(X)\times\mathfrak{F}(Y) \end{array}$$

Since $\mathfrak{A}$ has finite products, $\mathfrak{F}(X)\times\mathfrak{F}(Y)$ is representable: $\mathfrak{F}(X)\times\mathfrak{F}(Y)\simeq\mathfrak{F}(X\times Y)$. By hypothesis (b), the diagonal morphism $G\xrightarrow{\Delta_G}G\times G$ is affine. Therefore, the presheaf $\mathfrak{F}(X)\times_G\mathfrak{F}(Y)$ is representable too; hence the assertion. ∎

2.6.3. Proposition. *Suppose that the category $\mathfrak{A}$ has limits of diagrams $\delta\mathfrak{D}\xrightarrow{\mathfrak{D}}\mathfrak{A}$ for $\delta\mathfrak{D}$ from a certain class Ξ, and the functor $\mathfrak{A}\xrightarrow{\mathfrak{F}}\mathfrak{B}$ is full and preserves the limits of these diagrams. Let $\delta\mathfrak{D}\xrightarrow{\mathfrak{D}}\mathfrak{B}$ be a diagram with $\delta\mathfrak{D}\in\Xi$ which maps objects of $\delta\mathfrak{D}$ to semi-separated objects. Then the limit of $\mathfrak{D}$ (if any) is semi-separated.*

Proof. Let $G=\lim\mathfrak{D}$ and let $G\xrightarrow{\xi_{\mathfrak{b}}}\mathfrak{D}(\mathfrak{b})$, $\mathfrak{b}\in Ob\delta\mathfrak{D}$, be the universal cone. Then the diagonal morphism $\mathcal{G}\xrightarrow{\Delta_{\mathcal{G}}}\mathcal{G}\times\mathcal{G}$ is the limit of the diagram $\mathfrak{D}\xrightarrow{\Delta_{\mathfrak{D}}}\mathfrak{D}\times\mathfrak{D}$ of diagonal morphisms. Let $\mathfrak{F}(X)\xrightarrow{\gamma}G\times G$ be an arbitrary morphism and let $\mathfrak{F}(X)\xrightarrow{\gamma_{\mathfrak{b}}}\mathfrak{D}(\mathfrak{b})\times\mathfrak{D}(\mathfrak{b})$ denote its composition with the projections. $G\times G\xrightarrow{\xi_{\mathfrak{b}}\times\xi_{\mathfrak{b}}}\mathfrak{D}(\mathfrak{b})\times\mathfrak{D}(\mathfrak{b})$, $\mathfrak{b}\in Ob\delta\mathfrak{D}$. By hypothesis, the diagonal morphism $\mathfrak{D}(\mathfrak{b})\xrightarrow{\Delta_{\mathfrak{D}(\mathfrak{b})}}\mathfrak{D}(\mathfrak{b})\times\mathfrak{D}(\mathfrak{b})$ is representable. So that there exists a cartesian square of the form

$$\begin{array}{ccc} \mathfrak{F}(\mathfrak{X}_{\mathfrak{b}}) & \xrightarrow{\mathfrak{F}(\lambda_{\mathfrak{b}})} & \mathfrak{F}(X) \\ \tilde{\gamma}_{\mathfrak{b}}\big\downarrow & cart & \big\downarrow\gamma_{\mathfrak{b}} \\ \mathfrak{D}(\mathfrak{b}) & \xrightarrow{\Delta_{\mathfrak{D}(\mathfrak{b})}} & \mathfrak{D}(\mathfrak{b})\times\mathfrak{D}(\mathfrak{b}) \end{array}$$

By the universal property of cartesian squares, for every arrow $\mathfrak{a} \xrightarrow{\mathfrak{u}} \mathfrak{b}$ of $\delta\mathfrak{D}$, there exists a unique morphism $\mathfrak{F}(\mathfrak{X}_\mathfrak{a}) \longrightarrow \mathfrak{F}(\mathfrak{X}_\mathfrak{b})$ making the diagram

$$\begin{array}{ccc} \mathfrak{F}(\mathfrak{X}_\mathfrak{a}) & \longrightarrow & \mathfrak{F}(\mathfrak{X}_\mathfrak{b}) \\ {\scriptstyle \widetilde{\gamma}_\mathfrak{a}}\downarrow & & \downarrow{\scriptstyle \widetilde{\gamma}_\mathfrak{b}} \\ \mathfrak{D}(\mathfrak{a}) & \xrightarrow{\mathfrak{D}(\mathfrak{u})} & \mathfrak{D}(\mathfrak{b}) \end{array}$$

commute. Since, by hypothesis, the functor $\mathfrak{F}$ is full, the morphism $\mathfrak{F}(\mathfrak{X}_\mathfrak{a}) \longrightarrow \mathfrak{F}(\mathfrak{X}_\mathfrak{b})$ is of the image of some arrow $\mathfrak{X}_\mathfrak{a} \xrightarrow{\beta_\mathfrak{u}} \mathfrak{X}_\mathfrak{b}$. The correspondence

$$\mathfrak{c} \longmapsto \mathfrak{X}_\mathfrak{c}, \ (\mathfrak{a} \xrightarrow{\mathfrak{u}} \mathfrak{b}) \longmapsto (\mathfrak{X}_\mathfrak{a} \xrightarrow{\beta_\mathfrak{u}} \mathfrak{X}_\mathfrak{b})$$

is a diagram $\delta\mathfrak{D} \xrightarrow{\widetilde{\mathfrak{D}}} \mathfrak{A}$. By hypothesis, there exists a limit of the diagram $\widetilde{\mathfrak{D}}$ and the natural morphism $\mathfrak{F}(\lim \widetilde{\mathfrak{D}}) \longrightarrow \lim(\mathfrak{F}\circ\widetilde{\mathfrak{D}})$ is an isomorphism. Thus, we have a cartesian square

$$\begin{array}{ccc} \mathfrak{F}(\lim \widetilde{\mathfrak{D}}) & \longrightarrow & \mathfrak{F}(X) \\ \downarrow & cart & \downarrow{\scriptstyle \gamma} \\ G & \xrightarrow{\Delta_G} & G \times G \end{array}$$

Since the functor $\mathfrak{F}$ is full, its upper horizontal arrow is the image of a morphism of the category $\mathfrak{A}$. This shows the representability of the diagonal morphism $\mathcal{G} \xrightarrow{\Delta_\mathcal{G}} \mathcal{G} \times \mathcal{G}$. ■

2.6.3.1. Corollary. *Let $\mathfrak{A}$ be a category with finite limits and $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ a full functor preserving finite limits. Then limit of any finite diagram of semi-separated objects of $\mathfrak{B}$ is a semi-separated object.*

Proof. The assertion follows from 2.6.3. ■

2.6.4. Semiseparated morphisms. A morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ of the category $\mathfrak{B}$ will be called *semi-separated* if it has a kernel pair, $K_2(\mathfrak{f}) = \mathcal{X} \times_\mathcal{Y} \mathcal{X}$, and the diagonal morphism $\mathcal{X} \xrightarrow{\Delta_\mathfrak{f}} K_2(\mathfrak{f})$ is representable.

2.6.5. Proposition. *Let $\mathfrak{A}$ be a category with finite limits; and let the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ be full and preserve finite products and pull-backs. Let $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ be a morphism of $\mathfrak{B}$ such that there is a kernel pair $\mathcal{X} \times_\mathcal{Y} \mathcal{X}$. The following conditions are equivalent:*

(a) The morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is semi-separated.

(b) Any morphism $(\mathfrak{F}(\mathcal{Z}), \gamma) \longrightarrow (\mathcal{X}, \mathfrak{f})$ *of the category* $\mathfrak{B}/\mathcal{Y}$ *is representable for the local data*

$$\mathfrak{F}/\mathcal{Y} \xrightarrow{\mathfrak{F}_{\mathcal{Y}}} \mathfrak{B}/\mathcal{Y}, \quad (\mathcal{Z}, \mathfrak{F}(\mathcal{Z}) \to \mathcal{Y}) \longmapsto (\mathfrak{F}(\mathcal{Z}), \mathfrak{F}(\mathcal{Z}) \to \mathcal{Y}).$$

Proof. The assertion follows from 2.6.2. ∎

2.6.6. Proposition. *Suppose that the category* $\mathfrak{A}$ *has limits of diagrams* $\delta\mathfrak{D} \xrightarrow{\mathfrak{D}} \mathfrak{A}$ *for* $\delta\mathfrak{D}$ *from a class* Ξ, *and the functor* $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ *is full and preserves the limits of these diagrams. Let* $\delta\mathfrak{D} \xrightarrow{\mathfrak{D}} \mathfrak{B}/\mathcal{Y}$ *be a diagram with* $\delta\mathfrak{D} \in \Xi$ *which maps objects of* $\delta\mathfrak{D}$ *to semi-separated morphisms. Then the limit of* $\mathfrak{D}$ *(if any) is a semi-separated morphism.*

Proof. The assertion follows from 2.6.3. ∎

2.6.7. Proposition. *Pull-backs of semi-separated morphisms are semi-separated.*

Proof. Let the lower horizontal arrow of a cartesian square

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\mathfrak{f}} & \mathfrak{Y} \\ {\scriptstyle \phi_1}\big\downarrow & cart & \big\downarrow{\scriptstyle \phi} \\ \mathcal{X} & \xrightarrow{\mathfrak{t}} & \mathcal{Y} \end{array} \qquad (1)$$

is a $\mathcal{P}$-separated morphism. The claim is that its pull-back – the upper horizontal arrow, is $\mathcal{P}$-separated too. In fact, we can insert the cartesian square to the diagram

$$\begin{array}{ccccccc} \mathfrak{X} & \xrightarrow{\Delta_{\mathfrak{f}}} & K_2(\mathfrak{f}) & \overset{\mathfrak{f}_1}{\underset{\mathfrak{f}_2}{\rightrightarrows}} & \mathfrak{X} & \xrightarrow{\mathfrak{f}} & \mathfrak{Y} \\ {\scriptstyle \phi_1}\big\downarrow & & {\scriptstyle \phi_2}\big\downarrow & cart & {\scriptstyle \phi_1}\big\downarrow & cart & \big\downarrow{\scriptstyle \phi} \\ \mathcal{X} & \xrightarrow{\Delta_{\mathfrak{t}}} & K_2(\mathfrak{t}) & \overset{\mathfrak{t}_1}{\underset{\mathfrak{t}_2}{\rightrightarrows}} & \mathcal{X} & \xrightarrow{\mathfrak{t}} & \mathcal{Y} \end{array}$$

whose middle square is the morphism of kernel pairs. This diagram yields the decomposition of (1) into two commutative squares

$$\begin{array}{ccccc} \mathfrak{X} & \xrightarrow{\Delta_{\mathfrak{f}}} & K_2(\mathfrak{f}) & \xrightarrow{\mathfrak{f} \circ \mathfrak{f}_1} & \mathfrak{Y} \\ {\scriptstyle \phi_1}\big\downarrow & & {\scriptstyle \phi_2}\big\downarrow & & \big\downarrow{\scriptstyle \phi} \\ \mathcal{X} & \xrightarrow{\Delta_{\mathfrak{t}}} & K_2(\mathfrak{t}) & \xrightarrow{\mathfrak{t} \circ \mathfrak{t}_1} & \mathcal{Y} \end{array} \qquad (2)$$

It follows from the fact that the square (1) is cartesian that the left square of (2) is cartesian. ■

2.7. A generalization: $\mathcal{P}$-separated morphisms and $\mathcal{P}$-separated objects. Fix a local data $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ and a class $\mathcal{P}$ of arrows of the category $\mathfrak{A}$.

2.7.1. Definitions. (a) A morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ of the category $\mathfrak{B}$ will be called $\mathcal{P}$-*separated*, if the diagonal morphism $\mathcal{X} \xrightarrow{\Delta_{\mathfrak{f}}} K_2(\mathfrak{f}) = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (exists and) is representable by morphisms of $\mathcal{P}$.

(b) An object $\mathfrak{X}$ of the category $\mathfrak{B}$ is called $\mathcal{P}$-separated, if $\mathfrak{X} \times \mathfrak{X}$ exists and the diagonal morphism $\mathfrak{X} \xrightarrow{\Delta_{\mathfrak{X}}} \mathfrak{X} \times \mathfrak{X}$ is representable by morphisms of $\mathcal{P}$.

2.7.2. Proposition. *Let $\mathcal{U} \xrightarrow{\pi} \mathfrak{X}$ be a morphism of the category $\mathfrak{B}$ having a kernel pair $K_2(\pi) = \mathcal{U} \times_{\mathfrak{X}} \mathcal{U} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{U}$. If the object $\mathfrak{X}$ is $\mathcal{P}$-separated, then the natural embedding*

$$K_2(\pi) = \mathcal{U} \times_{\mathfrak{X}} \mathcal{U} \xrightarrow{\mathfrak{j}_\pi} \mathcal{U} \times \mathcal{U}$$

is representable by morphisms of $\mathcal{P}$.

Proof. Notice that the square

$$\begin{array}{ccc} K_2(\pi) & \xrightarrow{\pi \circ p_1} & \mathfrak{X} \\ \mathfrak{j}_\pi \downarrow & cart & \downarrow \Delta_{\mathfrak{X}} \\ \mathcal{U} \times \mathcal{U} & \xrightarrow{\pi \times \pi} & \mathfrak{X} \times \mathfrak{X} \end{array} \qquad (1)$$

is cartesian by definition of the kernel pair $K_2(\pi) = \mathcal{U} \times_{\mathfrak{X}} \mathcal{U}$.

Suppose $\mathfrak{X}$ is $\mathcal{P}$-separated; that is the diagonal morphism $\mathfrak{X} \xrightarrow{\Delta_{\mathfrak{X}}} \mathfrak{X} \times \mathfrak{X}$ is representable by morphisms of $\mathcal{P}$. Since the class of arrows representable by morphisms of $\mathcal{P}$ is stable under pull-backs and the square (1) is cartesian, the embedding $K_2(\pi) \xrightarrow{\mathfrak{j}_\pi} \mathcal{U} \times \mathcal{U}$ is representable by morphisms of $\mathcal{P}$. ■

2.7.3. Proposition. *(a) Pull-backs of $\mathcal{P}$-separated morphisms are $\mathcal{P}$-separated.*

(b) Suppose that $\mathcal{P}$ is closed under composition.

(b1) If $\mathcal{X}$ is a $\mathcal{P}$-separated object and $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ is a $\mathcal{P}$-separated morphism, then the object $\mathcal{U}$ is $\mathcal{P}$-separated.

(b2) The composition of $\mathcal{P}$-separated morphisms is a $\mathcal{P}$-separated morphism.

Proof. (a) This follows from the argument of 2.6.7.

(b1) By 2.7.2, the fact that $\mathcal{X}$ is a $\mathcal{P}$-separated object implies that the canonical embedding $K_2(\pi) = \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \xrightarrow{\mathfrak{j}_\pi} \mathcal{U} \times \mathcal{U}$ is $\mathcal{P}$-representable. By definition, the morphism $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ is $\mathcal{P}$-semi-separated if the diagonal morphism $\mathcal{U} \xrightarrow{\Delta_\pi} K_2(\pi)$ is $\mathcal{P}$-representable. By 2.1(c), if the class $\mathcal{P}$ is closed under composition, then the class of $\mathcal{P}$-representable morphisms is closed under composition. In particular, the diagonal morphism $\mathcal{U} \xrightarrow{\Delta_{\mathcal{U}}} \mathcal{U} \times \mathcal{U}$, being the composition of two $\mathcal{P}$-representable morphisms,

$$\mathcal{U} \xrightarrow{\Delta_\pi} K_2(\pi) \quad \text{and} \quad K_2(\pi) = \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \xrightarrow{\mathfrak{j}_\pi} \mathcal{U} \times \mathcal{U},$$

is $\mathcal{P}$-representable. The latter means that $\mathcal{U}$ is a $\mathcal{P}$-separated object.

(b2) Let $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ and $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ be morphisms of the category $\mathfrak{B}$. It follows from definitions, that the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is $\mathcal{P}$-separated iff the object $(\mathcal{X}, \mathfrak{f})$ of the category $\mathfrak{B}/\mathcal{Y}$ is $\mathcal{P}_{\mathcal{Y}}$-separated, where $\mathcal{P}_{\mathcal{Y}} = F_Y^{-1}(\mathcal{P})$ – the preimage of the class $\mathcal{P}$ by the forgetful functor $\mathfrak{B}/\mathcal{Y} \longrightarrow \mathfrak{B}$. Therefore, if the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is $\mathcal{P}$-separated, then the canonical morphism

$$(\mathcal{U}, \mathfrak{f}\pi) \times_{(\mathcal{X},\mathfrak{f})} (\mathcal{U}, \mathfrak{f}\pi) \longrightarrow (\mathcal{U}, \mathfrak{f}\pi) \times (\mathcal{U}, \mathfrak{f}\pi) = (K_2(\mathfrak{f}\pi), K_2(\mathfrak{f}\pi) \to \mathcal{Y}) \tag{2}$$

is $\mathcal{P}_{\mathcal{Y}}$-representable. Notice that $(\mathcal{U}, \mathfrak{f}\pi) \times_{(\mathcal{X},\mathfrak{f})} (\mathcal{U}, \mathfrak{f}\pi) = (K_2(\pi), K_2(\pi) \to \mathcal{Y})$ and the morphism (2) is given by the canonical embedding $K_2(\pi) \longrightarrow K_2(\mathfrak{f}\pi)$, and $\mathcal{P}_{\mathcal{Y}}$-representability of (2) means that the morphism $K_2(\pi) \longrightarrow K_2(\mathfrak{f}\pi)$ is $\mathcal{P}$-representable. The diagonal morphism $\mathcal{U} \xrightarrow{\Delta_{\mathfrak{f}\pi}} K_2(\mathfrak{f}\pi)$ is the composition of the $\mathcal{P}$-representable morphism $K_2(\pi) \longrightarrow K_2(\mathfrak{f}\pi)$ and the diagonal morphism $\mathcal{U} \xrightarrow{\Delta_\pi} K_2(\pi) = \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$. So that if $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ is a $\mathcal{P}$-separated morphism, then the diagonal morphism $\mathcal{U} \xrightarrow{\Delta_{\mathfrak{f}\pi}} K_2(\mathfrak{f}\pi)$ is the composition of two $\mathcal{P}$-representable morphism. Since, by hypothesis, $\mathcal{P}$ is closed under composition, it follows from 2.1(c), that the morphism $\mathcal{U} \xrightarrow{\Delta_{\mathfrak{f}\pi}} K_2(\mathfrak{f}\pi)$ is $\mathcal{P}$-representable; that is the composition $\mathcal{U} \xrightarrow{\mathfrak{f}\pi} \mathcal{Y}$ is a $\mathcal{P}$-separated morphism. ∎

2.8. Strict monomorphisms and closed immersions.

2.8.1. Strict monomorphisms. For a morphism $Y \xrightarrow{f} X$ of a category $\mathfrak{A}$, let Λ_f denote the class of all pairs of arrows $X \overset{u_1}{\underset{u_2}{\rightrightarrows}} V$ equalizing f.

A morphism $Y \xrightarrow{f} X$ is called a *strict monomorphism* if any morphism $Z \xrightarrow{g} X$ such that $\Lambda_f \subseteq \Lambda_g$ has a unique decomposition $g = f \circ g'$. We denote the class of strict monomorphisms of the category $\mathfrak{A}$ by $\mathfrak{M}_s(\mathfrak{A})$, or by $\mathfrak{M}_s$. The class $\mathfrak{E}_s = \mathfrak{E}_s(\mathfrak{A})$ of strict epimorphisms is defined dually.

2.8.2. Lemma. *(a) Any strict monomorphism is a monomorphism.*
(b) Pull-backs of strict monomorphisms are strict monomorphisms.

Proof. (a) Evidently, $\Lambda(f \circ g) \subseteq \Lambda(f)$. If f is a strict monomorphism, then the morphism g is uniquely determined by the composition $f \circ g$. So that if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

(b) In fact, consider the diagram

$$\begin{array}{ccccc} X \times_Y V & \xrightarrow{p_2} & V & & \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g} & & \\ X & \xrightarrow{f} & Y & \rightrightarrows & Z \end{array} \qquad (1)$$

where $Y \rightrightarrows Z$ is an arbitrary pair of arrows from the class Λ_f of arrows equalizing f. It follows from the universal property of cartesian squares that p_2 is a universal arrow equalizing all pairs $\Lambda_f \circ g = \{(u_1 g, u_2 g) | \ (u_1, u_2) \in \Lambda_f\}$. ■

2.8.3. Note. Suppose that a morphism $X \xrightarrow{f} Y$ is such there exists its cokernel pair $\mathcal{C}_2(f) = Y \coprod_X Y$. Then f is a strict monomorphism iff it is a kernel of the *cokernel pair* (that is the pair of coprojections) $Y \rightrightarrows \mathcal{C}_2(f)$. In particular, if the category $\mathcal{A}$ has cokernel pairs, then strict monomorphisms can be defined as morphisms $X \longrightarrow Y$ such that the diagram $X \longrightarrow Y \rightrightarrows \mathcal{C}_2(f)$ is exact.

2.8.3.1. Example. Let $\mathcal{A}$ be the category $\mathbf{Aff}_k = Alg_k^{op}$ of noncommutative affine schemes over k. Let $S \xrightarrow{f} T$ be a k-algebra morphism. The corresponding morphism of affine schemes $T^\vee \xrightarrow{f^\vee} S^\vee$ is a strict monomorphism iff the diagram

$$T \coprod_S T = K_2(f) \rightrightarrows T \xrightarrow{f} S$$

is exact. The latter means that f is a strict epimorphism, that is S is the quotient of T by the two-sided ideal $Ker(f)$.

2.8.4. Lemma. *(a) If the composition, gf, of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a strict monomorphism, then f is a strict monomorphism.*
(b) Any retraction is a strict monomorphism.

Proof. (a) If gf is a universal morphism with respect to the class of arrows

$$\Lambda_{gf} = \{Z \overset{u_1}{\underset{u_2}{\rightrightarrows}} V \mid u_1 gf = u_2 gf\},$$

then f is universal for the class of arrows $\Lambda_{gf} \circ g = \{(u_1 g, u_2 g) | \ (u_1, u_2) \in \Lambda_{gf}\}$.

(b) Let $X \xrightarrow{p} Y$ is a retraction, i.e. there exists a morphism $Y \xrightarrow{e} X$ such that $ep = id_X$. Then p is a kernel of the pair $X \overset{id_X}{\underset{pe}{\rightrightarrows}} X$.

In fact, if $Y \xrightarrow{f} X$ is a morphism equalizing the pair (id_X, pe), then $f = p \circ (ef)$; and this decomposition is unique because p is a monomorphism. ■

2.8.5. Closed immersions. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$.

We call a morphism of the category $\mathfrak{B}$ a *closed immersion*, if it is representable by strict monomorphisms of the category $\mathfrak{A}$.

2.8.6. Example. Let $\mathcal{A}$ be the category $\mathbf{CAff}_k$ of commutative affine schemes over a commutative unital ring k. Then strict monomorphisms are exactly closed immersions of affine schemes. If X and Y are arbitrary schemes identified with the corresponding sheaves of sets on the category $\mathbf{CAff}_k$, then a morphism $X \longrightarrow Y$ is a closed immersion in the sense of the definition 2.3 iff it is a closed immersion of schemes in the conventional sense.

2.8.7. Note. This example shows in particular that a strict monomorphism of (pre)sheaves is not necessarily a closed immersion. For instance, if $X \xrightarrow{f} Y$ is a scheme morphism, the diagonal morphism $X \xrightarrow{\Delta_f} X \times_Y X$ is a kernel of the pair of projections $X \times_Y X \rightrightarrows X$, hence it is a strict monomorphism of sheaves of sets. But, Δ_f is a closed immersion (in the sense of 2.3) only if the scheme morphism f is separated. Note that, in general, the diagonal morphism Δ_f is not representable.

2.9. Separated objects. We call an object $\mathcal{X}$ of the category $\mathfrak{B}$ *separated* if the product $\mathcal{X} \times \mathcal{X}$ exists and the diagonal morphism $\mathcal{X} \xrightarrow{\Delta_{\mathcal{X}}} \mathcal{X} \times \mathcal{X}$ is a closed immersion.

2.9.1. Proposition. *Let a morphism $\mathcal{U} \xrightarrow{\pi} \mathfrak{X}$ of the category $\mathfrak{B}$ have a kernel pair $K_2(\pi) = \mathcal{U} \times_{\mathfrak{X}} \mathcal{U} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathcal{U}$. If $\mathfrak{X}$ is separated, then the natural embedding*

$$K_2(\pi) = \mathcal{U} \times_{\mathfrak{X}} \mathcal{U} \xrightarrow{\mathfrak{j}_\pi} \mathcal{U} \times \mathcal{U}$$

is a closed immersion.

Proof. The assertion is a special case of 2.7.3. ∎

2.10. Separated morphisms. We call a morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ *separated* if there exists the kernel pair $K_2(\mathfrak{f}) = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ and the diagonal morphism $\mathcal{X} \xrightarrow{\Delta_{\mathfrak{f}}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is a closed immersion.

Closed immersions and separated morphisms are discussed in detail in III.2 in the case when $\mathfrak{F}$ is the Yoneda embedding of a category $\mathfrak{A}$ into the category $\mathfrak{A}^\wedge$ of presheaves of sets on $\mathfrak{A}$.

3. Formally smooth and formally étale objects and morphisms.

Fix a category $\mathfrak{B}$ and a class $\mathfrak{M}$ of morphisms of $\mathfrak{B}$, which will be referred sometimes as the class of *infinitesimal* morphisms.

3.0. Formally smooth and formally étale objects.

(i) We call an object $\mathcal{X}$ of the category $\mathfrak{B}$ *formally smooth*, or *formally $\mathfrak{M}$-smooth*, if, for any pair of arrows $\mathcal{S} \xleftarrow{\phi} \mathcal{T} \xrightarrow{g} \mathcal{X}$ with $\phi \in \mathfrak{M}$, there exists an arrow $\mathcal{S} \xrightarrow{\gamma} \mathcal{X}$ such that $\gamma \circ \phi = g$.

(ii) We call an object $\mathcal{X}$ of the category $\mathfrak{B}$ *formally unramified*, or *formally $\mathfrak{M}$-unramified*, if, for any pair of arrows $\mathcal{S} \xleftarrow{\phi} \mathcal{T} \xrightarrow{g} \mathcal{X}$ with $\phi \in \mathfrak{M}$, there exists at most one arrow $\mathcal{S} \xrightarrow{\gamma} \mathcal{X}$ such that $\gamma \circ \phi = g$.

(iii) We call an object $\mathcal{X}$ of $\mathfrak{B}$ *formally étale*, or *formally $\mathfrak{M}$-étale*, if it is both formally smooth and formally unramified.

3.0.1. Note. The notion of a formally $\mathfrak{M}$-smooth object coincides with the notion of $\mathfrak{M}$-injective object, which appeared in the context of additive categories (in homological algebra) about a half of century ago. If $\mathfrak{M}$ is the class of all monomorphisms of the category $\mathfrak{B}$, then formally $\mathfrak{M}$-smooth objects are precisely injective objects of the category $\mathfrak{B}$.

3.0.2. Proposition. *(a) The full subcategory $\mathfrak{B}^{\mathfrak{M}}_{fsm}$ of the category $\mathfrak{B}$ generated by formally $\mathfrak{M}$-smooth objects is closed under retracts and arbitrary products (taken in $\mathfrak{B}$).*

(b) Let $\mathfrak{I}$ be a class of objects of the category $\mathfrak{B}$ and $\mathfrak{M}_{\mathfrak{I}}$ the largest class of arrows of $\mathfrak{B}$ such that all objects of $\mathfrak{I}$ are $\mathfrak{M}_{\mathfrak{I}}$-formally smooth. The class $\mathfrak{M}_{\mathfrak{I}}$ contains all isomorphisms and is closed under compositions and push-forwards.

Proof. The argument is left to the reader. ∎

3.0.3. Formally smooth and formally étale affine schemes. Let $\mathfrak{A}$ be the category $\mathbf{Aff}_k = Alg_k^{op}$ of affine noncommutative schemes over a commutative ring k (see 1.0.3), $\mathfrak{B} = \mathbf{Aff}_k^\wedge$ the category of presheaves of sets on $\mathbf{Aff}_k$, or, what is the same, the category of functors $Alg_k \longrightarrow Sets$, and $\mathfrak{F}$ the Yoneda embedding $\mathbf{Aff}_k \longrightarrow \mathbf{Aff}_k^\wedge$.

For every k-algebra R, we denote by $R^\vee$ the presheaf of sets $Alg_k \longrightarrow Sets$ on $\mathbf{Aff}_k$ corepresentable by the k-algebra R. In what follows, we identify the category $\mathbf{Aff}_k$ of affine k-schemes with the full subcategory of $\mathbf{Aff}_k^\wedge$ formed by the functors $R^\vee$, $R \in ObAlg_k$.

3.0.3.1. Extreme example. Let the class $\mathfrak{M}$ of *infinitesimal* morphisms coincide with the class of all strict monomorphisms of affine noncommutative schemes. And let $\mathcal{R}$ be the tensor algebra, $T_k(\mathcal{M})$, of a k-module $\mathcal{M}$. Then $\mathcal{R}^\vee = T_k(\mathcal{M})^\vee$ is formally smooth iff $\mathcal{M}$ is a projective k-module.

3.0.3.2. The standard setting. Let $\mathfrak{M}_\mathfrak{n}$ denote the class of strict monomorphisms of affine noncommutative schemes such that the kernel of the corresponding strict epimorphism of algebras is a nilpotent ideal.

The formally $\mathfrak{M}_\mathfrak{n}$-smooth (resp. formally $\mathfrak{M}_\mathfrak{n}$-étale, resp. formally $\mathfrak{M}_\mathfrak{n}$-unramified) objects are called *formally smooth* (resp. *formally étale*, resp. *formally unramified*).

3.0.3.3. Proposition. *Let R be a unital associative k-algebra; and let R^e denote the k-algebra $R \otimes_k R^o$, where R^o is the algebra opposite to R.*

(a) The functor $R^\vee$ is formally smooth iff the algebra R is quasi-free in the sense of Quillen and Cuntz [CQ1]. The latter is equivalent to the condition:

(a1) The left R^e-module $\Omega^1_{R|k}$ of Kähler differentials of R (which is, by definition, the kernel of the multiplication $R^e = R \otimes_k R^o \longrightarrow R$) is projective.

(b) The presheaf $R^\vee$ is unramified iff $\Omega^1_{R|k} = 0$.

Proof. A standard argument shows that $R^\vee$ is formally smooth (resp. formally unramified) iff for any strict k-algebra epimorphism $S \xrightarrow{\phi} R$ such that $Ker(\phi)^2 = 0$, there exists a splitting (resp. at most one splitting), that is a k-algebra morphism $R \xrightarrow{\psi} S$ such that $\phi \circ \psi = id_R$.

(a) Thus $R^\vee$ is formally smooth iff $Ext^2_{R^e}(R, M) = 0$ for any R^e-module M. Consider the long exact sequence

$$... \longrightarrow Ext^i_{R^e}(R, M) \longrightarrow Ext^i_{R^e}(R^e, M) \longrightarrow Ext^i_{R^e}(\Omega^1_{R|k}, M) \longrightarrow Ext^{i+1}_{R^e}(R, M) \longrightarrow ...$$

corresponding to the short exact sequence $0 \longrightarrow \Omega^1_{R|k} \longrightarrow R^e \longrightarrow R \longrightarrow 0$. Since $Ext^i_{R^e}(R^e, M) = 0$ for all $i \geq 1$ and all R^e-modules M, $Ext^i_{R^e}(\Omega^1_{R|k}, M) \simeq Ext^{i+1}_{R^e}(R, M)$ for all $i \geq 1$ and all R^e-modules M. In particular, $Ext^2_{R^e}(R, M) = 0$ for all M iff $Ext^1_{R^e}(\Omega^1_{R|k}, M) = 0$ for all M. The latter means precisely that $\Omega^1_{R|k}$ is a projective R^e-module.

(b) Let $R \xrightarrow{\psi} S$ be a k-algebra morphism such that $\phi \circ \psi = id_R$. It gives a decomposition of S into a semidirect product of R and an R-bimodule, M, with multiplication defined by $(r,m)(r',m') = (rr', r\cdot m' + m\cdot r')$. Any other splitting, $R \xrightarrow{\psi'} S$, is (id_R, d), where $R \xrightarrow{d} M$ is a derivation sending k to zero. Thus, the set of splittings of ϕ is in one-to-one correspondence with $Der_{R|k}(M)$. But $Der_{R|k}(M) \simeq Hom_{R^e}(\Omega^1_{R|k}, M)$. Hence ϕ is unramified iff $\Omega^1_{R|k} = 0$. ∎

3.0.4. Quasi-free and separable algebras. Let $\mathfrak{Ass}_k$ be the category whose objects are associative unital k-algebras and morphisms from a k-algebra R to a k-algebra S are conjugation classes of algebra morphisms $R \longrightarrow S$; that is two algebra morphisms, $R \underset{g}{\overset{f}{\rightrightarrows}} S$, are equivalent if $g(-) = tf(-)t^{-1}$ for an invertible element t of the algebra S.

We take as $\mathfrak{A}$ the category $\mathfrak{Aff}_k = \mathfrak{Ass}_k^{op}$ and as $\mathfrak{B}$ the category $\mathfrak{Aff}_k^{\wedge}$ of presheaves of sets on $\mathfrak{Aff}_k$ or, what is the same, the category of functors $\mathfrak{Ass}_k \longrightarrow Sets$.

The functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the Yoneda embedding $R \longmapsto R^\vee = \mathfrak{Ass}_k(R,-)$.

Let $\bar{\mathfrak{M}}_\mathfrak{n}$ consist of morphisms $R^\vee \xrightarrow{[\varphi]^\vee} S^\vee$ corresponding to the conjugacy class of a strict algebra epimorphism $S \xrightarrow{\varphi} R$ with a nilpotent kernel – the image in $\mathfrak{Aff}_k^{\wedge}$ of the class $\mathfrak{M}_\mathfrak{n}$ of morphisms of $\mathbf{Aff}_k^{\wedge}$ (see 3.0.3).

A k-algebra R is called *separable* if R is a projective left R^e-module, $R^e = R \otimes_k R^o$. It follows from the exact sequence of R^e-modules

$$0 \longrightarrow \Omega^1_{R|k} \longrightarrow R^e \longrightarrow R \longrightarrow 0$$

that if R is separable, then $\Omega^1_{R|k}$ is a projective R^e-module, i.e. R is quasi-free [CQ1].

3.0.4.1. Proposition. *Let R be an associative unital k-algebra.*
(a) The following conditions are equivalent:
(i) The affine scheme $R^\vee$ is formally $\bar{\mathfrak{M}}_\mathfrak{n}$-smooth.
(ii) The left R^e-module of Kähler differentials, $\Omega^1_{R/k} \overset{\text{def}}{=} Ker(R^e \to R)$, is projective.
(b) The following conditions are equivalent:
(iii) $R^\vee$ is formally $\bar{\mathfrak{M}}_\mathfrak{n}$-étale.
(iv) $R^\vee$ is formally $\bar{\mathfrak{M}}_\mathfrak{n}$-unramified.
(v) the algebra R is separable.

Proof. (a) Let $S \xrightarrow{\phi} R$ be a k-algebra morphism such that there exists a k-algebra morphism $R \xrightarrow{\psi} S$ right inverse to ϕ in the category $\mathfrak{A}ss_k$. The latter means, in particular, that $\phi\circ\psi$ is conjugate to id_R; i.e. there exists an invertible element t of R such that for any $r \in R$, $\phi\circ\psi(r) = trt^{-1}$. The composition, ψ_t, of ψ with the inner automorphism $r \longmapsto t^{-1}rt$ is a right inverse to ϕ in the category Alg_k. This shows that R is formally $\bar{\mathfrak{M}}_\mathfrak{n}$-smooth iff it is formally smooth (cf. 4.2). The assertion follows from 3.0.3.3 (or [CQ1]).

(b) The implication (iii)⇒(iv) is true by definition.

(iv)⇒(v). Let M be an R^e-module, S a semidirect product of R and M, $S \xrightarrow{\phi} R$ the canonical epimorphism. It follows from (a) that any right inverse to ϕ in the sense of $\mathfrak{Ass}_k$ is conjugate to a right inverse, $R \xrightarrow{\psi} S$ to ϕ in the sense of Alg_k. The morphism ψ is of the form $r \longmapsto r + D(r)$ for some (any) derivation $R \xrightarrow{D} M$ which sends k to zero. If R is $\bar{\mathfrak{M}}_{\mathfrak{n}}$-unramified, the morphism ψ is equivalent to the morphism $R \longrightarrow S$, $r \longmapsto r$. This means that there exists an invertible element u of S such that $\psi(r) = uru^{-1}$ for all $r \in R$. The element u can be written as $t(1_R + z)$, where 1_R is the unit of R, t is an invertible element of R, and $z \in M$. Then

$$uru^{-1} = trt^{-1} + (tzt^{-1})(trt^{-1}) - (trt^{-1})(tzt^{-1}) \tag{1}$$

In particular, $\phi \circ \psi(r) = trt^{-1}$ for all $r \in R$. But $\phi \circ \psi = id_R$, hence the element t is central. Thus $\psi(r) = r + z_t r - r z_t$, where $z_t = tzt^{-1}$, i.e. D is an inner derivation. It is known [CQ1] (and easy to prove) that *R is a separable k-algebra iff any derivation of R in any R^e-module M is inner*, hence the implication.

(v)⇒(iii). Let R be a separable k-algebra. Let $T \xrightarrow{\phi} S$ be a k-algebra morphism with a nilpotent kernel and $R \xrightarrow{f} S$ an arbitrary algebra morphism. It follows from the argument in [CQ1] that any two liftings of f to a morphism $R \longrightarrow T$ are conjugate by an element t of T such that $1 - t$ belongs to the kernel of ϕ, in particular it is nilpotent. Conversely, such a lifting property implies that R is separable. ∎

3.1. Formally smooth morphisms. Fix a class $\mathfrak{M}$ of morphisms of a category $\mathfrak{B}$.

(i) We call a morphism $X \xrightarrow{f} Y$ of the category $\mathfrak{B}$ *formally $\mathfrak{M}$-smooth* if any commutative square

$$\begin{array}{ccc} T & \xrightarrow{g} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g'} & Y \end{array} \tag{1}$$

whose left vertical arrow belongs to $\mathfrak{M}$ extends to a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ {\scriptstyle\phi}\downarrow & \nearrow_{\gamma} & \downarrow{\scriptstyle f} \\ S & \xrightarrow{g} & Y \end{array} \tag{2}$$

(ii) We call $X \xrightarrow{f} Y$ *formally $\mathfrak{M}$-unramified* if for any commutative diagram (1) such that $\phi \in \mathfrak{M}$, there exists at most one morphism $S \xrightarrow{\gamma} X$ making the diagram (2) commute.

(iii) We call $X \xrightarrow{f} Y$ *formally* $\mathfrak{M}$*-étale* if it is both formally $\mathfrak{M}$-smooth and formally $\mathfrak{M}$-unramified (that is the diagonal morphism in (2) exists and is unique).

We denote by $\mathfrak{M}_{fsm}$ (resp. $\mathfrak{M}_{fnr}$, resp. $\mathfrak{M}_{fet}$) the class of all formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) morphisms.

3.2. $\mathfrak{N}$-infinitesimal morphisms. On the other hand, given a class $\mathfrak{N}$ of morphisms of $\mathfrak{B}$, denote by $\mathfrak{N}_{inf}$ the class of all morphisms $T \xrightarrow{\phi} S$ of $\mathfrak{B}$ such that any commutative diagram (1) such that $X \xrightarrow{f} Y$ belongs to $\mathfrak{N}$ extends to a commutative diagram (2). Morphisms of $\mathfrak{N}_{inf}$ will be called $\mathfrak{N}$-*infinitesimal morphisms.*

3.3. Remarks. (a) There is a natural "duality"

$$\mathfrak{M} \longmapsto \mathfrak{M}_{fsm}, \quad \mathfrak{N} \longmapsto \mathfrak{N}_{inf}.$$

If follows from the definitions that $\mathfrak{M} \subseteq \mathfrak{N}_{inf}$ iff $\mathfrak{N} \subseteq \mathfrak{M}_{fsm}$. In a more symmetric way, the latter relations can be expressed as follows: any commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g} & X \\ \phi \big\downarrow & & \big\downarrow f \\ S & \xrightarrow{g'} & Y \end{array}$$

such that $\phi \in \mathfrak{M}$ and $f \in \mathfrak{N}$ extends to a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ \phi \big\downarrow & \nearrow_{\gamma} & \big\downarrow f \\ S & \xrightarrow{g} & Y \end{array}$$

(b) The definitions of a formally smooth and a formally étale) morphism are special cases of the corresponding definitions of formally smooth and étale objects. Namely, a morphism $X \xrightarrow{f} Y$ is identified with the object (X, f) of the category $\mathfrak{B}/Y$ and the class $\mathfrak{M}$ of infinitesimal morphisms is replaced by its preimage, $\mathfrak{M}_Y$ with respect to the forgetful functor $\mathfrak{B}/Y \longrightarrow \mathfrak{B}$. There is, however, another important aspect – the base change, which is more convenient to deal with using direct definitions above.

3.4. Proposition. *Let* $\mathfrak{M}$ *be a family of arrows of a category* $\mathfrak{B}$.

(a) The class $\mathfrak{M}_{fsm}$ *(resp.* $\mathfrak{M}_{fnr}$*, resp.* $\mathfrak{M}_{fet}$*) of formally* $\mathfrak{M}$*-smooth (resp. formally* $\mathfrak{M}$*-unramified, resp. formally* $\mathfrak{M}$*-étale) morphisms is closed under composition and contains all isomorphisms of the category* $\mathfrak{B}$.

(b) Let $X \xrightarrow{f} Y,\ Y \xrightarrow{h} Z$ *be morphisms of* $\mathfrak{B}$.

(i) If $h \circ f$ *is formally* $\mathfrak{M}$*-unramified, then* f *is formally* $\mathfrak{M}$*-unramified.*

(ii) Suppose that h *is formally* $\mathfrak{M}$*-unramified. If* $X \xrightarrow{h\circ f} Z$ *is formally* $\mathfrak{M}$*-smooth (resp. formally* $\mathfrak{M}$*-étale), then* f *is formally* $\mathfrak{M}$*-smooth (resp. formally* $\mathfrak{M}$*-étale).*

(c) Let $X \xrightarrow{\xi} S \xleftarrow{\xi'} X'$ *and* $Y \xrightarrow{\nu} S \xleftarrow{\nu'} Y'$ *be morphisms such that there exist* $X \times_S X'$ *and* $Y \times_S Y'$*. Let* $(X,\xi) \xrightarrow{f} (Y,\nu)$ *and* $(X',\xi') \xrightarrow{f'} (Y',\nu)$ *be morphisms of objects over* S*. The morphisms* $f,\ f'$ *are formally* $\mathfrak{M}$*-smooth (resp. formally* $\mathfrak{M}$*-étale) iff the morphism* $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$ *has the respective property.*

(d) Let $X \xrightarrow{f} S \xleftarrow{h} Y$ *be such a diagram that there exists a fiber product* $X \times_S Y$*. If* f *is formally* $\mathfrak{M}$*-smooth (resp. formally* $\mathfrak{M}$*-unramified, resp. formally étale), then the canonical projection* $X \times_S Y \xrightarrow{f'} Y$ *is formally* $\mathfrak{M}$*-smooth (resp. formally* $\mathfrak{M}$*-unramified, resp. formally* $\mathfrak{M}$*-étale).*

Proof. (a) Let $X \xrightarrow{f} Y,$ and $Y \xrightarrow{h} Z$ be formally $\mathfrak{M}$-smooth morphisms and

$$\begin{array}{ccc} T & \xrightarrow{\ g\ } & X \\ \phi \big\downarrow & & \big\downarrow h\circ f \\ S & \xrightarrow{\ g'\ } & Z \end{array}$$

be a commutative diagram with $\phi \in \mathfrak{M}$. Since the morphism $Y \xrightarrow{h} Z$ is formally $\mathfrak{M}$-smooth, there exists a morphism $S \xrightarrow{\beta} Y$ such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{f\circ g'} & Y \\ \phi \big\downarrow & \nearrow_{\beta} & \big\downarrow h \\ S & \xrightarrow{\ g\ } & Z \end{array}$$

commutes. Since $X \xrightarrow{f} Y$ is formally $\mathfrak{M}$-smooth, there exists a morphism $S \xrightarrow{\gamma} X$ such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{\ g'\ } & X \\ \phi \big\downarrow & \nearrow_{\gamma} & \big\downarrow f \\ S & \xrightarrow{\ \beta\ } & Y \end{array}$$

commutes. Therefore, the diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ \phi \Big\downarrow & \nearrow_{\gamma} & \Big\downarrow h \circ f \\ S & \xrightarrow{g'} & Y \end{array}$$

commutes. This shows that the class $\mathfrak{M}_{fsm}$ of formally $\mathfrak{M}$-smooth morphisms is closed under composition.

(a1) Suppose that $X \xrightarrow{f} Y$ and $Y \xrightarrow{h} Z$ are formally $\mathfrak{M}$-unramified morphisms; and let

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ \phi \Big\downarrow & \nearrow_{\gamma} & \Big\downarrow h \circ f \\ S & \xrightarrow{g} & Z \end{array} \qquad (1)$$

be a commutative diagram with the left vertical arrow from the class $\mathfrak{M}$. Since the morphism $Y \xrightarrow{h} Z$ is formally $\mathfrak{M}$-unramified, the arrow $S \xrightarrow{f\circ\gamma} Y$ is uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} T & \xrightarrow{f\circ g'} & Y \\ \phi \Big\downarrow & {\scriptstyle f\circ\gamma} \nearrow & \Big\downarrow h \\ S & \xrightarrow{g} & Z \end{array}$$

and the morphism $S \xrightarrow{\gamma} X$ is uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ \phi \Big\downarrow & \nearrow_{\gamma} & \Big\downarrow f \\ S & \xrightarrow{f\circ\gamma} & Y \end{array}$$

because the morphism $X \xrightarrow{f} Y$ is formally $\mathfrak{M}$-unramified. Therefore, the morphism $S \xrightarrow{\gamma} X$ is uniquely determined by the commutativity of the diagram (1).

(a2) Since the class $\mathfrak{M}_{fsm}$ morphism and the class $\mathfrak{M}_{fnr}$ are closed under composition, their intersection, $\mathfrak{M}_{fet}$, of $\mathfrak{M}_{fsm}$ and $\mathfrak{M}_{fnr}$ is closed under composition.

(b) Let

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ \phi \Big\downarrow & \nearrow_{\gamma} & \Big\downarrow f \\ S & \xrightarrow{g} & Y \end{array} \qquad (2)$$

be a commutative diagram. In particular,

$$\begin{array}{ccc} T & \xrightarrow{g} & X \\ \phi \downarrow & \nearrow_{\gamma} & \downarrow h \circ f \\ S & \xrightarrow{h \circ g'} & Z \end{array} \tag{3}$$

It $h \circ f$ is unramified, then the morphism $S \xrightarrow{\gamma} X$ is uniquely determined by the commutativity of the diagram (3). Therefore, γ is uniquely determined by the commutativity of the diagram (2). All together means that the morphism $X \xrightarrow{f} Y$ is unramified.

(c) Let $(X, \xi) \xrightarrow{f} (Y, \nu)$ and $(X', \xi') \xrightarrow{f'} (Y', \nu)$ be morphisms of the category $\mathfrak{B}/S$. The commutativity of a diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \times_S X' \\ \phi \downarrow & \nearrow_{\gamma} & \downarrow f \times_S f' \\ V & \xrightarrow{g} & Y \times_S Y' \end{array} \tag{4}$$

is equivalent to the commutativity of two diagrams

$$\begin{array}{ccc} T & \xrightarrow{g'_1} & X \\ \phi \downarrow & \nearrow_{\gamma_1} & \downarrow f \\ V & \xrightarrow{g_1} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} T & \xrightarrow{g'_2} & X' \\ \phi \downarrow & \nearrow_{\gamma_2} & \downarrow f' \\ V & \xrightarrow{g_2} & Y' \end{array} \tag{5}$$

where (g'_1, g'_2) and (g_1, g_2) are determined by the the morphisms respectively g' and g in the diagram (4). Therefore, the morphisms f, f' are formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-étale) iff the morphism $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$ has the respective property.

(d) Consider a commutative diagram

$$\begin{array}{ccccc} T & \xrightarrow{g} & \mathfrak{X} & \xrightarrow{\widetilde{h}} & X \\ \phi \downarrow & & \downarrow f' & cart & \downarrow f \\ V & \xrightarrow{g'} & Y & \xrightarrow{h} & S \end{array} \tag{6}$$

whose right square is cartesian. Suppose that the morphism $X \xrightarrow{f} S$ is formally $\mathfrak{M}$-smooth and $T \xrightarrow{\phi} V$ belongs to $\mathfrak{M}$. Then there exists an arrow $V \xrightarrow{\gamma} X$ such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{\widetilde{h}\circ g} & X \\ {\scriptstyle\phi}\downarrow & \nearrow_{\gamma} & \downarrow{\scriptstyle f} \\ V & \xrightarrow{h\circ g'} & Y \end{array}$$

commutes. The commutativity of the square

$$\begin{array}{ccc} V & \xrightarrow{\gamma} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g'} & S \end{array}$$

and the fact that the right square of the diagram (6) is cartesian, implies that there exists a unique morphism $V \xrightarrow{\lambda} \mathfrak{X}$ such that $\widetilde{h} \circ \lambda = \gamma$ and $f' \circ \lambda = g'$.

It follows that

$$\widetilde{h} \circ (\lambda \circ \phi) = \gamma \circ \phi = \widetilde{h} \circ g \quad \text{and} \quad f' \circ (\lambda \circ \phi) = (f' \circ \lambda) \circ \phi = g' \circ \phi = f' \circ g.$$

Since the right square of (6) is cartesian, the equalities $\widetilde{h} \circ (\lambda \circ \phi) = \widetilde{h} \circ g$ and $f' \circ (\lambda \circ \phi) = f' \circ g$ imply that $\lambda \circ \phi = g$. All together shows that the diagram

$$\begin{array}{ccc} T & \xrightarrow{g} & \mathfrak{X} \\ {\scriptstyle\phi}\downarrow & \nearrow_{\lambda} & \downarrow{\scriptstyle f'} \\ V & \xrightarrow{g'} & Y \end{array} \qquad (7)$$

commutes. Therefore, the pull-back $\mathfrak{X} \xrightarrow{f'} Y$ of the morphism $X \xrightarrow{f} S$ is formally $\mathfrak{M}$-smooth. If the morphism $X \xrightarrow{f} S$ is formally $\mathfrak{M}$-étale, then the morphism γ in the argument above is uniquely defined, which implies that λ is uniquely determined by the commutativity of (7). So that the morphism $\mathfrak{X} \xrightarrow{f'} Y$ is formally $\mathfrak{M}$-étale.

Finally, the morphisms $V \xrightarrow{\gamma} X$ and $V \xrightarrow{\lambda} \mathfrak{X}$ exist simultaneously. Therefore, if $X \xrightarrow{f} S$ is formally $\mathfrak{M}$-unramified, then $\mathfrak{X} \xrightarrow{f'} Y$ is formally $\mathfrak{M}$-unramified. ■

3.4.1. Corollary. *Let $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Z$ be morphisms of $\mathfrak{B}$. Suppose that h is formally $\mathfrak{M}$-étale. Then $h \circ f$ is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) iff f belongs to the same class.*

Proof. It follows from 3.4(b) that the morphism f is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified) if $h \circ f$ is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified). Conversely, since the morphism h is formally $\mathfrak{M}$-étale, the composition of h with a formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified) is formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified). ■

3.5. Proposition. *Let $\mathfrak{N}$ be a family of arrows of $\mathfrak{B}$.*
(a) Any split monomorphism (in particular, any isomorphism) belongs to $\mathfrak{N}_{inf}$.
(b) The class $\mathfrak{N}_{inf}$ of $\mathfrak{N}$-infinitesimal morphisms is closed under composition.
(c) Let $T \xleftarrow{\phi} U \xrightarrow{\psi} S$ be morphisms such that there exists $T \coprod_U S$. If ϕ belongs to $\mathfrak{N}_{inf}$, then the coprojection $S \longrightarrow T \coprod_U S$ belongs to $\mathfrak{N}_{inf}$.

Proof. (a) Obvious.

(b) Let $T \xrightarrow{\phi} U$ and $U \xrightarrow{\psi} V$ be $\mathfrak{N}$-infinitesimal morphisms; and let

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ {\scriptstyle \psi\circ\phi}\downarrow & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & Y \end{array}$$

be a commutative square whose right vertical arrow, $X \xrightarrow{f} Y$, belongs to $\mathfrak{N}$. Since $T \xrightarrow{\phi} U$ is $\mathfrak{N}$-infinitesimal, there exists an arrow $U \xrightarrow{\gamma} X$ such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ {\scriptstyle \phi}\downarrow & \nearrow_{\gamma} & \downarrow{\scriptstyle f} \\ U & \xrightarrow{g\circ\psi} & Y \end{array}$$

commutes. Since $U \xrightarrow{\psi} V$ is $\mathfrak{N}$-infinitesimal, there exists an arrow $V \xrightarrow{\lambda} X$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\gamma} & X \\ {\scriptstyle \psi}\downarrow & \nearrow_{\lambda} & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & Y \end{array}$$

commutes. Therefore, the diagram

$$\begin{array}{ccc} T & \xrightarrow{g'} & X \\ {\scriptstyle \psi\circ\phi}\downarrow & \nearrow_{\lambda} & \downarrow{\scriptstyle f} \\ V & \xrightarrow{g} & Y \end{array}$$

commutes.

(c) Consider the commutative diagram

$$\begin{array}{ccccc} T & \xrightarrow{\mathfrak{t}} & \mathfrak{T} & \xrightarrow{g} & X \\ \phi \downarrow & cocart & \downarrow \widetilde{\phi} & & \downarrow f \\ V & \xrightarrow{\mathfrak{t}'} & \mathfrak{V} & \xrightarrow{g'} & Y \end{array} \qquad (8)$$

whose left square is cocartesian, the left vertical arrow is $\mathfrak{N}$-infinitesimal and the right vertical arrow belongs to $\mathfrak{N}$. So that there exists an arrow $V \xrightarrow{\gamma} X$ such that the diagram

$$\begin{array}{ccc} T & \xrightarrow{g\circ\mathfrak{t}} & X \\ \phi \downarrow & \nearrow_{\gamma} & \downarrow f \\ V & \xrightarrow{g'\circ\mathfrak{t}'} & Y \end{array}$$

commutes. Since the left square of the diagram (8) is cocartesian, there exists a unique arrow $\mathfrak{V} \xrightarrow{\lambda} X$ such that the diagram

$$\begin{array}{ccc} \mathfrak{T} & \xrightarrow{g} & X \\ \widetilde{\phi} \downarrow & \nearrow_{\gamma} & \downarrow f \\ \mathfrak{V} & \xrightarrow{g'} & Y \end{array}$$

commutes (see the argument of 3.4(d)). ■

3.6. Formally smooth and formally étale morphisms of noncommutative affine schemes.

3.6.1. Proposition. *Let R, S be associative unital k-algebras, and let $R \xrightarrow{\phi} S$ be a k-algebra morphism.*

(a) The morphism $S^\vee \xrightarrow{\phi^\vee} R^\vee$ is formally unramified iff the morphism

$$S \otimes_R S^o \longrightarrow S, \quad s \otimes t \longmapsto st,$$

is an isomorphism, or, equivalently, $\Omega^1_{S|R} \stackrel{\mathrm{def}}{=} Ker(S \otimes_R S^o \longrightarrow S) = 0$.

(b) Suppose that the k-algebra R is separable. Then the morphism $S^\vee \xrightarrow{\phi^\vee} R^\vee$ is formally smooth iff the left S^e-module of relative differential forms $\Omega^1_{S|R} = \Omega_\phi$ is projective.

Proof. A standard argument shows that the morphism $S^\vee \xrightarrow{\phi^\vee} R^\vee$ is formally smooth (resp. formally unramified) iff for any R-ring epimorphism $T \xrightarrow{\alpha} S$ such that $Ker(\alpha)^2 = 0$, there exists an R-ring morphism (resp. at most one R-ring morphism) $S \xrightarrow{\beta} T$ such that $\alpha \circ \beta = id_S$.

(a) Let $T \xrightarrow{\alpha} S$ be an R-ring epimorphism such that $Ker(\alpha)^2 = 0$; and let $S \xrightarrow{\beta} T$ be a right inverse to α; that is $\alpha \circ \beta = id_S$. The morphism β gives a decomposition of the algebra T into a semidirect product of S and an S^e-module, M, with multiplication defined by $(s,m)(s',m') = (ss', s \cdot m' + m \cdot s')$. Any other right inverse to α, is of the form (id_S, D), where $S \xrightarrow{D} M$ is a derivation sending R to zero. The latter means precisely that D is an R^e-module morphism, $R^e = R \otimes_k R^o$. Thus, the set of splittings of α is in one-to-one correspondence with the set $Der_{S|R}(M)$ of derivations $S \xrightarrow{D} M$ which are R^e-module morphisms. But $Der_{S|R}(M)$ is naturally isomorphic to $Hom_{S^e}(\Omega^1_{S|R}, M)$. Hence ϕ is unramified iff $\Omega^1_{S|R} = 0$.

(b) Suppose the k-algebra R is separable, i.e. R is a projective R^e-module. Then the S^e-module $S \otimes_R S^o$ is projective.

In fact, for any S^e-module M, there is a functorial isomorphism $Hom_{S^e}(S \otimes_R S^o, M) \simeq Hom_{R^e}(R, \phi_*(M))$. Here ϕ_* is the pull-back functor $S^e - mod \longrightarrow R^e - mod$ induced by the morphism ϕ. Since R is a projective R^e-module and the functor ϕ_* is exact, the functor $M \longmapsto Hom_{R^e}(R, \phi_*(M))$ is exact. Therefore the functor $M \longmapsto Hom_{S^e}(S \otimes_R S^o, M)$ is exact, i.e. $S \otimes_R S^o$ is a projective S^e-module.

By 3.0.4.1, the algebra R is separable iff the corresponding affine scheme $R^\vee$ is $\bar{\mathfrak{M}}_{\mathfrak{n}}$-étale. The latter means that the morphism $R^\vee \longrightarrow k^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-étale. If follows from 3.4(ii) that the morphism $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-smooth iff the composition of $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ and $R^\vee \longrightarrow k^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-smooth, i.e. the affine scheme $S^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-smooth. By 3.0.4.1, the affine k-scheme $S^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-smooth iff it is formally smooth. On the other hand, $S^\vee$ is formally smooth iff $Ext^2_{S^e}(S, M) = 0$ for any S^e-module M. Consider the long exact sequence

$$\begin{array}{ccc} ... \longrightarrow Ext^i_{S^e}(S,M) \longrightarrow Ext^i_{S^e}(S \otimes_R S^o, M) \longrightarrow & Ext^i_{S^e}(\Omega^1_{S|R}, M) & \\ & \downarrow & \\ & Ext^{i+1}_{S^e}(S,M) & \longrightarrow \; ... \end{array} \qquad (1)$$

corresponding to the short exact sequence

$$0 \longrightarrow \Omega^1_{S|R} \longrightarrow S \otimes_R S^o \longrightarrow S \longrightarrow 0.$$

Since $S^e = S \otimes_R S^o$ is a projective S^e-module, $Ext^i_{S^e}(S \otimes_R S^o, M) = 0$ for all $i \geq 1$ and all S^e-modules M. Therefore $Ext^i_{S^e}(\Omega^1_{S|R}, M) \simeq Ext^{i+1}_{S^e}(S, M)$ for all $i \geq 1$ and all

S^e-modules M. In particular, $Ext^2_{S^e}(S,M)=0$ for all M iff $Ext^1_{S^e}(\Omega^1_{S|R},M)=0$ for all M. The latter means precisely that $\Omega^1_{S|R}$ is a projective S^e-module. ■

3.6.1.1. Corollary. *Let $R \xrightarrow{\varphi} S$ be a k-algebra morphism. Suppose R is a separable k-algebra. Then the morphism $S^\vee \xrightarrow{\varphi^\vee} R^\vee$ is formally unramified iff it is formally étale.*

Proof. By 3.6.1(a), the morphism $S^\vee \xrightarrow{\varphi^\vee} R^\vee$ is unramified iff $\Omega^1_{S|R}=0$. By 3.6.1(b), $S^\vee \xrightarrow{\varphi^\vee} R^\vee$ is formally smooth iff the S^e-module $\Omega^1_{S|R}$ is projective. In particular, $S^\vee \xrightarrow{\varphi^\vee} R^\vee$ is formally smooth (hence étale), if $\Omega^1_{S|R}=0$. ■

3.6.2. Proposition. *Let R, S be associative unital k-algebras and $R \xrightarrow{\varphi} S$ a k-algebra morphism. The following conditions are equivalent:*

(i) The morphism $S^\vee \xrightarrow{\varphi^\vee} R^\vee$ is formally unramified and flat.
(ii) $R \xrightarrow{\varphi} S$ is a flat monomorphism.
(iii) The functor $R-mod \xrightarrow{\varphi^*} S-mod$ is an exact localization.
(iv) If the conditions above hold, then $S^\vee \xrightarrow{\varphi^\vee} R^\vee$ is formally étale.

Proof. (ii)⇒(i), because every monomorphism is formally unramified.

(i)⇒(iii). By 3.6.1(a), the canonical morphism $S\otimes_R S^o \longrightarrow S$, $s\otimes t \longmapsto st$, is an isomorphism. Since $\varphi^*\varphi_* \simeq (S\otimes_R S^o)\otimes_S -$ and $Id_{S-mod}\simeq S\otimes_S -$, this means precisely that the adjunction morphism $\varphi^*\varphi_* \longrightarrow Id_{S-mod}$ is an isomorphism. The latter is equivalent to the full faithfulness of the direct image functor φ_*. By [GZ], Proposition I.1.3, φ^* is a localization.

(iii)⇒(ii) follows from the fact that any morphisms $R \xrightarrow{\varphi} S$ such that its inverse image functor, φ^*, is a localization, is an algebra epimorphism.

In fact, let $S \overset{f_1}{\underset{f_2}{\rightrightarrows}} T$ be a pair of algebra morphisms such that $f_1\circ\psi = f_2\circ\varphi$, i.e. we have the diagram of algebra morphisms over R:

$$\begin{array}{ccc} S & \overset{f_1}{\underset{f_2}{\rightrightarrows}} & T \\ {\scriptstyle\varphi}\nwarrow & & \nearrow{\scriptstyle\gamma} \\ & R & \end{array} \qquad (1)$$

Here $\gamma = f_1\circ\varphi$. Applying to (1) first scalar restriction functor and then the functor $\varphi_*\varphi^*$, we obtain the diagram $\varphi_*\varphi^*(R) \longrightarrow \varphi_*\varphi^*\varphi_*(S) \rightrightarrows \varphi_*\varphi^*\gamma_*(T)$ which is isomorphic to the diagram

$$\varphi_*\varphi^*(R) \longrightarrow \varphi_*(S) \rightrightarrows \gamma_*(T), \qquad (2)$$

because, due to the fact that φ^* is a localization, φ_* is a fully faithful functor, or, equivalently, $\varphi^*\varphi_* \simeq Id_{S-mod}$. Notice that the morphism $\varphi_*\varphi^*(R) \longrightarrow \varphi_*(S)$ in (2) is an isomorphism. Since it equalizes the pair $\varphi_*(S) \rightrightarrows \gamma_*(T)$, this pair is trivial. Hence the initial pair of morphisms is trivial: $f_1 = f_2$.

{(iii),(i)} ⇒(iv). It suffices to show that if $R \xrightarrow{\varphi} S$ is an exact localization, then φ is formally smooth. A standard argument shows that a morphism $R \xrightarrow{\varphi} S$ is smooth iff any R-ring strict epimorphism (i.e. a surjection) $T \xrightarrow{g} S$ such that the square of the kernel of g is zero, has right inverse. Denote the kernel of g by J. Thus we have an exact sequence of R-bimodules

$$0 \longrightarrow J \longrightarrow T \longrightarrow S \longrightarrow 0. \tag{3}$$

Denote by Φ^* the functor

$$R^e - mod \longrightarrow S^e - mod, \quad M \longmapsto S \otimes_R M \otimes_R S.$$

Notice that this functor is an exact localization having a (necessarily fully faithful) right adjoint, Φ_*. In particular, it maps the exact sequence (3) into exact sequence. Applying the functor Φ^* to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & T & \longrightarrow & S & \longrightarrow & 0 \\ & & & & \nwarrow & & \nearrow & & \\ & & & & & R & & & \end{array} \tag{4}$$

we obtain the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi^*(J) & \longrightarrow & \Phi^*(T) & \longrightarrow & \Phi^*(S) & \longrightarrow & 0 \\ & & & & \nwarrow & & \nearrow & & \\ & & & & & \Phi^*(R) & & & \end{array} \tag{5}$$

Since Φ^* is a localization, the natural morphism $S \longrightarrow \Phi_*\Phi^*(S)$ is an isomorphism, $\Phi^*(R) = S \otimes_R S^o \simeq S$, and the k-algebra morphism $\Phi^*(\varphi) : \Phi^*(R) \longrightarrow \Phi^*(S)$ is an isomorphism.

Note that J is an S-bimodule. This implies that $\Phi_*\Phi^*(J) \simeq J$. Thus we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & T & \longrightarrow & S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Phi_*\Phi^*(J) & \longrightarrow & \Phi_*\Phi^*(T) & \longrightarrow & \Phi_*\Phi^*(S) & \longrightarrow & 0 \\ & & & & \nwarrow & & \nearrow & & \\ & & & & & S & & & \end{array} \tag{6}$$

whose left and right vertical arrows are isomorphisms. Since both rows are exact sequences, it follows that the adjunction morphism $T \longrightarrow \Phi_*\Phi^*(T)$ is an isomorphism too, hence the assertion. ■

3.7. Formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-unramified and formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-étale morphisms. The following assertion is a relative version of 3.0.4.1.

3.7.1. Proposition. *Let R, S be associative k-algebras, and let $R \xrightarrow{\phi} S$ be a k-algebra morphism.*

1) The following conditions are equivalent:

(i) The morphism $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-unramified.

(ii) Any derivation $S \xrightarrow{D} M$ which is an R^e-module morphism is inner.

(iii) The canonical S^e-module epimorphism $S \otimes_R S^o \longrightarrow S$ has a right inverse.

2) Suppose that the k-algebra R is separable. Then

(a) The morphism $R \xrightarrow{\phi} S$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-smooth iff $\Omega^1_{S|R}$ is a projective S^e-module.

(b) The following conditions are equivalent:

(iv) The morphism $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-unramified.

(v) $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-étale.

(vi) S is a separable k-algebra (i.e. S is a projective S^e-module).

Proof. 1) (i)⇔(ii). Let T be a semidirect product of S and an S^e-bimodule M, and let $T \xrightarrow{\alpha} S$ the natural projection, $(s,z) \longmapsto s$. Any k-algebra morphism $S \longrightarrow T$ which is right inverse to α in category $\mathfrak{A}ss_k$ is conjugate to a k-algebra morphism, $S \xrightarrow{\beta} T$, which is right inverse to α in Alg_k. Any such morphism β is of the form $s \longmapsto (s, D(s)$, where $S \xrightarrow{D} M$ is an $S|R$-derivation. If $R \xrightarrow{\phi} S$ is $\bar{\mathfrak{M}}_{\mathfrak{n}}$-unramified, β is of the form $s \longmapsto usu^{-1}$. The argument of 3.0.4.1 shows that this (together with the equality $\alpha \circ \beta = id_S$) implies that D is an inner derivation.

Conversely, if the morphism $S \xrightarrow{\beta} T$ is given by $s \longmapsto (s, D(s))$, where D is an inner derivation, i.e. $D(s) = s\cdot z - z\cdot s$ for some element z of M and all $s \in S$, then $\beta(s) = usu^{-1}$, where $u = 1_S - z$.

(ii)⇒(iii). The functor $Der_{S|R} : S^e - mod \longrightarrow \mathbf{Sets}$, $M \longmapsto Der_{S|R}(M)$, is representable by the S^e-module $\Omega^1_{S|R} = Ker(S \otimes_R S^o \longrightarrow S)$. The canonical monomorphism $\Omega^1_{S|R} \xrightarrow{i_\phi} S \otimes_R S^o$ induces a map

$$Hom_{S^e}(S \otimes_R S^o, M) \longrightarrow Hom_{S^e}(\Omega^1_{S|R}, M) \qquad (1)$$

Notice that $Hom_{S^e}(S \otimes_R S^o, M) \simeq Hom_{R^e}(R, \phi_*(M))$, and $Hom_{R^e}(R, \phi_*(M))$ is naturally isomorphic to the center, $\mathfrak{z}(\phi_*(M)) = \{v \in M | \ r \cdot v = v \cdot r \text{ for all } r \in R\}$, of the R^e-module $\phi_*(M)$. The composition of the bijection $\mathfrak{z}(\phi_*(M)) \longrightarrow Hom_{S^e}(S \otimes_R S^o, M)$ and the map (1) assigns to each central element, z, of $\phi_*(M)$ the corresponding inner derivation, $s \longmapsto s \cdot z - z \cdot s$. Thus, each derivation of $Der_{S|R}(M)$ is inner iff the map (1) is surjective. In the case $M = \Omega^1_{S|R}$, this implies the existence of an S^e-module morphism $S \otimes_R S^o \xrightarrow{p} \Omega^1_{S|R}$ such that $p \circ i_\phi = id$. Or, equivalently, the canonical S^e-module morphism $S \otimes_R S^o \longrightarrow S$ has a right inverse.

The implication (iii)⇒(ii) follows from the argument above.

2) (a) The morphism $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-smooth iff it is formally smooth. By 3.6.1, if R is a separable k-algebra, then $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ is formally smooth iff $\Omega^1_{S|R}$ is a projective S^e-module.

(b) By the argument of 3.6.1, if R is a separable k-algebra, then the S^e-module $S \otimes_R S^o$ is projective. By 1), the morphism $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ is $\bar{\mathfrak{M}}_{\mathfrak{n}}$-unramified iff the S^e-module morphism $S \otimes_R S^o \longrightarrow S$ has a right inverse. Since the S^e-module $S \otimes_R S^o$ is projective, the latter implies that S is a projective S^e-module, hence (equivalently) $\Omega^1_{S|R}$ is a projective S^e-module, i.e. the morphism $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-smooth. This proves the implications (iv)⇒(v)⇒(vi)⇒(v). The implication (v)⇒(iv) is true by definition. ∎

3.7.2. Corollary. *The following conditions on a k-algebra morphism $R \xrightarrow{\phi} S$ are equivalent:*

(a) The morphism $S^\vee \xrightarrow{\bar{\phi}^\vee} R^\vee$ is formally $\bar{\mathfrak{M}}_{\mathfrak{n}}$-étale.

(b) The adjunction morphism $\phi^*\phi_* \xrightarrow{\epsilon_\phi} Id_{S-mod}$ has a right inverse.

Proof. (a)⇒(b). By 3.7.1, the canonical S^e-module epimorphism $S \otimes_R S^o \xrightarrow{\mu} S$ has a right inverse, $S \xrightarrow{tau'} S \otimes_R S^o$. The morphism τ' defines a morphism, $Id_{S-mod} \xrightarrow{\tau} \phi^*\phi_*$. The equality $\mu \circ \tau = id_S$ implies that the composition of τ with the adjunction morphism, $\phi^*\phi_* \xrightarrow{\epsilon_\phi} Id_{S-mod}$ is the identity morphism.

(b)⇒(a). Conversely, any morphism, $Id_{S-mod} \xrightarrow{\tau'} \phi^*\phi_*$, is induced by an S^e-module morphism, $S \xrightarrow{\tau} S \otimes_R S^o$. The morphism τ' is a right inverse to the adjunction morphism $\phi^*\phi_* \xrightarrow{\epsilon_\phi} Id_{S-mod}$ iff the composition of the bimodule morphism τ with the canonical morphism $S \otimes_R S^o \longrightarrow S$ equals to id_S. ∎

3.8. Other choices of infinitesimal morphisms. The most important are the class $\mathfrak{M}_{\mathfrak{J}}$ of *radical closed immersions* and its subclass $\mathfrak{M}^{\mathfrak{c}}_{\mathfrak{J}}$ of *complete radical closed immersions.* They are defined below.

3.8.1. Radical closed immersions. We denote by $\mathfrak{M}_{\mathfrak{J}}$ the class of strict monomorphisms of $\mathbf{Aff}_k$ such that the kernel of the corresponding algebra morphism is contained in the Jacobson radical. Since $\mathfrak{M}_{\mathfrak{n}} \subset \mathfrak{M}_{\mathfrak{J}}$, the class of formally $\mathfrak{M}_{\mathfrak{J}}$-smooth (resp. formally $\mathfrak{M}_{\mathfrak{J}}$-étale, resp. formally $\mathfrak{M}_{\mathfrak{J}}$-unramified) morphisms is contained in the class of formally smooth (resp. formally étale, resp. formally unramified) morphisms.

3.8.2. Complete radical closed immersions. We denote by $\mathfrak{M}_{\mathfrak{J}}^{\mathfrak{c}}$ the class of strict monomorphisms $S^\vee \xrightarrow{\varphi^\vee} T^\vee$ of the category $\mathbf{Aff}_k$ such that the kernel of the corresponding algebra morphism is contained in the Jacobson radical and the natural algebra morphism $T \longrightarrow \lim_{n\geq 1} T/(Ker(\varphi)^n)$ is an isomorphism. Since $\mathfrak{M}_{\mathfrak{n}} \subset \mathfrak{M}_{\mathfrak{J}}^{\mathfrak{c}} \subseteq \mathfrak{M}_{\mathfrak{J}}$, the class of formally $\mathfrak{M}_{\mathfrak{J}}$-smooth (resp. formally $\mathfrak{M}_{\mathfrak{J}}$-étale, resp. formally $\mathfrak{M}_{\mathfrak{J}}$-unramified) morphisms is contained in the class of $\mathfrak{M}_{\mathfrak{J}}^{\mathfrak{c}}$-formally smooth (resp. $\mathfrak{M}_{\mathfrak{J}}^{\mathfrak{c}}$-formally étale, resp. $\mathfrak{M}_{\mathfrak{J}}^{\mathfrak{c}}$-formally unramified) morphisms, and the latter class is contained in the class of formally smooth (resp. formally étale, resp. formally unramified) morphisms.

3.9. Example: separated, universally closed, and proper morphisms of schemes. Let A be the category $CAlg_k$ of commutative k-algebras. Let $\mathfrak{M}'_v$ be the family of canonical injections of valuation rings to their fields of fractions; and let $\mathfrak{M}_v$ denote the image of $\mathfrak{M}'_v$ in the category $A^\vee$ of functors $A \longrightarrow \mathbf{Sets}$.

3.9.1. Proposition. *Let $X \xrightarrow{f} Y$ be a quasi-separated scheme morphism. Then*
(a) The morphism f is separated iff it is formally $\mathfrak{M}_v$-unramified.
(b) The morphism f is universally closed iff it is formally $\mathfrak{M}_v$-smooth.
(c) The morphism f is proper iff it is formally $\mathfrak{M}_v$-étale.

Proof. The assertions (a) and (c) are equivalent resp. to the Grothendieck's criterion of separateness and properness (see EGA, Ch.II, 7.2.3 and 7.2.8). A proof of the assertion (b) can be extracted from the argument of Theorem 7.2.8, EGA, Ch.II. ∎

Standard properties of separated and proper morphisms become special cases of assertions on formally $\mathfrak{M}$-unramified and formally $\mathfrak{M}$-étale morphisms (cf. 3.4):

3.9.2. Corollary. *(a) Any monomorphism is a separated morphism.*

(b) A composition of two separated (resp. proper) morphisms is separated (resp. proper).

(c) Separated (resp. proper) morphisms are stable under base change.

(d) If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two morphisms such that $g \circ f$ is separated, then f is separated.

(e) If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ are two morphisms such that g is separated and $g \circ f$ is proper, then f is proper.

(f) If $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ are separated (resp. proper) morphisms over S, then their product, $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$, is also separated (resp. proper).

3.9.3. Remarks. (a) One can introduce the notions of *formally separated* and *formally proper* morphisms by omitting the condition that the morphism in question is quasi-compact. In terms of the family $\mathfrak{M}_v$, a morphism is formally separated (resp. formally proper) iff they are formally $\mathfrak{M}_v$-unramified (resp. formally $\mathfrak{M}_v$-étale). It follows that the assertions obtained from 3.9.1 and 3.9.2 by dropping the quasi-compactness condition and inserting 'formally' at appropriate places, are corollaries of 3.4.

(b) The notions of a (formally) proper morphism and a (formally) separated morphism make sense for morphisms of arbitrary presheaves of sets on the category A^{op}, not only for scheme morphisms, because the notions of a (formally) $\mathfrak{M}$-smooth and (formally) $\mathfrak{M}$-unramified morphisms make sense for morphisms of presheaves of sets on A^{op}.

(c) At the moment, it is not clear what might be an adequate noncommutative version of the family $\mathfrak{M}_v$.

4. Smooth, unramified, and étale morphisms. Open immersions.

4.1. Smooth, unramified, and étale morphisms. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ and a family $\mathfrak{M}$ of morphisms of $\mathfrak{B}$. We say that a morphism $X \xrightarrow{f} Y$ of the category $\mathfrak{B}$ is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-*étale*, resp. $\mathfrak{M}$-*unramified*) if it is $(\mathfrak{A}, \mathfrak{F})$-finitely presentable and formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale).

4.1.1. Notations. We denote by $\mathfrak{M}_{sm}$ (resp. $\mathfrak{M}_{nr}$, resp. $\mathfrak{M}_{et}$) the family of all $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale) morphisms.

4.1.2. Open immersions. We call $\mathfrak{M}$-smooth monomorphisms $\mathfrak{M}$-*open immersions* and denote the class of $\mathfrak{M}$-open immersions by $\mathfrak{M}_{zar}$.

4.2. Proposition. *(a) Each monomorphism is $\mathfrak{M}$-unramified and each isomorphism is $\mathfrak{M}$-open immersion.*

(b) Composition of $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale) morphisms is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale).

(c) Let $X \xrightarrow{f} Y$, $Y \xrightarrow{h} Z$ be morphisms of $\mathfrak{B}$.

(i) If $g \circ f$ is formally $\mathfrak{M}$-unramified and g is of $\mathfrak{M}$-finite type, then f is $\mathfrak{M}$-unramified.

(ii) Suppose g is $\mathfrak{M}$-unramified. If $X \xrightarrow{g \circ f} Z$ is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-étale), then f is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-étale).

(d) Let $X \xrightarrow{\xi} S \xleftarrow{\xi'} X'$ and $Y \xrightarrow{\nu} S \xleftarrow{\nu'} Y'$ be morphisms such that there exist $X \times_S X'$ and $Y \times_S Y'$. Let $(X, \xi) \xrightarrow{f} (Y, \nu)$ and $(X', \xi') \xrightarrow{f'} (Y', \nu)$ be morphisms of

objects over S. The morphisms f, f' are $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale) iff the morphism $f \times_S f' : X \times_S X' \longrightarrow Y \times_S Y'$ has the respective property.

(e) Let $X \xrightarrow{f} S \xleftarrow{h} Y$ be such a diagram that there exists a fiber product $X \times_S Y$. If f is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. étale), then the projection $X \times_S Y \xrightarrow{f'} Y$ is $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-unramified, resp. $\mathfrak{M}$-étale).

Proof. The assertion follows from 1.11.2.2 and 1.11.2. ∎

4.3. Note. In the known examples, the class of morphisms $\mathfrak{M}$ is the image of a class of morphisms $\widetilde{\mathfrak{M}}$ of the category $\mathfrak{A}$.

4.4. Standard examples.

4.4.1. Let $\mathfrak{A}$ be the category $\mathbf{CAff}_k$ of commutative affine k-schemes, which is, by definition, the category $CAlg_k^{op}$ opposite to the category of commutative unital k-algebras, and let $\mathfrak{B}$ be the category $\mathcal{E}sp$ of *spaces* in the sense of Grothendieck (and [DG]). That is $\mathfrak{B}$ is the category of sheaves of sets on $\mathfrak{A}$ for the flat (**fpqc**) topology. In other words, objects of $\mathfrak{B}$ are functors $CAlg_k \longrightarrow Sets$ which preserve finite products, and for any faithfully flat k-algebra morphism $R \longrightarrow T$, the diagram

$$X(R) \longrightarrow X(T) \rightrightarrows X(T \otimes_R T) \qquad (1)$$

is exact. The functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is (induced by) the Yoneda embedding, which maps every object R of $\mathfrak{A}$ to the functor $\mathfrak{A}(-, R) = CAlg_k(R, -)$ represented by R (here we identify objects of $\mathfrak{A}$ with the corresponding objects of $CAlg_k$).

Then $(\mathfrak{A}, \mathfrak{F})$-finitely presentable morphisms (resp. morphisms of $(\mathfrak{A}, \mathfrak{F})$-finite type) are precisely locally finitely presentable morphisms (resp. morphisms of locally finite type) in the conventional sense. We take as $\mathfrak{M}$ the (image of the) family of all morphisms of $\mathfrak{A}$ such that the corresponding k-algebra morphism is a strict epimorphism with a nilpotent kernel. The formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) morphisms are formally smooth (resp. formally unramified, resp. formally étale) in the usual sense. Therefore, $\mathfrak{M}$-smooth, $\mathfrak{M}$-unramified, $\mathfrak{M}$-étale morphisms are resp. smooth, unramified and étale. In particular, $\mathfrak{M}$-open immersions are precisely open immersions in the usual sense.

4.4.2. Let $\mathfrak{A} = \mathbf{Aff}_k = Alg_k^{op}$ and $\mathfrak{B}$ the category of presheaves of sets on $\mathfrak{A}$, i.e. functors $Alg_k \longrightarrow Sets$, which are *local* in the following sense: they preserve finite products, and for any faithfully flat k-algebra morphism $R \longrightarrow T$, the diagram

$$X(R) \longrightarrow X(T) \rightrightarrows X(T \star_R T) \qquad (1)$$

is exact. Here $\star_R$ denote the 'star'-product of k-algebras over R (which is a traditional name for a push-forward of associative k-algebras). We denote this category by $\mathcal{NE}sp_k$ and call its objects 'noncommutative spaces', or simply 'spaces'. The functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the Yoneda embedding, $R \longmapsto \mathfrak{A}(-, R^\vee) = Alg_k(R, -)$ (here $R^\vee$ is the affine scheme corresponding to the k-algebra R).

We call $(\mathfrak{A}, \mathfrak{F})$-finitely presentable morphisms (resp. morphisms of $(\mathfrak{A}, \mathfrak{F})$-finite type) simply *locally finitely presentable* morphisms (resp. *morphisms of locally finite type*).

We take as $\mathfrak{M}$ the family of all morphisms of $\mathfrak{A}$ such that the corresponding k-algebra morphism is a strict epimorphism with a nilpotent kernel. Then formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-unramified, resp. formally $\mathfrak{M}$-étale) morphisms are called *formally smooth* (resp. *formally unramified*, resp. *formally étale*) morphisms.

5. Pretopologies and classes of morphisms.

5.1. Preliminaries on pretopologies and right exact structures. By a *pretopology* on a category $\mathfrak{B}$, we understand the usual notion of Grothendieck pretopology formed by a collection of *covers* $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that

(a) every {isomorphism} is a cover;

(b) if $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover of X and $\{U_{ij} \xrightarrow{u_{ij}} U_i \mid j \in J_i\}$ is a cover for every $i \in J$, then their composition, $\{U_{ij} \xrightarrow{u_i \circ u_{ij}} X \mid i \in J,\ j \in J_i\}$, is a cover of X;

(c) if $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover and $Y \longrightarrow X$ an arbitrary morphism, then all pull-backs $U_i \times_X Y \longrightarrow Y$ of arrows $U_i \xrightarrow{u_i} X$ along $Y \longrightarrow X$ exist and form a cover.

A category equipped with a pretopology is called a *presite*.

5.1.1. Subcanonical and canonical pretopologies.

(a) Recall that a pretopology τ on a category $\mathfrak{B}$ is called *subcanonical* if every representable presheaf of sets on $\mathfrak{B}$ is a sheaf on the presite $(\mathfrak{B}, \tau)$.

(a1) Equivalently, every cover $\{\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ is a *strictly epimorphic* family. The latter means that if $\{\mathcal{U}_i \xrightarrow{\mathfrak{v}_i} \mathcal{Y} \mid i \in J\}$ a set of morphisms such that, for every $i \in J$, the morphism $\mathcal{U}_i \xrightarrow{\mathfrak{v}_i} \mathcal{Y}$ equalizes all pairs of arrows $\mathcal{Z} \rightrightarrows \mathcal{U}_i$ equalized by $\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathcal{Y}$, then there exists a unique morphism $\mathcal{X} \xrightarrow{\lambda} \mathcal{Y}$ such that $\mathfrak{v}_i = \lambda \circ \mathfrak{u}_i$.

(b) The *canonical* pretopology on $\mathfrak{B}$ is the *finest* subcanonical pretopology.

5.1.2. Right exact structures. By definition, *right exact structures* are subcanonical pretopologies with all covers consisting of one arrow. The requirement that the pretopology is subcanonical means precisely that all the arrows forming covers are strict epimorphisms: they are cokernels of their kernel pairs.

Thus, a right exact structure on a category $\mathfrak{B}$ is a class $\mathfrak{E}$ of strict epimorphisms containing all isomorphisms and stable under arbitrary pull-backs and composition.

5.1.3. The canonical right exact structure. Right exact structures on a category $\mathfrak{B}$ form a filtered family: if $\{\mathfrak{E}_i \mid i \in I\}$ is a set of right exact structures, then all possible compositions of arrows from $\mathfrak{E}_i$, $i \in I$, form a right exact structure which we denote by $\sup\limits_{i\in I} \mathfrak{E}_i$. This is the coarsest common refinement of all right exact structures $\mathfrak{E}_i$, $i \in I$.

In particular, it follows that the union of all right exact structures on the category $\mathfrak{B}$ is a right exact structure, which we call *canonical* and denote by $\mathfrak{E}^{\mathfrak{c}}_{\mathfrak{B}}$.

The canonical right exact structure can be described directly as follows.

5.1.4. Proposition. *The canonical right exact structure $\mathfrak{E}^{\mathfrak{c}}_{\mathfrak{B}}$ on the category $\mathfrak{B}$ consists of universally strict epimorphisms; that is morphisms whose arbitrary pull-backs (in particular, themselves) are strict epimorphisms.*

Proof. (i) Let $\mathcal{L}_1 \xrightarrow{\mathfrak{t}_1} \mathcal{L}_2$ and $\mathcal{L}_2 \xrightarrow{\mathfrak{t}_2} \mathcal{L}_3$ be strict epimorphisms whose pull-backs along strict epimorphisms are strict epimorphisms. Then their composition, $\mathcal{L}_1 \xrightarrow{\mathfrak{j}_2\circ\mathfrak{j}_1} \mathcal{L}_3$, is a strict epimorphism.

The kernel pair of the composition $\mathcal{L}_1 \xrightarrow{\mathfrak{t}_2\circ\mathfrak{t}_1} \mathcal{L}_3$ is naturally decomposed into the diagram

$$\begin{array}{ccccc}
\mathcal{K}_2(\mathfrak{t}_2\circ\mathfrak{t}_1) & \xrightarrow{\mathfrak{t}''_1} & \mathcal{K}_{12} & \xrightarrow{\mathfrak{t}''_2} & \mathcal{L}_1 \\
{\scriptstyle \widetilde{\mathfrak{p}}_1}\big\downarrow & cart & {\scriptstyle \mathfrak{p}_1}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{t}_1} \\
\mathcal{K}_{12} & \xrightarrow{\mathfrak{t}'_1} & \mathcal{K}_2(\mathfrak{t}_2) & \xrightarrow{\mathfrak{t}'_2} & \mathcal{L}_2 \\
{\scriptstyle \widetilde{\mathfrak{p}}_2}\big\downarrow & cart & {\scriptstyle \pi_2}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{t}_2} \\
\mathcal{L}_1 & \xrightarrow{\mathfrak{t}_1} & \mathcal{L}_2 & \xrightarrow{\mathfrak{t}_2} & \mathcal{L}_3
\end{array} \qquad (1)$$

whose all squares are cartesian.

For any morphism $\mathcal{M} \xrightarrow{\mathfrak{f}} \mathcal{N}$, let $\Lambda^o(\mathfrak{f})$ denote the class of all pairs of arrows $\mathcal{V} \rightrightarrows \mathcal{M}$ which are equalized by the morphism $\mathfrak{f}$.

Let $\mathcal{L}_1 \xrightarrow{\xi} \mathcal{V}$ be a morphism such that $\Lambda^o(\mathfrak{t}_2\circ\mathfrak{t}_1) \subseteq \Lambda^o(\xi)$. In particular, $\Lambda^o(\mathfrak{t}_1) \subseteq \Lambda^o(\xi)$. The latter implies that $\xi = \xi_1 \circ \mathfrak{t}_1$ for a uniquely defined morphism $\mathcal{L}_1 \xrightarrow{\xi_1} \mathcal{V}$. The inclusion $\Lambda^o(\mathfrak{t}_2\circ\mathfrak{t}_1) \subseteq \Lambda^o(\xi) = \Lambda^o(\xi_1\circ\mathfrak{t}_1)$ implies (actually, means) that

$$\xi_1 \circ \mathfrak{t}_1 \circ (\widetilde{\mathfrak{p}}_2 \circ \widetilde{\mathfrak{p}}_1) = \xi_1 \circ \mathfrak{t}_1 \circ (\mathfrak{t}''_2 \circ \mathfrak{t}''_1).$$

It follows from the commutativity of the diagram (1) that

$$\xi_1 \circ \mathfrak{t}_1 \circ (\widetilde{\mathfrak{p}}_2 \circ \widetilde{\mathfrak{p}}_1) = (\xi_1 \circ \mathfrak{p}_2) \circ (\mathfrak{p}_1 \circ \mathfrak{t}''_1) \quad \text{and}$$
$$\xi_1 \circ \mathfrak{t}_1 \circ (\mathfrak{t}''_2 \circ \mathfrak{t}''_1) = (\xi_1 \circ \mathfrak{t}'_2) \circ (\mathfrak{p}_1 \circ \mathfrak{t}''_1).$$

So that

$$(\xi_1 \circ \mathfrak{p}_2) \circ (\mathfrak{p}_1 \circ \mathfrak{t}_1'') = (\xi_1 \circ \mathfrak{t}_2') \circ (\mathfrak{p}_1 \circ \mathfrak{t}_1''). \tag{2}$$

Since $\mathfrak{p}_1$ and $\mathfrak{t}_1''$ are (strict) epimorphisms, their composition $\mathfrak{p}_1 \circ \mathfrak{t}_1''$ is an epimorphism. Therefore, it follows from the equality (2) that $\xi_1 \circ \mathfrak{p}_2 = \xi_1 \circ \mathfrak{t}_2'$. By hypothesis, $\mathfrak{t}_2$ is a strict epimorphism, that is the cokernel of the pair of arrows $\mathcal{K}_2(\mathfrak{t}_2) \underset{\mathfrak{p}_2}{\overset{\mathfrak{t}_2'}{\rightrightarrows}} \mathcal{L}_2$ (see the lower right square of the diagram (1)). Therefore, $\xi_1 = \xi_2 \circ \mathfrak{t}_2$ for a unique morphism $\mathcal{L}_2 \xrightarrow{\xi_2} \mathcal{V}$.

(ii) Since a pull-back of a composition of morphisms having pull-backs is the composition of pull-backs, it follows from (i) that the composition of universal strict epimorphisms is a strict epimorphism. ■

5.2. Expanding a class of arrows via a pretopology. Let $(\mathcal{A}, \tau)$ be a presite; and let $\mathfrak{P}$ be a class of arrows of the category $\mathcal{A}$. We denote by $\mathfrak{P}^\tau$ the class of all arrows $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ of the category $\mathcal{A}$ for which there exists a cover $\{\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in I\}$ such that $\mathfrak{f} \circ \mathfrak{u}_i \in \mathfrak{P}$ for all $i \in I$.

5.2.1. Proposition. *Let $(\mathcal{A}, \tau)$ be a presite and $\mathfrak{P}$ a class of arrows of $\mathcal{A}$.*

(a) $\mathfrak{P} \subseteq \mathfrak{P}^\tau$ and $(\mathfrak{P}^\tau)^\tau = \mathfrak{P}^\tau$.

(b) If $\mathfrak{P}$ is stable under pull-backs, then the class $\mathfrak{P}^\tau$ is closed under pull-backs.

(c) Suppose that the class $\mathfrak{P}$ is closed under composition and pull-backs along the elements of covers of τ. Then the class $\mathfrak{P}^\tau$ is closed under composition.

Proof. (a) Obviously, $\mathfrak{P} \subseteq \mathfrak{P}^\tau$. In particular, $(\mathfrak{P})^\tau \subseteq (\mathfrak{P}^\tau)^\tau$. The inverse inclusion follows from the fact that the composition of covers is a cover.

(b) Let $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ be a morphism from $\mathfrak{P}^\tau$ and

$$\begin{array}{ccc}
\widetilde{\mathcal{X}} & \xrightarrow{\widetilde{\mathfrak{f}}} & \widetilde{\mathcal{Y}} \\
{\scriptstyle \xi'} \downarrow & cart & \downarrow {\scriptstyle \xi} \\
\mathcal{Y} & \xrightarrow{\mathfrak{f}} & \mathcal{Y}
\end{array}$$

a cartesian square. Let $\{\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in I\}$ be a cover of $\mathcal{X}$ such that $\mathfrak{f} \circ \mathfrak{u}_i \in \mathfrak{P}$ for all $i \in I$. We have a family of diagrams

$$\begin{array}{ccccc}
\widetilde{\mathcal{U}}_i & \xrightarrow{\widetilde{\mathfrak{u}}_i} & \widetilde{\mathcal{X}} & \xrightarrow{\widetilde{\mathfrak{f}}} & \widetilde{\mathcal{Y}} \\
{\scriptstyle \xi_i} \downarrow & cart & \downarrow {\scriptstyle \xi'} & cart & \downarrow {\scriptstyle \xi} \\
\mathcal{U}_i & \xrightarrow{\mathfrak{u}_i} & \mathcal{X} & \xrightarrow{\mathfrak{f}} & \mathcal{Y}
\end{array} \qquad i \in I, \tag{2}$$

with cartesian squares. So that we have a cover $\{\widetilde{\mathcal{U}}_i \xrightarrow{\widetilde{\mathfrak{u}}_i} \widetilde{\mathcal{X}} \mid i \in I\}$ of the object $\widetilde{\mathcal{X}}$ such that each of the compositions $\widetilde{\mathfrak{f}} \circ \widetilde{\mathfrak{u}}_i$ is a pull-back of the composition $\mathfrak{f} \circ \mathfrak{u}_i$; and the latter, by hypothesis, belongs to the class $\mathfrak{P}$. Therefore, $\widetilde{\mathfrak{f}} \circ \widetilde{\mathfrak{u}}_i \in \mathfrak{P}$ for all $i \in I$, which means that the pull-back $\widetilde{\mathfrak{f}}$ of the morphism $\mathfrak{f}$ belongs to $\mathfrak{P}^\tau$.

(c) Let $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ and $\mathcal{Y} \xrightarrow{g} \mathcal{Z}$ be morphisms of $\mathfrak{P}^\tau$. Let $\{\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in I\}$ and $\{\mathcal{V}_j \xrightarrow{\mathfrak{v}_j} \mathcal{Y} \mid i \in J\}$ be covers such that $\mathfrak{f} \circ \mathfrak{u}_i \in \mathfrak{P}$ for all $i \in I$ and $g \circ \mathfrak{v}_j \in \mathfrak{P}$ for all $j \in J$. Consider the diagrams

$$\begin{array}{ccccccc}
\mathcal{U}_{ij} & \xrightarrow{\mathfrak{u}_{ij}} & \widetilde{\mathcal{V}}_j & \xrightarrow{\mathfrak{f}_j} & \mathcal{V}_j & & \\
{\scriptstyle \mathfrak{v}_{ij}}\big\downarrow & cart & \big\downarrow{\scriptstyle \widetilde{\mathfrak{v}}_j} & cart & \big\downarrow{\scriptstyle \mathfrak{v}_j} & & \\
\mathcal{U}_i & \xrightarrow{\mathfrak{u}_i} & \mathcal{X} & \xrightarrow{\mathfrak{f}} & \mathcal{Y} & \xrightarrow{g} & \mathcal{Z}, \qquad j \in J,
\end{array} \tag{1}$$

with cartesian squares. The family of arrows $\{\mathcal{U}_{ij} \xrightarrow{\mathfrak{u}_i \mathfrak{v}_{ij}} \mathcal{X} \mid (i,j) \in I \times J\}$ is the composition of covers, hence a cover of the object $\mathcal{X}$. It follows from the commutativity of the diagram (1) that

$$(g \circ \mathfrak{f}) \circ (\mathfrak{u}_i \circ \mathfrak{v}_{ij}) = (g \circ \mathfrak{v}_j) \circ (\mathfrak{f}_i \circ \mathfrak{u}_{ij}).$$

By hypothesis, $g \circ \mathfrak{v}_j \in \mathfrak{P}$ for all $j \in J$. Since the composition of cartesian squares is a cartesian square, the morphism $\mathfrak{f}_i \circ \mathfrak{u}_{ij}$ is a pull-back of the morphism $\mathfrak{f} \circ \mathfrak{u}_i$ along the morphism $\mathcal{V}_j \xrightarrow{\mathfrak{v}_j} \mathcal{Y}$. By hypothesis, $\mathfrak{f} \circ \mathfrak{u}_i \in \mathfrak{P}$ for all $i \in I$ and the class $\mathfrak{P}$ is stable under pull-backs along the elements of covers of the pretopology τ. Therefore, $\mathfrak{f}_i \circ \mathfrak{u}_{ij} \in \mathfrak{P}$ for all $i \in I$ and $j \in J$. This shows that the composition $\mathcal{X} \xrightarrow{g \circ \mathfrak{f}} \mathcal{Z}$ belongs to $\mathfrak{P}^\tau$. ∎

5.2.2. Corollary. *Let $(\mathcal{A}, \tau)$ be a presite. Suppose that a class $\mathfrak{P}$ of morphisms of the category $\mathcal{A}$ contains all isomorphisms of $\mathcal{A}$ and is closed under composition and pull-backs. Then the class $\mathfrak{P}^\tau$ has the same properties.*

Proof. It follows from 5.2.1(b) and 5.2.1(c) that the class $\mathfrak{P}^\tau$ is closed under pull-backs and composition. By 5.2.1(a), $\mathfrak{P} \subseteq \mathfrak{P}^\tau$. So that if $\mathfrak{P}$ contains all isomorphisms of the category $\mathcal{A}$, same holds for $\mathfrak{P}^\tau$. ∎

5.3. Some special cases. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ (– a local data) and a pretopology τ on the category $\mathfrak{B}$.

5.3.1. Locally finitely presentable morphisms. By 1.11.2, the class $\Sigma^1_{\mathfrak{A}}$ of $(\mathfrak{A}, \mathfrak{F})$-finitely presentable morphisms of the category $\mathfrak{B}$ has all the required properties:

it is closed under compositions and pull-backs and contains all isomorphisms. We define *locally* $(\mathfrak{A}, \mathfrak{F})$-finitely presentable morphisms as morphisms of the class $(\Sigma^1_{\mathfrak{A}})^\tau$.

5.3.2. Locally representable morphisms. Let $\mathfrak{P}$ be the class of representable morphisms of the category $\mathfrak{B}$. By 2.1, the class $\mathfrak{P}$ contains all isomorphisms and is stable under composition and pull-backs. The class $\mathfrak{P}^\tau$ consists of *locally affine morphisms.*

5.3.3. Formally $(\mathfrak{M}, \tau)$-smooth and $(\mathfrak{M}, \tau)$-étale morphisms. Let $\mathfrak{M}$ a class of arrows of the category $\mathfrak{B}$. Taking as $\mathfrak{P}$ the class $\mathfrak{M}_{fsm}$ of formally $\mathfrak{M}$-smooth morphisms, we obtain the class $\mathfrak{M}^\tau_{fsm}$ of *locally* (with respect to the pretopology τ) formally $\mathfrak{M}$-smooth morphisms. We call them *formally* $(\mathfrak{M}, \tau)$*-smooth* morphisms.

Taking as $\mathfrak{P}$ the class $\mathfrak{M}_{fet}$ of formally $\mathfrak{M}$-étale morphisms, we construct the class $\mathfrak{M}^\tau_{fet}$ of *formally* $(\mathfrak{M}, \tau)$*-étale* morphisms.

5.3.4. $(\mathfrak{M}, \tau)$-smooth and $(\mathfrak{M}, \tau)$-étale morphisms. Let $\mathfrak{M}$ a class of arrows of the category $\mathfrak{B}$. Starting from the class $\mathfrak{M}_{fsm} \stackrel{\text{def}}{=} \mathfrak{M}_{fsm} \bigcap \Sigma^1_{\mathfrak{A}}$ of $\mathfrak{M}$-smooth morphisms of $\mathcal{A}$, we get the class $\mathfrak{M}^\tau_{sm}$ of $(\mathfrak{M}, \tau)$*-smooth* morphisms.

Similarly, we obtain the class $\mathfrak{M}^\tau_{et}$ of $(\mathfrak{M}, \tau)$-étale morphisms.

5.4. The classes $\mathfrak{P}^\tau$ and the sheafification.

5.4.1. Notations. Let $(\mathcal{A}, \tau)$ be a presite. We denote by $\mathcal{A}^\wedge$ the category of presheaves of sets on $\mathcal{A}$ and by $(\mathcal{A}, \tau)^\wedge$ the category of sheaves of sets on the presite $(\mathcal{A}, \tau)$.

Let $\mathcal{A}^\wedge \xrightarrow{\mathfrak{q}^*} (\mathcal{A}, \tau)^\wedge$ be the sheafification functor and $\mathfrak{q}_*$ its right adjoint. For every $\mathcal{X} \in Ob\mathcal{A}^\wedge$, we denote $\mathfrak{q}_*\mathfrak{q}^*(\mathcal{X})$ by $\mathcal{X}^\mathfrak{a}$ and the adjunction morphism $\mathcal{X} \longrightarrow \mathcal{X}^\mathfrak{a}$ by $\eta_\mathcal{X}$.

5.4.2. Proposition. *Fix a class $\mathfrak{P}$ of arrows of the category $\mathcal{A}$. Let $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ be a morphism of presheaves of sets on $\mathcal{A}$.*

(a) Suppose that the sheafification $\mathcal{X}^\mathfrak{a} \xrightarrow{\mathfrak{f}^\mathfrak{a}} \mathcal{Y}^\mathfrak{a}$ of the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ belongs to the class $\mathfrak{P}^\tau$. If the square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathfrak{f}} & \mathcal{Y} \\ {\scriptstyle \eta_\mathcal{X}}\downarrow & & \downarrow{\scriptstyle \eta_\mathcal{Y}} \\ \mathcal{X}^\mathfrak{a} & \xrightarrow{\mathfrak{f}^\mathfrak{a}} & \mathcal{Y}^\mathfrak{a} \end{array}$$

is cartesian, then $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ belongs to $\mathfrak{P}^\tau$ too.

(b) Suppose that every τ-cover has a refinement whose arrows belong to $\mathfrak{P}^\tau$.

Then the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ belongs to the class $\mathfrak{P}^\tau$ iff $\mathcal{X}^\mathfrak{a} \xrightarrow{\mathfrak{f}^\mathfrak{a}} \mathcal{Y}^\mathfrak{a}$ belongs to $\mathfrak{P}^\tau$.

Proof. (a) Consider the decomposition

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{\phi} & \mathfrak{X} & \xrightarrow{\widetilde{\mathfrak{f}}} & \mathcal{Y} \\ & & {\scriptstyle\gamma_{\mathfrak{f}}}\big\downarrow & cart & \big\downarrow{\scriptstyle\eta_{\mathcal{Y}}} \\ & & \mathcal{X}^{\mathfrak{a}} & \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} & \mathcal{Y}^{\mathfrak{a}} \end{array} \quad \text{of the square} \quad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\mathfrak{f}} & \mathcal{Y} \\ {\scriptstyle\eta_{\mathcal{X}}}\big\downarrow & & \big\downarrow{\scriptstyle\eta_{\mathcal{Y}}} \\ \mathcal{X}^{\mathfrak{a}} & \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad (1)$$

into a cartesian square and a uniquely defined morphism $\mathcal{X} \xrightarrow{\phi} \mathfrak{X}$.

If $\mathcal{X} \xrightarrow{\phi} \mathfrak{X}$ is an isomorphism and $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ belongs to $\mathfrak{P}^{\tau}$, then it follows from 5.2.1(b) that the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ belongs to $\mathfrak{P}^{\tau}$.

(b) Let $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ belong to $\mathfrak{P}^{\tau}$ and the condition of (b) hold. Then there is a cover $\{\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathfrak{X} \mid i \in I\}$ such that $\mathfrak{u}_i \in \mathfrak{P}^{\tau}$ and factors through $\mathcal{X} \xrightarrow{\phi} \mathfrak{X}$ for all $i \in I$.

On the other hand, the morphism $\mathfrak{X} \xrightarrow{\widetilde{\mathfrak{f}}} \mathcal{Y}$, being a pull-back of the arrow $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ from $\mathfrak{P}^{\tau}$, belongs to $\mathfrak{P}^{\tau}$ as well. So that there exists a cover $\{\mathcal{V}_j \xrightarrow{\mathfrak{v}_j} \mathfrak{X} \mid j \in J\}$ such that $\widetilde{\mathfrak{f}} \circ \mathfrak{v}_j \in \mathfrak{P}^{\mathfrak{t}}$ for all $j \in J$. Then we have the cartesian squares

$$\begin{array}{ccc} \mathcal{U}_{ij} & \xrightarrow{\mathfrak{v}_{ij}} & \mathcal{V}_j \\ {\scriptstyle\mathfrak{u}_{ij}}\big\downarrow & cart & \big\downarrow{\scriptstyle\mathfrak{v}_j} \\ \mathcal{U}_i & \xrightarrow{\mathfrak{u}_i} & \mathfrak{X} \end{array} \qquad i \in I,\ j \in J,$$

describing the smallest common refinement $\{\mathcal{U}_{ij} \xrightarrow{\mathfrak{u}_i \circ \mathfrak{u}_{ij}} \mathfrak{X} \mid (i, j \in I \times J\}$ of the two covers. This refinement is a composition of a cover $\{\mathcal{U}_{ij} \xrightarrow{\widetilde{\mathfrak{u}}_{ij}} \mathcal{X} \mid (i, j \in I \times J\}$ of $\mathcal{X}$ and the morphism $\mathcal{X} \xrightarrow{\phi} \mathfrak{X}$, because the cover $\{\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathfrak{X} \mid i \in I\}$ has this property.

For every $(i, j) \in I \times J$, we have:

$$\mathfrak{f} \circ \widetilde{\mathfrak{u}}_{ij} = (\widetilde{\mathfrak{f}} \circ \phi) \circ \widetilde{\mathfrak{u}}_{ij} = \widetilde{\mathfrak{f}} \circ (\mathfrak{u}_i \circ \mathfrak{u}_{ij}) = (\widetilde{\mathfrak{f}} \circ \mathfrak{v}_j) \circ \mathfrak{v}_{ij}$$

By hypothesis, $\widetilde{\mathfrak{f}} \circ \mathfrak{v}_j \in \mathfrak{P}^{\tau}$; and, by 5.2.1(b), $\mathfrak{v}_{ij} \in \mathfrak{P}^{\tau}$, because $\mathfrak{v}_{ij}$ is a pull-back of the morphism $\mathfrak{u}_i$, which belongs to $\mathfrak{P}^{\tau}$ for all $i \in I$. Since, by 5.2.1(c), the class $\mathfrak{P}^{\tau}$ is closed under composition, this shows that $\mathfrak{f} \circ \widetilde{\mathfrak{u}}_{ij} \in \mathfrak{P}^{\tau}$ for all $(i, j) \in I \times J$. The latter means that the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ belongs to $(\mathfrak{P}^{\tau})^{\tau} = \mathfrak{P}^{\tau}$ (see 5.2.1(a)).

(b1) Suppose now that $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is a morphism from $\mathfrak{P}^{\tau}$. Then the morphism $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ belong to $\mathfrak{P}^{\tau}$.

There is a cover $\{\mathcal{U}_i \xrightarrow{\mathfrak{u}_i} \mathcal{Y} |\ i \in I\}$ such that $\eta_{\mathcal{Y}} \circ \mathfrak{u}_i \in \mathfrak{P}^\tau$ for all $i \in I$. We associate with this data commutative diagrams

$$\begin{array}{ccccc} \mathcal{V}_i & \xrightarrow{\phi_i} & \widetilde{\mathcal{U}}_i & \xrightarrow{\mathfrak{f}_i} & \mathcal{U}_i \\ {\scriptstyle\widetilde{\mathfrak{v}}_i}\big\downarrow & cart & {\scriptstyle\mathfrak{v}_i}\big\downarrow & cart & \big\downarrow{\scriptstyle\mathfrak{u}_i} \\ \mathcal{X} & \xrightarrow{\phi} & \mathfrak{X} & \xrightarrow{\widetilde{\mathfrak{f}}} & \mathcal{Y} \\ & & {\scriptstyle\gamma_{\mathfrak{f}}}\big\downarrow & cart & \big\downarrow{\scriptstyle\eta_{\mathcal{Y}}} \\ & & \mathcal{X}^{\mathfrak{a}} & \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad i \in I, \tag{2}$$

with cartesian squares (which is derived from the diagram (1)). Since $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is, by hypothesis, a morphism from $\mathfrak{P}^\tau$ and $\mathfrak{f} = \widetilde{\mathfrak{f}} \circ \phi$, the composition $\mathcal{V}_i \xrightarrow{\mathfrak{f}_i \circ \phi_i} \mathcal{U}_i$ of the upper horizontal arrows – a pull-back of $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$, belongs to $\mathfrak{P}^\tau$. It follows from the commutativity of the diagram (2) that

$$\mathfrak{f}^{\mathfrak{a}} \circ (\eta_{\mathcal{X}} \circ \widetilde{\mathfrak{v}}_i) = \mathfrak{f}^{\mathfrak{a}} \circ (\gamma_{\mathfrak{f}} \circ \phi \circ \widetilde{\mathfrak{v}}_i) = (\eta_{\mathcal{Y}} \circ \mathfrak{u}_i) \circ (\mathfrak{f}_i \circ \phi_i) \tag{3}$$

for all $i \in I$. Since both $\eta_{\mathcal{Y}} \circ \mathfrak{u}_i$ and $\mathfrak{f}_i \circ \phi_i$ are morphisms from $\mathfrak{P}^\tau$, by 5.2.1(c), their composition is a morphism from $\mathfrak{P}^\tau$. Therefore, by (3), $\mathfrak{f}^{\mathfrak{a}} \circ (\eta_{\mathcal{X}} \circ \widetilde{\mathfrak{v}}_i) \in \mathfrak{P}^\tau$ for all $i \in I$. Since $\{\mathcal{V}_i \xrightarrow{\eta_{\mathcal{X}} \circ \widetilde{\mathfrak{v}}_i} \mathcal{X}^{\mathfrak{a}} \mid i \in I\}$ is a cover of $\mathcal{X}^{\mathfrak{a}}$, the latter means that the morphism $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ belongs to $(\mathfrak{P}^\tau)^\tau$ and, by 5.2.1(a), $(\mathfrak{P}^\tau)^\tau = \mathfrak{P}^\tau$. ∎

6. Locally affine spaces and schemes.

6.1. Classes of morphisms and pretopologies. Let τ be a pretopology on a category $\mathfrak{B}$ and $\mathcal{P}$ a class of morphisms of $\mathfrak{B}$. We denote by $\tau^{\mathcal{P}}$ the family of all covers of the pretopology τ formed by arrows from $\mathcal{P}$. If the class $\mathcal{P}$ contains all isomorphisms and is closed under pull-backs and compositions, then $\tau^{\mathcal{P}}$ is a pretopology.

6.1.1. Note. For any class of arrows $\mathcal{P}$ in the category $\mathfrak{B}$ which contains all isomorphisms and is stable under pull-backs and composition and any pretopology τ on $\mathfrak{B}$, the pretopologies $\tau^{\mathcal{P}}$ and $\tau^{\mathcal{P}^\tau}$ are equivalent.

In fact, $\tau^{\mathcal{P}} \subseteq \tau^{\mathcal{P}^\tau}$, because $\mathcal{P} \subseteq \mathcal{P}^\tau$. On the other hand, it follows from the definition of the class $\mathcal{P}^\tau$ that every cover $\{\mathcal{U}_i \longrightarrow \mathcal{X} \mid i \in J\}$ in the pretopology $\tau^{\mathcal{P}^\tau}$ has a refinement (– a composition with τ-covers of each object $\mathcal{U}_i$), which is a cover in $\tau^{\mathcal{P}}$.

6.2. Canonical choices of a pretopology. In what follows, we take as τ
– either the *canonical right exact structure* $\mathfrak{E}^{\mathfrak{c}}_{\mathfrak{B}}$ on $\mathfrak{B}$,

– or the *canonical* pretopology, $\tau_{\mathfrak{c}}$,
– or associated with $\tau_{\mathfrak{c}}$ quasi-compact subpretopology, $\tau_{\mathfrak{c},\mathfrak{f}}$.

Recall that a quasi-compact pretopology $\tau_{\mathfrak{f}}$ (where $\mathfrak{f}$ stands for 'finite') associated with a pretopology τ is formed by all covers of τ which have a finite subcover.

6.3. Étale, Zariski and smooth pretopologies. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ and a class $\mathfrak{M}$ of morphisms of the category $\mathfrak{B}$.

6.3.1. Étale pretopology. We define the *étale* pretopology, $\tau_{et}^{\mathfrak{M}}$, on $\mathfrak{B}$ as the pretopology $\tau_{\mathfrak{c}}^{\mathcal{P}}$, where $\mathcal{P}$ is the class $\mathfrak{M}_{et}$ of $\mathfrak{M}$-étale morphisms. In other words, the covers of the pretopology $\tau_{et}^{\mathfrak{M}}$ are strictly epimorphic families of $\mathfrak{M}$-étale morphisms.

6.3.2. Zariski pretopology. Zariski pretopology, $\tau_{\mathfrak{z}}^{\mathfrak{M}}$, on the category $\mathfrak{B}$ is defined in a similar way, as the pretopology $\tau_{\mathfrak{c}}^{\mathcal{P}}$. Only this time, $\mathcal{P}$ is the class $\mathfrak{M}_{zar}$ of $\mathfrak{M}$-open immersions.

6.3.3. Smooth pretopology. We define the *smooth* pretopology, $\tau_{sm}^{\mathfrak{M}}$, on $\mathfrak{B}$ as the pretopology $\tau_{\mathfrak{c}}^{\mathcal{P}}$, where $\mathcal{P}$ is the class of $\mathfrak{M}$-smooth morphisms.

6.4. Semiseparated pretopologies. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$. We call a pretopology τ on the category $\mathfrak{B}$ *semi-separated* if the class Σ_τ of elements of covers of the pretopology τ consists of $(\mathfrak{A}, \mathfrak{F})$-representable morphisms.

In other words, $\tau = \tau^{\mathcal{P}}$, where $\mathcal{P}$ is the class of all $(\mathfrak{A}, \mathfrak{F})$-representable morphisms.

6.4.1. Canonical semi-separated pretopology. Taking the canonical pretopology $\tau_{\mathfrak{c}}$ on $\mathfrak{B}$, we obtain a pretopology $\tau_{\mathfrak{c}}^{\mathfrak{A},\mathfrak{F}}$ which we call the *canonical* semi-separated pretopology. Its covers are strictly epimorphic families of representable morphisms.

6.4.2. Semiseparated étale, Zariski and smooth pretopologies. They are defined by taking semi-separated version of each of the pretopologies. That is covers of the semi-separated étale (resp. Zariski, resp. smooth) pretopology are strictly epimorphic families of $\mathfrak{M}$-étale (resp. $\mathfrak{M}$-open immersions, resp. $\mathfrak{M}$-smooth) representable morphisms.

6.5. Noncommutative locally affine spaces, algebraic spaces and schemes. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ and a class of morphisms $\mathfrak{M}$ of the category $\mathfrak{B}$.

6.5.1. Algebraic spaces. We call an object $\mathcal{X}$ of $\mathfrak{B}$ an *algebraic space* if there is a cover in $\mathfrak{M}$-étale pretopology of the form $\{\mathfrak{F}(U_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$.

6.5.2. Schemes. We call an object $\mathcal{X}$ of $\mathfrak{B}$ a *scheme* if there is a cover in $\mathfrak{M}$-Zariski pretopology of the form $\{\mathfrak{F}(U_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$.

6.5.3. Locally affine spaces. Similarly, a *locally affine* space is defined as an object $\mathcal{X}$ of the category $\mathfrak{B}$ which has a strictly epimorphic family $\{\mathfrak{F}(U_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ formed by $\mathfrak{M}$-smooth morphisms.

6.6. Semi-separated locally affine spaces and schemes.

6.6.1. Semi-separated algebraic spaces. We call an object $\mathcal{X}$ of $\mathfrak{B}$ a *semi-separated algebraic space* if there is a cover in $\mathfrak{M}$-étale pretopology of the form $\{\mathfrak{F}(U_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ formed by representable morphisms.

6.6.2. Semi-separated schemes. We call an object $\mathcal{X}$ of $\mathfrak{B}$ a *semi-separated scheme* if there is a cover in $\mathfrak{M}$-Zariski pretopology of the form $\{\mathfrak{F}(U_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ formed by representable morphisms.

6.6.3. Semi-separated locally affine spaces. Similarly, a *semi-separated locally affine* space is defined as an object $\mathcal{X}$ of the category $\mathfrak{B}$ which has a strictly epimorphic family $\{\mathfrak{F}(U_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ formed by $\mathfrak{M}$-smooth, representable morphisms.

6.6.4. Semi-separated objects and semi-separated locally affine spaces. Recall that an object $\mathcal{G}$ of the category $\mathfrak{B}$ is *semi-separated*, if there exists a product $\mathcal{G} \times \mathcal{G}$ and the diagonal morphism $\mathcal{G} \xrightarrow{\Delta_{\mathcal{G}}} \mathcal{G} \times \mathcal{G}$ is representable.

6.6.4.1. Proposition. *Let the category* $\mathfrak{A}$ *have finite limits and the functor* $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ *preserves products and pull-backs. If* G *is an object of* $\mathfrak{B}$ *such that the diagonal morphism* $G \xrightarrow{\Delta_G} G \times G$ *is representable, then all structures of a locally affine space on* $\mathcal{X}$ *(if any) are semi-separated.*

Proof. The assertion follows from definitions and 2.6.2. ■

6.6.5. Proposition. *Suppose that the functor* $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ *is full and the category* $\mathfrak{B}$ *has pull-backs. Let*

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \big\downarrow & cart & \big\downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array} \qquad (1)$$

be a cartesian square in the category $\mathfrak{B}$*. If* $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ *are semi-separated schemes (resp. algebraic spaces, resp. locally affine spaces), then* $\widetilde{\mathcal{X}}$ *is a semi-separated scheme (resp. a semi-separated algebraic space, resp. a semi-separated locally affine space).*

Proof. (a) Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be semi-separated locally affine spaces. Fix affine covers $\{\mathfrak{F}(\mathcal{V}_\alpha) \xrightarrow{\mathfrak{v}_\alpha} \mathcal{Z} \mid \alpha \in J_{\mathcal{Z}}\}$, $\{\mathfrak{F}(\mathcal{U}_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J_{\mathcal{X}}\}$, $\{\mathfrak{F}(\mathcal{W}_j) \xrightarrow{\mathfrak{w}_j} \mathcal{Y} \mid j \in J_{\mathcal{Y}}\}$ of the objects $\mathcal{Z}$, $\mathcal{X}$ and $\mathcal{Y}$ formed by smooth representable morphisms. With these covers and the cartesian square (1), we associate a commutative diagram

$$\begin{array}{ccccccc} & & & & \mathfrak{F}(\mathcal{U}_i) & \longleftarrow & \mathfrak{F}(\mathcal{U}_{i,\alpha}) \\ & & & & \big\downarrow & cart & \big\downarrow \\ & & \widetilde{\mathcal{X}} & \longrightarrow & \mathcal{X} & \longleftarrow & \widetilde{\mathcal{V}}_\alpha \\ & & \big\downarrow & cart & \big\downarrow & cart & \big\downarrow \\ \mathfrak{F}(\mathcal{W}_j) & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} & \longleftarrow & \mathfrak{F}(\mathcal{V}_\alpha) \\ \big\uparrow & cart & \big\uparrow & cart & \big\uparrow & & \\ \mathfrak{F}(\mathcal{W}_{j,\alpha}) & \longrightarrow & \mathcal{V}''_\alpha & \longrightarrow & \mathfrak{F}(\mathcal{V}_\alpha) & & \end{array} \tag{2}$$

built of cartesian squares. Since pull-backs of representable and smooth morphisms are representable and smooth, all left and right horizontal arrows and all upper and lower vertical arrows of the diagram (2) are representable and smooth. Since representable and smooth morphisms are closed under composition, it follows from the right upper and left lower cartesian squares, that

$$\{\mathfrak{F}(\mathcal{U}_{i,\alpha}) \longrightarrow \mathcal{X} \mid i \in J_{\mathcal{X}},\ \alpha \in J_{\mathcal{Z}}\} \quad \text{and} \quad \{\mathfrak{F}(\mathcal{W}_{j,\alpha}) \longrightarrow \mathcal{Y} \mid j \in J_{\mathcal{Y}},\ \alpha \in J_{\mathcal{Z}}\}$$

are covers of the corresponding objects formed by smooth representable morphisms.

By the universal property of cartesian squares, we have a morphism of the cartesian square

$$\begin{array}{ccc} \mathfrak{F}(\mathcal{U}^j_{i,\alpha}) & \longrightarrow & \mathfrak{F}(\mathcal{U}_{i,\alpha}) \\ \big\downarrow & cart & \big\downarrow \\ \mathfrak{F}(\mathcal{W}_{j,\alpha}) & \longrightarrow & \mathfrak{F}(\mathcal{V}_\alpha) \end{array} \tag{3}$$

to the cartesian square (1), which is uniquely determined by the commutative diagram

$$\begin{array}{ccccc} \mathfrak{F}(\mathcal{U}_{i,\alpha}) & \longrightarrow & \mathfrak{F}(\mathcal{V}_\alpha) & \longleftarrow & \mathfrak{F}(\mathcal{W}_{j,\alpha}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathcal{X} & \longrightarrow & \mathcal{Z} & \longleftarrow & \mathcal{Y} \end{array}$$

whose vertical arrows are representable and smooth. The claim is that the unique arrow $\mathfrak{F}(\mathcal{U}^j_{i,\alpha}) \longrightarrow \widetilde{\mathcal{X}}$ completing the morphism of squares is also representable and smooth.

In order to see this, we continue to expand the diagram (2) by adding cartesian squares until we get the diagram

$$
\begin{array}{ccccccccc}
\mathfrak{F}(\mathcal{U}^j_{i,\alpha}) & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & \mathfrak{F}(\mathcal{U}_{i,\alpha}) & & \\
\Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow & & \\
? & \longrightarrow & ? & \longrightarrow & ? & \longrightarrow & \mathfrak{F}(\mathcal{U}_i) & \longleftarrow & \mathfrak{F}(\mathcal{U}_{i,\alpha}) \\
\Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow \\
? & \longrightarrow & ? & \longrightarrow & \widetilde{\mathcal{X}} & \longrightarrow & \mathcal{X} & \longleftarrow & \widetilde{\mathcal{V}}_\alpha \\
\Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow \\
\mathfrak{F}(\mathcal{W}_{j,\alpha}) & \longrightarrow & \mathfrak{F}(\mathcal{W}_j) & \longrightarrow & \mathcal{Y} & \longrightarrow & \mathcal{Z} & \longleftarrow & \mathfrak{F}(\mathcal{V}_\alpha) \\
 & & \Big\uparrow & cart & \Big\uparrow & cart & \Big\uparrow & & \\
 & & \mathfrak{F}(\mathcal{W}_{j,\alpha}) & \longrightarrow & \mathcal{V}''_\alpha & \longrightarrow & \mathfrak{F}(\mathcal{V}_\alpha) & &
\end{array}
\qquad (4)
$$

which is a decomposition of the cartesian square (3) into smaller cartesian squares. Here we use the fact that a square built of cartesian squares is cartesian. The diagram (4) contains also a decomposition of the morphism of the cartesian square (3) to the cartesian square (1). All arrows of the diagram (4), except for horizontal arrows over and under horizontal arrows of the square (1) and the vertical arrows in the same row as the vertical arrows of (1), are smooth and representable. In particular, the morphism from $\mathfrak{F}(\mathcal{U}^j_{i,\alpha})$ to $\widetilde{\mathcal{X}}$ is the composition of arrows which are smooth and representable, hence it is smooth and representable. In other words, each morphism $\mathfrak{F}(\mathcal{U}^j_{i,\alpha}) \longrightarrow \widetilde{\mathcal{X}}$ is an element of a cover of the object $\widetilde{\mathcal{X}}$ in the smooth semi-separated pretopology. The set of all these morphisms, $\{\mathfrak{F}(\mathcal{U}^j_{i,\alpha}) \longrightarrow \widetilde{\mathcal{X}} \mid (i,j,\alpha) \in J_{\mathcal{X}} \times J_{\mathcal{Y}} \times J_{\mathcal{Z}}\}$, is a cover of $\widetilde{\mathcal{X}}$ in the smooth semi-separated pretopology. This follows from the construction, which consisted of several pull-backs of covers along morphisms and refinements.

(b) If the covers we started with, $\{\mathfrak{F}(\mathcal{V}_\alpha) \xrightarrow{\mathfrak{v}_\alpha} \mathcal{Z} \mid \alpha \in J_{\mathcal{Z}}\}$, $\{\mathfrak{F}(\mathcal{U}_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J_{\mathcal{X}}\}$, $\{\mathfrak{F}(\mathcal{W}_j) \xrightarrow{\mathfrak{w}_j} \mathcal{Y} \mid j \in J_{\mathcal{Y}}\}$, belong to some subpretopology of the smooth semi-separated pretopology, then it follows from the argument above that the final result, the cover $\{\mathfrak{F}(\mathcal{U}^j_{i,\alpha}) \longrightarrow \widetilde{\mathcal{X}} \mid (i,j,\alpha) \in J_{\mathcal{X}} \times J_{\mathcal{Y}} \times J_{\mathcal{Z}}\}$, belongs to this pretopology too.

In particular, if the initial covers are formed by étale morphisms (resp. open immersions), then the cover $\{\mathfrak{F}(\mathcal{U}^j_{i,\alpha}) \longrightarrow \widetilde{\mathcal{X}} \mid (i,j,\alpha) \in J_{\mathcal{X}} \times J_{\mathcal{Y}} \times J_{\mathcal{Z}}\}$ of the object $\widetilde{\mathcal{X}}$ consists of étale morphisms (resp. open immersions).

This shows that the full subcategory $\mathcal{A}\mathfrak{lg}^{\mathfrak{ss}}_{\mathfrak{F},\mathfrak{M}}$ of the category $\mathfrak{B}$ formed by semi-separated algebraic spaces is closed under pull-backs and same holds for the full subcategory $\mathcal{S}\mathfrak{ch}^{\mathfrak{ss}}_{\mathfrak{F},\mathfrak{M}}$ of the category $\mathfrak{B}$ formed by semi-separated schemes. ■

6.6.5.1. Note. The argument of 6.6.5 becomes shorter, if $\mathcal{Z}$ is a semi-separated scheme covered by representable open immersions, because in this case, thanks to the fact that each $\mathfrak{F}(\mathcal{V}_\alpha) \longrightarrow \mathcal{Z}$ is a monomorphism, it is immediate that the square

$$\begin{array}{ccc} \mathfrak{F}(\mathcal{U}_{i,\alpha}^{j}) & \longrightarrow & \mathfrak{F}(\mathcal{U}_{i,\alpha}) \\ \Big\downarrow & cart & \Big\downarrow \\ \mathfrak{F}(\mathcal{W}_{j,\alpha}) & \longrightarrow & \mathcal{Z} \end{array}$$

is cartesian.

6.7. Remark about general case. The main reason why we considered semi-separated locally affine spaces is that all our examples belong to this class. But, of course, there exist non-semi-separated schemes. For them, the invariance under cartesian squares holds under additional conditions on the initial data.

6.7.1. Proposition. *Suppose that the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is full, the category $\mathfrak{B}$ has pull-backs, and, for any open immersion $\mathcal{U} \longrightarrow \mathfrak{F}(\mathcal{V})$, the object $\mathcal{U}$ is a scheme. Let*

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \Big\downarrow & cart & \Big\downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array} \qquad (1)$$

be a cartesian square in the category $\mathfrak{B}$. If $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are schemes, then $\widetilde{\mathcal{X}}$ is a scheme.

Proof. The argument follows the same steps (– uses the same diagrams) as the argument of 6.6.5. Only the object in the upper left corner of the diagram 6.6.5(4) is not affine in general. But, it follows from the hypothesis that it has an affine cover, which is enough. Details are left to the reader. ■

6.8. Relative locally affine spaces and schemes. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ and a class of morphisms $\mathfrak{M}$ of the category $\mathfrak{B}$.

6.8.1. Relative locally affine spaces. We call a morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ of the category $\mathfrak{B}$ a *locally affine* space over $\mathcal{Y}$, if there exists a strictly epimorphic family $\{\mathfrak{F}(U_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ formed by smooth morphisms such that each composition $\mathfrak{F}(U_i) \xrightarrow{\mathfrak{f}\circ\mathfrak{u}_i} \mathcal{Y}$ is a representable morphism.

6.8.2. Relative schemes and algebraic spaces. We call a morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ of the category $\mathfrak{B}$ a *scheme* (resp. an *algebraic space*) over $\mathcal{Y}$ if there exists a strictly epimorphic family $\{\mathfrak{F}(U_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ formed by open immersions (resp. by étale) morphisms such that each composition $\mathfrak{F}(U_i) \xrightarrow{\mathfrak{f}\circ\mathfrak{u}_i} \mathcal{Y}$ is a representable morphism.

6.9. Relative semi-separated schemes and locally affine spaces. A morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ of the category $\mathfrak{B}$ is called a *semi-separated locally affine* space (resp. a semi-separated algebraic space, resp. a semi-separated scheme) over $\mathcal{Y}$), if there exists a strictly epimorphic family $\{\mathfrak{F}(U_i) \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ formed by $\mathfrak{M}$-smooth (resp. $\mathfrak{M}$-étale, resp. $\mathfrak{M}$-open immersions) representable morphisms such that each composition $\mathfrak{F}(U_i) \xrightarrow{\mathfrak{f} \circ \mathfrak{u}_i} \mathcal{Y}$ is a representable morphism.

6.9.1. Proposition. *Let $\mathcal{S}$ be an object of the category $\mathfrak{B}$; and let*

$$\begin{array}{ccc} \widetilde{\mathcal{X}} & \longrightarrow & \mathcal{X} \\ \big\downarrow & cart & \big\downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array} \qquad (1)$$

be a cartesian square in the category $\mathfrak{B}/\mathcal{S}$. If $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are semi-separated schemes (resp. algebraic spaces, resp. locally affine spaces) over $\mathcal{S}$, then $\widetilde{\mathcal{X}}$ is a semi-separated scheme (resp. a semi-separated algebraic space, resp. a semi-separated locally affine space) over $\mathcal{S}$.

Proof. The assertion follows from (the argument of) 6.6.5. Indeed, the forgetful functor $\mathfrak{B}/\mathcal{S} \longrightarrow \mathfrak{B}$ preserves limits (in particular, it preserves pull-backs) and the argument of 6.6.5 is valid in the relative case. Details are left to the reader. ■

7. Quasi-topologies, quasi-pretopologies and gluing.

Quasi-topologies (in particular, topologies) and (quasi-)pretopologies play different role in our story. Quasi-topologies and topologies serve to define the categories of sheaves, while (quasi-)pretopologies serve for gluing spaces and studying their local structure.

7.0. Quasi-topologies and topologies.

7.0.1. Quasi-topologies. A quasi-topology τ on a category $\mathfrak{A}$ is determined by the category $(\mathfrak{A},\tau)^\wedge$ of sheaves on the quasi-site $(\mathfrak{A},\tau)$. The map $(\mathfrak{A},\tau) \longmapsto (\mathfrak{A},\tau)^\wedge$ is a bijective correspondence between quasi-topologies on $\mathfrak{A}$ and strictly full reflective subcategories of the category $\mathfrak{A}^\wedge$ of presheaves of sets on $\mathfrak{A}$ whose objects are sheaves on the quasi-site $(\mathfrak{A},\tau)$.

7.0.1.1. Topologies. A quasi-topology τ on a category $\mathfrak{A}$ is a topology iff the sheafification functor $\mathfrak{A}^\wedge \longrightarrow (\mathfrak{A},\tau)^\wedge$ is exact.

7.0.1.2. Subcanonical quasi-topologies. A quasi-topology τ is called *subcanonical* if every representable presheaf is a sheaf.

7.0.1.3. Quasi-topologies and topologies. For every quasi-topology $\widetilde{\tau}$ on a category $\mathfrak{A}$, there exists the finest topology among the topologies which are coarser than τ. The topology $\widetilde{\tau}$ is formed by all refinements $\mathcal{R} \hookrightarrow \widehat{\mathcal{X}}$, $\mathcal{X} \in Ob\mathfrak{A}$, of the quasi-topology τ whose pull-backs are refinements for the quasi-topology τ.

7.0.1.4. The decomposition of the sheafification functor. The sheafification functor $\mathfrak{A}^\wedge \longrightarrow (\mathfrak{A}, \tau)^\wedge$ is the composition of the sheafification functor $\mathfrak{A}^\wedge \longrightarrow (\mathfrak{A}, \widetilde{\tau})^\wedge$ and a continuous localization $(\mathfrak{A}, \widetilde{\tau})^\wedge \longrightarrow (\mathfrak{A}, \tau)^\wedge$. The localization $(\mathfrak{A}, \widetilde{\tau})^\wedge \longrightarrow (\mathfrak{A}, \tau)^\wedge$ is the sheafification functor for a quasi-topology τ' on the category $(\mathfrak{A}, \widetilde{\tau})^\wedge$, which is not a refinement of any non-trivial topology.

7.0.2. Quasi-topology associated with a local data. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ regarded as a 'local data'. Suppose that the category $\mathfrak{B}$ has colimits of small diagrams. Then the functor

$$\mathfrak{B} \xrightarrow{\mathfrak{F}_*} \mathfrak{A}^\wedge, \quad \mathcal{X} \longmapsto \mathfrak{B}(\mathfrak{F}(-), \mathcal{X}),$$

has a left adjoint, $\mathfrak{A}^\wedge \xrightarrow{\mathfrak{F}^*} \mathfrak{B}$ whose composition with the Yoneda embedding $\mathfrak{A} \xrightarrow{h_{\mathfrak{A}}} \mathfrak{A}^\wedge$ coincides with the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$.

The functor $\mathfrak{F}^*$ is the composition of the continuous (that is having a right adjoint) localization $\mathfrak{A}^\wedge \xrightarrow{\mathfrak{q}^*_{\mathfrak{F}}} \mathfrak{B}_{\mathfrak{F}}$ and a continuous conservative functor $\mathfrak{B}_{\mathfrak{F}} \xrightarrow{\mathfrak{F}^*_{\mathfrak{c}}} \mathfrak{B}$.

One can see that the a right adjoint to the localization $\mathfrak{q}^*_{\mathfrak{F}}$ induces an equivalence between the category $\mathfrak{B}_{\mathfrak{F}}$ and the smallest reflective strictly full subcategory $\widetilde{\mathfrak{B}}_{\mathfrak{F}}$ of the category $\mathfrak{A}^\wedge$ containing the family of presheaves $\mathfrak{B}(\mathfrak{F}(-), \mathcal{X})$, $\mathcal{X} \in Ob\mathfrak{B}$.

Notice that the category $\widetilde{\mathfrak{B}}_{\mathfrak{F}}$ is well defined independently on the condition that the category $\mathfrak{B}$ has colimits. A left adjoint to the embedding $\widetilde{\mathfrak{B}}_{\mathfrak{F}}$ is a localization functor $\mathfrak{A}^\wedge \xrightarrow{\widetilde{\mathfrak{q}}^*_{\mathfrak{F}}} \widetilde{\mathfrak{B}}_{\mathfrak{F}}$ which is the sheafification functor for a uniquely defined conservative quasi-topology $\tau_{\mathfrak{F}}$ on the category $\mathfrak{A}$. The subcategory $\widetilde{\mathfrak{B}}_{\mathfrak{F}}$ coincides with the category $(\mathfrak{A}, \tau_{\mathfrak{F}})^\wedge$ of sheaves of sets on the quasi-site $(\mathfrak{A}, \tau_{\mathfrak{F}})$.

7.1. Quasi-pretopologies. Fix a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$. We assume that the category $\mathfrak{B}$ is endowed with a quasi-pretopology, τ. The latter is a function which assigns to each object X of $\mathfrak{B}$ a family, τ_X, of covers of X. An element of τ_X is set of arrows $\{U_i \xrightarrow{u_i} X \mid i \in J\}$. We assume that any isomorphism forms a cover, and the composition of covers is a cover.

A cover of the form $\{\mathfrak{F}(U_i) \xrightarrow{u_i} X \mid i \in J\}$ of an object X is called a $(\mathfrak{A}, \tau)$-*cover*, or simply $\mathfrak{A}$-*cover*, if τ is fixed.

An object X of $\mathfrak{B}$ is called *locally* $(\mathfrak{A}, \tau)$-*affine* (or locally $(\mathfrak{A}, \mathfrak{F})$-affine, if no ambiguity arises) if it has an $(\mathfrak{A}, \tau)$-cover.

We denote by $Sp_{\mathfrak{A},\tau}$ the full subcategory of the category $\mathfrak{B}$ whose objects are locally $(\mathfrak{A}, \tau)$-affine.

7.1.1. Quasi-finite locally $(\mathfrak{A}, \mathfrak{F})$-affine objects. Given a quasi-pretopology τ on $\mathfrak{B}$, let $\tau_{\mathfrak{f}}$ denote the quasi-pretopology formed by all finite covers of τ. We call an object X of $\mathfrak{B}$ *quasi-finite locally* $(\mathfrak{A}, \tau)$*-affine* if it is locally $(\mathfrak{A}, \tau_{\mathfrak{f}})$-affine.

7.1.2. 2-$(\mathfrak{A}, \mathfrak{F})$-covers and 2-locally $(\mathfrak{A}, \mathfrak{F})$-affine objects. Let X be an object of $\mathfrak{B}$. We call an $(\mathfrak{A}, \tau)$-cover $\{\mathfrak{F}(U_i) \overset{u_i}{\to} X \mid i \in J\}$ a *2-*$(\mathfrak{A}, \tau)$*-cover* (or *2-*$\mathfrak{A}$*-cover*), if for any $i, j \in J$, there exists a set of morphisms $(U_i, u_i) \longleftarrow (U_{ij}^\nu, u_{ij}^\nu) \longrightarrow (U_j, u_j),\ \nu \in J_{ij}$ in the category $\mathfrak{F}/X$ such that the corresponding set of morphisms

$$\{\mathfrak{F}(U_{ij}^\nu) \longrightarrow \mathfrak{F}(U_i) \times_X \mathfrak{F}(U_j) \mid \nu \in J_{ij}\}$$

is a cover for any $i, j \in J$. We call the diagram

$$(U_i, u_i) \longleftarrow (U_{ij}^\nu, u_{ij}^\nu) \longrightarrow (U_j, u_j),\ \nu \in J_{ij},\ i, j \in J, \tag{1}$$

(in the category $\mathfrak{F}/X$) a *diagram of relations* of the 2-cover $\mathfrak{U} = \{\mathfrak{F}(U_i) \overset{u_i}{\to} X \mid i \in J\}$.

We call an object X of $\mathfrak{B}$ *2-locally* $(\mathfrak{A}, \mathfrak{F})$*-affine* if it has a 2-locally $(\mathfrak{A}, \mathfrak{F})$-affine cover.

7.1.3. Weakly semi-separated covers. We call an $\mathfrak{A}$-cover $\{\mathfrak{F}(U_i) \overset{u_i}{\to} X \mid i \in J\}$ *weakly semi-separated* if, for any $i, j \in J$, there exists a diagram

$$(U_i, u_i) \longleftarrow (U_{ij}, u_{ij}) \longrightarrow (U_j, u_j)$$

in $\mathfrak{F}/X$ such that the square

$$\begin{array}{ccc} \mathfrak{F}(U_{ij}) & \longrightarrow & \mathfrak{F}(U_j) \\ \downarrow & cart & \downarrow \\ \mathfrak{F}(U_i) & \longrightarrow & X \end{array}$$

is cartesian. In particular, the object $\mathfrak{F}(U_i) \times_X \mathfrak{F}(U_j)$ is isomorphic to an object of the form $\mathfrak{F}(U_{ij})$. It follows that any weakly semi-separated $\mathfrak{A}$-cover is a 2-$\mathfrak{A}$-cover.

We say that an object X of $\mathfrak{B}$ is $\mathfrak{A}$*-weakly semi-separated* if it has a weakly semi-separated $\mathfrak{A}$-cover.

7.2. $(\mathfrak{A}, \mathfrak{F})$-Representable morphisms and covers. If $\mathcal{E}$ is a subcategory of $\mathfrak{B}$ such that $Ob\mathcal{E} = Ob\mathfrak{B}$, we denote by $\tau^{\mathcal{E}}$ the quasi-pretopology on $\mathfrak{B}$ formed by all covers $\{U_i \overset{u_i}{\to} X \mid i \in J\}$ in τ such that all morphisms u_i belong to $\mathcal{E}$. Given a functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$, we have a natural choice of the subcategory $\mathcal{E}$, which is the subcategory of $(\mathfrak{A}, \mathfrak{F})$*-representable morphisms* described below.

7.2.2. Representable covers. We call a cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ $(\mathfrak{A}, \mathfrak{F})$*-representable* if each morphism u_i of the cover is $(\mathfrak{A}, \mathfrak{F})$-representable. We denote by

$\tau^{\mathfrak{A}}$ the function which assigns to each object X of the category $\mathfrak{B}$ the set, $\tau_X^{\mathfrak{A}}$, of $(\mathfrak{A},\mathfrak{F})$-representable covers of X.

7.2.2.1. Lemma. *The function $\tau^{\mathfrak{A}}$ is a quasi-pretopology on $\mathfrak{B}$. If τ is a pretopology, then $\tau^{\mathfrak{A}}$ is a pretopology.*

Proof. The assertion is a corollary of 2.1. ∎

7.2.3. Locally $(\mathfrak{A},\mathfrak{F})$-representable objects. Evidently, every representable $(\mathfrak{A},\mathfrak{F})$-cover is weakly semi-separated (cf. 7.1.3.). In particular, it is a 2-$(\mathfrak{A},\mathfrak{F})$-cover.

We say that an object X of $\mathfrak{B}$ is *locally $(\mathfrak{A},\mathfrak{F})$-representable* if it has a representable $(\mathfrak{A},\mathfrak{F})$-cover. Thus, every locally $(\mathfrak{A},\mathfrak{F})$-representable object is locally $(\mathfrak{A},\mathfrak{F})$-affine.

7.3. Coinduced pretopology. Let $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ be a functor, and let $\mathfrak{T}$ be a quasi-pretopology on $\mathfrak{A}$. The coinduced quasi-pretopology, $\mathfrak{T}^{\mathfrak{F}}$, on $\mathfrak{B}$ is defined as follows: a set of arrows $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover of X iff for any morphism $\mathfrak{F}(V) \xrightarrow{g} X$, there exists a cover $\{V_j \xrightarrow{v_j} V \mid j \in I\} \in \mathfrak{T}_V$ such that for every $j \in I$, the morphism $g \circ \mathfrak{F}(v_j) : \mathfrak{F}(V_j) \longrightarrow X$ factors through u_i for some $i \in J$.

7.3.1. Proposition. *Suppose $\mathfrak{B}$ is a category with fiber products. Then the coinduced quasi-pretopology $\mathfrak{T}^{\mathfrak{F}}$ on $\mathfrak{B}$ is a pretopology.*

Proof. Let $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a cover in $\mathfrak{T}^{\mathfrak{F}}$ and $Y \xrightarrow{g} X$ an arbitrary morphism. The claim (equivalent to the proposition) is that the set of arrows $\{U_i \times_X Y \xrightarrow{\bar{u}_i} Y \mid i \in J\}$ is a cover in $\mathfrak{T}^{\mathfrak{F}}$.

In fact, let $\mathfrak{F}(V) \xrightarrow{v} Y$ be an arbitrary morphism. Since $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is a cover, there exists a cover $\{V_j \xrightarrow{v_i} V \mid j \in I\}$ in $\mathfrak{T}$ such that for any $j \in I$, there exists $i_j \in J$ and a morphism $\mathfrak{F}(V_j) \xrightarrow{\bar{v}_j} U_{i_j}$ which make the diagram

$$\begin{array}{ccc} \mathfrak{F}(V_j) & \xrightarrow{\bar{v}_j} & U_{i_j} \\ {\scriptstyle \mathfrak{F}(v_j)}\downarrow & & \downarrow{\scriptstyle u_{i_j}} \\ \mathfrak{F}(V) & \xrightarrow{g\circ v} & X \end{array} \qquad (1)$$

commute. The commutativity of (1) implies the existence if a unique morphism

$$\mathfrak{F}(V_j) \xrightarrow{v'_j} U_{i_j} \times_X Y$$

such that the diagram

$$\begin{array}{ccc} \mathfrak{F}(V_j) & \xrightarrow{\bar{v}'_j} & U_{i_j} \times_X Y \\ {\scriptstyle \mathfrak{F}(v_j)}\downarrow & & \downarrow{\scriptstyle u_{i_j}} \\ \mathfrak{F}(V) & \xrightarrow{v} & Y \end{array} \qquad (2)$$

commutes and the morphism $\mathfrak{F}(V_j) \xrightarrow{\bar{v}_j} U_{i_j}$ is the composition of $\mathfrak{F}(V_j) \xrightarrow{v'_j} U_{i_j} \times_X Y$ and the canonical projection $U_{i_j} \times_X Y \longrightarrow U_{i_j}$. This shows that $\{U_i \times_X Y \xrightarrow{\bar{u}_i} Y \mid i \in J\}$ is a cover in $\mathfrak{T}^{\mathfrak{F}}$. ■

7.3.2. Note. Let $\mathfrak{A}$ be a category with fiber products and $\mathfrak{T}$ a quasi-pretopology on $\mathfrak{A}$. Taking $\mathfrak{F} = Id_{\mathfrak{A}}$, we obtain the coinduced pretopology, $\mathfrak{T}^g$, on $\mathfrak{B} = \mathfrak{A}$. The pretopology $\mathfrak{T}^g$ is the finest pretopology among those pretopologies on $\mathfrak{A}$ which are coarser than $\mathfrak{T}$.

7.4. Standard commutative examples.

7.4.1. Geometric spaces and schemes. Let $\mathfrak{B}$ be the category of locally ringed topological spaces which we call otherwise *geometric spaces*, $\mathfrak{A}$ the category opposite to the category $CAlg_k$ of commutative unital k-algebras, $\mathfrak{F}$ the functor $\mathfrak{A} \longrightarrow \mathfrak{B}$ which assigns to every commutative k-algebra its spectrum. The pretopology on $\mathfrak{B}$ is the standard *Zariski* pretopology given by families of open immersions covering the underlying space: a set $\{(U_i, \mathcal{O}_{U_i}) \xrightarrow{u_i} (X, \mathcal{O}_X) \mid i \in J\}$ of open immersions is a cover iff $\bigcup_{i \in J} U_i = X$.

Then locally $(\mathfrak{A}, \mathfrak{F})$-affine objects of $\mathfrak{B}$ are arbitrary schemes over k.

7.4.1.1. Semi-separated schemes. Locally $(\mathfrak{A}, \mathfrak{F})$-representable objects of $\mathfrak{B}$ are precisely *semi-separated schemes.* Recall that a scheme $\mathcal{X} = (X, \mathcal{O})$ is called *semi-separated* if it has an affine cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ such that each morphism $U_i \xrightarrow{u_i} X$ is representable. Clearly, every semi-separated scheme is weakly separated.

7.4.2. Quasi-finite $(\mathfrak{A}, \mathfrak{F})$-objects. Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ are same as in 7.4.1. Then quasi-finite $(\mathfrak{A}, \mathfrak{F})$-objects (i.e. locally $(\mathfrak{A}, \tau_{\mathfrak{Z}_{\mathfrak{f}}})$-affine objects, where $\tau_{\mathfrak{Z}_{\mathfrak{f}}}$ is the sub-pretopology of $\tau_{\mathfrak{Z}}$ formed by finite covers, cf. 7.1.1) are exactly quasi-compact schemes.

Notice that 2-locally $(\mathfrak{A}, \tau_{\mathfrak{Z}_{\mathfrak{f}}})$-affine objects are quasi-compact quasi-separated schemes.

7.4.3. Spaces as sheaves of sets. Let $\mathfrak{A}$ be the category $CAlg_k^{op}$, as in 2.1. Let $\mathfrak{B}$ be category $\mathcal{E}sp$ of sheaves of sets on $CAlg_k^{op}$ for the **fpqc** topology, and let $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ be the Yoneda embedding: $R \longmapsto CAlg_k(R, -)$.

Zariski covers in $CAlg_k^{op} = \mathfrak{A}$ are given by sets of morphisms $\{R \longrightarrow R_i \mid i \in J\}$ such that R_i is a localization of R at an element of R (that is at the multiplicative set generated by this element), and $\bigcup_{i \in J} Spec(R_i) = Spec(R)$. Zariski covers form a (Zariski) pretopology, $\mathfrak{T}_{\mathfrak{Z}}$. We define *Zariski* pretopology on $\mathfrak{B} = \mathcal{E}sp$ as the pretopology coinduced by $\mathfrak{T}_{\mathfrak{Z}}$ (cf. 7.3).

Locally affine $(\mathfrak{A}, \mathfrak{F})$-objects in this setting are schemes in the sense of [DG], that is schemes realized as functors $CAlg_k \longrightarrow Sets$. The functor $\mathcal{S}$ which assigns to each geometric space $\mathbf{X} = (X, \mathcal{O})$ the functor $R \longmapsto Hom((SpecR, \mathcal{O}_R), \mathbf{X})$ establishes an equivalence between *geometric* schemes and *functorial* schemes.

Representable morphisms in $\mathfrak{B}$ are corepresentable functors $CAlg_k \longrightarrow Sets$. The functor $\mathcal{S}$ induces an equivalence of the category of semi-separated schemes (cf. 7.4.1.1) and the category of locally $(\mathfrak{A}, \mathfrak{F})$-representable objects of $\mathfrak{B}$.

Replacing the Zariski pretopology $\mathfrak{T}_{\mathfrak{Z}}$ by its finite version, $\mathfrak{T}_{\mathfrak{Z}_{\mathfrak{f}}}$, we obtain a full subcategory of the category of locally $\mathfrak{A}$-affine objects formed by quasi-finite locally $\mathfrak{A}$-affine objects. The functor $\mathcal{S}$ induces an equivalence of this category and the category of quasi-compact geometric schemes. The functor $\mathcal{S}$ induces an equivalence of the category of 2-locally $(\mathfrak{A}, \mathfrak{T}_{\mathfrak{Z}_{\mathfrak{f}}})$-affine objects and the category of quasi-compact, quasi-separated geometric schemes.

7.5. Standard noncommutative examples. We take as $\mathfrak{A}$ the category $\mathbf{Aff}_k = Alg_k^{op}$ opposite to the category of associative unital k-algebras, together with one of the canonical quasi-pretopologies defined below.

7.5.1. The fpqc quasi-pretopology on $\mathbf{Aff}_k = Alg_k^{op}$. We call the image in $\mathbf{Aff}_k$ of a set of k-algebra morphisms $\{R \longrightarrow R_i \mid i \in J\}$ an **fpqc** *cover* if all morphisms $R \longrightarrow R_i$ are flat (i.e. R_i is a flat right R-module), and there is a finite subset I of J such that the family of functors $\{R_i \otimes_R \ \mid i \in I\}$ is conservative. The composition of **fpqc** covers is an **fpqc** cover, and any faithfully flat k-algebra morphism $R \longrightarrow S$ forms an **fpqc** cover. Thus, **fpqc** covers form a quasi-pretopology which we denote by $\tau_{\mathbf{fpqc}}$.

7.5.1.1. The lqc pretopology. We call an **fpqc** cover $\{R_i^\vee \longrightarrow R^\vee \mid i \in J\}$ an **lqc** *cover* if the corresponding k-algebra morphisms $R \longrightarrow R_i$ are localizations, or, equivalently, the corresponding 'restriction of scalars' functors $R_i - mod \longrightarrow R - mod$ are full (hence fully faithful). It follows from [R4, 2.6.3.1] that **lqc** covers form pretopology on $\mathbf{Aff}_k$ which we denote by $\tau_{\mathbf{lqc}}$.

7.5.2. The fppf quasi-pretopology and Zariski pretopology. We call an **fpqc** cover $\{R \longrightarrow R_i \mid i \in J\}$ an **fppf** *cover* if it consists of *finitely presentable* morphisms. We denote the **fppf** quasi-pretopology by $\tau_{\mathbf{fppf}}$.

A set of algebra morphisms $\{R \longrightarrow R_i \mid i \in J\}$ defines a *Zariski cover* if it consists of finitely presentable localizations and the family of functors $\{R_i \otimes_R \mid i \in J\}$ is conservative. Zariski covers form a pretopology which we denote by $\tau_{\mathfrak{Z}}$ and call it the *Zariski pretopology.*

7.5.3. Noncommutative schemes as presheaves of sets. Let $\mathfrak{B}$ be the category $\mathcal{NE}sp_k$ of sheaves of sets on $\mathbf{Aff}_k$ for **fpqc** quasi-pretopology. In other words, objects of $\mathfrak{B}$ are functors $Alg_k \longrightarrow Sets$ which preserve finite products, and for any faithfully flat k-algebra morphism $R \longrightarrow T$, the diagram

$$X(R) \longrightarrow X(T) \rightrightarrows X(T \star_R T) \qquad (1)$$

is exact. The functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the Yoneda embedding, $R \longmapsto Alg_k(R, -)$.

Let $\tau_3^{\mathbf{o}}$ denote the pretopology on $\mathfrak{B}$ coinduced by the Zariski pretopology τ_3 via the functor $\mathfrak{F}$. We define *schemes* as locally $(\mathfrak{A}, \tau_3^{\mathbf{o}})$-affine objects of $\mathfrak{B}$.

7.5.4. A remark on fpqc-locally affine spaces. Let the category $\mathfrak{B}$ and the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ be the same as in 7.5.3. But, we take the **fpqc** quasi-pretopology on $\mathfrak{A} = \mathbf{Aff}_k$ instead of the Zariski pretopology. Let $\tau_{\mathbf{fpqc}}^{\mathbf{o}}$ denote the pretopology coinduced on $\mathfrak{B}$ by the **fpqc** quasi-pretopology on $\mathfrak{A} = A^{op}$. Applying the formalism of 7.1, we obtain *locally* $(\mathfrak{A}, \tau_{\mathbf{fpqc}}^{\mathbf{o}})$*-affine spaces.* This approach, however, is less satisfactory in the case of general **fpqc** covers, than in the case of Zariski covers. The reason is that **fpqc** covers do not form a pretopology; hence the operation of coinduction decimates the original quasi-pretopology on $\mathbf{Aff}_k$. Fortunately, this inconvenience is easily avoided by defining **fpqc** quasi-pretopology directly on the category $\mathfrak{B}$.

8. Locally affine 'spaces' and schemes.

8.1. Flat quasi-pretopologies in $|Cat|^o$. Let $\mathfrak{B} = |Cat|^o$. We call a set of morphisms $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ in $|Cat|^o$ a *weakly flat cover* if all u_i are weakly flat and the set of their inverse image functors, $\{u_i^* | \ i \in J\}$, is conservative. This defines a *weakly flat* quasi-pretopology, τ^w, on the category $|Cat|^o$.

8.1.1. Naive finiteness conditions. We call a weakly flat cover an **fpqc** cover, if it contains a finite subcover. We denote the corresponding quasi-pretopology by $\tau_{\mathbf{fpqc}}$.

Let $\mathfrak{E}$ be a set of types of diagrams. We denote by $\tau_{\mathbf{fpqc}}^{\mathfrak{E}}$ the quasi-pretopology defined as follows: $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ belongs to $\tau_{\mathbf{fpqc}}^{\mathfrak{E}}$ iff it is a weakly flat **fpqc** cover such that all direct image functors, u_{i*} preserve colimits from $\mathfrak{E}$.

We denote by $\tau_{\mathbf{fpqc}}^{af}$ the quasi-pretopology formed by weakly flat **fpqc** covers which consist of affine morphisms. We denote by τ_1^w the quasi-pretopology generated by weakly flat covers which consist of one morphism.

Finally, we denote by τ_1^{af} the quasi-pretopology generated by weakly flat covers which consist of one *affine* morphism.

8.1.2. Semi-separated covers. A weakly flat cover $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ is called *semi-separated* if all morphisms $U_i \xrightarrow{u_i} X$ are affine. We denote the corresponding quasi-pretopology by τ^{af}.

8.1.3. Proposition. *Let $\mathfrak{U} = \{U_i \xrightarrow{u_i} X \mid i \in J\}$ be a weakly flat cover and $\mathcal{U} = \coprod_{i \in J} U_i \xrightarrow{u} X$ the canonical morphism corresponding to the cover $\mathfrak{U}$.*

(a) If the category C_X has products of J objects, then the morphism $\mathcal{U} \xrightarrow{u} X$ is weakly flat and conservative.

(b) If the category C_X is additive and the cover $\mathfrak{U}$ is finite and semi-separated (i.e. every morphism u_i is affine), then the morphism $\mathcal{U} \xrightarrow{u} X$ is affine.

Proof. The family of inverse image functors $\mathfrak{U} = \{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is conservative iff the corresponding functor

$$C_X \xrightarrow{u^*} \prod_{i \in J} C_{U_i} = C_{\mathcal{U}}, \quad x \longmapsto (u_i^*(x) | i \in J),$$

is conservative. Similarly, the functors u_i^* preserve kernels of coreflexive pairs of arrows for all $i \in J$ iff the functor u^* has the same property.

(a) Suppose the category C_X has products of J objects. Then the functor

$$\prod_{i \in J} C_{U_i} = C_{\mathcal{U}} \xrightarrow{u_*} C_X, \quad (a_i | i \in J) \longmapsto \prod_{i \in J} u_{i*}(a_i),$$

is a right adjoint to the functor u^*.

(b) If every direct image functor u_{i*} is conservative, then the functor u_* is conservative. If the category C_X is additive and the cover $\mathfrak{U} = \{C_X \xrightarrow{u_i^*} C_{U_i} \mid i \in J\}$ is finite, then $u_*(a_i | i \in J) = \coprod_{i \in J} u_{i*}(a_i)$ for any object $(a_i | i \in J)$ of the category $C_{\mathcal{U}}$, and for any object x of the category C_X, we have:

$$C_X\big(u_*(a_i | i \in J), x\big) = C_X\big(\coprod_{i \in J} u_{i*}(a_i), x\big) \simeq \prod_{i \in J} C_X\big(u_{i*}(a_i), x\big) \simeq$$

$$\prod_{i \in J} C_X(a_i, u_i^!(x)) = C_{\mathcal{U}}\big((a_i | i \in J), (u_i^!(x) | i \in J)\big).$$

Here $u_i^!$ is a right adjoint to the direct image functor u_{i*}. This shows that the functor

$$C_X \xrightarrow{u^!} C_{\mathcal{U}}, \quad x \longmapsto (u_i^!(x) | i \in J),$$

is a right adjoint to the functor u_*. ■

8.2. Locally affine morphisms. Relative schemes. Fix a 'space' S, and consider the category $\mathfrak{B} = |Cat|_S^o$. Recall that $|Cat|_S^o$ is a full subcategory of $|Cat|^o/S$ whose objects are pairs (X, f), where $X \xrightarrow{f} S$ is a continuous morphism.

8.2.0. The choices of 'local' objects. We assume that the category C_S has cokernels of reflexive pairs of arrows. There are two extreme choices of the category of 'local' (or 'affine') objects $\mathfrak{A}$. The largest (in a certain sense) choice is the category $\mathfrak{A} = \mathbf{Aff}_S^w$ of weakly affine morphisms to S. The other extremity is the category $\mathbf{Aff}_S$ of affine morphisms to S (as in 8.9.3).

8.2.0.1. Generic intermediate choices. These are categories $\mathbf{Aff}_S^{\mathfrak{S}}$, where $\mathfrak{S}$ is a set of types of diagrams: objects of $\mathbf{Aff}_S^{\mathfrak{S}}$ are pairs (X, f), where f is a weakly affine morphism $X \longrightarrow S$ such that f_* is preserves colimits of diagrams of $\mathfrak{S}$.

In each of these cases, the functor $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the inclusion functor.

8.2.0.2. Important special case. We assume that the category C_S has cokernels of reflexive pairs of arrows and countable coproducts and 'local' objects are pair (X, f), where f is a weakly affine morphism $X \longrightarrow S$ such that f_* is preserves countable coproducts.

8.2.1. Canonical flat quasi-pretopologies. Any quasi-pretopology on $|Cat|^o$ induces a quasi–pretopology on $|Cat|^o/S$. In particular, each of the canonical quasi-pretopologies on $|Cat|^o$ induces a canonical quasi-pretopology on the category $\mathfrak{B} = |Cat|^o_S$. Thus, we have the quasi-pretopology $\tau_{\mathbf{fpqc}}$ given by finite weakly flat covers and its versions, $\tau^{\mathfrak{E}}_{\mathbf{fpqc}}$ and $\tau^{af}_{\mathbf{fpqc}}$ (cf. 8.1.1). Taking covers formed by $\mathfrak{F}$-finitely presentable morphism, we obtain the quasi-pretopologies respectively $\tau_{\mathbf{fppf}}$, $\tau^{\mathfrak{E}}_{\mathbf{fppf}}$ and $\tau^{af}_{\mathbf{fppf}}$.

8.2.2. Flat Zariski covers. We call a conservative set $\{U_i \xrightarrow{u_i} X \mid i \in J\}$ of morphisms of $|Cat|^o_S$ a *flat Zariski cover*, if all morphisms u_i are locally finitely presentable exact localizations. Zariski covers form a quasi-pretopology, τ_3.

8.2.3. Locally affine S-'spaces'. In this subsection, $\mathfrak{A} = \mathbf{Aff}_S$.

We call an object $(X, X \xrightarrow{f} S)$ of the category $\mathfrak{B} = |Cat|^o_S$ a *locally affine S-'space'*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|^o_S$ for the quasi-pretopology $\tau_{\mathbf{fppf}}$.

An object $(X, X \xrightarrow{f} S)$ of $|Cat|^o_S$ is called a *semi-separated locally affine S-'space'*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|^o_S$ with respect to the quasi-pretopology $\tau^{af}_{\mathbf{fppf}}$.

In other words, the S-'space' $(X, X \xrightarrow{f} S)$ has a finite weakly flat $\mathbf{Aff}_S$-affine **fppf** cover, which consists of affine morphisms.

8.2.4. Relative schemes. We call an object $(X, X \xrightarrow{f} S)$ of the category $|Cat|^o_S$ an *S-scheme*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|^o_S$ with respect to the quasi-pretopology τ_3 and $\mathfrak{A} = \mathbf{Aff}_S$.

8.2.5. Special cases. Taking Zariski covers which are covers in resp. $\tau^{\mathfrak{E}}_{\mathbf{fppf}}$ and $\tau^{af}_{\mathbf{fppf}}$, we obtain the corresponding versions of Zariski quasi-pretopology resp. $\tau^{\mathfrak{E}}_3$ and τ^{af}_3.

We call an object $(X, X \xrightarrow{f} S)$ of $|Cat|^o/S$ a *semi-separated S-scheme*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|^o/S$ with respect to the quasi-pretopology $\tau_{\mathfrak{Z}}^{af}$ and $\mathfrak{A} = \mathbf{Aff}_S$. In other words, $(X, X \xrightarrow{f} S)$ has a weakly flat $\mathbf{Aff}_S$-affine Zariski cover which consists of finitely presentably affine localizations.

8.3. Locally S-affine 'spaces' and schemes. Fix an object S of the category $\mathfrak{B} = |Cat|^o$. One can take as $\mathfrak{A}$ the category $\mathbf{Aff}_S$, $\mathbf{Aff}_S^w$, or $\mathbf{Aff}_S^{\mathfrak{S}}$ for some set $\mathfrak{S}$ of types of diagrams. This time, $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ is the natural functor which maps an object $(X, X \to S)$ of the category $\mathfrak{A}$ to X. We call an object X of $\mathfrak{B}$ *S-locally affine 'space'* if it is a locally $(\mathfrak{A}, \mathfrak{F})$-affine object with respect to the quasi-pretopology $\tau_{\mathbf{fppf}}$.

We call a conservative set $\{\mathfrak{F}(U_i) \xrightarrow{u_i} X \mid i \in J\}$ of morphisms of $|Cat|_S^o$ a *Zariski cover*, if all morphisms u_i are $(\mathfrak{A}, \mathfrak{F})$-finitely presentable localizations. Zariski covers form a quasi-pretopology which we denote by $\tau_{\mathfrak{Z}}$, as in the relative case sketched in 8.2.4.

We call an object $(X, X \xrightarrow{f} S)$ of the category $|Cat|_S^o$ an *S-scheme*, if it is a locally $\mathfrak{A}$-affine object of $|Cat|_S^o$ with respect to the quasi-pretopology $\tau_{\mathfrak{Z}}$ and $\mathfrak{A} = \mathbf{Aff}_S$.

8.3.1. Locally affine $\mathbb{Z}$-spaces and $\mathbb{Z}$-schemes. Let $S = \mathbf{Sp}\mathbb{Z}$ (i.e. C_S is the category of abelian groups). We call an object X of $\mathfrak{B}$ *locally affine $\mathbb{Z}$-space* (resp. a *$\mathbb{Z}$-scheme*) if it is a locally $\mathfrak{A}$-affine object with respect to the quasi-pretopology $\tau_{\mathbf{fpqc}}$ (resp. the Zariski quasi-pretopology $\tau_{\mathfrak{Z}}$).

8.3.2. Note. Let $\mathfrak{A}$ be the category $\mathbf{Aff}_k = Alg_k^{op}$, and let $\mathfrak{F}$ be the functor $\mathbf{Sp}$ which assigns to each associative unital k-algebra its categoric spectrum. Then the category of locally $(\mathfrak{A}, \mathfrak{F})$-affine 'spaces' is isomorphic to the category of locally affine $\mathbb{Z}$-'spaces' defined in 8.3.1. Similarly, the category of $(\mathfrak{A}, \mathfrak{F})$-schemes (defined in an obvious way) is isomorphic to the category of $\mathbb{Z}$-schemes.

8.4. The structure of locally affine 'spaces'. Fix a 'space' S. Let $\mathfrak{A}$ be the category $\mathbf{Aff}_S$ of affine morphisms to S, or the category $\mathbf{Aff}_S^{\mathfrak{S}}$ for some set of diagram types $\mathfrak{S}$, and let $\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B} = |Cat|^o$ be the forgetful functor $(X, X \to S) \longmapsto X$.

Suppose that C_S has finite products and cokernels of reflexive pairs of arrows. Let $\{\mathfrak{F}(U_i, \bar{u}_i) = U_i \xrightarrow{u_i} X \mid i \in J\}$ be a finite $\mathfrak{A}$-cover of X, and let $U = \coprod_{i \in J} U_i \xrightarrow{u} X$ be the corresponding morphism. If C_X has finite products, then u is a continuous morphism (see 8.1.3), hence u is weakly flat and conservative. By Beck's theorem, X is isomorphic to $\mathbf{Sp^o}(U\backslash\mathcal{G}_u)$; i.e. the category C_X is equivalent to the category of $(U\backslash\mathcal{G}_u)$-comodules.

8.5. Examples: NC schemes associated with a semi-simple Lie algebra. Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field k of zero characteristic and $U_q(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$. Let $\mathcal{G}$ denote the lattice of integral weights of $\mathfrak{g}$ and $\mathcal{R}$ the $\mathcal{G}$-graded algebra of "functions" on quantum base affine 'space': $R = \bigoplus_{\lambda \in \mathcal{G}_+} R_\lambda$ (see I.1.7).

8.5.1. The quantum base affine 'space'. The quantum base affine 'space', $\mathbf{Cone}(R_+)$, is a noncommutative scheme over k. Its canonical affine cover

$$\mathbf{Sp}(S_w^{-1}R) \longrightarrow \mathbf{Cone}(R_+), \quad w \in W, \tag{1}$$

is described in I.1.7.1.

8.5.2. The natural D-scheme over the base affine 'space'. Let $\widetilde{U}_q(\mathfrak{g})$ denote the *extended* quantized enveloping algebra of the Lie algebra $\mathfrak{g}$ defined by

$$\widetilde{U}_q(\mathfrak{g}) \stackrel{\text{def}}{=} U_q(\mathfrak{g}) \otimes_k U_q(\mathfrak{h}),$$

where $U_q(\mathfrak{h})$ is the quantized enveloping algebra of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. There is a natural Hopf action of $\widetilde{U}_q(\mathfrak{g})$ on each simple $U_q(\mathfrak{g})$-module R_λ, which defines a Hopf action of the algebra R_+. So that we have the crossed product $R_+\#\widetilde{U}_q(\mathfrak{g})$ and the associated quasi-affine 'space' $\mathbf{Cone}(R_+\#\widetilde{U}_q(\mathfrak{g}))$. The natural embedding $R_+ \longrightarrow R_+\#\widetilde{U}_q(\mathfrak{g})$ induces an affine morphism of 'spaces'

$$\mathbf{Cone}(R_+\#\widetilde{U}_q(\mathfrak{g})) \longrightarrow \mathbf{Cone}(R_+). \tag{2}$$

The Hopf action of $\widetilde{U}_q(\mathfrak{g})$ on the algebra R_+ is compatible with localizations at the Ore sets S_w, $w \in W$, which means that $S_w^{-1}(R_+\#\widetilde{U}_q(\mathfrak{g})) \simeq S_w^{-1}R_+\#\widetilde{U}_q(\mathfrak{g})$ for all $w \in W$. Therefore, the cover 8.5.1(1) induces an affine cover

$$\mathbf{Sp}(S_w^{-1}R\#\widetilde{U}_q(\mathfrak{g})) \longrightarrow \mathbf{Cone}(R_+\#\widetilde{U}_q(\mathfrak{g})), \quad w \in W, \tag{3}$$

such that the diagrams

$$\begin{array}{ccc} \mathbf{Sp}(S_w^{-1}R\#\widetilde{U}_q(\mathfrak{g})) & \longrightarrow & \mathbf{Cone}(R_+\#\widetilde{U}_q(\mathfrak{g})) \\ \big\downarrow & & \big\downarrow \\ \mathbf{Sp}(S_w^{-1}R) & \longrightarrow & \mathbf{Cone}(R_+) \end{array} \qquad w \in W, \tag{4}$$

commute for all $w \in W$.

8.5.3. The quantum flag variety and the associated D-scheme. The cover 8.5.1(1) induces a canonical affine cover of the *quantum flag variety* $\mathbf{Proj}_{\mathcal{G}}(R)$ of $\mathfrak{g}$ turning it into a noncommutative scheme. The same cover induces also an affine cover of the associated quantum D-scheme $\mathbf{Proj}_{\mathcal{G}}(R\#U_{\mathfrak{q}}(\mathfrak{g}))$, where $R\#U_{\mathfrak{q}}(\mathfrak{g})$ is the crossed product corresponding to the natural Hopf action of the quantized enveloping algebra $U_{\mathfrak{q}}(\mathfrak{g})$ on the $\mathcal{G}$-graded algebra R. All together is expressed by the diagram (4) and the commutative diagram

$$\begin{array}{ccccc}
\mathbf{Sp}(S_w^{-1}R\#\widetilde{U}_q(\mathfrak{g})) & \longrightarrow & \mathbf{Sp}((S_w^{-1}R)_0\#\widetilde{U}_q(\mathfrak{g})) & & \\
\downarrow & & \downarrow & & \\
\mathbf{Cone}(R_+\#\widetilde{U}_q(\mathfrak{g})) & \longrightarrow & \mathbf{Proj}_{\mathcal{G}}(R\#U_{\mathfrak{q}}(\mathfrak{g})) & \longrightarrow & \mathbf{Sp}((S_w^{-1}R)_0\#\widetilde{U}_q(\mathfrak{g})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Cone}(R_+) & \longrightarrow & \mathbf{Proj}_{\mathcal{G}}(R) & \longrightarrow & \mathbf{Sp}((S_w^{-1}R)_0) \\
\uparrow & & \uparrow & & \\
\mathbf{Sp}(S_w^{-1}R) & \longrightarrow & \mathbf{Sp}((S_w^{-1}R)_0) & & w \in W,
\end{array} \tag{5}$$

whose all vertical arrows are affine morphisms.

Chapter III
Geometry of Presheaves of Sets.

The first three sections outline the generalities, which start with specialization of the notions and facts of Chapter II to the case of the categories of (pre)sheaves of sets on an arbitrary category followed by a more detailed study of their properties. In Section 1, we discuss representable morphisms of (pre)sheaves of sets (interpreted as *affine* morphisms). In Section 2, we specialize to presheaves of sets the notions of *closed immersions* and *semi-separated* and *separated* morphisms introduced in Chapter II and study the invariance properties of these classes under sheafification functors. Section 3 is dedicated to (formally) smooth and étale morphisms and *open immersions* of presheaves and sheaves of sets.

In Section 4, we look at locally affine presheaves of sets on the category of noncommutative affine schemes. Namely, we apply the formalism of Section 3 to three natural pretopologies, whose covers are strictly epimorphic families of respectively smooth morphisms, étale morphisms, and open immersions. As a result, we obtain the notions of noncommutative schemes and algebraic spaces in this context.

The remaining sections are devoted to several important examples which illustrate and motivate general notions. In Section 5, we introduce several important affine (that is representable) constructions – vector fibers, inner homs, isomorphisms, which serve as building blocks for the locally affine 'spaces' considered here. In Section 6, we introduce and study the *noncommutative Grassmannian*, which is one of important examples of a noncommutative locally affine space. In Section 7, we introduce noncommutative flag varieties. In Section 8, we follow with the *generic* Grassmannians associated to modules. The generic Grassmannians are noncommutative versions of the Grothendieck's Quot schemes. Section 9 gives an introduction to *generic* flags. In Section 10, we apply some results on generic flags to the 'usual' noncommutative flag varieties – those introduced in Section 7.

In connection with generic flags, we discuss, in Section 11, Stiefel schemes and observe that they coincide with the canonical affine cover of generic flags of certain type.

In Section 12, "Remarks and observations", we start with a short summary of common properties of all our examples – generic and non-generic flag varieties and (as their special cases) Grassmannians and then continue with functorial properties of the canonical covers of these varieties and the action of GL. We conclude the Chapter with passing from the "toy" noncommutative Grassmannian and flag varieties to the "real" ones; that is we pass from the presheaves described in the previous sections to their associated sheaves for a relevant topology. Apparently, the most relevant topology for the examples of this chapter is the smooth topology.

1. Representable and semi-separated morphisms of (pre)sheaves.

Here we discuss in greater detail the notions introduced at the end of Section II.2 in the case when the "local data" is the Yoneda embedding of a category $\mathcal{A}$ into the category $\mathcal{A}^\wedge$ of presheaves of sets on $\mathcal{A}$. For readers' convenience, some notions are reformulated directly in this setting.

1.1. Representable morphisms. Let $\mathcal{P}$ be a class of morphisms of the category $\mathcal{A}$ closed under arbitrary pull-backs. A morphism $F \longrightarrow G$ of presheaves of sets on $\mathcal{A}$ is called *representable by morphisms of* $\mathcal{P}$, if for any $\widehat{X} \longrightarrow G$, the projection $F \times_G \widehat{X} \longrightarrow \widehat{X}$ is of the form $\widehat{u}$ for a morphism $u \in \mathcal{P}$. In particular, the presheaf $F \times_G \widehat{X}$ is representable.

We denote by $\mathcal{P}^\wedge$ the class of all morphisms of $\mathcal{A}^\wedge$ representable by morphisms of $\mathcal{P}$. Clearly, a morphism $\widehat{X} \longrightarrow \widehat{Y}$ belongs to $\mathcal{P}^\wedge$ iff it is of the form $\widehat{w}$ for $w \in \mathcal{P}$.

1.1.1. Proposition. *(i) The class $\mathcal{P}^\wedge$ is invariant under the base change: if a morphism $F \longrightarrow G$ belongs to $\mathcal{P}^\wedge$ and $H \longrightarrow G$ is an arbitrary morphism, then the projection $H \times_G F \longrightarrow H$ belongs to $\mathcal{P}^\wedge$.*

(ii) If $\mathcal{P}$ is closed under composition, then $\mathcal{P}^\wedge$ has the same property.

Proof. The assertion is a special case of II.2.1. ■

1.1.2. Standard examples. 1) The class $\mathfrak{M} = \mathfrak{M}(\mathcal{A})$ of all monomorphisms of the category $\mathcal{A}$ is closed under pull-backs and composition.

2) Same holds for the class $\mathcal{E}^u = \mathcal{E}^u(\mathcal{A})$ of universal epimorphisms. Recall that a morphism $X \xrightarrow{f} Y$ is called a *universal epimorphism* if for any morphism $V \longrightarrow Y$, there exists a fiber product $X \times_Y V$ and the canonical projection $X \times_Y V \longrightarrow V$ is an epimorphism.

1.2. Representable morphisms of presheaves of sets. Let $\mathcal{A}$ be a category with pull-backs, and let $\mathcal{P}$ be the class of all morphisms of $\mathcal{A}$. In this case, we call $\mathcal{P}$-representable morphisms of presheaves simply *representable*, or, sometimes, *affine*. It follows that *a presheaf morphism $F \longrightarrow G$ is representable iff for any object X of $\mathcal{A}$ and for any morphism $\widehat{X} \longrightarrow G$, the presheaf $F \times_G \widehat{X}$ is representable.*

1.2.1. Lemma. *Let $\mathcal{A}$ be a category with finite limits, and let G be a presheaf of sets on a category $\mathcal{A}$. The following conditions are equivalent:*

(a) For every object X of $\mathcal{A}$, any morphism $\widehat{X} \longrightarrow G$ is representable.

(b) The diagonal morphism

$$G \xrightarrow{\Delta_G} G \times G$$

is representable.

Proof. The assertion is a special case of II.2.6.2. ∎

1.2.2. Semiseparated presheaves and morphisms of presheaves. Specializing the notions of II.2.6.1 and II.2.6.3, we obtain the following:

(a) A presheaf of sets G on a category $\mathcal{A}$ is *semi-separated*, if the diagonal morphism

$$G \xrightarrow{\Delta_G} G \times G$$

is representable.

(b) A presheaf morphism $\mathcal{G} \xrightarrow{\mathfrak{f}} \mathcal{F}$ is *semi-separated*, if the diagonal morphism

$$\mathcal{G} \xrightarrow{\Delta_{\mathfrak{f}}} K_2(\mathfrak{f}) = \mathcal{G} \times_{\mathcal{F}} \mathcal{G}$$

is representable.

Let $\bullet$ denote the constant presheaf $\mathcal{A}^{op} \longrightarrow \mathbf{Sets}$ with values in a one point set – the final object of the category $\mathcal{A}^\wedge$ of presheaves of sets on $\mathcal{A}$. It follows that a presheaf of sets G is semi-separated iff the (unique) morphism $G \longrightarrow \bullet$ is semi-separated.

1.3. Proposition. *Let $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ be a presheaf morphism and*

$$\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{U}$$

its kernel pair. Consider the following conditions:

(a) the morphism $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ is representable;
(b) the morphisms p_1, p_2 are representable;
(c) one of the morphisms p_1, p_2 is representable.
(d) the induced epimorphism $\mathcal{U} \xrightarrow{\pi_1} \mathcal{X}_1$ onto the image of π is representable.

There are implications (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (b).

Proof. The implication (b)⇒(c) is true by a trivial reason. The implication (a)⇒(b) holds because representable morphisms are stable under pull-backs.

(d)⇒(b). The morphism $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ is the composition of $\mathcal{U} \xrightarrow{\pi_1} \mathcal{X}_1$ and a monomorphism $\mathcal{X}_1 \xrightarrow{\mathfrak{j}} \mathcal{X}$. Therefore, the kernel pair of π_1 is the same as the kernel pair of π. So that the implication (d)⇒(b) follows from the implication (a)⇒(b).

(c)⇒(d). Let

$$\begin{array}{ccc} \widetilde{\mathcal{V}} & \xrightarrow{\xi'} & \mathcal{U} \\ {\scriptstyle \widetilde{\pi}}\downarrow & cart & \downarrow{\scriptstyle \pi_1} \\ \mathcal{V} & \xrightarrow{\xi} & \mathcal{X}_1 \end{array} \qquad (1)$$

be a cartesian square with $\mathcal{V}$ representable. Suppose that $\mathcal{R} \xrightarrow{p_1} \mathcal{U}$ is representable. Since $\mathcal{U} \xrightarrow{\pi_1} \mathcal{X}_1$ is an epimorphism and $\mathcal{V}$, being a representable, is a projective object of the category of presheaves of sets, there exists a morphism $\mathcal{V} \xrightarrow{\gamma} \mathcal{U}$ such that $\pi_1 \circ \gamma = \xi$. Therefore, the cartesian square (1) can be decomposed into

$$\begin{array}{ccccc} \widetilde{\mathcal{V}} & \xrightarrow{\xi''} & \mathcal{R} & \xrightarrow{p_2} & \mathcal{U} \\ {\scriptstyle \widetilde{\pi}}\downarrow & cart & {\scriptstyle p_1}\downarrow & cart & \downarrow{\scriptstyle \pi_1} \\ \mathcal{V} & \xrightarrow{\gamma} & \mathcal{U} & \xrightarrow{\pi_1} & \mathcal{X}_1 \end{array}$$

with both squares cartesian (the right square is cartesian by the observation in the argument of $(d) \Rightarrow (b)$ above: the kernel pair of π_1 is isomorphic to the kernel pair of π).

By hypothesis, the projection $\mathcal{R} \xrightarrow{p_1} \mathcal{U}$ is a representable morphism. Therefore, the morphism $\widetilde{\mathcal{V}} \xrightarrow{\widetilde{\pi}} \mathcal{V}$, being a pull-back of p_1, is representable. ∎

1.3.1. Corollary. *Let $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ be a presheaf epimorphism and*

$$\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{U}$$

its kernel pair. The following conditions are equivalent:

(a) the morphism $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ is representable;

(b) the morphisms p_1, p_2 are representable;

(c) one of the morphisms p_1, p_2 is representable.

Proof. The assertion follows from 1.3. ∎

1.3.2. Corollary. *Let $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ be a presheaf epimorphism with representable $\mathcal{U}$. Then π is representable iff the object of relations $\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ is representable.*

Proof. By 1.3.1, a presheaf epimorphism $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ is representable iff the projections

$$\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathcal{U}$$

are representable. If $\mathcal{U}$ is representable, then the projections p_1, p_2 are representable iff $\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U}$ is representable. ∎

1.3.3. Note. In 1.3, 'representable' can be replaced by '$\mathfrak{P}$-representable', where $\mathfrak{P}$ is any class of morphisms of the category C_X stable under pull-backs along arbitrary morphisms. For instance, one can take as $\mathfrak{P}$ the class of formally $\mathfrak{M}$-smooth, or the class of formally $\mathfrak{M}$-étale morphisms for some class of morphisms $\mathfrak{M}$.

1.4. Representable morphisms of presheaves and their associated sheaves.

1.4.0. Right exact structures. Recall that a *right exact* structure on a category $\mathcal{A}$ is a class $\mathfrak{E}_{\mathcal{A}}$ of strict epimorphisms which contains all isomorphisms and is stable under compositions and arbitrary pull-backs. In other words, morphisms of $\mathfrak{E}_{\mathcal{A}}$ are covers of a subcanonical pretopology on $\mathcal{A}$. The elements of $\mathfrak{E}_{\mathcal{A}}$ are called *deflations*.

1.4.1. Proposition. *Let* $(\mathcal{A}, \mathfrak{E}_{\mathcal{A}})$ *be a right exact category.*

1) Suppose that every pair $M \overset{\mathfrak{t}_1}{\underset{\mathfrak{t}_2}{\rightrightarrows}} N$ *of deflations has a cokernel. Let* $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ *be an epimorphism of sheaves of sets on* $(\mathcal{A}, \mathfrak{E}_{\mathcal{A}})$ *and*

$$\mathcal{U} \xrightarrow{\pi_1} \mathcal{X}_1 \xrightarrow{\mathfrak{j}} \mathcal{X}$$

its natural decomposition into an epimorphism and a monomorphism of presheaves. The following conditions are equivalent:

(a) the morphism $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ *is representable;*

(b) the morphisms $\mathcal{R} = \mathcal{U} \times_{\mathcal{X}} \mathcal{U} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathcal{U}$ *are representable;*

(c) one of the morphisms p_1, p_2 *is representable.*

(d) the presheaf epimorphism $\mathcal{U} \xrightarrow{\pi_1} \mathcal{X}_1$ *onto the image of* π *is representable.*

2) The sheafification functor maps representable morphisms of presheaves of sets to representable morphisms of sheaves.

Proof. 1) The equivalence of the conditions (b), (c) and (d) and the fact that (a) implies them are established in 1.3. It remains to show that

(d)⇒(a). Let $\widehat{\mathcal{N}} = \mathcal{A}(-, \mathcal{N})$ for an object $\mathcal{N}$ of $\mathcal{A}$ and $\widehat{\mathcal{N}} \xrightarrow{\xi} \mathcal{X}$ an arbitrary sheaf morphism. The claim is that, if the presheaf epimorphism $\mathcal{U} \xrightarrow{\pi_1} \mathcal{X}_1$ is representable, then the pull-back of the sheaf epimorphism $\mathcal{U} \xrightarrow{\pi} \mathcal{X}$ along ξ is representable.

Consider the decomposition of this pull-back into two cartesian squares:

$$\begin{array}{ccccc}
\mathfrak{N} & \xrightarrow{\widetilde{\pi}_1} & \mathfrak{N}_1 & \xrightarrow{\widetilde{\mathfrak{j}}} & \widehat{\mathcal{N}} \\
{\scriptstyle \xi''}\downarrow & cart & \downarrow{\scriptstyle \xi'} & cart & \downarrow{\scriptstyle \xi} \\
\mathcal{U} & \xrightarrow{\pi_1} & \mathcal{X}_1 & \xrightarrow{\mathfrak{j}} & \mathcal{X}
\end{array}$$

The sheafification functor, $\mathfrak{q}^*$, maps $\mathfrak{j}$ to isomorphism, which implies, thanks to the exactness of $\mathfrak{q}^*$, that it maps $\widetilde{\mathfrak{j}}$ to an isomorphism. The latter means that $\mathfrak{N}_1 \xrightarrow{\widetilde{\mathfrak{j}}} \widehat{\mathcal{N}}$ is a *refinement* of $\widehat{\mathcal{N}}$. Therefore, there exists a deflation $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$ such that $\widehat{\mathcal{M}} \xrightarrow{\widehat{\mathfrak{t}}} \widehat{\mathcal{N}}$

factors through $\mathfrak{N}_1 \xrightarrow{\widetilde{\mathfrak{j}}} \widehat{\mathcal{N}}$. In other words, there is a morphism $\widehat{\mathcal{M}} \xrightarrow{\mathfrak{u}} \mathfrak{N}_1$ whose composition with $\mathfrak{N}_1 \xrightarrow{\widetilde{\mathfrak{j}}} \widehat{\mathcal{N}}$ coincides with $\widehat{\mathcal{M}} \xrightarrow{\widehat{\mathfrak{t}}} \widehat{\mathcal{N}}$. We have a diagram

$$\begin{array}{ccccc}
\mathfrak{M} & \xrightarrow{\pi_2} & \widehat{\mathcal{M}} & \xrightarrow{id} & \widehat{\mathcal{M}} \\
{\scriptstyle \mathfrak{u}'}\big\downarrow & cart & \big\downarrow{\scriptstyle \mathfrak{u}} & & \big\downarrow{\scriptstyle \widehat{\mathfrak{t}}} \\
\mathfrak{N} & \xrightarrow{\widetilde{\pi}_1} & \mathfrak{N}_1 & \xrightarrow{\widetilde{\mathfrak{j}}} & \widehat{\mathcal{N}} \\
{\scriptstyle \xi''}\big\downarrow & cart & \big\downarrow{\scriptstyle \xi'} & cart & \big\downarrow{\scriptstyle \xi} \\
\mathcal{U} & \xrightarrow{\pi_1} & \mathcal{X}_1 & \xrightarrow{\mathfrak{j}} & \mathcal{X}
\end{array} \tag{1}$$

whose lower squares and the left upper square are cartesian.

Notice that the kernel pair $K_2(\mathfrak{u}) = \widehat{\mathcal{M}} \times_{\mathfrak{N}_1} \widehat{\mathcal{M}} \rightrightarrows \widehat{\mathcal{M}}$ of the morphism $\widehat{\mathcal{M}} \xrightarrow{\mathfrak{u}} \mathfrak{N}_1$ is isomorphic to the kernel pair $K_2(\widehat{\mathfrak{t}}) = \widehat{\mathcal{M}} \times_{\widehat{\mathcal{N}}} \widehat{\mathcal{M}} \rightrightarrows \widehat{\mathcal{M}}$ of the morphism $\widehat{\mathcal{M}} \xrightarrow{\widehat{\mathfrak{t}}} \widehat{\mathcal{N}}$. The latter can be identified with the image of the kernel pair $K_2(\mathfrak{t}) \underset{\mathfrak{t}_2}{\overset{\mathfrak{t}_1}{\rightrightarrows}} \mathcal{M}$ of the morphism $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$. Thus, we have a commutative diagram

$$\begin{array}{ccccccc}
K_2(\mathfrak{u}') & \underset{\widehat{\mathfrak{u}}_2}{\overset{\widehat{\mathfrak{u}}_1}{\rightrightarrows}} & \mathfrak{M} & \xrightarrow{\mathfrak{u}'} & \mathfrak{N} & \xrightarrow{\xi''} & \mathcal{U} \\
{\scriptstyle K_2(\mathfrak{p}'_1)}\big\downarrow & cart & {\scriptstyle \pi_2}\big\downarrow & cart & {\scriptstyle \widetilde{\pi}_1}\big\downarrow & cart & \big\downarrow{\scriptstyle \pi_1} \\
K_2(\widehat{\mathfrak{u}}) & \underset{\widehat{\mathfrak{t}}_2}{\overset{\widehat{\mathfrak{t}}_1}{\rightrightarrows}} & \widehat{\mathcal{M}} & \xrightarrow{\mathfrak{u}} & \mathfrak{N}_1 & \xrightarrow{\xi'} & \mathcal{X}_1 \\
{\scriptstyle id}\big\downarrow & & {\scriptstyle id}\big\downarrow & & {\scriptstyle \widetilde{\mathfrak{j}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{j}} \\
K_2(\widehat{\mathfrak{t}}) & \underset{\widehat{\mathfrak{t}}_2}{\overset{\widehat{\mathfrak{t}}_1}{\rightrightarrows}} & \widehat{\mathcal{M}} & \xrightarrow{\widehat{\mathfrak{t}}} & \widehat{\mathcal{N}} & \xrightarrow{\xi} & \mathcal{X}
\end{array} \tag{2}$$

with four cartesian squares. The left upper cartesian square means that the squares

$$\begin{array}{ccc}
K_2(\mathfrak{u}') & \xrightarrow{K_2(\pi'_1)} & K_2(\widehat{\mathfrak{t}}) \\
{\scriptstyle \widehat{\mathfrak{u}}_i}\big\downarrow & cart & \big\downarrow{\scriptstyle \widehat{\mathfrak{t}}_i} \\
\mathfrak{M} & \xrightarrow{\pi_2} & \widehat{\mathcal{M}}
\end{array}$$

are cartesian, $i = 1, 2$.

Suppose now that the condition (d) holds; that is the morphism $\mathcal{U} \xrightarrow{\pi_1} \mathcal{X}_1$ is representable. Then all horizontal arrows of the diagram (2) above $\mathcal{U} \xrightarrow{\pi_1} \mathcal{X}_1$ are representable. Since $\widehat{\mathcal{M}}$ and $K_2(\widehat{\mathfrak{t}}) = \widehat{K_2(\mathfrak{t})}$ are representable presheaves, this implies that the presheaves $\mathfrak{M}$ and $K_2(\mathfrak{u}')$ are representable. That is we can set $\mathfrak{M} = \widehat{\mathcal{M}}_1$ and $K_2(\mathfrak{u}') = \widehat{\mathcal{K}}$ for some objects $\mathcal{M}_1$, $\mathcal{K}$ of the category $\mathcal{A}$; and the upper left cartesian double square of the diagram (2) is the image of the double cartesian square

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & K_2(\mathfrak{t}) \\ {\scriptstyle \mathfrak{u}_2}\downarrow\downarrow{\scriptstyle \mathfrak{u}_1} & cart & {\scriptstyle \mathfrak{t}_2}\downarrow\downarrow{\scriptstyle \mathfrak{t}_1} \\ \mathcal{M}_1 & \longrightarrow & \mathcal{M} \end{array} \qquad (3)$$

of the category $\mathcal{A}$. Notice that the vertical arrows $\mathfrak{u}_1$, $\mathfrak{u}_2$ are deflations, because they are pull-backs of deflations respectively $\mathfrak{t}_1$ and $\mathfrak{t}_2$. Therefore, by hypothesis, the pair of arrows $\mathcal{K} \underset{\mathfrak{u}_2}{\overset{\mathfrak{u}_1}{\rightrightarrows}} \mathcal{M}_1$ in the diagram (3) has a cokernel, $\mathcal{M}_1 \xrightarrow{\mathfrak{v}} \mathcal{N}_1$. So, we have a commutative diagram

$$\begin{array}{ccccc} \widehat{\mathcal{K}} & \underset{\widehat{\mathfrak{u}_2}}{\overset{\widehat{\mathfrak{u}_1}}{\rightrightarrows}} & \widehat{\mathcal{M}}_1 & \xrightarrow{\widehat{\mathfrak{v}}} & \widehat{\mathcal{N}}_1 \\ \downarrow & cart & \downarrow & & \downarrow \\ \widehat{K_2(\mathfrak{t})} & \underset{\widehat{\mathfrak{t}_2}}{\overset{\widehat{\mathfrak{t}_1}}{\rightrightarrows}} & \widehat{\mathcal{M}} & \xrightarrow{\widehat{\mathfrak{t}}} & \widehat{\mathcal{N}} \end{array}$$

of representable sheaves whose rows are exact diagrams. In particular, there exists a unique morphism $\widehat{\mathcal{N}}_1 \xrightarrow{\mathfrak{j}_1} \mathfrak{N}$ such that its composition with $\mathfrak{M} = \widehat{\mathcal{M}}_1 \xrightarrow{\widehat{\mathfrak{v}}} \widehat{\mathcal{N}}_1$ coincides with the morphism $\mathfrak{M} \xrightarrow{\mathfrak{u}'} \mathfrak{N}$ in the upper row of the diagram (2). The sheafification functor is exact. In particular, it maps the canonical morphism $\widehat{\mathcal{N}}_1 \xrightarrow{\mathfrak{j}_1} \mathfrak{N}$ to an isomorphism. Since the pretopology $\mathfrak{E}_{\mathcal{A}}$ is subcanonical, every representable presheaf is a sheaf. The presheaf $\mathfrak{N}$ is a sheaf, because it is a pull-back of a sheaf morphism along a sheaf morphism. Therefore, $\mathfrak{j}_1$ is an isomorphism; i.e. $\mathfrak{N}$ is a representable sheaf.

2) Let $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ be a representable morphisms of presheaves of sets and $\mathfrak{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathfrak{Y}^{\mathfrak{a}}$ the corresponding morphism of associated sheaves. The claim is that the pull-back of $\mathfrak{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathfrak{Y}^{\mathfrak{a}}$ along an arbitrary morphism $\widehat{\mathcal{N}} = \mathcal{A}(-, \mathcal{N}) \xrightarrow{\xi} \mathfrak{Y}^{\mathfrak{a}}$ is representable.

Consider the commutative diagram

$$\begin{array}{ccccccc}
\widetilde{\mathfrak{M}} & \xrightarrow{K_2(\pi_1')} & Ker(\mathfrak{u}\widehat{\mathfrak{t}}_1, \mathfrak{u}\widehat{\mathfrak{t}}_2) & \xrightarrow{\mathfrak{j}} & K_2(\widehat{\mathfrak{t}}) & \xrightarrow{id} & K_2(\widehat{\mathfrak{t}}) \\
\widetilde{\mathfrak{v}}_2 \Big\downarrow\Big\downarrow \widetilde{\mathfrak{v}}_1 & cart & \widehat{\mathfrak{t}}_2 \circ \mathfrak{j} \Big\downarrow\Big\downarrow \widehat{\mathfrak{t}}_1 \circ \mathfrak{j} & & \widehat{\mathfrak{t}}_2 \Big\downarrow\Big\downarrow \widehat{\mathfrak{t}}_1 & & \widehat{\mathfrak{t}}_2 \Big\downarrow\Big\downarrow \widehat{\mathfrak{t}}_1 \\
\mathfrak{M} & \xrightarrow{\mathfrak{f}} & \widehat{\mathcal{M}} & \xrightarrow{id} & \widehat{\mathcal{M}} & \xrightarrow{id} & \widehat{\mathcal{M}} \\
\mathfrak{v} \Big\downarrow & cart & \Big\downarrow \mathfrak{u} & & \Big\downarrow \mathfrak{t}' & & \Big\downarrow \widehat{\mathfrak{t}} \\
\widetilde{\mathfrak{X}} & \xrightarrow{\widetilde{\mathfrak{f}}} & \widetilde{\mathfrak{Y}} & \xrightarrow{\widetilde{\eta}_{\mathfrak{e}}} & \mathfrak{Y}_1 & \xrightarrow{\widetilde{\eta}_{\mathfrak{m}}} & \widehat{\mathcal{N}} \\
\Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow \xi \\
\mathfrak{X} & \xrightarrow{\mathfrak{f}} & \mathfrak{Y} & \xrightarrow{\eta_{\mathfrak{e}}} & \mathfrak{Y}_1 & \xrightarrow{\eta_{\mathfrak{m}}} & \mathfrak{Y}^{\mathfrak{a}}
\end{array} \qquad (4)$$

Here $\mathfrak{Y} \xrightarrow{\eta_{\mathfrak{e}}} \mathfrak{Y}_1 \xrightarrow{\eta_{\mathfrak{m}}} \mathfrak{Y}^{\mathfrak{a}}$ is the presentation of the adjunction morphism $\mathfrak{Y} \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{Y}}} \mathfrak{Y}^{\mathfrak{a}}$ as the composition of a presheaf epimorphism, $\eta_{\mathfrak{e}}$, and a presheaf monomorphism, $\eta_{\mathfrak{m}}$. The arrow $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$ is a deflation which factors through the monomorphism $\mathfrak{Y}_1 \xrightarrow{\eta_{\mathfrak{m}}} \mathfrak{Y}^{\mathfrak{a}}$ (its existence argued in 1) above). The arrow $\widehat{\mathcal{M}} \xrightarrow{\mathfrak{u}} \widetilde{\mathfrak{Y}}$ is due to the fact that $\widehat{\mathcal{M}}$ is a projective object in the category of presheaves of sets and $\widetilde{\mathfrak{Y}} \xrightarrow{\widetilde{\eta}_{\mathfrak{e}}} \mathfrak{Y}_1$ is an isomorphism. Finally, $Ker(\mathfrak{u}\widehat{\mathfrak{t}}_1, \mathfrak{u}\widehat{\mathfrak{t}}_2) \xrightarrow{\mathfrak{j}} K_2(\widehat{\mathfrak{t}})$ is the canonical monomorphism.

By hypothesis, the morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ is representable; so that all horizontal arrows above $\mathfrak{f}$ in the diagram (4) are representable. In particular, the presheaf $\mathfrak{M}$ is representable.

The sheafification functor maps the horizontal arrows of the central and right parts of the diagram (4), in particular, $Ker(\mathfrak{u}\widehat{\mathfrak{t}}_1, \mathfrak{u}\widehat{\mathfrak{t}}_2) \xrightarrow{\mathfrak{j}} K_2(\widehat{\mathfrak{t}})$, to isomorphisms. So that the sheaf $Ker(\mathfrak{u}\widehat{\mathfrak{t}}_1, \mathfrak{u}\widehat{\mathfrak{t}}_2)^{\mathfrak{a}}$ is representable. Since the sheafification functor is exact, in particular it preserves cartesian squares, the sheaf $\widetilde{\mathfrak{M}}^{\mathfrak{a}}$ is representable too.

Thus, the image of the diagram (4) by the sheafification functor is equivalent to the diagram

$$\begin{array}{ccccccc}
\widetilde{\mathfrak{M}}^{\mathfrak{a}} & \overset{\widehat{\mathfrak{v}}_1}{\underset{\widehat{\mathfrak{v}}_2}{\rightrightarrows}} & \mathfrak{M} & \xrightarrow{\widehat{\mathfrak{v}}} & \widetilde{\mathfrak{X}}^{\mathfrak{a}} & \longrightarrow & \mathfrak{X}^{\mathfrak{a}} \\
\Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow \mathfrak{f}^{\mathfrak{a}} \\
\widehat{K_2(\mathfrak{t})} & \overset{\widehat{\mathfrak{t}}_1}{\underset{\widehat{\mathfrak{t}}_2}{\rightrightarrows}} & \widehat{\mathcal{M}} & \xrightarrow{\widehat{\mathfrak{t}}} & \widehat{\mathcal{N}} & \xrightarrow{\xi} & \mathfrak{Y}^{\mathfrak{a}}
\end{array}$$

with cartesian squares. Therefore, the sheaf $\widetilde{\mathfrak{Y}}^{\mathfrak{a}}$ is representable. ■

1.4.2. Remarks. (i) If $\mathfrak{E}_\mathcal{A}$ is a *trivial* right exact structure, that is all deflations are isomorphisms, then the category of sheaves of sets on $(\mathcal{A}, \mathfrak{E}_\mathcal{A})$ coincides with the category of presheaves of sets on $\mathcal{A}$. In this case, the first assertion of 1.4.1 coincides with 1.3.1.

(ii) In 1.4.1, a right exact structure can be replaced by a subcanonical pretopology τ on the category $\mathcal{A}$ with the condition (in the first assertion) that colimits of (certain) diagrams formed by arrows of covers exist in $\mathcal{A}$. The argument follows the same lines.

1.5. Proposition. *Let τ be a subcanonical (pre)topology on a category $\mathcal{A}$. If a morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ of presheaves of sets on $\mathcal{A}$ is representable and $\mathfrak{Y}$ is a sheaf on $(\mathcal{A}, \tau)$, then $\mathfrak{X}$ is a sheaf on $(\mathcal{A}, \tau)$ too.*

Proof. (a) Let $\mathfrak{X}^\mathfrak{a}$ denote the sheaf associated with the presheaf $\mathfrak{X}$ and $\mathfrak{X} \xrightarrow{\eta^\mathfrak{a}_\mathfrak{X}} \mathfrak{X}^\mathfrak{a}$ the adjunction morphism. Let

$$\begin{array}{ccc} \widetilde{\mathfrak{X}} & \xrightarrow{\widetilde{\mathfrak{f}}} & \widetilde{\mathfrak{Y}} \\ {\scriptstyle \xi'}\downarrow & cart & \downarrow{\scriptstyle \xi} \\ \mathfrak{X} & \xrightarrow{\mathfrak{f}} & \mathfrak{Y} \end{array} \qquad (1)$$

be a cartesian square. If the morphism $\mathfrak{f}$ and the presheaf $\widetilde{\mathfrak{Y}}$ are representable, then the presheaf $\widetilde{\mathfrak{X}}$ is representable. By hypothesis, the (pre)topology τ is subcanonical, that is representable presheaves are sheaves. Therefore, the sheafification functor maps (1) to a cartesian square, which is isomorphic to

$$\begin{array}{ccc} \widetilde{\mathfrak{X}} & \xrightarrow{\widetilde{\mathfrak{f}}} & \widehat{\mathfrak{Y}} \\ {\scriptstyle \eta^\mathfrak{a}_\mathfrak{X} \circ \xi'}\downarrow & cart & \downarrow{\scriptstyle \eta^\mathfrak{a}_\mathfrak{Y} \circ \xi} \\ \mathfrak{X}^\mathfrak{a} & \xrightarrow{\mathfrak{f}^\mathfrak{a}} & \mathfrak{Y}^\mathfrak{a} \end{array} \qquad (2)$$

That is the pull-back of $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ along $\widetilde{\mathfrak{Y}} \xrightarrow{\xi} \mathfrak{Y}$ coincides with the pull-back of the associated sheaf morphism $\mathfrak{X}^\mathfrak{a} \xrightarrow{\mathfrak{f}^\mathfrak{a}} \mathfrak{Y}^\mathfrak{a}$ along the composition $\eta^\mathfrak{a}_\mathfrak{Y} \circ \xi$.

(b) Let $\widehat{\mathcal{M}} = \mathcal{A}(-, \mathcal{M})$ for some $\mathcal{M} \in Ob\mathcal{A}$, and let $\widehat{\mathcal{M}} \xrightarrow{\gamma} \mathfrak{X}^\mathfrak{a}$ be an arbitrary (pre)sheaf morphism. Consider the diagram

$$\begin{array}{ccccc} \widehat{\mathcal{M}} & \xrightarrow{\mathfrak{j}_\gamma} & \widetilde{\mathfrak{X}} & \xrightarrow{\widetilde{\mathfrak{f}}} & \widehat{\mathcal{M}} \\ & & {\scriptstyle \gamma''}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}^\mathfrak{a} \circ \gamma} \\ & & \mathfrak{X}^\mathfrak{a} & \xrightarrow{\mathfrak{f}^\mathfrak{a}} & \mathfrak{Y}^\mathfrak{a} \end{array} \qquad (3)$$

with cartesian square, where $\widehat{\mathcal{M}} \xrightarrow{\mathfrak{j}_\gamma} \widetilde{\mathfrak{X}}$ is a unique morphism satisfying the equalities

$$\gamma'' \circ \mathfrak{j}_\gamma = \gamma \quad \text{and} \quad \widetilde{\mathfrak{f}} \circ \mathfrak{j}_\gamma = id_{\widehat{\mathcal{M}}}.$$

Suppose that $\mathfrak{Y}$ is a sheaf and the morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ of presheaves of sets is representable. Then, identifying $\mathfrak{Y}$ with $\mathfrak{Y}^{\mathfrak{a}}$, we have, by the argument (a) above, a commutative diagram

$$\begin{array}{ccccc} \widehat{\mathcal{M}} & \xrightarrow{\mathfrak{j}_\gamma} & \widetilde{\mathfrak{X}} & \xrightarrow{\widetilde{\mathfrak{f}}} & \widehat{\mathcal{M}} \\ {\scriptstyle \gamma}\downarrow & & {\scriptstyle \mathfrak{u}_\gamma}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}} \circ \gamma} \\ \mathfrak{X}^{\mathfrak{a}} & \xleftarrow{\eta^{\mathfrak{a}}_{\mathfrak{X}}} & \mathfrak{X} & \xrightarrow{\mathfrak{f}} & \mathfrak{Y} \end{array} \qquad (4)$$

with cartesian right square. The diagram (4) shows that the (arbitrarily chosen) morphism $\widehat{\mathcal{M}} \xrightarrow{\gamma} \mathfrak{X}^{\mathfrak{a}}$ factors through the adjunction morphism $\mathfrak{X} \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{X}}} \mathfrak{X}^{\mathfrak{a}}$. This factorization is unique. Since every presheaf of sets is the colimit of (the canonical diagram of) representable presheaves, this shows that the adjunction morphism $\mathfrak{X} \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{X}}} \mathfrak{X}^{\mathfrak{a}}$ is an isomorphism. ■

1.6. Proposition. *Let τ be a subcanonical (pre)topology on a category $\mathcal{A}$. If a morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ of presheaves of sets on $\mathcal{A}$ is representable, then the square*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{X}}} & \mathfrak{X}^{\mathfrak{a}} \\ {\scriptstyle \mathfrak{f}}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ \mathfrak{Y} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{Y}}} & \mathfrak{Y}^{\mathfrak{a}} \end{array} \qquad (1)$$

is cartesian.

Proof. (a) For an arbitrary morphism $\widehat{\mathcal{N}} = \mathcal{A}(-, \mathcal{N}) \xrightarrow{\xi} \mathfrak{Y}$, consider the diagram

$$\begin{array}{ccccc} \widetilde{\mathfrak{X}} & \xrightarrow{\widetilde{\xi}} & \mathfrak{X} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{X}}} & \mathfrak{X}^{\mathfrak{a}} \\ {\scriptstyle \mathfrak{f}_1}\downarrow & cart & {\scriptstyle \mathfrak{f}}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ \widehat{\mathcal{N}} & \xrightarrow{\xi} & \mathfrak{Y} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{Y}}} & \mathfrak{Y}^{\mathfrak{a}} \end{array} \qquad (2)$$

of presheaf morphisms with a cartesian left square.

If the morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ is representable, then the square

$$\begin{array}{ccc} \widetilde{\mathfrak{X}} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{X}} \circ \widetilde{\xi}} & \mathfrak{X}^{\mathfrak{a}} \\ {\scriptstyle \mathfrak{f}_1} \downarrow & & \downarrow {\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ \widehat{\mathcal{N}} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{Y}} \circ \xi} & \mathfrak{Y}^{\mathfrak{a}} \end{array} \qquad (3)$$

is cartesian. In fact, since the left square in (2) is cartesian and the morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ is representable, the presheaf $\widetilde{\mathfrak{X}}$ is representable, hence it is a sheaf. The sheafification functor maps cartesian squares to cartesian squares; and it maps the left square of the diagram (1) to the square isomorphic to (2).

(b) Let

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\gamma_1} & \mathcal{X} \\ {\scriptstyle \widetilde{\mathfrak{f}}} \downarrow & & \downarrow {\scriptstyle \mathfrak{f}} \\ \mathcal{Y}_1 & \xrightarrow{\gamma} & \mathcal{Y} \end{array} \qquad (4)$$

be a diagram of presheaves of sets such that the composition of (3) with every cartesian square

$$\begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{\psi_1} & \mathcal{X}_1 \\ {\scriptstyle \mathfrak{f}'} \downarrow & cart & \downarrow {\scriptstyle \widetilde{\mathfrak{f}}} \\ \widehat{\mathcal{N}} & \xrightarrow{\psi} & \mathcal{Y}_1 \end{array}$$

(where $\widehat{\mathcal{N}} = \mathcal{A}(-, \mathcal{N})$) is a cartesian square. Then the square (4) is cartesian.

(b1) Since every presheaf of sets is the colimit of a canonical diagram of representable presheaves, it suffices to show that for any commutative square

$$\begin{array}{ccc} \widehat{\mathcal{N}} & \xrightarrow{\lambda} & \mathcal{X} \\ {\scriptstyle \psi} \downarrow & & \downarrow {\scriptstyle \mathfrak{f}} \\ \mathcal{Y}_1 & \xrightarrow{\gamma} & \mathcal{Y} \end{array} \qquad (5)$$

with a representable presheaf in the left upper corner, *there exist a unique morphism* $\widehat{\mathcal{N}} \xrightarrow{\xi} \mathcal{X}_1$ *such that* $\widetilde{\mathfrak{f}} \circ \xi = \psi$ *and* $\gamma_1 \circ \xi = \lambda$.

(b2) We start with the existence of such morphism $\widehat{\mathcal{N}} \xrightarrow{\xi} \mathcal{X}_1$.

By hypothesis, the composition of the squares

$$\begin{array}{ccccc} \mathcal{X}_2 & \xrightarrow{\psi_1} & \mathcal{X}_1 & \xrightarrow{\gamma_1} & \mathcal{X} \\ {\scriptstyle \mathfrak{f}'}\downarrow & cart & \downarrow{\scriptstyle \widetilde{\mathfrak{f}}} & & \downarrow{\scriptstyle \mathfrak{f}} \\ \widehat{\mathcal{N}} & \xrightarrow{\psi} & \mathcal{Y}_1 & \xrightarrow{\gamma} & \mathcal{Y} \end{array} \qquad (6)$$

is a cartesian square. Therefore, it follows from the commutativity of the square (5) that there exists a unique morphism $\widehat{\mathcal{N}} \xrightarrow{\mathfrak{j}} \mathcal{X}_2$ such that

$$\mathfrak{f}' \circ \mathfrak{j} = id_{\widehat{\mathcal{N}}} \quad \text{and} \quad (\gamma_1 \circ \psi_1) \circ \mathfrak{j} = \lambda.$$

So that

$$\widetilde{\mathfrak{f}} \circ (\psi_1 \circ \mathfrak{j}) = \lambda \quad \text{and} \quad \widetilde{\mathfrak{f}} \circ (\psi_1 \circ \mathfrak{j}) = (\widetilde{\mathfrak{f}} \circ \psi_1) \circ \mathfrak{j} = (\psi \circ \mathfrak{f}') \circ \mathfrak{j} = \psi \circ (\mathfrak{f}' \circ \mathfrak{j}) = \psi,$$

which shows that $\xi = \psi_1 \circ \mathfrak{j}$ satisfies the required conditions.

(b3) It remains to show that a morphism $\widehat{\mathcal{N}} \xrightarrow{\xi} \mathcal{X}_1$ satisfying the equalities $\widetilde{\mathfrak{f}} \circ \xi = \psi$ and $\gamma_1 \circ \xi = \lambda$ is unique.

In fact, the equality $\widetilde{\mathfrak{f}} \circ \xi = \psi$ applied to the left, cartesian, square of the diagram (8) implies the there exists a unique morphism $\widehat{\mathcal{N}} \xrightarrow{\mathfrak{j}_\xi} \mathfrak{X}_2$ such that $\xi = \psi_1 \circ \mathfrak{j}_\xi$ and $\mathfrak{f}' \circ \mathfrak{j}_\xi = id_{\widehat{\mathcal{N}}}$. So that we have the equalities:

$$\lambda = \gamma_1 \circ \xi = (\gamma_1 \circ \psi_1) \circ \mathfrak{j}_\xi \quad \text{and} \quad \mathfrak{f}' \circ \mathfrak{j}_\xi = id_{\widehat{\mathcal{N}}}.$$

since, by hypothesis, the composition

$$\begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{\gamma_1 \circ \psi_1} & \mathcal{X} \\ {\scriptstyle \mathfrak{f}'}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}} \\ \widehat{\mathcal{N}} & \xrightarrow{\gamma \circ \psi} & \mathcal{Y} \end{array} \qquad (7)$$

of the squares (6) is cartesian, the morphism $\widehat{\mathcal{N}} \xrightarrow{\mathfrak{j}_\xi} \mathfrak{X}_2$ is uniquely determined by the arrows of the square (7).

(c) It follows from (a) and (b) that if a morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ of presheaves of sets is a representable, then the square (1) is cartesian. ■

1.6.1. Corollary. *Let τ be a subcanonical (pre)topology on a category $\mathcal{A}$. The following conditions on a morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ of presheaves of sets on $\mathcal{A}$ are equivalent:*

(a) The morphism $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ *is representable.*

(b) The associated sheaf morphism, $\mathfrak{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathfrak{Y}^{\mathfrak{a}}$, *is representable and the square*

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{X}}} & \mathfrak{X}^{\mathfrak{a}} \\ {\scriptstyle \mathfrak{f}}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ \mathfrak{Y} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathfrak{Y}}} & \mathfrak{Y}^{\mathfrak{a}} \end{array} \qquad (1)$$

is cartesian.

Proof. (a)⇒(b). If $\mathfrak{X} \xrightarrow{\mathfrak{f}} \mathfrak{Y}$ is a representable morphism, then, by 1.4.1 2), the associated sheaf morphism $\mathfrak{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathfrak{Y}^{\mathfrak{a}}$, is representable, and, by 1.6, the square (1) is cartesian.

(b)⇒(a). The implication follows from the fact that the class of representable morphisms is stable under pull-backs: if the right vertical arrow of the square (1) is representable and the square is cartesian, then the left vertical arrow of the square (1) is also representable. ■

2. Closed immersions and separated morphisms

2.1. Closed immersions of presheaves of sets. Let F, G be presheaves of sets on $\mathcal{A}$. We call a morphism $F \longrightarrow G$ a *closed immersion*, if it belongs to $\mathfrak{M}_s^{\wedge}$, i.e. if it is representable by strict monomorphisms. In particular, a closed immersion $\widehat{X} \longrightarrow \widehat{Y}$ of representable presheaves is of the form $\widehat{u}$, where u is a strict monomorphism.

2.2. Separated morphisms and separated presheaves. Let X, Y be presheaves of sets on a category $\mathcal{A}$. A morphism $X \xrightarrow{f} Y$ is called *separated* if the natural morphism $X \xrightarrow{\Delta_f} X \times_Y X$ is a closed immersion. A presheaf of sets X on $\mathcal{A}$ is *separated* if the diagonal morphism $X \longrightarrow X \times X$ is a closed immersion.

It follows that a presheaf of sets X is separated iff the (unique) morphism X to the final object • is separated (see 1.2.2).

2.3. Proposition. *Let* $\mathcal{U} \xrightarrow{\pi} \mathfrak{X}$ *be a presheaf morphism and*

$$K_2(\pi) = \mathcal{U} \times_{\mathfrak{X}} \mathcal{U} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathcal{U}$$

its kernel pair.

(a) If $\mathfrak{X}$ is separated, then the natural embedding

$$K_2(\pi) = \mathcal{U} \times_{\mathfrak{X}} \mathcal{U} \xrightarrow{\mathfrak{j}_\pi} \mathcal{U} \times \mathcal{U}$$

is a closed immersion.

(b) If $\mathcal{U} \xrightarrow{\pi} \mathfrak{X}$ is an epimorphism, then the converse is true.

Proof. (a) The assertion is a special case of II.2.7.2.

(b) Suppose now that $\mathcal{U} \xrightarrow{\pi} \mathfrak{X}$ is an epimorphism and $K_2(\pi) \xrightarrow{\mathfrak{j}_\pi} \mathcal{U} \times \mathcal{U}$ is a closed immersion. The claim is that $\mathfrak{X}$ is separated, i.e. $\mathfrak{X} \xrightarrow{\Delta_{\mathfrak{X}}} \mathfrak{X} \times \mathfrak{X}$ is a closed immersion.

In fact, let $\mathcal{Z}$ be a representable presheaf and $\mathcal{Z} \xrightarrow{\xi} \mathfrak{X} \times \mathfrak{X}$ a presheaf morphism. Since $\mathcal{Z}$ is a projective object of the category of presheaves and, by hypothesis, $\mathcal{U} \xrightarrow{\pi} \mathfrak{X}$ is a presheaf epimorphism, the morphism $\mathcal{Z} \xrightarrow{\xi} \mathfrak{X} \times \mathfrak{X}$ factors through the epimorphism $\mathcal{U} \times \mathcal{U} \xrightarrow{\pi \times \pi} \mathfrak{X} \times \mathfrak{X}$; that is $\xi = (\pi \times \pi) \circ \widetilde{\xi}$. So that the pull-back of the diagonal morphism $\mathfrak{X} \xrightarrow{\Delta_{\mathfrak{X}}} \mathfrak{X} \times \mathfrak{X}$ along $\mathcal{Z} \xrightarrow{\xi} \mathfrak{X} \times \mathfrak{X}$ is the left vertical arrow in the commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{Z}} & \longrightarrow & K_2(\pi) & \xrightarrow{\pi \circ p_1} & \mathfrak{X} \\ \big\downarrow & cart & \mathfrak{j}_\pi \big\downarrow & cart & \big\downarrow \Delta_{\mathfrak{X}} \\ \mathcal{Z} & \xrightarrow{\widetilde{\xi}} & \mathcal{U} \times \mathcal{U} & \xrightarrow{\pi \times \pi} & \mathfrak{X} \times \mathfrak{X} \end{array} \qquad (3)$$

whose both squares are cartesian. Since, by hypothesis, $K_2(\pi) \xrightarrow{\mathfrak{j}_\pi} \mathcal{U} \times \mathcal{U}$ is a closed immersion and the left square of (3) is cartesian, the left vertical arrow of (3) is a closed immersion of representable presheaves. ■

2.4. Proposition. *Let $(\mathcal{A}, \tau)$ be a right exact category. Suppose that every monomorphism of the category $\mathcal{A}$ has a cokernel pair. Then the sheafification functor maps closed immersions to closed immersions and separated morphisms to separated morphisms.*

In particular, if $\mathcal{X}$ is a separated presheaf of sets on $\mathcal{A}$, then its associated sheaf $\mathcal{X}^{\mathfrak{a}}$ is separated.

Proof. (a) Let $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ be a closed immersion of presheaves and $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ the corresponding morphism of associated sheaves. By 1.4.1, the morphism $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ is representable. The claim is that it is representable by strict monomorphisms.

For any morphism $\widehat{\mathcal{N}} \xrightarrow{\xi} \mathcal{Y}^{\mathfrak{a}}$, we have a commutative diagram

$$\begin{array}{ccccccc} K_2(\widehat{\mathfrak{v}}) & \overset{\widehat{\mathfrak{v}}_1}{\underset{\widehat{\mathfrak{v}}_2}{\rightrightarrows}} & \widehat{\mathfrak{M}} & \xrightarrow{\widehat{\mathfrak{v}}} & \widehat{\mathfrak{N}} & \xrightarrow{\xi'} & \mathcal{X}^{\mathfrak{a}} \\ \widehat{\mathfrak{f}}_3 \downarrow & cart & \widehat{\mathfrak{f}}_2 \downarrow & cart & \downarrow \widehat{\mathfrak{f}}_1 & cart & \downarrow \mathfrak{f}^{\mathfrak{a}} \\ K_2(\widehat{\mathfrak{t}}) & \overset{\widehat{\mathfrak{t}}_1}{\underset{\widehat{\mathfrak{t}}_2}{\rightrightarrows}} & \widehat{\mathcal{M}} & \xrightarrow{\widehat{\mathfrak{t}}} & \widehat{\mathcal{N}} & \xrightarrow{\xi} & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad (4)$$

with cartesian squares. Here $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$ is a deflation such that $\xi \circ \widehat{\mathfrak{t}}$ factors through the adjunction morphism $\mathcal{Y} \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{Y}}} \mathcal{Y}^{\mathfrak{a}}$, and $\widehat{\mathfrak{M}} \xrightarrow{\xi' \circ \widehat{\mathfrak{v}}} \mathcal{X}^{\mathfrak{a}}$ factors through the adjunction morphism $\mathcal{X} \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{X}}} \mathcal{X}^{\mathfrak{a}}$. The existence of such diagram follows from the proof of the assertion 2) in 1.4.1. Moreover, it follows from the diagram 1.4.1(3) and the surrounding it argument that the morphisms $\widehat{\mathfrak{f}}_2$ and $\widehat{\mathfrak{f}}_3$ in the diagram (4) are pull-backs of the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$. Since, by hypothesis, $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is a closed immersion, the morphisms $\mathfrak{M} \xrightarrow{\mathfrak{f}_2} \mathcal{M}$ and $K_2(\mathfrak{v}) \xrightarrow{\mathfrak{f}_3} K_2(\mathfrak{t})$ are strict monomorphisms. Thus, we have a diagram

$$\begin{array}{ccccc} K_2(\mathfrak{v}) & \overset{\mathfrak{v}_1}{\underset{\mathfrak{v}_2}{\rightrightarrows}} & \mathfrak{M} & \xrightarrow{\mathfrak{v}} & \mathfrak{N} \\ \mathfrak{f}_3 \downarrow & cart & \mathfrak{f}_2 \downarrow & cart & \downarrow \mathfrak{f}_1 \\ K_2(\mathfrak{t}) & \overset{\mathfrak{t}_1}{\underset{\mathfrak{t}_2}{\rightrightarrows}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \end{array} \qquad (5)$$

with cartesian squares whose horizontal arrows are deflations and two vertical arrows, $\mathfrak{f}_2$ and $\mathfrak{f}_3$, are strict monomorphisms. The claim is that the remaining vertical arrow, $\mathfrak{N} \xrightarrow{\mathfrak{f}_1} \mathcal{N}$, is a strict monomorphism too.

(a1) Notice that $\mathfrak{N} \xrightarrow{\mathfrak{f}_1} \mathcal{N}$ is a monomorphism. In fact, let $\mathcal{V} \overset{g_1}{\underset{g_2}{\rightrightarrows}} \mathfrak{N}$ be a pair of arrows equalized by $\mathfrak{N} \xrightarrow{\mathfrak{f}_1} \mathcal{N}$. Then we have a commutative diagram

$$\begin{array}{ccccc} \widetilde{\mathcal{V}} & \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} & \mathfrak{M} & \xrightarrow{\mathfrak{f}_2} & \mathcal{M} \\ \widetilde{\mathfrak{v}} \downarrow & cart & \mathfrak{v} \downarrow & cart & \downarrow \mathfrak{t} \\ \mathcal{V} & \overset{g_1}{\underset{g_2}{\rightrightarrows}} & \mathfrak{N} & \xrightarrow{\mathfrak{f}_1} & \mathcal{N} \end{array} \qquad (6)$$

with cartesian squares. In particular, all vertical arrows of the diagram are deflations, hence epimorphisms. The morphism $\mathfrak{M} \xrightarrow{\mathfrak{f}_2} \mathcal{M}$ equalizes the pair of arrows $\widetilde{\mathcal{V}} \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} \mathfrak{M}$. Therefore, since $\mathfrak{f}_2$ is a monomorphism, $\gamma_1 = \gamma_2$. By the commutativity of the left (double) square of (6), $g_1 \circ \widetilde{\mathfrak{v}} = g_2 \circ \widetilde{\mathfrak{v}}$. Since $\widetilde{\mathfrak{v}}$ is an epimorphism, this implies that $g_1 = g_2$.

(a2) By hypothesis, monomorphisms of the category $\mathcal{A}$ have cokernel pairs. So that we the diagram

$$\begin{array}{ccccc}
K_2(\mathfrak{v}) & \overset{\mathfrak{v}_1}{\underset{\mathfrak{v}_2}{\rightrightarrows}} & \mathfrak{M} & \xrightarrow{\mathfrak{v}} & \mathfrak{N} \\
\mathfrak{f}_3 \downarrow & cart & \mathfrak{f}_2 \downarrow & cart & \downarrow \mathfrak{f}_1 \\
K_2(\mathfrak{t}) & \overset{\mathfrak{t}_1}{\underset{\mathfrak{t}_2}{\rightrightarrows}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\mathcal{C}_2(\mathfrak{f}_3) & \rightrightarrows & \mathcal{C}_2(\mathfrak{f}_2) & \longrightarrow & \mathcal{C}_2(\mathfrak{f}_1)
\end{array} \qquad (7)$$

obtained by adjoining to (5) the cokernel pairs of vertical arrows and canonical morphisms between them. Notice that the lower squares of the diagram (7) are cartesian.

Let $\mathcal{V} \xrightarrow{\gamma_1} \mathcal{N}$ be a morphism equalizing the pair $\mathcal{N} \rightrightarrows \mathcal{C}_2(\mathfrak{f}_1)$. Then we have a diagram

$$\begin{array}{ccccc}
K_2(\mathfrak{s}) & \overset{\mathfrak{s}_1}{\underset{\mathfrak{s}_2}{\rightrightarrows}} & \mathcal{W} & \xrightarrow{\mathfrak{s}} & \mathcal{V} \\
\gamma_3 \downarrow & cart & \gamma_2 \downarrow & cart & \downarrow \gamma_1 \\
K_2(\mathfrak{t}) & \overset{\mathfrak{t}_1}{\underset{\mathfrak{t}_2}{\rightrightarrows}} & \mathcal{M} & \xrightarrow{\mathfrak{t}} & \mathcal{N} \\
\downarrow\downarrow & cart & \downarrow\downarrow & cart & \downarrow\downarrow \\
\mathcal{C}_2(\mathfrak{f}_3) & \rightrightarrows & \mathcal{C}_2(\mathfrak{f}_2) & \longrightarrow & \mathcal{C}_2(\mathfrak{f}_1)
\end{array} \qquad (8)$$

formed by cartesian squares. Since the right squares of (8) are cartesian and the arrow $\mathcal{V} \xrightarrow{\gamma_1} \mathcal{N}$ equalizes the pair $\mathcal{N} \rightrightarrows \mathcal{C}_2(\mathfrak{f}_1)$, the morphism $\mathcal{W} \xrightarrow{\gamma_2} \mathcal{M}$ equalizes the pair $\mathcal{M} \rightrightarrows \mathcal{C}_2(\mathfrak{f}_2)$. By a similar reason, the morphism $K_2(\mathfrak{s}) \xrightarrow{\gamma_3} K_2(\mathfrak{t})$ equalizes the pair $K_2(\mathfrak{t}) \rightrightarrows \mathcal{C}_2(\mathfrak{f}_3)$. Since $\mathfrak{f}_2$ and $\mathfrak{f}_3$ are strict monomorphisms, they are kernels of respectively the pair $\mathcal{M} \rightrightarrows \mathcal{C}_2(\mathfrak{f}_2)$ and $K_2(\mathfrak{t}) \rightrightarrows \mathcal{C}_2(\mathfrak{f}_3)$. Therefore, the morphism γ_2 factors through $\mathfrak{f}_2$ and γ_3 factors through $\mathfrak{f}_3$. Since the diagram $K_2(\mathfrak{s}) \overset{\mathfrak{s}_1}{\underset{\mathfrak{s}_2}{\rightrightarrows}} \mathcal{W} \xrightarrow{\mathfrak{s}} \mathcal{V}$ is exact, this implies that $\mathcal{V} \xrightarrow{\gamma_1} \mathcal{N}$ factors through $\mathfrak{N} \xrightarrow{\mathfrak{f}_1} \mathcal{N}$.

(b) Let $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ be a separated morphism of presheaves. By definition, this means that the diagonal morphism $\mathcal{X} \xrightarrow{\Delta_{\mathfrak{f}}} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is a closed immersion. Since the sheafification functor is left exact, it maps $\Delta_{\mathfrak{f}}$ to the diagonal morphism $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\Delta_{\mathfrak{f}^{\mathfrak{a}}}} \mathcal{X}^{\mathfrak{a}} \times_{\mathcal{Y}^{\mathfrak{a}}} \mathcal{X}^{\mathfrak{a}}$ corresponding to $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$. Therefore, by (a), the fact that $\Delta_{\mathfrak{f}}$ is a closed immersion implies that $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\Delta_{\mathfrak{f}^{\mathfrak{a}}}} \mathcal{X}^{\mathfrak{a}} \times_{\mathcal{Y}^{\mathfrak{a}}} \mathcal{X}^{\mathfrak{a}}$ is a closed immersion. So that $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ is a separated morphism. ■

2.5. Universally strict monomorphisms. A morphism $\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M}$ of a category $\mathcal{A}$ is called *universally strict*, if there exists a push-forward of $\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M}$ along any arrow $\mathcal{L} \xrightarrow{\mathfrak{f}} \mathcal{N}$, and this push-forward a strict monomorphism. We denote by $\mathfrak{M}_{\mathfrak{st}}(\mathcal{A})$ the class of all universally strict monomorphisms the category $\mathcal{A}$.

We denote by $\mathfrak{M}_{\mathfrak{sm}}(\mathcal{A})$ the class of all strict monomorphisms such that their push-forward along any strict monomorphism exists and is a strict monomorphism.

2.5.1. Note. It follows from the definition of $\mathfrak{M}_{\mathfrak{sm}}(\mathcal{A})$ that every morphism $\mathcal{L} \xrightarrow{\mathfrak{j}} \mathcal{M}$ from $\mathfrak{M}_{\mathfrak{sm}}(\mathcal{A})$, in particular every universally strict monomorphism, has a cokernel pair, $\mathcal{M} \rightrightarrows C(\mathfrak{j})$ and, therefore, is isomorphic to the kernel of its cokernel pair.

2.5.2. Proposition. *(a) The class $\mathfrak{M}_{\mathfrak{sm}}(\mathcal{A})$ contains all isomorphisms of $\mathcal{A}$ and is closed under push-forwards along strict monomorphisms and composition.*

(b) The class $\mathfrak{M}_{\mathfrak{st}}(\mathcal{A})$ of the universally strict monomorphisms contains all isomorphisms of $\mathcal{A}$ and is closed under push-forwards and composition.

Proof. The argument is the dualization of the argument of II.4.1.4. ■

2.5.3. Example. Let $\mathcal{A} = \mathbf{Aff}_k = Alg_k^{op}$. So that strict monomorphisms of the category $\mathcal{A}$ correspond to strict epimorphisms of k-algebras. The latter are stable under pull-backs, which means that the class $\mathfrak{M}_{\mathfrak{s}}(\mathcal{A})$ of strict monomorphisms coincides, in this case, with the class $\mathfrak{M}_{\mathfrak{st}}(\mathcal{A})$ of universally strict monomorphisms.

3. Smooth and étale morphisms. Open immersions.

3.0. The setting and the notions. Here we consider as "local data" the canonical embedding of a category $\mathcal{A}$ into the category $(\mathcal{A},\tau)^\wedge$ of sheaves of set on a presite $(\mathcal{A},\tau)$ for a subcanonical pretopology τ. We identify the category $\mathcal{A}$ with the full subcategory of $(\mathcal{A},\tau)^\wedge$ generated by representable presheaves. For every object $\mathcal{L}$ of the category $\mathcal{A}$, we denote by $\widehat{\mathcal{L}}$ the presheaf $\mathcal{A}(-,\mathcal{L})$; and for every morphism $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$ of the category $\mathcal{A}$, we denote by $\widehat{\phi}$ the corresponding morphism $\widehat{\mathcal{L}} \longrightarrow \widehat{\mathcal{M}}$.

We also fix a class $\mathfrak{M}$ of "inifinitesimal" morphisms of the category $\mathcal{A}$.

3.0.1. The notions. Applying to this setting the machinary of Sections II.3 and II.4, we obtain the notions of (formally) smooth and (formally) étale morphisms and open immersions of sheaves of sets, together with their general properties – invariance with respect to base change, composition and fiber products and some other facts.

3.0.2. Formally smooth presheaves. In particular, we obtain the notions of open immersions and smooth and étale morphisms of *presheaves* of sets on a category $\mathcal{A}$. These notions become considerably more convenient in the case of the category of presheaves. The main reason for that is that the category of presheaves has enough projectives: every representable presheaf is a projective. The following simple observation is one of the illustrations of this thesis.

3.0.2.1. Proposition. *Let* $\mathcal{F} \xrightarrow{\mathfrak{f}} \mathcal{G}$ *be a presheaf epimorphism. If the presheaf* $\mathcal{F}$ *is formally smooth, then* $\mathcal{G}$ *is formally smooth.*

Proof. In fact, for every morphism $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(\mathcal{L}) & \xrightarrow{\mathcal{F}(\phi)} & \mathcal{F}(\mathcal{M}) \\ {\scriptstyle \mathfrak{f}(\mathcal{L})}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}(\mathcal{M})} \\ \mathcal{G}(\mathcal{L}) & \xrightarrow{\mathcal{G}(\phi)} & \mathcal{G}(\mathcal{M}) \end{array}$$

whose vertical arrows are epimorphisms. If the morphism $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$ belongs to the class $\mathfrak{M}$ and the presheaf $\mathcal{F}$ is $\mathfrak{M}$-smooth, then the upper arrow of the diagram is an epimorphism. Therefore, its lower arrow is an epimorphism too. ∎

3.0.3. Effects of sheafification. On the other hand, the presheaves of sets on affine schemes represented by spaces of (both commutative and noncommutative) algebraic geometry are sheaves for a non-trivial topology. So that, in order to be able to use the advantages of presheaves for studying noncommutative locally affine spaces and schemes, we need to understand what happens with these classes of morphisms of presheaves when we apply the sheafification functors. The rest of the section contains some observations in this direction.

3.1. Proposition. *Let* τ *be a subcanonical (pre)topology on a category* $\mathcal{A}$ *and* $\mathfrak{M}$ *a class of arrows of* $\mathcal{A}$. *Let* $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ *be a representable morphism of presheaves of sets on* $\mathcal{A}$ *and* $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ *the associated sheaf morphism. If the morphism* $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ *is formally* $\mathfrak{M}$*-smooth (resp. formally* $\mathfrak{M}$*-étale, resp. formally* $\mathfrak{M}$*-unramified), then the morphism* $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ *has the same property.*

Proof. By 1.6, if $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is a representable presheaf morphism, then the canonical square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{X}}} & \mathcal{X}^{\mathfrak{a}} \\ {\scriptstyle \mathfrak{f}}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ \mathcal{Y} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{Y}}} & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad (1)$$

is cartesian. The class of formally $\mathfrak{M}$-smooth (resp. formally $\mathfrak{M}$-étale, resp. formally $\mathfrak{M}$-unramified) morphisms is closed under pull-backs. ∎

3.2. Proposition. *Let $(\mathcal{A}, \mathfrak{E}_{\mathcal{A}})$ be a right exact category and $\mathfrak{M}$ a class of arrows of the category $\mathcal{A}$, which is stable under pull-backs along deflations. Let $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ be a morphism of presheaves of sets on $\mathcal{A}$ such that the canonical square*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{X}}} & \mathcal{X}^{\mathfrak{a}} \\ {\scriptstyle \mathfrak{f}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ \mathcal{Y} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{Y}}} & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad (1)$$

is cartesian. Then the presheaf morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is formally $\mathfrak{M}$-étale iff the morphism $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ of associated sheaves is formally $\mathfrak{M}$-étale.

Proof. (a) If $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ is formally $\mathfrak{M}$-smooth, then the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is formally $\mathfrak{M}$-smooth, because, by hypothesis, the square (1) is cartesian, and pull-backs of formally $\mathfrak{M}$-smooth morphisms are $\mathfrak{M}$-smooth. Same consideration works for formally $\mathfrak{M}$-unramified and, therefore, for formally $\mathfrak{M}$-étale morphisms.

(b) Suppose now that the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is formally $\mathfrak{M}$-étale. Let

$$\begin{array}{ccc} \widehat{\mathfrak{N}} & \xrightarrow{g'} & \mathcal{X}^{\mathfrak{a}} \\ {\scriptstyle \widehat{\phi}}\downarrow & & \downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ \widehat{\mathcal{N}} & \xrightarrow{g} & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad (2)$$

be a commutative diagram with $\phi \in \mathfrak{M}$. The claim is that there exists a unique morphism $\widehat{\mathcal{N}} \xrightarrow{\xi} \mathcal{X}^{\mathfrak{a}}$ such that the diagram

$$\begin{array}{ccc} \widehat{\mathfrak{N}} & \xrightarrow{g'} & \mathcal{X}^{\mathfrak{a}} \\ {\scriptstyle \widehat{\phi}}\downarrow & {\scriptstyle \xi}\nearrow & \downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ \widehat{\mathcal{N}} & \xrightarrow{g} & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad (3)$$

commutes.

(b1) By the argument similar to the one used in the proof of 1.4.1, there exists a commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{M}} & \xrightarrow{\ \widehat{\mathfrak{t}}\ } & \widehat{\mathcal{N}} \\ {\scriptstyle \gamma}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathcal{Y} & \xrightarrow{\ \eta^{\mathfrak{a}}_{\mathcal{Y}}\ } & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad (4)$$

where $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$ is a deflation.

(b2) Using the deflation $\mathcal{M} \xrightarrow{\mathfrak{t}} \mathcal{N}$ from (b1) (see the diagram (4)), we extend the commutative square (2) to the diagram

$$\begin{array}{ccccccc} K_2(\widehat{\mathfrak{u}}) & \overset{\widehat{\mathfrak{u}}_1}{\underset{\widehat{\mathfrak{u}}_2}{\rightrightarrows}} & \widehat{\mathfrak{L}} & \xrightarrow{\ \widehat{\mathfrak{u}}\ } & \widehat{\mathfrak{N}} & \xrightarrow{\ g'\ } & \mathcal{X}^{\mathfrak{a}} \\ {\scriptstyle \widehat{\phi}_2}\big\downarrow & cart & {\scriptstyle \widehat{\phi}_1}\big\downarrow & cart & \big\downarrow{\scriptstyle \widehat{\phi}} & & \big\downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ K_2(\widehat{\mathfrak{t}}) & \overset{\widehat{\mathfrak{t}}_1}{\underset{\widehat{\mathfrak{t}}_2}{\rightrightarrows}} & \widehat{\mathcal{M}} & \xrightarrow{\ \widehat{\mathfrak{t}}\ } & \widehat{\mathcal{N}} & \xrightarrow{\ g\ } & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad (5)$$

whose central square and the left (double) square are cartesian. Since the composition $\widehat{\mathcal{M}} \xrightarrow{g \circ \widehat{\mathfrak{t}}} \mathcal{Y}^{\mathfrak{a}}$ equals to the composition of the morphisms $\widehat{\mathcal{M}} \xrightarrow{\gamma} \mathcal{Y} \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{Y}}} \mathcal{Y}^{\mathfrak{a}}$ (see the diagram (4) above) and the square (1) is cartesian, there exists a unique morphism $\mathfrak{M} \xrightarrow{\widetilde{\gamma}} \mathcal{X}$ such that the diagram

$$\begin{array}{ccc} \widehat{\mathfrak{L}} & \xrightarrow{\ \widehat{\mathfrak{u}}\ } & \widehat{\mathfrak{N}} \\ {\scriptstyle \widetilde{\gamma}}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\ \mathcal{X} & \xrightarrow{\ \eta^{\mathfrak{a}}_{\mathcal{X}}\ } & \mathcal{X}^{\mathfrak{a}} \end{array} \qquad (6)$$

commutes. Therefore, we can associate with (5) the commutative diagram

$$\begin{array}{ccccccc} K_2(\widehat{\mathfrak{u}}) & \overset{\widehat{\mathfrak{u}}_1}{\underset{\widehat{\mathfrak{u}}_2}{\rightrightarrows}} & \widehat{\mathfrak{L}} & \xrightarrow{\ \widetilde{\gamma}\ } & \mathcal{X} & \xrightarrow{\ \eta^{\mathfrak{a}}_{\mathcal{X}}\ } & \mathcal{X}^{\mathfrak{a}} \\ {\scriptstyle \widehat{\phi}_2}\big\downarrow & cart & {\scriptstyle \widehat{\phi}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{f}} & & \big\downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ K_2(\widehat{\mathfrak{t}}) & \overset{\widehat{\mathfrak{t}}_1}{\underset{\widehat{\mathfrak{t}}_2}{\rightrightarrows}} & \widehat{\mathcal{M}} & \xrightarrow{\ \gamma\ } & \mathcal{Y} & \xrightarrow{\ \eta^{\mathfrak{a}}_{\mathcal{Y}}\ } & \mathcal{Y}^{\mathfrak{a}} \end{array} \qquad (7)$$

(b3) By hypothesis, the class $\mathfrak{M}$ is stable under pull-backs along deflations. Therefore, the morphisms $\mathfrak{L} \xrightarrow{\phi_1} \widehat{\mathcal{M}}$ and $K_2(\mathfrak{u}) \xrightarrow{\phi_2} K_2(\mathfrak{t})$ belong to $\mathfrak{M}$ (see the diagram (5) above).

Since, by hypothesis, the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is formally $\mathfrak{M}$-smooth, there exists a morphism $\widehat{\mathcal{M}} \xrightarrow{\lambda} \mathcal{X}$ such that the diagram

$$\begin{array}{ccc} \widehat{\mathfrak{L}} & \xrightarrow{\widetilde{\gamma}} & \mathcal{X} \\ {\scriptstyle \widehat{\phi}_1}\downarrow & {\scriptstyle \lambda}\nearrow & \downarrow{\scriptstyle \mathfrak{f}} \\ \widehat{\mathcal{M}} & \xrightarrow{\gamma} & \mathcal{Y} \end{array} \qquad (8)$$

commutes. The sheafification $\lambda^{\mathfrak{a}}$ of λ can be identified with the composition $\widehat{\mathcal{M}} \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{X}} \circ \lambda} \mathcal{X}^{\mathfrak{a}}$.

(b4) Suppose that the morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is étale. The morphism $K_2(\mathfrak{u}) \xrightarrow{\phi_2} K_2(\mathfrak{t})$, being a pull-back of $\widehat{\mathfrak{L}} \xrightarrow{\phi_1} \widehat{\mathcal{M}}$, belongs to the class $\mathfrak{M}$. Since there is only one morphism $K_2(\widehat{\mathfrak{t}}) \xrightarrow{\beta} \mathcal{X}$ which makes the diagram

$$\begin{array}{ccc} K_2(\widehat{\mathfrak{u}}) & \xrightarrow{\widetilde{\gamma}} & \mathcal{X} \\ {\scriptstyle \widehat{\phi}_2}\downarrow & {\scriptstyle \beta}\nearrow & \downarrow{\scriptstyle \mathfrak{f}} \\ K_2(\widehat{\mathfrak{t}}) & \xrightarrow{\gamma} & \mathcal{Y} \end{array}$$

commute, $\lambda \circ \widehat{\mathfrak{t}}_1 = \lambda \circ \mathfrak{t}_2$. Therefore, $\lambda^{\mathfrak{a}} \circ \widehat{\mathfrak{t}}_1 = \eta^{\mathfrak{a}}_{\mathcal{X}} \circ \lambda \circ \widehat{\mathfrak{t}}_1 = \eta^{\mathfrak{a}}_{\mathcal{X}} \circ \lambda \circ \widehat{\mathfrak{t}}_2 = \lambda^{\mathfrak{a}} \circ \mathfrak{t}_2$.

Since $K_2(\widehat{\mathfrak{t}}) \underset{\widehat{\mathfrak{t}}_2}{\overset{\widehat{\mathfrak{t}}_1}{\rightrightarrows}} \widehat{\mathcal{M}} \xrightarrow{\widehat{\mathfrak{t}}} \widehat{\mathcal{N}}$ is an exact diagram and the sheafification functor is exact, there exists a unique morphism $\widehat{\mathcal{N}} \xrightarrow{\xi} \mathcal{X}^{\mathfrak{a}}$ such that $\lambda^{\mathfrak{a}} = \xi \circ \widehat{\mathfrak{t}}$. ■

3.3. Proposition. *Let $(\mathcal{A}, \mathfrak{E}_{\mathcal{A}})$ be a right exact category and $\mathfrak{M}$ a class of arrows of the category $\mathcal{A}$, which is stable under pull-backs along deflations. A representable morphism $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ of presheaves of sets on $\mathcal{A}$ is formally $\mathfrak{M}$-étale iff the morphism $\mathcal{X}^{\mathfrak{a}} \xrightarrow{\mathfrak{f}^{\mathfrak{a}}} \mathcal{Y}^{\mathfrak{a}}$ of associated sheaves is formally $\mathfrak{M}$-étale.*

Proof. By 1.6, if $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$ is a representable morphism, then the canonical square

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{X}}} & \mathcal{X}^{\mathfrak{a}} \\ {\scriptstyle \mathfrak{f}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{f}^{\mathfrak{a}}} \\ \mathcal{Y} & \xrightarrow{\eta^{\mathfrak{a}}_{\mathcal{Y}}} & \mathcal{Y}^{\mathfrak{a}} \end{array}$$

is cartesian. The assertion follows from 3.2. ■

4. Locally affine spaces represented by (pre)sheaves of sets.

4.1. Local data. The 'local data' is the canonical full embedding of the category $\mathfrak{A}$ into the category $\mathfrak{B} = (\mathfrak{A}, \mathfrak{T})^\wedge$ of sheaves on $(\mathfrak{A}, \mathfrak{T})$, where $\mathfrak{T}$ is a subcanonical quasi-topology. We fix a class of arrows $\mathfrak{M}$ of the category $\mathfrak{A}$.

4.2. Pretopologies. We endow the category $\mathfrak{B} = (\mathfrak{A}, \mathfrak{T})^\wedge$ with the *smooth* pretopology $\tau^{\mathfrak{M}}_{sm}$. We shall also consider étale pretopology $\tau^{\mathfrak{M}}_{et}$ and Zariski topology $\tau^{\mathfrak{M}}_{\mathfrak{Z}}$.

4.3. Locally affine spaces, schemes and algebraic spaces.

4.3.1. Algebraic spaces. An object $\mathcal{X}$ of the category $\mathfrak{B} = (\mathfrak{A}, \mathfrak{T})^\wedge$ is an *algebraic space* if there is a cover in $\mathfrak{M}$-étale pretopology of the form $\{\widehat{U_i} \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$.

4.3.2. Schemes. An object $\mathcal{X}$ of $\mathfrak{B}$ is a *scheme* if there is a cover in Zariski pretopology of the form $\{\widehat{U_i} \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$.

4.3.3. Locally affine spaces for the smooth pretopology. A *locally affine* space for the smooth pretopology is an object $\mathcal{X}$ of the category $\mathfrak{B}$ which has a strictly epimorphic family $\{\widehat{U_i} \xrightarrow{\mathfrak{u}_i} \mathcal{X} \mid i \in J\}$ formed by $\mathfrak{M}$-smooth morphisms.

4.4. Note. All examples of locally affine spaces which fill the rest of this Chapter and Chapter V are semi-separated, or even separated object of $\mathfrak{B}$. In particular, all covers will be semi-separated, that is formed by representable morphisms.

5. Vector fibers, inner homs, isomorphisms.

Fix an associative unital k-algebra R. Let $\mathcal{A}$ be the category $R\backslash Alg_k$ of associative *k-algebras over R* (i.e. pairs $(S, R \to S)$, where S is a k-algebra and $R \to S$ a k-algebra morphism) which we call for convenience R-rings. We denote by R^o the k-algebra opposite to R and by R^e the k-algebra $R \otimes_k R^o$. We shall identify (via a natural category isomorphism) the category $(R\backslash Alg_k)^{op}$ with the category $\mathbf{Aff}_k/R^\vee$ of affine k-schemes over $R^\vee = Alg_k(R, -)$.

5.1. Vector fiber associated with a bimodule. Let $\mathcal{M}$ be a left R^e-module. We denote by $\mathbb{V}_R(\mathcal{M})$ the affine k-scheme $T_R(\mathcal{M})^\vee$ represented by the tensor algebra $T_R(\mathcal{M}) = \bigoplus_{n\geq 0} \mathcal{M}^{\otimes n}$ of the R^e-module $\mathcal{M}$. Here $\mathcal{M}^{\otimes 0} = R$ and $\mathcal{M}^{\otimes (n+1)} = \mathcal{M} \otimes_R \mathcal{M}^{\otimes n}$ for $n \geq 0$. We call $\mathbb{V}_R(\mathcal{M})$ the *vector fiber* of the R^e-module $\mathcal{M}$ and identify it with the object $(\mathbb{V}_R(\mathcal{M}), \mathbb{V}_R(\mathcal{M}) \to R^\vee)$ of the category $\mathbf{Aff}_k/R^\vee$, where the morphism $\mathbb{V}_R(\mathcal{M}) \longrightarrow R^\vee$ corresponds to the natural embedding $R \longrightarrow T_R(\mathcal{M})$.

5.1.1. Proposition. *For any unital ring morphism* $R \xrightarrow{s} S$, *there is a natural isomorphism*

$$(S,s)^\vee \prod_{R^\vee} \mathbb{V}_R(\mathcal{M}) \xrightarrow{\sim} \mathbb{V}_S(\bar{\varphi}^*(\mathcal{M}))$$

over $S^\vee$. *Here* $\bar{\varphi}^*(\mathcal{M})) = S \otimes_R \mathcal{M} \otimes_R S$.

Proof. Consider arbitrary commutative square

$$\begin{array}{ccc} (A,g)^\vee & \longrightarrow & \mathbb{V}_R(\mathcal{M}) \\ \downarrow & & \downarrow \\ (S,s)^\vee & \longrightarrow & R^\vee \end{array},$$

or, the corresponding commutative square of k-algebra morphisms

$$\begin{array}{ccc} T_R(\mathcal{M}) & \longrightarrow & A \\ \uparrow & & \uparrow \\ R & \xrightarrow{s} & S \end{array} \qquad (1)$$

The algebra morphism $T_R(\mathcal{M}) \longrightarrow A$ is uniquely determined by an R^e-module morphism $\mathcal{M} \longrightarrow A$. The commutativity of the diagram (1) implies that the pair of morphisms $\mathcal{M} \longrightarrow A \longleftarrow S$ defines an S^e-module morphism $\bar{s}^*(\mathcal{M}) = S^e \otimes_{R^e} \mathcal{M} \longrightarrow A$ which, in turn, uniquely determines a k-algebra morphism $T_S(\bar{s}^*(\mathcal{M})) \longrightarrow A$. Therefore $T_S(\bar{s}^*(\mathcal{M})) \simeq T_R(\mathcal{M}) \star_R S$ as algebras over S; hence the assertion. ∎

5.2. Proposition. *Let* $\mathcal{M}$ *be a left* R^e*-module. The space* $\mathbb{V}_R(\mathcal{M})$ *is locally of cofinite type (resp. locally finitely copresentable) over* R *iff the* R^e*-module* $\mathcal{M}$ *is of finite type (resp. locally finitely presentable).*

Proof. Let $\mathcal{A} = R\backslash Alg_k$, and let $D \xrightarrow{\mathfrak{D}} \mathcal{A}$ be a filtered inductive system. Then we have a commutative diagram of canonical morphisms

$$\begin{array}{ccc} \operatorname{colim} Hom_{\mathcal{A}}(T_R(\mathcal{M}), \mathfrak{D}) & \xrightarrow{\lambda} & Hom_{\mathcal{A}}(T_R(\mathcal{M}), \operatorname{colim} \mathfrak{D}) \\ \wr\downarrow & & \downarrow\wr \\ \operatorname{colim} Hom_{R^e}(\mathcal{M}, \Phi_R \mathfrak{D}) & \xrightarrow{\widetilde{\lambda}} & Hom_{R^e}(\mathcal{M}, \Phi_R(\operatorname{colim} \mathfrak{D})) \end{array} \qquad (2)$$

Here $\mathcal{A} \xrightarrow{\Phi_R} R^e - mod$ is the functor which maps any R-ring $(S, R \longrightarrow S)$ to the left R^e-module S. The functor Φ_R is a right adjoint to the functor

$$R^e - mod \xrightarrow{T_R} \mathcal{A}, \quad \mathcal{M} \longmapsto (T_R(\mathcal{M}), R \to T_R(\mathcal{M}).$$

The functor Φ_R preserves colimits of filtered inductive systems, i.e. the canonical morphism colim $\Phi_R\mathfrak{D} \longrightarrow \Phi_R(\text{colim } \mathfrak{D})$ is an isomorphism. Thus, the map $\widetilde{\lambda}$ in (2) is the composition of a canonical map

$$\text{colim } Hom_{R^e}(\mathcal{M}, \Phi_R\mathfrak{D}) \xrightarrow{\widehat{\lambda}} Hom_{R^e}(\mathcal{M}, \text{colim } \Phi_R\mathfrak{D})$$

and an isomorphism $Hom_{R^e}(\mathcal{M}, \text{colim } \Phi_R\mathfrak{D}) \xrightarrow{\sim} Hom_{R^e}(\mathcal{M}, \Phi_R(\text{colim } \mathfrak{D}))$. Together with the commutativity of (2), this means that λ is injective (resp. bijective) iff $\widehat{\lambda}$ is injective (resp. bijective). Therefore, the morphism $\mathbb{V}_R(\mathcal{M}) \longrightarrow R^\vee$ is locally of finite type (resp. locally finitely presentable) if the R^e-module $\mathcal{M}$ is of finite type (resp. locally finitely presentable).

Let now $\widetilde{D} \xrightarrow{\widetilde{\mathfrak{D}}} R^e - mod$ be an inductive (that is filtered) system. Let $\mathcal{E}x_R$ denote the functor $R^e - mod \longrightarrow \mathcal{A} = R\backslash Alg_k$ which assigns to every left R^e-module $\mathcal{L}$ the pair $(\mathcal{L}_R, R \to \mathcal{L}_R)$, where $\mathcal{L}_R$ is the extension R by $\mathcal{L}$. Then for every R^e-module L, the composition $\Phi_R \circ \mathcal{E}x_R$ transfers $\mathcal{L}$ into $R \oplus \mathcal{L}$. Taking in (2) $\mathfrak{D} = \mathcal{E}x_R \circ \widetilde{\mathfrak{D}}$, we obtain (from the following (2) discussion) a commutative diagram

$$\begin{array}{ccc} \text{colim } Hom_{\mathcal{A}}(T_{R}(\mathcal{M}), \mathfrak{D}) & \xrightarrow{\lambda} & Hom_{\mathcal{A}}(T_{R}(\mathcal{M}), \text{colim } \mathfrak{D}) \\ \wr\downarrow & & \downarrow\wr \\ \text{colim } Hom_{R^e}(\mathcal{M}, \Phi_R\mathfrak{D}) & \xrightarrow{\widehat{\lambda}} & Hom_{R^e}(\mathcal{M}, \text{colim } \Phi_R\mathfrak{D}) \end{array} \qquad (3)$$

If $\mathbb{V}_R(\mathcal{M})$ is locally of finite type (resp. locally finitely presentable) over R, then the map λ in (3) is injective (resp. bijective). And λ is injective (resp. bijective) iff $\widehat{\lambda}$ has the same property. Since there are natural isomorphisms

$$\begin{aligned} \text{colim } Hom_{R^e}(\mathcal{M}, \Phi_R\mathfrak{D}) &\simeq \text{colim } Hom_{R^e}(\mathcal{M}, R \oplus \widetilde{\mathfrak{D}}) \\ &\simeq Hom_{R^e}(\mathcal{M}, R) \oplus \text{colim } Hom_{R^e}(\mathcal{M}, \widetilde{\mathfrak{D}}) \end{aligned}$$

and

$$\begin{aligned} Hom_{R^e}(\mathcal{M}, \text{colim } \Phi_R\mathfrak{D}) &\simeq Hom_{R^e}(\mathcal{M}, \text{colim } (R \oplus \widetilde{\mathfrak{D}}) \\ &\simeq Hom_{R^e}(\mathcal{M}, R) \oplus Hom_{R^e}(\mathcal{M}, \text{colim } \widetilde{\mathfrak{D}}) \end{aligned}$$

compatible with the map $\widehat{\lambda}$ in (3), $\widehat{\lambda}$ is injective (resp. bijective) iff the canonical map colim $Hom_{R^e}(\mathcal{M}, \widetilde{\mathfrak{D}}) \longrightarrow Hom_{R^e}(\mathcal{M}, \text{colim } \widetilde{\mathfrak{D}})$ is injective (resp. bijective). This shows that if $\mathbb{V}_R(\mathcal{M})$ is locally of finite type (resp. locally finitely presentable) over R, then the R^e-module $\mathcal{M}$ is of finite type (resp. locally finitely presentable). ■

5.3. Inner hom. Let M, V be left R-modules. Consider the presheaf

$$\mathcal{H}_R(M,V) : (\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \longrightarrow Sets$$

which assigns to each algebra $(S, R \xrightarrow{\phi} S)$ over R the set $Hom_S(\phi^*(M), \phi^*(V))$.

5.3.1. Proposition. *Let M and V be left R-modules. For any unital ring morphism $R \xrightarrow{s} S$, there is a natural isomorphism*

$$(S,s)^\vee \prod_{R^\vee} \mathcal{H}_R(M,V) \xrightarrow{\sim} \mathcal{H}_S(s^*(M), s^*(V))$$

over $(S,s)^\vee$ (cf. 5.2(5)).

Proof. Consider a commutative square

$$\begin{array}{ccc}
(A,g)^\vee & \xrightarrow{\xi} & \mathcal{H}_{M,V} \\
\gamma \big\downarrow & & \big\downarrow \\
(S,s)^\vee & \xrightarrow{s} & R^\vee
\end{array}$$

The morphism ξ corresponds to an element of $\mathcal{H}_{M,V}(R \to A)$, i.e. to an A-module morphism $A \otimes_R M \longrightarrow A \otimes_R V$. Since $A \otimes_R - \simeq A \otimes_S (S \otimes_R -)$, the latter morphism defines an element of $\mathcal{H}_{s^*(M),s^*(V)}(S \to A)$ which uniquely determines a morphism $(A,g)^\vee \xrightarrow{\bar{\xi}} \mathcal{H}_{s^*(M),s^*(V)}$ over $(S,s)^\vee$. This implies the assertion. ∎

5.3.2. Proposition. *If V is a projective R-module of finite type, then the presheaf $\mathcal{H}_R(M,V)$ is representable.*

Proof. In fact, for any k-algebra morphism $R \xrightarrow{\phi} S$, we have:

$$Hom_S(\phi^*(M), \phi^*(V) \simeq Hom_R(M, \phi^*(V)) = Hom_R(M, S \otimes_R V).$$

If V is a projective R-module of finite type, then $S \otimes_R V \simeq Hom^R(V_R^*, S)$, where V_R^* is the right R-module dual to V, i.e. $V_R^* = Hom_R(V,R)$; and $Hom^R(-,-)$ denotes the functor of right R-module morphisms. Thus,

$$Hom_R(M, S \otimes_R V) \simeq Hom_R(M, Hom^R(V_R^*, S)) \simeq Hom_{R^e}(M \otimes_k V_R^*, S)$$

and

$$Hom_{R^e}(M \otimes_k V_R^*, S) \simeq R\backslash Alg_k(\mathbf{T}_R(M \otimes_k V_R^*), S),$$

hence the assertion. ■

5.3.3. Corollary. *Let M be a left R-module and V a projective left R-module of finite type. Then, for any unital algebra morphism $R \xrightarrow{\varphi} S$, there is a natural isomorphism*

$$S \star_R \mathbf{T}_R(M \otimes_k V_R^*) \xrightarrow{\sim} \mathbf{T}_S(\varphi^*(M) \otimes_k \varphi^*(V)_S^*)$$

over S. Here $\varphi^*(V)_S^* = Hom_S(\varphi^*(V), S) \simeq Hom_R(V, \varphi_*(S))$.

Proof. By (the argument of) 5.3.2, the presheaf $\mathcal{H}_R(M,V)$ is representable by the vector fiber of the tensor algebra $T_R(M \otimes_k V_R^*)$ of the R^e-module $M \otimes_k V_R^*$. In particular, the functor $\mathcal{H}_S(\varphi^*(M), \varphi^*(V))$ is representable by the vector fiber of the tensor algebra of the S^e-module

$$\varphi^*(M) \otimes_k \varphi^*(V_R^*) = S \otimes_R M \otimes_k V_R^* \otimes_R S \simeq S^e \otimes_{R^e} (M \otimes_k V_R^*).$$

The assertion follows now from 5.1.1 (see also 5.2). ■

5.3.4. Corollary. *Let M be a left R-module and V a projective left R-module of finite type. If the R-module M is locally of finite type (resp. locally finitely presentable), then the presheaf $\mathcal{H}_R(M,V)$ is locally of finite type (resp. locally finitely presentable) over R. If the module V is a generator of the category $R - mod$, then the converse holds; i.e. the presheaf $\mathcal{H}_R(M,V)$ is of finite type (resp. finitely presentable) over R iff the R-module M is of finite type (resp. finitely presentable).*

Proof. Since V is a projective module of finite type, by 5.3.2, the presheaf $\mathcal{H}_R(M,V)$ is representable by the tensor algebra $T_R(M \otimes_k V_R^*)$ of the R^e-module $M \otimes_k V_R^*$, i.e. $\mathcal{H}_R(M,V) \simeq \mathbb{V}_R(M \otimes_k V_R^*)$. If the R-module M is of finite type (resp. finitely presentable), then the R^e-module $M \otimes_k V_R^*$ is of finite type (resp. finitely presentable). If V is a generator of the category $R - mod$ (that is $Hom_R(V, -)$ is a faithful functor), then the R^e-module $M \otimes_k V_R^*$ is of finite type (resp. finitely presentable) iff the R-module M has this property. The assertion follows now from 5.2. ■

5.4. Isomorphisms. Fix left R-modules $\mathcal{M}$ and $\mathcal{N}$. We denote by $\mathfrak{Iso}_R(\mathcal{M}, \mathcal{N})$ the presheaf $(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \longrightarrow Sets$, which maps every R-ring (S,s) to the set $Iso_S(s^*(\mathcal{M}), s^*(\mathcal{N}))$ of S-module isomorphisms $s^*(\mathcal{M}) \xrightarrow{\sim} s^*(\mathcal{N})$.

5.4.1. Proposition. *If $\mathcal{M}$ and $\mathcal{N}$ are projective R-modules of finite type, then the presheaf $\mathfrak{Iso}_R(\mathcal{M}, \mathcal{N})$ is representable.*

Proof. The presheaf $\mathfrak{Iso}_R(\mathcal{M},\mathcal{N})$ is the limit of the diagram

$$\begin{array}{ccc} \mathcal{H}_R(\mathcal{N},\mathcal{M}) \times \mathcal{H}_R(\mathcal{M},\mathcal{N}) & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{M}}}{\rightrightarrows}} & \mathcal{H}_R(\mathcal{M},\mathcal{M}) \\ \mathfrak{S} \downarrow \wr & & \\ \mathcal{H}_R(\mathcal{M},\mathcal{N}) \times \mathcal{H}_R(\mathcal{N},\mathcal{M}) & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{N}}}{\rightrightarrows}} & \mathcal{H}_R(\mathcal{N},\mathcal{N}) \end{array} \tag{1}$$

where $\mathfrak{S}$ is the standard symmetry, $(x,y) \longmapsto (y,x)$, $\mathfrak{m}$ is the composition; and $\mathfrak{p}_{\mathcal{M}}(S,s)$ maps $\mathcal{H}_R(\mathcal{N},\mathcal{M}) \times \mathcal{H}_R(\mathcal{M},\mathcal{N})(S,s) = Hom_S(s^*(\mathcal{N}), s^*(\mathcal{M})) \times Hom_S(s^*(\mathcal{M}), s^*(\mathcal{N}))$ to the identical endomorphism of $s^*(\mathcal{M})$. If the R-modules $\mathcal{M}$ and $\mathcal{N}$ are of finite type, then all presheaves in the diagram (1) are representable; hence the limit of the diagram is representable too. The R-ring representing $\mathfrak{Iso}_R(\mathcal{M},\mathcal{N})$ is easily read from the diagram (1): it is the colimit of the diagram of morphisms of tensor algebras

$$\begin{array}{ccc} T_R(\mathcal{M} \otimes_k \mathcal{M}_R^*) & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{M}}}{\rightrightarrows}} & T_R(\mathcal{N} \otimes_k \mathcal{M}_R^* \oplus \mathcal{M} \otimes_k \mathcal{N}_R^*) \\ & & T_R(\sigma) \downarrow \wr \\ T_R(\mathcal{N} \otimes_k \mathcal{N}_R^*) & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{N}}}{\rightrightarrows}} & T_R(\mathcal{M} \otimes_k \mathcal{N}_R^* \oplus \mathcal{N} \otimes_k \mathcal{M}_R^*) \end{array} \tag{2}$$

dual to (1). ■

5.4.2. The presheaf $GL_{\mathcal{V}}$. We shall write $GL_{\mathcal{V}}$ instead of $\mathfrak{Iso}_R(\mathcal{V},\mathcal{V})$. By 5.4.1, the presheaf $GL_{\mathcal{V}}$ is representable, if $\mathcal{V}$ is a projective R-module of finite type.

6. Grassmannians.

6.0. The Grassmannian of a pair of modules. Let M be a left R-module and V a projective left R-module. Consider the presheaf

$$(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \xrightarrow{Gr_{M,V}} \mathbf{Sets}$$

which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set of isomorphism classes of S-module epimorphisms $s^*(M) \longrightarrow s^*(V)$ (where $s^*(M) = S \otimes_R M$) and to any R-ring morphism $(S, R \xrightarrow{s} S) \xrightarrow{\phi} (T, R \xrightarrow{t} T)$ the map

$$Gr_{M,V}(S,s) \longrightarrow Gr_{M,V}(T,t)$$

induced by the inverse image functor $S-mod \xrightarrow{\phi^*} T-mod$, $\mathcal{N} \longmapsto T\otimes_S \mathcal{N}$.

6.1. The presheaf $G_{M,V}$**.** Let

$$(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \longrightarrow \mathbf{Sets}$$

denote the presheaf which assigns to any R-ring $(S, R \stackrel{s}{\to} S)$ the set of pairs of morphisms $s^*(V) \xrightarrow{v} s^*(M) \xrightarrow{u} s^*(V)$ such that $u\circ v = id_{s^*(V)}$ and acts naturally on morphisms. Since V is a projective module, the map

$$\pi = \pi_{M,V} : G_{M,V}(S,s) \longrightarrow Gr_{M,V}(S,s), \quad (v,u) \longmapsto [u], \tag{1}$$

is a (strict) presheaf epimorphism.

6.2. Relations. Denote by $\mathfrak{R}_{M,V}$ the "presheaf of relations" $G_{M,V} \prod\limits_{Gr_{M,V}} G_{M,V}$. By definition, $\mathfrak{R}_{M,V}$ is a subpresheaf of $G_{M,V}\times G_{M,V}$ which assigns to each R-ring, $(S, R \stackrel{s}{\to} S)$, the set of all 4-tuples $(u_1,v_1;u_2,v_2) \in G_{M,V}\times G_{M,V}$ such that the epimorphisms $u_1,u_2 : s^*(M) \longrightarrow s^*(V)$ are equivalent. The latter means that there exists an isomorphism $s^*(V) \xrightarrow{\varphi} s^*(V)$ such that $u_2 = \varphi\circ u_1$, or, equivalently, $\varphi^{-1}\circ u_2 = u_1$. Since $u_i\circ v_i = id$, $i=1,2$, these equalities imply that

$$\varphi = u_2\circ v_1 \quad \text{and} \quad \varphi^{-1} = u_1\circ v_2.$$

Thus $\mathfrak{R}_{M,V}(S,s)$ is a subset of all $(u_1,v_1;u_2,v_2) \in G_{M,V}(S,s)\prod G_{M,V}(S,s)$ satisfying the following relations:

$$u_2 = (u_2\circ v_1)\circ u_1, \quad u_1 = (u_1\circ v_2)\circ u_2 \tag{2}$$

in addition to the relations describing $G_{M,V}(S,s)\prod G_{M,V}(S,s)$:

$$u_1\circ v_1 = id_{S\otimes_R V} = u_2\circ v_2 \tag{3}$$

Note that the relations (2) and (3) imply that $\varphi = u_2\circ v_1$ and $\varphi^{-1} = u_1\circ v_2$ are, indeed, mutually inverse morphisms:

$$\begin{aligned}\varphi^{-1}\circ\varphi =&(u_1\circ v_2)\circ(u_2\circ v_1) = (u_1\circ(v_2\circ u_2))\circ v_1 = u_1\circ v_1 = id,\\ \varphi\circ\varphi^{-1} =&(u_2\circ v_1)\circ(u_1\circ v_2) = (u_2\circ(v_1\circ u_1)\circ v_2 = u_2\circ v_2 = id.\end{aligned}$$

Let $\mathfrak{R}_{M,V} \underset{p_2}{\overset{p_1}{\rightrightarrows}} G_{M,V}$ be the canonical projections. It follows from the surjectivity of $G_{M,V} \longrightarrow Gr_{M,V}$ that the diagram

$$\mathfrak{R}_{M,V} \underset{p_2}{\overset{p_1}{\rightrightarrows}} G_{M,V} \xrightarrow{\pi} Gr_{M,V} \tag{4}$$

is exact.

6.3. Proposition. *If both M and V are projective R-modules of a finite type, then the presheaves $G_{_{M,V}}$ and $\mathfrak{R}_{_{M,V}}$ are representable.*

Proof. (a) Suppose that the R-module V is projective of finite type. For any algebra morphism $R \xrightarrow{\phi} S$, we have the following functorial isomorphisms:

$$Hom_S(\phi^*(M), \phi^*(V)) \simeq Hom_R(M, \phi_*\phi^*(V)) = Hom_R(M, S \otimes_R V) \simeq$$
$$Hom_R(M, Hom^R(V_R^*, S)) \simeq Hom_{R^e}(M \otimes_k V_R^*, S) \simeq R\backslash Alg_k(\mathbf{T}_R(M \otimes_k V_R^*), S)$$

Here $Hom^R(V_R^*, S)$ is the (left) R-module of right R-module morphisms from V_R^* to S, $R^e = R \otimes_k R^{op}$, and $\mathbf{T}_R(M \otimes_k V_R^*)$ is the tensor algebra of the R-bimodule $M \otimes_k V_R^*$.

(b) The set $G_{_{M,V}}(S)$ is the kernel of the pair of morphisms

$$Hom_S(\phi^*(M), \phi^*(V)) \times Hom_S(\phi^*(V), \phi^*(M)) \rightrightarrows Hom_S(\phi^*(V), \phi^*(V)) \qquad (5)$$

where one arrow assigns to each pair (u, v) the composition, $u \circ v$, of morphisms u and v, and the other one maps each pair (u, v) to the identity morphism, $id_{\phi^*(V)}$. Since the modules M and V are finite, we have canonical functorial isomorphisms:

$$Hom_S(\phi^*(M), \phi^*(V)) \times Hom_S(\phi^*(V), \phi^*(M)) \simeq$$

$$Hom_{R^e}(M \otimes_k V_R^*, S) \times Hom_{R^e}(V \otimes_k M_R^*, S) \simeq$$

$$Hom_{R^e}(M \otimes_k V_R^* \oplus V \otimes_k M_R^*, S) \simeq R\backslash Alg_k(\mathbf{T}_R(M \otimes_k V_R^* \oplus V \otimes_k M_R^*), S)$$

and

$$Hom_S(\phi^*(V), \phi^*(V)) \simeq R\backslash Alg_k(\mathbf{T}_R(V \otimes_k V_R^*), S)$$

Thus, to the diagram (1), there corresponds a diagram

$$\mathbf{T}_R(V \otimes_k V_R^*) \rightrightarrows \mathbf{T}_R(M \otimes_k V_R^* \oplus V \otimes_k M_R^*) \qquad (6)$$

of algebra morphisms. The cokernel, $\mathcal{G}_{_{M,V}}$, of the pair of morphisms (6) corepresents the kernel of the pair of morphisms (5). This proves the corepresentability of $G_{_{M,V}}$.

(c) A similar argument proves the representability of $\mathfrak{R}_{_{M,V}}$. Details are left to the reader. ∎

6.3.1. Proposition. *If M and V are projective R-modules of a finite type, then the presheaves $G_{_{M,V}}$, $\mathfrak{R}_{_{M,V}}$, and $Gr_{_{M,V}}$ are locally finitely copresentable over R.*

Proof. It follows from the argument of 6.3 that $G_{M,V}$ is isomorphic to the kernel of a pair of arrows

$$\mathbb{V}_R(M \otimes_k V_R^* \oplus V \otimes_k M_R^*) \rightrightarrows \mathbb{V}_R(V \otimes_k V_R^*) \qquad (7)$$

(see (6)). Since the R-modules M and V are projective of finite type, the R^e-modules $V \otimes_k V_R^*$ and $M \otimes_k V_R^* \oplus V \otimes_k M_R^*$ are projective of finite type; in particular, they are finitely presentable. Therefore, by 5.2, both presheaves in (7) are locally finitely presentable over $R^\vee$. The kernel of a pair of arrows between locally finitely presentable over $R^\vee$ presheaves is locally finitely presentable over $R^\vee$; hence $G_{M,V}$ is locally finitely presentable over $R^\vee$. By a similar reason the presheaf of relations $\mathfrak{R}_{M,V}$ is locally finitely presentable over $R^\vee$. Since $Gr_{M,V}$ is a cokernel of a pair of arrows between locally finitely presentable over $R^\vee$ presheaves (see 6.2(4)), it is locally finitely presentable too. ∎

6.4. Universality with respect to the base change.

6.4.1. Proposition. *Let M be an R-module and V a projective R-module. For any unital k-algebra morphism $R \xrightarrow{\phi} S$, there is a natural isomorphism between the diagram*

$$(S,s)^\vee \prod_{R^\vee} \big(\mathfrak{R}_{M,V} \underset{p_2}{\overset{p_1}{\rightrightarrows}} G_{M,V} \xrightarrow{\ \pi\ } Gr_{M,V}\big) \qquad (1)$$

and the diagram

$$\mathfrak{R}_{\phi^*(M),\phi^*(V)} \underset{p_2}{\overset{p_1}{\rightrightarrows}} G_{\phi^*(M),\phi^*(V)} \xrightarrow{\ \pi\ } Gr_{\phi^*(M),\phi^*(V)} \qquad (2)$$

In particular, $(S,s)^\vee \prod_{R^\vee} Gr_{M,V}$ *is isomorphic to* $Gr_{\phi^*(M),\phi^*(V)}$.

Proof. Consider a commutative square

$$\begin{array}{ccc} (A,g)^\vee & \xrightarrow{\ \xi\ } & Gr_{M,V} \\ {\scriptstyle \gamma^\vee}\downarrow & & \downarrow \\ (S,s)^\vee & \xrightarrow{\ s^\vee\ } & R^\vee \end{array}$$

The morphism ξ corresponds to an element of $Gr_{M,V}(A,g)$, i.e. to the equivalence class of an A-module epimorphism $g^*(M) \longrightarrow g^*(V)$. Since $g^* \simeq \gamma^* s^*$, this epimorphism defines an element of $Gr_{\phi^*(M),\phi^*(V)}(A,\gamma)$ which corresponds to a morphism

$$(A,\gamma)^\vee \xrightarrow{\ \bar{\xi}\ } Gr_{\phi^*(M),\phi^*(V)}$$

over $S^\vee$. The latter means that the diagram

$$\begin{array}{ccc} (A,\gamma)^\vee & \xrightarrow{\bar{\xi}} & Gr_{\phi^*(M),\phi^*(V)} \\ \gamma \searrow & & \swarrow \\ & S^\vee & \end{array}$$

commutes. This implies that $(S,s)^\vee \prod_{R^\vee} Gr_{M,V}$ is isomorphic to $Gr_{\phi^*(M),\phi^*(V)}$.

Similarly, one can show that $(S,s)^\vee \prod_{R^\vee} G_{M,V}$ is isomorphic to $G_{s^*(M),s^*(V)}$.

It follows from the universality of these constructions that the isomorphisms can be chosen in such a way that the diagram

$$\begin{array}{ccc} (S,s)^\vee \prod_{R^\vee} G_{M,V} & \longrightarrow & (S,s)^\vee \prod_{R^\vee} Gr_{M,V} \\ \downarrow & & \downarrow \\ G_{s^*(M),s^*(V)} & \xrightarrow{\pi} & Gr_{\phi^*(M),\phi^*(V)} \end{array} \qquad (3)$$

commutes. Notice that the functor $(S,s)^\vee \prod_{R^\vee} -$ preserves fibred products. Since

$$\mathfrak{R}_{M,V} = G_{M,V} \prod_{Gr_{M,V}} G_{M,V},$$

the diagram (3) induces an isomorphism

$$(S,s)^\vee \prod_{R^\vee} \mathfrak{R}_{M,V} \longrightarrow \mathfrak{R}_{s^*(M),s^*(V)}.$$

Hence the assertion. ∎

6.4.2. Note. Let M and V be projective left R-modules of finite type. By 6.3.1, the presheaves $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are representable. Let $\mathcal{G}_{M,V}$ denote a k-algebra representing the presheaf $G_{M,V}$ and $\mathcal{R}_{M,V}$ a k-algebra representing the presheaf $\mathfrak{R}_{M,V}$. Then, for any unital k-algebra morphism $R \xrightarrow{s} S$, the presheaf $(S,s)^\vee \prod_{R^\vee} G_{M,V}$ is represented by the k-algebra $S \star_R \mathcal{G}_{M,V}$. Similarly, the presheaf $(S,s)^\vee \prod_{R^\vee} R_{M,V}$ is represented by the k-algebra $S \star_R \mathcal{G}_{M,V}$. It follows from 6.4.1 that there is a natural isomorphism between the k-algebras

$$S \star_R \mathcal{G}_{M,V} \longrightarrow \mathcal{G}_{s^*(M),s^*(V)} \quad \text{and} \quad S \star_R \mathcal{R}_{M,V} \longrightarrow \mathcal{R}_{s^*(M),s^*(V)}. \qquad (4)$$

6.5. Smoothness.

6.5.0. The choice of infinitesimal morphisms. In this Section, we take as *infinitesimal* morphisms the class $\mathfrak{M}_{\mathfrak{J}}$ of *radical closed immersions* (cf. II.3.8.1), which is the largest among reasonable candidates, and consider (formal) smoothness with respect to this class. Recall that a morphism $(S,s)^\vee \xrightarrow{\varphi^\vee} (T,t)^\vee$ of the category $\mathbf{Aff}_k/R^\vee$ is called a *radical closed immersion*, if the corresponding algebra morphism $T \xrightarrow{\varphi} S$ is surjective and its kernel is contained in the Jacobson radical of the algebra T.

6.5.1. Proposition. *Let M be a projective R-module and V a projective R-module of finite type. Then all presheaves and all morphisms of the canonical diagram*

$$\mathfrak{R}_{M,V} \underset{p_2}{\overset{p_1}{\rightrightarrows}} G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V} \qquad (1)$$

are formally smooth.

If M is a projective module of finite type, then the morphisms and presheaves of the diagram (1) are smooth.

Proof. Fix strict (that is surjective) epimorphism $(T,t) \xrightarrow{\varphi} (S,s)$ of R-rings, whose kernel is contained in the Jacobson radical of the algebra T.

(a) *The presheaf $G_{M,V}$ is formally smooth.*

By Yoneda's lemma, a morphism $(S,s)^\vee \longrightarrow G_{M,V}$ is uniquely defined by an element of $G_{M,V}(S,s)$, i.e. by a pair of S-module morphisms

$$S \otimes_R V \xrightarrow{g} S \otimes_R M \xrightarrow{h} S \otimes_R V \qquad (2)$$

such that $h \circ g = id$. Since M and V are projective R-modules and the algebra morphism $T \xrightarrow{\varphi} S$ is an epimorphism, the diagram (2) can be lifted to a commutative diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{g'} & T \otimes_R M & \xrightarrow{h'} & T \otimes_R V \\ \downarrow & & \downarrow & & \downarrow \\ S \otimes_R V & \xrightarrow{g} & S \otimes_R M & \xrightarrow{h} & S \otimes_R V \end{array}$$

Since V is a module of finite type and the kernel of the surjective morphism $T \xrightarrow{\varphi} S$ is contained in the Jacobson's radical of the algebra T, the fact that the composition $h \circ g$ is an isomorphism implies (by Nakayama's Lemma) that the composition $h' \circ g'$ is an isomorphism. Set $\bar{g} = g'$ and $\bar{h} = (h' \circ g')^{-1} \circ h'$. It follows that $\bar{h} \circ \bar{g} = id_{t^*(V)} = id_{T \otimes_R V}$.

Therefore, the presheaf $G_{M,V}$ is formally smooth.

(b) *The presheaf* $Gr_{M,V}$ *is formally smooth.*

A morphism $(S,s)^\vee \longrightarrow Gr_{M,V}$ is given by an element ξ of $Gr_{M,V}(S,s)$. Since the map $G_{M,V}(S,s) \longrightarrow Gr_{M,V}(S,s)$ is surjective, the element ξ is the image of an element ξ' of $G_{M,V}(S,s)$. By (a), the element ξ' can be lifted to an element, ξ'_T, of $G_{M,V}(T,t)$. The image of ξ'_T in $Gr_{M,V}(T,t)$ is a preimage of ξ.

(c) *The presheaf* $\mathfrak{R}_{M,V}$ *is formally smooth.*

A morphism $(S,s)^\vee \longrightarrow \mathfrak{R}_{M,V}$ is given by a pair of elements, (u_1,v_1), (u_2,v_2) of the set $G_{M,V}(S,s)$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2 \tag{3}$$

in addition to the relations describing $G_{M,V}(S,s)$:

$$u_1 \circ v_1 = id = u_2 \circ v_2 \tag{4}$$

(see 6.2). By (a), each of the pairs (u_1,v_1) and (u_2,v_2) can be lifted to an element resp. (u'_1,v'_1) and (u'_2,v'_2) of $G_{M,V}(T,t)$. Set $\phi' = u'_2 \circ v'_1 : T \otimes_R V \longrightarrow T \otimes_R V$.

It follows that $u'_2 = \phi' \circ u'_1$. Since $\varphi^*(\phi') = S \otimes_T \phi' = u_2 \circ v_1$ is invertible and the kernel of the algebra morphism $T \xrightarrow{\varphi} S$ is contained in the Jacobson radical of T, the morphism $t^*(V) \xrightarrow{\phi'} t^*(V)$ is invertible too. This shows that the presheaf of relations $\mathfrak{R}_{M,V}$ is formally smooth.

(d) *The presheaf morphism* $G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V}$ *is formally smooth.*

Consider the commutative diagram

$$\begin{array}{ccc} G_{M,V} & \xrightarrow{\pi_{M,V}} & Gr_{M,V} \\ g \uparrow & & \uparrow g_1 \\ (S,s)^\vee & \xrightarrow{\varphi^\vee} & (T,t)^\vee \end{array} \tag{5}$$

whose lower horizontal arrow is a *radical closed immersion*; i.e. $T \xrightarrow{\varphi} S$ is a surjective algebra epimorphism with the kernel contained in the Jacobson radical of the algebra T.

The left vertical arrow g_1 in (5) is uniquely determined by an element of $Gr_{M,V}(T,t)$, i.e. by a T-module epimorphism $T \otimes_R M = t^*(M) \xrightarrow{u} t^*(V) = T \otimes_R V$. By the same Yoneda's lemma, the left vertical arrow in (5) is uniquely determined by an element of $G_{M,V}(S,s)$, i.e. a pair of S-module morphisms

$$S \otimes_R V \xrightarrow{v'} S \otimes_R M \xrightarrow{u'} S \otimes_R V$$

such that $u' \circ v' = id$. The commutativity of the diagram (5) is equivalent to the commutativity of the diagram

$$\begin{array}{ccccc} T \otimes_R V & & T \otimes_R M & \xrightarrow{u} & T \otimes_R V \\ \varphi_V \downarrow & & \varphi_M \downarrow & & \downarrow \varphi_V \\ S \otimes_R V & \xrightarrow{v'} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{6}$$

in which the vertical arrows correspond to the surjective algebra epimorphism $T \xrightarrow{\varphi} S$, hence they are epimorphisms. Since $T \otimes_R V$ is a projective T-module and ϕ_M is a T-module epimorphism, there exists a T-module morphism $T \otimes_R V \xrightarrow{w} T \otimes_R M$ such that the diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{w} & T \otimes_R M & \xrightarrow{u} & T \otimes_R V \\ \varphi_V \downarrow & & \varphi_M \downarrow & & \downarrow \varphi_V \\ S \otimes_R V & \xrightarrow{v'} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{6$'$}$$

commutes. Since $t^*(V) = T \otimes_R V$ is a projective T-module of finite type and $u' \circ v' = id$, it follows from Nakayama's Lemma that $u \circ w$ is an isomorphism. Set $v = w \circ (u \circ w)^{-1}$. Then $u \circ v = id$ and the diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{v} & T \otimes_R M & \xrightarrow{u} & T \otimes_R V \\ \varphi_V \downarrow & & \varphi_M \downarrow & & \downarrow \varphi_V \\ S \otimes_R V & \xrightarrow{v'} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{6$''$}$$

commutes. The pair of arrows

$$T \otimes_R V \xrightarrow{v} T \otimes_R M \xrightarrow{u} T \otimes_R V \tag{7}$$

is an element of $G_{M,V}(T,t)$ which corresponds to a morphism $(T,t)^\vee \xrightarrow{\gamma} G_{M,V}$.

Since the pair (7) is a preimage of the element $T \otimes_R M \xrightarrow{u} T \otimes_R V$ of $Gr_{M,V}(T,t)$ corresponding to the morphism $(T,t)^\vee \xrightarrow{g_1} Gr_{M,V}$ (by definition of the projection $\pi_{M,V}$), we have the equality: $\pi \circ \gamma = g_1$. The commutativity of the diagram (6") means precisely that the diagram

$$\begin{array}{ccc} & G_{M,V} & \\ g \nearrow & & \nwarrow \gamma \\ (S,s)^\vee & \xrightarrow{\varphi^\vee} & (T,t)^\vee \end{array}$$

commutes.

(e) Since the morphism $\pi_{M,V}$ in the cartesian square

$$\begin{array}{ccc} \mathfrak{R}_{M,V} & \xrightarrow{p_1} & G_{M,V} \\ p_2 \downarrow & cart & \downarrow \pi \\ G_{M,V} & \xrightarrow{\pi_{M,V}} & Gr_{M,V} \end{array}$$

is formally smooth, the morphisms p_1 and p_2 are formally smooth (see 4.6).

(f) Suppose now that the R-module M is also of finite type. Then, by 6.3.1, all presheaves in the diagram (1) are locally finitely presentable over $R^\vee$. It follows from II.1.11.2(d) that all morphisms of the diagram (1) are locally finitely presentable. By the argument above, they are formally smooth; hence they are smooth. ∎

We need a slightly stronger version of a part of Proposition 6.5.1:

6.5.2. Proposition. *Let M and V be projective left R-modules of finite type. Then all morphisms of the canonical diagram*

$$\mathfrak{R}_{M,V} \underset{p_2}{\overset{p_1}{\rightrightarrows}} G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V}$$

are coverings for the smooth topology.

Proof. Let $T \xrightarrow{\xi} Gr_{M,V}$ be a morphism with affine T. Since the presheaf morphism $G_{M,V} \xrightarrow{\pi} Gr_{M,V}$ is surjective, there exists a morphism $T \xrightarrow{\xi'} Gr_{M,V}$ such that $\pi \circ \xi' = \xi$. This implies that the canonical projection $T \prod_{Gr_{M,V}} G_{M,V} \xrightarrow{\pi'} T$ has a splitting; in particular, it is surjective. Since by 6.5.1, $G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V}$ is a smooth morphism, and smooth morphisms are stable under pull-backs, the projection π' is smooth too, hence the assertion. ∎

6.6. Functoriality of Grassmannians and some of its consequences.

6.6.1. Proposition. *Let V be a projective R-module of finite type. Every R-module epimorphism $M' \xrightarrow{\phi} M$ gives rise to a morphism*

$$Gr_{M,V} \xrightarrow{Gr_{\phi,V}} Gr_{M',V},$$

which is a closed immersion.

Proof. (a) Let $R \xrightarrow{g} A$ be a k-algebra morphism and $[\xi]$ an element of $Gr_{M,V}(A,g)$ – the isomorphism class of an epimorphism $g^*(M) \xrightarrow{\xi} g^*(V)$ of A-modules. The map

$$Gr_{M,V}(A,g) \xrightarrow{Gr_{\phi,V}(A,g)} Gr_{M',V}(A,g)$$

assigns to $[\xi]$ the isomorphism class of the composition $\xi \circ g^*(\phi)$. The map is well defined, because $g^*(\phi)$ is a module epimorphism; and

$$Gr_{\phi,V} = (Gr_{\phi,V}(A,g) \mid (A,g) \in ObR\backslash Alg_k)$$

is a presheaf morphism $Gr_{M,V} \xrightarrow{Gr_{\phi,V}} Gr_{M',V}$.

(b) The claim is that the morphism $Gr_{\phi,V}$ is a closed immersion; that is, for any R-ring (S,s), the pull-back of $Gr_{\phi,V}$ along any morphism

$$(S,s)^\vee \xrightarrow{\zeta} Gr_{M',V}$$

is a closed immersion (that is a strict monomorphism) of representable presheaves.

Let $Epi_R(M,V)$ denote the presheaf $(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \longrightarrow Sets$ which maps every R-ring $(A, R \xrightarrow{g} A)$ to the set of epimorphisms $g^*(M) \longrightarrow g^*(V)$.

Let $Epi_R(M,V) \xrightarrow{\gamma_{M,V}} Gr_{M,V}$ be the natural epimorphism The R-module epimorphism $M' \xrightarrow{\phi} M$ gives rise to a presheaf monomorphism

$$Epi_R(M,V) \xrightarrow{\phi_{\mathfrak{e}}} Epi_R(M',V)$$

such that the diagram

$$\begin{array}{ccc} Epi_R(M,V) & \xrightarrow{\phi_{\mathfrak{e}}} & Epi_R(M',V) \\ \gamma_{M,V}\downarrow & & \downarrow\gamma_{M',V} \\ Gr_{M,V} & \xrightarrow{Gr_{\phi,V}} & Gr_{M',V} \end{array} \qquad (1)$$

commutes. Since representable presheaves are projective objects and $\gamma_{M',V}$ is an epimorphism, the morphism $(S,s)^\vee \xrightarrow{\zeta} Gr_{M',V}$ is the composition of a morphism

$$(S,s)^\vee \xrightarrow{\widetilde{\zeta}} Epi_R(M,V) \quad \text{and} \quad Epi_R(M,V) \xrightarrow{\gamma_{M',V}} Gr_{M',V}.$$

Let ζ' denote the composition of $\widetilde{\zeta}$ and the embedding $Epi_R(M',V) \longrightarrow \mathcal{H}_R(M',V)$. The cartesian square

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathcal{H}_R(M,V) \\ {\scriptstyle \phi_S}\downarrow & cart & \downarrow {\scriptstyle \mathcal{H}_R(\phi,V)} \\ (S,s)^\vee & \xrightarrow{\zeta'} & \mathcal{H}_R(M',V) \end{array} \qquad (2)$$

is decomposed into the commutative diagram

$$\begin{array}{ccccc} \mathfrak{X} & \longrightarrow & Epi_R(M,V) & \longrightarrow & \mathcal{H}_R(M,V) \\ {\scriptstyle \phi_S}\downarrow & cart & \downarrow & & \downarrow \\ (S,s)^\vee & \xrightarrow{\widetilde{\zeta}} & Epi_R(M',V) & \longrightarrow & \mathcal{H}_R(M',V) \end{array} \qquad (3)$$

with cartesian left square (due to the fact that the square (2) is cartesian).

Notice that the square (1) is cartesian. So that we have the diagram

$$\begin{array}{ccccc} \mathfrak{X} & \longrightarrow & Epi_R(M,V) & \xrightarrow{\gamma_{M,V}} & Gr_{M,V} \\ \downarrow & cart & \downarrow & cart & \downarrow {\scriptstyle Gr_{\phi,V}} \\ (S,s)^\vee & \xrightarrow{\widetilde{\zeta}} & Epi_R(M',V) & \xrightarrow{\gamma_{M',V}} & Gr_{M',V} \end{array} \qquad (4)$$

which consists of two cartesian squares. Therefore, their composition

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & Gr_{M,V} \\ {\scriptstyle \phi_S}\downarrow & cart & \downarrow {\scriptstyle Gr_{\phi,V}} \\ (S,s)^\vee & \xrightarrow{\zeta} & Gr_{M',V} \end{array}$$

is a cartesian square. Thus, the pull-back of the morphism $Gr_{M,V} \xrightarrow{Gr_{\phi,V}} Gr_{M',V}$ along $(S,s)^\vee \xrightarrow{\zeta} Gr_{M',V}$ coincides with the pull-back of the morphism

$$\mathcal{H}_R(M,V) \xrightarrow{\mathcal{H}_R(\phi,V)} \mathcal{H}_R(M',V)$$

along $(S,s)^\vee \xrightarrow{\zeta'} \mathcal{H}_R(M',V)$.

(b1) By hypothesis, the projective module V is of finite type. Therefore, by 5.3.2, the presheaves $\mathcal{H}_R(M',V)$ and $\mathcal{H}_R(M,V)$ are naturally isomorphic to vector fibers respectively $\mathbb{V}_R(M'\otimes_k V_R^*)$ and $\mathbb{V}_R(M\otimes_k V_R^*)$, and the morphism $\mathcal{H}_R(\phi,V)$ corresponds to the morphism $\mathbb{V}_R(\phi\otimes_k V_R^*)$. Since the map

$$\phi\otimes_k V_R^* : M'\otimes_k V_R^* \longrightarrow M\otimes_k V_R^*$$

is a module epimorphism, the corresponding morphism of tensor algebras

$$T_R(M'\otimes_k V_R^*) \longrightarrow T_R(M\otimes_k V_R^*) \tag{4}$$

representing $\mathbb{V}_R(\phi\otimes_k V_R^*)$ is a strict epimorphism of R-rings. Therefore, $\mathbb{V}_R(\phi\otimes_k V_R^*)$ is a closed immersion, which implies that its pull-back, $\mathfrak{X} \xrightarrow{\phi_S} (S,s)^\vee$, is a closed immersion. In particular, $\mathfrak{X}\simeq (A,g)^\vee$ and the morphism ϕ_S is represented by the push-forward of the strict epimorphism (4) of the R-rings. ■

6.6.2. Proposition. *Let V be a projective R-module of finite type and M an R-module of finite type. Then*

(a) The presheaf $Gr_{M,V}$ is locally of strictly cofinite type.

(b) The presheaf $Gr_{M,V}$ is locally affine for the smooth pretopology. More precisely, there exists an exact diagram

$$\mathfrak{R}^{\phi}_{M,V} \underset{\mathfrak{p}_2^{\phi}}{\overset{\mathfrak{p}_1^{\phi}}{\rightrightarrows}} G^{\phi}_{M,V} \xrightarrow{\mathfrak{p}_\phi} Gr_{M,V} \tag{5}$$

whose arrows are representable coverings for the smooth topology. The presheaves $G^{\phi}_{M,V}$ and (therefore) $\mathfrak{R}^{\phi}_{M,V} = K_2(\mathfrak{p}_\phi)$ are representable and locally of strictly cofinite type.

Proof. Since M is an R-module of finite type, there exists an R-module epimorphism $\mathcal{L}\xrightarrow{\phi} M$ with $\mathcal{L}$ a projective R-module of finite type. This epimorphism ϕ appears as a parameter in the diagram (5). The diagram itself is defined (uniquely up to isomorphism) via the diagram

$$\begin{array}{ccccc}
\mathfrak{R}^{\phi}_{M,V} & \underset{\mathfrak{p}_2^{\phi}}{\overset{\mathfrak{p}_1^{\phi}}{\rightrightarrows}} & G^{\phi}_{M,V} & \xrightarrow{\mathfrak{p}_\phi} & Gr_{M,V} \\
\downarrow & cart & \downarrow & cart & \downarrow Gr_{\phi,V} \\
\mathfrak{R}_{\mathcal{L},V} & \underset{p_2}{\overset{p_1}{\rightrightarrows}} & G_{\mathcal{L},V} & \xrightarrow{\pi} & Gr_{\mathcal{L},V}
\end{array} \tag{6}$$

with cartesian squares. By 6.5.1, the presheaves of the bottom of the diagram are smooth and, by 6.5.2, all three arrows of the bottom of the diagram are coverings for the smooth pretopology. It follows from 1.3.1 and the fact that the presheaves $G_{\mathcal{L},V}$ and $\mathfrak{R}_{\mathcal{L},V}$ are representable that these morphisms are also representable. By 6.6.1, the right vertical arrow of the diagram (6) is a closed immersion. Therefore, since the squares of the diagram are cartesian, the remaining two arrows are closed immersions. Since the presheaves of the lower row of (6) are locally finitely copresentable, this means that presheaves of the upper row are locally of strictly cofinite type. ■

6.6.3. Remark. (a) One can see that the presheaf $G^{\phi}_{M,V}$ is naturally identified with the subpresheaf of $G_{\mathcal{L},V}$ which assigns to every R-ring (S,s) the set of all elements of $G_{\mathcal{L},V}(S,s)$ of the form $s^*(V) \xrightarrow{\mathfrak{v}} s^*(\mathcal{L}) \xrightarrow{\mathfrak{u}\circ s^*(\phi)} s^*(V)$. So that we have a map

$$G^{\phi}_{M,V}(S,s) \longrightarrow G_{M,V}(S,s)$$

which assigns to every element $s^*(V) \xrightarrow{\mathfrak{v}} s^*(\mathcal{L}) \xrightarrow{\mathfrak{u}\circ s^*(\phi)} s^*(V)$ of $G^{\phi}_{M,V}(S,s)$ the element $s^*(V) \xrightarrow{s^*(\phi)\circ\mathfrak{v}} s^*(\mathcal{L}) \xrightarrow{\mathfrak{u}} s^*(V)$ of $G_{M,V}(S,s)$. This map is surjective, because $s^*(V)$ is a projective S-module and $s^*(\mathcal{L}) \xrightarrow{s^*(\phi)} s^*(M)$ is an epimorphism of S-modules. So that we have a presheaf epimorphism $G^{\phi}_{M,V} \xrightarrow{\lambda_\phi} G_{M,V}$ which determines an epimorphism of diagrams

$$\begin{array}{ccccc}
\mathfrak{R}^{\phi}_{M,V} & \overset{\mathfrak{p}^{\phi}_1}{\underset{\mathfrak{p}^{\phi}_2}{\rightrightarrows}} & G^{\phi}_{M,V} & \xrightarrow{\mathfrak{p}_\phi} & Gr_{M,V} \\
\lambda^{\mathfrak{r}}_{\phi}\downarrow & & \downarrow\lambda_\phi & & \downarrow id \\
\mathfrak{R}_{M,V} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & G_{M,V} & \xrightarrow{\pi} & Gr_{M,V}
\end{array} \qquad (7)$$

(b) If M is a projective R-module of finite type, then the epimorphism $\mathcal{L} \xrightarrow{\phi} M$ splits which gives an isomorphism between $\mathcal{L}$ and $M \oplus Ker(\phi)$. It is easy to see that, in this case, $G^{\phi}_{M,V}$ is naturally isomorphic to

$$G_{M,V} \times \mathcal{H}_R(V, Ker(\phi)) \simeq G_{M,V} \times \mathbb{V}_R(V \otimes_k Ker(\phi)^\vee).$$

This shows, in particular, that λ_ϕ is an isomorphism iff ϕ is an isomorphism.

6.7. Affine Zariski subschemes of a Grassmannian. Noncommutative Grassmannians have affine Zariski subschemes (constructed below), which, being restricted to commutative algebras, produce a Zariski affine cover of the corresponding commutative Grassmannian (when the latter is not empty).

Fix a projective R-module V and an R-module morphism $V \xrightarrow{\phi} M$. For any R-ring $(S, R \xrightarrow{s} S)$, consider the set $F_{\phi;M,V}(S,s)$ of equivalence classes of all morphisms $s^*(M) \xrightarrow{v} V'$ such that $v \circ s^*(\phi)$ is an isomorphism. Here two morphisms, $s^*(M) \xrightarrow{v} V'$ and $s^*(M) \xrightarrow{v'} V''$, are equivalent iff $v' = \psi \circ v$ for some S-module isomorphism $V' \xrightarrow{\psi} V''$.

6.7.1. Proposition. *Let* $V \xrightarrow{\phi} M$ *be an R-module morphism.*
(a) The map $(S,s) \longmapsto F_{\phi;M,V}(S,s)$ *is naturally extended to a presheaf*

$$F_{\phi;M,V} : (\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \longrightarrow \mathbf{Sets},$$

which is a subpresheaf of the presheaf $Gr_{M,V}$.
(b) Suppose that the R-module V is projective of finite type. Then the presheaf $F_{\phi;M,V}$ *and the presheaf monomorphism* $F_{\phi;M,V} \xrightarrow{\mathfrak{j}_\phi} Gr_{M,V}$ *are representable.*

Proof. (a) (i) Fix an object $(S, R \xrightarrow{s} S)$ of $R\backslash Alg_k$. If $s^*(M) \xrightarrow{v} V'$ belongs to $F_{\phi;M,V}(S,s)$, i.e. $v \circ s^*(\phi)$ is an isomorphism, then for any morphism $(S,s) \xrightarrow{h} (T,t)$, the composition $h^*(v) \circ h^*s^*(\phi)$ is an isomorphism, and $h^*s^*(\phi) \simeq t^*(\phi)$. There is a natural morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$.

(ii) Note that one can identify $F_{\phi;M,V}(S,s)$ with the set of all S-module epimorphisms $s^*(M) \xrightarrow{v} s^*(V)$ such that $v \circ s^*(\phi) = id_{s^*(V)}$.

In fact, if $s^*(M) \xrightarrow{v'} V'$ is such that $w = v' \circ s^*(\phi) : s^*(V) \longrightarrow V'$ is an isomorphism, then $v = w^{-1} \circ v'$ is a morphism $s^*(M) \longrightarrow s^*(V)$ which is equivalent to v' and has the required property: $v \circ s^*(\phi) = id_{s^*(V)}$. Evidently, such morphism v is unique.

(iii) One of the consequences of the observation (ii) is that the canonical presheaf morphism $F_{\phi;M,V} \xrightarrow{\mathfrak{j}_\phi} Gr_{M,V}$ is a monomorphism.

In fact, the morphism $\mathfrak{j}_\phi(S,s)$ mapping the elements $\mathfrak{u}_1$, $\mathfrak{u}_2$ of $F_{\phi;M,V}(S,s)$ to the same element of $Gr_{M,V}(S,s)$ means that $\mathfrak{u}_2 = \mathfrak{t} \circ \mathfrak{u}_1$ for some automorphism $\mathfrak{t}$ of the S-module $s^*(V)$. So that $\mathfrak{t} = \mathfrak{t} \circ (\mathfrak{u}_1 \circ s^*(\phi)) = (\mathfrak{t} \circ (\mathfrak{u}_1) \circ s^*(\phi) = id_{s^*(V)}$.

(b) There are two maps,

$$Hom_S(s^*(M), s^*(V)) \underset{\beta_S}{\overset{\alpha_S}{\rightrightarrows}} Hom_S(s^*(V), s^*(V)),$$

defined by $v \overset{\alpha_S}{\longmapsto} v \circ s^*(\phi)$, $\quad v \overset{\beta_S}{\longmapsto} id_{s^*(V)}$. The maps α_S and β_S are functorial in (S, s), hence they define morphisms, resp. α and β, from the presheaf

$$(S, s) \longmapsto Hom_S(s^*(M), s^*(V)) \simeq Hom_R(M, s_*s^*(V)) = Hom_R(M, S \otimes_R V) \qquad (1)$$

to the presheaf

$$(S, s) \longmapsto Hom_S(s^*(V), s^*(V)) \simeq Hom_R(V, s_*s^*(V)) = Hom_R(V, S \otimes_R V). \qquad (2)$$

(iv) Suppose now that V is a projective R-module of finite type. Then, by 5.3.2, the presheaf (1) is representable by the vector fiber $\mathbb{V}_R(M \otimes_k V_R^*)$ of the left R^e-module $M \otimes_k V_R^*$, and the presheaf (2) is representable by the vector fiber $\mathbb{V}_R(V \otimes_k V_R^*)$ of the projective R^e-module of finite type $V \otimes_k V_R^*$. Let α' and β' be morphisms from $\mathbb{V}_R(M \otimes_k V_R^*)$ to $\mathbb{V}_R(V \otimes_k V_R^*)$ corresponding to resp. α and β. The presheaf $F_{\phi;M,V}$ is the kernel of the pair (α, β), hence it is representable by the kernel, $\mathbf{F}_{\phi;M,V}$, of the pair (α', β').

(v) *The presheaf morphism* $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ *is representable*; i.e. for any R-ring $(S, R \xrightarrow{s} S)$ and any presheaf morphism $(S, s)^\vee \longrightarrow Gr_{M,V}$, the presheaf

$$(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \longrightarrow \mathbf{Sets}, \quad (T, t) \longmapsto F_{\phi;M,V}(T, t) \prod_{Gr_{M,V}(T,t)} (S, s)^\vee(T, t)$$

is representable by an affine subscheme of $(S, s)^\vee$.

In fact, by the Yoneda's lemma, any morphism $(S, s)^\vee \longrightarrow Gr_{M,V}$ is uniquely determined by an element of $Gr_{M,V}(S, s)$, i.e. by the equivalence class, $[v]$, of a locally split epimorphism $s^*(M) \xrightarrow{v} V'$. The corresponding map $(S, s)^\vee(T, t) \longrightarrow Gr_{M,V}(T, t)$ sends any morphism $(S, s) \xrightarrow{f} (T, t)$ to the equivalence class $[f^*(v)]$. The fiber product $F_{\phi;M,V}(T, t) \prod_{Gr_{M,V}(T,t)} (S, s)^\vee(T, t)$ consists of all pairs (w, γ), where $\gamma \in (S, s)^\vee(T, t)$ and $[T \otimes_k M \xrightarrow{w} T \otimes_k V]$ are such that $w \circ (T \otimes_k \phi) = id_{T \otimes_k V}$ and $w = \gamma^*(v)$.

Since v and ϕ here are fixed, the fiber product

$$F_{\phi;M,V}(T, t) \prod_{Gr_{M,V}(T,t)} (S, s)^\vee(T, t)$$

can be identified with the set of all morphisms $(S, s) \xrightarrow{\gamma} (T, t)$ of $R\backslash Alg_k$ (i.e. k-algebra morphisms $S \xrightarrow{\gamma} T$ satisfying $t = \gamma \circ s$) such that $\gamma^*(v \circ (T \otimes_k \phi)) = id_{T \otimes_k V}$. In other words, this fiber product is identified with the kernel of the pair of morphisms

$$(S, s)^\vee(T, t) \underset{\beta_{(T,t)}}{\overset{\alpha_{(T,t)}}{\rightrightarrows}} Hom_T(T \otimes_R V, T \otimes_R V)$$

defined by

$$\beta_{(T,t)} : \gamma \longmapsto id_{t^*(V)} = id_{T\otimes_k V}, \quad \alpha_{(T,t)} : \gamma \longmapsto \gamma^*(v) \circ (T \otimes_k \phi).$$

The morphisms $\beta_{(T,t)},\ \alpha_{(T,t)}$ are functorial in (T,t), and

$$Hom_T(T \otimes_k V, T \otimes_k V) \simeq \mathbb{V}_R(V \otimes_k V_R^*)(T,t).$$

Hence the morphisms $\beta = (\beta_{(T,t)}),\ \alpha = (\alpha_{(T,t)})$ define a pair of arrows

$$(S,s)^\vee \underset{\beta'}{\overset{\alpha'}{\rightrightarrows}} \mathbb{V}_R(V \otimes_k V_R^*),$$

and the presheaf

$$(T,t) \longmapsto F_{\phi;M,V}(T,t) \prod_{Gr_{M,V}(T,t)} (S,s)^\vee(T,t)$$

is representable by the kernel of the pair (α', β'). ∎

6.7.2. Proposition. *Let* $V \xrightarrow{\phi} M$ *be an R-module morphism, and let V be a projective R-module of finite type. If M is a finitely presentable R-module (resp. an R-module of finite type), then* $F_{\phi;M,V}$ *is locally finitely presentable (resp. locally of finite type) over R.*

Proof. By the part (iv) of the argument of 6.7.1, the presheaf $F_{\phi;M,V}$ is isomorphic to the kernel of a pair of arrows $\mathbb{V}_R(M \otimes_k V_R^*) \rightrightarrows \mathbb{V}_R(V \otimes_k V_R^*)$ over R. By 5.3.4, $\mathbb{V}_R(V \otimes_k V_R^*)$ is locally finitely presentable over R, and $\mathbb{V}_R(M \otimes_k V_R^*)$ is locally finitely presentable (resp. locally of finite type), if the R-module M is finitely presentable (resp. of finite type). The kernel of a pair of morphisms between locally finitely presentable presheaves (resp. presheaves locally of finite type) is locally finitely presentable (resp. locally of finite type); hence the assertion. ∎

6.7.2.1. Corollary. *Suppose M and V are projective R-modules of finite type. Then the canonical morphism* $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ *is locally finitely presentable.*

Proof. By 6.3.1, $Gr_{M,V}$ is locally finitely presentable, and by 6.7.2, $F_{\phi;M,V}$ has the same property. By II.1.11.2(d), the morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is locally finitely presentable. ∎

6.7.3. Proposition. *Let M be a projective R-module and V a projective R-module of finite type. Then the presheaf* $F_{\phi;M,V}$ *is formally smooth and the canonical morphism* $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ *is a formally open immersion.*

If M is a projective module of finite type, then the presheaf $F_{\phi;M,V}$ is smooth and the morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is an open immersion.

Proof. (a) Let M be a projective R-module. Since by 6.5.1, $Gr_{M,V}$ is formally smooth and the composition of formally smooth morphisms is formally smooth, the formal smoothness of $F_{\phi;M,V}$ is a consequence of the formal smoothness of the canonical morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$.

Fix an R-ring surjective epimorphism $(T,t) \xrightarrow{\alpha} (S,s)$ whose kernel is contained in the Jacobson radical of T and which is a part of a commutative diagram

$$\begin{array}{ccc} (S,s)^\vee & \longrightarrow & F_{\phi;M,V} \\ \alpha \downarrow & & \downarrow \\ (T,t)^\vee & \longrightarrow & Gr_{M,V} \end{array} \tag{1}$$

By Yoneda's lemma, the morphism $(S,s)^\vee \longrightarrow F_{\phi;M,V}$ in (1) is uniquely defined by an element of $F_{\phi;M,V}(S,s)$, i.e. by an S-module morphism $s^*(M) \xrightarrow{u'} s^*(V)$ such that $u' \circ s^*(\phi) = id_{s^*(V)}$, and the morphism $(T,t)^\vee \longrightarrow Gr_{M,V}$ is uniquely determined by an element of $Gr_{M,V}(T,t)$. The commutativity of (1) is equivalent to the commutativity of the diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{t^*(\phi)} & T \otimes_R M & \xrightarrow{u} & T \otimes_R V \\ \alpha_V \downarrow & & \alpha_M \downarrow & & \downarrow \alpha_V \\ S \otimes_R V & \xrightarrow{s^*(\phi)} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{2}$$

whose vertical arrows are induced by the ring epimorphism $T \xrightarrow{\alpha} S$, hence they are epimorphisms. Since $T \otimes_R V$ is a projective T-module of finite type, and the kernel of $T \xrightarrow{\alpha} S$ is contained in the Jacobson radical of T, it follows from Nakayama's lemma that $u \circ t^*(\phi)$ is an isomorphism. Set $\widetilde{u} = (u \circ t^*(\phi))^{-1} \circ u$. Then $\widetilde{u} \circ t^*(\phi) = id_{t^*(V)}$ and the diagram

$$\begin{array}{ccccc} T \otimes_R V & \xrightarrow{t^*(\phi)} & T \otimes_R M & \xrightarrow{\widetilde{u}} & T \otimes_R V \\ \alpha_V \downarrow & & \alpha_M \downarrow & & \downarrow \alpha_V \\ S \otimes_R V & \xrightarrow{s^*(\phi)} & S \otimes_R M & \xrightarrow{u'} & S \otimes_R V \end{array} \tag{3}$$

commutes. The pair of arrows

$$T \otimes_R V \xrightarrow{t^*(\phi)} T \otimes_R M \xrightarrow{\widetilde{u}} T \otimes_R V \tag{4}$$

is an element of $F_{\phi;M,V}(T,t)$ which corresponds to a morphism $(T,t)^\vee \xrightarrow{\gamma} F_{\phi;M,V}$. It follows from the construction that adjoining the morphism γ to the diagram (1) makes a commutative diagram. This shows that the canonical monomorphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is formally smooth, hence a formally open immersion.

(b) Suppose now that M is a projective R-module of a finite type. Then by 6.7.2, the presheaf $F_{\phi;M,V}$ is locally finitely presentable over R, and, by 6.7.2.1, the morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is locally finitely presentable. Therefore, the presheaf $F_{\phi;M,V}$ is smooth and the presheaf morphism $F_{\phi;M,V} \longrightarrow Gr_{M,V}$ is an open immersion. ■

6.7.4. Functorialities. Let $\mathcal{L} \xrightarrow{\varphi} M$ be an R-module epimorphism, V a projective R-module and $V \xrightarrow{\psi} M$ an R-module morphism. Since V is projective and $\mathcal{L} \xrightarrow{\varphi} M$ is an epimorphism, there exists an R-module morphism $V \xrightarrow{\phi} \mathcal{L}$ such that $\varphi \circ \phi = \psi$.

With this data, we associate a commutative diagram

$$\begin{array}{ccc} F_{\psi;M,V} & \xrightarrow{\mathfrak{j}_\psi} & Gr_{M,V} \\ {\scriptstyle F_{\varphi,V}}\big\downarrow & & \big\downarrow {\scriptstyle Gr_{\varphi,V}} \\ F_{\phi;\mathcal{L},V} & \xrightarrow{\mathfrak{j}_\phi} & Gr_{\mathcal{L},V} \end{array} \tag{5}$$

6.7.4.1. Lemma. *The square (5) is cartesian.*

Proof. Let (S,s) be an R-ring and

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\zeta} & Gr_{M,V} \\ {\scriptstyle \xi}\big\downarrow & & \big\downarrow {\scriptstyle Gr_{\varphi,V}} \\ F_{\phi;\mathcal{L},V} & \xrightarrow{\mathfrak{j}_\phi} & Gr_{\mathcal{L},V} \end{array} \tag{6}$$

a commutative diagram. The upper horizontal arrow is determined by an element of $Gr_{M,V}(S,s)$ represented by an S-module epimorphism $s^*(M) \xrightarrow{\widehat{\zeta}} s^*(V)$ and the left vertical arrow is determined by an element of $F_{\phi;\mathcal{L},V}(S,s)$, that is by an S-module epimorphism $s^*(\mathcal{L}) \xrightarrow{\widehat{\xi}} s^*(V)$ such that $\widehat{\xi} \circ s^*(\phi) = id_{s^*(V)}$. The commutativity of the diagram (6) means that the morphism $s^*(\mathcal{L}) \xrightarrow{\widehat{\xi}} s^*(V)$ is isomorphic to the composition $s^*(\mathcal{L}) \xrightarrow{\widehat{\zeta}\circ s^*(\varphi)} s^*(V)$. This means that the composition

$$\widehat{\zeta} \circ s^*(\varphi) \circ s^*(\phi) = \widehat{\zeta} \circ s^*(\varphi \circ \phi) = \widehat{\zeta} \circ s^*(\psi)$$

is an isomorphism. Therefore, the morphism $(S,s)^\vee \xrightarrow{\lambda} F_{\psi;M,V}$ determined by the element $(\widehat{\zeta} \circ s^*(\psi))^{-1} \circ \widehat{\zeta}$ of $F_{\psi;M,V}$ satisfies the equations $F_{\varphi,V} \circ \lambda = \xi$ and $\mathfrak{j}_\psi \circ \lambda = \zeta$.

Any of these equations determines the morphism $(S,s)^\vee \xrightarrow{\lambda} F_{\psi;M,V}$ uniquely, because both $\mathfrak{j}_\psi$ and $F_{\varphi,V}$ are monomorphisms. Since every presheaf of sets is the colimit of representable presheaves, this shows that the square (5) is cartesian. ■

6.7.4.2. Corollary. *Let $\mathcal{L} \xrightarrow{\varphi} M$ be an R-module epimorphism, V a projective R-module of finite type, $V \xrightarrow{\psi} \mathcal{L}$ an R-module morphism, and $\phi = \varphi \circ \psi$. Then the canonical morphism of representable presheaves*

$$F_{\psi;M,V} \xrightarrow{F_{\varphi,V}} F_{\phi;M,V} \tag{7}$$

is a closed immersion; that is $F_{\varphi,V}$ is the image of a strict epimorphism of algebras representing these presheaves.

Proof. By 6.6.1, the right vertical arrow of the square (5) above is a closed immersion. By 6.7.4, the square (5) is cartesian, and closed immersions are stable under pull-backs. Therefore, the left vertical arrow of the square (5) – the morphism (7), is a closed immersion. By 6.7.1, the presheaves $F_{\psi;M,V}$ and $F_{\phi;M,V}$ are representable; and closed immersions of representable presheaves on $\mathbf{Aff}_k/R^\vee$ are represented by strict epimorphisms of the corresponding R-rings. ■

6.7.4.3. Proposition. *Let M be an R-module of finite type and V a projective R-module of finite type. Then, for any morphism $V \xrightarrow{\psi} M$, the canonical embedding $F_{\psi;M,V} \xrightarrow{\mathfrak{j}_\psi} Gr_{M,V}$ is an open immersion.*

Proof. Since the R-module M is of finite type, there is an R-module epimorphism $\mathcal{L} \xrightarrow{\varphi} M$ with $\mathcal{L}$ a projective R-module of finite type. Since the R-module V is projective, there exists an R-module morphism $V \xrightarrow{\phi} \mathcal{L}$ such that $\varphi \circ \phi = \psi$. By 6.7.4.1, the square

$$\begin{array}{ccc} F_{\psi;M,V} & \xrightarrow{\mathfrak{j}_\psi} & Gr_{M,V} \\ {\scriptstyle F_{\varphi,V}}\big\downarrow & cart & \big\downarrow {\scriptstyle Gr_{\varphi,V}} \\ F_{\phi;\mathcal{L},V} & \xrightarrow{\mathfrak{j}_\phi} & Gr_{\mathcal{L},V} \end{array}$$

is cartesian and, by 6.7.3, its lower horizontal arrow is an open immersion. Therefore, its upper horizontal arrow, the morphism $F_{\psi;M,V} \xrightarrow{\mathfrak{j}_\psi} Gr_{M,V}$ is an open immersion. ■

6.7.5. Projective completion of a vector bundle. Let $M' = M \oplus V$, and let $V \xrightarrow{j_V} M'$ be the canonical morphism. The presheaf $F_{j_V;M',V}$ is isomorphic to the presheaf, which assigns to any R-ring $(S, R \xrightarrow{s} S)$ the set $Hom_S(s^*(M), s^*(V)$ (see (ii) and (b) in the argument of 6.7.1). The latter presheaf is representable by the vector bundle $\mathbb{V}_R(M \otimes_k V_R^*)$. If the R-modules M and V (hence M') are projective of finite type, then, by 6.7.3, we have a canonical affine open immersion $\mathbb{V}_R(M \otimes_k V_R^*) \longrightarrow Gr_{_{M',V}}$.

6.7.5.1. Projective completion. In particular, if $R = k$, than taking $V = R^1 = k^1$ and using the fact that $M \otimes_k k^1 \simeq M$, we obtain a canonical embedding

$$\mathbb{V}_R(M) \longrightarrow \mathbb{P}_{M \oplus k^1}. \tag{8}$$

If M is a k-module of finite type, then, by 6.7.4.3, the morphism (8) is an open immersion. The projective space $\mathbb{P}_{M \oplus k^1}$ can be regarded (like in the commutative case) as the *projective completion* of the vector bundle $\mathbb{V}_k(M)$.

6.7.6. Zero section and the hyperplane at infinity. Let M be an R-module and V a projective R-module of finite type. Set $M' = M \oplus V$, and let $V \xleftarrow{p_V} M \oplus V \xrightarrow{p_M} M$ be canonical projections. The projection p_V determines a canonical section $R^\vee \longrightarrow Gr_{M',V}$ which (following the commutative tradition) will be called the *zero section.* By 6.6.1, the projection $M' \xrightarrow{p_M} M$ induces a closed immersion

$$Gr_{_{M,V}} \longrightarrow Gr_{_{M',V}}$$

called the *hyperplane at infinity.*

6.8. Grassmannians are separated. Recall that a presheaf of sets X on a category C is *separated* if the diagonal morphism $X \xrightarrow{\Delta_X} X \times X$ is a closed immersion (cf. 2.2).

Here C is the category $\mathbf{Aff}_k/R^\vee$ of affine k-schemes over $R^\vee$ for an associative unital k-algebra R. In other words, $C = (R\backslash Alg_k)^{op}$.

6.8.1. Proposition. *For any pair* M, V *of projective* R*-modules of finite type, the presheaf* $(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \xrightarrow{Gr_{_{M,V}}}$ **Sets** *is separated.*

Proof. Let $(S, R \xrightarrow{s} S)$ be an R-ring, and let $(S, s)^\vee \underset{u_2}{\overset{u_1}{\rightrightarrows}} Gr_{_{M,V}}$ be a pair of morphisms over R. The claim is that the kernel of the pair (u_1, u_2) is representable by a closed immersion of affine schemes.

Let $s^*(M) \xrightarrow{\xi_i} s^*(V)$ be an epimorphism corresponding to u_i, $i = 1, 2$. Since $s^*(V)$ is a projective S-module, there exists an S-module morphism $s^*(V) \xrightarrow{\nu_i} s^*(M)$ such that $\xi_i \nu_i = id_{s^*(V)}$. Set $p_i = \nu_i \xi_i$. Then the diagram

$$s^*(M) \underset{p_i}{\overset{id}{\rightrightarrows}} s^*(M) \xrightarrow{\xi_i} s^*(V)$$

is exact. Consider the pairs of morphisms

$$s^*(M) \underset{\xi_1 p_2}{\overset{\xi_1}{\rightrightarrows}} s^*(V) \quad \text{and} \quad s^*(M) \underset{\xi_2 p_1}{\overset{\xi_2}{\rightrightarrows}} s^*(V). \tag{1}$$

There exists a universal R-ring morphism $(S, s) \xrightarrow{\psi} (T, t)$ such that the image by ψ^* of each of the pairs (1) belongs to the diagonal. We leave to the reader arguing that the morphism ψ is a strict epimorphism. ■

7. Noncommutative flag varieties.

7.1. Definition. Let M be an R-module and $\bar{V} = (V_i \mid 1 \le i \le n)$ projective R-modules. For any R-ring (S, s), we denote by $\mathfrak{F}\ell_{M,\bar{V}}(S, s)$ the subset of $\prod_{1 \le i \le n} Gr_{M,V_i}(S, s)$ formed by all n-tuples $([u_1], \dots, [u_n],\ s^*(M) \xrightarrow{u_i} s^*(V_i)$, such that $Ker(u_j) \subseteq Ker(u_i)$ if $j \ge i$. This means that $u_i = \xi_i \circ u_{i+1}$ for all $1 \le i \le n-1$.

7.2. Proposition. *(a) The map* $(S, s) \longmapsto \mathfrak{F}\ell_{M,\bar{V}}(S, s)$ *is a presheaf*

$$(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \xrightarrow{\ \mathfrak{F}\ell_{M,\bar{V}}\ } Sets,$$

and there is a natural presheaf monomorphism

$$\mathfrak{F}\ell_{M,\bar{V}} \longrightarrow \prod_{1 \le i \le n} Gr_{M,V_i} \tag{2}$$

(b) If the projective modules $V_i,\ 1 \le i \le n$, *are of finite type, then the morphism (2) is a closed immersion.*

Proof. (a) Straightforward.

(b) The second assertion will be proven in 10.1 using some simple considerations, which involve *generic* flag varieties. ■

7.2.1. Note. If $n = 1$, that is $\bar{V} = (V)$ for some projective R-module V, then the flag variety $\mathfrak{F}\ell_{M,\bar{V}}$ coincides with the Grassmannian $Gr_{M,V}$.

7.3. An alternative description and a canonical cover.

7.3.1. Lemma. *Let* (S, s) *be an* R*-ring and* $\bar{V} = (V_i \mid 1 \le i \le n)$ *projective* R*-modules. Every element of* $\mathfrak{F}\ell_{M,\bar{V}}(S, s)$ *is represented by a chain of* S*-module epimorphisms*

$$s^*(M) \xrightarrow{\xi_n} s^*(V_n) \xrightarrow{\xi_{n-1}} s^*(V_{n-1}) \xrightarrow{\xi_{n-2}} \dots \xrightarrow{\xi_1} s^*(V_1).$$

It defines the same element of $\mathfrak{Fl}_{M,\bar V}(S,s)$ *as another chain of epimorphisms,*

$$s^*(M) \xrightarrow{\zeta_n} s^*(V_n) \xrightarrow{\zeta_{n-1}} s^*(V_{n-1}) \xrightarrow{\zeta_{n-2}} \dots \xrightarrow{\zeta_1} s^*(V_1),$$

iff these chains are isomorphic.

Proof. In fact, by definition of the flag variety, these chains define the same element of $\mathfrak{Fl}_{M,\bar V}(S,s)$ iff there exist isomorphisms $s^*(V_i) \xrightarrow{\psi_i} s^*(V_i)$, $1 \le i \le n$, making diagrams

$$\begin{array}{ccccccc} s^*(M) & \xrightarrow{\xi_n} & \cdots & \xrightarrow{\xi_m} & s^*(V_m) & \xrightarrow{\xi_{m-1}} & s^*(V_{m-1}) \\ id\downarrow & & & & & & \wr\downarrow \psi_{m-1} \\ s^*(M) & \xrightarrow{\zeta_n} & \cdots & \xrightarrow{\zeta_m} & s^*(V_m) & \xrightarrow{\zeta_{m-1}} & s^*(V_{m-1}) \qquad 2 \le m \le n, \end{array} \tag{1}$$

commute. In other words, for every $2 \le m \le n$, we have the equalities

$$\zeta_m \circ \cdots \circ \zeta_n = \psi_m \circ \xi_m \circ \cdots \circ \xi_n \quad \text{and}$$
$$\zeta_{m-1} \circ \zeta_m \circ \cdots \circ \zeta_n = \psi_{m-1} \circ \xi_{m-1} \circ \xi_m \circ \cdots \circ \xi_n$$

So that

$$\zeta_{m-1} \circ \psi_m \circ (\xi_m \circ \cdots \circ \xi_n) = \psi_{m-1} \circ \xi_{m-1} \circ (\xi_m \circ \cdots \circ \xi_n).$$

Since $\xi_m \circ \cdots \circ \xi_n$ is an epimorphism, the latter equality implies that

$$\zeta_{m-1} \circ \psi_m = \psi_{m-1} \circ \xi_{m-1}.$$

That is the diagram

$$\begin{array}{ccccccccc} s^*(M) & \xrightarrow{\xi_n} & s^*(V_n) & \xrightarrow{\xi_{n-1}} & s^*(V_{n-1}) & \xrightarrow{\xi_{n-2}} & \dots & \xrightarrow{\xi_1} & s^*(V_1) \\ id\downarrow & & \psi_n\downarrow\wr & & \psi_{n-1}\downarrow\wr & & & & \wr\downarrow\psi_1 \\ s^*(M) & \xrightarrow{\zeta_n} & s^*(V_n) & \xrightarrow{\zeta_{n-1}} & s^*(V_{n-1}) & \xrightarrow{\zeta_{n-2}} & \dots & \xrightarrow{\zeta_1} & s^*(V_1) \end{array}$$

commutes. ■

7.3.2. A canonical cover. Thus, we have a surjective map

$$Epi_S(s^*(M), s^*(V_n)) \times \cdots \times Epi_S(s^*(V_2), s^*(V_1)) \longrightarrow \mathfrak{Fl}_{M,\bar V}(S,s)$$

which assigns to every chain of epimorphisms its isomorphism class. This map is functorial in (S,s). Since the S-modules $s^*(V_i)$, $1 \le i \le n$, are projective, the obvious maps

$$G_{V_i,V_{i-1}}(S,s) \longrightarrow Epi_S(s^*(V_i), s^*(V_{i-1}))$$

are surjective for $2 \le i \le n+1$. Here $V_{n+1} = M$. They are also functorial in (S,s).

Thus, we obtain an epimorphism of presheaves

$$\prod_{1\le i\le n} G_{V_{i+1},V_i} \xrightarrow{\pi_{M,\bar V}} \mathfrak{Fl}_{M,\bar V}. \tag{2}$$

7.3.3. Relations. The kernel pair

$$\mathfrak{R}_{M,\bar V} = Ker_2(\pi_{M,\bar V}) \overset{p^1_{M,\bar V}}{\underset{p^2_{M,\bar V}}{\rightrightarrows}} \prod_{2\le i\le n+1} G_{V_i,V_{i-1}} \tag{3}$$

of the cover (2) admits the following description.

For every R-ring (S,s), the subset $\mathfrak{R}_{M,\bar V}(S,s)$ of

$$\Big(\prod_{1\le i\le n+1} G_{V_i,V_{i-1}}(S,s)\Big)\prod\Big(\prod_{1\le i\le n+1} G_{V_i,V_{i-1}}(S,s)\Big)$$

is formed by the sequences of pairs of S-module morphisms

$$s^*(V_m) \xrightarrow{\mathfrak{v}_m} s^*(V_{m+1}) \xrightarrow{\mathfrak{u}_m} s^*(V_m), \quad 1 \le m \le n,$$
$$s^*(V_m) \xrightarrow{\widetilde{\mathfrak{v}}_m} s^*(V_{m+1}) \xrightarrow{\widetilde{\mathfrak{u}}_m} s^*(V_m), \quad 1 \le m \le n,$$

satisfying the following relations:

$$\begin{aligned}
&\mathfrak{u}_m \circ \mathfrak{v}_m = id_{s^*(V_m)} = \widetilde{\mathfrak{u}}_m \circ \widetilde{\mathfrak{v}}_m \\
&\mathfrak{u}_m \circ \ldots \circ \mathfrak{u}_n = (\mathfrak{u}_m \circ \ldots \circ \mathfrak{u}_n) \circ (\widetilde{\mathfrak{v}}_n \circ \ldots \circ \widetilde{\mathfrak{v}}_1) \circ (\widetilde{\mathfrak{u}}_m \circ \ldots \circ \widetilde{\mathfrak{u}}_n) \\
&\widetilde{\mathfrak{u}}_m \circ \ldots \circ \widetilde{\mathfrak{u}}_n = (\widetilde{\mathfrak{u}}_m \circ \ldots \circ \widetilde{\mathfrak{u}}_n) \circ (\mathfrak{v}_n \circ \ldots \circ \mathfrak{v}_1) \circ (\mathfrak{u}_m \circ \ldots \circ \mathfrak{u}_n) \\
&\text{for} \quad 1 \le m \le n.
\end{aligned} \tag{4}$$

The first line of (4) reflects the fact that $(\mathfrak{v}_m, \mathfrak{u}_m)$ and $(\widetilde{\mathfrak{v}}_m, \widetilde{\mathfrak{u}}_m)$ are elements of $G_{V_{m+1},V_m}(S,s)$ for $1 \le m \le n$. The remaining two lines express the fact that the pair of arrows (3) equalizes the projection (2).

In fact, by the definition of the map

$$\prod_{1\le m\le n+1} G_{V_m,V_{m-1}}(S,s) \xrightarrow{\pi_{M,\bar V}} \mathfrak{Fl}_{M,\bar V}(S,s),$$

it sends the elements $(\mathfrak{v}_m, \mathfrak{u}_m \mid 1\le m\le n)$ and $(\widetilde{\mathfrak{v}}_m, \widetilde{\mathfrak{u}}_m \mid 1\le m\le n)$ to the same element of $\mathfrak{Fl}_{M,\bar V}(S,s)$ iff there exists a commutative diagram

$$\begin{array}{ccccccccc}
s^*(M) & \xrightarrow{\mathfrak{u}_n} & s^*(V_n) & \xrightarrow{\mathfrak{u}_{n-1}} & s^*(V_{n-1}) & \xrightarrow{\mathfrak{u}_{n-2}} & \dots & \xrightarrow{\mathfrak{u}_1} & s^*(V_1) \\
{\scriptstyle id}\downarrow & & {\scriptstyle \psi_n}\downarrow\wr & & {\scriptstyle \psi_{n-1}}\downarrow\wr & & & & \wr\downarrow{\scriptstyle \psi_1} \\
s^*(M) & \xrightarrow{\widetilde{\mathfrak{u}}_n} & s^*(V_n) & \xrightarrow{\widetilde{\mathfrak{u}}_{n-1}} & s^*(V_{n-1}) & \xrightarrow{\widetilde{\mathfrak{u}}_{n-2}} & \dots & \xrightarrow{\widetilde{\mathfrak{u}}_1} & s^*(V_1)
\end{array}$$

whose vertical arrows are isomorphisms. In other words, we have equalities

$$\begin{aligned}
&\widetilde{\mathfrak{u}}_m\circ\ldots\circ\widetilde{\mathfrak{u}}_n = \psi_m\circ(\mathfrak{u}_m\circ\ldots\circ\mathfrak{u}_n)\\
&\mathfrak{u}_m\circ\ldots\circ\mathfrak{u}_n = \psi_m^{-1}\circ(\widetilde{\mathfrak{u}}_m\circ\ldots\circ\widetilde{\mathfrak{u}}_n) \qquad \text{for all} \quad 1\le m\le n.
\end{aligned} \tag{5}$$

Composing both sides of the first equality with $(\mathfrak{v}_n\circ\ldots\circ\mathfrak{v}_m)$ and both sides of the second equality with $(\widetilde{\mathfrak{v}}_n\circ\ldots\circ\widetilde{\mathfrak{v}}_m)$, we obtain

$$\begin{aligned}
&(\widetilde{\mathfrak{u}}_m\circ\ldots\circ\widetilde{\mathfrak{u}}_n)\circ(\mathfrak{v}_n\circ\ldots\circ\mathfrak{v}_m) = \psi_m\circ(\mathfrak{u}_m\circ\ldots\circ\mathfrak{u}_n)\circ(\mathfrak{v}_n\circ\ldots\circ\mathfrak{v}_m) = \psi_m\\
&(\mathfrak{u}_m\circ\ldots\circ\mathfrak{u}_n)\circ(\widetilde{\mathfrak{v}}_n\circ\ldots\circ\widetilde{\mathfrak{v}}_m) = \psi_m^{-1}\circ(\widetilde{\mathfrak{u}}_m\circ\ldots\circ\widetilde{\mathfrak{u}}_n)\circ(\widetilde{\mathfrak{v}}_n\circ\ldots\circ\widetilde{\mathfrak{v}}_m) = \psi_m^{-1}\\
&\text{for} \quad 1\le m\le n.
\end{aligned} \tag{6}$$

Replacing ψ_m and ψ_m^{-1} in (5) by the corresponding left hand side expressions in (6), we obtain the last two equations of (4).

7.4. Proposition. *Suppose that* $M = V_{n+1}$ *and* V_m, $1\le m\le n$, *are projective* R*-modules of finite type. Then the presheaf* $\prod_{1\le m\le n} G_{V_{m+1},V_m}$ *and the presheaf of relations* $\mathfrak{R}_{M,\bar V} = Ker_2(\pi_{M,\bar V})$ *are representable and locally finitely copresentable.*

Proof. (a1) By 6.3, if V_m are projective R-modules of finite type for $1\le m\le n$, then the presheaves G_{V_{m+1},V_m} are representable. Therefore, their product, $\prod_{1\le m\le n} G_{V_{m+1},V_m}$, is representable by the coproduct of R-rings representing the presheaves G_{V_{m+1},V_m}.

(a2) By 6.3.1, the presheaves G_{V_{m+1},V_m} are locally finitely corepresentable. Therefore, their product, $\prod_{1\le m\le n} G_{V_{m+1},V_m}$, is locally finitely corepresentable.

(b1) Let (S,s) be an R-ring. The equations (4) describing the values $\mathfrak{R}_{M,\bar V}(S,s)$ of the presheaf of relations at (S,s) can be encoded in the diagram

$$\begin{array}{ccc} G_{M,\bar V}(S,s)\times G_{M,\bar V}(S,s) & \overset{\mathfrak{c}^m_{M,\bar V}}{\underset{\tilde{\mathfrak{c}}^m_{M,\bar V}}{\rightrightarrows}} & Hom_S(s^*(M),s^*(V_m)) \overset{\mathrm{def}}{=} \mathcal{H}_R(M,V_m)(S,s) \\ \mathfrak{S}\downarrow\wr & & \\ G_{M,\bar V}(S,s)\times G_{M,\bar V}(S,s) & \overset{\mathfrak{c}^m_{M,\bar V}}{\underset{\tilde{\mathfrak{c}}^m_{M,\bar V}}{\rightrightarrows}} & Hom_S(s^*(M),s^*(V_m)), \qquad 1\le m\le n, \end{array} \tag{7}$$

where $G_{M,\bar V}=\prod\limits_{1\le m\le n} G_{V_{m+1},V_m}$, the vertical arrow is the standard symmetry, the upper horizontal arrow maps any element $((\mathfrak{u}_1,\mathfrak{v}_1,\dots,\mathfrak{u}_n,\mathfrak{v}_n;\widetilde{\mathfrak{u}}_1,\widetilde{\mathfrak{v}}_1,\dots,\widetilde{\mathfrak{u}}_n,\widetilde{\mathfrak{v}}_n)$ of $G_{M,\bar V}\times G_{M,\bar V}$ to the composition $\mathfrak{u}_m\circ\ldots\circ\mathfrak{u}_n$, and the lower horizontal arrow maps this element to the composition $(\mathfrak{u}_m\circ\ldots\circ\mathfrak{u}_n)\circ(\widetilde{\mathfrak{v}}_n\circ\ldots\circ\widetilde{\mathfrak{v}}_1)\circ(\widetilde{\mathfrak{u}}_m\circ\ldots\circ\widetilde{\mathfrak{u}}_n)$.

The diagram (7) depends functorially on (S,s); i.e. it is the value of the diagram

$$\begin{array}{ccc} G_{M,\bar V}\times G_{M,\bar V} & \overset{\mathfrak{c}^m_{M,\bar V}}{\underset{\tilde{\mathfrak{c}}^m_{M,\bar V}}{\rightrightarrows}} & \mathcal{H}_R(M,V_m) \\ \mathfrak{S}\downarrow\wr & & \\ G_{M,\bar V}\times G_{M,\bar V} & \overset{\mathfrak{c}^m_{M,\bar V}}{\underset{\tilde{\mathfrak{c}}^m_{M,\bar V}}{\rightrightarrows}} & \mathcal{H}_R(M,V_m), \qquad 1\le m\le n. \end{array} \tag{8}$$

It follows from the construction of the diagram (7) (hence (8)) and the equations 7.3.3(4) describing the functor of relations $\mathfrak{R}_{M,\bar V}$ that it is the limit of the diagram (8).

If the R-modules V_m, $1\le m\le n$, then the functors $\mathcal{H}_R(M,V_m)$ are representable; namely $\mathcal{H}_R(M,V_m)\simeq \mathbb{V}_R(M\otimes_k V_m^\wedge)$. If, in addition, the R-module M is a projective R-module of finite type, then $M\otimes_k V^\wedge$ is a projective R^e-module of finite type, hence, by 5.2, its vector fiber is locally finitely copresentable. By (a2) above, in this case, $G_{M,\bar V}$ is both presentable and locally finitely copresentable. Therefore, the presheaf $G_{M,\bar V}\times G_{M,\bar V}$ is presentable and locally finitely copresentable. Thus the functor of relations $\mathfrak{R}_{M,\bar V}$ is the limit of a finite diagram formed by representable and locally finitely corepresentable presheaves. Therefore, $\mathfrak{R}_{M,\bar V}$ is also presentable and locally finitely copresentable. ∎

7.5. Functoriality.

7.5.1. Proposition. *Let V_i, $1 \le i \le n$, be projective modules of finite type. Then any R-module epimorphism $M' \xrightarrow{\phi} M$ induces a closed immersion*

$$\mathfrak{Fl}_{M,\bar{V}} \longrightarrow \mathfrak{Fl}_{M',\bar{V}}$$

such that the diagram

$$\begin{array}{ccc} \mathfrak{Fl}_{M,\bar{V}} & \longrightarrow & \mathfrak{Fl}_{M',\bar{V}} \\ \downarrow & & \downarrow \\ \prod_{1\le i\le n} Gr_{M,V_i} & \longrightarrow & \prod_{1\le i\le n} Gr_{M',V_i} \end{array} \qquad (3)$$

commutes.

Proof. By 6.6.1, each morphism $Gr_{M,V_i} \longrightarrow Gr_{M',V_i}$ is a closed immersion. Therefore, their product,

$$\prod_{1\le i\le n} Gr_{M,V_i} \longrightarrow \prod_{1\le i\le n} Gr_{M',V_i},$$

is a closed immersion. Notice that the square (3) is cartesian. So that

$$\mathfrak{Fl}_{M,\bar{V}} \longrightarrow \mathfrak{Fl}_{M',\bar{V}},$$

being a pull-back of a closed immersion is a closed immersion. ∎

7.6. Base change.

7.6.1. Proposition. *Let M be an R-module and V_m, $1 \le m \le n$ projective R-modules. For any R-ring (S,s), there is a natural isomorphism*

$$(S,s)^\vee \prod_{R^\vee} \mathfrak{Fl}_{M,\bar{V}} \xrightarrow{\sim} \mathfrak{Fl}_{s^*(M),s^*(\bar{V})}$$

of presheaves of sets on $\mathbf{Aff}_k/S^\vee$.

Proof. One can see that the diagram

$$\begin{array}{ccc} \mathfrak{Fl}_{s^*(\mathcal{M}),s^*(\bar{V})} & \longrightarrow & \mathfrak{Fl}_{\mathcal{M},\bar{V}} \\ \downarrow & cart & \downarrow \\ \prod_{1\le i\le n} Gr_{s^*(M),s^*(V_i)} & \longrightarrow & \prod_{1\le i\le n} Gr_{M,V_i} \end{array}$$

is cartesian. On the other hand, the diagram

$$\begin{array}{ccc} \prod_{1\le i\le n} Gr_{s^*(M),s^*(V_i)} & \longrightarrow & \prod_{1\le i\le n} Gr_{M,V_i} \\ \downarrow & cart & \downarrow \\ (S,s)^\vee & \longrightarrow & R^\vee \end{array}$$

is cartesian by 6.4.1. Since the composition of cartesian squares is a cartesian square, the square

$$\begin{array}{ccc} \mathfrak{F}\ell_{s^*(\mathcal{M}),s^*(\bar V)} & \longrightarrow & \mathfrak{F}\ell_{\mathcal{M},\bar V} \\ \downarrow & cart & \downarrow \\ (S,s)^\vee & \longrightarrow & R^\vee \end{array}$$

is cartesian, hence the assertion. ■

7.6.2. Proposition. *Let M be an R-module and V_m, $1 \le m \le n$ projective R-modules. For any R-ring (S,s), there is a natural isomorphism between the diagram*

$$(S,s)^\vee \prod_{R^\vee} \big(\mathfrak{R}_{M,\bar V} = Ker_2(\pi_{M,\bar V}) \overset{p^1_{M,\bar V}}{\underset{p^2_{M,\bar V}}{\rightrightarrows}} G_{M,\bar V} = \prod_{1\le i\le n+1} G_{V_i,V_{i-1}} \xrightarrow{\pi_{M,\bar V}} \mathfrak{F}\ell_{M,\bar V}\big)$$

and

$$\mathfrak{R}_{s^*(M),s^*(\bar V)} \overset{p^1_{s^*(M),s^*(\bar V)}}{\underset{p^2_{s^*(M),s^*(\bar V)}}{\rightrightarrows}} G_{s^*(M),s^*(\bar V)} \xrightarrow{\pi_{s^*(M),s^*(\bar V)}} \mathfrak{F}\ell_{s^*(M),s^*(\bar V)},$$

where $s^*(\bar V)$ denotes $(s^*(V_m) \mid 1 \le m \le n)$.

Proof. The isomorphism $(S,s)^\vee \prod_{R^\vee} G_{M,\bar V} \xrightarrow{\sim} G_{s^*(M),s^*(\bar V)}$ is the composition of the obvious isomorphism

$$(S,s)^\vee \prod_{R^\vee} G_{M,\bar V} = (S,s)^\vee \prod_{R^\vee} \Big(\prod_{1\le i\le n+1} G_{V_i,V_{i-1}}\Big) \xrightarrow{\sim} \prod_{1\le i\le n+1} \Big((S,s)^\vee \prod_{R^\vee} G_{V_i,V_{i-1}}\Big)$$

and the the product of the isomorphisms

$$(S,s)^\vee \prod_{R^\vee} G_{V_i,V_{i-1}} \xrightarrow{\sim} G_{s^*(V_i),s^*(V_{i-1})}$$

(see 6.4.1). One can see that the diagram

$$\begin{array}{ccc} (S,s)^{\vee}\prod_{R^{\vee}} G_{M,\bar{V}} & \longrightarrow & (S,s)^{\vee}\prod_{R^{\vee}} \mathfrak{Fl}_{M,\bar{V}} \\ \wr\downarrow & & \downarrow\wr \\ G_{s^*(M),s^*(\bar{V})} & \xrightarrow{\pi_{s^*(M),s^*(\bar{V})}} & \mathfrak{Fl}_{s^*(M),s^*(\bar{V})} \end{array} \qquad (1)$$

commutes. The remaining isomorphism,

$$(S,s)^{\vee}\prod_{R^{\vee}} \mathfrak{R}_{M,\bar{V}} \xrightarrow{\sim} \mathfrak{R}_{s^*(M),s^*(\bar{V})},$$

is induced by the commutative diagram (1), because the base change functor $(S,s)^{\vee}\prod_{R^{\vee}} -$ preserves fibred products. ■

7.7. Smoothness.

7.7.1. Proposition. *Let M be a projective R-module.*

(a) If $\bar{V} = (V_i \mid 1 \le i \le n)$ are projective R-modules of finite type, then all presheaves and all morphisms in the canonical diagram

$$\mathfrak{R}_{M,\bar{V}} = Ker_2(\pi_{M,\bar{V}}) \overset{p^1_{M,\bar{V}}}{\underset{p^2_{M,\bar{V}}}{\rightrightarrows}} G_{M,\bar{V}} = \prod_{1\le i\le n+1} G_{V_i,V_{i-1}} \xrightarrow{\pi_{M,\bar{V}}} \mathfrak{Fl}_{M,\bar{V}}$$

are formally smooth. In particular, the flag variety $\mathfrak{Fl}_{M,\bar{V}}$ is formally smooth.

(b) If M is a projective module of finite type, then the morphisms and presheaves of the diagram are smooth.

Proof. (a1) Consider the canonical cover

$$\prod_{1\le i\le n+1} G_{V_i,V_{i-1}} \xrightarrow{\pi_{M,\bar{V}}} \mathfrak{Fl}_{M,\bar{V}}. \qquad (1)$$

of the flag variety (see 7.3.2). If the projective R-modules V_i are of finite type for $1 \le i \le n$, then the presheaves $G_{V_i,V_{i-1}}$ are smooth for $2 \le i \le n$ and the presheaf $G_{V_{n+1},V_n} = G_{M,V_n}$ is formally smooth. Therefore, their product, $\prod_{1\le i\le n+1} G_{V_i,V_{i-1}}$ is formally smooth. Since (1) is a presheaf epimorphism, this implies the formal smoothness of $\mathfrak{Fl}_{M,\bar{V}}$.

(a2) *The morphism (1) is formally smooth.*

Consider a commutative diagram

$$\begin{array}{ccc}
(S,s)^\vee & \xrightarrow{\varphi^\vee} & (T,t)^\vee \\
\zeta \big\downarrow & & \big\downarrow \xi \\
\prod_{1\le i\le n+1} G_{V_i,V_{i-1}} & \xrightarrow{\pi_{M,\bar V}} & \mathfrak{F}\ell_{M,\bar V}
\end{array} \qquad (2)$$

Its right vertical arrow is determined by an element of $\mathfrak{F}\ell_{M,\bar V}(T,t)$ represented by a chain

$$t^*(M) \xrightarrow{\xi_n} t^*(V_n) \xrightarrow{\xi_{n-1}} t^*(V_{n-1}) \xrightarrow{\xi_{n-2}} \dots \xrightarrow{\xi_1} t^*(V_1)$$

of T-module epimorphisms. The left vertical arrow of (2) is determined by an element of $\prod_{1\le i\le n+1} G_{V_i,V_{i-1}}(S,s)$, that is a set of pairs of S-module morphisms

$$s^*(V_{i-1}) \xrightarrow{\mathfrak{v}_i} s^*(V_i) \xrightarrow{\mathfrak{u}_i} s^*(V_{i-1}), \quad 2\le i\le n+1,$$

where $V_{n+1}=M$ and $\mathfrak{u}_i\circ\mathfrak{v}_i = id_{V_{i-1}}$ for $2\le i\le n$.

Using the equality $s=\varphi\circ t$, we choose $s^*=\varphi^*t^*$. The commutativity of the diagram (2) means that the sequence

$$\varphi^*(t^*(M) \xrightarrow{\xi_n} t^*(V_n) \xrightarrow{\xi_{n-1}} t^*(V_{n-1}) \xrightarrow{\xi_{n-2}} \dots \xrightarrow{\xi_1} t^*(V_1))$$

defines the same element of $\mathfrak{F}\ell_{M,\bar V}(S,s)$ as the sequence

$$\varphi^*t^*(M) \xrightarrow{\mathfrak{u}_n} \varphi^*t^*(V_n) \xrightarrow{\mathfrak{u}_{n-1}} \varphi^*t^*(V_{n-1}) \xrightarrow{\mathfrak{u}_{n-2}} \dots \xrightarrow{\mathfrak{u}_1} \varphi^*t^*(V_1).$$

By (a0), this means, precisely, that these two sequences are isomorphic, i.e. there is a commutative diagram

$$\begin{array}{ccccccccc}
\varphi^*t^*(M) & \xrightarrow{\varphi^*(\xi_n)} & \varphi^*t^*(V_n) & \xrightarrow{\varphi^*(\xi_{n-1})} & \varphi^*t^*(V_{n-1}) & \xrightarrow{\varphi^*(\xi_{n-2})} & \dots & \xrightarrow{\varphi^*(\xi_1)} & \varphi^*t^*(V_1) \\
id \big\downarrow & & \psi_n \big\downarrow \wr & & \psi_{n-1} \big\downarrow \wr & & & & \wr \big\downarrow \psi_1 \\
\varphi^*t^*(M) & \xrightarrow{\mathfrak{u}_n} & \varphi^*t^*(V_n) & \xrightarrow{\mathfrak{u}_{n-1}} & \varphi^*t^*(V_{n-1}) & \xrightarrow{\mathfrak{u}_{n-2}} & \dots & \xrightarrow{\mathfrak{u}_1} & \varphi^*t^*(V_1)
\end{array}$$

whose vertical arrows are isomorphisms of S-modules. Set

$$\widetilde{\mathfrak{u}}_i = \psi_i^{-1}\circ\mathfrak{u}_i \quad \text{and} \quad \widetilde{\mathfrak{v}}_i = \mathfrak{v}_i\circ\psi_i, \quad 1\le i\le n.$$

We replace this diagram by the diagram

$$\begin{array}{ccccccccc}
\varphi^*t^*(M) & \xrightarrow{\varphi^*(\xi_n)} & \varphi^*t^*(V_n) & \xrightarrow{\varphi^*(\xi_{n-1})} & \varphi^*t^*(V_{n-1}) & \xrightarrow{\varphi^*(\xi_{n-2})} & \dots & \xrightarrow{\varphi^*(\xi_1)} & \varphi^*t^*(V_1) \\
{\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow & & {\scriptstyle id}\downarrow & & & & \downarrow{\scriptstyle id} \\
\varphi^*t^*(M) & \xrightarrow{\widetilde{\mathfrak{u}}_n} & \varphi^*t^*(V_n) & \xrightarrow{\widetilde{\mathfrak{u}}_{n-1}} & \varphi^*t^*(V_{n-1}) & \xrightarrow{\widetilde{\mathfrak{u}}_{n-2}} & \dots & \xrightarrow{\widetilde{\mathfrak{u}}_1} & \varphi^*t^*(V_1)
\end{array}$$

whose vertical arrows are identical. The latter diagram is equivalent to the diagram

$$\begin{array}{ccccccc}
t^*(M) & \xrightarrow{\xi_n} & t^*(V_n) & \xrightarrow{\xi_{n-1}} & \dots & \xrightarrow{\xi_1} & t^*(V_1) \\
\downarrow & & \downarrow & & & & \downarrow \\
\varphi_*\varphi^*t^*(M) & \xrightarrow{\varphi_*(\widetilde{\mathfrak{u}}_n)} & \varphi_*\varphi^*t^*(V_n) & \xrightarrow{\varphi_*(\widetilde{\mathfrak{u}}_{n-1})} & \dots & \xrightarrow{\varphi_*(\widetilde{\mathfrak{u}}_1)} & \varphi_*\varphi^*t^*(V_1)
\end{array}$$

whose vertical arrows are adjunction morphisms. For every $1 \le m \le n$, we have a commutative diagram

$$\begin{array}{ccccc}
t^*(V_m) & & t^*(V_{m+1}) & \xrightarrow{\xi_m} & t^*(V_m) \\
\downarrow & & \downarrow & & \downarrow \\
\varphi_*\varphi^*t^*(V_m) & \xrightarrow{\varphi_*(\widetilde{\mathfrak{v}}_m)} & \varphi_*\varphi^*t^*(V_{m+1}) & \xrightarrow{\varphi_*(\widetilde{\mathfrak{u}}_m)} & \varphi_*\varphi^*t^*(V_m)
\end{array} \qquad (3)$$

whose vertical arrows are adjunction morphisms.

Suppose that the algebra morphism $T \xrightarrow{\varphi} S$ is a strict epimorphism. Then the vertical arrows in the diagram (3) are epimorphisms of T-modules. Therefore, since $t^*(V_m)$ is a projective T-module for every $1 \le m \le n$, there exists a T-module morphism $t^*(V_m) \xrightarrow{\beta_n} t^*(V_{m+1})$ such that the diagram

$$\begin{array}{ccccc}
t^*(V_m) & \xrightarrow{\beta_m} & t^*(V_{m+1}) & \xrightarrow{\xi_m} & t^*(V_m) \\
\downarrow & & \downarrow & & \downarrow \\
\varphi_*\varphi^*t^*(V_m) & \xrightarrow{\varphi_*(\widetilde{\mathfrak{v}}_m)} & \varphi_*\varphi^*t^*(V_{m+1}) & \xrightarrow{\varphi_*(\widetilde{\mathfrak{u}}_m)} & \varphi_*\varphi^*t^*(V_m)
\end{array} \qquad (4)$$

is commutative (that is its left square commutes).

By construction the composition of the lower horizontal arrows of (4),

$$\varphi_*(\widetilde{\mathfrak{u}}_m) \circ \varphi_*(\widetilde{\mathfrak{v}}_m) = \varphi_*(\widetilde{\mathfrak{u}}_m \circ \mathfrak{v}_m) = \varphi_*(id_{t^*(V_m)}),$$

is the identical isomorphism. And $t^*(V)$ is a projective T-module of finite type. Therefore, if the kernel of the algebra morphism $T \xrightarrow{\varphi} S$ is contained in the Jacobson radical of the algebra T, then, by Nakayama Lemma, the composition $\xi_m \circ \beta_m$ of the upper horizontal arrows of the diagram (4), is an isomorphism. Replacing in the diagram (4) the morphism β_m by the morphism $\widetilde{\beta}_m = \beta_m \circ (\xi_m \circ \beta_m)^{-1}$, we obtain a commutative diagram

$$\begin{array}{ccccc} t^*(V_m) & \xrightarrow{\widetilde{\beta}_m} & t^*(V_{m+1}) & \xrightarrow{\xi_m} & t^*(V_m) \\ \downarrow & & \downarrow & & \downarrow \\ \varphi_*\varphi^*t^*(V_m) & \xrightarrow{\varphi_*(\widetilde{\mathfrak{v}}_m)} & \varphi_*\varphi^*t^*(V_{m+1}) & \xrightarrow{\varphi_*(\widetilde{\mathfrak{u}}_m)} & \varphi_*\varphi^*t^*(V_m) \end{array} \quad (5)$$

such that the composition of its upper horizontal arrows, $\xi_m \circ \widetilde{\beta}_m$, is an identical morphism. The upper row of the diagram (5) is an element of $G_{V_{m+1},V_m}(T,t)$ which defines a morphism $(T,t)^\vee \xrightarrow{\gamma_m} G_{V_{m+1},V_m}$ for every $1 \le m \le n$, or, what is the same, a morphism

$$(T,t)^\vee \xrightarrow{\gamma} \prod_{1\le m\le n} G_{V_{m+1},V_m}$$

making the diagram

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\varphi^\vee} & (T,t)^\vee \\ \zeta \downarrow & {}^{\gamma}\swarrow & \downarrow \xi \\ \prod_{1\le m\le n} G_{V_{m+1},V_m} & \xrightarrow{\pi_{M,\bar V}} & \mathfrak{F}\ell_{M,\bar V} \end{array} \quad (7)$$

commute.

(a3) The canonical morphisms

$$\mathfrak{R}_{M,\bar V} = Ker_2(\pi_{M,\bar V}) \overset{p^1_{M,\bar V}}{\underset{p^2_{M,\bar V}}{\rightrightarrows}} \prod_{1\le i\le n+1} G_{V_i,V_{i-1}}$$

are formally smooth, because, by (a2), the cover

$$G_{M,\bar V} = \prod_{1\le m\le n} G_{V_{m+1},V_m} \xrightarrow{\pi_{M,\bar V}} \mathfrak{F}\ell_{M,\bar V}$$

is formally smooth and pull-backs of formally smooth morphisms are formally smooth.

(a4) Since composition of formally smooth morphisms is a formally smooth morphism, it follows from (a1) and (a3) that the presheaves $\mathfrak{R}_{M,\bar{V}}$, $G_{M,\bar{V}}$, and $\mathfrak{F}\ell_{M,\bar{V}}$ are formally smooth.

(b1) If the R-module M is also projective of finite type, then, by 7.4, the presheaves $G_{M,\bar{V}}$ and $\mathfrak{R}_{M,\bar{V}}$ are locally finitely copresentable. Since the diagram

$$\mathfrak{R}_{M,\bar{V}} = Ker_2(\pi_{M,\bar{V}}) \underset{p^2_{M,\bar{V}}}{\overset{p^1_{M,\bar{V}}}{\rightrightarrows}} G_{M,\bar{V}} \xrightarrow{\pi_{M,\bar{V}}} \mathfrak{F}\ell_{M,\bar{V}} \qquad (8)$$

is exact, the presheaf $\mathfrak{F}\ell_{M,\bar{V}}$ is locally finitely copresentable too. By (a4), all presheaves of the diagram (8) are formally smooth. Therefore, they are smooth.

(b2) It follows from II.1.11.2(d) that all morphisms of the diagram (8) are locally finitely presentable. By the argument above, they are formally smooth; hence they are smooth. ■

8. Generic Grassmannians.

8.1. The presheaf $\mathfrak{Gr}_{\mathcal{M}}$. Let $\mathcal{M}$ be an R-module. For any R-ring $(S, R \xrightarrow{s} S)$, we denote by $\mathfrak{Gr}_{\mathcal{M}}(S,s)$ the set of isomorphism classes of epimorphisms $s^*(\mathcal{M}) \longrightarrow V'$ of S-modules, where V' runs through the class of projective S-modules of finite type.

8.1.1. Proposition. *(a) The map* $(S,s) \longmapsto \mathfrak{Gr}_{\mathcal{M}}(S,s)$ *naturally extends to a presheaf*

$$(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \xrightarrow{\mathfrak{Gr}_{\mathcal{M}}} Sets.$$

(b) For every projective R-module of finite type V, there is a presheaf monomorphism

$$Gr_{\mathcal{M},V} \xrightarrow{\rho_V} \mathfrak{Gr}_{\mathcal{M}}. \qquad (1)$$

Proof. The argument is left to the reader. ■

8.2. The presheaf $\mathfrak{Pr}_{\mathcal{M}}$. Let $\mathcal{M}$ be an R-module. For any R-ring (S,s), we denote by $\mathfrak{Pr}_{\mathcal{M}}(S,s)$ the set of projectors $s^*(\mathcal{M}) \xrightarrow{p} s^*(\mathcal{M})$, $p^2 = p$, such that $p(s^*(\mathcal{M}))$ is a projective S-module of finite type.

8.2.1. Proposition. *(a) The map* $(S,s) \longmapsto \mathfrak{Pr}_{\mathcal{M}}(S,s)$ *extends to a presheaf*

$$(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \xrightarrow{\mathfrak{Pr}_{\mathcal{M}}} Sets. \qquad (2)$$

(b) There is a natural presheaf epimorphism

$$\mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}. \tag{3}$$

Proof. (a) Straightforward.

(b) For any R-ring (S,s) the map

$$\mathfrak{Pr}_{\mathcal{M}}(S,s) \xrightarrow{\pi_{\mathcal{M}}(S,s)} \mathfrak{Gr}_{\mathcal{M}}(S,s)$$

assigns to every projector $s^*(\mathcal{M}) \xrightarrow{p} s^*(\mathcal{M})$ the isomorphism class of the natural epimorphism of $s^*(\mathcal{M})$ onto the image of p. Evidently, the map $\pi_{\mathcal{M}}(S,s)$ is surjective. ■

8.3. Relations. Let $\mathcal{M}$ be an R-module. There is an exact diagram of presheaves

$$\mathfrak{R}_{\mathcal{M}} = \mathfrak{Pr}_{\mathcal{M}} \prod_{\mathfrak{Gr}_{\mathcal{M}}} \mathfrak{Pr}_{\mathcal{M}} \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} \mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}. \tag{4}$$

By definition, $\mathfrak{R}_{\mathcal{M}}(S,s)$ consists of pairs projectors (p_1,p_2) of $s^*(\mathcal{M})$ such that the S-module $p_1(s^*(\mathcal{M}))$ is projective of finite type and the corresponding epimorphisms $s^*(\mathcal{M}) \xrightarrow{\mathfrak{e}_{p_i}} p_i(s^*(\mathcal{M}))$ are isomorphic; that is there exists an S-module isomorphism $p_1(s^*(\mathcal{M})) \xrightarrow{\phi} p_2(s^*(\mathcal{M}))$ such that $\mathfrak{e}_{p_2} = \phi \circ \mathfrak{e}_{p_1}$. Taking the composition of both parts of the latter equality with the embedding $p_1(s^*(\mathcal{M})) \xrightarrow{\mathfrak{j}_{p_1}} s^*(\mathcal{M})$, we obtain the equality $\phi = \mathfrak{e}_{p_2} \circ \mathfrak{j}_{p_1}$. Replacing ϕ in $\mathfrak{e}_{p_2} = \phi \circ \mathfrak{e}_{p_1}$ by $\mathfrak{e}_{p_2} \circ \mathfrak{j}_{p_1}$, we get the equality

$$\mathfrak{e}_{p_2} = \mathfrak{e}_{p_2} \circ \mathfrak{j}_{p_1} \circ \mathfrak{e}_{p_1} = \mathfrak{e}_{p_2} \circ p_1.$$

Composing both parts of this equality with $\mathfrak{j}_{p_2}$, gives $p_2 = p_2 \circ p_1$. Applying similar routine to the equality $\phi^{-1} \circ \mathfrak{e}_{p_2} = \mathfrak{e}_{p_1}$, we obtain the symmetric equality: $p_1 = p_1 \circ p_2$.

Thus, $\mathfrak{R}_{\mathcal{M}}(S,s)$ consists of pairs (p_1,p_2) of endomorphisms of $s^*(\mathcal{M})$ satisfying the relations

$$p_2^2 = p_2 = p_2 \circ p_1 \quad \text{and} \quad p_1^2 = p_1 = p_1 \circ p_2. \tag{5}$$

8.4. Base change.

8.4.1. Proposition. *Let $\mathcal{M}$ be an R-module and (S,s) an R-ring. There is a natural isomorphism from the diagram*

$$(S,s)^{\vee} \prod_{R^{\vee}} \left(\mathfrak{R}_{\mathcal{M}} = \mathfrak{Pr}_{\mathcal{M}} \prod_{\mathfrak{Gr}_{\mathcal{M}}} \mathfrak{Pr}_{\mathcal{M}} \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} \mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}\right)$$

to the diagram

$$\mathfrak{R}_{s^*(\mathcal{M})} = \mathfrak{Pr}_{s^*(\mathcal{M})} \prod_{\mathfrak{Gr}_{s^*(\mathcal{M})}} \mathfrak{Pr}_{s^*(\mathcal{M})} \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} \mathfrak{Pr}_{s^*(\mathcal{M})} \xrightarrow{\pi_{s^*(\mathcal{M})}} \mathfrak{Gr}_{s^*(\mathcal{M})}.$$

Proof. Consider a commutative square

$$\begin{array}{ccc} (A,g)^\vee & \xrightarrow{\xi} & \mathfrak{Gr}_{\mathcal{M}} \\ {\scriptstyle\gamma^\vee}\downarrow & & \downarrow \\ (S,s)^\vee & \xrightarrow{s^\vee} & R^\vee \end{array}$$

The morphism ξ corresponds to an element of $\mathfrak{Gr}_{\mathcal{M}}(A,g)$, i.e. to the equivalence class of an A-module epimorphism from $g^*(\mathcal{M}) = A \otimes_R \mathcal{M}$ to a projective A-module of finite type. Since $g^* \simeq \gamma^* s^*$, this epimorphism defines an element of $\mathfrak{Gr}_{s^*(\mathcal{M})}(A,\gamma)$ which corresponds to a morphism

$$(A,\gamma)^\vee \xrightarrow{\bar{\xi}} \mathfrak{Gr}_{s^*(\mathcal{M})}$$

over $S^\vee$. The latter means that the diagram

$$\begin{array}{ccc} (A,\gamma)^\vee & \xrightarrow{\bar{\xi}} & \mathfrak{Gr}_{s^*(\mathcal{M})} \\ & {\scriptstyle\gamma}\searrow \quad \swarrow & \\ & S^\vee & \end{array}$$

commutes. Since every preseaf of sets is the colimit of the naturally associated with it diagram of representable presheaves, this implies that $(S,s)^\vee \prod_{R^\vee} \mathfrak{Gr}_{\mathcal{M}}$ is isomorphic to $\mathfrak{Gr}_{s^*(\mathcal{M})}$.

Similarly, one can show that $(S,s)^\vee \prod_{R^\vee} \mathfrak{Pr}_{\mathcal{M}}$ is isomorphic to $\mathfrak{Pr}_{s^*(\mathcal{M})}$. It follows from the universality of these constructions that the isomorphisms can be chosen in such a way that the diagram

$$\begin{array}{ccc} (S,s)^\vee \prod_{R^\vee} \mathfrak{Pr}_{\mathcal{M}} & \longrightarrow & (S,s)^\vee \prod_{R^\vee} \mathfrak{Gr}_{\mathcal{M}} \\ \downarrow & & \downarrow \\ \mathfrak{Pr}_{s^*(\mathcal{M})} & \xrightarrow{\pi} & \mathfrak{Gr}_{s^*(\mathcal{M})} \end{array} \qquad (3)$$

commutes. Notice that the functor $(S,s)^\vee \prod_{R^\vee} -$ preserves fibred products. Since

$$\mathfrak{R}_{\mathcal{M}} = \mathfrak{Pr}_{\mathcal{M}} \prod_{\mathfrak{Gr}_{\mathcal{M}}} \mathfrak{Pr}_{\mathcal{M}},$$

the diagram (3) induces an isomorphism

$$(S,s)^\vee \prod_{R^\vee} \mathfrak{R}_{\mathcal{M}} \longrightarrow \mathfrak{R}_{s^*(\mathcal{M})}.$$

Hence the assertion. ∎

8.5. Proposition. *If $\mathcal{M}$ is a projective module of finite type, then the presheaves $\mathfrak{Pr}_{\mathcal{M}}$ and $\mathfrak{R}_{\mathcal{M}}$ are representable and locally finitely copresentable.*

Proof. (a) For any R-ring (S,s), the set $\mathfrak{Pr}_{\mathcal{M}}(S,s)$ is the kernel of the pair of maps

$$End_S(s^*(M)) \overset{\mathfrak{s}}{\underset{id}{\rightrightarrows}} End_S(s^*(M)), \qquad (6)$$

where $\mathfrak{s}$ is the "taking square" – the composition of the diagonal map

$$End_S(s^*(M)) \xrightarrow{\ \Delta\ } End_S(s^*(M)) \times End_S(s^*(M)),$$

and the composition. The diagram (6) is functorial in (S,s). If $\mathcal{M}$ is a projective of finite type, then $(S,s) \longmapsto End_S(s^*(\mathcal{M}))$ is a presheaf $(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \longrightarrow Sets$ representable by the vector fiber of $T_R(\mathcal{M} \otimes_k \mathcal{M}^\vee)$. Therefore, the presheaf $\mathfrak{Pr}_{\mathcal{M}}$ is representable by the vector fiber of the cokernel of the pair of R-ring morphisms

$$T_R(\mathcal{M} \otimes_k \mathcal{M}^\vee) \overset{\sigma}{\underset{id}{\rightrightarrows}} T_R(\mathcal{M} \otimes_k \mathcal{M}^\vee),$$

where σ is an R-ring morphism corresponding to the presheaf morphism $\mathfrak{s}$ in (6).

(b) For any R-ring (S,s), the set $\mathfrak{R}_{\mathcal{M}}(S,s)$ is the limit of the diagram

$$\begin{array}{ccc} \mathfrak{Pr}_{\mathcal{M}}(S,s) \times \mathfrak{Pr}_{\mathcal{M}}(S,s) & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathfrak{Pr}_{\mathcal{M}}(S,s) \\ \mathfrak{S} \downarrow \wr & & \downarrow id \\ \mathfrak{Pr}_{\mathcal{M}}(S,s) \times \mathfrak{Pr}_{\mathcal{M}}(S,s) & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathfrak{Pr}_{\mathcal{M}}(S,s) \end{array} \qquad (7)$$

where $\mathfrak{m}$ is the composition map, π_2 is the projection on the first component, $(x, y) \longmapsto x$; and the left vertical map is the standard symmetry: $(x, y) \longmapsto (y, x)$.

If $\mathcal{M}$ is a projective R-module of finite type, then, by (a) above, the presheaf $\mathfrak{Pr}_{\mathcal{M}}$ is representable. Therefore, in this case, the presheaf $\mathfrak{R}_{\mathcal{M}}$ is representable.

(c) Since $\mathcal{M}$ is projective R-module of finite type, $\mathcal{M} \otimes_k \mathcal{M}^\vee$ is a projective R^e-module of finite type. In particular, it is a finitely presentable R^e-module which implies that the vector fiber of the R-ring $T_R(\mathcal{M} \otimes_k \mathcal{M}^\vee)$ is locally finitely copresentable. Since finite limits of locally finitely copresentable presheaves are locally finitely copresentable, it follows from the descriptions of the presheaf $\mathfrak{Pr}_{\mathcal{M}}$ in (a) and the description of $\mathfrak{R}_{\mathcal{M}}$ in (b) that both of them are locally finitely copresentable. ∎

8.6. Proposition. *Every R-module epimorphism* $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ *induces a closed immersion*

$$\mathfrak{Gr}_{\mathcal{M}} \xrightarrow{\mathfrak{Gr}_{\varphi}} \mathfrak{Gr}_{\mathcal{L}}.$$

For every projective R-module of finite type $\mathcal{V}$*, the square*

$$\begin{array}{ccc} Gr_{\mathcal{M},\mathcal{V}} & \xrightarrow{\rho_{\mathcal{V}}} & \mathfrak{Gr}_{\mathcal{M}} \\ {\scriptstyle Gr_{\varphi,V}}\big\downarrow & cart & \big\downarrow {\scriptstyle \mathfrak{Gr}_{\varphi}} \\ Gr_{\mathcal{L},\mathcal{V}} & \xrightarrow{\rho_{\mathcal{V}}} & \mathfrak{Gr}_{\mathcal{L}} \end{array} \tag{8}$$

is cartesian.

Proof. (a) For every R-ring (S, s), the map

$$\mathfrak{Gr}_{\mathcal{M}}(S, s) \xrightarrow{\mathfrak{Gr}_{\varphi}(S,s)} \mathfrak{Gr}_{\mathcal{L}}(S, s) \tag{8'}$$

assigns to an isomorphism class of any epimorphism $s^*(\mathcal{M}) \xrightarrow{\lambda} V$ (with a projective S-module of finite type V) the isomorphism class of the composition $s^*(\mathcal{M}) \xrightarrow{\lambda \circ s^*(\varphi)} V$.

Since $s^*(\mathcal{L}) \xrightarrow{s^*(\varphi)} s^*(\mathcal{M})$ is an epimorphism, the map (8') is well defined, and it is, evidently, functorial in (S, s).

(b1) Fix an arbitrary R-module epimorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$. Let (S, s) be an R-ring and $(S, s)^\vee \xrightarrow{\zeta} \mathfrak{Gr}_{\mathcal{L}}$ a presheaf morphism. It follows from the definition of the generic Grassmannian that

$$\mathfrak{Gr}_{\mathcal{N}}(S, s) = \mathfrak{Gr}_{s^*(\mathcal{N})}(S, id)$$

for any R-module $\mathcal{N}$. By Yoneda Lemma, the morphism $(S, s)^\vee \xrightarrow{\zeta} \mathfrak{Gr}_{\mathcal{L}}$ is determined by an element of $\mathfrak{Gr}_{\mathcal{L}}(S, s) = \mathfrak{Gr}_{s^*(\mathcal{L})}(S, id)$; hence it is determined by a morphism

$$(S, id)^\vee \xrightarrow{\zeta_s} \mathfrak{Gr}_{s^*(\mathcal{L})}.$$

Let $\widehat{\zeta}_s$ be the corresponding element of $\mathfrak{Gr}_{s^*(\mathcal{L})}(S,s)$, that is the isomorphism class of an epimorphism $s^*(\mathcal{L}) \longrightarrow V$ for some projective S-module of finite type V. This means that the morphism ζ_s factors through the monomorphism

$$Gr_{s^*(\mathcal{M}),V} \xrightarrow{\rho_V} \mathfrak{Gr}_{s^*(\mathcal{L})}.$$

Therefore, the pull-back $\mathfrak{X} \xrightarrow{\widetilde{\varphi}} (S,s)^\vee$ of the morphism $\mathfrak{Gr}_{\mathcal{M}} \xrightarrow{\mathfrak{Gr}_\varphi} \mathfrak{Gr}_{\mathcal{L}}$ appeares in the commutative diagram

$$\begin{array}{ccccc} \mathfrak{X} & \xrightarrow{\widetilde{\zeta}_s} & Gr_{s^*(\mathcal{M}),V} & \xrightarrow{\rho_V} & \mathfrak{Gr}_{s^*(\mathcal{M})} \\ \widetilde{\varphi}\big\downarrow & cart & \big\downarrow Gr_{s^*(\varphi),V} & cart & \big\downarrow \mathfrak{Gr}_{s^*(\varphi)} \\ (S,s)^\vee & \longrightarrow & Gr_{s^*(\mathcal{L}),V} & \xrightarrow{\rho_V} & \mathfrak{Gr}_{s^*(\mathcal{L})} \end{array} \qquad (9)$$

whose both squares are cartesian.

(b2) Suppose now that the R-module $\mathcal{M}$ is of finite type. By 6.6.1, the central vertical arrow of the diagram (9) is a closed immersion. Therefore, the left vertical arrow of (9), $\mathfrak{X} \longrightarrow (S,s)^\vee$, being a pull-back of a closed immersion, is a closed immersion of representable presheaves. Since, by the argument in (b1) above, the arrow $\mathfrak{X} \longrightarrow (S,s)^\vee$ is a pull-back of $\mathfrak{Gr}_{\mathcal{M}} \xrightarrow{\mathfrak{Gr}_\varphi} \mathfrak{Gr}_{\mathcal{L}}$ along an arbitrarily chosen morphism $(S,s)^\vee \longrightarrow \mathfrak{Gr}_{\mathcal{L}}$, this shows that $\mathfrak{Gr}_\varphi$ is a closed immersion. ■

8.7. Proposition. *Let $\mathcal{M}$ be an R-module of finite type. Then the presheaf $\mathfrak{Gr}_{\mathcal{M}}$ is separated.*

Proof. Let $(S, R \xrightarrow{s} S)$ be an R-ring, and let $(S,s)^\vee \underset{u_2}{\overset{u_1}{\rightrightarrows}} \mathfrak{Gr}_{\mathcal{M}}$ be a pair of morphisms over R. The claim is that the kernel

$$Ker(u_1,u_2) \longrightarrow (S,s)^\vee$$

of the pair (u_1,u_2) is representable by a closed immersion of affine schemes.

(a) For any R-module epimorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$, the kernel of the pair (u_1,u_2) coincides with the kernel of its composition with the monomorphism $\mathfrak{Gr}_{\mathcal{M}} \xrightarrow{\mathfrak{Gr}_\varphi} \mathfrak{Gr}_{\mathcal{L}}$. Since $\mathcal{M}$ is an R-module of finite type, there is an R-module epimorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$, where $\mathcal{L}$ is a projective R-module of finite type.

(b) Replacing $\mathcal{M}$ by $\mathcal{L}$, we assume that $\mathcal{M}$ is a projective R-module. In this case, we have an exact diagram

$$\mathfrak{R}_{\mathcal{M}} = \mathfrak{Pr}_{\mathcal{M}} \prod_{\mathfrak{Gr}_{\mathcal{M}}} \mathfrak{Pr}_{\mathcal{M}} \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} \mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}$$

(see 8.3). Since $(S,s)^\vee$ is a projective objects of the category of presheaves from $\mathbf{Aff}_k/R^\vee = (R\backslash Alg_k)^{op}$ to *Sets* and $\mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}$ is a presheaf epimorphism, each of the morphisms u_i can be lifted to a morphism $(S,s)^\vee \xrightarrow{\mathfrak{v}_i} \mathfrak{Pr}_{\mathcal{M}}$, $i = 1, 2$.

(c) Consider the cartesian square

$$\begin{array}{ccc}
\mathfrak{R}_{\mathcal{M}} & \xrightarrow{\pi_{\mathcal{M}} \circ \mathfrak{p}_1} & \mathfrak{Gr}_{\mathcal{M}} \\
{\scriptstyle \mathfrak{j}_{\mathcal{M}}} \downarrow & cart & \downarrow {\scriptstyle \Delta} \\
\mathfrak{Pr}_{\mathcal{M}} \times \mathfrak{Pr}_{\mathcal{M}} & \longrightarrow & \mathfrak{Gr}_{\mathcal{M}} \times \mathfrak{Gr}_{\mathcal{M}}
\end{array}$$

(whose left vertical arrow is the natural embedding) and complete it to the diagram

$$\begin{array}{ccccc}
Ker(u_1, u_2) & \longrightarrow & \mathfrak{R}_{\mathcal{M}} & \longrightarrow & \mathfrak{Gr}_{\mathcal{M}} \\
\downarrow & cart & {\scriptstyle \mathfrak{j}_{\mathcal{M}}} \downarrow & cart & \downarrow {\scriptstyle \Delta} \\
(S,s)^\vee & \longrightarrow & \mathfrak{Pr}_{\mathcal{M}} \times \mathfrak{Pr}_{\mathcal{M}} & \longrightarrow & \mathfrak{Gr}_{\mathcal{M}} \times \mathfrak{Gr}_{\mathcal{M}}
\end{array} \qquad (10)$$

whose left square is cartesian too.

Since $\mathcal{M}$ is a projective R-module of finite type, the presheaf $\mathfrak{Pr}_{\mathcal{M}}$ is representable by the cokernel of the pair

$$T_R(\mathcal{M} \otimes_k \mathcal{M}^\vee) \underset{id}{\overset{\sigma}{\rightrightarrows}} T_R(\mathcal{M} \otimes_k \mathcal{M}^\vee)$$

of k-algebra morphisms (see the part (a) of the argument of 8.5).

By the part (b) of the argument of 8.5, the presheaf $\mathfrak{R}_{\mathcal{M}}$ is the limit of the diagram

$$\begin{array}{ccc}
\mathfrak{Pr}_{\mathcal{M}} \times \mathfrak{Pr}_{\mathcal{M}} & \underset{\pi_1}{\overset{\mathfrak{m}}{\rightrightarrows}} & \mathfrak{Pr}_{\mathcal{M}} \\
{\scriptstyle \mathfrak{S}} \downarrow \wr & & \downarrow {\scriptstyle id} \\
\mathfrak{Pr}_{\mathcal{M}} \times \mathfrak{Pr}_{\mathcal{M}} & \underset{\pi_1}{\overset{\mathfrak{m}}{\rightrightarrows}} & \mathfrak{Pr}_{\mathcal{M}}
\end{array} \qquad (11)$$

where $\mathfrak{m}$ is the composition map, π_1 is the projection on the first component, and the left vertical map is the standard symmetry. The natural embedding of $\mathfrak{R}_{\mathcal{M}}$ into $\mathfrak{Pr}_{\mathcal{M}} \times \mathfrak{Pr}_{\mathcal{M}}$ – the middle vertical arrow of the diagram (10), coincides with the canonical morphism

$$\mathfrak{R}_{\mathcal{M}} \longrightarrow \mathfrak{Pr}_{\mathcal{M}} \times \mathfrak{Pr}_{\mathcal{M}}$$

coming from the diagram (11). The latter is, evidently, a strict monomorphism of representable presheaves, hence a closed immersion. Since the middle vertical arrow of the diagram (10) is a closed immersion, the left vertical arrow is a closed immersion too, whence the assertion. ■

8.8. Smoothness.

8.8.0. The choice of infinitesimal morphisms. In this Section, we take as *infinitesimal* morphisms the class $\mathfrak{M}_{\mathfrak{J}}^{\mathfrak{c}}$ of *complete radical closed immersions* (cf. II.3.8.2 and consider (formal) smoothness with respect to this class. Recall that a morphism $(S,s)^\vee \xrightarrow{\varphi^\vee} (T,t)^\vee$ of the category $\mathbf{Aff}_k/R^\vee$ is called a *complete radical closed immersion*, if the corresponding algebra morphism $T \xrightarrow{\varphi} S$ is surjective, its kernel is contained in the Jacobson radical of the algebra T, and the canonical morphism $T \longrightarrow \lim_{n\geq 1} T/(Ker(\varphi)^n)$ is an isomorphism. Since the class $\mathfrak{M}_{\mathfrak{n}}$ of closed immersions corresponding to strict epimorphisms with nilpotent kernel is contained in the class $\mathfrak{M}_{\mathfrak{J}}^{\mathfrak{c}}$, every (formally) $\mathfrak{M}_{\mathfrak{J}}^{\mathfrak{c}}$-smooth presheaf morphism is (formally) smooth in the standard sense.

In this Section, we shall write (formally) smooth meaning (formally) $\mathfrak{M}_{\mathfrak{J}}^{\mathfrak{c}}$-smooth.

8.8.1. Proposition. *(a) Let $\mathcal{M}$ be a projective R-module. Then all presheaves and all presheaf morphisms of the canonical diagram*

$$\mathfrak{R}_{\mathcal{M}} = \mathfrak{Pr}_{\mathcal{M}} \prod_{\mathfrak{Gr}_{\mathcal{M}}} \mathfrak{Pr}_{\mathcal{M}} \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} \mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}} \qquad (1)$$

are formally smooth.

(b) If $\mathcal{M}$ is a projective R-module of finite type, then the presheaves and presheaf morphisms in (1) are smooth.

Proof. (a1) *The presheaf $\mathfrak{Pr}_{\mathcal{M}}$ is formally smooth.*

(i) Consider a commutative diagram

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\xi} & \mathfrak{Pr}_{\mathcal{M}} \\ {\scriptstyle \varphi^\vee}\downarrow & & \downarrow \\ (T,t)^\vee & \xrightarrow{t^\vee} & R^\vee \end{array}$$

where $S^\vee \xrightarrow{\varphi^\vee} T^\vee$ is a *complete radical closed immersion* (see 8.8.0). The morphism $(S,s)^\vee \xrightarrow{\xi} \mathfrak{Pr}_{\mathcal{M}}$ is determined by a projector $s^*(\mathcal{M}) = \varphi^* t^*(\mathcal{M}) \xrightarrow{\mathfrak{p}} s^*(\mathcal{M})$ whose image is a projective S-module of finite type. Since $T \xrightarrow{\varphi} S$ is a strict algebra epimorphism such that the algebra T is $Ker(\varphi)$*-adically complete*, i.e. $T \longrightarrow \lim_{n\ge 1} T/(Ker(\varphi)^n)$ is an isomorphism, the projective S-module $\mathfrak{p}(s^*(\mathcal{M}))$ is isomorphic to $\phi^*(\mathcal{V})$, where $\mathcal{V}$ is a projective T-module of finite type (see [Ba], III.2.12]).

Therefore, the projector $\mathfrak{p}$ has a decomposition $\varphi^*(\mathcal{V}) \xrightarrow{\mathfrak{v}} \varphi^* t^*(\mathcal{M}) \xrightarrow{\mathfrak{u}} \varphi^*(\mathcal{V})$, that is $\mathfrak{p} = \mathfrak{v} \circ \mathfrak{u}$ and $\mathfrak{u} \circ \mathfrak{v} = id_{\varphi^*(\mathcal{V})}$. Choosing this decomposition can be interpreted as the commutative diagram

$$\begin{array}{ccccc}
(S,s)^\vee & \xrightarrow{\xi} & \mathfrak{Pr}_{\mathcal{M}} & \longrightarrow & R^\vee \\
\widetilde{\xi}\downarrow & & \uparrow & cart & \uparrow s^\vee \\
G_{s^*(\mathcal{M}),\varphi^*(\mathcal{V})} & \xrightarrow{\widetilde{\rho}_{\varphi^*(\mathcal{V})}} & \mathfrak{Pr}_{s^*(\mathcal{M})} & \longrightarrow & S^\vee
\end{array} \qquad (2)$$

whose left square is the corresponding decomposition of $(S,s)^\vee \xrightarrow{\xi} \mathfrak{Pr}_{\mathcal{M}}$, and the right, cartesian, square is the base change. The left lower arrow is the canonical presheaf morphism which assigns to every element $g^*\varphi^*(\mathcal{V}) \xrightarrow{\mathfrak{v}''} g^* s^*(\mathcal{M}) \xrightarrow{\mathfrak{u}''} g^*\varphi^*(\mathcal{V})$, of $G_{s^*(\mathcal{M}),\varphi^*(\mathcal{V})}(A, g\circ s)$ the corresponding projection $\mathfrak{v}''\circ\mathfrak{u}''$ of the A-module $g^*s^*(\mathcal{M})$.

(ii) The vertical arrows of the diagram

$$\begin{array}{ccccc}
\mathcal{V} & & t^*(\mathcal{M}) & & \mathcal{V} \\
\downarrow & & \downarrow & & \downarrow \\
\varphi_*\varphi^*(\mathcal{V}) & \xrightarrow{\varphi_*(\mathfrak{v})} & \varphi_*\varphi^* t^*(\mathcal{M}) & \xrightarrow{\varphi_*(\mathfrak{u})} & \varphi_*\varphi^*(\mathcal{V})
\end{array}$$

– adjunction morphisms, are epimorphisms, because $T \xrightarrow{\varphi} S$ is a strict epimorphism of algebras. Therefore, since $\mathcal{V}$ and $t^*(\mathcal{M})$ are projective modules, there exist morphisms $\mathcal{V} \xrightarrow{\mathfrak{v}'} t^*(\mathcal{M}) \xrightarrow{\mathfrak{u}'} \mathcal{V}$ such that the diagram

$$\begin{array}{ccccc}
\mathcal{V} & \xrightarrow{\mathfrak{v}'} & t^*(\mathcal{M}) & \xrightarrow{\mathfrak{u}'} & \mathcal{V} \\
\downarrow & & \downarrow & & \downarrow \\
\varphi_*\varphi^*(\mathcal{V}) & \xrightarrow{\varphi_*(\mathfrak{v})} & \varphi_*\varphi^* t^*(\mathcal{M}) & \xrightarrow{\varphi_*(\mathfrak{u})} & \varphi_*\varphi^*(\mathcal{V})
\end{array} \qquad (3)$$

commutes. Since the T-module $\mathcal{V}$ is projective of finite type and the kernel of $T \xrightarrow{\varphi} S$ is contained in the Jacobson radical of the algebra T, the fact that the composition of lower horizontal arrows of (3) is an isomorphism, implies, by [Ba], III.2.12], that the composition $\mathfrak{u}' \circ \mathfrak{v}'$ of the upper horizontal arrows is an isomorphism too. Replacing $\mathfrak{v}'$ by $\widetilde{\mathfrak{v}} = \mathfrak{v}' \circ (\mathfrak{u}' \circ \mathfrak{v}')^{-1}$, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{\ \widetilde{\mathfrak{v}}\ } & t^*(\mathcal{M}) & \xrightarrow{\ \mathfrak{u}'\ } & \mathcal{V} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \varphi_*\varphi^*(\mathcal{V}) & \xrightarrow{\ \varphi_*(\mathfrak{v})\ } & \varphi_*\varphi^*t^*(\mathcal{M}) & \xrightarrow{\ \varphi_*(\mathfrak{u})\ } & \varphi_*\varphi^*(\mathcal{V}) \end{array} \tag{4}$$

such that the composition $\mathfrak{u}' \circ \widetilde{\mathfrak{v}}$ of its upper horizontal arrows is an identical morphism; that is the upper horizontal row of the diagram (3) is an element of $G_{t^*(\mathcal{M}),\mathcal{V}}(T,t)$, which determines, by Yoneda Lemma, a morphism $(T,t) \xrightarrow{\ \widetilde{\gamma}\ } G_{t^*(\mathcal{M}),\mathcal{V}}$.

By adjunction, the diagram (4) corresponds to the commutative diagram

$$\begin{array}{ccccc} \varphi^*(\mathcal{V}) & \xrightarrow{\ \varphi^*(\widetilde{\mathfrak{v}})\ } & \varphi^*t^*(\mathcal{M}) & \xrightarrow{\ \varphi^*(\mathfrak{u}')\ } & \varphi^*(\mathcal{V}) \\ id \big\downarrow & & id \big\downarrow & & \big\downarrow id \\ \varphi^*(\mathcal{V}) & \xrightarrow{\ \mathfrak{v}\ } & \varphi^*t^*(\mathcal{M}) & \xrightarrow{\ \mathfrak{u}\ } & \varphi^*(\mathcal{V}) \end{array} \tag{$\widehat{4}$}$$

In other words,

$$\varphi^*(\mathcal{V}) \xrightarrow{\mathfrak{v}} \varphi^*t^*(\mathcal{M}) \xrightarrow{\mathfrak{u}} \varphi^*(\mathcal{V}) = \varphi^*\big(\mathcal{V} \xrightarrow{\widetilde{\mathfrak{v}}} t^*(\mathcal{M}) \xrightarrow{\mathfrak{u}'} \mathcal{V}\big).$$

(iii) All together can be expressed in the commutative diagram

$$\begin{array}{ccccccccc} (S,s)^\vee & \xrightarrow{\ \widetilde{\xi}\ } & G_{s^*(\mathcal{M}),\varphi^*(\mathcal{V})} & \xrightarrow{\ \widetilde{\rho}_{\varphi^*(\mathcal{V})}\ } & \mathfrak{Pr}_{s^*(\mathcal{M})} & \longrightarrow & \mathfrak{Pr}_{\mathcal{M}} & & \\ \varphi^\vee \big\downarrow & & \big\uparrow & & \big\uparrow & & \big\downarrow id & & \\ (T,t)^\vee & \xrightarrow{\ \widetilde{\gamma}\ } & G_{t^*(\mathcal{M}),\mathcal{V}} & \xrightarrow{\ \widetilde{\rho}_{\mathcal{V}}\ } & \mathfrak{Pr}_{t^*(\mathcal{M})} & \longrightarrow & \mathfrak{Pr}_{\mathcal{M}} & \longrightarrow & R^\vee \end{array} \tag{5}$$

which can be regarded as a decomposition of the commutative diagram

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\ \xi\ } & \mathfrak{Pr}_{\mathcal{M}} \\ \varphi^\vee \big\downarrow & {}^{\gamma}\nearrow & \big\downarrow \\ (T,t)^\vee & \xrightarrow{\ t^\vee\ } & R^\vee \end{array}$$

This shows that the morphism $\mathfrak{Pr}_{\mathcal{M}} \longrightarrow R^\vee$ is formally smooth.

(a2) *The presheaf* $\mathfrak{Gr}_{\mathcal{M}}$ *is formally smooth.*

By 3.0.2.1, this follows from the fact that $\mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}$ is an epimorphism of presheaves and the presheaf $\mathfrak{Pr}_{\mathcal{M}}$ is formally smooth.

(a3) *The presheaf morphism* $\mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}$ *is formally smooth.*

Consider a commutative diagram

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\varphi^\vee} & (T,t)^\vee \\ \xi \big\downarrow & & \big\downarrow \xi_1 \\ \mathfrak{Pr}_{\mathcal{M}} & \xrightarrow{\pi_{\mathcal{M}}} & \mathfrak{Gr}_{\mathcal{M}} \end{array} \qquad (6)$$

The morphism $(T,t)^\vee \xrightarrow{\xi_1} \mathfrak{Gr}_{\mathcal{M}}$ is uniquely defined by an element of $\mathfrak{Gr}_{\mathcal{M}}(T,t)$, that is by the isomorphism class of a T-module epimorphism $t^*(\mathcal{M}) \xrightarrow{\widehat{\xi_1}} \mathcal{V}$ for some projective T-module of finite type $\mathcal{V}$. Similarly, the morphism $(S,s)^\vee \xrightarrow{\xi} \mathfrak{Pr}_{\mathcal{M}}$ is uniquely determined by an element of $\mathfrak{Pr}_{\mathcal{M}}(S,s)$, a projector $s^*(\mathcal{M}) \xrightarrow{\mathfrak{p}_\xi} s^*(\mathcal{M})$, which can be represented by its canonical decomposition – a pair of S-module morphisms $\mathcal{W} \xrightarrow{\mathfrak{v}} s^*(\mathcal{M}) \xrightarrow{\mathfrak{u}} \mathcal{W}$ with a projective S-module of finite type $\mathcal{W}$ such that $\mathfrak{u} \circ \mathfrak{v} = id_{\mathcal{W}}$. The commutativity of the diagram (6) means that there is a commutative diagram

$$\begin{array}{ccccc} \varphi^*(\mathcal{V}) & & \varphi^* t^*(\mathcal{M}) & \xrightarrow{\varphi^*(\widehat{\xi_1})} & \varphi^*(\mathcal{V}) \\ \wr\big\downarrow & & \wr\big\downarrow & & \big\downarrow\wr \\ \mathcal{W} & \xrightarrow{\mathfrak{v}} & s^*(\mathcal{M}) & \xrightarrow{\varphi^*(\mathfrak{u})} & \mathcal{W} \end{array} \qquad (7)$$

whose vertical arrows are S-module isomorphisms. Therefore, the diagram (7) can be replaced by the commutative diagram of T-module morphisms

$$\begin{array}{ccccc} \mathcal{V} & & t^*(\mathcal{M}) & \xrightarrow{\widehat{\xi_1}} & \mathcal{V} \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \varphi_*\varphi^*(\mathcal{V}) & \xrightarrow{\widetilde{\mathfrak{v}}} & \varphi_*\varphi^* t^*(\mathcal{M}) & \xrightarrow{\widetilde{\mathfrak{u}}} & \varphi_*\varphi^*(\mathcal{V}) \end{array} \qquad (8)$$

whose vertical arrows are adjunction morphisms and the composition $\widetilde{\mathfrak{u}} \circ \widetilde{\mathfrak{v}}$ is the identical morphism. If the algebra morphism $T \xrightarrow{\varphi} S$ is a strict epimorphism, then the vertical arrows of the diagram (8) are T-module epimorphisms. Since $\mathcal{V}$ is a projective module, the fact that the middle vertical arrow is an epimorphism implies the existence of a T-module morphism $\mathcal{V} \xrightarrow{\gamma_1} t^*(\mathcal{M})$ such that the diagram

$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{\gamma_1} & t^*(\mathcal{M}) & \xrightarrow{\widehat{\xi}_1} & \mathcal{V} \\ \downarrow & & \downarrow & & \downarrow \\ \varphi_*\varphi^*(\mathcal{V}) & \xrightarrow{\widetilde{\mathfrak{v}}} & \varphi_*\varphi^*t^*(\mathcal{M}) & \xrightarrow{\widetilde{\mathfrak{u}}} & \varphi_*\varphi^*(\mathcal{V}) \end{array} \qquad (9)$$

commutes. If $T \xrightarrow{\varphi} S$ is a strict epimorphism whose kernel is contained in the Jacobson radical of the algebra T, then, since $\mathcal{V}$ is a projective T-module of finite type, it follows from the Nakayama Lemma that the composition $\widetilde{\xi}_1 \circ \gamma_1$ is an isomorphism, because the composition $\widetilde{\mathfrak{u}} \circ \widetilde{\mathfrak{v}}$ is an isomorphism. Replacing the arrow $\mathcal{V} \xrightarrow{\gamma_1} t^*(\mathcal{M})$ in the diagram (9) with $\gamma_2 = \gamma_1 \circ (\widetilde{\xi}_1 \circ \gamma_1)^{-1}$, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{V} & \xrightarrow{\gamma_2} & t^*(\mathcal{M}) & \xrightarrow{\widehat{\xi}_1} & \mathcal{V} \\ \downarrow & & \downarrow & & \downarrow \\ \varphi_*\varphi^*(\mathcal{V}) & \xrightarrow{\widetilde{\mathfrak{v}}} & \varphi_*\varphi^*t^*(\mathcal{M}) & \xrightarrow{\widetilde{\mathfrak{u}}} & \varphi_*\varphi^*(\mathcal{V}) \end{array} \qquad (9')$$

such that $\widetilde{\xi}_1 \circ \gamma_2 = id_{\mathcal{V}}$. The projector $\gamma_2 \circ \widetilde{\xi}_1$ of the T-module $t^*(\mathcal{M})$ is an element of $\mathfrak{Pr}_{\mathcal{M}}(T,t)$ defining a morphism $(T,t)^\vee \xrightarrow{\gamma} \mathfrak{Pr}_{\mathcal{M}}$ which makes the diagram

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\varphi^\vee} & (T,t)^\vee \\ \xi \downarrow & {}^{\gamma}\swarrow & \downarrow \xi_1 \\ \mathfrak{Pr}_{\mathcal{M}} & \xrightarrow{\pi_{\mathcal{M}}} & \mathfrak{Gr}_{\mathcal{M}} \end{array}$$

commute. This shows that $\mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}$ is a formally smooth morphism.

(a4) Since formally smooth morphisms are stable under pull-backs, the formal smoothness of $\mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}$ implies that the projections $\mathfrak{R}_{\mathcal{M}} \overset{\mathfrak{p}_1}{\underset{\mathfrak{p}_2}{\rightrightarrows}} \mathfrak{Pr}_{\mathcal{M}}$ are formally smooth as well.

(b) Suppose now that $\mathcal{M}$ is a projective R-module of finite type. Then, by 8.5, all presheaves in the diagram (1) are locally finitely copresentable. It follows from II.1.11.2(d) that all morphisms of the diagram (1) are locally finitely copresentable. By (a) above, they are formally smooth; hence they are smooth. ∎

8.9. Proposition. *(a) For every projective R-module of finite type $\mathcal{V}$, the canonical morphism*

$$Gr_{\mathcal{M},\mathcal{V}} \xrightarrow{\rho_{\mathcal{V}}} \mathfrak{Gr}_{\mathcal{M}}. \tag{1}$$

is formally smooth; i.e. it is a formally open immersion.

(b) If both R-modules $\mathcal{M}$ and $\mathcal{V}$ are projective objects of finite type, then the morphism (1) is an open immersion.

Proof. (a) Consider a commutative diagram

$$\begin{array}{ccccc} & & (S,s)^{\vee} & \xrightarrow{\varphi^{\vee}} & (T,t)^{\vee} \\ & & \xi \downarrow & & \downarrow \xi_1 \\ G_{\mathcal{M},\mathcal{V}} & \xrightarrow{\pi} & Gr_{\mathcal{M},\mathcal{V}} & \xrightarrow{\rho_V} & \mathfrak{Gr}_{\mathcal{M}} \end{array} \tag{2}$$

Since $(S,s)^{\vee}$ is a projective object of the category of presheaves on $\mathbf{Aff}_k/R^{\vee}$ and $G_{\mathcal{M},\mathcal{V}} \xrightarrow{\pi} Gr_{\mathcal{M},\mathcal{V}}$ is a presheaf epimorphism, there is a morphism $(S,s)^{\vee} \xrightarrow{\zeta} G_{\mathcal{M},\mathcal{V}}$ such that $\xi = \pi \circ \zeta$. So that we have a commutative diagram

$$\begin{array}{ccc} (S,s)^{\vee} & \xrightarrow{\varphi^{\vee}} & (T,t)^{\vee} \\ \zeta \downarrow & & \downarrow \xi_1 \\ G_{\mathcal{M},\mathcal{V}} & \xrightarrow{\rho_V \circ \pi} & \mathfrak{Gr}_{\mathcal{M}} \end{array} \tag{3}$$

The morphism $(T,t)^{\vee} \xrightarrow{\xi_1} \mathfrak{Gr}_{\mathcal{M}}$ is uniquely defined by an element of $\mathfrak{Gr}_{\mathcal{M}}(T,t)$, that is by a T-module epimorphism $t^*(\mathcal{M}) \xrightarrow{\widehat{\xi_1}} \mathcal{W}$ for some projective T-module $\mathcal{W}$. The morphism $(S,s)^{\vee} \xrightarrow{\zeta} G_{\mathcal{M},\mathcal{V}}$ is uniquely determined by an element of $G_{\mathcal{M},\mathcal{V}}(S,s)$, – a pair of S-module morphisms $s^*(\mathcal{V}) \xrightarrow{\mathfrak{v}} s^*(\mathcal{M}) \xrightarrow{\mathfrak{u}} s^*(\mathcal{V})$ such that $\mathfrak{u} \circ \mathfrak{v} = id_{s^*(\mathcal{V})}$.

The commutativity of the diagram (3) means that there is a commutative diagram

$$\begin{array}{ccccc} \varphi^*(\mathcal{W}) & & \varphi^* t^*(\mathcal{M}) & \xrightarrow{\varphi^*(\widehat{\xi_1})} & \varphi^*(\mathcal{W}) \\ \wr\downarrow & & \wr\downarrow & & \downarrow\wr \\ s^*(\mathcal{V}) & \xrightarrow{\mathfrak{v}} & s^*(\mathcal{M}) & \xrightarrow{\mathfrak{u}} & s^*(\mathcal{V}) \end{array} \tag{4}$$

whose vertical arrows are S-module isomorphisms. Therefore, since $s^* \simeq \varphi^* \circ t^*$, the diagram (4) can be replaced by the commutative diagram of T-module morphisms

$$\begin{array}{ccccc} t^*(\mathcal{V}) & & t^*(\mathcal{M}) & \xrightarrow{\widehat{\xi}_1} & t^*(\mathcal{V}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \varphi_*\varphi^*t^*(\mathcal{V}) & \xrightarrow{\widetilde{\mathfrak{v}}} & \varphi_*\varphi^*t^*(\mathcal{M}) & \xrightarrow{\widetilde{\mathfrak{u}}} & \varphi_*\varphi^*t^*(\mathcal{V}) \end{array} \qquad (5)$$

whose vertical arrows are adjunction morphisms and the composition $\widetilde{\mathfrak{u}} \circ \widetilde{\mathfrak{v}}$ is the identical morphism. If the algebra morphism $T \xrightarrow{\varphi} S$ is a strict epimorphism, then the vertical arrows of the diagram (4) are T-module epimorphisms. Since $\mathcal{V}$ is a projective R-module, $t^*(\mathcal{V})$ is a projective T-module. Therefore, the fact that the middle vertical arrow is an epimorphism implies the existence of a T-module morphism $\mathcal{V} \xrightarrow{\gamma_1} t^*(\mathcal{M})$ such that the diagram

$$\begin{array}{ccccc} t^*(\mathcal{V}) & \xrightarrow{\gamma_1} & t^*(\mathcal{M}) & \xrightarrow{\widehat{\xi}_1} & t^*(\mathcal{V}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \varphi_*\varphi^*(\mathcal{V}) & \xrightarrow{\widetilde{\mathfrak{v}}} & \varphi_*\varphi^*t^*(\mathcal{M}) & \xrightarrow{\widetilde{\mathfrak{u}}} & \varphi_*\varphi^*(\mathcal{V}) \end{array} \qquad (6)$$

commutes. Since $\mathcal{V}$ is a projective R-module of finite type, $t^*(\mathcal{V})$ is a projective T-module of finite type. Therefore, if $T \xrightarrow{\varphi} S$ is a strict epimorphism whose kernel is contained in the Jacobson radical of the algebra T, then the fact that the composition $\widetilde{\mathfrak{u}} \circ \widetilde{\mathfrak{v}}$ is an isomorphism implies, by Nakayama Lemma, that the composition $\widetilde{\xi}_1 \circ \gamma_1$ is an isomorphism. Replacing γ_1 in the diagram (6) with $\gamma_2 = \gamma_1 \circ (\widetilde{\xi}_1 \circ \gamma_1)^{-1}$, we obtain a commutative diagram

$$\begin{array}{ccccc} t^*(\mathcal{V}) & \xrightarrow{\gamma_2} & t^*(\mathcal{M}) & \xrightarrow{\widehat{\xi}_1} & t^*(\mathcal{V}) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \varphi_*\varphi^*t^*(\mathcal{V}) & \xrightarrow{\widetilde{\mathfrak{v}}} & \varphi_*\varphi^*t^*(\mathcal{M}) & \xrightarrow{\widetilde{\mathfrak{u}}} & \varphi_*\varphi^*t^*(\mathcal{V}) \end{array} \qquad (7)$$

such that $\widetilde{\xi}_1 \circ \gamma_2 = id_{t^*(\mathcal{V})}$. The upper row of the diagram (7) is an element of $G_{\mathcal{M},\mathcal{V}}(T,t)$ which defines a morphism $(T,t)^\vee \xrightarrow{\gamma} G_{\mathcal{M},\mathcal{V}}$ making the diagram

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\varphi^\vee} & (T,t)^\vee \\ \zeta \big\downarrow & {}^{\gamma}\swarrow & \big\downarrow \xi_1 \\ G_{\mathcal{M},\mathcal{V}} & \xrightarrow{\rho_{\mathcal{V}} \circ \pi} & \mathfrak{Gr}_{\mathcal{M}} \end{array}$$

commute. Therefore, the diagram

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\varphi^\vee} & (T,t)^\vee \\ \xi \downarrow & \stackrel{\pi\circ\gamma}{\swarrow} & \downarrow \xi_1 \\ Gr_{\mathcal{M},\mathcal{V}} & \xrightarrow{\rho_{\mathcal{V}}} & \mathfrak{Gr}_{\mathcal{M}} \end{array}$$

commutes (see the diagrams (2) and (3) above).

This shows that the embedding $Gr_{\mathcal{M},\mathcal{V}} \xrightarrow{\rho_{\mathcal{V}}} \mathfrak{Gr}_{\mathcal{M}}$ is a formally smooth morphism. So that it is a *formally open immersion.*

(b) Suppose now that both $\mathcal{V}$ and $\mathcal{M}$ are projective R-modules of finite type. Then, by 6.3.1, the presheaf $Gr_{\mathcal{M},\mathcal{V}}$ is locally finitely copresentable over R and, by 8.5, the presheaf $\mathfrak{Gr}_{\mathcal{M}}$ is locally finitely copresentable over R. By II.1.11.2(d), this implies that the morphism $Gr_{\mathcal{M},\mathcal{V}} \xrightarrow{\rho_{\mathcal{V}}} \mathfrak{Gr}_{\mathcal{M}}$ is locally finitely copresentable. By (a) above, it is formally smooth; hence it is a smooth monomorphism – an open immersion. ■

8.9.1. Corollary. *Let $\mathcal{M}$ be a projective R-module of finite type. Then, for every projective R-module of finite type $\mathcal{V}$ and any R-module morphism $\mathcal{V} \xrightarrow{\psi} \mathcal{M}$, there is a canonical open representable immersion of the representable presheaf $F_{\psi;\mathcal{M},\mathcal{V}}$ into the generic Grassmannian $\mathfrak{Gr}_{\mathcal{M}}$.*

Proof. The immersion in question is the composition of the canonical embedding $F_{\psi;\mathcal{M},V} \longrightarrow Gr_{\mathcal{M},\mathcal{V}}$, which is representable by 6.7.1(b) as well as the presheaf $F_{\psi;\mathcal{M},\mathcal{V}}$ (provided the projective R-module $\mathcal{V}$ is of finite type) and $Gr_{\mathcal{M},\mathcal{V}} \xrightarrow{\rho_{\mathcal{V}}} \mathfrak{Gr}_{\mathcal{M}}$. If both $\mathcal{M}$ and $\mathcal{V}$ are projective R-modules of finite type, then the first morphism is an open immersion by 6.7.2.1 and the second one is an open immersion by 8.9. ■

8.10. Proposition. *Let $\mathcal{M}$ be an R-module of finite type. Then*

(a) The presheaf $\mathfrak{Gr}_{\mathcal{M}}$ is of strictly cofinite type.

(b) The presheaf $\mathfrak{Gr}_{\mathcal{M}}$ is locally affine for the smooth pretopology. More precisely, there exists an exact diagram

$$\mathfrak{R}^{\phi}_{\mathcal{M}} \underset{\mathfrak{p}^{\phi}_2}{\overset{\mathfrak{p}^{\phi}_1}{\rightrightarrows}} \mathfrak{Pr}^{\phi}_{\mathcal{M}} \xrightarrow{\mathfrak{p}_{\phi}} \mathfrak{Gr}_{\mathcal{M}} \qquad (1)$$

whose arrows are representable coverings for the smooth topology. The presheaves $\mathfrak{Pr}^{\phi}_{\mathcal{M}}$ and (therefore) $\mathfrak{R}^{\phi}_{\mathcal{M}} = K_2(\mathfrak{p}_{\phi})$ are representable and of strictly cofinite type.

Proof. Since $\mathcal{M}$ is an R-module of finite type, there exists an R-module epimorphism $\mathcal{L} \xrightarrow{\phi} M$ with $\mathcal{L}$ a projective R-module of finite type. This epimorphism ϕ appears as a parameter in the diagram (1). The diagram itself is defined (uniquely up to isomorphism) via the diagram

$$\begin{array}{ccccc} \mathfrak{R}^{\phi}_{\mathcal{M}} & \overset{\mathfrak{p}^{\phi}_1}{\underset{\mathfrak{p}^{\phi}_2}{\rightrightarrows}} & \mathfrak{Pr}^{\phi}_{\mathcal{M}} & \xrightarrow{\mathfrak{p}_{\phi}} & \mathfrak{Gr}_{\mathcal{M}} \\ \downarrow & cart & \downarrow & cart & \downarrow \mathfrak{Gr}_{\phi} \\ \mathfrak{R}_{\mathcal{L}} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \mathfrak{Pr}_{\mathcal{L}} & \xrightarrow{\pi} & \mathfrak{Gr}_{\mathcal{L}} \end{array} \qquad (2)$$

with cartesian squares. By 8.8.1, the presheaves of the bottom of the diagram are smooth and all three arrows of the bottom of the diagram are coverings for the smooth pretopology. It follows from 1.3.1 and the fact that the presheaves $\mathfrak{Pr}_{\mathcal{L}}$ and $\mathfrak{R}_{\mathcal{L}}$ are representable that these morphisms are also representable. By 8.6, the right vertical arrow of the diagram (2) is a closed immersion. Therefore, since the squares of the diagram are cartesian, the remaining two arrows are closed immersions. Since the presheaves of the lower row of (2) are locally finitely presentable, this means that presheaves of the upper row are locally of strictly finite type. ■

8.11. Proposition. *Let $\mathcal{V}$ be a projective R-module of finite type and $\mathcal{V} \xrightarrow{\psi} \mathcal{M}$ an R-module morphism. If the R-module $\mathcal{M}$ is of finite type, then the canonical map*

$$F_{\psi;\mathcal{M},V} \longrightarrow \mathfrak{Gr}_{\mathcal{M}}$$

is an open immersion.

Proof. The canonical map in question is the composition of the canonical embedding $F_{\psi;\mathcal{M},V} \xrightarrow{\mathfrak{j}_{\psi}} Gr_{\mathcal{M},\mathcal{V}}$ and $Gr_{\mathcal{M},\mathcal{V}} \xrightarrow{\rho_{\mathcal{V}}} \mathfrak{Gr}_{\mathcal{M}}$. Let $\mathcal{M}$ be an R-module of finite type, and let $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ be an R-module epimorphism with $\mathcal{L}$ a projective R-module of finite type. Then we have a commutative diagram

$$\begin{array}{ccccc} F_{\psi;\mathcal{M},\mathcal{V}} & \xrightarrow{\mathfrak{j}_{\psi}} & Gr_{\mathcal{M},\mathcal{V}} & \xrightarrow{\rho_{\mathcal{V}}} & \mathfrak{Gr}_{\mathcal{M}} \\ F_{\varphi,\mathcal{V}} \downarrow & cart & \downarrow Gr_{\varphi,\mathcal{V}} & cart & \downarrow \mathfrak{Gr}_{\varphi} \\ F_{\phi;\mathcal{L},\mathcal{V}} & \xrightarrow{\mathfrak{j}_{\phi}} & Gr_{\mathcal{L},\mathcal{V}} & \xrightarrow{\rho_{\mathcal{V}}} & \mathfrak{Gr}_{\mathcal{L}} \end{array}$$

whose left square is cartesian by 6.7.3 and the right one is cartesian by 8.6. By 6.7.3, the left lower horizontal arrow is an open immersion; and, by 8.9, the right lower horizontal arrow is an open immersion. Therefore, both upper horizontal arrows are open immersions. And composition of open immersions is an open immersion. ∎

9. Generic flags.

9.1. The presheaf $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$. Let $\mathcal{M}$ be a left R-module and $\mathfrak{I}$ a preorder with an initial object $\bullet$. For every R-ring (S,s), we denote by $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ the set of isomorphism classes of functors $\mathfrak{I} \longrightarrow S-mod$ which map every arrow $\bullet \longrightarrow i$ to epimorphism $s^*(\mathcal{M}) \longrightarrow V_i$, where V_i is a projective S-module of finite type. The map $(S,s) \longmapsto \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ extends naturally to a presheaf

$$(\mathbf{Aff}_k/R^{\vee})^{op} = R\backslash Alg_k \xrightarrow{\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}} Sets.$$

9.1.1. Note. If the preorder $\mathfrak{I}$ consists of one arrow $\bullet \longrightarrow x$, then the generic flag variety $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ coincides with the generic Grassmannian $\mathfrak{Gr}_{\mathcal{M}}$.

9.2. The presheaf $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}$, cover of the generic flag variety, and relations. For an R-ring (S,s), we denote by $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ the set of projectors $\{p_i = p_i^2 \mid i \in \mathfrak{I}\}$ of $s^*(\mathcal{M})$ such that $p_\bullet = id_{s^*(\mathcal{M})}$, the image of p_i is a projective S-module of finite type, provided i is not an initial object; and $p_i p_j = p_j$ if $i \le j$.

The map $(S,s) \longmapsto \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ extends to a presheaf

$$(\mathbf{Aff}_k/R^{\vee})^{op} = R\backslash Alg_k \xrightarrow{\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}} Sets.$$

For studying the relations of the presheaf $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}$ with the generic flag variety, we need the following lemma, which shall be useful other constructions as well.

9.2.1. Lemma. *Let C_X be a category, $\mathfrak{E}_X$ a class of epimorphisms of C_X closed under composition and containing all isomorphisms. Let $\mathfrak{I}$ be a small category with an initial object $\bullet$ and $\mathfrak{I} \xrightarrow{\Phi} C_X$ a functor which maps all arrows of the category $\mathfrak{I}$ to $\mathfrak{E}_X$ and all non-initial objects to projective objects with respect to $\mathfrak{E}_X$. Then there exists a functor $\mathfrak{I}^{op} \xrightarrow{\Psi} C_X$ such that $\Phi(\gamma) \circ \Psi(\gamma) = id_{\Phi(y)}$ for every arrow $x \xrightarrow{\gamma} y$ of $\mathfrak{I}$.*

Proof. (i) The existence of an initial object in $\mathfrak{I}$ and the condition that the functor Φ maps all arrows to epimorphisms imply that, for any pair x, y of objects of the category $\mathfrak{I}$, the functor Φ maps $\mathfrak{I}(x,y)$ to one arrow. Therefore, we assume that $\mathfrak{I}$ is a preorder.

(ii) We shall call the morphisms of $\mathfrak{E}_X$ *deflations*.

Let Ξ_Φ denote all full subpreorders of $\mathfrak{I}$ which contain the initial object $\bullet$ and for which the assertion holds. Since the functor Φ maps all arrows to deflations and all non-initial objects to projective objects, for every arrow $x \xrightarrow{\gamma} y$, there exists a splitting of the morphism $\Phi(\gamma)$. This implies that the family Ξ_Φ contains all ordered subsets of the preorder $\mathfrak{I}$. Therefore, by Zorn Lemma, the set Ξ_Φ contains maximal elements with respect to the inclusion. Let $\mathfrak{I}_0$ be one of them, and let Ψ_0 denote the functor $\mathfrak{I}_0^{op} \longrightarrow C_X$, which is right inverse to the restriction of Φ to $\mathfrak{I}_0$, i.e. $\Phi(\gamma) \circ \Psi_0(\gamma) = id_{\Phi(x)}$ for any arrow $x \xrightarrow{\gamma} y$ of the subpreorder $\mathfrak{I}_0$. Suppose that $\mathfrak{I}_0 \neq \mathfrak{I}$, and let $\mathfrak{y}$ be an object of $\mathfrak{I}$ which does not belong to $\mathfrak{I}_0$. Since the morphism $\Phi(\bullet \xrightarrow{\mathfrak{e}_\mathfrak{y}} \mathfrak{y})$ is a deflation and the object $\Phi(\mathfrak{y})$ is projective, there exists a splitting $\Phi(\mathfrak{y}) \xrightarrow{\mathfrak{j}_\mathfrak{y}} \Phi(\bullet)$ of the deflation $\Phi(\mathfrak{e}_\mathfrak{y})$.

Let $\mathfrak{I}_\mathfrak{y}$ denote the full subpreorder of the preorder $\mathfrak{I}$ defined by $Ob\mathfrak{I}_\mathfrak{y} = Ob\mathfrak{I}_0 \cup \{\mathfrak{y}\}$. We denote by $\Psi_\mathfrak{y}$ the diagram $\mathfrak{I}_\mathfrak{y}^{op} \longrightarrow C_X$ defined as follows:

— the restriction of $\Psi_\mathfrak{y}$ to $\mathfrak{I}_0^{op}$ coincides with Ψ_0;

— $\Psi_\mathfrak{y}(\mathfrak{e}_\mathfrak{y}) = \mathfrak{j}_\mathfrak{y}$ – a splitting of the deflation $\Phi(\mathfrak{e}_\mathfrak{y})$;

— for every arrow $x \xrightarrow{\gamma} \mathfrak{y}$ of $\mathfrak{I}_\mathfrak{y}$ with $x \in Ob\mathfrak{I}_0$, we set $\Psi_\mathfrak{y}(\gamma) = \Phi(\mathfrak{e}_\mathfrak{y}) \circ \Psi_0(\mathfrak{e}_x)$;

— for every arrow $\mathfrak{y} \xrightarrow{\lambda} z$ of $\mathfrak{I}_\mathfrak{y}$ with $z \in Ob\mathfrak{I}_0$, we set $\Psi_\mathfrak{y}(\lambda) = \Phi(\mathfrak{e}_z) \circ \Psi_\mathfrak{y}(\mathfrak{e}_\mathfrak{y})$.

One can see that the diagram $\mathfrak{I}_\mathfrak{y}^{op} \xrightarrow{\Psi_\mathfrak{y}} C_X$ is a functor satisfying the condition $\Phi(\gamma) \circ \Psi_\mathfrak{y}(\gamma) = id_{\Phi(y)}$ for all $\gamma \in Hom\mathfrak{I}_\mathfrak{y}$, which contradicts to the maximality of $\mathfrak{I}_0$. ■

9.2.2. Proposition. *(a) There is a natural epimorphism*

$$\mathfrak{F}_\mathcal{M}^\mathfrak{I} \xrightarrow{\pi_\mathcal{M}^\mathfrak{I}} \mathfrak{Fl}_\mathcal{M}^\mathfrak{I}.$$

(b) The relations presheaf $\mathfrak{R}_\mathcal{M}^\mathfrak{I} = \mathfrak{F}_\mathcal{M}^\mathfrak{I} \prod_{\mathfrak{Fl}_\mathcal{M}^\mathfrak{I}} \mathfrak{F}_\mathcal{M}^\mathfrak{I}$ *maps every R-ring* (S,s) *to the set* $(p_i,\ p'_i \mid i \in \mathfrak{I})$ *of projectors of the S-module* $s^*(\mathcal{M})$ *satisfying the following relations:*

$$p_i p'_i = p_i, \quad p'_i p_i = p'_i; \quad p_i p_j = p_j,\ p'_i p'_j = p'_j, \quad if \quad i \le j. \tag{1}$$

and such that $p_i(s^*(\mathcal{M}))$ *and* $p'_i(s^*(\mathcal{M}))$ *are projective S-modules of finite type for all* $i \in \mathfrak{I}$.

(c) If $\mathcal{M}$ *is a projective module of finite type and* $\mathfrak{I}$ *is finite, then the presheaves* $\mathfrak{F}_\mathcal{M}^\mathfrak{I}$ *and* $\mathfrak{R}_\mathcal{M}^\mathfrak{I}$ *are representable and locally finitely corepresentable.*

Proof. (a1) For every R-ring (S,s), the map

$$\mathfrak{F}_\mathcal{M}^\mathfrak{I}(S,s) \xrightarrow{\pi_\mathcal{M}^\mathfrak{I}(S,s)} \mathfrak{Fl}_\mathcal{M}^\mathfrak{I}(S,s) \tag{2}$$

assigns to every element $\{p_i \mid i \in \mathfrak{I}\}$ the isomorphism class of the functor which maps every arrow $\bullet \longrightarrow i$ to the projection of S-modules $s^*(\mathcal{M}) \longrightarrow p_i(s^*(\mathcal{M}))$ and every arrow

$i \longrightarrow j$ to the S-module epimorphism $p_i(s^*(\mathcal{M})) \longrightarrow p_j(s^*(\mathcal{M}))$ uniquely determined by the commutativity of the diagram

$$\begin{array}{ccc} & s^*(\mathcal{M}) & \\ \swarrow & & \searrow \\ p_i(s^*(\mathcal{M})) & \longrightarrow & p_j(s^*(\mathcal{M})) \end{array}$$

The map (2) is functorial in (S,s); i.e it defines a presheaf morphism

$$\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}.$$

(a2) It remains to show that the map (2) is surjective.

Let ξ be an element of $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ and $\mathfrak{I} \xrightarrow{\Phi_\xi} S-mod$ a functor which represents the element ξ. The functor $\mathcal{V}_\xi$ maps the initial object $\bullet$ to the S-module $s^*(\mathcal{M})$ and every arrow of $\mathfrak{I}$ to an epimorphism to a projective S-module. Applying 9.2.1 to this case (with $C_X = S-mod$ and $\mathfrak{E}_X$ coinciding with the class of all epimorphisms of $S-mod$), we obtain an existence of a functor $\mathfrak{I}^{op} \xrightarrow{\Psi_\xi} S-mod$ which maps every arrow $i \to j$ of $\mathfrak{I}$ to a morphism splitting $\mathcal{V}_\xi(i \to j)$; that is $\mathcal{V}_\xi(i \to j) \circ \Psi_\xi(i \to j) = id_{\mathcal{V}_\xi(j)}$. The set of compositions $\{p_i = \Psi_\xi(\bullet \to i) \circ \mathcal{V}_\xi(\bullet \to i) \mid i \in \mathfrak{I}\}$ is an element of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ which the map (2) sends to the element ξ.

(b) The relations

$$p_i p'_i = p_i, \quad p'_i p_i = p'_i; \quad p_i p_j = p_j, \; p'_i p'_j = p'_j, \quad \text{if} \quad i \le j.$$

express the fact that the map (2) sends $(p_i \mid i \in \mathfrak{I})$ and $(p'_i \mid i \in \mathfrak{I})$ to the same element. The argument is similar to that in 8.3.

(c1) By definition, for every R-ring (S,s), the set $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ consists of all families $\{p_i|\ i \in \mathfrak{I}\}$ of endomorphisms of the S-module $s^*(\mathcal{M})$ such that $p_\bullet = id_{s^*(\mathcal{M})}$ and $p_i p_j = p_j$ if $i \le j$. In other words, $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ is the limit of the diagram

$$End_S(s^*(\mathcal{M}))^{\times \mathfrak{I}} \xrightarrow{\mathfrak{p}_i \times \mathfrak{p}_j} End_S(s^*(\mathcal{M})) \prod End_S(s^*(\mathcal{M})) \overset{\mathfrak{m}}{\underset{\mathfrak{p}_1}{\rightrightarrows}} End_S(s^*(\mathcal{M})) \qquad (6)$$

for all $i, j \in \mathfrak{I} - \{\bullet\}$ such that $i \le j$.

Here the left arrow is the projection of the product of $\mathfrak{I}$ copies of $End_S(s^*(\mathcal{M}))$ on the product of i^{th} and j^{th} components, the upper right arrow is the composition and the lower one is the projection on the first component. If $\mathcal{M}$ is a projective R-module of finite type, then, by 5.3, the presheaf $(S,s)^\vee \longmapsto End_S(s^*(\mathcal{M}))$ is isomorphic to the vector fiber

$\mathbb{V}_R(\mathcal{M}\otimes_k\mathcal{M}^\vee)$ of the projective R^e-module of finite type $\mathcal{M}\otimes_k\mathcal{M}^\vee$, which is presentable and, by 5.3.4, locally finitely copresentable. So that if $\mathfrak{I}$ is finite, then $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}$ is the limit of a finite diagram of presentable and locally finitely corepresentable presheaves. Therefore, it is presentable and, by II.1.4, it is locally finitely corepresentable.

(c2) A similar argument shows that the presheaf $\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}$ of relations (described by the equalities (1)) is the limit of the diagram

$$\begin{array}{ccccc}
\mathcal{E}_R(\mathcal{M})^{\times\mathfrak{I}} & \xrightarrow{\mathfrak{p}_i\times\mathfrak{p}_j} & \mathcal{E}_R(\mathcal{M})\times\mathcal{E}_R(\mathcal{M}) & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathcal{E}_R(\mathcal{M}) \\
{\scriptstyle \pi_1^{\mathfrak{I}}}\uparrow & & & & \\
\mathcal{E}_R(\mathcal{M})^{\times\mathfrak{I}}\prod\mathcal{E}_R(\mathcal{M})^{\times\mathfrak{I}} & \xrightarrow{\mathfrak{p}_i\times\mathfrak{p}_i} & \mathcal{E}_R(\mathcal{M})\times\mathcal{E}_R(\mathcal{M}) & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathcal{E}_R(\mathcal{M}) \\
\mathfrak{S}\downarrow\wr & & \sigma\downarrow\wr & & \\
\mathcal{E}_R(\mathcal{M})^{\times\mathfrak{I}}\prod\mathcal{E}_R(\mathcal{M})^{\times\mathfrak{I}} & \xrightarrow{\mathfrak{p}_i\times\mathfrak{p}_i} & \mathcal{E}_R(\mathcal{M})\times\mathcal{E}_R(\mathcal{M}) & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathcal{E}_R(\mathcal{M}) \\
{\scriptstyle \pi_1^{\mathfrak{I}}}\downarrow & & & & \\
\mathcal{E}_R(\mathcal{M})^{\times\mathfrak{I}} & \xrightarrow{\mathfrak{p}_i\times\mathfrak{p}_j} & \mathcal{E}_R(\mathcal{M})\times\mathcal{E}_R(\mathcal{M}) & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathcal{E}_R(\mathcal{M}) \\
 & & i\in\mathfrak{I}-\{\bullet\}\ni j & \text{and} & i<j.
\end{array} \tag{4}$$

Here $\mathcal{E}_R(\mathcal{M})\overset{\text{def}}{=}\mathcal{H}_R(\mathcal{M},\mathcal{M})$ – the presheaf $(S,s)^\vee\longmapsto End_S(s^*(\mathcal{M}))$ of endomorphisms of the R-module $\mathcal{M}$; $\mathfrak{S}$ denotes the standard symmetry; the vertical arrow $\pi_1^{\mathfrak{I}}$ and the (four times repeated) horizontal arrow π_1 in the right column are projections to the first component, and the (four times repeated) arrow $\mathcal{E}_R(\mathcal{M})\times\mathcal{E}_R(\mathcal{M})\xrightarrow{m}\mathcal{E}_R(\mathcal{M})$ is the composition map.

If the R-module $\mathcal{M}$ is projective and $\mathfrak{I}$ is finite, then the presheaf $\mathcal{E}_R(\mathcal{M})$ is isomorphic to the vector fiber $\mathbb{V}_R(\mathcal{M}\otimes_k\mathcal{M}^*_k)$. of the projective R^e-module of finite type $\mathcal{M}\otimes_k\mathcal{M}^*_k$. So that (4) becomes a finite diagram with values in representable presheaves and locally finitely corepresentable preshaves. Therefore, the limit of (4) is a representable and locally finitely corepresentable presheaf. ■

9.3. Proposition. *(a) There is a natural embedding*

$$\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}\longrightarrow\prod_{i\in\mathfrak{I}}\mathfrak{Gr}_{\mathcal{M}}=\mathfrak{Gr}^{\times\mathfrak{I}}_{\mathcal{M}}. \tag{1}$$

(b) If the module $\mathcal{M}$ is projective of finite type and $\mathfrak{I}$ is finite, then the morphism (1) is a closed immersion.

Proof. (a) The existence of the monomorphism (1) follows from the definitions of the presheaves of sets $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$ and $\mathfrak{Gr}_{\mathcal{M}}$.

(b) We have a commutative square of canonical covers

$$\begin{array}{ccc} \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}} & \xrightarrow{\pi^{\mathfrak{J}}_{\mathcal{M}}} & \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}} \\ \big\downarrow & cart & \big\downarrow \\ \prod\limits_{i\in\mathfrak{J}} \mathfrak{Pr}_{\mathcal{M}} & \xrightarrow{\pi^{\times\mathfrak{J}}_{\mathcal{M}}} & \prod\limits_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}} \end{array} \tag{2}$$

which is cartesian. Let (S,s) be an R-ring and $(S,s)^\vee \xrightarrow{\xi} \prod\limits_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}}$ and arbitrary presheaf morphism. Since $(S,s)^\vee$ is a projective object in the category of presheaves and $\prod\limits_{i\in\mathfrak{J}} \mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi^{\times\mathfrak{J}}_{\mathcal{M}}} \prod\limits_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}}$ a presheaf epimorphism, the morphism $(S,s)^\vee \xrightarrow{\xi} \prod\limits_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}}$ factors through the epimorphism $\pi^{\times\mathfrak{J}}_{\mathcal{M}}$; that is $\xi = \pi^{\times\mathfrak{J}}_{\mathcal{M}} \circ \widetilde{\xi}$ for some $(S,s)^\vee \xrightarrow{\widetilde{\xi}} \prod\limits_{i\in\mathfrak{J}} \mathfrak{Pr}_{\mathcal{M}}$. So that the pull-back of the morphism $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}} \longrightarrow \prod\limits_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}}$ along $(S,s)^\vee \xrightarrow{\xi} \prod\limits_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}}$ can be decomposed in two cartesian squares:

$$\begin{array}{ccccc} T,t)^\vee & \xrightarrow{\xi'} & \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}} & \xrightarrow{\pi^{\mathfrak{J}}_{\mathcal{M}}} & \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}} \\ \big\downarrow & cart & \big\downarrow & cart & \big\downarrow \\ (S,s)^\vee & \xrightarrow{\widetilde{\xi}} & \prod\limits_{i\in\mathfrak{J}} \mathfrak{Pr}_{\mathcal{M}} & \xrightarrow{\pi^{\times\mathfrak{J}}_{\mathcal{M}}} & \prod\limits_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}} \end{array} \tag{3}$$

It follows from the argument of 9.2.2(c1) that the presheaf $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}$ is the kernel of the pair of arrows

$$\prod_{i\in\mathfrak{J}} \mathfrak{Pr}_{\mathcal{M}} \xrightarrow{(\mathfrak{p}_i\times\mathfrak{p}_j)} \prod_{i<j} \big(\mathfrak{Pr}_{\mathcal{M}} \times \mathfrak{Pr}_{\mathcal{M}} \underset{\pi_1}{\overset{\mathfrak{m}}{\rightrightarrows}} \mathfrak{Pr}_{\mathcal{M}}\big), \tag{4}$$

which encodes the diagram (3) in the argument of 9.2.2. If $\mathcal{M}$ is a projective of finite type and $\mathfrak{J}$ is finite, then all presheaves in the diagram (4) are representable and the morphism $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}} \longrightarrow \prod\limits_{i\in\mathfrak{J}} \mathfrak{Pr}_{\mathcal{M}}$, being the kernel of a pair of representable morphisms is a closed immersion. Therefore, since pull-backs of closed immersions are closed immersions, the left vertical arrow of the diagram (3) is a closed immersion. ∎

9.4. Proposition. *(a) Every R-module epimorphism $\mathcal{N} \xrightarrow{\varphi} \mathcal{M}$ induces a presheaf monomorphism*

$$\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}} \xrightarrow{\mathfrak{Fl}^{\mathfrak{J}}_{\varphi}} \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{N}}. \tag{1}$$

(b) Suppose that $\mathcal{M}$ is an R-module of finite type and $\mathfrak{J}$ is finite. Then the morphism (1) is a closed immersion.

Proof. (a) This follows from the definitions of $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$ and the morphism $\mathfrak{Fl}^{\mathfrak{J}}_{\varphi}$.

(b) We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}} & \longrightarrow & \prod_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}} \\ {\scriptstyle \mathfrak{Fl}_{\varphi}}\big\downarrow & cart & \big\downarrow {\scriptstyle \mathfrak{Gr}^{\times\mathfrak{J}}_{\varphi}} \\ \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{N}} & \longrightarrow & \prod_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{N}} \end{array} \tag{2}$$

which is a cartesian square. If the R-module $\mathcal{M}$ is of finite type, then, by 8.6, the morphism $\mathfrak{Gr}_{\mathcal{M}} \xrightarrow{\mathfrak{Gr}_{\varphi}} \mathfrak{Gr}_{\mathcal{N}}$ is a closed immersion. Therefore, since $\mathfrak{J}$ is finite, the right vertical arrow of (2) is a closed immersion. Since closed immersions are stable under pull-backs, the left vertical arrow of (2) is a closed immersion too. ∎

9.4.1. Corollary. *If the module $\mathcal{M}$ is an R-module of finite type and $\mathfrak{J}$ is finite, then the canonical morphism*

$$\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}} \longrightarrow \prod_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}}. \tag{3}$$

is a closed immersion.

Proof. Since $\mathcal{M}$ is a module of finite type, there is an epimorphism $\mathcal{N} \xrightarrow{\varphi} \mathcal{M}$ for a projective R-module of finite type $\mathcal{N}$. By 9.3, the lower horizontal arrow of the cartesian square (2) is a closed immersion. Therefore, thanks to the invariance of closed immersions under pull-backs, the upper horizontal arrow of (2) is a closed immersion. ∎

9.5. Proposition. *If $\mathcal{M}$ is an R-module of finite type and $\mathfrak{J}$ is finite, then the presheaf $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$ is separated.*

Proof. If $\mathcal{M}$ is an R-module of finite type, then, by 8.7, the generic Grassmannian $\mathfrak{Gr}_{\mathcal{M}}$ is separated. Since $\mathfrak{J}$ is finite, the product $\prod_{i\in\mathfrak{J}} \mathfrak{Gr}_{\mathcal{M}}$ is a separated presheaf too.

By 9.3, the canonical morphism $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \longrightarrow \prod_{i\in\mathfrak{I}} \mathfrak{Gr}_{\mathcal{M}}$ is a monomorphism (actually, a closed immersion). Therefore, since any subpresheaf of a separated presheaf is separated, the presheaf $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ is separated. ∎

9.6. Base change.

9.6.1. Proposition. *For any R-ring (S,s), there is a natural isomorphism between*

$$(S,s)\prod_{R^\vee}\big(\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}} \underset{\mathfrak{q}^{\mathfrak{I}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{\mathfrak{I}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}\big) \quad \textit{and} \quad \mathfrak{R}^{\mathfrak{I}}_{s^*(\mathcal{M})} \underset{\mathfrak{q}^{\mathfrak{I}}_{s^*(\mathcal{M})}}{\overset{\mathfrak{p}^{\mathfrak{I}}_{s^*(\mathcal{M})}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{s^*(\mathcal{M})} \xrightarrow{\pi^{\mathfrak{I}}_{s^*(\mathcal{M})}} \mathfrak{Fl}^{\mathfrak{I}}_{s^*(\mathcal{M})}.$$

Proof. By the argument of 9.3, the square

$$\begin{array}{ccc}
\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} & \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} & \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \\
\big\downarrow & cart & \big\downarrow \\
\prod_{i\in\mathfrak{I}} \mathfrak{Pr}_{\mathcal{M}} & \xrightarrow{\pi^{\times\mathfrak{I}}_{\mathcal{M}}} & \prod_{i\in\mathfrak{I}} \mathfrak{Gr}_{\mathcal{M}}
\end{array}$$

is cartesian. In particular, the square

$$\begin{array}{ccc}
\mathfrak{F}^{\mathfrak{I}}_{s^*(\mathcal{M})} & \xrightarrow{\pi^{\mathfrak{I}}_{s^*(\mathcal{M})}} & \mathfrak{Fl}^{\mathfrak{I}}_{s^*(\mathcal{M})} \\
\big\downarrow & cart & \big\downarrow \\
\prod_{i\in\mathfrak{I}} \mathfrak{Pr}_{s^*(\mathcal{M})} & \xrightarrow{\pi^{\times\mathfrak{I}}_{s^*(\mathcal{M})}} & \prod_{i\in\mathfrak{I}} \mathfrak{Gr}_{s^*(\mathcal{M})}
\end{array}$$

is cartesian. On the other hand, the base change functor $(S,s)^\vee \prod_{R^\vee} -$ preserves cartesian squares, and, by 8.4.1, $(S,s)^\vee \prod_{R^\vee}\big(\mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}\big)$ is isomorphic to the morphism $\mathfrak{Pr}_{s^*(\mathcal{M})} \xrightarrow{\pi_{s^*(\mathcal{M})}} \mathfrak{Gr}_{s^*(\mathcal{M})}$. This gives a natural isomorphism between

$$(S,s)^\vee \prod_{R^\vee}\big(\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}\big) \quad \text{and} \quad \mathfrak{F}^{\mathfrak{I}}_{s^*(\mathcal{M})} \xrightarrow{\pi^{\mathfrak{I}}_{s^*(\mathcal{M})}} \mathfrak{Fl}^{\mathfrak{I}}_{s^*(\mathcal{M})}.$$

Since the functor $(S,s)^\vee \prod_{R^\vee} -$ preserves cartesian squares, an isomorphism between

$$(S,s)^\vee \prod_{R^\vee} \big(\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}\big) \quad \text{and} \quad \mathfrak{F}^{\mathfrak{I}}_{s^*(\mathcal{M})} \xrightarrow{\pi^{\mathfrak{I}}_{s^*(\mathcal{M})}} \mathfrak{Fl}^{\mathfrak{I}}_{s^*(\mathcal{M})}$$

determines an isomorphism between the exact diagrams

$$(S,s) \prod_{R^\vee} \big(\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}} \overset{\mathfrak{p}^{\mathfrak{I}}_{\mathcal{M}}}{\underset{\mathfrak{q}^{\mathfrak{I}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}\big) \quad \text{and} \quad \mathfrak{R}^{\mathfrak{I}}_{s^*(\mathcal{M})} \overset{\mathfrak{p}^{\mathfrak{I}}_{s^*(\mathcal{M})}}{\underset{\mathfrak{q}^{\mathfrak{I}}_{s^*(\mathcal{M})}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{s^*(\mathcal{M})} \xrightarrow{\pi^{\mathfrak{I}}_{s^*(\mathcal{M})}} \mathfrak{Fl}^{\mathfrak{I}}_{s^*(\mathcal{M})},$$

which is the claim. ■

9.7. Smoothness.

9.7.0. The choice of infinitesimal morphisms. In this Section, we take as *infinitesimal* morphisms the class $\mathfrak{M}^{\mathfrak{c}}_{\mathfrak{I}}$ of *complete radical closed immersions* (see 8.8.0 and II.3.8.2), same as in 8.8, and consider (formal) smoothness with respect to this class.

In this Section, we shall write (formally) smooth meaning (formally) $\mathfrak{M}^{\mathfrak{c}}_{\mathfrak{I}}$-smooth.

9.7.1. Proposition. *(a) All presheaves and morphisms in the canonical diagram*

$$\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}} \overset{\mathfrak{p}^{\mathfrak{I}}_{\mathcal{M}}}{\underset{\mathfrak{q}^{\mathfrak{I}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \tag{1}$$

are formally smooth.

(b) If $\mathcal{M}$ is a projective R-module of finite type and the preorder $\mathfrak{I}$ is finite, then all presheaves and morphisms of the diagram (1) are smooth.

Proof. As in the argument of 8.8.1, we start from the middle term of the diagram (1).

(a1) *The presheaf $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ is formally smooth.*

(i) Let $(T,t) \xrightarrow{\varphi} (S,s)$ be an *infinitesimal* morphism of R-rings and

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\xi} & \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \\ {\scriptstyle \varphi^\vee}\downarrow & & \downarrow \\ (T,t)^\vee & \xrightarrow{t^\vee} & R^\vee \end{array}$$

a commutative diagram of presheaves of sets. Its upper horizontal arrow is determined by an element $\widehat{\xi}$ of $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$, that is by the isomorphism class of a functor $\mathfrak{I} \xrightarrow{\mathcal{V}} S-mod$ which map every arrow $\bullet \longrightarrow i$ to an epimorphism $\varphi^* t^*(\mathcal{M}) = s^*(\mathcal{M}) \xrightarrow{\mathfrak{u}_i} \varphi^*(\mathcal{V}_i)$, where $\mathcal{V}_i$ is a projective T-module of finite type. Here we use the fact that, since the R-ring morphism $(T,t) \xrightarrow{\varphi} (S,s)$ is infinitesimal, every projective S-module of finitey type is isomorphic to the inverse image of a projective T-module of finite type.

Since $t^*(\mathcal{M})$ is a projective T-module and the adjunctions arrows $\mathcal{V}_i \longrightarrow \varphi_*\varphi^*(\mathcal{V}_i)$ are epimorphisms for all $i \in \mathfrak{I}$, there exists a morphism $t^*(\mathcal{M}) \xrightarrow{\widetilde{\mathfrak{u}}_i} \mathcal{V}_i$ such that the diagram

$$\begin{array}{ccc} t^*(\mathcal{M}) & \xrightarrow{\widetilde{\mathfrak{u}}_i} & \mathcal{V}_i \\ \big\downarrow & & \big\downarrow \\ \varphi_*\varphi^*t^*(\mathcal{M}) & \xrightarrow{\varphi_*(\mathfrak{u}_i)} & \varphi_*\varphi^*(\mathcal{V}_i) \end{array} \qquad (2)$$

commutes. By adjunction, this means precisely that $\mathfrak{u}_i = \varphi^*(\widetilde{\mathfrak{u}}_i)$.

Since $\mathcal{V}_i$ is a module of finite type and $T \xrightarrow{\varphi} S$ is a strict epimorphism whose kernel is contained in the Jacobson radical of the algebra T, it follows from Nakayama Lemma that $t^*(\mathcal{M}) \xrightarrow{\widetilde{\mathfrak{u}}_i} \mathcal{V}_i$ is an epimorphism (see [Ba, III.2.12]).

(ii) *The morphism* $\widetilde{\mathfrak{u}}_i$ *is uniquely defined by the equality* $\mathfrak{u}_i = \varphi^*(\widetilde{\mathfrak{u}}_i)$.

In fact, let $t^*(\mathcal{M}) \xrightarrow{\mathfrak{u}'_i} \mathcal{V}_i$ be another morphism such that $\mathfrak{u}_i = \varphi^*(\mathfrak{u}'_i)$. Let $\widetilde{\mathcal{M}}$ denote the quotient of the module $t^*(\mathcal{M})$ by the intersection $Ker(\widetilde{\mathfrak{u}}_i) \cap Ker(\mathfrak{u}'_i)$ of kernels of $\widetilde{\mathfrak{u}}_i$ and $\mathfrak{u}'_i$. Both morphisms factor through the canonical epimorphism $t^*(\mathcal{M}) \xrightarrow{\mathfrak{e}} \widetilde{\mathcal{M}}$; that is $\widetilde{\mathfrak{u}}_i = \widetilde{\xi}_i \circ \mathfrak{e}$ and $\mathfrak{u}'_i = \xi'_i \circ \mathfrak{e}$ for uniquely determined epimorphisms $\widetilde{\mathcal{M}} \overset{\widetilde{\xi}_i}{\underset{\xi'_i}{\rightrightarrows}} \mathcal{V}_i$.

Since the functor φ^* maps epimorphisms to epimorphisms, the equalities $\varphi^*(\widetilde{\mathfrak{u}}_i) = \mathfrak{u}_i = \varphi^*(\mathfrak{u}'_i)$ imply that $\varphi^*(\widetilde{\xi}_i) = \varphi^*(\xi'_i)$.

Let $\mathcal{V}_i \xrightarrow{\widetilde{\mathfrak{v}}_i} t^*(\mathcal{M})$ be a splitting of $t^*(\mathcal{M}) \xrightarrow{\widetilde{\mathfrak{u}}_i} \mathcal{V}_i$ and $\mathcal{V}_i \xrightarrow{\mathfrak{v}'_i} t^*(\mathcal{M})$ a splitting of $t^*(\mathcal{M}) \xrightarrow{\mathfrak{u}'_i} \mathcal{V}_i$; that is $\widetilde{\mathfrak{u}}_i \circ \widetilde{\mathfrak{v}}_i = id_{\mathcal{V}_i} = \mathfrak{u}'_i \circ \mathfrak{v}'_i$. One can see that the morphism

$$\mathcal{V}_i \oplus \mathcal{V}_i \oplus \big(Ker(\widetilde{\mathfrak{u}}_i) \cap Ker(\mathfrak{u}'_i)\big) \longrightarrow t^*(\mathcal{M}) \qquad (3)$$

corresponding to the morphisms $\mathcal{V}_i \xrightarrow{\widetilde{\mathfrak{v}}_i} t^*(\mathcal{M})$, $\mathcal{V}_i \xrightarrow{\mathfrak{v}'_i} t^*(\mathcal{M})$, and the embedding $Ker(\widetilde{\mathfrak{u}}_i) \cap Ker(\mathfrak{u}'_i) \longrightarrow t^*(\mathcal{M})$ is an epimorphism. Therefore, the composition of (3) with the epimorphism $t^*(\mathcal{M}) \xrightarrow{\mathfrak{e}} \widetilde{\mathcal{M}}$ induces an epimorphism $\mathcal{V}_i \oplus \mathcal{V}_i \longrightarrow \widetilde{\mathcal{M}}$. The latter implies that $\widetilde{\mathcal{M}}$ is a T-module of finite type.

Let $\mathcal{K} \xrightarrow{\mathfrak{k}} \widetilde{\mathcal{M}}$ be the kernel of the pair of arrows $\widetilde{\mathcal{M}} \overset{\widetilde{\xi}_i}{\underset{\xi'_i}{\rightrightarrows}} \mathcal{V}_i$. The functor φ^* is right exact, which implies that the canonical morphism $\varphi^*(\mathcal{K}) \longrightarrow Ker(\varphi^*(\widetilde{\mathcal{M}} \overset{\widetilde{\xi}_i}{\underset{\xi'_i}{\rightrightarrows}} \mathcal{V}_i))$ is an epimorphism. But, since $\varphi^*(\widetilde{\xi}_i) = \varphi^*(\xi'_i)$, the kernel of $\varphi^*(\widetilde{\mathcal{M}} \overset{\widetilde{\xi}_i}{\underset{\xi'_i}{\rightrightarrows}} \mathcal{V}_i)$ is isomorphic to $\varphi^*(\widetilde{\mathcal{M}})$. So that the morphism $\varphi^*(\mathcal{K} \xrightarrow{\mathfrak{k}} \widetilde{\mathcal{M}})$ is an epimorphism. Since $\widetilde{\mathcal{M}}$ is a module of finite type, this implies, by Nakayama Lemma, that $\mathcal{K} \xrightarrow{\mathfrak{k}} \widetilde{\mathcal{M}}$ is an epimorphism, hence an isomorphism. The latter means that $\widetilde{\xi}_i = \xi'_i$. Therefore, $\widetilde{\mathfrak{u}}_i = \widetilde{\xi}_i \circ \mathfrak{e} = \xi'_i \circ \mathfrak{e} = \mathfrak{u}'_i$.

(iii) By the same reason, for every $i \leq j$, there exists a unique morphism $\mathcal{V}_i \xrightarrow{\widetilde{\mathfrak{u}}_{ij}} \mathcal{V}_j$ such that $\varphi^*(\widetilde{\mathfrak{u}}_{ij}) = \mathfrak{u}_{ij}$. The uniqueness of the morphisms $\widetilde{\mathfrak{u}}_i$ and $\widetilde{\mathfrak{u}}_{ij}$ implies that $\widetilde{\mathfrak{u}}_{ij} \circ \widetilde{\mathfrak{u}}_i = \widetilde{\mathfrak{u}}_j$ for all $i \leq j$. In other words, the map which assigns to every $i \in \mathfrak{I}$ the epimorphism $t^*(\mathcal{M}) \xrightarrow{\widetilde{\mathfrak{u}}_i} \mathcal{V}_i$ and to every $i \leq j$ the epimorphism of T-modules $\mathcal{V}_i \xrightarrow{\widetilde{\mathfrak{u}}_{ij}} \mathcal{V}_j$ is a functor $\mathfrak{I} \xrightarrow{\widetilde{\mathcal{V}}} T - mod$ such that the initial functor $\mathfrak{I} \xrightarrow{\mathcal{V}} S - mod$ representing an element of $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(S, s)$, or, what is the same, a morphism $(S, s)^\vee \xrightarrow{\xi} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ which was arbitrarily chosen at the beginning of this argument (see (i) above) is equal to the composition $\varphi^* \circ \widetilde{\mathcal{V}}$. This means that the isomorphism class of the functor $\widetilde{\mathcal{V}}$ – an element of $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(T, t)$, defines a morphism $(T, t)^\vee \xrightarrow{\gamma} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ such that the diagram

$$\begin{array}{ccc} (S,s)^\vee & \xrightarrow{\ \xi\ } & \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \\ {\scriptstyle \varphi^\vee}\downarrow & {\scriptstyle \gamma}\nearrow & \downarrow \\ (T,t)^\vee & \xrightarrow{\ t^\vee\ } & R^\vee \end{array}$$

commutes. Equivalently, the map $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(T, t) \xrightarrow{\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(\varphi)} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(S, s)$ is surjective.

(a2) By the argument of 9.3, the square

$$\begin{array}{ccc} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} & \xrightarrow{\ \pi^{\mathfrak{I}}_{\mathcal{M}}\ } & \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \\ \downarrow & cart & \downarrow \\ \prod_{i \in \mathfrak{I}} \mathfrak{Pr}_{\mathcal{M}} & \xrightarrow{\ \pi^{\times \mathfrak{I}}_{\mathcal{M}}\ } & \prod_{i \in \mathfrak{I}} \mathfrak{Gr}_{\mathcal{M}} \end{array} \qquad (4)$$

is cartesian. By 8.8, the morphism $\mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}$ is formally smooth. Any product of formally smooth morphisms is a formally smooth morphism. In particular, the lower horizontal arrow of the square (4) is a formally smooth morphism. Since the square (2) is cartesian, its upper horizontal arrow, the morphism $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$, is formally smooth.

(a3) The pull-backs of formally smooth morphisms are formally smooth. So that formal smoothness of the morphism $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ implies formal smoothness of the arrows of the kernel pair $\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}} \underset{\mathfrak{q}^{\mathfrak{I}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{\mathfrak{I}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}$ of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$.

(b1) If $\mathcal{M}$ is a projective R-module of finite type, then, by 8.8, the morphism

$$\mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{Gr}_{\mathcal{M}}$$

and the presheaves $\mathfrak{Pr}_{\mathcal{M}}$ and $\mathfrak{Gr}_{\mathcal{M}}$ are smooth. Finite products of smooth morphisms are smooth morphisms. So that if $\mathfrak{I}$ is finite, then the morphism

$$\prod_{i\in\mathfrak{I}} \mathfrak{Pr}_{\mathcal{M}} \xrightarrow{\pi^{\times\mathfrak{I}}_{\mathcal{M}}} \prod_{i\in\mathfrak{I}} \mathfrak{Gr}_{\mathcal{M}}$$

– the lower horizontal arrow of the cartesian square (2), is smooth. Therefore, the upper horizontal arrow of (4), the morphism $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$, is smooth. By the invariance of smooth morphisms under pull-backs, this implies the smoothness of the morphisms

$$\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}} \underset{\mathfrak{q}^{\mathfrak{I}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{\mathfrak{I}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}. \tag{5}$$

(b2) By 9.2.2(c) the presheaves $\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}$ and $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}$ are locally finitely copresentable. Thanks to the exactness of the diagram

$$\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}} \underset{\mathfrak{q}^{\mathfrak{I}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{\mathfrak{I}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}},$$

the local finite corepresentability of $\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}$ and $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}$ implies that the presheaf $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ is locally finitelty corepresentable. Since, by (a), all three presheaves are formally smooth, they are smooth. ∎

9.7.2. Proposition. *Let $\mathcal{M}$ be an R-module of finite type; and let $\mathfrak{J}$ be finite. Then*

(a) The presheaf $\mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}}$ is of strictly cofinite type.

(b) The presheaf $\mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}}$ is locally affine for the smooth pretopology. More precisely, there exists an exact diagram

$$\mathfrak{R}_{\mathcal{M}}^{\mathfrak{J},\phi} \overset{\mathfrak{p}_1^{\phi}}{\underset{\mathfrak{p}_2^{\phi}}{\rightrightarrows}} \mathfrak{F}_{\mathcal{M}}^{\mathfrak{J},\phi} \xrightarrow{\ \mathfrak{p}_{\phi}\ } \mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}} \qquad (6)$$

whose arrows are representable coverings for the smooth topology. The presheaves $\mathfrak{F}_{\mathcal{M}}^{\mathfrak{J},\phi}$ and (therefore) $\mathfrak{R}_{\mathcal{M}}^{\mathfrak{J},\phi} = K_2(\mathfrak{p}_{\phi})$ are representable and strictly of finite type.

Proof. Since $\mathcal{M}$ is an R-module of finite type, there exists an R-module epimorphism $\mathcal{L} \xrightarrow{\phi} M$ with $\mathcal{L}$ a projective R-module of finite type. This epimorphism ϕ appears as a parameter in the diagram (6). The diagram itself is defined (uniquely up to isomorphism) via the diagram

$$\begin{array}{ccccc}
\mathfrak{R}_{\mathcal{M}}^{\mathfrak{J},\phi} & \overset{\mathfrak{p}_1^{\phi}}{\underset{\mathfrak{p}_2^{\phi}}{\rightrightarrows}} & \mathfrak{F}_{\mathcal{M}}^{\mathfrak{J},\phi} & \xrightarrow{\ \mathfrak{p}_{\phi}\ } & \mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}} \\
\big\downarrow & cart & \big\downarrow & cart & \big\downarrow \mathfrak{Fl}_{\phi} \\
\mathfrak{R}_{\mathcal{L}}^{\mathfrak{J}} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & \mathfrak{F}_{\mathcal{L}}^{\mathfrak{J}} & \xrightarrow{\ \pi_{\mathcal{L}}^{\mathfrak{J}}\ } & \mathfrak{Fl}_{\mathcal{L}}^{\mathfrak{J}}
\end{array} \qquad (7)$$

with cartesian squares. By 9.7.1(b), the presheaves of the bottom of the diagram are smooth and all three arrows of the bottom of the diagram are coverings for the smooth pretopology. It follows from 1.3.1 and the fact that the presheaves $\mathfrak{F}_{\mathcal{L}}^{\mathfrak{J}}$ and $\mathfrak{R}_{\mathcal{L}}^{\mathfrak{J}}$ are representable that these morphisms are also representable. By 9.4(b), the right vertical arrow of the diagram (7) is a closed immersion. Therefore, since the squares of the diagram are cartesian, the remaining two arrows are closed immersions. Since the presheaves of the lower row of (7) are locally finitely presentable, this means that presheaves of the upper row are locally of strictly finite type. ■

9.8. Limit and colimit decompositions.

9.8.1. Restriction morphisms. Let I be a full subpreorder of $\mathfrak{J}$ containing the initial object $\bullet$. For every R-ring (S,s), there is a natural map $\mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}}(S,s) \longrightarrow \mathfrak{Fl}_{\mathcal{M}}^{I}(S,s)$ which assigns to every element of $\mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}}(S,s)$ represented by a functor $\mathfrak{J} \xrightarrow{\mathcal{V}} S-mod$ the element of $\mathfrak{Fl}_{\mathcal{M}}^{I}(S,s)$ represented by the restriction of the functor $\mathcal{V}$ to I. This map is functorial in (S,s); hence it defines a presheaf morphism

$$\mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}} \xrightarrow{\ \mathfrak{p}_{\mathfrak{J}}^{I}\ } \mathfrak{Fl}_{\mathcal{M}}^{I}. \qquad (1)$$

9.8.2. Infinite and finite flags. Let $\mathfrak{S}_{\mathfrak{f}}^{o}(\mathfrak{I})$ denotes the preorder of finite subsets of $\mathfrak{I}$ with respect to the inverse inclusion. One can see that the natural morphism

$$\mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{I}} \longrightarrow \lim_{I\in\mathfrak{S}_{\mathfrak{f}}^{o}(\mathfrak{I})} (\mathfrak{Fl}_{\mathcal{M}}^{I} \mid \mathfrak{p}_{L}^{J})$$

is an isomorphism of presheaves.

9.8.3. Inductive system of finite flag subvarieties. Let I be a full subpreorder of $\mathfrak{I}$ containing the initial object $\bullet$ and having a final object, $\mathfrak{y}_I$. Suppose that, in addition, if i, m are elements of I and j is an element of $\mathfrak{I}$ such $i \le j \le m$, then $j \in I$. Then there is a map

$$\mathfrak{Fl}_{\mathcal{M}}^{I}(S,s) \longrightarrow \mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{I}}(S,s)$$

which assigns to every element of $\mathfrak{Fl}_{\mathcal{M}}^{I}(S,s)$ represented by a functor $I \xrightarrow{\mathcal{V}} S-mod$ the element of $\mathfrak{Fl}_{\mathcal{M}}^{I}(S,s)$ represented by the functor $\mathfrak{I} \xrightarrow{\widetilde{\mathcal{V}}} S-mod$ constructed as follows:

The restriction $\widetilde{\mathcal{V}}|_I$ of $\widetilde{\mathcal{V}}$ to I coincides with $\mathcal{V}$. If $i \in \mathfrak{I} - I$, then $\widetilde{\mathcal{V}} = \mathcal{V}_{\mathfrak{y}_I}$. If $i \le j$ and $i \in \mathfrak{I} - I$, then the morphism $\widetilde{\mathcal{V}}_i \longrightarrow \widetilde{\mathcal{V}}_j$ is identical. If $i \in I$, $i \le j$ and $i \in \mathfrak{I} - I$, then the morphism $\mathcal{V}_i = \widetilde{\mathcal{V}}_i \longrightarrow \widetilde{\mathcal{V}}_j$ coincides with the morphism $\mathcal{V}_i = \widetilde{\mathcal{V}}_i \longrightarrow \mathcal{V}_{\mathfrak{y}_I}$.

This map is functorial in (S,s); i.e. it is the value at (S,s) of a presheaf morphism

$$\mathfrak{Fl}_{\mathcal{M}}^{I} \xrightarrow{\mathfrak{q}_I^{\mathfrak{I}}} \mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{I}}. \qquad (2)$$

It follows that $\mathfrak{p}_{\mathfrak{I}}^{I} \circ \mathfrak{q}_{I}^{\mathfrak{I}} = id_{\mathfrak{Fl}_{\mathcal{M}}^{I}}$. In particular, (2) is a monomorphism.

9.8.4. Proposition. *The morphism (2) is formally smooth. If the R-module $\mathcal{M}$ is of finite type and the filtered subpreorder I is finite, then (2) is an open immersion.*

Proof. (a) Let $(T,t) \xrightarrow{\varphi} (S,s)$ be an *infinitesimal* morphism of R-rings and

$$\begin{array}{ccc} (S,s)^{\vee} & \xrightarrow{\xi} & \mathfrak{F}_{\mathcal{M}}^{I} \\ {\scriptstyle \varphi^{\vee}}\big\downarrow & & \big\downarrow {\scriptstyle \mathfrak{q}_I^{\mathfrak{I}}} \\ (T,t)^{\vee} & \xrightarrow{\zeta} & \mathfrak{F}_{\mathcal{M}}^{\mathfrak{I}} \end{array} \qquad (3)$$

a commutative diagram of presheaves of sets. Its the lower horizontal arrow is determined by an isomorphism class of a functor $\mathfrak{I} \xrightarrow{\mathcal{V}} T-mod$ which maps every arrow $\bullet \longrightarrow i$ to an epimorphism $t^*(\mathcal{M}) \xrightarrow{\widetilde{\mathfrak{u}}_i} \mathcal{V}_i$, where $\mathcal{V}_i$ is a projective T-module of finite type, and to every $\bullet \ne i \le j$, an epimorphism $\mathcal{V}_i \xrightarrow{\widetilde{\mathfrak{u}}_{ij}} \mathcal{V}_j$ such that $\widetilde{\mathfrak{u}}_j = \widetilde{\mathfrak{u}}_{ij} \circ \widetilde{\mathfrak{u}}_i$.

Similarly, the upper horizontal arrow of the diagram (3) is determined by an element $\widehat{\xi}$ of $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(S, s)$, that is by the isomorphism class of a functor $I \xrightarrow{\mathcal{V}'} S - mod$ which maps every arrow $\bullet \longrightarrow i$ to an epimorphism $\varphi^* t^*(\mathcal{M}) = s^*(\mathcal{M}) \xrightarrow{\mathfrak{u}_i} \mathcal{V}'_i$, where $\mathcal{V}'_i$ is a projective S-module of finite type, and to every $\bullet \neq i \leq j$, an epimorphism $\mathcal{V}'_i \xrightarrow{\mathfrak{u}_{ij}} \mathcal{V}'_j$ such that $\mathfrak{u}_j = \mathfrak{u}_{ij} \circ \mathfrak{u}_i$.

The commutativity of the diagram (3) means that the functor $\varphi^* \circ \mathcal{V}$ is isomorphic to the extension of $\mathcal{V}'$ described in 9.8.3. In particular, $\varphi^*(\mathcal{V}_i \xrightarrow{\widetilde{\mathfrak{u}}_{ij}} \mathcal{V}_j)$ is the identity isomorphism if $i \leq j$ and either $i \notin I$, or i is the final object of I. Since all $\mathcal{V}_i$ and $\mathcal{V}_j$ are projective module of finite type, it follows from Nakayama Lemma that $\mathcal{V}_i \xrightarrow{\widetilde{\mathfrak{u}}_{ij}} \mathcal{V}_j$ is an isomorphism under the same conditions on i and j. But, this means that the functor $\mathfrak{I} \xrightarrow{\mathcal{V}} T - mod$ is isomorphic to the extension (described in 9.8.3) of the restriction of $\mathcal{V}$ to the subpreorder I. This means, precisely, that the morphism $(T, t)^\vee \xrightarrow{\zeta} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}$ – the lower arrow of the diagram (3), coincides with $\mathfrak{q}^{\mathfrak{I}}_I \circ \mathfrak{p}^I_{\mathfrak{I}} \circ \zeta$.

Therefore, the morphism $\gamma = \mathfrak{p}^I_{\mathfrak{I}} \circ \zeta$ makes the diagram

$$\begin{array}{ccc} (S, s)^\vee & \xrightarrow{\xi} & \mathfrak{F}^I_{\mathcal{M}} \\ {\scriptstyle \varphi^\vee}\downarrow & {\scriptstyle \gamma}\nearrow & \downarrow{\scriptstyle \mathfrak{q}^{\mathfrak{I}}_I} \\ (T, t)^\vee & \xrightarrow{\zeta} & \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \end{array} \qquad (4)$$

commute. This proves the formal smoothness of the presheaf morphism $\mathfrak{Fl}^I_{\mathcal{M}} \xrightarrow{\mathfrak{q}^{\mathfrak{I}}_I} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$.

(b) Suppose now that $\mathcal{M}$ is a projective module of finite type and I is finite. Then, by 9.7.1(b), the presheaf $\mathfrak{Fl}^I_{\mathcal{M}}$ is smooth; in particular, it is locally finitely copresentable. Since $\mathfrak{Fl}^I_{\mathcal{M}} \xrightarrow{\mathfrak{q}^{\mathfrak{I}}_I} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ is a monomorphism, this implies that it is locally finitely copresentable. Since, by (a) above, the morphism $\mathfrak{Fl}^I_{\mathcal{M}} \xrightarrow{\mathfrak{q}^{\mathfrak{I}}_I} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ is formally smooth, it is smooth.

(c) Suppose that $\mathcal{M}$ is an R-module of finite type. Let $\mathcal{L} \xrightarrow{\psi} \mathcal{M}$ be an R-module epimorphism with $\mathcal{L}$ a projective R-module of finite type. Then we have a commutative square

$$\begin{array}{ccc} \mathfrak{Fl}^I_{\mathcal{M}} & \xrightarrow{\mathfrak{q}^{\mathfrak{I}}_{I,\mathcal{M}}} & \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \\ {\scriptstyle \mathfrak{Fl}^I_\psi}\downarrow & & \downarrow{\scriptstyle \mathfrak{Fl}^{\mathfrak{I}}_\psi} \\ \mathfrak{Fl}^I_{\mathcal{L}} & \xrightarrow{\mathfrak{q}^{\mathfrak{I}}_{I,\mathcal{L}}} & \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{L}} \end{array} \qquad (5)$$

whose vertical arrows are closed immersions corresponding to the epimorphism $\mathcal{L} \xrightarrow{\psi} \mathcal{M}$.

The square (5) is cartesian and, by the argument (b) above, its lower horizontal arrow is an open immersion. Therefore, its upper horizontal arrow is an open immersion too. ∎

9.8.5. Filtered preorders and colimits of finite generic flag varieties. Let $\Xi_{\mathfrak{I}}$ denote the preorder of all finite subpreorders I of the preorder $\mathfrak{I}$ containing the initial object $\bullet$, having the a final object and all intermediate elements of $\mathfrak{I}$.

Suppose that the preorder $\mathfrak{I}$ is filtered. Then, evidently, $\Xi_{\mathfrak{I}}$ is filtered with respect to the inclusion and $\mathfrak{I} = \bigcup_{I \in \Xi_{\mathfrak{I}}} I$. In this case, we have a presheaf monomorphism

$$\operatorname{colim}(\mathfrak{Fl}^I_{\mathcal{M}} \mid I \in \Xi_{\mathfrak{I}},\ \mathfrak{q}^J_I) \longrightarrow \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}. \tag{6}$$

Let $\mathfrak{Fl}^{\mathfrak{I},\mathfrak{f}}_{\mathcal{M}}$ denote the image of the morphism (6). One can see that, for every R-ring (S,s), the set $\mathfrak{Fl}^{\mathfrak{I},\mathfrak{f}}_{\mathcal{M}}(S,s)$ consists of isomorphism classes of functors $\mathfrak{I} \xrightarrow{\mathcal{V}} S-mod$ which map *almost* all (that is with a possible exception of a finite number) arrows of the preorder $\mathfrak{I}$ to isomorphisms. We call $\mathfrak{Fl}^{\mathfrak{I},\mathfrak{f}}_{\mathcal{M}}$ the *subvariety of locally finite flags.*

The isomorphism

$$\operatorname{colim}(\mathfrak{Fl}^I_{\mathcal{M}} \mid I \in \Xi_{\mathfrak{I}},\ \mathfrak{q}^J_I) \xrightarrow{\ \sim\ } \mathfrak{Fl}^{\mathfrak{I},\mathfrak{f}}_{\mathcal{M}} \tag{7}$$

induced by (6) implies that the set

$$\mathfrak{Fl}^I_{\mathcal{M}} \xrightarrow{\mathfrak{q}^{\mathfrak{I}}_I} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}, \quad I \in \Xi_{\mathfrak{I}}, \tag{8}$$

of formally open immersions forms a cover of the subvariety of locally finite flags.

If the R-module $\mathcal{M}$ is of finite type, then, by 9.8.4, the elements of this cover are open immersions; i.e. they form a *Zariski* cover of the subvariety $\mathfrak{Fl}^{\mathfrak{I},\mathfrak{f}}_{\mathcal{M}}$ of locally finite flags.

10. Generic flag varieties and the usual flag varieties. Let R be an associative unital k-algebra, $\mathcal{M}$ an R-module, $\bar{V} = (V_i \mid 1 \le i \le n)$ projective R-modules of finite type. Let $\mathfrak{F}\ell_{\mathcal{M},\bar{V}}$ be the flag variety corresponding to this data (cf. 7.1). It follows from the constructions, that there is a natural presheaf monomorphism

$$\mathfrak{F}\ell_{\mathcal{M},\bar{V}} \xrightarrow{\mathfrak{j}_{\mathcal{M},\bar{V}}} \mathfrak{Fl}^{[n]}_{\mathcal{M}},$$

where $[n] = \{1, 2, \dots, n\}$, such that the diagram

$$\begin{array}{ccc} \mathfrak{F}\ell_{\mathcal{M},\bar{V}} & \longrightarrow & \prod_{1\le i\le n} Gr_{\mathcal{M},V_i} \\ {\scriptstyle \mathfrak{j}_{\mathcal{M},\bar{V}}}\downarrow & & \downarrow{\scriptstyle (\rho_{V_i})} \\ \mathfrak{Fl}^{[n]}_{\mathcal{M}} & \longrightarrow & \prod_{n\ times} \mathfrak{Gr}_{\mathcal{M}} \end{array} \tag{1}$$

commutes.

10.1. Proposition. *The natural presheaf morphism*

$$\mathfrak{F}\ell_{M,\bar V} \longrightarrow \prod_{1\le i\le n} Gr_{M,V_i}$$

is a closed immersion.

Proof. (a) Suppose that $\mathcal{M}$ is an R-module of finite type. Then, by 9.4.1, the canonical morphism

$$\mathfrak{Fl}^{[n]}_{\mathcal{M}} \longrightarrow \prod_{n\ times} \mathfrak{Gr}_{\mathcal{M}}$$

is a closed immersion. Notice that the square (1) is cartesian. So that the morphism

$$\mathfrak{F}\ell_{M,\bar V} \longrightarrow \prod_{1\le i\le n} Gr_{M,V_i},$$

being a pull-back of a closed immersion, is a closed immersion.

(b) Let now $\mathcal{M}$ be an arbitrary R-module. Let (S,s) be an R-ring and

$$(S,s)^\vee \xrightarrow{\ \xi\ } \prod_{1\le i\le n} Gr_{M,V_i}$$

a presheaf morphism. By Yoneda Lemma, this presheaf morphism is determined by an element $\widehat{\xi} = (\xi_1,\dots,\xi_n)$ of the set $\prod_{1\le i\le n} Gr_{M,V_i}(S,s)$, which is given by S-module epimorphisms $s^*(\mathcal{M}) \xrightarrow{\xi_i} s^*(V_i)$, $1\le i\le n$. These morphisms determine a morphism

$$(S,s)^\vee \xrightarrow{\ \widetilde{\xi}\ } \prod_{1\le i\le n} Gr_{s^*(M),s^*(V_i)} \simeq (S,s)^\vee \prod_{R^\vee} \Big(\prod_{1\le i\le n} Gr_{M,V_i} \Big)$$

So that we have a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{X} & \longrightarrow & \mathfrak{F}\ell_{s^*(\mathcal{M}),s^*(\bar V)} & \longrightarrow & \mathfrak{F}\ell_{\mathcal{M},\bar V} \\
{\scriptstyle \xi'}\downarrow & cart & \downarrow & cart & \downarrow \\
(S,s)^\vee & \xrightarrow{\ \widetilde{\xi}\ } & \prod_{1\le i\le n} Gr_{s^*(M),s^*(V_i)} & \longrightarrow & \prod_{1\le i\le n} Gr_{M,V_i} \\
 & & \downarrow & cart & \downarrow \\
 & & (S,s)^\vee & \longrightarrow & R^\vee
\end{array} \qquad (2)$$

formed by cartesian squares. Since, by hypothesis, all S-modules $s^*(V_i)$ are of finite type, there exists an S-module epimorphism $s^*(\mathcal{M}) \xrightarrow{\psi} \mathcal{L}$ such that each of the S-module epimorphisms $s^*(\mathcal{M}) \xrightarrow{\xi_i} s^*(V_i)$, $1 \le i \le n$, determining $\widetilde{\xi}$ factor through $s^*(\mathcal{M}) \xrightarrow{\psi} \mathcal{L}$. This means that the morphism $(S,s)^\vee \xrightarrow{\widetilde{\xi}} \prod_{1\le i\le n} Gr_{s^*(M),s^*(V_i)}$ factors through the morphism

$$\prod_{1\le i\le n} Gr_{\mathcal{L},s^*(V_i)} \xrightarrow{\prod_{1\le i\le n} Gr_{\psi,s^*(V_i)}} \prod_{1\le i\le n} Gr_{s^*(\mathcal{M}),s^*(V_i)}.$$

Therefore, the upper left square of the diagram (2) is decomposed into two squares

$$\begin{array}{ccccc} \mathfrak{X} & \longrightarrow & \mathfrak{F}\ell_{\mathcal{L},s^*(\bar{V})} & \longrightarrow & \mathfrak{F}\ell_{s^*(\mathcal{M}),s^*(\bar{V})} \\ {\scriptstyle \xi'}\downarrow & cart & \downarrow & & \downarrow \\ (S,s)^\vee & \xrightarrow{\widetilde{\xi}} & \prod_{1\le i\le n} Gr_{\mathcal{L},s^*(V_i)} & \longrightarrow & \prod_{1\le i\le n} Gr_{s^*(M),s^*(V_i)} \end{array} \qquad (3)$$

and the left square is cartesian. Since $\mathcal{L}$ is an S-module of finite type, it follows from (a) above that the second vertical arrow of the diagram (3) is a closed immersion of presheaves of sets on the category $\mathbf{Aff}_k/S^\vee = (S\backslash Alg_k)^{op}$ of noncommutative affine schemes over $S^\vee$. Therefore, the left vertical arrow of (3) is a closed immersion of presheaves of sets on $\mathbf{Aff}_k/S^\vee$. In particular, $\mathfrak{X}$ is isomorphic to $(T, s\circ\varphi)^\vee$ for some k-algebra T and a strict epimorphism $S \xrightarrow{\varphi} T$; and the extreme left vertical arrow of the diagram (2) is isomorphic to the strict monomorphism (that is a closed immersion) $(T,t)^\vee \xrightarrow{\varphi^\vee} (S,s)^\vee$ of affine schemes over $R^\vee$. ■

10.2. Proposition. *Suppose that $\mathcal{M}$ is an R-module of finite type and R-modules $\{V_i \mid 1 \le i \le n\}$ are projective R-modules of finite type. Then*

(a) The presheaf $\mathfrak{F}\ell_{M,\bar{V}}$ is of strictly cofinite type.

(b) The presheaf $\mathfrak{F}\ell_{M,\bar{V}}$ is locally affine for the smooth pretopology. More precisely, there exists a closed immersion of the flag variety $\mathfrak{F}\ell_{M,\bar{V}}$ into a smooth locally affine presheaf which gives a rise to an exact diagram

$$\mathfrak{R}^\phi_{M,\bar{V}} \overset{\mathfrak{p}_1^\phi}{\underset{\mathfrak{p}_2^\phi}{\rightrightarrows}} \mathfrak{F}^\phi_{M,\bar{V}} \xrightarrow{\mathfrak{p}_\phi} \mathfrak{F}\ell_{M,\bar{V}} \qquad (4)$$

whose arrows are representable coverings for the smooth topology. The presheaves $\mathfrak{F}^{\phi}_{M,\bar{V}}$ *and (therefore)* $\mathfrak{R}^{\phi}_{M,\bar{V}} = K_2(\mathfrak{p}_{\phi})$ *are representable and of strictly finite type.*

Proof. By 10.1, the natural presheaf morphism $\mathfrak{F}\ell_{M,\bar{V}} \longrightarrow \prod_{1\le i\le n} Gr_{M,V_i}$ is a closed immersion. Since M is an R-module of finite type, there exists an epimorphism $\mathcal{L} \xrightarrow{\phi} M$ with $\mathcal{L}$ a projective R-module of finite type. This epimorphism ϕ appears as a parameter in the diagram (4), which is defined (uniquely up to isomorphism) via the diagram

$$
\begin{array}{ccccc}
\mathfrak{R}^{\phi}_{M,\bar{V}} & \overset{\mathfrak{p}^{\phi}_1}{\underset{\mathfrak{p}^{\phi}_2}{\rightrightarrows}} & \mathfrak{F}^{\phi}_{M,\bar{V}} & \xrightarrow{\mathfrak{p}_{\phi}} & \mathfrak{F}\ell_{M,\bar{V}} \\
\downarrow & cart & \downarrow & cart & \downarrow \\
\prod\limits_{1\le i\le n} \mathfrak{R}^{\phi}_{\mathcal{M},V_i} & \overset{\bar{p}_1}{\underset{\bar{p}_2}{\rightrightarrows}} & \prod\limits_{1\le i\le n} G^{\phi}_{\mathcal{M},V_i} & \xrightarrow{\pi} & \prod\limits_{1\le i\le n} Gr_{\mathcal{M},V_i} \\
\downarrow & cart & \downarrow & cart & \downarrow \\
\prod\limits_{1\le i\le n} \mathfrak{R}_{\mathcal{L},V_i} & \overset{\bar{p}_1}{\underset{\bar{p}_2}{\rightrightarrows}} & \prod\limits_{1\le i\le n} G_{\mathcal{L},V_i} & \xrightarrow{\pi} & \prod\limits_{1\le i\le n} Gr_{\mathcal{L},V_i}
\end{array}
\qquad (5)
$$

whose all squares are cartesian. The latter implies, together with the fact that the upper left vertical arrow is a closed immersion, that all upper vertical arrows are closed immersions.

The lower part of the diagram (5) is the product of the diagrams

$$
\begin{array}{ccccc}
\mathfrak{R}^{\phi}_{\mathcal{M},V_i} & \overset{\mathfrak{p}^{\phi}_1}{\underset{\mathfrak{p}^{\phi}_2}{\rightrightarrows}} & G^{\phi}_{\mathcal{M},V_i} & \xrightarrow{\mathfrak{p}_{\phi}} & Gr_{\mathcal{M},V_i} \\
\downarrow & cart & \downarrow & cart & \downarrow Gr_{\phi,V} \\
\mathfrak{R}_{\mathcal{L},V_i} & \overset{p_1}{\underset{p_2}{\rightrightarrows}} & G_{\mathcal{L},V_i} & \xrightarrow{\pi} & Gr_{\mathcal{L},V_i}
\end{array}
\qquad 1 \le i \le n. \qquad (6)
$$

from the argument of 6.6.2. By the argument of 6.6.2, all vertical arrows of the diagrams (6) are closed immersions and all horizontal arrows are covering in the *semi-separated* smooth pretopology. This *semi-separated* means that all horizontal arrows are representable. Therefore, the same holds for the product of these diagrams. The fact that the (upper two) squares are cartesian implies that the upper horizontal arrows are also representable coverings in smooth pretopology. The fact that the presheaves $\prod_{1\le i\le n} \mathfrak{R}_{\mathcal{L},V_i}$

and $\prod_{1\le i\le n} G_{\mathcal{L},V_i}$ are representable and the vertical arrows (over these presheaves) are closed immersions, imply that all presheaves in the diagram (5) except those in the right column, are representable. In particular $\mathfrak{F}^{\phi}_{M,\bar{V}}$ and $\mathfrak{R}^{\phi}_{M,\bar{V}}$ are representable. Since the presheaves of the lower row of (5) are locally finitely presentable and vertical arrows are closed immersions, the presheaves of the upper row are locally of strictly finite type. ■

11. Stiefel schemes and generic flag varieties. Fix a projective R-module $\mathcal{M}$ and a positive integer n. For every R-ring (S,s), we denote by $\mathfrak{Stief}^{n+1}_{\mathcal{M}}(S,s)$ the set of all possible decompositions

$$s^*(\mathcal{M}) = \bigoplus_{1\le i\le n} V_i \tag{1}$$

of the S-module $s^*(\mathcal{M})$ into a direct sum of its submodules. One can see that the map $(S,s) \longmapsto \mathfrak{Stief}^{n+1}_{\mathcal{M}}(S,s)$ defines a presheaf

$$(\mathbf{Aff}_k/R^\vee)^{op} = R\backslash Alg_k \xrightarrow{\ \mathfrak{Stief}^{n+1}_{\mathcal{M}}\ } Sets. \tag{2}$$

11.1. Proposition. *For any projective R-module $\mathcal{M}$, there is a natural isomorphism*

$$\mathfrak{Stief}^{n+1}_{\mathcal{M}} \xrightarrow{\ \sim\ } \mathfrak{F}^{[n]}_{\mathcal{M}},$$

where $[n] = \{0,1,2,\dots,n\}$. In particular, there is a natural epimorphism

$$\mathfrak{Stief}^{n+1}_{\mathcal{M}} \longrightarrow \mathfrak{Fl}^{[n]}_{\mathcal{M}}.$$

Proof. Let (S,s) be an R-ring. By definition, the elements of $\mathfrak{Stief}^{n+1}_{\mathcal{M}}(S,s)$ are decompositions (1) of the S-module $s^*(\mathcal{M})$. Every such decomposition is described by the system of endomorphisms $\mathfrak{e}_i$, $1\le i\le n$, of the S-module $s^*(\mathcal{M})$ such that $\mathfrak{e}_i\mathfrak{e}_j = \delta_{ij}$ and $\mathfrak{e}_i(\mathcal{M}) = V_i$ for all $1\le i,j\le n$. Thus, we identify $\mathfrak{Stief}^{n+1}_{\mathcal{M}}(S,s)$ with the set of all n-tuples $(\mathfrak{e}_i \mid 1\le i\le n)$ of endomorphisms of the S-module $s^*(\mathcal{M})$ such that $\mathfrak{e}_i\mathfrak{e}_j = \delta_{ij}$.

By definition, the elements of $\mathfrak{F}^{[n]}_{\mathcal{M}}(S,s)$ are sets $(p_i \in End_S(s^*(\mathcal{M})) \mid 0\le i\le n)$ of projectors such that $p_0 = id_{s^*(\mathcal{M})}$ and $p_ip_j = p_j$ if $i\le j$. The map which assigns to every element $(p_i \mid 0\le i\le n)$ of $\mathfrak{F}^{[n]}_{\mathcal{M}}(S,s)$ the set $(\mathfrak{e}_i = p_{i-1} - p_i \mid 1\le i\le n)$ is an isomorphism from $\mathfrak{F}^{[n]}_{\mathcal{M}}(S,s)$ to $\mathfrak{Stief}^{n+1}_{\mathcal{M}}(S,s)$. ■

11.2. Corollary. *If $\mathcal{M}$ is a projective R-module of finite type, then $\mathfrak{Stief}^{n+1}_{\mathcal{M}}$ is a representable presheaf.*

Proof. The assertion follows from 11.1 and 9.2.2(c). ■

11.3. Generalized Stiefel varieties. We call presheaves $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}$ *generalized Stiefel varieties*. They are representable, if the R-module $\mathcal{M}$ is projective of finite type and $\mathfrak{J}$ is finite. Proposition 11.1 provides a ground for such terminology.

12. Remarks and observations.

12.1. A partial summary of properties shared by these examples. So far, we have introduced two families of examples – flag varieties, with Grassmannians as a special case, and generic flag varieties, with generic Grassmannians as a special case. Both are presheaves of sets on the category $\mathbf{Aff}_k/R^\vee$ for some associative k-algebra R. Generic flag varieties, $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$, depend on an R-module $\mathcal{M}$. Non-generic flag varieties, $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$, (in particular, Grassmannians) depend on an R-module $\mathcal{M}$ and a finite sequence $\bar{\mathcal{V}}$ of projective R-modules. They both have canonical presheaf epimorphisms

$$\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{J}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}, \qquad G_{\mathcal{M},\bar{V}} \xrightarrow{\pi_{\mathcal{M},\bar{V}}} \mathfrak{F}\ell_{\mathcal{M},\bar{V}},$$

which give rise to exact diagrams of presheaves:

$$\mathfrak{R}^{\mathfrak{J}}_{\mathcal{M}} \underset{\mathfrak{q}^{\mathfrak{J}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{\mathfrak{J}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{J}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}} \tag{1}$$

$$\mathfrak{R}_{\mathcal{M},\bar{V}} \underset{p^2_{\mathcal{M},\bar{V}}}{\overset{p^1_{\mathcal{M},\bar{V}}}{\rightrightarrows}} G_{\mathcal{M},\bar{V}} \xrightarrow{\pi_{\mathcal{M},\bar{V}}} \mathfrak{F}\ell_{\mathcal{M},\bar{V}} \tag{2}$$

12.1.1. Base change. For any R-ring (S, s), there are natural isomorphisms between

$$(S,s)\prod_{R^\vee}\Big(\mathfrak{R}^{\mathfrak{J}}_{\mathcal{M}} \underset{\mathfrak{q}^{\mathfrak{J}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{\mathfrak{J}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{J}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}\Big) \quad \text{and} \quad \mathfrak{R}^{\mathfrak{J}}_{s^*(\mathcal{M})} \underset{\mathfrak{q}^{\mathfrak{J}}_{s^*(\mathcal{M})}}{\overset{\mathfrak{p}^{\mathfrak{J}}_{s^*(\mathcal{M})}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{J}}_{s^*(\mathcal{M})} \xrightarrow{\pi^{\mathfrak{J}}_{s^*(\mathcal{M})}} \mathfrak{Fl}^{\mathfrak{J}}_{s^*(\mathcal{M})}$$

and between

$$(S,s)^\vee\prod_{R^\vee}\Big(\mathfrak{R}_{\mathcal{M},\bar{\mathcal{V}}} \underset{p^2_{\mathcal{M},\bar{\mathcal{V}}}}{\overset{p^1_{\mathcal{M},\bar{\mathcal{V}}}}{\rightrightarrows}} G_{\mathcal{M},\bar{\mathcal{V}}} \xrightarrow{\pi_{\mathcal{M},\bar{\mathcal{V}}}} \mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}\Big)$$

and

$$\mathfrak{R}_{s^*(\mathcal{M}),s^*(\bar{\mathcal{V}})} \underset{p^2_{s^*(\mathcal{M}),s^*(\bar{\mathcal{V}})}}{\overset{p^1_{s^*(\mathcal{M}),s^*(\bar{\mathcal{V}})}}{\rightrightarrows}} G_{s^*(\mathcal{M}),s^*(\bar{\mathcal{V}})} \xrightarrow{\pi_{s^*(\mathcal{M}),s^*(\bar{\mathcal{V}})}} \mathfrak{F}\ell_{s^*(\mathcal{M}),s^*(\mathcal{M}),s^*(\bar{\mathcal{V}})},$$

12.1.2. Functoriality. Functoriality and some other properties depend only on the R-module $\mathcal{M}$. So that we denote by $\mathfrak{X}_{\mathcal{M}}$ any of the presheaves $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$ and $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$ and replace the canonical diagrams (1) and (2) with the diagram

$$\mathfrak{R}_{\mathcal{M}} \underset{\mathfrak{p}^2_{\mathcal{M}}}{\overset{\mathfrak{p}^1_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{X}_{\mathcal{M}} \tag{3}$$

(i) Let $R - mod_{\mathfrak{e}}$ denote the subcategory of the category R-modules formed by all epimorphisms of R-modules. The map $\mathcal{M} \longmapsto \mathfrak{X}_{\mathcal{M}}$ is a functor

$$R - mod_{\mathfrak{e}}^{op} \longrightarrow (\mathbf{Aff}_k/R^{\vee})^{\wedge}$$

which maps every R-module epimorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ to a closed immersion

$$\mathfrak{X}_{\mathcal{M}} \xrightarrow{\mathfrak{X}_{\varphi}} \mathfrak{X}_{\mathcal{L}}.$$

(ii) The closed immersion $\mathfrak{X}_{\mathcal{M}} \xrightarrow{\mathfrak{X}_{\varphi}} \mathfrak{X}_{\mathcal{L}}$ gives rise to a commutative diagram

$$\begin{array}{ccccc}
\mathfrak{R}^{\varphi}_{\mathcal{M}} & \underset{\mathfrak{p}^{2\varphi}_{\mathcal{M}}}{\overset{\mathfrak{p}^{1\varphi}_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{U}^{\varphi}_{\mathcal{M}} & \xrightarrow{\pi^{\varphi}_{\mathcal{M}}} & \mathfrak{X}_{\mathcal{M}} \\
\downarrow & cart & \downarrow & cart & \downarrow \mathfrak{X}_{\varphi} \\
\mathfrak{R}_{\mathcal{L}} & \underset{\mathfrak{p}^2_{\mathcal{L}}}{\overset{\mathfrak{p}^1_{\mathcal{L}}}{\rightrightarrows}} & \mathfrak{U}_{\mathcal{L}} & \xrightarrow{\pi_{\mathcal{L}}} & \mathfrak{X}_{\mathcal{L}}
\end{array} \tag{4}$$

with cartesian squares whose vertical arrows are (therefore) closed immersions and the rows are exact diagrams.

12.1.3. The choice of infinitesimal morphisms. Let $\mathcal{M}$ be an R-modules. There are two cases:

(a) If the presheaf $\mathfrak{X}_{\mathcal{M}}$ is the flag variety $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$ with $\bar{\mathcal{V}} = \{\mathcal{V}_i \mid 1 \le i \le n\}$ consisting of projective R-modules of finite type, then we take as infinitesimal morphisms the class $\mathfrak{M}_{\mathfrak{J}}$ of radical closed immersions.

(b) If $\mathfrak{X}_{\mathcal{M}}$ is the generic flag variety $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$, then our choice of infinitesimal morphisms is the class $\mathfrak{M}^{\mathfrak{c}}_{\mathfrak{J}}$ of *complete* radical closed immersions.

In each of the cases the (formally) smooth, étale, and unramified morphisms (in particular, (formally) open immersions) are understood as (formally) $\mathfrak{M}$-smooth, $\mathfrak{M}$-étale and $\mathfrak{M}$-unramified morphisms, where $\mathfrak{M} = \mathfrak{M}_{\mathfrak{J}}$, or $\mathfrak{M} = \mathfrak{M}^{\mathfrak{c}}_{\mathfrak{J}}$, depending on the case.

12.1.4. Formal smoothness.

(i) If $\mathcal{M}$ is a projective R-module, then all presheaves and morphisms of the diagram 12.1.2(3) are formally smooth.

(ii) In particular, if $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ is an epimorphism of R-modules and the R-module $\mathcal{L}$ is projective, then all horizontal arrows of the diagram 12.1.2(4) are formally smooth morphisms.

12.1.5. Smoothness and representability. Here we assume that $\mathfrak{X}_{\mathcal{M}}$ is either the flag variety $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$, or the generic flag variety $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$ with a finite $\mathfrak{J}$.

(i) If $\mathcal{M}$ is a projective R-module of finite type, then all presheaves and morphisms of the diagram 12.1.2(3) are smooth and representable, and the presheaves $\mathfrak{U}_{\mathcal{M}}$ and $\mathfrak{R}_{\mathcal{M}}$ are representable.

(ii) Let $\mathcal{M}$ be an R-module of finite type and let $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ be an epimorphism with $\mathcal{L}$ a projective module of finite type. Then all horizontal arrows of the diagram 12.1.2(4) are smooth representable morphisms and all presheaves of the diagram, except of $\mathfrak{X}_{\mathcal{L}}$ and $\mathfrak{X}_{\mathcal{M}}$ (that is all presheaves of the left square of 12.1.2(4)) are representable.

12.2. Functoriality for covers. We start with an observation about complementing the diagram 12.1.2(4).

12.2.1. Proposition. *To any R-module epimorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$, there corresponds* *a commutative diagram*

$$\begin{array}{ccccc}
\mathfrak{R}_{\mathcal{M}} & \overset{\mathfrak{p}^1_{\mathcal{M}}}{\underset{\mathfrak{p}^2_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{U}_{\mathcal{M}} & \xrightarrow{\pi_{\mathcal{M}}} & \mathfrak{X}_{\mathcal{M}} \\
\wp^{\varphi}_{\mathcal{M}}\Big\uparrow & & \sigma^{\varphi}_{\mathcal{M}}\Big\uparrow & & \Big\uparrow id_{\mathfrak{X}_{\mathcal{M}}} \\
\mathfrak{R}^{\phi}_{\mathcal{M}} & \overset{\mathfrak{p}^{1\phi}_{\mathcal{M}}}{\underset{\mathfrak{p}^{2\phi}_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{U}^{\phi}_{\mathcal{M}} & \xrightarrow{\pi^{\phi}_{\mathcal{M}}} & \mathfrak{X}_{\mathcal{M}} \\
\Big\downarrow & cart & \Big\downarrow & cart & \Big\downarrow \mathfrak{X}_{\phi} \\
\mathfrak{R}_{\mathcal{L}} & \overset{\mathfrak{p}^1_{\mathcal{L}}}{\underset{\mathfrak{p}^2_{\mathcal{L}}}{\rightrightarrows}} & \mathfrak{U}_{\mathcal{L}} & \xrightarrow{\pi_{\mathcal{L}}} & \mathfrak{X}_{\mathcal{L}}
\end{array} \qquad (1)$$

of presheaves of sets on $\mathbf{Aff}_k/R^{\vee}$ *whose lower two squares are cartesian.*

(i) The diagram morphism from the middle to the upper row of (1) splits, if the R-module morphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ *splits. In particular, it splits if the module* $\mathcal{M}$ *is projective.*

(ii) Suppose that $\mathfrak{X}_{-}$ *is either a non-generic flag variety* $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$, *or the generic flag variety* $\mathfrak{Fl}^{\mathfrak{J}}_{-}$ *such that* $\mathfrak{J} - \{\bullet\}$ *has an initial object. Then the upper vertical arrows are epimorphisms.*

Proof. (a1) Let $\mathfrak{X}_{-}$ be the generic flag variety $\mathfrak{F}^{\mathfrak{J}}_{-}$. For any R-ring (S,s), the pull-back of the maps

$$\mathfrak{F}^{\mathfrak{J}}_{\mathcal{L}}(S,s) \xrightarrow{\pi_{\mathcal{L}}(S,s)} \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{L}} \xleftarrow{\mathfrak{Fl}^{\mathfrak{J}}_{\varphi}(S,s)} \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}(S,s)$$

is determined by an element $\{p_i \mid i \in \mathfrak{J}\}$ of $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{L}}(S,s)$ such that $s^*(\mathcal{L}) \xrightarrow{p_i} s^*(\mathcal{L})$ factors through the epimorphism $s^*(\mathcal{L}) \xrightarrow{s^*(\varphi)} s^*(\mathcal{M})$ for every $i \in \mathfrak{J}$; that is $p_i = \mathfrak{p}_i \circ s^*(\varphi)$ for a unique morphism $s^*(\mathcal{M}) \xrightarrow{\mathfrak{p}_i} s^*(\mathcal{L})$. Thus, the element $\{p_i \mid i \in \mathfrak{J}\}$ determines an element $\{s^*(\varphi) \circ \mathfrak{p}_i | \ i \in \mathfrak{J}\}$ of $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}(S,s)$. The map $\{p_i \mid i \in \mathfrak{J}\} \longmapsto \{s^*(\varphi) \circ \mathfrak{p}_i | \ i \in \mathfrak{J}\}$ is functorial in (S,s); hence the presheaf morphism (1). The commutativity of the diagram

$$\begin{array}{ccc}
\mathfrak{F}^{\mathfrak{J},\varphi}_{\mathcal{M}} & \xrightarrow{\sigma^{\varphi}_{\mathcal{M}}} & \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}} \\
{\scriptstyle \pi^{\varphi}_{\mathcal{M}}} \searrow & & \swarrow {\scriptstyle \pi_{\mathcal{M}}} \\
& \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}} &
\end{array}$$

follows from the construction.

(a1) The commutativity of the right upper square of the diagram (1) and the fact that $\mathfrak{R}_{\mathcal{M}} \overset{\mathfrak{p}^1_{\mathcal{M}}}{\underset{\mathfrak{p}^2_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}_{\mathcal{M}}$ is the kernel pair of the morphism $\mathfrak{U}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{X}_{\mathcal{M}}$ imply the existence of a unique morphism $\mathfrak{R}^{\phi}_{\mathcal{M}} \xrightarrow{\wp^{\varphi}_{\mathcal{M}}} \mathfrak{R}_{\mathcal{M}}$ making the upper left square of the diagram (1) commute. It is useful to have an explicit description of the morphism $\wp^{\varphi}_{\mathcal{M}}$.

(a2) By general nonsense considerations, $\mathfrak{R}^{\mathfrak{I},\phi}_{\mathcal{M}} \overset{\mathfrak{p}^{1\phi}_{\mathcal{M}}}{\underset{\mathfrak{p}^{2\phi}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I},\phi}_{\mathcal{M}}$ is the kernel pair of the morphism $\mathfrak{F}^{\mathfrak{I},\phi}_{\mathcal{M}} \xrightarrow{\pi^{\phi}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$. It follows from the description of $\mathfrak{F}^{\mathfrak{I},\phi}_{\mathcal{M}}$ given at the beginning of the argument that $\mathfrak{R}^{\mathfrak{I},\varphi}_{\mathcal{M}}(S,s)$ consists of all elements $\{p_i, p'_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ of $\mathfrak{R}^{\mathfrak{I}}_{\mathcal{L}}(S,s)$ such that all p_i and p'_i factor through the epimorphism $s^*(\mathcal{L}) \xrightarrow{s^*(\varphi)} s^*(\mathcal{M})$; that is $p_i = \mathfrak{p}_i \circ s^*(\varphi)$ and $p'_i = \mathfrak{p}'_i \circ s^*(\varphi)$ for morphisms $\mathfrak{p}_i$, $\mathfrak{p}'_i$ uniquely determined by these equalities. Notice that $\{s^*(\varphi) \circ \mathfrak{p}_i, s^*(\varphi) \circ \mathfrak{p}'_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ is an element of $\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$.

In fact, it follows from the relation $p_i \circ p'_i = p_i$ that

$$(s^*(\varphi) \circ \mathfrak{p}_i) \circ (s^*(\varphi) \circ \mathfrak{p}'_i) \circ s^*(\varphi) = s^*(\varphi) \circ (\mathfrak{p}_i \circ s^*(\varphi)) \circ (\mathfrak{p}'_i \circ s^*(\varphi)) =$$
$$s^*(\varphi) \circ (p_i \circ p'_i) = s^*(\varphi) \circ p_i = (s^*(\varphi) \circ \mathfrak{p}_i) \circ s^*(\varphi),$$

which, thanks to the epimorphness of $s^*(\varphi)$, gives the equality

$$(s^*(\varphi) \circ \mathfrak{p}_i) \circ (s^*(\varphi) \circ \mathfrak{p}'_i) = s^*(\varphi) \circ \mathfrak{p}_i.$$

Symmetrically, the relation $p'_i \circ p_i = p'_i$ implies the equality

$$(s^*(\varphi) \circ \mathfrak{p}'_i) \circ (s^*(\varphi) \circ \mathfrak{p}_i) = s^*(\varphi) \circ \mathfrak{p}'_i.$$

The map $\mathfrak{R}^{\mathfrak{I},\phi}_{\mathcal{M}}(S,s) \xrightarrow{\wp^{\varphi}_{\mathcal{M}}(S,s)} \mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ assigns to the element $\{p_i, p'_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ the element $\{s^*(\varphi) \circ \mathfrak{p}_i, s^*(\varphi) \circ \mathfrak{p}'_i \mid i \in \mathfrak{I} - \{\bullet\}\}$.

(i) Suppose that the morphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ splits. that is there is an R-module morphism $\mathcal{M} \xrightarrow{\beta} \mathcal{L}$ such that $\varphi \circ \beta = id_{\mathcal{M}}$. For any R-ring (S,s), consider the map

$$End_S(s^*(\mathcal{M})) \longrightarrow End_S(s^*(\mathcal{L})), \quad \mathfrak{f} \longmapsto s^*(\beta) \circ \mathfrak{f} \circ s^*(\varphi). \qquad (2)$$

This map induces a morphism $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\mathfrak{t}^{\varphi}_{\beta}} \mathfrak{F}^{\mathfrak{I},\varphi}_{\mathcal{M}}$ such that $\sigma^{\varphi}_{\mathcal{M}} \circ \mathfrak{t}^{\varphi}_{\beta} = id_{\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}}$.

In fact, if $\{p_i \mid i \in \mathfrak{I}\}$ is an element of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$, then, for any $i \leq j$,

$$(s^*(\beta) \circ p_i \circ s^*(\varphi)) \circ (s^*(\beta) \circ p_j \circ s^*(\varphi)) =$$
$$s^*(\beta) \circ (p_i \circ p_j) \circ s^*(\varphi) = s^*(\beta) \circ p_j \circ s^*(\varphi),$$

which shows that (2) maps $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ to the subset of elements $\{\mathfrak{p}_i \mid i \in \mathfrak{I}\}$ of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{L}}(S,s)$ which factor through $s^*(\varphi)$. The latter subset is naturally isomorphic to $\mathfrak{F}^{\mathfrak{I},\varphi}_{\mathcal{M}}$ and it was already identified with $\mathfrak{F}^{\mathfrak{I},\varphi}_{\mathcal{M}}$ from the beginning of the argument.

The composition $(\sigma^{\varphi}_{\mathcal{M}} \circ \mathfrak{t}^{\varphi}_{\beta})(S,s)$ acts as follows:

$$\{p_i \mid i \in \mathfrak{I}\} \longmapsto \{s^*(\beta) \circ p_i \circ s^*(\varphi) \mid i \in \mathfrak{I}\} \longmapsto \{s^*(\varphi) \circ (s^*(\beta) \circ p_i) \mid i \in \mathfrak{I}\} = \{p_i \mid i \in \mathfrak{I}\},$$

which proves the identity $\sigma^{\varphi}_{\mathcal{M}} \circ \mathfrak{t}^{\varphi}_{\beta} = id_{\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}}$.

(i') It follows from the calculations above that the map (2) induces a morphism

$$\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\ \mathfrak{r}^{\varphi}_{\beta}\ } \mathfrak{R}^{\mathfrak{I},\varphi}_{\mathcal{M}}, \quad \{p_i, p'_i \mid i \in \mathfrak{I}\} \longmapsto \{s^*(\beta) \circ p_i \circ s^*(\varphi), s^*(\beta) \circ p'_i \circ s^*(\varphi) \mid i \in \mathfrak{I}\},$$

such that $\wp^{\varphi}_{\mathcal{M}} \circ \mathfrak{r}^{\varphi}_{\beta} = id_{\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}}$ and the diagram

$$\begin{array}{ccccc}
\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}} & \overset{\mathfrak{p}^1_{\mathcal{M}}}{\underset{\mathfrak{p}^2_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} & \xrightarrow{\ \pi_{\mathcal{M}}\ } & \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \\
{\scriptstyle \mathfrak{r}^{\varphi}_{\mathcal{M}}}\downarrow & & {\scriptstyle \mathfrak{t}^{\varphi}_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle id_{\mathfrak{X}_{\mathcal{M}}}} \\
\mathfrak{R}^{\mathfrak{I},\phi}_{\mathcal{M}} & \overset{\mathfrak{p}^{1\phi}_{\mathcal{M}}}{\underset{\mathfrak{p}^{2\phi}_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{F}^{\mathfrak{I},\phi}_{\mathcal{M}} & \xrightarrow{\ \pi^{\phi}_{\mathcal{M}}\ } & \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}
\end{array}$$

commutes.

(ii) Suppose that $\mathfrak{I} - \{\bullet\}$ has an initial object, $\mathfrak{x}$. Let (S,s) be an R-ring and $\{p_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ an element of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$. Let $\mathcal{V}_{\mathfrak{x}}$ denote the image of $p_{\mathfrak{x}}$, and let $s^*(\mathcal{M}) \xrightarrow{\mathfrak{e}_{\mathfrak{x}}} \mathcal{V}_{\mathfrak{x}} \xrightarrow{\mathfrak{i}_{\mathfrak{x}}} s^*(\mathcal{M})$ be the decomposition of the projector $p_{\mathfrak{x}}$ into epimorphism on its image followed by the embedding. Notice that the element $\{p_i \mid i \in \mathfrak{I}\}$ is represented by this splitting of $p_{\mathfrak{x}}$ and the element $\{\mathfrak{p}_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{V}_{\mathfrak{x}}}(S,s)$, where $\mathfrak{p}_i$ is a projector of $\mathcal{V}_{\mathfrak{x}}$ induced by p_i, $i \in \mathfrak{I} - \{\bullet\}$. In particular, $\mathfrak{p}_{\mathfrak{x}} = id_{\mathcal{V}_{\mathfrak{x}}}$.

Since $\mathcal{V}_{\mathfrak{x}}$ is a projective S-module and $s^*(\mathcal{L} \xrightarrow{\varphi} \mathcal{M})$ is an epimorphism, there is a morphism $\mathcal{V}_x \xrightarrow{\bar{\mathfrak{i}}_{\mathfrak{x}}} s^*(\mathcal{L})$ such that $s^*(\varphi) \circ \bar{\mathfrak{i}}_{\mathfrak{x}} = \mathfrak{i}_{\mathfrak{x}}$. By the argument of (i) above, the splitting $\mathcal{V}_{\mathfrak{x}} \xrightarrow{\bar{\mathfrak{i}}_{\mathfrak{x}}} s^*(\mathcal{L}) \xrightarrow{\mathfrak{e}_{\mathfrak{x}} \circ s^*(\varphi)} \mathcal{V}_{\mathfrak{x}}$ gives rise to a map from $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{V}_{\mathfrak{x}}}(S,s)$ to $\mathfrak{F}^{\mathfrak{I},\varphi}_{\mathcal{M}}(S,s)$ which assigns to the element $\{\mathfrak{p}_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ the element $\{\bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{p}_i \circ \mathfrak{e}_{\mathfrak{x}} \circ s^*(\varphi) \mid i \in \mathfrak{I} - \{\bullet\}\}$. One can see that (the value at (S,s) of) the canonical presheaf morphism (7) maps this element to the initially chosen element $\{\mathfrak{p}_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$.

(ii') Let $\{p_i, p'_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ be an element of $\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$. Let $\mathcal{V}_{\mathfrak{x}}$ denote the image of the projector $p_{\mathfrak{x}}$ and $\mathcal{V}'_{\mathfrak{x}}$ the image of the projector $p'_{\mathfrak{x}}$ of the S-module $s^*(\mathcal{M})$. Let

$$s^*(\mathcal{M}) \xrightarrow{\mathfrak{e}_{\mathfrak{x}}} \mathcal{V}_{\mathfrak{x}} \xrightarrow{\mathfrak{i}_{\mathfrak{x}}} s^*(\mathcal{M}) \quad \text{and} \quad s^*(\mathcal{M}) \xrightarrow{\mathfrak{e}'_{\mathfrak{x}}} \mathcal{V}'_{\mathfrak{x}} \xrightarrow{\mathfrak{i}'_{\mathfrak{x}}} s^*(\mathcal{M})$$

be the decompositions of respectively the projector $p_{\mathfrak{x}}$ and the projector $p'_{\mathfrak{x}}$ into epimorphism on its image followed by the embedding; and let $\mathcal{V}_x \xrightarrow{\bar{\mathfrak{i}}_{\mathfrak{x}}} s^*(\mathcal{L})$ be an S-module morphism such that $s^*(\phi) \circ \bar{\mathfrak{i}}_{\mathfrak{x}} = \mathfrak{i}_{\mathfrak{x}}$. We set $\bar{\mathfrak{i}}'_x = \bar{\mathfrak{i}}_x \circ (\mathfrak{e}_x \circ \mathfrak{i}'_x)$. Then

$$(\mathfrak{e}'_{\mathfrak{x}} \circ s^*(\phi)) \circ \bar{\mathfrak{i}}'_x = (\mathfrak{e}'_{\mathfrak{x}} \circ s^*(\phi)) \circ (\bar{\mathfrak{i}}_x \circ (\mathfrak{e}_x \circ \mathfrak{i}'_x)) =$$
$$\mathfrak{e}'_{\mathfrak{x}} \circ (s^*(\phi) \circ \bar{\mathfrak{i}}_x) \circ (\mathfrak{e}_x \circ \mathfrak{i}'_x) = (\mathfrak{e}'_{\mathfrak{x}} \circ \mathfrak{i}_x) \circ (\mathfrak{e}_x \circ \mathfrak{i}'_x)) = id_{\mathcal{V}'_{\mathfrak{x}}}.$$

The last equality here is due to the fact that

$$\mathcal{V}'_{\mathfrak{x}} \xrightarrow{\mathfrak{e}_{\mathfrak{x}} \circ \mathfrak{i}'_{\mathfrak{x}}} \mathcal{V}_{\mathfrak{x}} \qquad \text{and} \qquad \mathcal{V}_{\mathfrak{x}} \xrightarrow{\mathfrak{e}'_{\mathfrak{x}} \circ \mathfrak{i}_{\mathfrak{x}}} \mathcal{V}'_{\mathfrak{x}}$$

are mutually inverse isomorphisms. Indeed, it follows from the equality $p_{\mathfrak{x}} \circ p'_{\mathfrak{x}} = p_{\mathfrak{x}}$ that

$$(\mathfrak{e}_x \circ \mathfrak{i}'_{\mathfrak{x}}) \circ (\mathfrak{e}'_{\mathfrak{x}} \circ \mathfrak{i}_{\mathfrak{x}}) = (\mathfrak{e}_{\mathfrak{x}} \circ \mathfrak{i}_{\mathfrak{x}}) \circ ((\mathfrak{e}_x \circ \mathfrak{i}'_{\mathfrak{x}}) \circ (\mathfrak{e}'_{\mathfrak{x}} \circ \mathfrak{i}_{\mathfrak{x}})) = \mathfrak{e}_{\mathfrak{x}} \circ (\mathfrak{i}_{\mathfrak{x}} \circ \mathfrak{e}_x) \circ (\mathfrak{i}'_{\mathfrak{x}} \circ \mathfrak{e}'_{\mathfrak{x}}) \circ \mathfrak{i}_{\mathfrak{x}} =$$
$$\mathfrak{e}_{\mathfrak{x}} \circ (p_{\mathfrak{x}} \circ p'_{\mathfrak{x}}) \circ \mathfrak{i}_{\mathfrak{x}} = \mathfrak{e}_{\mathfrak{x}} \circ p_{\mathfrak{x}} \circ \mathfrak{i}_x = (\mathfrak{e}_{\mathfrak{x}} \circ \mathfrak{i}_{\mathfrak{x}}) \circ (\mathfrak{e}_{\mathfrak{x}} \circ \mathfrak{i}_{\mathfrak{x}}) = id_{\mathcal{V}_{\mathfrak{x}}}.$$

Symmetrically, the relation $p'_{\mathfrak{x}} \circ p_{\mathfrak{x}} = p'_{\mathfrak{x}}$ implies that $(\mathfrak{e}'_x \circ \mathfrak{i}_{\mathfrak{x}}) \circ (\mathfrak{e}_{\mathfrak{x}} \circ \mathfrak{i}'_{\mathfrak{x}}) = id_{\mathcal{V}'_{\mathfrak{x}}}$.

The assignment

$$\{p_i, p'_i \mid i \in \mathfrak{I} - \{\bullet\}\} \longmapsto \{\widetilde{\mathfrak{p}}_i = \bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{p}_i \circ \mathfrak{e}_{\mathfrak{x}} \circ s^*(\phi), \widetilde{\mathfrak{p}}'_i = \bar{\mathfrak{i}}'_{\mathfrak{x}} \circ \mathfrak{p}'_i \circ \mathfrak{e}'_{\mathfrak{x}} \circ s^*(\phi) \mid i \in \mathfrak{I} - \{\bullet\}\},$$

where $\mathfrak{p}_i$ and $\mathfrak{p}'_i$ are projections of respectively $\mathcal{V}_{\mathfrak{x}}$ and $\mathcal{V}'_{\mathfrak{x}}$ induced by the projectors respectively p_i and p'_i, $i \in \mathfrak{I} - \{\bullet\}$ (see (ii) above), produces an element of the kernel of the pair of arrows

$$\mathfrak{R}^{\mathfrak{I}}_{\mathcal{L}}(S,s) \underset{\mathfrak{R}^{\mathfrak{I}}_{\bar{\psi}_2}}{\overset{\mathfrak{R}^{\mathfrak{I}}_{\bar{\psi}_1}}{\rightrightarrows}} \mathfrak{R}^{\mathfrak{I}}_{\mathcal{N}}(S,s).$$

Taking into consideration (a5) above, it suffices to show that $\widetilde{\mathfrak{p}}_i \circ \widetilde{\mathfrak{p}}'_i = \widetilde{\mathfrak{p}}_i$ for all $i \in \mathfrak{I} - \{\bullet\}$. These relations follow from the relations $p_i \circ p'_i = p_i$ via the following sequence of identities:

$$\widetilde{\mathfrak{p}}_i \circ \widetilde{\mathfrak{p}}'_i = (\bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{p}_i \circ \mathfrak{e}_{\mathfrak{x}} \circ s^*(\phi)) \circ (\bar{\mathfrak{i}}'_{\mathfrak{x}} \circ \mathfrak{p}'_i \circ \mathfrak{e}'_{\mathfrak{x}} \circ s^*(\phi)) =$$
$$\bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{p}_i \circ \mathfrak{e}_{\mathfrak{x}} \circ (s^*(\phi) \circ \bar{\mathfrak{i}}_x) \circ (\mathfrak{e}_x \circ \mathfrak{i}'_x) \circ \mathfrak{p}'_i \circ \mathfrak{e}'_{\mathfrak{x}} \circ s^*(\phi) =$$
$$\bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{p}_i \circ (\mathfrak{e}_{\mathfrak{x}} \circ \mathfrak{i}_x) \circ (\mathfrak{e}_x \circ \mathfrak{i}'_x) \circ \mathfrak{p}'_i \circ \mathfrak{e}'_{\mathfrak{x}} \circ s^*(\phi) =$$
$$\bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{p}_i \circ (\mathfrak{e}_x \circ \mathfrak{i}'_x) \circ \mathfrak{p}'_i \circ \mathfrak{e}'_{\mathfrak{x}} \circ s^*(\phi) =$$
$$\bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{e}_{\mathfrak{x}} \circ (\mathfrak{i}_{\mathfrak{x}} \circ \mathfrak{p}_i \circ \mathfrak{e}_x) \circ (\mathfrak{i}'_x \circ \mathfrak{p}'_i \circ \mathfrak{e}'_{\mathfrak{x}}) \circ s^*(\phi) =$$
$$\bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{e}_{\mathfrak{x}} \circ (p_i \circ p'_i) \circ s^*(\phi) = \bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{e}_{\mathfrak{x}} \circ p_i \circ s^*(\phi) =$$
$$\bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{e}_{\mathfrak{x}} \circ (\mathfrak{i}_{\mathfrak{x}} \circ \mathfrak{p}_i \circ \mathfrak{e}_{\mathfrak{x}}) \circ s^*(\phi) = \bar{\mathfrak{i}}_{\mathfrak{x}} \circ \mathfrak{p}_i \circ \mathfrak{e}_{\mathfrak{x}} \circ s^*(\phi) = \widetilde{\mathfrak{p}}_i.$$

(b) In the case when $\mathfrak{X}_{\mathcal{M}}$ is a non-generic flag variety $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$, the argument is an easy adaptation of the argument above. Details are left to the reader. ■

12.2.2. Splittings. For any category $\mathcal{C}$, we denote by $\mathcal{C}_{spl}$ the category which has the same class of objects as the category $\mathcal{C}$, and, for any pair of objects M, L, the elements of $\mathcal{C}_{spl}(M, L)$ are pairs of arrows $L \xrightarrow{\mathfrak{v}} M \xrightarrow{\mathfrak{u}} L$ of the category $\mathcal{C}$ such that $\mathfrak{u} \circ \mathfrak{v} = id_L$. The composition is defined naturally:

$$(M_1 \xrightarrow{\mathfrak{v}_1} M \xrightarrow{\mathfrak{u}_1} M_1) \circ (M_2 \xrightarrow{\mathfrak{v}_2} M_1 \xrightarrow{\mathfrak{u}_2} M_2) = (M_2 \xrightarrow{\mathfrak{v}_1 \circ \mathfrak{v}_2} M \xrightarrow{\mathfrak{u}_2 \circ \mathfrak{u}_1} M_2).$$

12.2.3. Proposition. *The maps* $\mathcal{M} \longmapsto \mathfrak{X}_{\mathcal{M}}$, $\mathcal{M} \longmapsto \mathfrak{U}_{\mathcal{M}}$ *and* $\mathcal{M} \longmapsto \mathfrak{R}_{\mathcal{M}}$ *are functors*

$$(R - mod_{spl})^{op} \longrightarrow (\mathbf{Aff}_k/R^\vee)^\wedge.$$

Moreover, for every morphism $\mathcal{L} \xrightarrow{\psi} \mathcal{M}$ *of the category* $R - mod_{spl}$, *the diagram*

$$\begin{array}{ccccc} \mathfrak{R}_{\mathcal{M}} & \overset{\mathfrak{p}^1_{\mathcal{M}}}{\underset{\mathfrak{p}^2_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{U}_{\mathcal{M}} & \xrightarrow{\pi_{\mathcal{M}}} & \mathfrak{X}_{\mathcal{M}} \\ {\scriptstyle \mathfrak{R}_\psi}\downarrow & & \downarrow{\scriptstyle \mathfrak{U}_\psi} & & \downarrow{\scriptstyle \mathfrak{X}_\psi} \\ \mathfrak{R}_{\mathcal{L}} & \overset{\mathfrak{p}^1_{\mathcal{L}}}{\underset{\mathfrak{p}^2_{\mathcal{L}}}{\rightrightarrows}} & \mathfrak{U}_{\mathcal{L}} & \xrightarrow{\pi_{\mathcal{L}}} & \mathfrak{X}_{\mathcal{L}} \end{array} \qquad (3)$$

commutes, and all its vertical arrows are closed immersions.

Proof. A morphism $\mathcal{L} \xrightarrow{\psi} \mathcal{M}$ of the category $R-mod_{spl}$ is, by definition, a pair of R-module morphisms $\mathcal{M} \xrightarrow{\mathfrak{v}} \mathcal{L} \xrightarrow{\mathfrak{u}} \mathcal{M}$ such that $\mathfrak{u} \circ \mathfrak{v} = id_{\mathcal{M}}$.

(a) The right vertical arrow, $\mathfrak{X}_{\mathcal{M}} \xrightarrow{\mathfrak{X}_{\psi}} \mathfrak{X}_{\mathcal{L}}$, is the morphism $\mathfrak{X}_{\mathfrak{u}}$ corresponding to the strict epimorphism of R-modules $\mathcal{L} \xrightarrow{\mathfrak{u}} \mathcal{M}$.

To define the middle arrow we have to consider each of the cases: $\mathfrak{X}_{\mathcal{M}}$ is a generic flag variety $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$, and $\mathfrak{X}_{\mathcal{M}}$ is a flag variety $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$.

(b) Suppose that $\mathfrak{X}_{\mathcal{M}}$ is a generic flag variety $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$. Then $\mathfrak{U}_{\mathcal{M}} = \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}$. For every R-ring (S,s), the set $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}(S,s)$ consists of endomorphisms $\{p_i \mid i \in \mathfrak{J} - \{\bullet\}\}$ of $s^*(\mathcal{M})$ such that the image of p_i is a projective S-module of finite type and $p_i p_j = p_j$ if $i \le j$.

(b1) The map $\mathfrak{U}_{\psi}(S,s) = \mathfrak{F}^{\mathfrak{J}}_{\psi}(S,s)$ is induced by the map which assigns to each projector p of $s^*(\mathcal{M})$ the projector $s^*(\mathfrak{v}) \circ p_i \circ s^*(\mathfrak{u})$ of $s^*(\mathcal{L})$. If $\{p_i \mid i \in \mathfrak{J}\}$ is an element of $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}(S,s)$ and $i \le j$, then

$$\begin{aligned}
&(s^*(\mathfrak{v}) \circ p_i \circ s^*(\mathfrak{u})) \circ (s^*(\mathfrak{v}) \circ p_j \circ s^*(\mathfrak{u})) = \\
&s^*(\mathfrak{v}) \circ p_i \circ (s^*(\mathfrak{u}) \circ s^*(\mathfrak{v})) \circ p_j \circ s^*(\mathfrak{u}) = \\
&s^*(\mathfrak{v}) \circ (p_i \circ p_j) \circ s^*(\mathfrak{u}) = s^*(\mathfrak{v}) \circ p_j \circ s^*(\mathfrak{u}),
\end{aligned}$$

which shows that $\mathfrak{F}^{\mathfrak{J}}_{\psi}(S,s)$ maps $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}(S,s)$ to $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{L}}(S,s)$.

(b2) It is easy to see that the embedding

$$\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}(S,s) \times \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}(S,s) \xrightarrow{\mathfrak{F}^{\mathfrak{J}}_{\psi} \times \mathfrak{F}^{\mathfrak{J}}_{\psi}(S,s)} \mathfrak{F}^{\mathfrak{J}}_{\mathcal{L}}(S,s) \times \mathfrak{F}^{\mathfrak{J}}_{\mathcal{L}}(S,s)$$

induces a map

$$\mathfrak{R}^{\mathfrak{J}}_{\mathcal{M}}(S,s) \xrightarrow{\mathfrak{R}^{\mathfrak{J}}_{\psi}(S,s)} \mathfrak{R}^{\mathfrak{J}}_{\mathcal{M}}(S,s).$$

(b3) Both $\mathfrak{F}^{\mathfrak{J}}_{\psi}(S,s)$ and $\mathfrak{R}^{\mathfrak{J}}_{\psi}(S,s)$ are functorial in (S,s); that is they define presheaf morphisms respectively $\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}} \xrightarrow{\mathfrak{F}^{\mathfrak{J}}_{\psi}} \mathfrak{F}^{\mathfrak{J}}_{\mathcal{L}}$ and $\mathfrak{R}^{\mathfrak{J}}_{\mathcal{M}} \xrightarrow{\mathfrak{R}^{\mathfrak{J}}_{\psi}} \mathfrak{R}^{\mathfrak{J}}_{\mathcal{L}}$ making the diagram

$$\begin{array}{ccccc}
\mathfrak{R}^{\mathfrak{J}}_{\mathcal{M}} & \overset{\mathfrak{p}^1_{\mathcal{M}}}{\underset{\mathfrak{p}^2_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}} & \xrightarrow{\pi_{\mathcal{M}}} & \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}} \\
\mathfrak{R}^{\mathfrak{J}}_{\psi} \big\downarrow & & \big\downarrow \mathfrak{F}^{\mathfrak{J}}_{\psi} & & \big\downarrow \mathfrak{Fl}^{\mathfrak{J}}_{\psi} \\
\mathfrak{R}^{\mathfrak{J}}_{\mathcal{L}} & \overset{\mathfrak{p}^1_{\mathcal{L}}}{\underset{\mathfrak{p}^2_{\mathcal{L}}}{\rightrightarrows}} & \mathfrak{F}^{\mathfrak{J}}_{\mathcal{L}} & \xrightarrow{\pi_{\mathcal{L}}} & \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{L}}
\end{array} \qquad (4)$$

commute.

(c) In the case when $\mathfrak{X}_{\mathcal{M}}$ is the flag variety $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$, the argument is similar. Details are left to the reader. ■

12.2.4. Note. If $\mathfrak{X}_{\mathcal{M}}$ is the flag variety $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$ and the set $\bar{\mathcal{V}} = (V_i \mid 1 \leq i \leq n)$ consists of projective R-modules of finite type, then we can use the canonical embedding

$$\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}} \xrightarrow{\mathfrak{j}_{\mathcal{M},\bar{\mathcal{V}}}} \mathfrak{Fl}^{[n]}_{\mathcal{M}} \tag{5}$$

of the flag variety $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$ into the generic flag variety $\mathfrak{Fl}^{[n]}_{\mathcal{M}}$, where $[n] = \{1, 2, \dots, n\}$ (see 10). This embedding is naturally lifted to a map of covers

$$G_{\mathcal{M},\bar{\mathcal{V}}} \xrightarrow{\mathfrak{j}^0_{\mathcal{M},\bar{\mathcal{V}}}} \mathfrak{Fl}^{[n]}_{\mathcal{M}}.$$

It follows that the square

$$\begin{array}{ccc} G_{\mathcal{M},\bar{\mathcal{V}}} & \xrightarrow{\pi_{\mathcal{M},\bar{\mathcal{V}}}} & \mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}} \\ {\scriptstyle \mathfrak{j}^0_{\mathcal{M},\bar{\mathcal{V}}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{j}_{\mathcal{M},\bar{\mathcal{V}}}} \\ \mathfrak{F}^{[n]}_{\mathcal{M}} & \xrightarrow{\pi^{[n]}_{\mathcal{M}}} & \mathfrak{Fl}^{[n]}_{\mathcal{M}} \end{array}$$

is cartesian, which implies that the both squares of the diagram

$$\begin{array}{ccccc} \mathfrak{R}_{\mathcal{M},\bar{\mathcal{V}}} & \overset{p^1_{\mathcal{M},\bar{\mathcal{V}}}}{\underset{p^2_{\mathcal{M},\bar{\mathcal{V}}}}{\rightrightarrows}} & G_{\mathcal{M},\bar{\mathcal{V}}} & \xrightarrow{\pi_{\mathcal{M},\bar{\mathcal{V}}}} & \mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}} \\ {\scriptstyle \mathfrak{j}^1_{\mathcal{M},\bar{\mathcal{V}}}}\downarrow & cart & {\scriptstyle \mathfrak{j}^0_{\mathcal{M},\bar{\mathcal{V}}}}\downarrow & cart & \downarrow{\scriptstyle \mathfrak{j}_{\mathcal{M},\bar{\mathcal{V}}}} \\ \mathfrak{R}^{[n]}_{\mathcal{M}} & \overset{\mathfrak{p}^{[n]}_{\mathcal{M}}}{\underset{\mathfrak{q}^{[n]}_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{F}^{[n]}_{\mathcal{M}} & \xrightarrow{\pi^{[n]}_{\mathcal{M}}} & \mathfrak{Fl}^{[n]}_{\mathcal{M}} \end{array} \tag{6}$$

are cartesian. The functoriality of the flag varieties $\mathfrak{Fl}^{[n]}_{\mathcal{M}}$ and $\mathfrak{F}\ell_{\mathcal{M},\bar{\mathcal{V}}}$ with respect to morphisms of $R - mod_{spl}$ is compatible with the morphisms (6); that is the diagram (6) is functorial in $\mathcal{M}$.

12.3. Additional details.

12.3.1. A non-additive incarnation of a short resolution. Let

$$\mathcal{L}_1 \xrightarrow{\lambda} \mathcal{L} \xrightarrow{\phi} \mathcal{M} \longrightarrow 0 \qquad (1)$$

be an exact sequence of R-modules and $\mathcal{L}_1 \xrightarrow{\lambda_{\mathfrak{e}}} Ker(\phi)$ the epimorphism induced by $\mathcal{L}_1 \xrightarrow{\lambda} \mathcal{L}$. Let $Ker_2(\phi) = \mathcal{L} \times_{\mathcal{M}} \mathcal{L} \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} \mathcal{L}$ be the kernel pair of the epimorphism $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$. Notice that the canonical morphisms

$$\mathcal{L} \xrightarrow{\mathfrak{j}_\phi} Ker_2(\phi) \xleftarrow{\mathfrak{k}'_\phi} Ker(\phi)$$

determined by the commutative squares respectively

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{id_{\mathcal{L}}} & \mathcal{L} \\ {\scriptstyle id_{\mathcal{L}}}\downarrow & & \downarrow \phi \\ \mathcal{L} & \xrightarrow{\phi} & \mathcal{M} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Ker(\phi) & \longrightarrow & \mathcal{L} \\ \downarrow & & \downarrow \phi \\ \mathcal{L} & \xrightarrow{\phi} & \mathcal{M} \end{array}$$

induce an isomorphism

$$\mathcal{L} \oplus Ker(\phi) \xrightarrow{\sim} Ker_2(\phi).$$

Let $\mathcal{L} \oplus \mathcal{L}_1 \xrightarrow{\psi} Ker_2(\phi)$ be the composition of this isomorphism with the epimorphism $\mathcal{L} \oplus \mathcal{L}_1 \xrightarrow{id_{\mathcal{L}} \oplus \lambda_{\mathfrak{e}}} \mathcal{L} \oplus Ker(\phi)$. Since the diagram

$$Ker_2(\phi) = \mathcal{L} \times_{\mathcal{M}} \mathcal{L} \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} \mathcal{L} \xrightarrow{\phi} \mathcal{M}$$

is exact and the morphism $\mathcal{L} \oplus \mathcal{L}_1 \xrightarrow{\psi} Ker_2(\phi)$ is an epimorphism, the diagram

$$\mathcal{N} = \mathcal{L} \oplus \mathcal{L}_1 \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} \mathcal{L} \xrightarrow{\phi} \mathcal{M}, \qquad (2)$$

where $\psi_i = \phi_i \circ \psi$, $i = 1, 2$, is exact, and all its arrows are epimorphisms. So that (2) is a diagram in the category $R - mod_{\mathfrak{e}}$ formed by R-modules and their epimorphisms.

12.3.2. The corresponding diagram of varieties. Applying the functor

$$R-mod_{\mathfrak{c}}^{op} \xrightarrow{\ \mathfrak{X}_{-}\ } (\mathbf{Aff}_k/R^{\vee})^{\wedge}, \quad (\mathcal{L} \xrightarrow{\phi} \mathcal{M}) \longmapsto (\mathfrak{X}_{\mathcal{M}} \xrightarrow{\mathfrak{X}_{\phi}} \mathfrak{X}_{\mathcal{L}}),$$

to the diagram (2), we obtain the diagram

$$\mathfrak{X}_{\mathcal{M}} \xrightarrow{\ \mathfrak{X}_{\phi}\ } \mathfrak{X}_{\mathcal{L}} \overset{\mathfrak{X}_{\psi_1}}{\underset{\mathfrak{X}_{\psi_2}}{\rightrightarrows}} \mathfrak{X}_{\mathcal{N}}, \tag{3}$$

of presheaves of sets whose arrows are closed immersions.

12.3.3. The corresponding diagrams of covers. Notice that the pair of morphisms $\mathcal{N} = \mathcal{L} \oplus \mathcal{L}_1 \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} \mathcal{L}$ in the diagram (2) is canonically reflexive: the morphism $\mathcal{L} \xrightarrow{(id_{\mathcal{L}},0)} \mathcal{L} \oplus \mathcal{L}_1 = \mathcal{N}$ is right inverse to both ψ_1 and ψ_2. So that the pair of morphisms $\mathcal{N} \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} \mathcal{L}$ gives rise to a pair of morphisms $\mathcal{N} = \mathcal{L} \oplus \mathcal{L}_1 \overset{\bar{\psi}_1}{\underset{\bar{\psi}_2}{\rightrightarrows}} \mathcal{L}$ of the category $R-mod_{spl}$, where $\bar{\psi}_i = (\psi_i, (id_{\mathcal{L}}, 0))$, $i = 1, 2$.

By 12.2.3, the pair of morphisms $\mathcal{N} = \mathcal{L} \oplus \mathcal{L}_1 \overset{\bar{\psi}_1}{\underset{\bar{\psi}_2}{\rightrightarrows}} \mathcal{L}$ of the category $R-mod_{spl}$ determines the pairs of closed immersions

$$\mathfrak{U}_{\mathcal{L}} \overset{\mathfrak{U}_{\bar{\psi}_1}}{\underset{\mathfrak{U}_{\bar{\psi}_2}}{\rightrightarrows}} \mathfrak{U}_{\mathcal{N}} \quad \text{and} \quad \mathfrak{R}_{\mathcal{L}} \overset{\mathfrak{R}_{\bar{\psi}_1}}{\underset{\mathfrak{R}_{\bar{\psi}_2}}{\rightrightarrows}} \mathfrak{R}_{\mathcal{N}} \tag{4}$$

12.4. Proposition. *(a) For any exact sequence of R-modules*

$$\mathcal{L}_1 \xrightarrow{\ \lambda\ } \mathcal{L} \xrightarrow{\ \phi\ } \mathcal{M} \longrightarrow 0,$$

there are natural presheaf isomorphisms

$$\mathfrak{U}_{\mathcal{M}}^{\phi} \xrightarrow{\sim} Ker\big(\mathfrak{U}_{\mathcal{L}} \overset{\mathfrak{U}_{\bar{\psi}_1}}{\underset{\mathfrak{U}_{\bar{\psi}_2}}{\rightrightarrows}} \mathfrak{U}_{\mathcal{N}}\big) \quad \text{and} \quad \mathfrak{R}_{\mathcal{M}}^{\phi} \xrightarrow{\sim} Ker\big(\mathfrak{R}_{\mathcal{L}} \overset{\mathfrak{R}_{\bar{\psi}_1}}{\underset{\mathfrak{R}_{\bar{\psi}_2}}{\rightrightarrows}} \mathfrak{R}_{\mathcal{N}}\big),$$

which gives a commutative diagram

$$
\begin{array}{ccccc}
\mathfrak{R}^{\phi}_{\mathcal{M}} & \overset{\mathfrak{p}^{1\phi}_{\mathcal{M}}}{\underset{\mathfrak{p}^{2\phi}_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{U}^{\phi}_{\mathcal{M}} & \xrightarrow{\pi^{\phi}_{\mathcal{M}}} & \mathfrak{X}_{\mathcal{M}} \\
\downarrow & cart & \downarrow & cart & \downarrow \mathfrak{X}_{\phi} \\
\mathfrak{R}_{\mathcal{L}} & \overset{\mathfrak{p}^{1}_{\mathcal{L}}}{\underset{\mathfrak{p}^{2}_{\mathcal{L}}}{\rightrightarrows}} & \mathfrak{U}_{\mathcal{L}} & \xrightarrow{\pi_{\mathcal{L}}} & \mathfrak{X}_{\mathcal{L}} \\
\mathfrak{R}_{\bar{\psi}_1}\downarrow\downarrow\mathfrak{R}_{\bar{\psi}_2} & & \mathfrak{U}_{\bar{\psi}_1}\downarrow\downarrow\mathfrak{U}_{\bar{\psi}_2} & & \mathfrak{X}_{\psi_1}\downarrow\downarrow\mathfrak{X}_{\psi_2} \\
\mathfrak{R}_{\mathcal{N}} & \overset{\mathfrak{p}^{1}_{\mathcal{N}}}{\underset{\mathfrak{p}^{2}_{\mathcal{N}}}{\rightrightarrows}} & \mathfrak{U}_{\mathcal{N}} & \xrightarrow{\pi_{\mathcal{N}}} & \mathfrak{X}_{\mathcal{N}}
\end{array}
\qquad (5)
$$

whose middle squares are cartesian.

(b) All vertical arrows of the diagram (5) are closed immersions, and all its rows and the two left columns are exact diagrams.

(c) If $\mathcal{L}$ and $\mathcal{L}_1$ are projective R-modules, then all horizontal arrows of the diagram (5) are formally smooth morphisms.

(d) Suppose that $\mathcal{L}$ and $\mathcal{L}_1$ are projective R-modules of finite type and $\mathfrak{X}_-$ is either a non-generic flag variety, or a generic flag variety $\mathfrak{Fl}^{\mathfrak{I}}_-$ with finite $\mathfrak{I}$. Then all horizontal arrows of the diagram (5) are smooth morphisms, all horizontal arrows are representable, and all presheaves, except those in the right column, are representable too.

Proof. 1) Let $\mathfrak{X}_-$ be the generic flag variety $\mathfrak{Fl}^{\mathfrak{I}}_-$; so, $\mathfrak{U}_- = \mathfrak{F}^{\mathfrak{I}}_-$ and $\mathfrak{R}_- = \mathfrak{R}^{\mathfrak{I}}_-$.

(a) The map $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{L}}(S,s) \xrightarrow{\mathfrak{F}^{\mathfrak{I}}_{\bar{\psi}_m}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{N}}(S,s)$ assigns to every element $\{p_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{L}}(S,s)$ the element $\{s^*(\mathfrak{v}_\phi) \circ p_i \circ s^*(\psi_m) \mid i \in \mathfrak{I} - \{\bullet\}\}$ of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{N}}(S,s)$. So that an element $\{p_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ of $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{L}}(S,s)$ belongs to the kernel of the pair of maps

$$
\mathfrak{F}^{\mathfrak{I}}_{\mathcal{L}}(S,s) \overset{\mathfrak{F}^{\mathfrak{I}}_{\bar{\psi}_1}}{\underset{\mathfrak{F}^{\mathfrak{I}}_{\bar{\psi}_2}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{N}}(S,s) \qquad (6)
$$

iff $s^*(\mathfrak{v}_\phi) \circ p_i \circ s^*(\psi_1) = s^*(\mathfrak{v}_\phi) \circ p_i \circ s^*(\psi_2)$ for all $i \in \mathfrak{I} - \{\bullet\}$. Since $s^*(\mathfrak{v}_\phi)$ is a monomorphism, this equality implies that $p_i \circ s^*(\psi_1) = p_i \circ s^*(\psi_2)$ for all $i \in \mathfrak{I} - \{\bullet\}$.

The exactness of the diagram

$$
\mathcal{N} \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} \mathcal{L} \xrightarrow{\ \phi\ } \mathcal{M}
$$

implies the exactness of the diagram

$$s^*\big(\mathcal{N} \overset{\psi_1}{\underset{\psi_2}{\rightrightarrows}} \mathcal{L} \xrightarrow{\ \phi\ } \mathcal{M}\big).$$

Therefore, it follows from the equality $p_i \circ s^*(\psi_1) = p_i \circ s^*(\psi_2)$ that $p_i = \mathfrak{p}_i \circ s^*(\phi)$ for a unique S-module morphism $s^*(\mathcal{M}) \xrightarrow{\mathfrak{p}_i} s^*(\mathcal{L})$.

Same argument shows that an element $\{p_i, p'_i \mid i \in \mathfrak{I} - \{\bullet\}\}$ of $\mathfrak{R}^{\mathfrak{I}}_{\mathcal{L}}(S,s)$ belongs to the kernel of the pair of maps

$$\mathfrak{R}^{\mathfrak{I}}_{\mathcal{L}}(S,s) \overset{\mathfrak{R}^{\mathfrak{I}}_{\bar\psi_1}}{\underset{\mathfrak{R}^{\mathfrak{I}}_{\bar\psi_2}}{\rightrightarrows}} \mathfrak{R}^{\mathfrak{I}}_{\mathcal{N}}(S,s)$$

iff $p_i = \mathfrak{p}_i \circ s^*(\phi)$ and $p'_i = \mathfrak{p}'_i \circ s^*(\phi)$ for all $i \in \mathfrak{I} - \{\bullet\}$.

It follows from this and the descriptions of $\mathfrak{F}^{\mathfrak{I},\phi}_{\mathcal{M}}$ and $\mathfrak{R}^{\mathfrak{I},\phi}_{\mathcal{M}}$ used in the argument of 12.2.1 that we have established natural isomorphisms

$$\mathfrak{F}^{\mathfrak{I},\phi}_{\mathcal{M}} \xrightarrow{\sim} Ker\big(\mathfrak{F}^{\mathfrak{I}}_{\mathcal{L}} \overset{\mathfrak{F}^{\mathfrak{I}}_{\bar\psi_1}}{\underset{\mathfrak{F}^{\mathfrak{I}}_{\bar\psi_2}}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{N}}\big),$$

$$\mathfrak{R}^{\mathfrak{I},\phi}_{\mathcal{M}} \xrightarrow{\sim} Ker\big(\mathfrak{R}^{\mathfrak{I}}_{\mathcal{L}} \overset{\mathfrak{R}^{\mathfrak{I}}_{\bar\psi_1}}{\underset{\mathfrak{R}^{\mathfrak{I}}_{\bar\psi_2}}{\rightrightarrows}} \mathfrak{R}^{\mathfrak{I}}_{\mathcal{N}}\big).$$

(b) The exactness of two left columns of (5) follows from (a). The other assertions are established prior to this proposition.

(c) If $\mathcal{L}$ is a projective R-module then all arrows of the middle row are formally smooth, hence all arrows of the upper row are formally smooth, because the two upper squares are cartesian. If, in addition, the R-module $\mathcal{L}_1$ is projective, then $\mathcal{N} = \mathcal{L} \oplus \mathcal{L}_1$ is a projective R-module. Therefore, the lower horizontal arrows of the diagram (5) are formally smooth morphisms too.

(d) If $\mathcal{L}$ and $\mathcal{L}_1$ are projective R-modules of finite type, then $\mathcal{N} = \mathcal{L} \oplus \mathcal{L}_1$ is a projective R-module of finite type. So that if $\mathfrak{I}$ is finite, then all horizontal arrows of the two lower rows of the diagram (5) are smooth and representable, which implies that the upper horizontal arrows have these properties.

2) If $\mathfrak{X}_-$ is a non-generic flag variety, a simplified version of the above argument proves the assertion. Details are left to the reader. ∎

12.5. The action of GL_-**.** Let $\mathcal{M}$ be an R-module. Recall that $GL_{\mathcal{M}}$ is a group in the category of presheaves of sets on $\mathbf{Aff}_k/R^\vee$ defined by $GL_{\mathcal{M}}(S,s) = Aut_S(s^*(\mathcal{M}))$ for any R-ring (S,s).

12.5.1. Proposition. *The group* $GL_{\mathcal{M}}$ *acts on the diagram*

$$\mathfrak{R}_{\mathcal{M}} \overset{\mathfrak{p}^1_{\mathcal{M}}}{\underset{\mathfrak{p}^2_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{X}_{\mathcal{M}}. \tag{1}$$

Proof. We look at the case of the generic flag variety, $\mathfrak{X}_- = \mathfrak{Fl}^{\mathfrak{I}}_-$, $\mathfrak{U}_- = \mathfrak{F}^{\mathfrak{I}}_-$, and $\mathfrak{R}_{\mathcal{M}} = \mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}$, leaving the non-generic flag varieties to the reader.

(a) Let (S,s) be an R-ring. Every element of $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ is an isomorphism class $[\xi]$ of a functor $\mathfrak{I} \xrightarrow{\xi} S-mod$ which maps the initial object $\bullet$ to the S-module $s^*(\mathcal{M})$ and is determined by the S-module epimorphisms $s^*(\mathcal{M}) \xrightarrow{\xi_i} \mathcal{V}_i$ – the images of arrows $\bullet \longrightarrow i$, $i \in \mathfrak{I} - \{\bullet\}$. For any element g of the group $GL_{\mathcal{M}}(S,s) = Aut_S(s^*(\mathcal{M}))$, we denote by $g \cdot [\xi]$ the element of $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ determined by the S-module morphisms $\{s^*(\mathcal{M}) \xrightarrow{\xi_i \circ g^{-1}} \mathcal{V}_i \mid i \in \mathfrak{I} - \{\bullet\}\}$. This defines an action of the group $GL_{\mathcal{M}}(S,s)$ on $\mathfrak{Fl}^{\mathfrak{I}}_-(S,s)$ which is functorial in (S,s).

(b) The action of $GL_{\mathcal{M}}(S,s)$ on $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ is defined by

$$g \cdot (p_i \mid i \in \mathfrak{I}) = (g \circ p_i \circ g^{-1} \mid i \in \mathfrak{I})$$

for every $(p_i \mid i \in \mathfrak{I}) \in \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ and every $g \in GL_{\mathcal{M}}(S,s) = Aut_S(s^*(\mathcal{M}))$.

This action is, evidently, functorial in (S,s) and agrees with the defined above action of $GL_{\mathcal{M}}$ on $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ in the sense that the projection $\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ is a morphism of actions.

(c) The action of $GL_{\mathcal{M}}(S,s)$ on $\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ is defined by

$$g \cdot (p_i, p'_i \mid i \in \mathfrak{I}) = (g \circ p_i \circ g^{-1},\ g \circ p'_i \circ g^{-1} \mid i \in \mathfrak{I})$$

for every $(p_i, p'_i \mid i \in \mathfrak{I}) \in \mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}}(S,s)$ and every $g \in GL_{\mathcal{M}}(S,s) = Aut_S(s^*(\mathcal{M}))$. ■

12.6. From "toy" varieties to varieties.

The generic and non-generic flag varieties studied in this chapter are *presheaves* of sets on the category $\mathbf{Aff}_k/R^\vee$ of affine noncommutative k-schemes over $R^\vee$, and they are described via exact diagrams in the category of presheaves

$$\mathfrak{R}_{\mathcal{M}} \overset{\mathfrak{p}^1_{\mathcal{M}}}{\underset{\mathfrak{p}^2_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}_{\mathcal{M}} \xrightarrow{\pi_{\mathcal{M}}} \mathfrak{X}_{\mathcal{M}}, \tag{1}$$

or, more generally,

$$\mathfrak{R}^{\phi}_{\mathcal{M}} \underset{\mathfrak{p}^{2^{\phi}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{1^{\phi}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}^{\phi}_{\mathcal{M}} \xrightarrow{\pi^{\phi}_{\mathcal{M}}} \mathfrak{X}_{\mathcal{M}} \tag{2}$$

for some R-module epimorphism $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$. This is why we refer to them as "toy" varieties. Passing from the "toy" varieties to the "real" varieties is passing from presheaves to their associated sheaves for an appropriate subcanonical (quasi-)topology τ on $\mathbf{Aff}_k/R^{\vee}$.

In other words, we apply the sheafification functor with respect to τ to the exact diagram (2). Since the sheafification functor is exact, it maps the exact diagram (2) to exact diagram

$$(\mathfrak{R}^{\phi}_{\mathcal{M}})^{\tau} \underset{\mathfrak{p}^{2^{\phi,\tau}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{1^{\phi,\tau}}_{\mathcal{M}}}{\rightrightarrows}} (\mathfrak{U}^{\phi}_{\mathcal{M}})^{\tau} \xrightarrow{\pi^{\phi,\tau}_{\mathcal{M}}} \mathfrak{X}^{\tau}_{\mathcal{M}} \tag{3}$$

of the associated sheaves, and the pair of arrows $(\mathfrak{R}^{\phi}_{\mathcal{M}})^{\tau} \underset{\mathfrak{p}^{2^{\phi,\tau}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{1^{\phi,\tau}}_{\mathcal{M}}}{\rightrightarrows}} (\mathfrak{U}^{\phi}_{\mathcal{M}})^{\tau}$ is the kernel pair of the morphism $(\mathfrak{U}^{\phi}_{\mathcal{M}})^{\tau} \xrightarrow{\pi^{\phi,\tau}_{\mathcal{M}}} \mathfrak{X}^{\tau}_{\mathcal{M}}$.

12.6.1. Finiteness conditions. If $\mathcal{M}$ is a module of finite type and $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$ is an epimorphism from a projective module of finite type, then the presheaves $\mathfrak{R}^{\phi}_{\mathcal{M}}$ and $\mathfrak{U}^{\phi}_{\mathcal{M}}$ in the diagram (2) are representable, as well as all arrows of the diagram (2). The (quasi-)pretopology τ being subcanonical means that representable presheaves are sheaves. So that, in this case, the exact diagram (3) becomes

$$\mathfrak{R}^{\phi}_{\mathcal{M}} \underset{\mathfrak{p}^{2^{\phi}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{1^{\phi}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}^{\phi}_{\mathcal{M}} \xrightarrow{\pi^{\phi,\tau}_{\mathcal{M}}} \mathfrak{X}^{\tau}_{\mathcal{M}}, \tag{4}$$

where $\mathfrak{U}^{\phi}_{\mathcal{M}} \xrightarrow{\pi^{\phi,\tau}_{\mathcal{M}}} \mathfrak{X}^{\tau}_{\mathcal{M}}$ is the composition of the presheaf epimorphism $\mathfrak{U}^{\phi}_{\mathcal{M}} \xrightarrow{\pi^{\phi}_{\mathcal{M}}} \mathfrak{X}_{\mathcal{M}}$ and the adjunction morphism $\mathfrak{X}_{\mathcal{M}} \xrightarrow{\eta^{\tau}_{\mathfrak{X}_{\mathcal{M}}}} \mathfrak{X}^{\tau}_{\mathcal{M}}$ of the sheafification functor.

In other words, the variety $\mathfrak{X}^{\tau}_{\mathcal{M}}$ is the cokernel of the pair of smooth morphisms

$$\mathfrak{R}^{\phi}_{\mathcal{M}} \underset{\mathfrak{p}^{2^{\phi}}_{\mathcal{M}}}{\overset{\mathfrak{p}^{1^{\phi}}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}^{\phi}_{\mathcal{M}}$$

of representable sheaves on $(\mathbf{Aff}_k/R^\vee, \tau)$.

If an R-module $\mathcal{M}$ is not of finite type, then we have the following assertion.

12.6.2. Proposition. *Suppose that* $\mathfrak{X}_-$ *is either a generic flag variety* $\mathfrak{Fl}_-^{\mathfrak{I}}$, *with finite* $\mathfrak{I}$, *or a non-generic flag variety* $\mathfrak{F}\ell_{-,\bar{\mathcal{V}}}$ *such that all modules of* $\bar{\mathcal{V}} = (\mathcal{V}_1, \dots, \mathcal{V}_n)$ *are projective of finite type. Let* $\mathcal{M}$ *be an arbitrary* R*-module and* $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$ *an* R*-module epimorphism from a projective module* $\mathcal{L}$*. Then there is an inductive system of diagrams*

$$\begin{array}{ccccc}
\mathfrak{R}^{\phi_\nu}_{\mathcal{M}_\nu} & \overset{\mathfrak{p}1^{\phi_\nu}_{\mathcal{M}_\nu}}{\underset{\mathfrak{p}2^{\phi_\nu}_{\mathcal{M}_\nu}}{\rightrightarrows}} & \mathfrak{U}^{\phi_\nu}_{\mathcal{M}_\nu} & \xrightarrow{\pi^{\phi_\nu,\tau}_{\mathcal{M}_\nu}} & \mathfrak{X}^\tau_{\mathcal{M}_\nu} \\
\downarrow & & \downarrow & & \downarrow \mathfrak{X}^\tau_{\mathfrak{e}'_\nu} \\
\mathfrak{R}^{\phi,\tau}_{\mathcal{M}} & \overset{\mathfrak{p}1^{\phi}_{\mathcal{M}}}{\underset{\mathfrak{p}2^{\phi}_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{U}^{\phi,\tau}_{\mathcal{M}} & \xrightarrow{\pi^{\phi,\tau}_{\mathcal{M}}} & \mathfrak{X}^\tau_{\mathcal{M}},
\end{array} \qquad \nu \in \Xi, \qquad (5)$$

corresponding to a presentation of the epimorphism $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$ *as a limit of a filtered system of epimorphisms* $\{\mathcal{L}_\nu \xrightarrow{\phi_\nu} \mathcal{M}_\nu \mid \nu \in \Xi\}$, *where* $\mathcal{L}_\nu$ *are projective modules of finite type. These diagrams have the following properties:*

(i) Their rows are exact diagrams of sheaves.

(ii) The upper horizontal arrows are representable and smooth, and $\mathfrak{U}^{\phi_\nu}_{\mathcal{M}_\nu}$, $\mathfrak{R}^{\phi_\nu}_{\mathcal{M}_\nu}$ *are representable sheaves.*

(iii) The vertical arrows are closed immersions.

(iv) The cone (5) is universal; that is the low horizontal row is the colimit of the upper horizontal rows. In particular, the right vertical arrows of (5) induce an isomorphism of sheaves

$$\operatorname{colim}\big(\mathfrak{X}^\tau_{\mathcal{M}_\nu} \mid \nu \in \Xi\big) \xrightarrow{\sim} \mathfrak{X}^\tau_{\mathcal{M}}.$$

Proof. Let $\mathcal{M}$ be an arbitrary R-module and $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$ an R-module epimorphism from a projective module $\mathcal{L}$. Fix a filtered system $\{\mathfrak{p}_\nu \mid \nu \in \Xi\}$ of projectors of the R-module $\mathcal{L}$ such that $\mathfrak{p}_\nu(\mathcal{L})$ is a module of finite type and $\mathcal{L} = sup\{\mathfrak{p}_\nu(\mathcal{L}) \mid \nu \in \Xi\}$.

Let $\mathcal{L}_\nu = \mathfrak{p}_\nu(\mathcal{L}) \xrightarrow{\phi_\nu} \mathcal{M}_\nu$ be the push-forward of the epimorphism $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$ along the corestriction $\mathcal{L} \xrightarrow{\mathfrak{e}_\nu} \mathfrak{p}_\nu(\mathcal{L})$ of $\mathfrak{p}_\nu$. The cocartesian square

$$\begin{array}{ccc}
\mathcal{L} & \xrightarrow{\phi} & \mathcal{M} \\
e_\nu \downarrow & cocart & \downarrow \mathfrak{e}'_\nu \\
\mathcal{L}_\nu & \xrightarrow{\phi_\nu} & \mathcal{M}_\nu
\end{array}$$

together with the embedding $\mathcal{L}_\nu = \mathfrak{p}_\nu(\mathcal{L}) \xrightarrow{\mathfrak{j}_\nu} \mathcal{L}$ determined by $\mathfrak{j}_\nu \circ \mathfrak{e}_\nu = \mathfrak{p}_\nu$ gives rise to a morphism of diagrams

$$\begin{array}{ccccc}
\mathfrak{R}^{\phi_\nu}_{\mathcal{M}_\nu} & \overset{\mathfrak{p}1^{\phi_\nu}_{\mathcal{M}_\nu}}{\underset{\mathfrak{p}2^{\phi_\nu}_{\mathcal{M}_\nu}}{\rightrightarrows}} & \mathfrak{U}^{\phi_\nu}_{\mathcal{M}_\nu} & \xrightarrow{\pi^{\phi_\nu}_{\mathcal{M}_\nu}} & \mathfrak{X}_{\mathcal{M}_\nu} \\
\downarrow & & \downarrow & & \downarrow \mathfrak{X}_{\mathfrak{e}'_\nu} \\
\mathfrak{R}^{\phi}_{\mathcal{M}} & \overset{\mathfrak{p}1^{\phi}_{\mathcal{M}}}{\underset{\mathfrak{p}2^{\phi}_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{U}^{\phi}_{\mathcal{M}} & \xrightarrow{\pi^{\phi}_{\mathcal{M}}} & \mathfrak{X}_{\mathcal{M}}
\end{array} \tag{6}$$

formed by closed immersions. Applying the sheafification functor, we obtain an inductive system of commutative diagrams

$$\begin{array}{ccccc}
\mathfrak{R}^{\phi_\nu}_{\mathcal{M}_\nu} & \overset{\mathfrak{p}1^{\phi_\nu}_{\mathcal{M}_\nu}}{\underset{\mathfrak{p}2^{\phi_\nu}_{\mathcal{M}_\nu}}{\rightrightarrows}} & \mathfrak{U}^{\phi_\nu}_{\mathcal{M}_\nu} & \xrightarrow{\pi^{\phi_\nu,\tau}_{\mathcal{M}_\nu}} & \mathfrak{X}^{\tau}_{\mathcal{M}_\nu} \\
\downarrow & & \downarrow & & \downarrow \mathfrak{X}^{\tau}_{\mathfrak{e}'_\nu} \\
\mathfrak{R}^{\phi,\tau}_{\mathcal{M}} & \overset{\mathfrak{p}1^{\phi,\tau}_{\mathcal{M}}}{\underset{\mathfrak{p}2^{\phi,\tau}_{\mathcal{M}}}{\rightrightarrows}} & \mathfrak{U}^{\phi,\tau}_{\mathcal{M}} & \xrightarrow{\pi^{\phi,\tau}_{\mathcal{M}}} & \mathfrak{X}^{\tau}_{\mathcal{M}}
\end{array} \tag{7}$$

of associated sheaves whose vertical arrows are closed immersions.

The claim is that the canonical morphisms of presheves

$$\begin{array}{ll}
\operatorname{colim}\big(\mathfrak{X}_{\mathcal{M}_\nu} \mid \nu \in \Xi\big) \longrightarrow \mathfrak{X}_{\mathcal{M}} & \text{and} \\
\operatorname{colim}\big(\mathfrak{U}^{\phi_\nu}_{\mathcal{M}_\nu} \mid \nu \in \Xi\big) \longrightarrow \mathfrak{U}^{\phi}_{\mathcal{M}} &
\end{array} \tag{8}$$

induced by the respectively right and central vertical arrows of the diagrams (6) are isomorphisms. Since colimits of filtered diagrams commute with finite limits, this implies that the diagram

$$\mathfrak{R}^{\phi}_{\mathcal{M}} \overset{\mathfrak{p}1^{\phi}_{\mathcal{M}}}{\underset{\mathfrak{p}2^{\phi}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}^{\phi}_{\mathcal{M}} \xrightarrow{\pi^{\phi}_{\mathcal{M}}} \mathfrak{X}_{\mathcal{M}}$$

is the colimit of the exact diagrams

$$\mathfrak{R}^{\phi_\nu}_{\mathcal{M}_\nu} \overset{\mathfrak{p}1^{\phi_\nu}_{\mathcal{M}_\nu}}{\underset{\mathfrak{p}2^{\phi_\nu}_{\mathcal{M}_\nu}}{\rightrightarrows}} \mathfrak{U}^{\phi_\nu}_{\mathcal{M}_\nu} \xrightarrow{\pi^{\phi_\nu}_{\mathcal{M}_\nu}} \mathfrak{X}_{\mathcal{M}_\nu}, \quad \nu \in \Xi.$$

Since the sheafification functor preserves colimits, this implies that the diagram

$$\mathfrak{R}^{\phi,\tau}_{\mathcal{M}} \overset{\mathfrak{p}^{1\phi,\tau}_{\mathcal{M}}}{\underset{\mathfrak{p}^{2\phi,\tau}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}^{\phi,\tau}_{\mathcal{M}} \xrightarrow{\pi^{\phi,\tau}_{\mathcal{M}}} \mathfrak{X}^{\tau}_{\mathcal{M}}$$

is the colimit of the inductive system of exact diagrams

$$\mathfrak{R}^{\phi_\nu}_{\mathcal{M}_\nu} \overset{\mathfrak{p}^{1\phi_\nu}_{\mathcal{M}_\nu}}{\underset{\mathfrak{p}^{2\phi_\nu}_{\mathcal{M}_\nu}}{\rightrightarrows}} \mathfrak{U}^{\phi_\nu}_{\mathcal{M}_\nu} \xrightarrow{\pi^{\phi_\nu,\tau}_{\mathcal{M}_\nu}} \mathfrak{X}^{\tau}_{\mathcal{M}_\nu}, \quad \nu \in \Xi.$$

In particular, the canonical morphism of associated sheaves

$$\mathrm{colim}\big(\mathfrak{X}^{\tau}_{\mathcal{M}_\nu} \mid \nu \in \Xi\big) \longrightarrow \mathfrak{X}^{\tau}_{\mathcal{M}} \tag{9}$$

determined by the right vertical arrows of the diagrams (7) is an isomorphism.

(a) Suppose that $\mathfrak{X}_-$ is the generic flag variety $\mathfrak{Fl}^{\mathfrak{J}}_-$, with finite $\mathfrak{J}$.

Let (S,s) be an R-ring and $\mathfrak{J} \xrightarrow{\xi} S - mod$ a representative of an element of $\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$. The functor ξ is determined by the epimorphisms $\{s^*(\mathcal{M}) \xrightarrow{\xi_i} \mathcal{V}_i \mid i \in \mathfrak{J} - \{\bullet\}\}$ – the values of ξ on $\bullet \longrightarrow i$. Since $\mathfrak{J}$ is finite and each S-module $\mathcal{V}_i$ is of finite type, there is a $\nu \in \Xi$ such that $\xi_i \circ s^*(\phi \circ \mathfrak{p}_\nu)$ is an epimorphism for each $i \in \mathfrak{J}$. This implies that the canonical morphism

$$\mathrm{colim}\big(\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}_\nu} \mid \nu \in \Xi\big) \longrightarrow \mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}}$$

is an isomorphism, hence the canonical morphism

$$\mathrm{colim}\big((\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}_\nu})^\tau \mid \nu \in \Xi\big) \longrightarrow (\mathfrak{Fl}^{\mathfrak{J}}_{\mathcal{M}})^\tau$$

of associated sheaves is an isomorphism too.

(a1) By the same reason, the canonical morphism of presheaves

$$\mathrm{colim}\big(\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}_\nu} \mid \nu \in \Xi\big) \longrightarrow \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}}$$

is an isomorphism, hence the canonical morphism

$$\mathrm{colim}\big((\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}_\nu})^\tau = \mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}_\nu} \mid \nu \in \Xi\big) \longrightarrow (\mathfrak{F}^{\mathfrak{J}}_{\mathcal{M}})^\tau$$

of associated sheaves is an isomorphism.

(b) The same considerations show that the canonical morphisms (8) are isomorphisms in the case when $\mathfrak{X}_-$ is a non-generic flag variety $\mathfrak{F}\ell_{-,\bar{\mathcal{V}}}$. ■

12.6.2.1. Corollary. *Suppose that the (quasi-)topology τ is quasi-compact. then, under the conditions of 12.6.2, the presheaves $\mathfrak{U}^{\phi}_{\mathcal{M}}$ and $\mathfrak{R}^{\phi}_{\mathcal{M}}$ are sheaves.*

Proof. Let τ be a (quasi-)pretopology. By the argument of 12.6.2,

$$\mathrm{colim}\big(\mathfrak{U}^{\phi_\nu}_{\mathcal{M}_\nu} \mid \nu \in \Xi\big) \xrightarrow{\sim} \mathfrak{U}^{\phi}_{\mathcal{M}} \quad \text{and} \quad \mathrm{colim}\big(\mathfrak{R}^{\phi_\nu}_{\mathcal{M}_\nu} \mid \nu \in \Xi\big) \xrightarrow{\sim} \mathfrak{R}^{\phi}_{\mathcal{M}};$$

that is each of these presheaves is a colimit of a filtered diagram of sheaves. Since colimits of filtered diagrams of sets are compatible with limits of finite diagrams, the presheaves $\mathfrak{U}^{\phi}_{\mathcal{M}}$ and $\mathfrak{R}^{\phi}_{\mathcal{M}}$ satisfy the sheaf property for finite covers of τ. So that, if the (quasi-)pretopology τ is quasi-compact, they are sheaves. ■

12.6.3. An appropriate choice of topology. If $\mathcal{M}$ is a module of finite type and $\mathcal{L} \xrightarrow{\phi} \mathcal{M}$ is an epimorphism from a projective module of finite type, then the presheaves $\mathfrak{R}^{\phi}_{\mathcal{M}}$ and $\mathfrak{U}^{\phi}_{\mathcal{M}}$ in the diagram

$$\mathfrak{R}^{\phi}_{\mathcal{M}} \underset{\mathfrak{p}2^{\phi}_{\mathcal{M}}}{\overset{\mathfrak{p}1^{\phi}_{\mathcal{M}}}{\rightrightarrows}} \mathfrak{U}^{\phi}_{\mathcal{M}} \xrightarrow{\pi^{\phi}_{\mathcal{M}}} \mathfrak{X}_{\mathcal{M}} \qquad (2)$$

are representable and all morphisms of this diagram are representable smooth covers of the smooth subcanonical pretopology. So that the smooth topology looks as the most natural choice for the class of varieties introduced in this chapter, at least for the gluing purposes.

12.7. Morphisms to the flag varieties.

12.7.1. A general observation. Let $(\mathcal{B}, \tau)$ be a subcanonical presite; i.e. τ is a subcanonical pretopology on the category $\mathcal{B}$. Let $\mathfrak{X}$ be a presheaf of sets on $\mathcal{B}$ and $\mathfrak{X} \xrightarrow{\eta^{\tau}_{\mathfrak{X}}} \mathfrak{X}^{\tau}$ its adjunction morphism to the associated sheaf.

(a) Let $\mathfrak{Y} \xrightarrow{\mathfrak{f}} \mathfrak{X}^{\tau}$ be a presheaf morphism. We denote by $\tau^{\mathfrak{f}}$ the class of all elements of the form $\widehat{\mathcal{U}} \xrightarrow{\mathfrak{u}} \mathfrak{Y}$ of a cover of $\mathfrak{Y}$ in the coinduced by τ pretopology such that the composition $\widehat{\mathcal{U}} \xrightarrow{\mathfrak{f}\circ\mathfrak{u}} \mathfrak{X}^{\tau}$ factors through the adjunction morphism $\mathfrak{X} \xrightarrow{\eta^{\tau}_{\mathfrak{X}}} \mathfrak{X}^{\tau}$. That is $\mathfrak{f} \circ \mathfrak{u} = \eta^{\tau}_{\mathfrak{X}} \circ \mathfrak{f}_{\mathfrak{u}}$ for some $\widehat{\mathcal{U}} \xrightarrow{\mathfrak{f}_{\mathfrak{u}}} \mathfrak{X}$. By Yoneda Lemma, each morphism $\widehat{\mathcal{U}} \xrightarrow{\mathfrak{f}_{\mathfrak{u}}} \mathfrak{X}$ is identified with an element $\xi_{\mathfrak{f}_{\mathfrak{u}}}(\mathcal{U})$ of the set $\mathfrak{X}(\mathcal{U})$.

(b) If $\mathfrak{X}$ is a *monopresheaf*, which means, by definition, that $\mathfrak{X} \xrightarrow{\eta^{\tau}_{\mathfrak{X}}} \mathfrak{X}^{\tau}$ is a monomorphism, then the morphisms $\widehat{\mathcal{U}} \xrightarrow{\mathfrak{f}_{\mathfrak{u}}} \mathfrak{X}$, hence the elements $\xi_{\mathfrak{f}_{\mathfrak{u}}}(\mathcal{U})$ of the set $\mathfrak{X}(\mathcal{U})$, are uniquely determined by $\mathfrak{f}$ and $\mathfrak{u}$.

(c) Suppose that $\mathfrak{Y}$ is a locally representable (– locally affine) sheaf. Then, for every sheaf morphism $\mathfrak{Y} \xrightarrow{\mathfrak{f}} \mathfrak{X}^\tau$, the cone $\tau^{\mathfrak{f}}$ contains a cover of $\mathfrak{Y}$ (by representable presheaves). In particular, $\mathfrak{Y}$ is the colimit of the functor $\tau^{\mathfrak{f}} \longrightarrow \mathcal{B}^\wedge$, which is the composition of the forgetful functor $\tau^{\mathfrak{f}} \longrightarrow \mathcal{B}$ and the Yoneda embedding $\mathcal{B} \longrightarrow (\mathcal{B}, \tau)^\wedge$.

(c1) Suppose, in addition, that $\mathfrak{X}$ is a monopresheaf. Then every morphism $\mathfrak{Y} \xrightarrow{\xi} \mathfrak{X}^\tau$ is described by a pair (τ_ξ, ξ_τ), where τ^ξ is a refinement of the pretopology $\tau_{\mathfrak{Y}}$ on $\mathcal{B}/\mathfrak{Y}$ and ξ_τ is a morphism from this refinement to $\mathfrak{X}$. Two such pairs are equivalent if they both are parts of a third such pair. Evidently each equivalence class has the largest element, which is, precisely, the refinement $\tau_{\mathfrak{f}}$ corresponding to the morphism $\mathfrak{f}$. Thus, we have a natural bijective correspondence between sheaf morphisms $\mathfrak{Y} \longrightarrow \mathfrak{X}^\tau$ and the equivalence classes of pairs $(\tau_\xi, \tau_\xi \xrightarrow{\xi_\tau} \mathfrak{X})$.

(c2) Suppose that $\mathfrak{X}$ is a monopresheaf. Making $\mathfrak{Y}$ run through $\widehat{\mathcal{V}}$, $\mathcal{V} \in Ob\mathcal{B}$, we obtain a description of the sheaf $\mathfrak{X}^\tau$ associated with $\mathfrak{X}$ in terms of the presheaf $\mathfrak{X}$.

Namely, every element ξ of $\mathfrak{X}^\tau(\mathcal{V})$ is given by a pair (τ^ξ, ξ_τ), where τ^ξ is a refinement of the pretopology τ on $\mathcal{B}/\mathcal{V}$ and ξ_τ a morphism from the refinement τ^ξ to $\mathfrak{X}$.

(d) Suppose that $\mathfrak{X}$ (equivalently, $\mathfrak{X}^\tau$) is locally representable, and $\mathfrak{X}$ is a monopresheaf. Then we can apply the above considerations to the identical morphism $\mathfrak{X}^\tau \xrightarrow{id_{\mathfrak{X}^\tau}} \mathfrak{X}^\tau$.

By definition, the refinement $\tau^{id_{\mathfrak{X}^\tau}}$ consists of all elements of covers of $\mathfrak{X}^\tau$, which are compositions $\eta^\tau_{\mathfrak{X}} \circ \mathfrak{u}$, for some element $\widehat{\mathcal{U}} \xrightarrow{\mathfrak{u}} \mathfrak{X}$ of a cover. We call this refinement and its morphism to $\mathfrak{X}$ *tautological.*

12.7.2. A description of $(\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}})^\tau$. According to 12.7.1(c2), the set $(\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}})^\tau(\mathcal{S}, \mathfrak{s})$ can be identified with the set of equivalence classes of pairs (τ^ξ, ξ_τ), where τ^ξ is a refinement of a pretopology τ on $\mathbf{Aff}_k/(\mathcal{S}, \mathfrak{s})^\vee$ and ξ_τ a morphism from this refinement to $\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$. This means that, for every element $(\mathcal{U}, \mathfrak{u})^\vee \xrightarrow{\widetilde{\mathfrak{u}}^\vee} (\mathcal{S}, \mathfrak{s})^\vee$ of the refinement τ_ξ, there is a uniquely defined isomorphism class of functors $\mathfrak{I} \xrightarrow{\xi_\tau} \mathcal{U} - mod$, which map every arrow $\bullet \longrightarrow \mathfrak{i}$, $\mathfrak{i} \in \mathfrak{I} - \{\bullet\}$, to an epimorphism

$$\mathfrak{u}^*(\mathcal{M}) \xrightarrow{\xi^{\mathfrak{i}}_\tau(\widetilde{\mathfrak{u}})} \mathcal{W}_{\mathfrak{i}}(\mathcal{U}, \widetilde{\mathfrak{u}})$$

with $\mathcal{W}_{\mathfrak{i}}(\mathcal{U}, \widetilde{\mathfrak{u}})$ a projective $\mathcal{U}$-module of finite type. This map depends functorially on the element $(\mathcal{U}, \mathfrak{u})^\vee \xrightarrow{\widetilde{\mathfrak{u}}^\vee} (\mathcal{S}, \mathfrak{s})^\vee$ of the refinement τ_ξ.

12.7.2.1. Note. If $\mathfrak{I}$ is finite and $\mathcal{M}$ is an R-module of finite type, then it follows from 12.6.1 that, for every $\mathfrak{i} \in \mathfrak{I}$, $\mathfrak{i} \neq \bullet$, there exists a projective $\mathcal{S}$-module $\mathcal{L}_{\mathfrak{i}}$ of finite type such that $\mathcal{W}_{\mathfrak{i}}(\mathcal{U},\widetilde{\mathfrak{u}})$ is isomorphic to $\widetilde{\mathfrak{u}}^*(\mathcal{L}_{\mathfrak{i}})$ for any element $(\mathcal{U},\mathfrak{u})^\vee \xrightarrow{\widetilde{\mathfrak{u}}^\vee} (\mathcal{S},\mathfrak{s})^\vee$ of the refinement τ_ξ.

12.7.2.2. Tautological refinement. It is given by epimorphisms

$$\mathfrak{u}^*(\mathcal{M}) \xrightarrow{\xi_\tau^{\mathfrak{i}}(\widetilde{\mathfrak{u}})} \mathcal{W}_{\mathfrak{i}}(\mathcal{U},\widetilde{\mathfrak{u}}), \quad \mathfrak{i} \in \mathfrak{J}, \tag{1}$$

corresponding to the arrows $\bullet \longrightarrow \mathfrak{i}$, $\mathfrak{i} \in \mathfrak{J}$, with $\mathcal{W}_{\mathfrak{i}}(\mathcal{U},\widetilde{\mathfrak{u}})$ a projective $\mathcal{U}$-module of finite type. This map depends functorially on the object $(\mathcal{U},\mathfrak{u})^\vee \xrightarrow{\widetilde{\mathfrak{u}}} \mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}}$ of $\tau/\mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}}$.

12.7.3. A description of $\mathfrak{F}\ell^\tau_{\mathcal{M},\bar{\mathcal{V}}}$. Fix an R-module $\mathcal{M}$ and projective R-modules $\bar{\mathcal{V}} = (\mathcal{V}_i \mid 1 \le i \le n)$. The description of $\mathfrak{F}\ell^\tau_{\mathcal{M},\bar{\mathcal{V}}}(S,s)$ is like the description of $(\mathfrak{Fl}_{\mathcal{M}}^{\mathfrak{J}})^\tau(S,s)$ in 12.7.2 for $\mathfrak{J} = (\bullet \to n \to \dots \to 1)$ and with additional condition that, for every element $(\mathcal{U},\mathfrak{u})^\vee \xrightarrow{\widetilde{\mathfrak{u}}^\vee} (\mathcal{S},\mathfrak{s})^\vee$ of the refinement τ_ξ, and every $n \ge \mathfrak{i} \ge 1$, the projective $\mathcal{U}$-module $\mathcal{W}_{\mathfrak{i}}(\mathcal{U},\widetilde{\mathfrak{u}})$ is isomorphic to $\widetilde{\mathfrak{u}}^*(\mathcal{V}_{\mathfrak{i}})$.

Chapter IV
Quasi-Coherent Sheaves on Fibred Categories and Noncommutative Spaces.

Quasi-coherent sheaves on geometric (i.e. locally ringed topological) spaces were introduced in fifties. The notion of quasi-coherent modules was extended in an obvious way to ringed sites and toposes at the moment the latter appeared (in SGA), but it was not used much in this generality. At the end of nineties, the subject was revisited by D. Orlov in his work on quasi-coherent sheaves in commutative an noncommutative geometry [Or] and by G. Laumon an L. Moret-Bailly in their book on algebraic stacks [LM-B].

Slightly generalizing [R4], we associate with any functor F (regarded as a category over a category) the category of 'quasi-coherent presheaves' on F (otherwise called 'quasi-coherent presheaves of modules' or simply 'quasi-coherent modules') and study some basic properties of this correspondence in the case when the functor defines a fibred category. Imitating [Gir], we define the quasi-topology of 1-descent (or simply 'descent') and the quasi-topology of 2-descent (or 'effective descent') on the base of a fibred category (i.e. on the target of the functor F). If the base is endowed with a quasi-topology, τ, we introduce the notion of a 'sheaf of modules' on (F,τ). We define the category $Qcoh(F,\tau)$ of quasi-coherent sheaves on (F,τ) as the intersection of the category $Qcoh(F)$ of quasi-coherent presheaves on F and the category of sheaves of modules on (F,τ).

If the quasi-topology τ is coarser than the quasi-topology of 1-descent, than every quasi-coherent module on F is a sheaf of modules on (F,τ), i.e. $Qcoh(F,\tau) = Qcoh(F)$. In this case, we show, under certain natural conditions on a presheaf of sets on the base X, the existence of a 'coherator' which is, by definition, a right adjoint to the embedding of the category $Qcoh(F/X)$ of quasi-coherent modules on X into the category of sheaves of modules on X (that is on F/X). This fact is important, because the existence of the coherator on X guarantees the existence of the direct image functor (between quasi-coherent modules) for any morphism from a presheaf of sets to X.

The relation of this formalism with the classical notions and those used in [Or] is as follows. With any ringed category $(\mathcal{A},\mathcal{O})$, one can naturally associate a fibred category F: its fiber over an object T of $\mathcal{A}$ is the category opposite to the category of $\mathcal{O}(T)$-modules. The category $Qcoh(F)$ of quasi-coherent modules on the fibred category F is equivalent to the category of quasi-coherent $\mathcal{O}$-modules in the sense of [Or]. If τ is a topology on $\mathcal{A}$, then the category $Qcoh(F,\tau)$ is equivalent to the category of quasi-coherent sheaves of $\mathcal{O}$-modules in the classical (i.e. [SGA]) sense. In particular, if F is the fibred category of modules over (commutative) affine schemes and the presheaf X is represented by a scheme (or an algebraic space) X, then the category $Qcoh(F/X)$ is naturally equivalent to the category of quasi-coherent sheaves on the scheme (resp. on the algebraic space) X.

A standard noncommutative example is the ringed category $(\mathbf{Aff}_k, \mathcal{O})$, where $\mathbf{Aff}_k$ is the category opposite to the category of associative unital k-algebras and the presheaf of k-algebras $\mathcal{O}$ assigns to any object $R^\vee$ of $\mathbf{Aff}_k$ the corresponding to the algebra R. To any presheaf of sets X on $\mathbf{Aff}_k$, there corresponds a ringed category $(\mathbf{Aff}_k/X, \mathcal{O}_X)$. We denote the associated category of quasi-coherent modules by $Qcoh(X, \mathcal{O}_X)$ and call it the *category of quasi-coherent modules on* X. If the presheaf X is representable by $R^\vee$, then $Qcoh(X, \mathcal{O}_X)$ is equivalent to the category $R-mod$ of left R-modules. If X is the colimit of a diagram of representable presheaves, then $Qcoh(X, \mathcal{O}_X)$ is the limit (in pseudo-functorial sense) of the corresponding diagram of the categories of left modules.

In particular, if X is a locally representable presheaf of sets, then the category $Qcoh(X, \mathcal{O}_X)$ is described via affine covers and relations. We show, among other facts, that the canonical topology on $\mathbf{Aff}_k$ (i.e. the strongest topology such that every representable presheaf of sets on $\mathbf{Aff}_k$ is a sheaf) is precisely the topology of 1-descent. In the commutative case, this fact was established by D. Orlov [Or].

In Section 1, we introduce modules and quasi-coherent modules on a category over a category and study first properties of these notions in the case of fibred categories.

In Section 2 we introduce, imitating [Gir, II.1.1.1], quasi-topologies of 1- and 2-descent, and establish, under certain conditions, the existence of a coherator.

In Section 3, we define sheaves of modules and sheaves of quasi-coherent modules on fibred and cofibred categories whose base is endowed with a (quasi-)topology.

In Section 4, we apply the facts and constructions of the previous Sections to the fibred categories associated with ringed categories (in particular, to ringed sites and toposes). Section 5 contains preliminaries on representable fiber categories and representable cartesian functors. In Section 6, we define 'local constructions' on fibred categories which is a device to transfer certain functorial constructions of (noncommutative) 'varieties' defined over an affine base to constructions of 'varieties' over stacks, in particular, over arbitrary locally affine "spaces". Among them, there are affine and projective vector-fibers corresponding to a quasi-coherent module on a ringed category, and Grassmannians corresponding to a pair of locally projective quasi-coherent modules on a ringed category.

1. Quasi-coherent modules on a fibred category.

1.0. Preliminaries on fibred an cofibred categories. See Appendix 1.

1.1. Modules and quasi-coherent modules on a category over a category. Let $\mathcal{E}$ be a svelte category and $\mathfrak{F} = (\mathcal{F}, \mathcal{F} \xrightarrow{\pi} \mathcal{E})$ a category over $\mathcal{E}$. Denote by $\mathcal{M}od(\mathfrak{F})$ the category opposite to the category of all sections of $\mathfrak{F}$.

We call objects of the category $\mathcal{M}od(\mathfrak{F})$ *modules* on $\mathfrak{F}$.

1.1.1. Quasi-coherent modules. We denote by $Qcoh(\mathfrak{F})$ the category opposite to the category $Cart_{\mathcal{E}}(\mathcal{E},\mathfrak{F})$ of cartesian sections of $\mathfrak{F}$. In other words, $Qcoh(\mathfrak{F}) = (Lim\mathfrak{F})^{op}$ (cf. A1.5.5). Objects of $Qcoh(\mathfrak{F})$ will be called *quasi-coherent modules* on $\mathfrak{F}$.

Any morphism $\mathfrak{F} \longrightarrow \mathfrak{G}$ of $\mathcal{E}$-categories induces a functor $\mathcal{M}od(\mathfrak{F}) \longrightarrow \mathcal{M}od(\mathfrak{G})$. Thus we have a functor

$$\mathcal{M}od : Cat/\mathcal{E} \longrightarrow Cat$$

from the category of $\mathcal{E}$-categories to the category of categories.

Similarly, the map $\mathfrak{F} \longmapsto Qcoh(\mathfrak{F})$ extends to a functor

$$Qcoh : Cart_{\mathcal{E}} \longrightarrow Cat$$

from the category of cartesian functors over $\mathcal{E}$ to Cat.

1.1.2. Proposition. *The functor $Qcoh : Cart_{\mathcal{E}} \longrightarrow Cat$ preserves small products.*

Proof. In fact, by A1.6.6 and A1.6.6.2, the functor $Lim : Cart_{\mathcal{E}} \longrightarrow Cat$ preserves small products. The functor $Qcoh$ is, by definition, the composition of Lim and the canonical automorphism $Cat \longrightarrow Cat$, $C \longmapsto C^{op}$. ∎

1.2. Modules and quasi-coherent modules on a fibred category. Let $\mathfrak{F}$ be a fibred category corresponding to a pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$,

$$Ob\mathcal{E} \ni X \longmapsto \mathcal{F}_X,\; Hom\mathcal{E} \ni f \longmapsto f^*,\; Hom\mathcal{E} \times_{Ob\mathcal{E}} Hom\mathcal{E} \ni (f,g) \longmapsto c_{f,g} \qquad (1)$$

(cf. A1.7, A1.7.1). Then the category $\mathcal{M}od(\mathfrak{F})$ of modules on $\mathfrak{F}$ can be described as follows. An object of $\mathcal{M}od(\mathfrak{F})$ is a function which assigns to each $T \in Ob\mathcal{E}$ an object $M(T)$ of the fiber $\mathcal{F}_T$ and to each morphism $T \xrightarrow{f} T'$ a morphism $f^*(M(T')) \xrightarrow{\xi_f} M(T)$ such that $\xi_{gf} \circ c_{f,g} = \xi_f \circ f^*(\xi_g)$. Morphisms are defined in a natural way.

1.2.1. Quasi-coherent modules. An object (M,ξ) of $\mathcal{M}od(\mathfrak{F})$ belongs to the subcategory $Qcoh(\mathfrak{F})$ iff ξ_f is an isomorphism for all $f \in Hom\mathcal{E}$.

1.3. Proposition. *Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$. Suppose that the category $\mathcal{E}$ has a final object, $T_\bullet$. Then*

(a) The category $Qcoh(\mathfrak{F})$ is equivalent to the category $\mathcal{F}_{T_\bullet}^{op}$ dual to the fiber of $\mathfrak{F}$ at the final object $T_\bullet$.

(b) The inclusion functor $Qcoh(\mathfrak{F}) \longrightarrow \mathcal{M}od(\mathfrak{F})$ has a right adjoint.

Proof. (a) The equivalence is given by the functor $Qcoh(\mathfrak{F}) \longrightarrow \mathcal{F}_{T_\bullet}$ which assigns to every quasi-coherent module M on $\mathfrak{F}$ the object $M(T_\bullet)$ of $\mathcal{F}_{T_\bullet}$. The quasi-inverse functor maps any object L of $\mathcal{F}_{T_\bullet}$ to the quasi-coherent module $L^\sim$ which assigns to each object S of $\mathcal{E}$ the object $f^*(L)$. Here f is the unique morphism $S \longrightarrow T_\bullet$.

(b) The composition of the functor $\mathcal{M}od(\mathfrak{F}) \longrightarrow \mathcal{F}^{op}_{T_\bullet}$, $M \longmapsto M(T_\bullet)$, with the equivalence $\mathcal{F}^{op}_{T_\bullet} \longrightarrow Qcoh(\mathfrak{F})$ constructed in (a) is a right adjoint to the inclusion functor $Qcoh(\mathfrak{F}) \longrightarrow \mathcal{M}od(\mathfrak{F})$. ∎

1.4. Base change and quasi-coherent modules.

1.4.1. Proposition. *Let $\mathfrak{F} = (\mathcal{F}, \mathcal{F} \xrightarrow{\pi} \mathcal{E})$ be a category over $\mathcal{E}$ and $\mathcal{E}' \longrightarrow \mathcal{E}$ a functor. Let $\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}'$ denote the induced category $(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{E}')$.*

Then $Qcoh(\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}')$ is isomorphic to the full subcategory of $Hom_{\mathcal{E}}(\mathcal{E}', \mathfrak{F})^{op}$ whose objects are those $\mathcal{E}$-functors which transform any morphism into a cartesian morphism.

If $\mathfrak{F}$ is a fibred category over $\mathcal{E}$ and $\mathcal{F}_c$ is the subcategory of $\mathcal{F}$ formed by all cartesian morphisms of $\mathcal{F}$, then $ObQcoh(\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}') \simeq ObHom_{\mathcal{E}}(\mathcal{E}', \mathcal{F}_c)$.

Proof. The assertion follows from A1.6.7.2. ∎

1.4.2. The 2-categories $Cart_{\mathfrak{U},\mathfrak{V}}$ and $\mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}}$. Let $\mathfrak{U}$, $\mathfrak{V}$ be two universums such that $\mathfrak{U} \in \mathfrak{V}$. Let $Cart_{\mathfrak{U},\mathfrak{V}}$ denote the full 2-subcategory of $Cart_{\mathfrak{V}}$ (see A1.5.3.1), whose objects are categories over categories $\mathfrak{F} = (\mathcal{F} \xrightarrow{\pi} \mathcal{E})$ such that the base $\mathcal{E}$ belongs to $\mathfrak{V}$ and each fiber belongs to $\mathfrak{U}$. Let $\mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}}$ denote the 2-subcategory of the 2-category $Cart_{\mathfrak{U},\mathfrak{V}}$ generated by all cartesian functors (– 1-morphisms of $Cart_{\mathfrak{U},\mathfrak{V}}$)

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{u} & \mathcal{F} \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathcal{E}' & \xrightarrow{v} & \mathcal{E} \end{array} \tag{1}$$

such that the functors induced on fibers are category equivalences.

1.4.3. Proposition. *The map $\mathfrak{F} \longmapsto Qcoh(\mathfrak{F})$ extends to a pseudo-functor*

$$\mathfrak{M}Cart^{op}_{\mathfrak{U},\mathfrak{V}} \xrightarrow{Qcoh} Cat_{\mathfrak{V}}.$$

Proof. Let (1) be an arbitrary cartesian morphism. It can be decomposed in two cartesian morphisms

$$\begin{array}{ccccc} \mathcal{F}' & \xrightarrow{u'} & \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' & \xrightarrow{v^\sim} & \mathcal{F} \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \pi}, \\ \mathcal{E}' & \xrightarrow{Id_{\mathcal{E}'}} & \mathcal{E}' & \xrightarrow{v} & \mathcal{E} \end{array} \tag{2}$$

where the right square is the canonical pull-back. By 1.4.1, the right square of (2) induces a functor

$$Qcoh(\mathfrak{F}) \longrightarrow Qcoh(\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}') \tag{3}$$

and by 1.1.2, the left square of (2) induces a functor

$$Qcoh(\mathfrak{F}') \longrightarrow Qcoh(\mathfrak{F} \times_{\mathcal{E}} \mathcal{E}'). \tag{4}$$

The morphism (1) belongs to $\mathfrak{M}Cart$ iff u' in (2) is a category equivalence, which implies that (4) is a category equivalence. Taking the composition of (3) with a quasi-inverse to (4), we assign to the morphism (1) a functor $Qcoh(\mathfrak{F}) \longrightarrow Qcoh(\mathfrak{F}')$. This correspondence defines a pseudo-functor $\mathfrak{M}Cart^{op} \longrightarrow Cat$. ■

1.5. Quasi-coherent modules on presheaves of sets. Let X be a presheaf of sets on the base $\mathcal{E}$. Then we have a functor $\mathcal{E}/X \longrightarrow \mathcal{E}$ and the category $\mathfrak{F}/X \stackrel{\mathrm{def}}{=} \mathfrak{F} \times_{\mathcal{E}} \mathcal{E}/X$ over $\mathcal{E}/X$ obtained via a base change (as usual, we identify $\mathcal{E}$ with a full subcategory of the category $\mathcal{E}^\wedge$ of presheaves of sets on $\mathcal{E}$ formed by representable presheaves). Notice that any morphism of the category $\mathcal{E}/X$ over $\mathcal{E}$ is cartesian. Therefore, by 1.4.1, the category $Qcoh(\mathfrak{F}/X)$ is equivalent to the category $Cart_{\mathcal{E}}(\mathcal{E}/X, \mathfrak{F})^{op}$ opposite to the category of cartesian functors $\mathcal{E}/X \longrightarrow \mathfrak{F}$.

1.5.1. The canonical extension of a fibred category. Following [Gir], we denote the category $Cart_{\mathcal{E}}(\mathcal{E}/X, \mathfrak{F}) = Qcoh(\mathfrak{F}/X)^{op}$ by $\mathfrak{F}^+(X)$. The correspondence $X \longmapsto \mathfrak{F}^+(X)$ extends to a pseudo-functor, hence defines a fibred category over $\mathcal{E}^\wedge$ which is called (in [Gir]) the *canonical extension* of $\mathfrak{F}$ onto $\mathcal{E}^\wedge$.

1.5.2. Proposition. *Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$ and X an object of the category $\mathcal{E}^\wedge$ of presheaves of sets on $\mathcal{E}$.*

(a) If the functor X is a representable by an object, x, of the category $\mathcal{E}$, then the category $Qcoh(\mathfrak{F}/X)$ is equivalent to the category $\mathfrak{F}_x^{op}$ opposite to the fiber $\mathfrak{F}_x$ over x.

(b) Suppose $X = \mathrm{colim}(X_i)$ for some diagram $I \longrightarrow \mathcal{E}^\wedge$, $i \longmapsto X_i$. Then the natural functor $Qcoh(\mathfrak{F}/X) \longrightarrow \lim Qcoh(\mathfrak{F}/X_i)$ is an isomorphism.

(c) $X \longmapsto Qcoh(\mathfrak{F}/X)$ is a sheaf of categories on $\mathcal{E}^\wedge$ for the canonical topology.

Proof. (a) This fact is a consequence of 1.3.

(b) The assertion follows from the isomorphism $Cart(\mathcal{E}/X, \mathfrak{F}) \xrightarrow{\sim} lim(Cart(\mathcal{E}/X_i, \mathfrak{F})$ proven in [Gir] 3.2.4.

The assertion (c) follows from (b). ■

2. The quasi-topology and topology of $\mathfrak{F} - i$-descent.

2.0. N-faithful functors. Recall that a functor is called *0-faithful* (resp. *1-faithful*, resp. *2-faithful*) if it is faithful (resp. fully faithful, resp. an equivalence ([Gir], 0.5.1.1).

2.1. Definition. Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$ and $\widehat{X}$ the presheaf represented by an object X of $\mathcal{E}$. A subpresheaf T of $\widehat{X}$ is called a subpresheaf of $\mathfrak{F}-i$*-descent*, $i = 0,\ 1,\ 2,$ if the corresponding functor $Qcoh(\mathfrak{F}/X) \longrightarrow Qcoh(\mathfrak{F}/T)$ (or, equivalently, the natural functor $\mathfrak{F}_X^{op} \longrightarrow Qcoh(\mathfrak{F}/T)$) is i-faithful.

2.1.1. The families of morphisms of $\mathfrak{F} - i$-descent. Let $\mathfrak{X} = \{X_i \to X|\ i \in I\}$ be a family of morphisms of the category $\mathcal{A}$ and $\Im_{\mathfrak{X}}$ the subpresheaf of the presheaf $\widehat{X}$ associated with the family $\mathfrak{X}$: for every $Y \in Ob\mathcal{A}$, the set $\Im_{\mathfrak{X}}(Y)$ consists of all morphisms $Y \longrightarrow X$, which factor through some morphism of the family $\mathfrak{X}$.

A family of morphisms $\mathfrak{X} = \{X_i \to X|\ i \in I\}$ is said to be of $\mathfrak{F} - i$-descent if the subpresheaf $\Im_{\mathfrak{X}}$ of $\widehat{X}$ associated with $\mathfrak{X}$ is of $\mathfrak{F} - i$-descent.

2.1.2. Note. The difinition 2.1 is equivalent to the usual definition of a sieve of $\mathfrak{F} - i$-descent (cf. [Gir], II.1.1.1 and II.1.1.1.1). Another terminology: $\mathfrak{F} - 1$-descent is called simply $\mathfrak{F}$*-descent* and $\mathfrak{F} - 2$-descent is called also *effective descent*.

2.2. The quasi-topology of $\mathfrak{F}-i$-descent. For any $X \in Ob\mathcal{E}$, we denote by $\mathfrak{T}_{\mathfrak{F},i}(X)$ the set of all subpresheaves of $\widehat{X}$ which are of $\mathfrak{F} - i$-descent. This defines a quasi-topology, $\mathfrak{T}_{\mathfrak{F},1}$, which we call the *quasi-topology of $\mathfrak{F} - i$-descent.*

2.2.1. Proposition. *The quasi-topology of $\mathfrak{F} - 1$-descent is the finest quasi-topology such that for any $X \in Ob\mathcal{E}$ and any $x, y \in Ob\mathfrak{F}_X$, the presheaf*

$$Hom_X(x, y) : \mathcal{E}/X \longrightarrow \mathbf{Sets}, \quad (Y \xrightarrow{f} X) \longmapsto Hom_Y(f^*(x), f^*(y)), \tag{1}$$

is a sheaf on $\mathcal{E}/X$ for the induced quasi-topology.

Proof. The presheaf $Hom_X(x, y)$ being a sheaf for all $x, y \in Ob\mathfrak{F}_X$ is equivalent to the full faithfulness of the functor $\mathfrak{F}_X^{op} \longrightarrow Qcoh(\mathfrak{F}/T)$ for any $T \in \mathfrak{T}(X)$. The assertion follows now from the definition of the quasi-topology of $\mathfrak{F} - 1$-descent (see also 2.1). ■

2.3. Definition. Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$, $\widehat{X}$ the presheaf represented by an object X of $\mathcal{E}$. A subpresheaf T in $\widehat{X}$ is called a subpresheaf of *universal $\mathfrak{F} - i$-descent*, $i = 0,\ 1,\ 2,$ if for any morphism $Y \longrightarrow X$ in $\mathcal{E}$, the subpresheaf $T \times_{\widehat{X}} \widehat{Y}$ of $\widehat{Y}$ is of $\mathfrak{F} - i$-descent.

2.4. The topology of $\mathfrak{F} - i$-descent. For any $X \in Ob\mathcal{E}$, denote by $\mathfrak{T}^u_{\mathfrak{F},i}(X)$ the set of all subpresheaves of $\widehat{X}$ which are of universal $\mathfrak{F} - i$-descent. This defines a topology which is called the *topology of $\mathfrak{F} - i$-descent.*

2.4.1. Proposition. *The topology of $\mathfrak{F} - 1$-descent is the finest topology such that for any $X \in Ob\mathcal{E}$ and any $x, y \in Ob\mathfrak{F}_X$, the presheaf*

$$Hom_X(x, y) : \mathcal{E}/X \longrightarrow \mathbf{Sets}, \quad (Y \xrightarrow{f} X) \longmapsto Hom_Y(f^*(x), f^*(y)), \tag{1}$$

is a sheaf on $\mathcal{E}/X$ for the induced topology.

Proof. The assertion follows from 2.2.1. ∎

2.5. Coinduced topologies and $\mathfrak{F} - i$-descent. Let $\mathfrak{T}$ be a topology on the category $\mathcal{A}$, and let $\mathfrak{T}^\wedge$ denote the *coinduced* topology on the category $\mathcal{A}^\wedge$ of presheaves. Recall that the topology $\mathfrak{T}^\wedge$ is defined as follows: for any presheaf of sets X, a subpresheaf $V \hookrightarrow \widehat{X} = Hom(-, X)$ is a *refinement of* X for $\mathfrak{T}^\wedge$ iff for any $S \in Ob\mathcal{A}$ and any morphism $S \longrightarrow X$, the subpresheaf $V \times_{\widehat{X}} \widehat{S} \hookrightarrow \widehat{S}$ is a refinement of S for $\mathfrak{T}$.

2.5.1. Example. If $\mathfrak{T}$ is the discrete topology on $\mathcal{E}$, then the coinduced topology on $\mathcal{E}^\wedge$ coincides with the canonical topology on $\mathcal{E}^\wedge$, which is, by definition, the finest topology for which all representable presheaves are sheaves.

2.5.2. Coverings and bicoverings. A morphism $X \xrightarrow{f} Y$ is called *covering* (resp. *bicovering*), if the induced morphism of associated sheaves, $X^a \xrightarrow{f^a} Y^a$, is an epimorphism (resp. an isomorphism).

2.5.3. Proposition. *Let $\mathfrak{F}$ be a fibred category over a category $\mathcal{E}$ and i an integer $0 \leq i \leq 2$. Let $\mathfrak{F}^+$ denote the canonical extension of $\mathfrak{F}$ onto $\mathcal{E}^\wedge$ (cf. 1.5.1).*

(a) The topology of $\mathfrak{F}^+ - i$-descent is the topology coinduced by the topology of $\mathfrak{F} - i$-descent.

(b) A morphism $X \xrightarrow{f} Y$ in $\mathcal{E}^\wedge$ is bicovering for the topology of $\mathfrak{F}^+ - i$-descent iff for any morphism $Y' \longrightarrow Y$, the corresponding functor

$$Qcoh(\mathfrak{F}/Y') \longrightarrow Qcoh(\mathfrak{F}/X \times_Y Y') \qquad (1)$$

is i-faithful. The converse is true when $i = 2$ (i.e. the functor (1) is a category equivalence), or when the presheaf morphism $X \xrightarrow{f} Y$ is a monomorphism.

Proof. The assertions (a) and (b) are equivalent to the assertions resp. (iii) and (iv) of II.11.3 in [Gir]. ∎

2.6. Proposition. *Let $\mathfrak{X} = \{X_i \longrightarrow X \mid i \in I\}$ be a family of arrows in $\mathcal{E}^\wedge$, and let $\mathfrak{X}_\Im \xrightarrow{\Im_\mathfrak{X}} X$ be the image of $\mathfrak{X}$. Then $\mathfrak{X}$ is of $\mathfrak{F} - i$-descent iff the corresponding inverse image functor $\Im_\mathfrak{X}^* = Qcoh(\Im_\mathfrak{X}) : Qcoh(\mathfrak{F}/X) \longrightarrow Qcoh(\mathfrak{F}/\mathfrak{X}_\Im)$ is i-faithful.*

Proof. The assertion is equivalent to the assertion II.1.1.3.1 in [Gir] (which is a part of the argument of II.1.1.3). ∎

2.7. Canonical topology on presheaves of sets and the effective descent. If $\mathfrak{X} = \{X_i \to X \mid i \in I\}$ is a cover for the canonical topology on $\mathcal{E}^\wedge$, then the image $\mathfrak{X}_\Im$ of $\mathfrak{X}$ coincides with X. By 2.6, the family $\mathfrak{X}$ is a cover for the effective $\mathfrak{F}$-descent (or

$\mathfrak{F}$ − 2-descent) topology. In particular, the topology of the effective $\mathfrak{F}^+$-descent is finer than the canonical topology (hence any subcanonical topology) on $\mathcal{E}^\wedge$.

2.7.1. Remark. One can deduce the latter fact directly from the part (a) of 2.5.3 as follows. Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$. If the topology of $\mathfrak{F} - i$-descent is finer than a topology $\mathfrak{T}$ than, evidently, the coinduced topology $\mathfrak{T}^\wedge$ on $\mathcal{E}^\wedge$ is coarser than the topology of $\mathfrak{F}^+ - i$-descent, because by 2.5.3 the topology of $\mathfrak{F}^+ - i$-descent is coinduced by the topology of $\mathfrak{F} - i$-descent). In particular, the topology $\mathfrak{T}_d^\wedge$ coinduced by the discrete topology is coarser than the topology of the effective $\mathfrak{F}^+$-descent. But, as it has been already observed (in 2.5.1), $\mathfrak{T}_d^\wedge$ coincides with the canonical topology on $\mathcal{E}^\wedge$.

3. Sheaves of modules.

3.1. Sheaves of modules on a cofibred category. Let $\mathfrak{F}$ be a cofibred category over a category $\mathcal{E}$ corresponding to a pseudo-functor $\mathcal{E} \longrightarrow Cat$,

$$Ob\mathcal{E} \ni X \longmapsto \mathcal{F}_X,\ Hom\mathcal{E} \ni f \longmapsto f_\bullet,\ Hom\mathcal{E} \times_{Ob\mathcal{E}} Hom\mathcal{E} \ni (f,g) \longmapsto ((fg)_\bullet \xrightarrow{c_{f,g}} f_\bullet g_\bullet) \tag{1}$$

(cf. A1.7, A1.7.1). Then the category $\mathcal{M}od(\mathfrak{F})$ can be described as follows. An object of $\mathcal{M}od(\mathfrak{F})$ is a function which assigns to each $T \in Ob\mathcal{E}$ an object $M(T)$ of the fiber $\mathcal{F}_T$ and to each morphism $T \xrightarrow{f} T'$ a morphism $M(T') \xrightarrow{\xi_f} f_\bullet(M(T))$ such that $c_{f,g} \circ \xi_{gf} = f_\bullet(\xi_g) \circ \xi_f$. Morphisms are defined in a natural way.

Let M be an object of $\mathcal{M}od(\mathfrak{F})$. For any object X of the category $\mathcal{E}$ and any subpresheaf $\Re \hookrightarrow \widehat{X}$, we have a cone

$$\{M(X) \xrightarrow{\xi_f} f_\bullet(M(Y))|\ Y \xrightarrow{f} X \text{ factors through } \Re \hookrightarrow \widehat{X}\}. \tag{2}$$

Denote by $\mathfrak{T}_M(X)$ the set of all subpresheaves $\Re \hookrightarrow \widehat{X}$ such that the cone (2) is terminal. The correspondence $\mathfrak{T}_M : X \longmapsto \mathfrak{T}_M(X)$ is a quasi-topology on $\mathcal{E}$.

Let $\mathfrak{T}$ be a quasi-topology on $\mathcal{E}$. We say that $M \in Ob\mathcal{M}od(\mathfrak{F})$ is a sheaf, or a *sheaf of modules* on $(\mathfrak{F}, \mathfrak{T})$, if the quasi-topology $\mathfrak{T}_M$ is finer than $\mathfrak{T}$; i.e. for any $X \in Ob\mathcal{E}$ and any $\Re \in \mathfrak{T}(X)$, the cone (2) is terminal. We denote by $\mathcal{M}od(\mathfrak{F}, \mathfrak{T})$ the full subcategory of $\mathcal{M}od(\mathfrak{F})$ formed by sheaves of modules.

3.2. Sheaves of modules on a fibred category. Notice that the cone 3.1(2) above is terminal iff for any $z \in Ob\mathfrak{F}_X$, the cone

$$\left\{Hom_{\mathfrak{F}_X}(z, M(X)) \xrightarrow{\mathfrak{F}_X(z,\xi_f)} Hom_{\mathfrak{F}_X}(z, f_\bullet(M(Y)))|\ Y \xrightarrow{f} X \text{ factors through } \Re \hookrightarrow \widehat{X}\right\} \tag{3}$$

is terminal. But, the cone (3) is naturally isomorphic to the cone

$$\left\{Hom_{\mathfrak{F}_X}(z, M(X)) \longrightarrow Hom_{\mathfrak{F}_Y}(f^*(z), M(Y))|\ Y \xrightarrow{f} X \text{ factors through } \Re \hookrightarrow \widehat{X}\right\} \tag{4}$$

Here the map $Hom_{\mathfrak{F}_X}(z, M(X)) \longrightarrow Hom_{\mathfrak{F}_X}(f^*(z), M(Y))$ sends a $z \xrightarrow{\alpha} M(X)$ to the composition of $f^*(z) \xrightarrow{f^*(\alpha)} f^*(M(X))$ and the morphism $f^*(M(X)) \longrightarrow M(Y)$.

This observation gives rise to the following

3.2.1. Definition. Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$, and let $\mathfrak{T}$ be a quasi-topology on $\mathcal{E}$. We call a presheaf of modules M on $\mathfrak{F}$ a *sheaf* if for any $X \in Ob\mathcal{E}$ and any $\Re \in \mathfrak{T}(X)$, the cone 3.2(4) is terminal for all $z \in Ob\mathfrak{F}_X$.

If $\mathfrak{F}$ is a bifibred category, then this definition is equivalent to that of 3.1. We denote by $\mathcal{M}od(\mathfrak{F}, \mathfrak{T})$ the category of sheaves of modules on the fibred quasi-site $(\mathfrak{F}, \mathfrak{T})$.

3.2.2. Quasi-coherent sheaves. We denote by $Qcoh(\mathfrak{F}, \mathfrak{T})$ the intersection of the category $Qcoh(\mathfrak{F})$ of quasi-coherent modules on $\mathfrak{F}$ with the category $\mathcal{M}od(\mathfrak{F}, \mathfrak{T})$ of sheaves of modules on $(\mathfrak{F}, \mathfrak{T})$ and call the objects of this category *quasi-coherent sheaves on* $(\mathfrak{F}, \mathfrak{T})$.

3.3. Proposition. *Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$, and let $\mathfrak{T}$ be a quasi-topology on $\mathcal{E}$. The following conditions are equivalent:*

(a) The quasi-topology of $\mathfrak{F} - 1$-descent is finer than $\mathfrak{T}$.

(b) For any $X \in Ob\mathcal{E}$, the category $Qcoh(\mathfrak{F}/X)$ is a subcategory of the category $\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X)$ of sheaves of modules on $\mathcal{E}/X$ (with the induced quasi-topology $\mathfrak{T}/X$).

Proof. By definition, a subpresheaf $\Re$ of $\widehat{X}$ is of $\mathfrak{F}$-descent if the inverse image functor $Qcoh(\mathfrak{F}/\widehat{X}) \longrightarrow Qcoh(\mathfrak{F}/\Re)$ of the embedding $\Re \hookrightarrow \widehat{X}$ is a fully faithful functor. By 1.5.2(a), the category $Qcoh(\mathfrak{F}/\widehat{X})$ is equivalent to $\mathfrak{F}_X^{op}$. Let M be a quasi-coherent module on $\mathfrak{F}$. And let z be any object of $\mathfrak{F}_X$. Since M is quasi-coherent, for any morphism $f : Y \longrightarrow X$ in $\mathcal{E}$, the object $M(Y)$ is isomorphic to $f^*(M(X)$. Thus the cone 3.2(4) is isomorphic to the cone

$$\{Hom_{\mathfrak{F}_X}(z, M(X)) \longrightarrow Hom_{\mathfrak{F}_X}(f^*(z), f^*(M(X)))|\ Y \xrightarrow{f} X \text{ factors through } \Re \hookrightarrow \widehat{X}\}. \tag{5}$$

The cone (5) is terminal, because if $\mathfrak{T}$ is of $\mathfrak{F}$-descent, then, for any $X \in Ob\mathcal{E}$ and any $z, x \in Ob\mathfrak{F}_X$, the presheaf of sets

$$Hom_{\mathfrak{F}_X}(z, y) : (Y, Y \xrightarrow{f} X) \longmapsto Hom_{\mathfrak{F}_Y}(f^*(z), f^*(x))$$

is a sheaf on $\mathcal{E}/X$ for the induced quasi-topology (see 2.2.1).

This implies also the assertion (b). ■

3.3.1. The quasi-topology (resp. topology) of $\mathfrak{F} - 1$-descent is the finest among quasi-topologies (resp. topologies) $\mathfrak{T}$ on $\mathcal{E}$ such that, for any $X \in Ob\mathcal{E}$, quasi-coherent modules on $\mathcal{E}/X$ are sheaves on $(\mathcal{E}/X, \mathfrak{T}/X)$. In particular, if $\mathfrak{T}$ is coarser than the quasi-topology of $\mathfrak{F} - 1$-descent, then all quasi-coherent modules on $\mathfrak{F}$ are sheaves on the quasi-site $(\mathcal{E}, \mathfrak{T})$.

3.4. Fibred category of sheaves of modules over presheaves of sets. Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$ and $\mathfrak{T}$ a quasi-topology on $\mathcal{E}$. To any $X \in Ob\mathcal{E}^\wedge$, we assign the category $\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X)$ of sheaves of modules on $\mathcal{E}/X$ (with the induced quasi-topology $\mathfrak{T}/X$). This correspondence extends to a functor $(\mathcal{E}^\wedge)^{op} \longrightarrow Cat$, hence defines a fibred category, $\mathfrak{Mod}^+(\mathfrak{F}, \mathfrak{T})$.

3.4.1. Note. Let $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})$ be the restriction of the fibred category $\mathfrak{Mod}^+(\mathfrak{F}, \mathfrak{T})$ to $\mathcal{E}$. Recall that a canonical extension, $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})^+$, of the fibred category $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})$ onto $\mathcal{E}^\wedge$ is defined by $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})^+_X = Qcoh(\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})/X)$. It follows from definitions that this extension coincides with the fibred category $\mathfrak{Mod}^+(\mathfrak{F}, \mathfrak{T})$.

3.4.2. Lemma. *The quasi-topology $\mathfrak{T}$ is of effective $\mathfrak{Mod}(\mathfrak{F}, \mathfrak{T})$-descent.*

Proof. The argument is left to the reader. ∎

3.5. Proposition. *Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$ and $\mathfrak{T}$ a quasi-topology on $\mathcal{E}$ which is coarser than the $\mathfrak{F} - 1$ descent quasi-topology. Let*

$$\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{U} \xrightarrow{f} X \qquad (1)$$

be a diagram in $\mathcal{E}^\wedge$ such that $\mathfrak{R}$ and $\mathfrak{U}$ are representable, $f \circ p_1 = f \circ p_2$, and the morphism $Cok(p_1, p_2) \longrightarrow X$ corresponding to f is bicovering. Then the inclusion functor

$$Qcoh(\mathfrak{F}/X) \xrightarrow{\mathfrak{q}_X^*} \mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X)$$

has a right adjoint.

Proof. The condition that the canonical morphism $Cok(p_1, p_2) \longrightarrow X$ is bicovering means that the inverse image functor $Qcoh(\mathfrak{F}/X) \longrightarrow Qcoh(\mathfrak{F}/Cok(p_1, p_2))$ is a category equivalence. Thus, we can (and will) assume that the diagram of presheaves of sets (1) is exact. Consider the quasi-commutative diagram

$$\begin{array}{ccccc}
Qcoh(\mathfrak{F}/X) & \xrightarrow{f^*} & Qcoh(\mathfrak{F}/\mathfrak{U}) & \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} & Qcoh(\mathfrak{F}/\mathfrak{R}) \\
\mathfrak{q}_X^* \downarrow & & \downarrow \mathfrak{q}_{\mathfrak{U}}^* & & \downarrow \mathfrak{q}_{\mathfrak{R}}^* \\
\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X) & \xrightarrow{f^\bullet} & \mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U}) & \underset{p_2^\bullet}{\overset{p_1^\bullet}{\rightrightarrows}} & \mathcal{M}od(\mathfrak{F}/\mathfrak{R}, \mathfrak{T}/\mathfrak{R})
\end{array} \qquad (2)$$

corresponding to the diagram (1). Since the functors $\mathfrak{U}$ and $\mathfrak{R}$ are representable, the functors $\mathfrak{q}_{\mathfrak{U}}^*$ and $\mathfrak{q}_{\mathfrak{R}}^*$ have right adjoints, resp. $\mathfrak{q}_{\mathfrak{U}*}$ and $\mathfrak{q}_{\mathfrak{R}*}$. Recall that the functor $\mathfrak{q}_{\mathfrak{U}*}$ (resp. $\mathfrak{q}_{\mathfrak{R}*}$ assigns to every sheaf M its value at $\mathfrak{U}$ (resp. at $\mathfrak{R}$) (cf. 1.3). This implies that

$$\mathfrak{q}_{\mathfrak{R}*} \circ p_i^\bullet = p_i^* \circ \mathfrak{q}_{\mathfrak{U}*}. \tag{3}$$

By 1.5.2, the subdiagram

$$Qcoh(\mathfrak{F}/X) \xrightarrow{f^*} Qcoh(\mathfrak{F}/\mathfrak{U}) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} Qcoh(\mathfrak{F}/\mathfrak{R})$$

of (2) is exact. This means that the category $Qcoh(\mathfrak{F}/X)$ can be identified with a category whose objects are pairs (M, ϕ), where $M \in Ob\mathfrak{F}_{\mathfrak{U}}$ and ϕ an isomorphism $p_1^*(M) \xrightarrow{\sim} p_2^*(M)$. Morphisms from (M, ϕ) to (M', ϕ') are given by arrows $M \xrightarrow{g} M'$ such that $p_2^*(g) \circ \phi = \phi' \circ p_1^*(g)$. The functor f^* maps every object (M, ϕ) to the object M and every morphism $(M, \phi) \xrightarrow{g} (M', \phi')$ to $M \xrightarrow{g} M'$.

Similarly, it follows from 3.4.2 and 1.5.2 that the subdiagram

$$\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X) \xrightarrow{f^\bullet} \mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U}) \overset{p_1^\bullet}{\underset{p_2^\bullet}{\rightrightarrows}} \mathcal{M}od(\mathfrak{F}/\mathfrak{R}, \mathfrak{T}/\mathfrak{R})$$

is exact, hence the category $\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X)$ of sheaves on X admits an analogous description: its objects are pairs (L, ψ), where $L \in Ob\mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U})$ and ψ an isomorphism $p_1^\bullet(L) \xrightarrow{\sim} p_2^\bullet(L)$, etc.. The functor $\mathfrak{q}_X^*$ maps an object (M, ϕ) of the category $Qcoh(\mathfrak{F}/X)$ to the object $(\mathfrak{q}_{\mathfrak{U}}^*(M), \mathfrak{q}_{\mathfrak{R}}^*(\phi))$. A right adjoint to $\mathfrak{q}_X^*$ is induced by a right adjoint $\mathfrak{q}_{\mathfrak{U}*}$ to the inclusion functor $\mathfrak{q}_{\mathfrak{U}}^*$.

In fact, let $(L, p_1^\bullet(L) \overset{\psi}{\to} p_2^\bullet(L))$ be an object of $\mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U})$. Thanks to (3), we have isomorphisms:

$$p_1^*\mathfrak{q}_{\mathfrak{U}*}(L) \xrightarrow{\sim} \mathfrak{q}_{\mathfrak{R}*}p_1^\bullet(L) \xrightarrow{\mathfrak{q}_{\mathfrak{R}*}(\psi)} \mathfrak{q}_{\mathfrak{R}*}p_2^\bullet(L) \xrightarrow{\sim} p_2^*\mathfrak{q}_{\mathfrak{U}*}(L)$$

the composition of which, ψ', defines an object $(\mathfrak{q}_{\mathfrak{U}*}(L), \psi')$ of the category $Qcoh(\mathfrak{F}/X)$. The map $(L, \psi) \longmapsto (\mathfrak{q}_{\mathfrak{U}*}(L), \psi')$ extends to functor

$$\mathcal{M}od(\mathfrak{F}/X, \mathfrak{T}/X) \xrightarrow{\mathfrak{q}_{X*}} Qcoh(\mathfrak{F}/X).$$

It is left to the reader to check that the functor $\mathfrak{q}_{X*}$ is a right adjoint to $\mathfrak{q}_X^*$. ∎

3.5.1. Remarks. (i) If the quasi-topology in 3.5 is of effective descent, it suffices to require that the canonical morphism $Cok(p_1, p_2) \xrightarrow{\hat{f}} X$ is a cover.

(ii) The argument of 3.5 is valid in a more general setting. Namely, one can replace representability of $\mathfrak{U}$ and $\mathfrak{R}$ by the existence of right adjoints to the inclusion functors

$$Qcoh(\mathfrak{F}/\mathfrak{U}) \xrightarrow{\mathfrak{q}^*_{\mathfrak{U}}} \mathcal{M}od(\mathfrak{F}/\mathfrak{U}, \mathfrak{T}/\mathfrak{U}) \quad \text{and} \quad Qcoh(\mathfrak{F}/\mathfrak{R}) \xrightarrow{\mathfrak{q}^*_{\mathfrak{R}}} \mathcal{M}od(\mathfrak{F}/\mathfrak{R}, \mathfrak{T}/\mathfrak{R})$$

which satisfy the condition (3). Notice that in this case a right adjoint, $\mathfrak{q}_{X*}$, to $\mathfrak{q}^*_X$ satisfies this condition too: $f^* \circ \mathfrak{q}_{X*} \simeq \mathfrak{q}_{\mathfrak{U}*} \circ f^\bullet$ (cf. the argument of 3.5).

3.5.2. Corollary. *Let $\mathfrak{F}$ be a fibred category over $\mathcal{E}$, and let*

$$\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{U} \xrightarrow{f} X \qquad (1)$$

be an exact diagram in $\mathcal{E}^\wedge$ such that $\mathfrak{R}$ and $\mathfrak{U}$ are representable. Then the inclusion functor $\mathfrak{q}^*_X : Qcoh(\mathfrak{F}/X) \hookrightarrow \mathcal{M}od(\mathfrak{F}/X)$ has a right adjoint.

Proof. Let C be a category with the discrete topology, $\mathfrak{T}_d$. The corresponding coinduced topology $\mathfrak{T}^\wedge_d$ on the category $C^\wedge$ of presheaves of sets on C can be described in terms of covers as follows. A set of morphisms $\{U_i \longrightarrow X \mid i \in J\}$ is a cover iff the corresponding presheaf morphism $\coprod_{i \in J} U_i \longrightarrow X$ is surjective; in particular, any surjective presheaf morphism $U \longrightarrow X$ is a cover in the topology $\mathfrak{T}^\wedge_d$. This shows that the topology $\mathfrak{T}^\wedge_d$ is the canonical topology on $C^\wedge$.

Take $C = \mathcal{E}$ endowed with the discrete topology. Notice that, for any $S \in Ob\mathcal{E}$, every quasi-coherent module on S is a sheaf; i.e. the condition of 3.5 holds for the discrete topology on $\mathcal{E}$. Moreover, by 2.7, the canonical topology on $\mathcal{E}^\wedge$ is of effective descent. Thus, the assertion follows from 3.5 and 3.5.1(i). ∎

4. Modules and quasi-coherent modules over a ringed category.

4.0. Ringed categories. By a *ringed* category, we understand a pair $(\mathcal{A}, \mathcal{O})$, where $\mathcal{A}$ is a category and $\mathcal{O}$ is a presheaf of k-algebras on $\mathcal{A}$. For any arrow $T \xrightarrow{f} T'$, let

$$\mathcal{O}(T) - mod \xrightarrow{f_*} \mathcal{O}(T') - mod$$

denote the restriction of scalars functor corresponding to the k-algebra morphism

$$\mathcal{O}(T') \xrightarrow{\mathcal{O}(f)} \mathcal{O}(T).$$

The map, which assigs to every object T of the category $\mathcal{A}$ the category $\mathcal{O}(T) - mod^{op}$ opposite to the category of left $\mathcal{O}(T)$-modules and to each morphism $T \xrightarrow{f} T'$ the functor

$$(\mathcal{O}(T) - mod)^{op} \xrightarrow{f_*^{op}} (\mathcal{O}(T') - mod)^{op}$$

is a functor $\mathcal{A} \longrightarrow Cat$. This functor defines a cofibred category, $\mathfrak{M}(\mathcal{A}, \mathcal{O})$ over $\mathcal{A}$ with the fiber $(\mathcal{O}(T) - mod)^{op}$ at $T \in Ob\mathcal{A}$. The category $\mathfrak{M}(\mathcal{A}, \mathcal{O})$ is fibred (hence bifibred), because for any morphism $T \xrightarrow{f} T'$, the functor f_* has a left adjoint,

$$\mathcal{O}(T') - mod \xrightarrow{f^*} \mathcal{O}(T) - mod, \quad M \longmapsto \mathcal{O}(T) \otimes_{\mathcal{O}(T')} M,$$

or, equivalently, f_*^{op} has a right adjoint, $(f^*)^{op}$.

4.1. Modules over noncommutative affine schemes. Our standard example of a ringed category is the category $\mathbf{Aff}_k = Alg_k^{op}$ of affine k-schemes endowed with the presheaf $\mathcal{O}$ which assigns to $R^\vee$ the k-algebra R. The corresponding bifibred category $\mathfrak{M}(\mathcal{A}, \mathcal{O})$ will be called the bifibred *category of modules over noncommutative affine k-schemes.*

4.2. Presheaves of modules. Let $\mathcal{O} - mod$ denote the category $\mathcal{M}od(\mathfrak{M}(\mathcal{A}, \mathcal{O}))$ of modules on the fibred category $\mathfrak{M}(\mathcal{A}, \mathcal{O})$. An object of $\mathcal{O} - mod$ is a function which assigns to each $T \in Ob\mathcal{A}$ an $\mathcal{O}$-module $M(T)$ and to each morphism $T \xrightarrow{f} T'$ an $\mathcal{O}(T')$-module morphism $M(T') \xrightarrow{\gamma_f} f_*(M(T))$ such that $\gamma_{gf} = g_*(\gamma_f) \circ \gamma_g$. Objects of the category $\mathcal{O} - mod$ are called *presheaves of $\mathcal{O}$-modules* on $\mathcal{A}$.

4.3. Quasi-coherent modules. We define the category $Qcoh(\mathcal{A}, \mathcal{O})$ of *quasi-coherent modules on* $(\mathcal{A}, \mathcal{O})$ as the category $Qcoh(\mathfrak{M}(\mathcal{A}, \mathcal{O}))$ of quasi-coherent modules on the fibred category $\mathfrak{M}(\mathcal{O})$. It follows from definitions that an object of the category $Qcoh(\mathcal{A}, \mathcal{O})$ is a presheaf M of $\mathcal{O}$-modules such that for any morphism $T \xrightarrow{f} T'$, the dual to γ_f morphism

$$f^*(M(T')) = \mathcal{O}(T) \otimes_{\mathcal{O}(T')} M(T', \xi') \xrightarrow{\gamma_f^\vee} M(T, \xi)$$

is an isomorphism.

4.4. Proposition. *Suppose the category $\mathcal{A}$ has a final object, $T_\bullet$. Then the category $Qcoh(\mathcal{A}, \mathcal{O})$ is equivalent to the category $\mathcal{O}(T_\bullet) - mod$ of left $\mathcal{O}(T_\bullet)$-modules.*

Proof. The assertion is a special case of 1.3. ∎

4.5. Example. Let $(\mathcal{A},\mathcal{O})$ be as in 4.1; i.e. $\mathcal{A} = \mathbf{Aff}_k$ and $\mathcal{O}(R^\vee) = R$ for any k-algebra R. Notice that $k^\vee$ is a final object of the category $\mathbf{Aff}_k$. It follows from 4.4 (or 1.3) that the category $Qcoh(\mathcal{A},\mathcal{O})$ is equivalent to the category $k-mod$ of left k-modules.

4.6. Modules on presheaves. Fix a ringed category $(\mathcal{A},\mathcal{O})$. Consider the category $\mathcal{A}^\wedge$ of presheaves of sets on $\mathcal{A}$. For any $X \in Ob\mathcal{A}^\wedge$, we have the category $\mathcal{A}/X$ of objects over X and the canonical functor $\mathcal{A}/X \xrightarrow{p_X} \mathcal{A}$. The presheaf of k-algebras $\mathcal{O}$ induces a presheaf of k-algebras $\mathcal{O}_X$ on $\mathcal{A}/X$, so that p_X becomes a morphism of ringed categories. The bifibred category $\mathfrak{M}(\mathcal{A}/X,\mathcal{O}_X)$ is naturally isomorphic to the category obtained from $\mathfrak{M}(\mathcal{A},\mathcal{O})$ via the base change along the functor p_X.

Thus, we have the category $\mathcal{O}_X - mod$ of presheaves of $\mathcal{O}_X$-modules, which we denote by $\mathcal{M}od_X$, and its full subcategory of quasi-coherent $\mathcal{O}_X$-modules, which we denote by $Qcoh_X$ (instead of $Qcoh(\mathcal{A}/X,\mathcal{O}_X)$). If the presheaf X is representable by an object T_X of the category $\mathcal{A}$, then the category $\mathcal{A}/X$ has a final object; so that, by 4.4, the category $Qcoh_X$ is equivalent to the category $\mathcal{O}(T_X) - mod$ of left $\mathcal{O}(T_X)$-modules.

4.6.1. Fibred category of modules over presheaves. Let X and Y be presheaves of sets on $\mathcal{A}$. To any presheaf morphism $X \xrightarrow{f} Y$, there corresponds a functor

$$\mathcal{A}/X \xrightarrow{f^a} \mathcal{A}/Y, \quad (T,\xi) \longmapsto (T, f\circ\xi), \tag{1}$$

which lifts to a fibred category morphism

$$\mathfrak{M}(\mathcal{O}_X) \longrightarrow \mathfrak{M}(\mathcal{O}_Y),\ (M,(T,\xi)) \longmapsto (M,(T,f\circ\xi)).$$

The latter induces the 'pull-back' functor

$$\mathcal{M}od_Y \xrightarrow{f^\bullet} \mathcal{M}od_X,$$

which maps each module M on Y to the module $f^\bullet(M)$ on X defined by $f^\bullet(M)(R.\xi) = M(R, f\circ\xi)$. The map assigning to any presheaf morphism $X \xrightarrow{f} Y$ the functor

$$\mathcal{M}od_Y^{op} \xrightarrow{f^{\bullet op}} \mathcal{M}od_X^{op}$$

extends to a pseudo-functor, hence defines a *fibred category* $\mathfrak{Mod}(\mathcal{A},\mathcal{O})$ *of modules over presheaves.* This fibred category is, actually, bifibred, because, for any presheaf morphism f, the functor $f^\bullet$ has a right adjoint, $f_\bullet$.

4.6.2. The fibred category of quasi-coherent modules. For any presheaf morphism $X \xrightarrow{f} Y$, the corresponding functor

$$\mathfrak{M}(\mathcal{A}/Y, \mathcal{O}_Y) \longrightarrow \mathfrak{M}(\mathcal{A}/X, \mathcal{O}_X)$$

(defined in 4.6.1) is cartesian, hence the functor $\mathcal{M}od_Y \xrightarrow{f^\bullet} \mathcal{M}od_X$ maps quasi-coherent modules to quasi-coherent modules, i.e. it induces a right exact functor

$$Qcoh_Y \xrightarrow{f^*} Qcoh_X$$

which we call *inverse image functor* of f. The map

$$X \longmapsto Qcoh_X^{op}, \ f \longmapsto (f^*)^{op}$$

extends to a pseudo-functor from the category $\mathcal{A}^\wedge$ of presheaves of sets on $\mathcal{A}$ to Cat which defines the fibred category $\mathfrak{Qcoh}(\mathcal{A}, \mathcal{O})$ of quasi-coherent modules on $\mathcal{A}^\wedge$.

4.6.3. Note. Suppose X, Y are objects of $\mathcal{A}$. Then a morphism $X \xrightarrow{f} Y$ determines a ring morphism $\mathcal{O}(Y) \xrightarrow{\mathcal{O}(f)} \mathcal{O}(X)$, $Qcoh_X \simeq \mathcal{O}(X) - mod$, $Qcoh_Y \simeq \mathcal{O}(Y) - mod$ (cf. 4.4), and the functor f^* is equivalent to

$$\mathcal{O}(X) \otimes_{\mathcal{O}(Y)} : \mathcal{O}(Y) - mod \longrightarrow \mathcal{O}(X) - mod.$$

In general, one might interpret f^* as the functor $M \longmapsto \mathcal{O}_X \otimes_{\mathcal{O}_Y} M$.

4.7. Quasi-topology and topology of descent.

4.7.1. Lemma. *Let $(\mathcal{A}, \mathcal{O})$ be a ringed category, and let $\mathfrak{T}$ be a quasi-topology on $\mathcal{A}$.*

(a) A presheaf M of $\mathcal{O}$-modules is a sheaf on $(\mathfrak{M}(\mathcal{O}), \mathfrak{T})$ (cf. Section 3) iff M is a sheaf of abelian groups.

(b) The quasi-topology $\mathfrak{T}$ is coarser than the quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent iff, for any $X \in Ob\mathcal{A}$ and any $\mathcal{O}(X)$-module L, the presheaf on $\mathcal{A}/X$, which assigns to any object $(S, S \xrightarrow{f} X)$ of $\mathcal{A}/X$ the $\mathcal{O}(X)$-module $f_*f^*(L) = f_*(\mathcal{O}(S) \otimes_{\mathcal{O}(X)} L)$, is a sheaf of $\mathcal{O}(X)$-modules.

In particular, if $\mathfrak{T}$ is coarser than the quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent, than $\mathcal{O}$ is a sheaf of rings on $(\mathcal{A}, \mathfrak{T})$.

Proof. (a) The assertion follows from definitions.

(b) By 3.3, the (quasi-)topology of $\mathfrak{M}(\mathcal{O})-1$-descent is the finest (quasi-)topology such that, for any $X \in Ob\mathcal{A}$, quasi-coherent modules on $\mathfrak{M}(\mathcal{O}_X)$ are sheaves on $(\mathcal{A}, \mathfrak{T})$. Quasi-coherent modules on $\mathfrak{M}(\mathcal{O}_X)$ map each object $(S, S \xrightarrow{f} X)$ of $\mathcal{A}/X$ to the $\mathcal{O}(X)$-module $f_*f^*(L) = f_*(\mathcal{O}(S) \otimes_{\mathcal{O}(X)} L)$ for some $\mathcal{O}(X)$-module L (see 4.4), hence the assertion. ∎

4.7.2. Canonical topology and the descent topology on the category of commutative affine schemes. Let $\mathcal{A}$ be the category $\mathbf{CAff}_k$ of commutative affine schemes over k and $\mathcal{O}$ the presheaf of k-algebras on $\mathcal{A}$, which assigns to every affine scheme $(X, \mathcal{O}_X)$ the algebra $\Gamma\mathcal{O}_X$ of global sections of the structure sheaf $\mathcal{O}_X$.

4.7.2.1. Lemma. *The quasi-topology of* $\mathfrak{M}(\mathcal{O}) - 1$*-descent is a topology.*

Proof. Let

$$\{R_i^\vee \xrightarrow{\phi_i^\vee} R^\vee \mid i \in I\} \tag{1}$$

be a family of scheme morphisms. It follows from 4.7.1(b) (and from the isomorphism $(R_i \otimes_R R_j)^\vee \simeq X_i \times_X X_j$, where $X_i = R_i^\vee$ and $X = R^\vee$) that (1) is a cover for the quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent iff for any R-module M the diagram

$$M \longrightarrow \prod_{i\in I} R_i \otimes_R M \rightrightarrows \prod_{i,j\in I} R_i \otimes_R R_j \otimes_R M \tag{2}$$

is exact. In particular, for any morphism $S^\vee \longrightarrow R^\vee$, the diagram

$$S \otimes_R M \longrightarrow \prod_{i\in I} R_i \otimes_R (S \otimes_R M) \rightrightarrows \prod_{i,j\in I} R_i \otimes_R R_j \otimes_R (S \otimes_R M) \tag{3}$$

is exact. The latter means that the family of morphisms $\{S^\vee \times_X X_i \longrightarrow S^\vee \mid i \in I\}$ is a cover for the quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent, hence the quasi-topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent is a topology. ∎

4.7.2.2. Proposition. *The canonical topology on* $\mathbf{CAff}_k$ *coincides with the topology of* $\mathfrak{M}(\mathcal{O}) - 1$*-descent.*

Proof. The family (1) is a cover for the canonical topology iff for any morphism $S^\vee \longrightarrow R^\vee$ the diagram of R-modules

$$S \longrightarrow \prod_{i\in I} R_i \otimes_R S \rightrightarrows \prod_{i,j\in I} R_i \otimes_R R_j \otimes_R S \tag{4}$$

is exact. If (1) is a cover in the topology of $\mathfrak{M}(\mathcal{O})-1$-descent, then the diagram (4) is exact. This shows that the canonical topology on $\mathcal{A}$ is finer than the topology of $\mathfrak{M}(\mathcal{O})-1$-descent.

On the other hand, to any R-module M there corresponds an augmented R-algebra $S_M = R \oplus M$ with zero multiplication on M. One can easily check that the exactness of the diagram (4) for $S = S_M$ is equivalent to the exactness of the diagram (2). ∎

4.7.3. Canonical topology and the descent topology on the category of noncommutative affine schemes. Proposition 4.7.2.2 extends to the noncommutative case. Namely, there is the following

4.7.3.1. Proposition. *Let* $(\mathcal{A}, \mathcal{O})$ *be the ringed category of (noncommutative) affine k-schemes; i.e.* $\mathcal{A}$ *is the category* $\mathbf{Aff}_k = Alg_k^{op}$ *of affine k-schemes and the presheaf* $\mathcal{O}$ *is defined by* $\mathcal{O}(R^\vee) = R$ *for any associative k-algebra R (cf. 4.1). Then the topology of* $\mathfrak{M}(\mathcal{O}) - 1$*-descent coincides with the canonical topology on* $\mathcal{A} = \mathbf{Aff}_k$*.*

Proof. A family

$$\{R_i^\vee \xrightarrow{\phi_i} R^\vee \mid i \in I\} \tag{1}$$

of morphisms of $\mathbf{Aff}_k$ is a cover for the descent quasi-topology iff for any R-module M the diagram

$$M \longrightarrow \prod_{i\in I} R_i \otimes_R M \rightrightarrows \prod_{i,j\in I} R_i \star_R R_j \otimes_R M \tag{2}$$

is exact. On the other hand, a family (1) is a cover for the canonical topology iff for any morphism $S^\vee \longrightarrow R^\vee$, the diagram

$$S \longrightarrow \prod_{i\in I} R_i \star_R S \rightrightarrows \prod_{i,j\in I} R_i \star_R R_j \star_R S \tag{3}$$

is exact.

(a) Suppose M is an R-bimodule. Let S_M be an algebra which is isomorphic to $R\oplus M$ as R-bimodule with zero multiplication on M. Then we have a commutative diagram

$$\begin{array}{ccccc} M & \longrightarrow & \prod_{i\in I} R_i \otimes_R M & \rightrightarrows & \prod_{i,j\in I} R_i \star_R R_j \otimes_R M \\ \downarrow & & \downarrow & & \downarrow \\ S_M & \longrightarrow & \prod_{i\in I} R_i \star_R S_M & \rightrightarrows & \prod_{i,j\in I} R_i \star_R R_j \star_R S_M \end{array} \tag{4}$$

If (1) is a cover for the canonical topology, then lower row in (4) is an exact diagram. Vertical arrows in (4) define a morphism from the diagram (2) to the diagram

$$S_M \longrightarrow \prod_{i\in I} R_i \star_R S_M \rightrightarrows \prod_{i,j\in I} R_i \star_R R_j \star_R S_M \tag{5}$$

which is a retraction (in particular, vertical arrows are split monomorphisms). This implies that the diagram (2) is exact too.

(b) Let M be an arbitrary left R-module. Denote by S_M the algebra $S_{M'}$, where M' is the R-bimodule $M \otimes_k R$. The argument (a) applied to $S_{M'}$ proves that the diagram (2) is exact, i.e. (1) is a cover for the $\mathfrak{M}(\mathcal{O}) - 1$ descent topology.

(c) Let $\{R_i^\vee \xrightarrow{\phi_i^\vee} R^\vee \mid i \in I\}$ be a cover for the $\mathfrak{M}(\mathcal{O}) - 1$ descent topology. Then for any morphism $S^\vee \longrightarrow R^\vee$, the family $(R_i \star S)^\vee \longrightarrow S^\vee \mid i \in I\}$ is a cover for the $\mathfrak{M}(\mathcal{O}) - 1$ descent topology; i.e. for any S-module M, the diagram

$$M \longrightarrow \prod_{i \in I} R_i \star_R S \otimes_S M \;\substack{\longrightarrow \\ \longrightarrow}\; \prod_{i,j \in I} R_i \star_R R_j \star_R S \otimes_S M$$

is exact. Taking $M = S$, we obtain the exact diagram (3). Thus, $\{R_i^\vee \xrightarrow{\phi_i} R^\vee \mid i \in I\}$ is a cover for the canonical topology. ∎

4.7.3.2. Corollary. *A subpresheaf $T \xrightarrow{\iota} X$ of a representable presheaf X on $\mathbf{Aff}_k$ is a refinement in the canonical topology iff the inverse image functor*

$$Qcoh_X \xrightarrow{\iota^*} Qcoh_T$$

is fully faithful.

Proof. This follows from the fact that the canonical topology on $\mathbf{Aff}_k$ is the topology of $\mathfrak{M}(\mathcal{O}) - 1$-descent. ∎

4.7.3.3. Corollary. *Every quasi-coherent module on $S \in Ob\mathbf{Aff}_k$ is a sheaf for the canonical topology on $\mathbf{Aff}_k/S$. In particular, for any subcanonical topology $\mathfrak{T}_S$ on $\mathbf{Aff}_k/S$, all quasi-coherent modules on S are sheaves.*

Proof. The fact follows from 4.7.3.1 and 3.3. ∎

4.7.3.4. Note. The assertion 4.7.3.2 is proven in [Or] for the commutative case. The corollary 4.7.3.3 is also a result by D. Orlov [Or, Proposition 4.9].

4.8. Sheaves of modules. Let $\mathfrak{T}$ be a topology on the category $\mathcal{A}$. The category $\mathcal{M}od(\mathfrak{M}(\mathcal{A}, \mathcal{O}), \mathfrak{T})$ of sheaves of modules on the fibred category $\mathfrak{M}(\mathcal{A}, \mathcal{O})$ coincides with the category $\mathcal{M}od(\mathcal{A}, \mathfrak{T}; \mathcal{O})$ of sheaves of left $\mathcal{O}$-modules on the site $(\mathcal{A}, \mathfrak{T})$ in the conventional sense. The category $\mathcal{M}od(\mathcal{A}, \mathfrak{T}; \mathcal{O})$ is a Grothendieck category with small products, i.e. an abelian category satisfying the Grothendieck's conditions AB5 and AB3*, which has a generator (cf. [SGA4], II).

For any presheaf of sets X, we denote by $\mathcal{M}od_X^{\mathfrak{T}}$ the full subcategory of $\mathcal{M}od_X$ whose objects are *sheaves* of modules with respect to the topology $\mathfrak{T}_X$ induced by $\mathfrak{T}$ on $\mathcal{A}/X$.

Let $X \in Ob\mathcal{A}^\wedge$ and $\mathcal{A}/X \xrightarrow{p_X} \mathcal{A}$ the canonical functor. A presheaf M of $\mathcal{O}_X$-modules is a sheaf on $(X, \mathfrak{T}_X)$ (i.e. on the site $(\mathcal{A}/X, \mathfrak{T}_X)$) iff, for any $S \in Ob\mathcal{A}$ and $S \xrightarrow{\xi} X$, the presheaf $\xi^\bullet(M)$ is a sheaf on S.

In particular, a quasi-coherent module M on X is a sheaf on $(X, \mathfrak{T}_X)$ iff for any $S \in Ob\mathcal{A}$ and $S \xrightarrow{\xi} X$, the inverse image $\xi^*(M)$ of M is a sheaf on S.

The functor $\mathcal{A}/X \xrightarrow{p_X} \mathcal{A}$ induces a functor $\mathcal{M}od(\mathcal{A}, \mathfrak{T}) \xrightarrow{p_X^\bullet} \mathcal{M}od_X^{\mathfrak{T}}$. The functor $p_X^\bullet$ has a right adjoint, $p_{X\bullet}$, and a left adjoint, $p_{X!}$.

Similarly, for any presheaf morphism $X \xrightarrow{f} Y$, the functor $\mathcal{A}/X \xrightarrow{p_f} \mathcal{A}/Y$ induces an 'inverse image' functor $\mathcal{M}od_Y^{\mathfrak{T}} \xrightarrow{p_f^\bullet} \mathcal{M}od_X^{\mathfrak{T}}$ which has a right adjoint, $p_{f\bullet}$, and a left adjoint, $p_{f!}$.

4.8.1. Coherator. Suppose every quasi-coherent module on $(\mathcal{A}, \mathcal{O})$ is a sheaf on the site $(\mathcal{A}, \mathfrak{T})$, i.e. $Qcoh(\mathcal{A}, \mathcal{O})$ is a full subcategory of the category $\mathcal{M}od(\mathcal{A}, \mathfrak{T}; \mathcal{O})$ of the sheaves of $\mathcal{O}$-modules on $(\mathcal{A}, \mathfrak{T})$. A right adjoint (if any) to the inclusion functor $Qcoh(\mathcal{A}, \mathcal{O}) \xrightarrow{\psi_X^{\mathfrak{T}}} \mathcal{M}od(\mathcal{A}, \mathfrak{T}; \mathcal{O})$ is called a *coherator on the ringed site* $(\mathcal{A}, \mathfrak{T}; \mathcal{O})$.

Since $\mathcal{M}od(\mathcal{A}, \mathfrak{T}; \mathcal{O})$ is a Grothendieck category with small products, the existence of the coherator implies that the category $Qcoh(\mathcal{A}, \mathcal{O})$ of quasi-coherent modules is a Grothendieck category with small products too (see [BD], 5.39).

If every quasi-coherent module on a presheaf X is a sheaf on X, i.e. $Qcoh_X$ is a (full) subcategory of $\mathcal{M}od_X^{\mathfrak{T}}$, we have the notion of a *coherator on* $(X, \mathfrak{T}_X)$.

4.8.2. Proposition. *Let $\mathfrak{T}$ be a pretopology on $\mathcal{A}$ such that, for any $S \in Ob\mathcal{A}$, quasi-coherent modules on S are sheaves on $(\mathcal{A}/S, \mathfrak{T}_S)$. If Y is a presheaf of sets on $\mathcal{A}$ such that there exists a coherator on $(Y, \mathfrak{T}_Y)$, then, for any presheaf morphism $X \xrightarrow{f} Y$, its inverse image functor $Qcoh_Y \xrightarrow{f^*} Qcoh_X$ has a right adjoint, f_* – a direct image functor of f.*

In particular, any morphism to an affine space has a direct image functor.

Proof. In fact, the functor $\mathcal{M}od_Y \xrightarrow{f^\bullet} \mathcal{M}od_X$ has a right adjoint, $f_\bullet$. The pair of adjoint functors $f^\bullet, f_\bullet$ induces a pair of adjoint functors

$$\mathcal{M}od_Y^{\mathfrak{T}} \xrightarrow{f_{\mathfrak{T}}^\bullet} \mathcal{M}od_X^{\mathfrak{T}}, \quad \mathcal{M}od_X^{\mathfrak{T}} \xrightarrow{f_{\mathfrak{T}\bullet}} \mathcal{M}od_Y^{\mathfrak{T}}.$$

The composition of $f_{\mathfrak{T}}^\bullet$ with the inclusion functor $Qcoh_Y \xrightarrow{j_Y} \mathcal{M}od_Y^{\mathfrak{T}}$ equals to the composition of $Qcoh_Y \xrightarrow{f^*} Qcoh_X$ and the inclusion functor $Qcoh_X \xrightarrow{j_X} \mathcal{M}od_X^{\mathfrak{T}}$. Since the functor j_Y has a right adjoint, a coherator $\mathcal{M}od_Y^{\mathfrak{T}} \xrightarrow{\psi_Y} Qcoh_Y$, the functor

$f^{\bullet}_{\mathfrak{T}} \circ j_Y = j_X \circ f^*$ is left adjoint to the functor $\psi_Y \circ f_{\mathfrak{T}\bullet}$. Denote by f_* the composition $\psi_Y \circ f_\bullet \circ j_X : Qcoh_X \longrightarrow Qcoh_Y$. Thus defined functor f_* is a right adjoint to f^*. In fact, for any $L \in ObQcoh_Y$ and $M \in ObQcoh_X$, we have functorial isomorphisms:

$$Qcoh_Y(L, \psi_Y \circ f_{\mathfrak{T}\bullet} \circ j_X(M)) \simeq \mathcal{M}od_X(f^{\bullet}_{\mathfrak{T}} \circ j_Y(L), j_X(M))$$

$$= \mathcal{M}od_X(j_X \circ f^*(L), j_X(M)) \simeq Qcoh_X(f^*(L), M),$$

hence the assertion. ■

4.8.3. A formula for the coherator. Assume that the inclusion functor j_X has a right adjoint, $\mathcal{M}od^{\mathfrak{T}}_X \xrightarrow{\psi_X} Qcoh_X$. Since ψ_X is left exact, it preserves kernels of pairs of morphisms. In particular, it maps the exact diagram (3) to the exact diagram

$$\psi_X(M) \longrightarrow \psi_X\pi_\bullet(M(\mathcal{U}, \pi)) \rightrightarrows \psi_X\nu_\bullet(M(\mathcal{R}, \nu)). \qquad (4)$$

The equality $j_U \circ \pi^* = \pi^\bullet \circ j_X$ (reflecting the fact that $\pi^\bullet$ maps quasi-coherent modules to quasi-coherent modules) implies that $\pi_* \circ \psi_{\mathcal{U}} \simeq \psi_X \circ \pi_\bullet$. Similarly, $\nu_* \circ \psi_{\mathcal{R}} \simeq \psi_X \circ \nu_\bullet$. Therefore the diagram (4) is isomorphic to the diagram

$$\psi_X(M) \longrightarrow \pi_*\psi_{\mathcal{U}}(M(\mathcal{U}, \pi)) \rightrightarrows \nu_*\psi_{\mathcal{R}}(M(\mathcal{R}, \nu)). \qquad (5)$$

Since the diagram (4) is exact, the diagram (5) is exact too; i.e. $\psi_X(M)$ is isomorphic to the kernel of the pair of arrows $\pi_*\psi_{\mathcal{U}}\pi^\bullet(M) \rightrightarrows \nu_*\psi_{\mathcal{R}}\nu^\bullet(M)$.

4.8.4. Proposition. *Suppose the topology $\mathfrak{T}$ on $\mathcal{A}$ is such that for any $S \in Ob\mathcal{A}$, quasi-coherent modules on S are sheaves on $(\mathcal{A}/S, \mathfrak{T}_S)$. Let X be a presheaf such that there exists a diagram*

$$\mathfrak{R} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \mathfrak{U} \xrightarrow{\pi} X \qquad (1)$$

where $\mathcal{R}$ and $\mathcal{U}$ are representable, $\pi \circ p_1 = \pi \circ p_2$, and the morphism $Cok(p_1, p_2) \longrightarrow X$ induced by π is bicovering. Then the inclusion functor

$$Qcoh_X \xrightarrow{j_X} \mathcal{M}od^{\mathfrak{T}}_X$$

has a right adjoint.

Proof. The assertion follows from 3.5 and 3.5.2. ■

4.8.5. Quasi-coherent modules on presheaves and quasi-coherent modules on the associated sheaves. Fix a ringed site $(\mathcal{A}, \mathfrak{T}; \mathcal{O})$.

4.8.5.1. Lemma. *Let $(\mathcal{A}, \mathfrak{T})$ be a site, X a presheaf on $\mathcal{A}$ and X^a the associated space (i.e. the associated sheaf of sets). The canonical morphism $X \longrightarrow X^a$ is a cover in the coinduced topology $\mathfrak{T}^\wedge$.*

Proof. Let $H_{\mathfrak{T}}$ denote the corresponding Heller's functor $\mathcal{A}^\wedge \longrightarrow \mathcal{A}^\wedge$ defined by $H_{\mathfrak{T}}(X)(T) = \mathrm{colim}(\mathcal{A}^\wedge(S, X) | \; S \in \mathfrak{T}(T))$ for all $T \in Ob\mathcal{A}$. It follows from the definition of $H_{\mathfrak{T}}$ that that the canonical morphism $X \stackrel{\tau}{\longrightarrow} H_{\mathfrak{T}}(X)$ is a cover in the topology $\mathfrak{T}^\wedge$. The associated sheaf, X^a, is isomorphic to $H^2_{\mathfrak{T}}(X)$ and the canonical morphism $X \longrightarrow X^a$ corresponds to the composition $\tau H_{\mathfrak{T}}(X) \circ \tau(X)$ of two covers, hence it is a cover itself. ∎

The following fact is well known (see [SGA4], II, or [Or, 2.4]).

4.8.5.2. Proposition. *For any presheaf X, the canonical morphism $X \stackrel{\mathfrak{j}_X}{\longrightarrow} X^a$ induces an equivalence of categories $\mathcal{M}od^{\mathfrak{T}}_{X^a} \stackrel{\mathfrak{j}^\bullet_X}{\longrightarrow} \mathcal{M}od^{\mathfrak{T}}_X$.*

Proof. The assertion follows from the fact that $X \longrightarrow X^a$ is a cover in the coinduced topology $\mathfrak{T}^\wedge$. Details are left to the reader. ∎

4.8.5.3. Corollary. *Suppose the topology $\mathfrak{T}$ on $\mathcal{A}$ is of 1-descent, i.e. quasi-coherent modules on S are sheaves on $(\mathcal{A}/S, \mathfrak{T}_S)$. Let X be a presheaf and $X \stackrel{\mathfrak{j}_X}{\longrightarrow} X^a$ the canonical morphism. Then the inverse image functor $Qcoh_{X^a} \stackrel{\mathfrak{j}^*_X}{\longrightarrow} Qcoh_X$ is fully faithful.*

Proof. In the commutative diagram

$$\begin{array}{ccc} Qcoh_{X^a} & \stackrel{\mathfrak{j}^*_X}{\longrightarrow} & Qcoh_X \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}od^{\mathfrak{T}}_{X^a} & \stackrel{\mathfrak{j}^\bullet_X}{\longrightarrow} & \mathcal{M}od^{\mathfrak{T}}_X \end{array}$$

the vertical arrows are full embeddings and the lower horizontal arrow, $\mathfrak{j}^\bullet_X$, is a category equivalence, hence $\mathfrak{j}^*_X$ is fully faithful. ∎

4.9. A description of the category of quasi-coherent modules. Let $(\mathcal{A}, \mathcal{O})$ be a ringed category. Let X be a presheaf of sets on $\mathcal{A}$ such that there exists a diagram

$$\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{U} \stackrel{\pi}{\longrightarrow} X \qquad (1)$$

where $\pi \circ p_1 = \pi \circ p_2$, and the morphism $Cok(p_1, p_2) \longrightarrow X$ induced by π is bicovering. Then the category $Qcoh_X$ of quasi-coherent modules on X is equivalent to the category $Ker(p_1^*, p_2^*)$ whose objects are pairs (M, ϕ), where M is an object of $Qcoh_{\mathfrak{U}}$ and ϕ is an isomorphism $p_1^*(M) \xrightarrow{\sim} p_2^*(M)$. Morphisms from (M,ϕ) to (M',ϕ') are given by arrows $M \xrightarrow{g} M'$ which make the diagram

$$\begin{array}{ccc} p_1^*(M) & \xrightarrow{\phi} & p_2^*(M) \\ {\scriptstyle p_1^*(g)}\downarrow & & \downarrow{\scriptstyle p_2^*(g)} \\ p_1^*(M') & \xrightarrow{\phi'} & p_2^*(M') \end{array}$$

commute. If $\mathfrak{U}$ and $\mathfrak{R}$ are representable by objects resp. $\mathcal{U}$ and $\mathcal{R}$, then $Qcoh_{\mathfrak{U}} = \mathcal{O}(\mathcal{U}) - mod$, $Qcoh_{\mathfrak{R}} = \mathcal{O}(\mathcal{R}) - mod$, and inverse image functor p_i^*, $i = 1, 2$, is the tensoring $L \longmapsto \mathcal{O}(\mathcal{R}) \otimes_{\mathcal{O}(\mathcal{U})} L$ corresponding to the ring morphism $\mathcal{O}(\mathcal{U}) \xrightarrow{\mathcal{O}(p_i)} \mathcal{O}(\mathcal{R})$.

This follows from the argument of 3.5. In particular, we have the following

4.9.1. Proposition. *Let $(\mathcal{A}, \mathcal{O})$ be a ringed category and $\mathfrak{T}$ a quasi-topology on $\mathcal{A}$, which is coarser than the quasi-topology of effective descent. Let*

$$\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{U} \xrightarrow{\pi} X$$

be an exact diagram of sheaves of sets on $\mathcal{A}$. Then the category $Qcoh_X$ of quasi-coherent modules on X is equivalent to the category $Ker(p_1^*, p_2^*)$.

If the sheaves $\mathfrak{R}$, $\mathfrak{U}$ are representable by objects resp. $\mathcal{R}$ and $\mathcal{U}$ of the category $\mathcal{A}$, then the category $Ker(p_1^*, p_2^*)$ is described by a linear algebra data: its objects are pairs (L, ϕ), where L is an $\mathcal{O}(\mathcal{U})$-module and ϕ is an $\mathcal{O}(\mathcal{R})$-module isomorphism $p_1^*(L) \xrightarrow{\sim} p_2^*(L)$.

4.9.2. Remark. The category $Ker(p_1^*, p_2^*)$ in 4.9.1 is equivalent to the category of quasi-coherent modules on the cokernel of the pair $\mathfrak{R} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{U}$. If $Cok(p_1, p_2) \xrightarrow{g} X$ is a presheaf morphism such that the corresponding map of associated sheaves is an isomorphism (and $\mathfrak{T}$ is coarser than the quasi-topology of 2-descent), than g^* is an equivalence of $Qcoh_X$ and $Qcoh_{Cok(p_1,p_2)}$. This allows to find the category of quasi-coherent modules on a space without finding the space itself. We illustrate the latter observation in the following examples.

4.9.3. Application: quasi-coherent modules on Grassmannians. Let R be a k-algebra, M and V projective left R-modules of finite type. Then the functors $G_{M,V}$ and $\mathfrak{R}_{M,V}$ in the exact diagram

$$\mathfrak{R}_{M,V} \underset{p_2}{\overset{p_1}{\rightrightarrows}} G_{M,V} \xrightarrow{\pi} Gr_{M,V} \qquad (1)$$

defining the presheaf $Gr_{M,V}$ are representable by resp. $(\mathcal{G}^\vee_{M,V}, \mathcal{G}^\vee_{M,V} \longrightarrow R^\vee)$ and $(\mathcal{R}^\vee_{M,V}, \mathcal{R}^\vee_{M,V} \longrightarrow R^\vee)$ (cf. III.6.3), which implies that the category $Qcoh_{Gr_{M,V}}$ of quasi-coherent modules on the presheaf $Gr_{M,V}$ is defined by a linear algebra data: it is equivalent to the category $Ker(p_1^*, p_2^*)$ whose objects are pairs (L, ϕ), where L is a $\mathcal{G}_{M,V}$-module and ϕ is an $\mathcal{R}_{M,V}$-module isomorphism $p_1^*(L) \xrightarrow{\sim} p_2^*(L)$. The inverse image functor p_i^*, $i = 1, 2$, is isomorphic to the tensoring $L \longmapsto \mathcal{R}_{M,V} \otimes_{\mathcal{G}_{M,V}} L$ corresponding to an algebra morphism $\mathcal{G}_{M,V} \xrightarrow{\tilde{p}_i} \mathcal{R}_{M,V}$ representing p_i.

Let $\mathfrak{T}$ be a quasi-topology on the category $\mathbf{Aff}_k/R^\vee$ of affine k-schemes over $R^\vee$. Let $Gr^{\mathfrak{T}}_{M,V}$ be the $\mathfrak{T}$-Grassmannian corresponding to $Gr_{M,V}$, i.e. a sheaf of sets (a 'space') associated to $Gr_{M,V}$. If $\mathfrak{T}$ is coarser than the quasi-topology of effective descent, then the category $Qcoh_{Gr_{M,V}}$ of quasi-coherent modules on $Gr_{M,V}$ is naturally equivalent to the category $Qcoh_{Gr^{\mathfrak{T}}_{M,V}}$ of quasi-coherent modules on $Gr^{\mathfrak{T}}_{M,V}$.

4.9.4. Noncommutative projective space. Let M be the free R-module of the rank n+1, V the free R-module of the rank 1. In this case, we denote the functor $Gr_{M,V}$ by $\mathcal{P}^n_R$. If a quasi-topology $\mathfrak{T}$ on the category $\mathbf{Aff}_k/R^\vee$ of affine k-schemes over $R^\vee$ is coarser than the quasi-topology of 2-descent, than the category $Qcoh_{\mathcal{P}^n_R}$ of quasi-coherent modules on $\mathcal{P}^n_R$ is equivalent to the category of quasi-coherent modules on the associated projective space ${\mathcal{P}^n_R}^{\mathfrak{T}}$.

4.9.5. The commutative case. Let $R = k$. Denote by $Gr^c_{M,V}$ the restriction of the presheaf $Gr_{M,V}$ to the subcategory $\mathbf{CAff}_k$ of commutative affine k-schemes (i.e. the opposite category to the category $CAlg_k$ of commutative k-algebras). We assume that the rank of the k-module M at each point of $Spec(k)$ is greater than, or equal to the rank of the k-module V at this point; otherwise the functor $Gr^c_{M,V}$ maps every commutative k-algebra to the empty set. The exact diagram (1) induces an exact diagram

$$\mathfrak{R}^c_{M,V} \underset{\mathfrak{p}_2}{\overset{\mathfrak{p}_1}{\rightrightarrows}} G^c_{M,V} \xrightarrow{\ \pi\ } Gr^c_{M,V}, \qquad (2)$$

where $\mathfrak{R}^c_{M,V}$ and $G^c_{M,V}$ denote the restrictions of presheaves resp. $\mathfrak{R}_{M,V}$ and $G_{M,V}$ to the subcategory $\mathbf{CAff}_k$. If $\mathfrak{R}_{M,V}$ and $G_{M,V}$ are representable by the algebras resp. $\mathcal{R}_{M,V}$ and $\mathcal{G}_{M,V}$, then the presheaves $\mathfrak{R}^c_{M,V}$ and $G^c_{M,V}$ are representable by *abelianizations* (quotients by the commutant) of these algebras, $\mathcal{R}^c_{M,V}$ and $\mathcal{G}^c_{M,V}$. By 4.9.2, the category of quasi-coherent modules on $Gr^c_{M,V}$ is isomorphic to the kernel $Ker(\mathfrak{p}_1^*, \mathfrak{p}_2^*)$ of the pair of the inverse image functors

$$\mathcal{G}^c_{M,V} - mod \underset{\mathfrak{p}_2^*}{\overset{\mathfrak{p}_1^*}{\rightrightarrows}} \mathcal{R}^c_{M,V} - mod. \qquad (3)$$

Now we regard (2) as an exact sequence of presheaves of sets on the ringed site $(\mathbf{CAff}_k, \mathcal{O})$ of commutative k-schemes for a flat (**fpqc**, or **fppf**) topology. The presheaves $\mathfrak{R}^c_{_{M,V}}$ and $G^c_{_{M,V}}$ are sheaves because the flat topology is subcanonical (all representable presheaves are sheaves). The presheaf $Gr^c_{_{M,V}}$ is not a sheaf, but the sheaf associated to $Gr^c_{_{M,V}}$ is isomorphic to the Grassmannian $Grassm_{_{M,V}}$. Since the **fpqc** topology (hence the **fppf** topology) is coarser than the 2-descent topology, the category of quasi-coherent modules on the Grassmannian $Grassm_{_{M,V}}$ is equivalent to the category of quasi-coherent modules on the presheaf $Gr^c_{_{M,V}}$, i.e. to the kernel of the pair of functors (3).

5. Representable fibred categories and representable cartesian functors.

5.1. Categories over $\mathcal{E}$ representable by a presheaf of sets. Fix a category $\mathcal{E}$. For any presheaf $\mathcal{E}^{op} \xrightarrow{S} \mathbf{Sets}$, we have a category $\mathcal{E}/S$ over $\mathcal{E}$. For any $X \in Ob\mathcal{E}$, the fiber $(\mathcal{E}/S)_X$ is a discrete category formed by all objects $(X, X \to S)$. In particular, it is empty if $S(X) = \emptyset$. Any morphism $X \xrightarrow{f} Y$ of the category $\mathcal{E}$ induces a functor

$$(\mathcal{E}/S)_Y \xrightarrow{f^*} (\mathcal{E}/S)_X, \quad (Y, Y \overset{\xi}{\to} S) \longmapsto (X, \xi \circ f).$$

The map $f \longmapsto f^*$ is a functor $\mathcal{E}^{op} \longrightarrow Cat$, and $\mathcal{E}/S$ is a fibred category corresponding to this functor. Note that every morphism of the category $\mathcal{E}/S$ is cartesian.

5.1.1. Proposition. *The map*

$$S \longmapsto \mathcal{E}/S,\ S \in Ob\mathcal{E}^\wedge, \quad (S \overset{g}{\to} T) \longmapsto \Big(\mathcal{E}/S \longrightarrow \mathcal{E}/T,\ (X,\xi) \longmapsto (X, g \circ \xi),\Big)$$

is a fully faithful functor, $\mathfrak{h}^{\mathcal{E}}$, from $\mathcal{E}^\wedge$ to the category $Cat/\mathcal{E}$ of categories over $\mathcal{E}$. The functor $\mathfrak{h}^{\mathcal{E}}$ preserves finite limits.

Proof is left to the reader. ∎

5.2. Definition. Let $\mathcal{E}'$ be a full subcategory of $\mathcal{E}^\wedge$. A category $\mathcal{F}$ over $\mathcal{E}$ is called *$\mathcal{E}'$-representable* if it is $\mathcal{E}$-equivalent to the category $\mathcal{E}/S$ for some object S of $\mathcal{E}'$.

In particular, any $\mathcal{E}'$-representable category over $\mathcal{E}$ is fibred.

5.2.1. Standard choices of $\mathcal{E}'$. For an arbitrary category $\mathcal{E}$, the standard choices of $\mathcal{E}'$ are the category $\mathcal{E}$ itself, the category $\mathcal{E}^\wedge$ of presheaves on $\mathcal{E}$, and the subcategory of left exact functors $\mathcal{E}^{op} \longrightarrow \mathbf{Sets}$.

5.3. Relatively representable cartesian functors. Fix a full subcategory $\mathcal{E}'$ of the category $\mathcal{E}^\wedge$. A cartesian functor $\mathcal{F} \xrightarrow{\Phi} \mathcal{G}$ between categories over $\mathcal{E}$ is called *$\mathcal{E}'$-representable*, if, for any $T \in Ob\mathcal{E}'$ and any cartesian functor $\mathcal{E}/T \longrightarrow \mathcal{G}$, the fibred product $\mathcal{E}/T \times_{\mathcal{G}} \mathcal{F}$ is $\mathcal{E}'$-representable.

5.3.1. Affine cartesian functors. We call a cartesian functor between categories over $\mathcal{E}$ *affine*, or *representable*, if it is $\mathcal{E}$-representable.

5.3.2. Proposition. *(a) Any $\mathcal{E}$-equivalence is $\mathcal{E}'$-representable for any full subcategory $\mathcal{E}'$ of $\mathcal{E}^\wedge$. In particular it is affine.*

(b) Suppose $\mathcal{E}'$ has finite products taken in $\mathcal{E}^\wedge$. Then, for any $\mathcal{E}'$-representable category $\mathcal{F}$ over $\mathcal{E}$, the structure morphism of $\mathcal{F}$ is $\mathcal{E}'$-representable.

(c) Suppose $\mathcal{E}'$ has a final object. Then a structure morphism of a category $\mathcal{F}$ over $\mathcal{E}$ is $\mathcal{E}'$-representable iff $\mathcal{F}$ is $\mathcal{E}'$-representable.

(c') Suppose $\mathcal{E}$ has a final object. Then a structure morphism of a category $\mathcal{F}$ over $\mathcal{E}$ is affine iff $\mathcal{F}$ is affine.

Proof. (a) The assertion follows from definitions.

(b) Let $X \in Ob\mathcal{E}'$. The only $\mathcal{E}$-functor $\mathcal{E}/X \longrightarrow \mathcal{E}$ is the canonical functor,

$$(V,\xi) \longmapsto V, \; ((V,\xi) \stackrel{f}{\to} (V',\xi')) \longmapsto (V \stackrel{f}{\to} V').$$

If $\mathcal{F} \simeq \mathcal{E}/S$ for some $S \in Ob\mathcal{E}'$, then $\mathcal{F} \times_{\mathcal{E}} \mathcal{E}/X \simeq \mathcal{E}/S \times_{\mathcal{E}} \mathcal{E}/X \simeq \mathcal{E}/(X \times S)$, hence the structure morphism $\mathcal{F} \longrightarrow \mathcal{E}$ is $\mathcal{E}'$-representable.

(c) Suppose the structure morphism $\mathcal{F} \stackrel{F}{\longrightarrow} \mathcal{E}$ is $\mathcal{E}'$-representable. This means that $\mathcal{F} \times_{\mathcal{E}} \mathcal{E}/X$ is $\mathcal{E}'$-representable for any $X \in Ob\mathcal{E}'$. Taking as X a final object of the category $\mathcal{E}'$, we obtain that $\mathcal{F} \times_{\mathcal{E}} \mathcal{E} \simeq \mathcal{F}$ is representable.

(c') This assertion is a special case of (c). ∎

5.3.3. Proposition. *Let $\mathcal{F}$ be a category over $\mathcal{E}$. Suppose $\mathcal{E}'$ is closed under finite limits taken in $\mathcal{E}^\wedge$. Then the following conditions are equivalent:*

(i) Any cartesian morphism $\mathcal{E}/S \longrightarrow \mathcal{F}$, $S \in Ob\mathcal{E}'$ is $\mathcal{E}'$-representable.

(ii) The diagonal morphism $\mathcal{F} \xrightarrow{\Delta_{\mathcal{F}}} \mathcal{F} \times_{\mathcal{E}} \mathcal{F}$ is $\mathcal{E}'$-representable.

Proof. (i) $\Rightarrow$ (ii). Let $X \in Ob\mathcal{E}'$, and let $f, g : \mathcal{E}/X \longrightarrow \mathcal{F}$ be arbitrary cartesian morphisms over $\mathcal{E}$. Consider the following canonical commutative diagram

$$\begin{array}{ccccc}
\mathcal{F} \times_{\mathcal{F} \times_{\mathcal{E}} \mathcal{F}} \mathcal{E}/X & \longrightarrow & \mathcal{E}/X \times_{\mathcal{F}} \mathcal{E}/X & \longrightarrow & \mathcal{F} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{E}/X & \xrightarrow{\Delta_{\mathcal{E}/X}} & \mathcal{E}/X \times_{\mathcal{E}} \mathcal{E}/X & \xrightarrow{(f,g)} & \mathcal{F} \times_{\mathcal{E}} \mathcal{F}
\end{array} \qquad (1)$$

formed by two universal squares. By (i), the fiber product $\mathcal{E}/X \times_{\mathcal{F}} \mathcal{E}/X$ is $\mathcal{E}'$-representable. Since $\mathcal{E}'$ has products, $X \times X$ exists in $\mathcal{E}'$, and $\mathcal{E}/X \times_{\mathcal{E}} \mathcal{E}/X \simeq \mathcal{E}/(X \times X)$, i.e. it is representable by $X \times X$. Since $\mathcal{E}'$ has fibred products and the embedding

$$\mathcal{E} \longrightarrow Cat/\mathcal{E}, \; X \longmapsto \mathcal{E}/X,$$

preserves limits (cf. 2.2), the category $\mathcal{F} \times_{\mathcal{F}\times_{\mathcal{E}}\mathcal{F}} \mathcal{E}/X$ over $\mathcal{E}$ is $\mathcal{E}'$-representable.

(ii) ⇒ (i). For any two morphisms, $\mathcal{E}/X \xrightarrow{f} \mathcal{F} \xleftarrow{g} \mathcal{E}/Y$, of $Cart_{\mathcal{E}}$, the square

$$\begin{array}{ccc} \mathcal{E}/X \times_{\mathcal{F}} \mathcal{E}/Y & \longrightarrow & \mathcal{E}/X \times_{\mathcal{E}} \mathcal{E}/Y \\ \big\downarrow & & \big\downarrow f \times g \\ \mathcal{F} & \xrightarrow{\Delta_{\mathcal{F}}} & \mathcal{F} \times_{\mathcal{E}} \mathcal{F} \end{array}$$

is cartesian, hence the assertion. ■

5.4. Definition. Let $\mathcal{P}$ be a class of morphisms of the category $\mathcal{E}'$ stable under base change. We say that a morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ of categories over $\mathcal{E}$ is $\mathcal{P}$-representable, if for any $X \in Ob\mathcal{E}'$ and any morphism $\mathcal{E}/X \longrightarrow \mathcal{G}$, the (morphism of $\mathcal{E}'$ representing the) projection $\mathcal{E}/X \times_{\mathcal{G}} \mathcal{F} \longrightarrow \mathcal{E}/X$ belongs to $\mathcal{P}$.

5.4.1. Proposition. *Let $\mathcal{P}$ be a class of morphisms of the subcategory $\mathcal{E}' \subseteq \mathcal{E}^\wedge$.*

(a) The class of $\mathcal{P}$-representable morphisms of is stable under base change.

(b) If the class $\mathcal{P}$ of morphisms of $\mathcal{E}'$ is stable under composition, then same holds for the class $\widetilde{\mathcal{P}}$ of $\mathcal{P}$-representable morphisms of $Cat/\mathcal{E}$.

Proof is left to the reader. ■

5.5. Finitely presentable and locally finitely presentable cartesian functors. Let $\mathcal{E}$ be a category and $\mathcal{E}'$ a full subcategory of $\mathcal{E}^\wedge$. We assume that $\mathcal{E}'$ contains all representable functors and is closed under limits of filtered projective systems.

Let $\mathcal{F}$ and $\mathcal{G}$ be fibred categories over $\mathcal{E}$. A cartesian functor $\mathcal{F} \xrightarrow{\Phi} \mathcal{G}$ over $\mathcal{E}$ will be called *$\mathcal{E}'$-finitely presentable*, if for any filtered projective system $D \xrightarrow{\mathfrak{D}} \mathcal{E}'$, the canonical square

$$\begin{array}{ccc} \operatorname{colim} \mathcal{F}^+_{\mathfrak{D}_\mu} & \longrightarrow & \mathcal{F}^+_{\lim \mathfrak{D}} \\ \Phi' \big\downarrow & & \big\downarrow \Phi_{\lim \mathfrak{D}} \\ \operatorname{colim} \mathcal{G}^+_{\mathfrak{D}_\mu} & \longrightarrow & \mathcal{G}^+_{\lim \mathfrak{D}} \end{array}$$

is 2-cartesian. Here $\mathcal{F}^+$ and $\mathcal{G}^+$ are the canonical extensions of the fibred categories resp. $\mathcal{F}$ and $\mathcal{G}$ onto $\mathcal{E}^\wedge$ (cf. 1.5.1); and $\lim \mathfrak{D}$ is taken in $\mathcal{E}^\wedge$.

We call an $\mathcal{E}'$-finitely presentable cartesian functor *locally finitely presentable* if $\mathcal{E}' = \mathcal{E}$. We call it *finitely presentable* if $\mathcal{E}' = \mathcal{E}^\wedge$.

6. Representable cartesian functors and local constructions.

6.0. Fix a category $\mathcal{E}$ and a full subcategory $\mathcal{E}'$ of the category $\mathcal{E}^\wedge$ of presheaves of sets on $\mathcal{E}$ which contains (the image of) $\mathcal{E}$ and is closed under fibred products. We denote by $\widetilde{\mathcal{E}}'$ the fibred category over $\mathcal{E}$ defined as follows: for any $\mathcal{V} \in Ob\mathcal{E}$, the fiber $\widetilde{\mathcal{E}}'_{\mathcal{V}}$ is the category $\mathcal{E}'/\widehat{\mathcal{V}}$. For any morphism $\mathcal{U} \xrightarrow{\phi} \mathcal{V}$, its inverse image functor, ϕ^*, assigns to any object $(X, X \to \widehat{\mathcal{V}})$ of the category $\mathcal{E}'/\widehat{\mathcal{V}}$ its pull-back $(X \times_{\phi,\mathcal{V}} \widehat{\mathcal{U}}, X \times_{\phi,\mathcal{V}} \widehat{\mathcal{U}} \to \widehat{\mathcal{U}})$.

6.1. Proposition. *Let $\mathfrak{X}$ be a fibred category over $\mathcal{E}$. There is a natural equivalence between the category of $\mathcal{E}'$-representable cartesian functors $\mathcal{Y} \longrightarrow \mathfrak{X}$ and the category of cartesian functors $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}'$.*

Proof. The argument is left to the reader. ∎

6.1.1. Local constructions. We call any cartesian functor $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}'$ a *local $\mathcal{E}'$-construction on $\mathfrak{X}$*. Local $\mathcal{E}$-constructions will be called *affine.*

6.2. Relative local constructions. Fix a functor $\mathcal{A} \xrightarrow{\Phi} \mathcal{E}$. Let $\widetilde{\mathcal{E}}'_{\Phi}$ denote the fibred category $\widetilde{\mathcal{E}}' \times_{\mathcal{E}} \mathcal{A} = \widetilde{\mathcal{E}}' \times_{\Phi,\mathcal{E}} \mathcal{A}$ over $\mathcal{A}$.

Let $\mathfrak{X}$ be a fibred category over $\mathcal{A}$. We call any cartesian functor $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}' \times_{\mathcal{E}} \mathcal{A}$ a *local* $(\mathcal{E}', \Phi)$*-construction on* $\mathfrak{X}$, or simply a *local construction on* $\mathfrak{X}$, if it is clear what are the subcategory $\mathcal{E}'$ and the functor Φ.

Local $(\mathcal{E}, \Phi)$-constructions will be called *affine.*

6.3. Local constructions over a ringed category. Let $(\mathcal{A}, \mathcal{O})$ be a ringed category. Let $\mathcal{E}$ be the category $\mathbf{Aff_k}$ of noncommutative affine k-schemes. Let τ be a pretopology on $\mathcal{E}$ and $\mathcal{E}'$ the category of τ-locally affine spaces. The presheaf of rings $\mathcal{O}$ induces a functor

$$\Phi = \Phi_{\mathcal{O}} : \mathcal{A} \longrightarrow \mathcal{E}, \quad U \longmapsto \mathcal{O}(U)^{\vee}.$$

A local $(\mathcal{E}, \Phi)$-construction on a fibred category $\mathfrak{X}$ over $\mathcal{A}$ is a family of functors $\mathfrak{X}_U \xrightarrow{F_U} (\widetilde{\mathcal{E}}' \times_{\mathcal{E}} \mathcal{A})_U$, $U \in Ob\mathcal{A}$, such that for any $U \in Ob\mathcal{A}$ and any object x of the fiber $\mathfrak{X}_U$ over U, $F_U(x)$ is a τ-locally affine space over $\mathcal{O}(U)^{\vee}$; and for any morphism $U \xrightarrow{\phi} \mathcal{V}$ and any $x \in Ob\mathfrak{X}_{\mathcal{V}}$, we are given an isomorphism of spaces

$$F_U(\phi^*(x)) \xrightarrow{\sim} F_{\mathcal{V}}(x) \prod_{\mathcal{O}(\mathcal{V})^{\vee}} \mathcal{O}(\mathcal{V})^{\vee}.$$

6.3.1. Affine local constructions. A local construction $\mathfrak{X} \xrightarrow{F} \mathcal{E}' \times_{\mathcal{E}} \mathcal{A}$ is affine iff for any $U \in Ob\mathcal{A}$ and any $x \in Ob\mathfrak{X}_U$, the object $F_U(x)$ is an affine space over $\mathcal{O}(U)^{\vee}$, i.e. $F_U(x)$ is isomorphic to a pair $(\mathcal{R}(U,x)^{\vee}, \mathcal{R}(U,x)^{\vee} \longrightarrow \mathcal{O}(U)^{\vee})$ corresponding to a ring morphism $\mathcal{O}(U) \longrightarrow \mathcal{R}(U,x)$ defined uniquely up to isomorphism.

Thus, an affine local construction on $\mathfrak{X}$ can be described as a function which assigns to every pair (U, x), where $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}_U$, a ring morphism $\mathcal{O}(\mathcal{V}) \longrightarrow \mathcal{R}(U,x)$ and to any morphism $U \xrightarrow{\phi} \mathcal{V}$ of $\mathcal{A}$ and any $x \in Ob\mathfrak{X}_{\mathcal{V}}$, a morphism $\mathcal{R}(\mathcal{V}, x) \xrightarrow{\xi_\phi} \mathcal{R}(U, \phi^*(x))$ such that the square

$$\begin{array}{ccc} \mathcal{R}(\mathcal{V}, x) & \xrightarrow{\xi_\phi} & \mathcal{R}(U, \phi^*(x)) \\ \uparrow & cocart & \uparrow \\ \mathcal{O}(\mathcal{V}) & \xrightarrow{\mathcal{O}(\phi)} & \mathcal{O}(U) \end{array}$$

is cocartesian. In other words, the induced ring morphism

$$\mathcal{R}(\mathcal{V}, x) \star_{\mathcal{O}(\mathcal{V})} \mathcal{O}(U) \xrightarrow{\xi'_\phi} \mathcal{R}(U, \phi^*(x))$$

is an isomorphism. The morphisms ξ_ϕ should satisfy standard compatibility conditions with respect to the composition of morphisms of $\mathcal{A}$.

6.3.2. Vector fibers over a ringed category. Let $\mathfrak{Bmod}(\mathcal{A}, \mathcal{O})$ denote the *fibred category of* $(\mathcal{A}, \mathcal{O})$*-bimodules* determined by the pseudo-functor $\mathcal{A}^{op} \longrightarrow Cat$ which assigns to each object $\mathcal{V}$ of $\mathcal{A}$ the category opposite to the category of $\mathcal{O}(\mathcal{V})$-bimodules and to any morphism $U \xrightarrow{\phi} \mathcal{V}$ the functor opposite to the functor

$$\widetilde{\phi}^* : \mathcal{M} \longmapsto \mathcal{O}(U) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{M} \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{O}(U).$$

Let $\mathfrak{X}$ be a fibred category over $\mathcal{A}$. We define $\mathcal{O}$*-bimodules on* $\mathfrak{X}$ as cartesian functors $\mathfrak{X} \xrightarrow{\mathcal{M}} \mathfrak{Bmod}(\mathcal{A}, \mathcal{O})$. This means that $\mathcal{M}$ is a function which assigns to every pair $(\mathcal{V}, x)$, where $\mathcal{V} \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}_{\mathcal{V}}$, an $\mathcal{O}(\mathcal{V})$-bimodule $\mathcal{M}(\mathcal{V}, x)$ and to every morphism $U \xrightarrow{\phi} \mathcal{V}$ a bimodule isomorphism

$$\mathcal{O}(U) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{M}(\mathcal{V}, x) \otimes_{\mathcal{O}(\mathcal{V})} \mathcal{O}(U) \xrightarrow{\zeta_\phi} \mathcal{M}(U, \phi^*(x)) \tag{1}$$

satisfying the usual compatibility conditions.

For any $\mathcal{V} \in Ob\mathcal{A}$ and $x \in \mathfrak{X}_{\mathcal{V}}$, let $\mathcal{R}(\mathcal{V}, x)$ be the tensor algebra, $T_{\mathcal{O}(\mathcal{V})}(\mathcal{M}(\mathcal{V}, x))$, of the $\mathcal{O}(\mathcal{V})$-bimodule $\mathcal{M}(\mathcal{V}, x)$. Fix a morphism $U \xrightarrow{\phi} \mathcal{V}$ of $\mathcal{A}$. The $\mathcal{O}(\mathcal{V})$-bimodule morphism $\mathcal{M}(\mathcal{V}, x) \longrightarrow \mathcal{M}(U, \phi^*(x))$ and the ring morphism $\mathcal{O}(\mathcal{V}) \xrightarrow{\mathcal{O}(\phi)} \mathcal{O}(U)$ induce ring morphisms $T_{\mathcal{O}(\mathcal{V})}(\mathcal{M}(\mathcal{V}, x)) \longrightarrow T_{\mathcal{O}(U)}(\mathcal{M}(U, \phi^*(x))) \longleftarrow \mathcal{O}(U)$ which, in turn, determines a morphism

$$\mathcal{O}(U) \star_{\mathcal{O}(\mathcal{V})} T_{\mathcal{O}(\mathcal{V})}(\mathcal{M}(\mathcal{V}, x)) \longrightarrow T_{\mathcal{O}(U)}(\mathcal{M}(U, \phi^*(x))). \tag{2}$$

Since the $\mathcal{O}$-bimodule $\mathcal{M}$ is quasi-coherent, (1) is a bimodule isomorphism, which implies that the ring morphism (2) is an isomorphism.

We denote the affine scheme $T_{\mathcal{O}(\mathcal{V})}(\mathcal{M}(\mathcal{V},x))^\vee$ by $\mathbb{V}_{\mathcal{O}(\mathcal{V})}(\mathcal{M}(\mathcal{V},x))$ and the local construction on $\mathfrak{X}$ given by $(\mathcal{V},x)\longmapsto \mathbb{V}_{\mathcal{O}(\mathcal{V})}(\mathcal{M}(\mathcal{V},x))$ by $\mathbb{V}_{\mathcal{O}}(\mathcal{M})$.

6.3.3. Vector fibers associated with pairs of quasi-coherent modules. Let $\mathfrak{M}(\mathcal{A},\mathcal{O})$ be the fibred category of $(\mathcal{A},\mathcal{O})$-modules (cf. 2). Let $\mathfrak{X}$ be a fibred category over $\mathcal{A}$. We define *$\mathcal{O}$-modules on $\mathfrak{X}$* as cartesian functors $\mathcal{M}:\mathfrak{X}\longrightarrow\mathfrak{M}(\mathcal{A},\mathcal{O})$. This means that $\mathcal{M}$ is a function which assigns to each pair $(\mathcal{V},x)$, where $\mathcal{V}\in Ob\mathcal{A}$ and $x\in Ob\mathfrak{X}_{\mathcal{V}}$, an $\mathcal{O}(\mathcal{V})$-module $\mathcal{M}(\mathcal{V},x)$ and to each morphism $U\xrightarrow{\phi}\mathcal{V}$ an $\mathcal{O}(U)$-module isomorphism

$$\mathcal{O}(U)\otimes_{\mathcal{O}(\mathcal{V})}\mathcal{M}(\mathcal{V},x)\ \xrightarrow{\ \zeta_\phi\ }\ \mathcal{M}(U,\phi^*(x)) \tag{3}$$

which satisfies the usual compatibility conditions.

Let $\mathcal{M}$ and $\mathcal{P}$ be two $\mathcal{O}$-modules on $\mathfrak{X}$. The pair $(\mathcal{M},\mathcal{P})$ defines a functor

$$Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P}):\mathfrak{X}\longrightarrow\widetilde{\mathcal{E}}^\wedge\times_{\mathcal{E}}\mathcal{A} \tag{4}$$

which assigns to each pair (U,x), $U\in Ob\mathcal{A}$, $x\in Ob\mathfrak{X}_U$, the functor

$$\mathcal{O}(U)\backslash Rings\longrightarrow Sets,\quad (\mathcal{O}(U)\xrightarrow{\phi}\mathcal{R})\longmapsto Hom_{\mathcal{R}}(\phi^*(\mathcal{M}(U,x)),\phi^*(\mathcal{P}(U,x))) \tag{5}$$

naturally defined on morphisms.

6.3.3.1. Proposition. *Suppose that for every $U\in Ob\mathcal{A}$ and $x\in Ob\mathfrak{X}_U$, the $\mathcal{O}(U)$-module $\mathcal{P}(U,x)$ is projective of finite type. Then the functor (4) is an affine local construction.*

Proof. Set for convenience $M=\mathcal{M}(U,x)$ and $P=\mathcal{P}(U,x)$. Then

$$Hom_{\mathcal{R}}(\phi^*(M),\phi^*(P))\simeq Hom_{\mathcal{O}(U)}(M,\phi_*\phi^*(P))=Hom_{\mathcal{O}(U)}(M,\mathcal{R}\otimes_{\mathcal{O}(U)}P)\simeq$$

$$Hom_{\mathcal{O}(U)}(M,Hom^{\mathcal{O}(U)}(P^\vee,\mathcal{R}))\simeq Hom_{\mathcal{O}(U)^e}(M\otimes P^\vee,\mathcal{R})\simeq$$
$$\mathcal{O}(U)\backslash Alg_k(T_{\mathcal{O}(U)}(M\otimes P^\vee),\mathcal{R})$$

Here $Hom^{\mathcal{O}(U)}(P^\vee,S)$ is the (left) $\mathcal{O}(U)$-module of right $\mathcal{O}(U)$-module morphisms from $P^\vee$ to S, $\mathcal{O}(U)^e:=\mathcal{O}(U)\otimes\mathcal{O}(U)^o$, and $T_{\mathcal{O}(U)}(M\otimes P^\vee)$ is the tensor algebra of the $\mathcal{O}(U)$-bimodule $M\otimes P^\vee$. This shows that (4) is isomorphic to the vector fiber $\mathbb{V}_{\mathcal{O}}(\mathcal{M}\otimes\mathcal{P}^\vee)$ of the quasi-coherent $\mathcal{O}$-bimodule $\mathcal{M}\otimes\mathcal{P}^\vee$. ■

It is useful to have an analogue of 6.3.3.1 for a family of modules. Let $\{\mathcal{M}_i,\mathcal{P}_i\mid i\in J\}$ be a family of $\mathcal{O}$-modules on $\mathfrak{X}$. These family defines a functor

$$\prod_{i\in J}Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M}_i,\mathcal{P}_i):\mathfrak{X}\longrightarrow\widetilde{\mathcal{E}}^\wedge\times_{\mathcal{E}}\mathcal{A} \tag{6}$$

which assigns to each pair $(U,x),\ U \in Ob\mathcal{A},\ x \in Ob\mathfrak{X}_U$, the functor

$$\mathcal{O}(U)\backslash Rings \longrightarrow Sets, \quad (\mathcal{O}(U) \xrightarrow{\phi} \mathcal{R}) \longmapsto \prod_{i \in J} Hom_{\mathcal{R}}(\phi^*(\mathcal{M}_i(U,x)), \phi^*(\mathcal{P}_i(U,x)))$$

naturally defined on morphisms.

6.3.3.2. Proposition. *Suppose that for every $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}_U$ and for all $i \in J$, the $\mathcal{O}(U)$-module $\mathcal{P}(U,x)$ is projective of finite type. Then the functor (6) is an affine local construction on $\mathfrak{X}$ isomorphic to $\mathbb{V}\big(\bigoplus_{i \in J}(\mathcal{M}_i \otimes \mathcal{P}_i^{\vee})\big)$.*

Proof. The argument is similar to that of 6.3.3.1. Details are left to the reader. ∎

6.3.4. The construction of isomorphisms. Fix a ringed category $(\mathcal{A},\mathcal{O})$ and a fibred category $\mathfrak{X}$ over $\mathcal{A}$. Let $\mathcal{M}$ and $\mathcal{P}$ be $\mathcal{O}$-modules on $\mathfrak{X}$. We denote by $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ the functor $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}^{\wedge} \times_{\mathcal{E}} \mathcal{A}$ which assigns to every pair $(U,x),\ U \in Ob\mathcal{A},\ x \in Ob\mathfrak{X}_U$, the functor

$$G^{\mathcal{O}(U)}_{\mathcal{M}(U,x),\mathcal{V}(U,x)} : \mathcal{O}(U)\backslash Rings \longrightarrow Sets \tag{7}$$

defined as follows: to every ring morphism $\mathcal{O}(U) \xrightarrow{\phi} \mathcal{R}$, the functor (7) assigns the set of all pairs of $\mathcal{R}$-module morphisms $\phi^*(\mathcal{P}(U,x)) \longrightarrow \phi^*(\mathcal{M}(U,x)) \longrightarrow \phi^*(\mathcal{P}(U,x))$ the composition of which is the identical morphism.

6.3.4.1. Proposition. *Let $\mathcal{M}(U,x)$ and $\mathcal{P}(U,x)$ be projective $\mathcal{O}(U)$-modules of finite type for all $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}$. Then the functor $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ is an affine local construction on $\mathfrak{X}$.*

Proof. (a) For convenience, we set $M = \mathcal{M}(U,x)$ and $P = \mathcal{P}(U,x)$. For any ring morphism $\mathcal{O}(U) \xrightarrow{\phi} \mathcal{R}$, the set $G^{\mathcal{O}(U)}_{M,P}(\mathcal{R},\phi)$ is the kernel of the pair of morphisms

$$Hom_{\mathcal{R}}(\phi^*(M),\phi^*(P)) \times Hom_{\mathcal{R}}(\phi^*(P),\phi^*(M)) \rightrightarrows Hom_{\mathcal{R}}(\phi^*(P),\phi^*(P)) \tag{8}$$

where one arrow assigns to each pair (u,v) the composition, $u \circ v$, of morphisms u and v, and the other one maps each pair (u,v) to the identity morphism, $id_{\phi^*(P)}$. Since (8) depends functorially on everything, the functor $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ is the kernel of a pair of functor morphisms

$$Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P}) \times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M}) \rightrightarrows Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{P}) \tag{9}$$

Since $\mathcal{M}(U,x)$ and $\mathcal{P}(U,x)$ are projective $\mathcal{O}(U)$-modules of finite type for all $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}$, the functors $Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P}),\ Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M})$, and $Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{P})$ are affine local constructions on $\mathfrak{X}$ resp. $\mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{M}\otimes\mathcal{P}^\vee)$, $\mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{P}\otimes\mathcal{M}^\vee)$, and $\mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{P}\otimes\mathcal{P}^\vee)$. Thus the diagram (9) is equivalent to the diagram

$$\mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{M}\otimes\mathcal{P}^\vee\oplus\mathcal{P}\otimes\mathcal{M}^\vee) \rightrightarrows \mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{P}\otimes\mathcal{P}^\vee) \tag{10}$$

(see 6.3.3.2). The kernel of a pair of morphisms between two affine local constructions is an affine local construction. ■

Let $\mathcal{M}$ and $\mathcal{P}$ be $\mathcal{O}$-modules on $\mathfrak{X}$. We denote by $Iso^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}}$ the functor $\mathfrak{X}\longrightarrow\widetilde{\mathcal{E}}^\wedge\times_{\mathcal{E}}\mathcal{A}$ which assigns to every pair (U,x), $U\in Ob\mathcal{A}$, $x\in Ob\mathfrak{X}_U$, the functor

$$Iso^{\mathcal{O}(U)}_{\mathcal{M}(U,x),\mathcal{P}(U,x)} : \mathcal{O}(U)\backslash Alg_k \longrightarrow Sets$$

that assigns every ring morphism $\mathcal{O}(U)\xrightarrow{\phi}\mathcal{R}$ the set of isomorphisms

$$\phi^*(\mathcal{M}(U,x)) \xrightarrow{\sim} \phi^*(\mathcal{P}(U,x)).$$

6.3.4.2. Proposition. *Let $\mathcal{M}(U,x)$ and $\mathcal{P}(U,x)$ be projective $\mathcal{O}(U)$-modules of finite type for all $U\in Ob\mathcal{A}$ and $x\in Ob\mathfrak{X}$. Then the functor $Iso^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ is an affine local construction on $\mathfrak{X}$.*

Proof. The functor $Iso^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ is naturally identified with the fiber product of the pair of morphisms

$$G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}} \xrightarrow{\ \varphi\ } Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P})\times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M}) \xleftarrow{\ \psi\ } G^{\mathcal{O},\mathfrak{X}}_{\mathcal{P},\mathcal{M}}, \tag{11}$$

where φ is the natural embedding, ψ is the composition of the natural imbedding

$$G^{\mathcal{O},\mathfrak{X}}_{\mathcal{P},\mathcal{M}} \xrightarrow{\ \varphi\ } Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M})\times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P})$$

and the isomorphism

$$Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M})\times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P}) \xrightarrow{\sim} Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{M},\mathcal{P})\times Hom_{\mathcal{O},\mathfrak{X}}(\mathcal{P},\mathcal{M})$$

defined by $(u,v)\longmapsto(v,u)$. By 6.3.3.2, the diagram (11) is isomorphic to the diagram

$$G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}} \longrightarrow \mathbb{V}_{\mathcal{O},\mathfrak{X}}(\mathcal{M}\otimes\mathcal{P}^\vee\oplus\mathcal{P}\otimes\mathcal{M}^\vee) \longleftarrow G^{\mathcal{O},\mathfrak{X}}_{\mathcal{P},\mathcal{M}}. \tag{12}$$

By 6.3.4.1, the functors $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}}$ and $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{P},\mathcal{M}}$ are affine local constructions on $\mathfrak{X}$. Therefore the pull-back of (5) is an affine local construction on $\mathfrak{X}$. ■

6.4. Grassmannians. For any associative unital ring S and any pair $M,\ P$ of left S-modules, we define the 'Grassmannian' $Gr^S_{M,P}$ as the functor $S\backslash Alg_k \longrightarrow Sets$ which assigns to every ring morphism $S \xrightarrow{\phi} T$ the isomorphism class of coretractions (– splittable epimorphisms) $\phi^*(M) \longrightarrow \phi^*(P)$. We have a canonical exact sequence of functors

$$\mathfrak{R}^S_{M,P} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G^S_{M,P} \xrightarrow{\ \pi\ } Gr^S_{M,P}. \tag{1}$$

Here π is the natural epimorphism (sending a split pair of arrows $\phi^*(P) \xrightarrow{v} \phi^*(M) \xrightarrow{u} \phi^*(P)$ to the class of the epimorphism u) and $\mathfrak{R}^S_{M,P}$ is the 'functor of relations', i.e.

$$\mathfrak{R}^S_{M,P} = G^S_{M,P} \prod_{Gr^S_{M,P}} G^S_{M,P}.$$

Fix a ringed category $(\mathcal{A}, \mathcal{O})$ and a fibred category $\mathfrak{X}$ over $\mathcal{A}$. Let $\mathcal{M}$ and $\mathcal{P}$ be $\mathcal{O}$-modules on $\mathfrak{X}$. Since the functors $Gr^S_{M,P}$, $G^S_{M,P}$, and $\mathfrak{R}^S_{M,P}$ in the diagram (1) depend functorially on S, M, and P, they determine functors $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}^\wedge \times_{\mathcal{E}} \mathcal{A}$ which we denote resp. by $Gr^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ and $\mathfrak{R}^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$. Thus, the diagram (1) induces an exact diagram

$$\mathfrak{R}^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}} \xrightarrow{\ \pi\ } Gr^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}} \tag{2}$$

of functors $\mathfrak{X} \longrightarrow \widetilde{\mathcal{E}}^\wedge \times_{\mathcal{E}} \mathcal{A}$.

6.4.1. Proposition. *Let $\mathcal{M}(U,x)$ and $\mathcal{P}(U,x)$ be projective $\mathcal{O}(U)$-modules of finite type for all $U \in Ob\mathcal{A}$ and $x \in Ob\mathfrak{X}$. Then (2) is the exact diagram of local constructions on $\mathfrak{X}$ (in particular, $Gr^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{P}}$ is a local construction), and the local constructions $\mathfrak{R}^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ and $G^{\mathcal{O},\mathfrak{X}}_{\mathcal{M},\mathcal{V}}$ are affine.*

Proof. If the S-modules M and V are projective of finite type, then, by III.6.3, the functor $\mathfrak{R}^S_{M,P}$ (and $G^S_{M,P}$) is representable. By III.6.4.1, all three functors in (1) are compatible with the base change. The latter means that if $S \xrightarrow{\phi} T$ is a ring morphism, and $\mathfrak{G}^S_{M,P}$ denotes any of the functors in the diagram (1), then

$$\mathfrak{G}^T_{\phi^*(M),\phi^*(P)} \simeq \mathfrak{G}^S_{M,P} \prod_{S^\vee} T^\vee.$$

This implies the assertion. ■

6.5. Flag varieties and generic flag varieties. We leave to the reader the generalization of 6.4, in particular, 6.4.1, to flag varieties and the versions for generic flag varieties, in particular, the generic Grassmannians.

Chapter V
Noncommutative Grassmannians and Related Constructions.

The main purpose of this chapter is presenting a general construction of noncommutative spaces and studying its first properties. This construction can be regarded as a far-reaching generalization of Grassmannians and flag varieties discussed in Chapter III.

There is an important aspect of noncommutative geometry which we take into consideration here. Local objects of commutative algebraic geometry, affine schemes, form a category dual to the category of commutative rings. The latter might be regarded as commutative algebras in the symmetric monoidal category of $\mathbb{Z}$-modules. The category $\mathbf{Aff}_k$ of noncommutative affine schemes over a commutative algebra k is, by definition, the category dual to the category of associative unital k-algebras, which are, precisely, associative unital algebras in the symmetric monoidal category of k-modules. One of particularities of noncommutative geometry is that some interesting noncommutative spaces 'live' in non-trivial, non-symmetric monoidal categories. For instance, the quantum flag variety of a simple Lie algebra $\mathfrak{g}$ is a scheme in the monoidal category of $\mathbb{Z}^r$-graded vector spaces endowed with a braiding determined by the Cartan matrix of $\mathfrak{g}$ (cf. [LR2]). Here we adopt a framework which allows to take these phenomena into account and gives to our constructions an appropriate level of generality. The framework is as follows: instead of the monoidal category of modules over a commutative unital ring k, we take an arbitrary monoidal category $\mathcal{A}^\sim$ together with its action on a category $\mathcal{C}_X$ and define the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of *affine schemes in* $\mathcal{A}^\sim$ as the category opposite to the category $Alg\mathcal{A}^\sim$ of associative unital algebras in $\mathcal{A}^\sim$. The formalism of the first sections of Chapter III singles out, among presheaves of sets on $\mathbf{Aff}_{\mathcal{A}^\sim}$, the locally affine spaces and schemes. For any associative unital algebra $\mathcal{S}$ in $\mathcal{A}^\sim$, we define the category $\mathcal{S}-mod_X$ of $\mathcal{S}$*-modules in* C_X, which is viewed as the category of *quasi-coherent modules* on the affine scheme $\mathcal{S}^\vee$ corresponding to the algebra $\mathcal{S}$. Applying the generalities of Chapter IV, we obtain, for an an arbitrary presheaf of sets $\mathfrak{X}$ on the category $\mathbf{Aff}_{\mathcal{A}^\sim}$, the category $Qcoh_{\mathfrak{X}}^{\Phi^\sim}$ of *quasi-coherent modules* on $\mathfrak{X}$ related to the action $\Phi^\sim$ of the monoidal category $\mathcal{A}^\sim$ on the category $\mathcal{C}_X$.

A standard noncommutative example is $\mathcal{C}_X = R - mod$ for an associative unital k-algebra R and $\mathcal{A}^\sim = R^e - mod^\sim = (R^e - mod, \otimes_R, R)$ – the monoidal category of left modules over $R^e = R \otimes_k R^o$, acting on $R - mod$ by tensoring over R. A unital associative algebra S in the monoidal category $R^e - mod^\sim$, that is a unital k-algebra morphism $R \longrightarrow S$; and the category of S-modules in $\mathcal{C}_X$ is naturally equivalent to the category $S - mod$ of left S-modules. This is the setting behind the constructions of Chapter III.

A straightforward extension of the previous example to non-affine non-commutative setting is the monoidal category of continuous (that is having a right adjoint) k-linear endofunctors of a k-linear category naturally acting on this category. This setting is implicitly used in Chapter I while studying affine morphisms.

A curious non-trivial example is the monoidal category $\mathcal{A}^{\sim} = \mathbf{S} - Vec_k^{\sim}$ of $\mathbf{S}$-*spaces* whose objects are families of representations of all symmetric groups, $\mathbf{S}_n,\ n \geq 1$, in vector spaces over a field k, and the tensor product is the so called *plethysm product.* Algebras in the monoidal category $\mathbf{S} - Vec_k^{\sim}$ are called k-*linear operads.* The monoidal category of $\mathbf{S}$-spaces acts canonically on the category $C = Vec_k$ of k-vector spaces. Modules in Vec_k over an operad $\mathcal{P}$ are traditionally called $\mathcal{P}$-*algebras.*

Note that some basic notions of *commutative* algebraic geometry in symmetric monoidal categories (starting with the site of affine schemes with **fpqc** topology) were sketched by P. Deligne in connection with the characterization of Tannakian categories [Dl]. Our monoidal categories are usually not symmetric and, even in the case of symmetric monoidal categories, our algebras are usually non-commutative.

Appendix 2 and Section 1 summarize preliminaries on monoidal categories and their actions, as well as related properties of algebras in monoidal categories and modules over them. In Section 2, we define affine schemes in monoidal categories and introduce vector fibers of bimodules and pair of "admissible" objects. The latter are the main building blocks for constructing of affine and non-affine varieties of this Chapter. In Section 3, we introduce and study presheaves of morphisms and isomorphisms between functors to a category endowed with an action of a monoidal category. Under certain finiteness conditions, these presheaves are representable by affine schemes, which are limits of finite diagrams of vector fibers. In Section 4, we describe a certain combinatorial construction of affine schemes, which is used later for constructing non-affine varieties. One of its special cases is the *universal localization.* In the next two Sections, we make a step towards non-affine varieties sketching natural generalizations of varieties discussed in Chapter III. Section 5 contains the construction of Grassmannians and generic Grassmannians. In Section 6, we extend to this setting generic flag varieties and Stiefel schemes. In Section 7, we introduce a construction of generalized Grassmannian type spaces. All generic and non-generic flag varieties of Chapter III and their generalizations appeared in the previous sections of this chapter are special cases of this construction. In Section 8, we discuss the conditions on our combinatorial data, which guarantee (formal) smoothness of the corresponding varieties. In Section 9, we apply the generalities of Chapter IV to introduce the categories of quasi-coherent modules and bimodules associated with an action of a monoidal category.

An aesthetically nice feature is that passing from affine to non-affine objects, we still stay inside of our initial pattern: the monoidal category of quasi-coherent bimodules acts on the category of quasi-coherent modules.

1. Preliminaries on algebras in monoidal categories.

1.0. Monoidal categories and their actions. Recall that an action of a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ on a category $\mathcal{C}_X$ is a monoidal functor $\Phi^\sim = (\Phi, \phi, \phi_0)$ from $\mathcal{A}^\sim$ to the monoidal category $(End(\mathcal{C}_X), \circ, Id_{\mathcal{C}_X})$ of endofunctors of $\mathcal{C}_X$.

1.0.1. Assumptions.

1.0.1.1. We usually assume that the categories $\mathcal{A}$ and $\mathcal{C}_X$ have cokernels of reflexive pairs of arrows (see I.4.2.1) and, for every $M \in Ob\mathcal{A}$, the functors

$$\mathcal{A} \xrightarrow{M\odot -} \mathcal{A} \xleftarrow{-\odot M} \mathcal{A} \quad \text{and} \quad \mathcal{C}_X \xrightarrow{\Phi(M)} \mathcal{C}_X \tag{1}$$

are *weakly* continuous, i.e. they preserve cokernels of reflexive pairs.

1.0.1.2. In addition, we assume, starting from the middle of Section 1, that the categories $\mathcal{A}$ and $\mathcal{C}_X$ have countable coproducts and the functors (1) preserve them.

1.0.2. The invariance of the assumptions under base change. Suppose that the assumption 1.0.1.1 holds. Then, for any associative unital algebra $\mathcal{R}$ in the monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$, an action $\Phi^\sim = (\Phi, \phi, \phi_0)$ of $\mathcal{A}^\sim$ on a category $\mathcal{C}_X$ gives rise to an action $\Phi^\sim_{\mathcal{R}} = (\Phi^{\mathcal{R}}, \phi^{\mathcal{R}}, \phi_0^{\mathcal{R}})$ of the monoidal category

$$\mathfrak{End}_{\mathcal{A}^\sim}(\mathcal{R}) = \mathcal{R} - bim^\sim = (\mathcal{R} - bim, \odot_{\mathcal{R}}, \mathfrak{a}; \mathcal{R}, \mathfrak{l}^{\mathcal{R}}, \mathfrak{r}^{\mathcal{R}})$$

of $\mathcal{R}$-bimodules on the category $\mathcal{R} - mod_X$ of $\mathcal{R}$-modules in $\mathcal{C}_X$ (see A2.10).

1.0.2.1. Notice that the monoidal category $\mathcal{R} - bim^\sim$, the category $\mathcal{R} - mod_X$ and the action $\Phi^\sim_{\mathcal{R}}$ inherit the property 1.0.1.1.

1.0.2.2. If the triple $(\mathcal{A}^\sim, \Phi^\sim, \mathcal{C}_X)$ satisfies the condition 1.0.1.2, then same holds for $(\mathcal{R} - bim^\sim, \Phi^\sim_{\mathcal{R}}, \mathcal{R} - mod_X)$.

1.0.3. Notations and basic facts. We refer to Appendix 2 for notations, notions and basic facts on monoidal categories and their actions, which are used in the chapter. One might browse Appendix 2 and then return to its specific parts when needed.

1.1. Limits and colimits of algebras in a monoidal category. Fix a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$. Let $\mathfrak{Alg}\mathcal{A}^\sim \xrightarrow{\widetilde{\mathfrak{f}}_*} \mathcal{A}$ be the forgetful functor

$$\mathcal{R} = (R, \mu_{\mathcal{R}}) \longmapsto R, \quad (\mathcal{R} \xrightarrow{\varphi} \mathcal{S}) \longmapsto (\widetilde{\mathfrak{f}}_*(\mathcal{R}) \xrightarrow{\varphi} \widetilde{\mathfrak{f}}_*(\mathcal{S})).$$

from the category $\mathfrak{Alg}\mathcal{A}^\sim$ of algebras in $\mathcal{A}^\sim$ (cf. A2.5) to the category $\mathcal{A}$.

1.1.1. Proposition. *Let* $\mathcal{D} \xrightarrow{\mathfrak{D}} \mathfrak{Alg}\mathcal{A}^{\sim}$ *be a diagram such that there exists the limit of* $\mathcal{D} \xrightarrow{\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}} \mathcal{A}$. *Then there exists the limit of the diagram* $\mathfrak{D}$.

Proof. Let $\mathfrak{p}_x$ denote the projection $\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}) \longrightarrow \widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x)$, $x \in Ob\mathcal{D}$. We have the set of morphisms

$$(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \odot (\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \xrightarrow{\mathfrak{p}_x \odot \mathfrak{p}_x} (\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x)) \odot (\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x)) \xrightarrow{\mu_{\mathfrak{D}(x)}} \widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x), \quad x \in Ob\mathcal{D}.$$

such that, for any morphism $x \xrightarrow{\xi} y$ of $\mathcal{D}$, the diagram

$$\begin{array}{ccccc}
(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \odot (\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) & \xrightarrow{\mathfrak{p}_x \odot \mathfrak{p}_x} & (\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x)) \odot (\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x)) & \xrightarrow{\mu_{\mathfrak{D}(x)}} & \widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x) \\
id \downarrow & & \widetilde{\mathfrak{f}}_*\mathfrak{D}(\xi) \odot \widetilde{\mathfrak{f}}_*\mathfrak{D}(\xi) \downarrow & & \downarrow \widetilde{\mathfrak{f}}_*\mathfrak{D}(\xi) \\
(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \odot (\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) & \xrightarrow{\mathfrak{p}_y \odot \mathfrak{p}_y} & (\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(y)) \odot (\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(y)) & \xrightarrow{\mu_{\mathfrak{D}(y)}} & \widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(y)
\end{array}$$

commutes: the right square commutes, because $\mathfrak{D}(\xi)$ is an algebra morphism; and the left square commutes, because $\big(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}) \xrightarrow{\mathfrak{p}_x} \widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x) \mid x \in Ob\mathcal{D}\big)$ is a cone, that is $\mathfrak{p}_y = \widetilde{\mathfrak{f}}_*\mathfrak{D}(\xi) \circ \mathfrak{p}_x$. This shows that the compositions

$$(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \odot (\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \xrightarrow{\mu_{\mathfrak{D}(y)} \circ (\mathfrak{p}_x \odot \mathfrak{p}_x)} \widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x), \qquad x \in Ob\mathcal{D},$$

form a cone. Since the cone $\big(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}) \xrightarrow{\mathfrak{p}_x} \widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x) \mid x \in Ob\mathcal{D}\big)$ is universal, there exists a unique morphism

$$(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \odot (\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \xrightarrow{\mu_{\mathfrak{D}}} \lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})$$

such that the diagram

$$\begin{array}{ccc}
(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \odot (\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) & \xrightarrow{\mu_{\mathfrak{D}}} & \lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}) \\
id \downarrow & & \downarrow \mathfrak{p}_x \\
(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) \odot (\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D})) & \xrightarrow{\mu_{\mathfrak{D}(y)} \circ (\mathfrak{p}_x \odot \mathfrak{p}_x)} & \widetilde{\mathfrak{f}}_* \circ \mathfrak{D}(x)
\end{array}$$

commutes for every $x \in Ob\mathcal{D}$. If follows from the argument above that the algebra $(\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}), \mu_{\mathfrak{D}})$ is the limit of the diagram $\mathfrak{D}$. ∎

1.1.2. Note. If the diagram $\mathfrak{D}$ in 1.1 takes values in the subcategory of unital (resp. associative) algebras, then the algebra $\lim \mathfrak{D} = (\lim(\widetilde{\mathfrak{f}}_* \circ \mathfrak{D}), \mu_{\mathfrak{D}})$ is unital (resp. associative). In particular, the limit of a diagram $\mathfrak{D}$ taking values in the category $Alg\mathcal{A}^\sim$ of associative unital algebras exists and is an associative unital algebra, provided the limit of its composition with the forgetful functor exists.

1.2. Tensor algebras. Let $Alg\mathcal{A}^\sim \xrightarrow{\mathfrak{f}_*} \mathcal{A}$ be the forgetful functor – the restriction of the functor $\mathfrak{Alg}\mathcal{A}^\sim \xrightarrow{\widetilde{\mathfrak{f}}_*} \mathcal{A}$ to the subcategory $Alg\mathcal{A}^\sim$ of associative unital algebras in $\mathcal{A}^\sim$ (see A2.5.2).

1.2.1. Proposition. *(a) Let an object E of the category $\mathcal{A}$ be such that the functors*

$$\mathcal{A} \xrightarrow{E\odot -} \mathcal{A} \xrightarrow{-\odot E} \mathcal{A}$$

preserve countable coproducts; and let there exists a coproduct $\coprod_{n\geq 0} E^{\odot n}$. Then the functor $\mathcal{A}(E, \mathfrak{f}_*(-))$ from $Alg\mathcal{A}^\sim$ to Sets is corepresentable.

(b) Let $\{E_i \mid i \in J\}$ be a set of objects of the category $\mathcal{A}$ such that there exists a coproduct $E_J = \coprod_{i\in J} E_i$ and the objects E_J and E_i, $i \in J$, satisfy the conditions (a) above; so that the tensor algebras $T(E_J)$, $T(E_i)$, $i \in J$, exist. Then the algebra $T(E_J)$ is the coproduct of the set of algebras $\{T(E_i) \mid i \in J\}$.

Proof. (a) Denote by $T(E)$ the algebra $(\coprod_{n\geq 0} E^{\odot n}, \mu_E)$, where the multiplication μ_E is given by the identical morphisms $E^{\odot n} \odot E^{\odot m} \longrightarrow E^{\odot(m+n)}$. For any associative unital algebra (R, μ) in $\mathcal{A}^\sim$, the natural map $\mathcal{A}(E, R) \longrightarrow Alg\mathcal{A}^\sim(T(E), (R, \mu))$ is an isomorphism functorially depending on the algebra (R, μ).

(b) It follows from (a) that

$$Alg\mathcal{A}^\sim(T(\coprod_{i\in J} E_i), -) \simeq \mathcal{A}(\coprod_{i\in J} E_i, \mathfrak{f}_*(-)) \simeq$$
$$\prod_{i\in J} \mathcal{A}(E_i, \mathfrak{f}_*(-)) \simeq \prod_{i\in J} Alg\mathcal{A}^\sim(T(E_i), -),$$

hence the assertion. ∎

1.2.2. Corollary. *Assume that the category $\mathcal{A}$ has countable coproducts and that the multiplication $\mathcal{A} \times \mathcal{A} \xrightarrow{\odot} \mathcal{A}$ preserves countable coproducts in both arguments. Then the forgetful functor $Alg\mathcal{A}^\sim \xrightarrow{\mathfrak{f}_*} \mathcal{A}$ has a left adjoint.*

Proof. The assertion follows from 1.2.1(a): the left adjoint to the functor $\mathfrak{f}_*$ maps each object L of the category $\mathcal{A}$ to the tensor algebra $T(L)$. of L. ∎

1.3. Proposition. *Suppose that the category $\mathcal{A}$ has countable coproducts and that the multiplication $\mathcal{A} \times \mathcal{A} \xrightarrow{\odot} \mathcal{A}$ preserves countable coproducts in both arguments. Then the category $Alg\mathcal{A}^\sim$ of associative unital algebras in $\mathcal{A}^\sim$ has finite coproducts.*

Proof. (a) Let $\mathcal{S}$ be a unital associative algebra in $\mathcal{A}^\sim$. By 1.0.2, for every $\mathcal{S}$-bimodule $\mathcal{M}$, there exists the tensor algebra $T_{\mathcal{S}}(\mathcal{M})$. In particular, for any object $\mathcal{L}$ of the category $\mathcal{A}$, there exists the tensor algebra $T_{\mathcal{S}}(\widetilde{\eta}^*_{\mathcal{S}}(\mathcal{L}))$ of the $\mathcal{S}$-bimodule $\widetilde{\eta}^*_{\mathcal{S}}(\mathcal{L}) = \mathcal{S} \odot \mathcal{L} \odot \mathcal{S}$.

The commutative square

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{\eta_{\mathcal{S}}} & \mathcal{S} \\ \big\downarrow & cocart & \big\downarrow \\ T(\mathcal{L}) & \longrightarrow & T_{\mathcal{S}}(\widetilde{\eta}^*_{\mathcal{S}}(\mathcal{L})) \end{array} \qquad (1)$$

is universal, that is $T_{\mathcal{S}}(\widetilde{\eta}^*_{\mathcal{S}}(\mathcal{L})) \simeq \mathcal{S} \star T(\mathcal{L})$.

This follows from the fact that the functor $T_{\mathcal{S}}(-)$ is left adjoint to the forgetful functor from the category of associative unital algebras in $\mathcal{S} - bim^\sim$ to the category $\mathcal{S} - bim$ of $\mathcal{S}$-bimodules the observation that every morphism $\mathcal{S} \longrightarrow \mathcal{B}$ of unital associative algebras makes $\mathcal{B}$ a unital associative algebra in the monoidal category $\mathcal{S} - bim^\sim$.

Of course, this can be easily seen directly: every pair of unital algebra morphisms $T(\mathcal{L}) \longrightarrow \mathcal{B} \longleftarrow \mathcal{S}$ is uniquely determined by the pair of morphisms $\mathcal{L} \longrightarrow \mathfrak{f}_*(\mathcal{B})$, $\mathcal{S} \longrightarrow \mathcal{B}$, which, in turn determines (and is determined by) an $\mathcal{S}$-bimodule morphism $\mathcal{S}\odot\mathcal{L}\odot\mathcal{S} \longrightarrow \mathcal{B}$.

(b1) Under the assumptions, the forgetful functor $Alg\mathcal{A}^\sim \xrightarrow{\mathfrak{f}_*} \mathcal{A}$ has a left adjoint, $\mathfrak{f}^*$. The pair of adjoint functors $\mathfrak{f}^*$, $\mathfrak{f}_*$ satisfies the conditions of Beck's theorem; that is the forgetful functor $Alg\mathcal{A}^\sim \xrightarrow{\mathfrak{f}_*} \mathcal{A}$ is isomorphic to the forgetful functor from the category of modules over the associated monad $\mathcal{F}_{\mathfrak{f}} = (F_{\mathfrak{f}}, \mu_{\mathfrak{f}})$, where $F_{\mathfrak{f}} = \mathfrak{f}_*$.

In particular, for every algebra $\mathcal{R}$ in $\mathcal{A}^\sim$, we have a canonical exact diagram

$$\mathfrak{f}^*\mathfrak{f}_*\mathfrak{f}^*\mathfrak{f}_*(\mathcal{R}) \;\underset{\mathfrak{f}^*\mu_{\mathcal{R}}}{\overset{\mathfrak{f}^*\mathfrak{f}_*\epsilon_{\mathfrak{f}}(\mathcal{R})}{\rightrightarrows}}\; \mathfrak{f}^*\mathfrak{f}_*(\mathcal{R}) \longrightarrow \mathcal{R} \qquad (2)$$

which the forgetful functor $Alg\mathcal{A}^\sim \xrightarrow{\mathfrak{f}_*} \mathcal{A}$ maps to an exact diagram.

(b2) It follows from (a) above that the pair of arrows

$$\mathcal{S} \star \big(\mathfrak{f}^*\mathfrak{f}_*\mathfrak{f}^*\mathfrak{f}_*(\mathcal{R}) \;\underset{\mathfrak{f}^*\mu_{\mathcal{R}}}{\overset{\mathfrak{f}^*\mathfrak{f}_*\epsilon_{\mathfrak{f}}(\mathcal{R})}{\rightrightarrows}}\; \mathfrak{f}^*\mathfrak{f}_*(\mathcal{R})\big) \qquad (3)$$

is well defined. Since this pair of arrows is reflexive (because the pair of arrows in (2) is reflexive), it has, by hypothesis, a cokernel. The latter is isomorphic to $\mathcal{S} \star \mathcal{R}$. ∎

1.3.1. Remarks. (i) It follows from the part (a) of the argument that the diagram (3) is isomorphic to the diagram

$$T_{\mathcal{S}} \circ \widetilde{\eta}_{\mathcal{S}}^{*}\big(\mathfrak{f}_*\mathfrak{f}^*\mathfrak{f}_*(\mathcal{R}) \overset{\mathfrak{f}_*\epsilon_{\mathfrak{f}}(\mathcal{R})}{\underset{\mu_{\mathcal{R}}}{\rightrightarrows}} \mathfrak{f}_*(\mathcal{R})\big) \tag{4}$$

(ii) The role of the algebras $\mathcal{S}$ and $\mathcal{R}$ in the presentation of their star product in the argument of 1.3 is not symmetric. If $\mathcal{S}$ is another algebra, then the coproduct $\mathcal{S} \star \mathcal{R}$ of $\mathcal{S}$ and $\mathcal{R}$ is the colimit of the coproduct of the diagrams

$$\mathfrak{f}^*\mathfrak{f}_*\mathfrak{f}^*\mathfrak{f}_*(\mathcal{R}) \overset{\mathfrak{f}^*\mathfrak{f}_*\epsilon_{\mathfrak{f}}(\mathcal{R})}{\underset{\mathfrak{f}^*\mu_{\mathcal{R}}}{\rightrightarrows}} \mathfrak{f}^*\mathfrak{f}_*(\mathcal{R}) \quad \text{and} \quad \mathfrak{f}^*\mathfrak{f}_*\mathfrak{f}^*\mathfrak{f}_*(\mathcal{S}) \overset{\mathfrak{f}^*\mathfrak{f}_*\epsilon_{\mathfrak{f}}(\mathcal{S})}{\underset{\mathfrak{f}^*\mu_{\mathcal{S}}}{\rightrightarrows}} \mathfrak{f}^*\mathfrak{f}_*(\mathcal{S}).$$

which is isomorphic to the diagram

$$\begin{array}{c} \mathfrak{f}^*\big(\mathfrak{f}_*\mathfrak{f}^*\mathfrak{f}_*(\mathcal{R}) \coprod \mathfrak{f}_*\mathfrak{f}^*\mathfrak{f}_*(\mathcal{S})\big) \\ \downarrow\downarrow \qquad \downarrow\downarrow \\ \mathfrak{f}^*\big(\mathfrak{f}_*(\mathcal{R}) \coprod \mathfrak{f}_*(\mathcal{S})\big) \end{array} \tag{5}$$

The colimit of the diagram (5) is isomorphic to $\mathcal{S} \star \mathcal{R}$.

2. Affine schemes in a monoidal category. Vector fibers.

2.0. The setting. We fix a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ and its action on a svelte category $\mathcal{C}_X$ – that is a monoidal functor $\Phi^{\sim} = (\Phi, \phi, \phi_0)$ from $\mathcal{A}^{\sim}$ to the strict monoidal category $\mathit{End}(\mathcal{C}_X)^{\sim} = (\mathit{End}(\mathcal{C}_X), \circ, Id_{\mathcal{C}_X})$.

We assume that they satisfy the following conditions:

(a) The categories $\mathcal{A}$ and $\mathcal{C}_X$ have cokernels of reflexive pairs of arrows and countable coproducts, and the functors $\mathcal{A} \xrightarrow{\Phi} \mathit{End}(\mathcal{C}_X)$ and $\mathcal{A} \times \mathcal{A} \xrightarrow{\odot} \mathcal{A}$ preserve countable coproducts and cokernels of reflexive pairs of arrows.

(b) The functor $\mathcal{A} \xrightarrow{\Phi} \mathit{End}(\mathcal{C}_X)$ takes values in the category $\mathit{End}^w(\mathcal{C}_X)$ of weakly continuous (i.e. preserving cokernels of reflexive pairs of arrows) endofunctors of $\mathcal{C}_X$.

2.1. Affine schemes. Fix a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$. The category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ of affine schemes in $\mathcal{A}^{\sim}$ is the category opposite to the category $\mathit{Alg}\mathcal{A}^{\sim}$ of associative unital algebras in $\mathcal{A}^{\sim}$. For an associative unital algebra $\mathcal{R}$, we denote by $\mathcal{R}^{\vee}$ the corresponding affine scheme.

2.2. Vector fibers of objects. Let $\mathcal{L}$ be an object of the category $\mathcal{A}$ such that there exists the coproduct of $\coprod_{n\geq 0} \mathcal{L}^{\odot n}$, or, what is the same, there exists the tensor algebra of the object $\mathcal{L}$. The affine scheme $T(\mathcal{L})^\vee$ of the tensor algebra $T(\mathcal{L})$ is called the *vector fiber* of $\mathcal{L}$ and is denoted by $\mathbb{V}(\mathcal{L})$. It follows from 1.2.1(b) that

$$\mathbb{V}(\coprod_{i\in J} \mathcal{L}_i) \simeq \prod_{i\in J} \mathbb{V}(\mathcal{L}_i),$$

provided that the vector fibers $\mathbb{V}(\coprod_{i\in J} \mathcal{L}_i)$ and $\mathbb{V}(\mathcal{L}_i)$, $i \in J$, are defined.

2.3. Admissible pairs of objects. Finite objects. We say that a pair (M, L) of objects of the category $\mathcal{C}_X$ is $\Phi^\sim$-*admissible*, if the functor $\mathcal{C}_X(M, \Phi(-)(L))$ from $\mathcal{A}$ to *Sets* is corepresentable, i.e. there is an object $L^\wedge M$ of $\mathcal{A}$ (defined uniquely up to isomorphism) and an isomorphism $\mathcal{C}_X(M, \Phi(-)(L)) \simeq \mathcal{A}(L^\wedge M, -)$.

2.3.0. Finite objects. We call an object L of the category $\mathcal{C}_X$ *finite,* if (M, L) is an admissible pair for any object M of the category $\mathcal{C}_X$.

2.3.1. Example. Let $\mathcal{C}_X$ be the category of left modules over an associative k-algebra R and $\mathcal{A}^\sim$ the monoidal category of R^e-modules, $R^e = R \otimes_k R^o$, acting on the category $\mathcal{C}_X$ via $\otimes_R$. Then the finite objects of the category $\mathcal{C}_X$ are projective R-modules of finite type. If L is a projective R-module of finite type and M an arbitrary left R-module, then $L^\wedge M \simeq M \otimes_k L^*$, where L^* denotes the dual to L (right) R-module: $L^* = Hom_R(L, R)$. In particular, if $L = R$, then $L^\wedge M$ is isomorphic to the R-bimodule $M \otimes_k R_r$, where R_r is R regarded as a right R-module.

2.3.2. Example. Let $\Phi^\sim$ be the standard left action of $\mathcal{A}^\sim$ on $\mathcal{A}$ (see A6.1.). Let $L, M \in Ob\mathcal{A}$. By definition, the pair (M, L) is admissible iff the functor

$$\mathcal{A} \longrightarrow \mathbf{Sets}, \quad F \longmapsto \mathcal{A}(M, F \odot L))$$

is corepresentable. Suppose that L is a *finite object*, i.e. there exists an object $L^!$ such that the functor $L^! \odot -$ is a right adjoint to $- \odot L$ (see A6.5.2). Then $F \odot L \simeq \mathfrak{Hom}(L^!, F)$, hence

$$\mathcal{A}(M, F \odot L)) \simeq \mathcal{A}(M, \mathfrak{Hom}(L^!, F)) \simeq \mathcal{A}(M \odot L^!, F)$$

functorially in F. In other words, L is a finite object of the monoidal category $\mathcal{A}^\sim$ in the conventional sense (see A6.5.2) iff it is a finite object in the sense of 2.3.

2.3.3. Proposition. *Let $\mathcal{S}$ and $\mathcal{R}$ be associative unital algebras in $\mathcal{A}^{\sim}$ and $\mathcal{R} \xrightarrow{\varphi} \mathcal{S}$ a unital algebra morphism. The inverse image functor*

$$\mathcal{R} - mod_X \xrightarrow{\varphi_X^*} \mathcal{S} - mod_X$$

maps $\Phi_{\mathcal{R}}^{\sim}$-admissible pairs of objects of the category $\mathcal{R} - mod_X$ to $\Phi_{\mathcal{S}}^{\sim}$-admissible pairs of objects of the category $\mathcal{S} - mod_X$.

The functor φ_X^* maps finite objects of $\mathcal{R} - mod_X$ to finite objects of $\mathcal{S} - mod_X$.

Proof. (a) Let $(\mathcal{L}, \mathcal{M})$ be a $\Phi_{\mathcal{R}}^{\sim}$-admissible pair of objects of the category $\mathcal{R} - mod_X$. Then, for every $\mathcal{S}$-bimodule $\mathfrak{N}$, we have the following canonical isomorphisms:

$$\begin{array}{ccc}
\mathcal{S} - mod_X(\varphi_X^*(\mathcal{M}), \Phi_{\mathcal{S}}^{\sim}(\mathfrak{N})(\varphi_X^*(\mathcal{L}))) & \xrightarrow{\sim} & \mathcal{R} - mod_X(\mathcal{M}, \varphi_{X*}\Phi_{\mathcal{S}}^{\sim}(\mathfrak{N})(\varphi_X^*(\mathcal{L}))) \\
 & & \downarrow \wr \\
\mathcal{R} - bim(\mathcal{L}^\wedge\mathcal{M}, \widetilde{\varphi}_*(\mathfrak{N})) & \xleftarrow{\sim} & \mathcal{R} - mod_X(\mathcal{M}, \Phi_{\mathcal{R}}^{\sim}(\widetilde{\varphi}_*(\mathfrak{N}))(\mathcal{L}))) \\
\wr\downarrow & & \\
\mathcal{S} - bim(\widetilde{\varphi}^*(\mathcal{L}^\wedge\mathcal{M}), \mathfrak{N}) & &
\end{array}$$

where $\widetilde{\varphi}_*$ is the restriction of scalars functor $\mathcal{S} - bim \longrightarrow \mathcal{R} - bim$ (see A10.1) and $\widetilde{\varphi}^*$ its left adjoint, which maps an $\mathcal{R}$-bimodule $\mathfrak{L}$ to an $\mathcal{S}$-bimodule $\mathcal{S} \odot_{\mathcal{R}} \mathfrak{L} \odot_{\mathcal{R}} \mathcal{S}$.

This shows that the object $(\varphi_X^*(\mathcal{L}))^\wedge\varphi_X^*(\mathcal{M})$ exists and is isomorphic to $\widetilde{\varphi}^*(\mathcal{L}^\wedge\mathcal{M})$.

(b) Suppose that $\mathcal{L}$ is a finite object of the category $\mathcal{R} - mod_X$ and $\mathcal{N}$ an arbitrary object of $\mathcal{S} - mod_X$. We have a canonical exact diagram

$$(\varphi_X^*\varphi_{X*})^2(\mathcal{N}) \underset{\epsilon\varphi_X^*\varphi_{X*}(\mathcal{N})}{\overset{\varphi_X^*\varphi_{X*}\epsilon(\mathcal{N})}{\rightrightarrows}} \varphi_X^*\varphi_{X*}(\mathcal{N}) \xrightarrow{\epsilon(\mathcal{N})} \mathcal{N}$$

which give rise, according to (a), to a commutative diagram

$$\begin{array}{ccc}
\mathcal{S} - mod_X(\mathcal{N}, \Phi_{\mathcal{S}}^{\sim}(-)(\varphi_X^*(\mathcal{L}))) & & \\
\downarrow & & \\
\mathcal{S} - mod_X(\varphi_X^*(\varphi_{X*}(\mathcal{N})), \Phi_{\mathcal{S}}^{\sim}(-)(\varphi_X^*(\mathcal{L}))) & \xrightarrow{\sim} & \mathcal{S} - bim(\widetilde{\varphi}^*(\mathcal{L}^\wedge(\varphi_{X*}(\mathcal{N}))), -) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathcal{S} - mod_X((\varphi_X^*\varphi_{X*})^2(\mathcal{N})), \Phi_{\mathcal{S}}^{\sim}(-)(\varphi_X^*(\mathcal{L}))) & \xrightarrow{\sim} & \mathcal{S} - bim(\widetilde{\varphi}^*(\mathcal{L}^\wedge\varphi_{X*}\varphi_X^*\varphi_{X*}(\mathcal{N})), -)
\end{array} \qquad (1)$$

which is functorial in all arguments. The right pair of vertical arrows in (1) corresponds to a reflexive pair of arrows

$$\widetilde{\varphi}^*(\mathcal{L}^\wedge \varphi_{X*}\varphi_X^*\varphi_{X*}(\mathcal{N})) \rightrightarrows \widetilde{\varphi}^*(\mathcal{L}^\wedge(\varphi_{X*}(\mathcal{N}))). \tag{2}$$

Therefore, there exists a cokernel of the pair (2), which we denote by $\varphi_X^*(\mathcal{L})^\wedge\mathcal{N}$. Since the right column of the diagram (1) is an exact diagram, there exists a unique isomorphism

$$\mathcal{S}-mod_X(\mathcal{N}, \Phi_{\mathcal{S}}^{\sim}(-)(\varphi_X^*(\mathcal{L}))) \xrightarrow{\sim} \mathcal{S}-bim(\varphi_X^*(\mathcal{L})^\wedge\mathcal{N}, -)$$

making the diagram

$$\begin{array}{ccc}
\mathcal{S}-mod_X(\mathcal{N}, \Phi_{\mathcal{S}}^{\sim}(-)(\varphi_X^*(\mathcal{L}))) & \xrightarrow{\sim} & \mathcal{S}-bim(\varphi_X^*(\mathcal{L})^\wedge\mathcal{N}, -) \\
\downarrow & & \downarrow \\
\mathcal{S}-mod_X(\varphi_X^*(\varphi_{X*}(\mathcal{N})), \Phi_{\mathcal{S}}^{\sim}(-)(\varphi_X^*(\mathcal{L}))) & \xrightarrow{\sim} & \mathcal{S}-bim(\widetilde{\varphi}^*(\mathcal{L}^\wedge(\varphi_{X*}(\mathcal{N}))), -) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathcal{S}-mod_X((\varphi_X^*\varphi_{X*})^2(\mathcal{N})), \Phi_{\mathcal{S}}^{\sim}(-)(\varphi_X^*(\mathcal{L}))) & \xrightarrow{\sim} & \mathcal{S}-bim(\widetilde{\varphi}^*(\mathcal{L}^\wedge\varphi_{X*}\varphi_X^*\varphi_{X*}(\mathcal{N})), -)
\end{array} \tag{3}$$

commute. ∎

2.3.4. Proposition. *Let $(\mathcal{M}, \mathcal{L})$ be an admissible pair of objects of the category $\mathcal{C}_X$. Suppose that the functors $\mathcal{C}_X(\mathcal{M}, -)$ and Φ preserve colimits of a certain type. Then the functor $\mathcal{A}(\mathcal{L}^\wedge\mathcal{M}, -)$ has the same property.*

Proof. Let $\mathfrak{D} \xrightarrow{\mathcal{D}} \mathcal{A}$ be a functor having a colimit, which is preserved by the functor $\mathcal{A} \xrightarrow{\Phi} End(\mathcal{C}_X)$. Then we have the following canonical maps

$$\begin{array}{ccccc}
\mathrm{colim}\mathcal{A}(\mathcal{L}^\wedge\mathcal{M}, \mathcal{D}(-)) & & & & \mathcal{A}(\mathcal{L}^\wedge\mathcal{M}, \mathrm{colim}(\mathcal{D})) \\
\wr\downarrow & & & & \uparrow\wr \\
\mathrm{colim}\mathcal{C}_X(\mathcal{M}, \Phi\circ\mathcal{D}(-)(\mathcal{L})) & \longrightarrow & \mathcal{C}_X(\mathcal{M}, \mathrm{colim}(\Phi\circ\mathcal{D})(\mathcal{L})) & \xrightarrow{\sim} & \mathcal{C}_X(\mathcal{M}, \Phi(\mathrm{colim}(\mathcal{D}))(\mathcal{L}))
\end{array}$$

three of which are bijections. If the functor $\mathcal{C}_X(\mathcal{M}, -)$ preserves colimits of functors from $\mathfrak{D}$, then the remaining map is bijective too. ∎

2.3.5. Proposition. *Let $(\mathcal{M}, \mathcal{L})$ be an admissible pair of objects of the category $\mathcal{C}_X$.*

(a) Suppose that the functor Φ preserves colimits. Then $\mathcal{L}^\wedge\mathcal{M}$ is a finite object of the category $\mathcal{A}$, provided that $\mathcal{M}$ is a finite object of the category $\mathcal{C}_X$.

(b) If the functor Φ preserves colimits of filtered diagrams and $\mathcal{M}$ is a finitely presentable object of the category $\mathcal{C}_X$, then $\mathcal{L}^\wedge\mathcal{M}$ is a finitely presentable object of $\mathcal{A}$.

(c) Suppose that the functor Φ preserves the colimits of filtered diagrams and cokernels of reflexive pairs of arrows. Then $\mathcal{L}^\wedge\mathcal{M}$ is a weakly finite object of the category $\mathcal{A}$ (cf. II.1.7.1), if $\mathcal{M}$ is a weakly finite object of the category $\mathcal{C}_X$.

Proof. The assertion follows from 2.3.4. ∎

2.4. Vector fibers associated with (families of) admissible pairs. Imposing relations. For an associative algebra $\mathcal{S} = (S, \mu_{\mathcal{S}})$ with the unit $\mathbb{I} \xrightarrow{\eta_{\mathcal{S}}} S$, we denote by $\mathfrak{s}$ (instead of $(\eta_{\mathcal{S}})_X$) the unit of the monad $\Phi^{\sim}_{Alg}(\mathcal{S})$ and, therefore, by $\mathfrak{s}_*$ (instead of $(\eta_{\mathcal{S}})_{X*}$) the forgetful functor

$$\mathcal{S} - mod_X = \Phi^{\sim}_{Alg}(\mathcal{S}) - mod \longrightarrow \mathcal{C}_X$$

and by $\mathfrak{s}^*$ (instead of $(\eta_{\mathcal{S}})^*_X$) its standard left adjoint.

The following proposition is a generalization of Theorem 3.1 in [B].

2.4.1. Proposition. *(a) Let (M, P) be an admissible pair of objects of the category $\mathcal{C}_X$. Then the presheaf*

$$\mathbf{Aff}^{op}_{\mathcal{A}^\sim} = Alg\mathcal{A}^\sim \xrightarrow{\mathcal{H}_{M,P}} Sets, \quad \mathcal{S} \longmapsto \mathcal{S} - mod_X(\mathfrak{s}^*(M), \mathfrak{s}^*(P)), \tag{1}$$

is representable.

(b) More generally, if $\{(M_i, P_i),\ i \in J\}$ is a family of admissible pairs of objects of $\mathcal{C}_X$ for which there exists a coproduct of $\{P_i^\wedge M_i \mid i \in J\}$, then the presheaf

$$\mathbf{Aff}^{op}_{\mathcal{A}^\sim} = Alg\mathcal{A}^\sim \longrightarrow Sets, \quad \mathcal{S} \longmapsto \prod_{i\in J} \mathcal{S} - mod_X(\mathfrak{s}^*(M_i), \mathfrak{s}^*(P_i)), \tag{2}$$

is representable.

Proof. (a) We have the following isomorphisms functorial in $\mathcal{S}$:

$$\mathcal{S} - mod_X(\mathfrak{s}^*(M), \mathfrak{s}^*(P)) \simeq \mathcal{A}(M, \mathfrak{s}_*(\mathfrak{s}^*(P))) = \mathcal{A}(M, \Phi(S)(P)) \simeq$$
$$\mathcal{A}(P^\wedge M, \mathfrak{s}_*(S)) \simeq Alg\mathcal{A}^\sim(\mathbf{T}(P^\wedge M), S) \simeq \mathbf{Aff}_{\mathcal{A}^\sim}(\mathcal{S}^\vee, \mathbb{V}(P^\wedge M)),$$

i.e. the functor (1) is representable by the vector fiber $\mathbb{V}(P^\wedge M)$.

(b) By (a), we have functorial isomorphisms:

$$\prod_{i\in J} \mathcal{S} - mod_X(\mathfrak{s}^*(M_i), \mathfrak{s}^*(P_i)) \simeq \prod_{i\in J} \mathcal{A}(P_i^\wedge M_i, \mathfrak{s}_*(S)) \simeq$$
$$\mathcal{A}(\coprod_{i\in J}(P_i^\wedge M_i), \mathfrak{s}_*(S)) \simeq Alg\mathcal{A}^\sim(\mathbf{T}(\coprod_{i\in J} P_i^\wedge M_i), S) \simeq \mathbf{Aff}_{\mathcal{A}^\sim}(\mathcal{S}^\vee, \mathbb{V}(\coprod_{i\in J} P_i^\wedge M_i)),$$

hence the assertion. ■

The next assertion is a generalization of Theorem 3.2 in [B].

2.4.2. Proposition. (Imposing relations) *(a) Let (M, P) be an admissible pair of objects of the category $\mathcal{C}_X$, and let $M \overset{f}{\underset{g}{\rightrightarrows}} P$ be a pair of arrows. There exists a unique up to isomorphism algebra $\mathcal{R}$ such that $\Phi(R)(f) = \Phi(R)(g)$ and universal for this property: given any algebra S with $\mathfrak{s}^*(f) = \mathfrak{s}^*(g)$, there exists a unique algebra morphism $\mathcal{R} \longrightarrow \mathcal{S}$.*

(b) More generally, given a family of pairs of morphisms, $M_i \overset{f_i}{\underset{g_i}{\rightrightarrows}} P_i$, $i \in J$, such that all pairs (M_i, P_i) are admissible and there exists a coproduct of the family $\{P_i^\wedge M_i | \ i \in J\}$, there is an algebra $\mathcal{R}$ universal for the property $\Phi(R)(f_i) = \Phi(R)(g_i)$ for all $i \in J$.

Proof. (a) Since $\mathcal{S} - mod_X(\mathfrak{s}^*(M), \mathfrak{s}^*(P)) \simeq Alg\mathcal{A}^\sim(T(P^\wedge M), \mathcal{S})$ (see the argument of 2.4.1(a)), to every morphism $M \xrightarrow{f} P$, there corresponds a unital algebra morphism $T(P^\wedge M) \xrightarrow{f^a} \mathbb{I}$. Therefore, the universal algebra $\mathcal{R}$ is the cokernel of the pair of algebra morphisms

$$\mathbf{T}(P^\wedge M) \overset{f^a}{\underset{g^a}{\rightrightarrows}} \mathbb{I}$$

corresponding to the morphisms f and g. Notice that, by the conditions 2.0, this cokernel exists, because any pair of unital algebra morphisms to $\mathbb{I}$ is reflexive.

(b) It follows from the functorial isomorphism

$$\prod_{i \in J} \mathcal{S} - mod_X(\mathfrak{s}^*(M_i), \mathfrak{s}^*(P_i)) \simeq Alg\mathcal{A}^\sim(\mathbf{T}(\coprod_{i \in J}(P_i^\wedge M_i), S)$$

(see the argument of 2.4.1(b)) that the universal algebra R is a cokernel of the pair of algebra morphisms

$$\mathbf{T}(\coprod_{i \in J} P_i^\wedge M_i) \overset{f^a}{\underset{g^a}{\rightrightarrows}} \mathbb{I}$$

corresponding to the family of pairs of morphisms $\{f_i, g_i, \ i \in J\}$. ■

2.4.2.1. Note. Let the conditions of 2.4.2(a) hold. Let $\mathfrak{K}_2$ denote the Kronecker quiver $x_1 \rightrightarrows x_2$, $\mathfrak{A}_1$ the quiver $a_1 \longrightarrow a_2$; and let G be the diagram $\mathfrak{K}_2 \longrightarrow \mathfrak{A}_1$, which maps both arrows of $\mathfrak{K}_2$ to the unique arrow of $\mathfrak{A}_1$. Let E be the diagram $\mathfrak{K}_2 \longrightarrow \mathcal{C}_X$ which maps $\mathfrak{K}_2$ to $M \overset{f}{\underset{g}{\rightrightarrows}} P$. Consider the presheaf of sets on $\mathbf{Aff}_{\mathcal{A}^\sim}$ which maps every unital associative algebra $\mathcal{S}$ in $\mathcal{A}^{\sim}$ to the set of all diagrams $\mathfrak{A}_1 \xrightarrow{H} \mathcal{S} - mod_X$ such that the diagram

$$\begin{array}{ccc} \mathfrak{K}_2 & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \eta_S^*} \\ \mathfrak{A}_2 & \xrightarrow{H} & S - mod_X \end{array}$$

commutes. It is easy to see that the map is functorial. The assertion 2.4.2(a) means that this functor is (representable by) an affine scheme in the monoidal category $\mathcal{A}^{\sim}$.

We leave to the reader the corresponding reformulation for 2.4.2(b).

2.4.3. Interpretation: the enriched category of finite objects. The category $\mathcal{B}^{\wedge}$ of presheaves of sets on a svelte category $\mathcal{B}$ is a monoidal category with respect to product, whose unit object is the presheaf $\bullet$ with values in one element set – the final object of $\mathcal{B}^{\wedge}$. If the category $\mathcal{B}$ has finite products and a final object, then the full subcategory of $\mathcal{B}^{\wedge}$ generated by representable presheaves is a monoidal subcategory of $(\mathcal{B}^{\wedge}, \times, \bullet)$.

In particular, the category $\mathbf{Aff}^{\wedge}_{\mathcal{A}^{\sim}}$ of presheaves of sets on the category of affine schemes in $\mathcal{A}^{\sim}$ is a monoidal category, and the Yoneda embedding is the monoidal functor from the monoidal category $\mathbf{Aff}^{\sim}_{\mathcal{A}^{\sim}} = (\mathbf{Aff}_{\mathcal{A}^{\sim}}, \times, \mathbb{I}^{\vee})$ to $(\mathbf{Aff}^{\wedge}_{\mathcal{A}^{\sim}})^{\sim} = (\mathbf{Aff}^{\wedge}_{\mathcal{A}^{\sim}}, \times, \bullet)$.

The assignment to each pair of objects, $\mathcal{L}$, $\mathcal{M}$, of the category $\mathcal{C}_X$ the presheaf of sets

$$\mathbf{Aff}^{op}_{\mathcal{A}^{\sim}} = Alg\mathcal{A}^{\sim} \xrightarrow{\mathcal{H}_{\mathcal{L},\mathcal{M}}} Sets, \quad \mathcal{S} \longmapsto \mathcal{S} - mod_X(\mathfrak{s}^*(\mathcal{L}), \mathfrak{s}^*(\mathcal{M})),$$

turns $\mathcal{C}_X$ into $(\mathbf{Aff}^{\wedge}_{\mathcal{A}^{\sim}})^{\sim}$-enriched category.

2.4.3.1. Finite objects and vector fibers. We denote by $\mathcal{C}_{X_{\mathfrak{f}},\Phi^{\sim}}$, or simply by $\mathcal{C}_{X_{\mathfrak{f}}}$ (if the action $\Phi^{\sim}$ of the monoidal category $\mathcal{A}^{\sim}$ on the category $\mathcal{C}_X$ is fixed) the full subcategory of the category $\mathcal{C}_X$ generated by $\Phi^{\sim}$-finite objects.

It follows from 2.4.1 that the structure of $(\mathbf{Aff}^{\wedge}_{\mathcal{A}^{\sim}})^{\sim}$-enriched category on $\mathcal{C}_X$ induces the structure of $\mathbf{Aff}^{\sim}_{\mathcal{A}^{\sim}}$-enriched category on its full subcategory $\mathcal{C}_{X_{\mathfrak{f}}}$. Moreover, the category $\mathcal{C}_{X_{\mathfrak{f}}}$ is, actually, enriched by the monoidal subcategory of $\mathbf{Aff}^{\sim}_{\mathcal{A}^{\sim}}$ formed by vector fibers of objects of $\mathcal{A}$ and their morphisms induced by morphisms of objects – the essential image of the functor

$$\mathcal{A}^{op} \xrightarrow{\mathbb{V}} \mathbf{Aff}_{\mathcal{A}^{\sim}}$$

which assigns to every object of $\mathcal{A}$ its vector fiber.

2.5. Isomorphisms and "splittings"; the group scheme GL_V. Fix a pair of objects V, W of the category $\mathcal{C}_X$. We have a presheaf of sets

$$\mathbf{Aff}^{op}_{\mathcal{A}^{\sim}} = Alg\mathcal{A}^{\sim} \xrightarrow{Iso_{V,W}} Sets, \tag{1}$$

on the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine schemes in $\mathcal{A}^\sim$, which maps every associative unital algebra $\mathcal{S}$ in $\mathcal{A}^\sim$ to the set $Iso(\mathfrak{s}^*(V), \mathfrak{s}^*(W))$ of isomorphisms from $\mathfrak{s}^*(V)$ to $\mathfrak{s}^*(W)$.

2.5.1. The presheaf $G_{W,V}$. For any pair V, W of objects of the category $\mathcal{C}_X$, we denote by $G_{V,W}$ the subpresheaf of the presheaf $\mathcal{H}_{W,V} \times \mathcal{H}_{V,W}$ which assigns to every associative unital algebra $\mathcal{S}$ in $\mathcal{A}^\sim$ the subset of all pairs $(u, v) \in \mathcal{H}_{V,W} \times \mathcal{H}_{W,V}(\mathcal{S})$ such that $u \circ v = id$. In other words, $G_{W,V}$ is the kernel of the pair of maps

$$\mathcal{H}_{W,V} \times \mathcal{H}_{V,W}(\mathcal{S}) \overset{\alpha_{\mathcal{S}}}{\underset{\beta_{\mathcal{S}}}{\rightrightarrows}} \mathcal{S} - mod_X(\mathfrak{s}^*(V), \mathfrak{s}^*(V)) = \mathcal{H}_{V,V}(\mathcal{S}) \tag{2}$$

defined by $\alpha_{\mathcal{S}}(v, u) = u \circ v$ and $\beta_{\mathcal{S}}(v, u) = id_{\mathfrak{s}^*(V)}$.

2.5.2. Proposition. *Let (V, W) be a pair of objects of the category $\mathcal{C}_X$.*

(a) Let the category $\mathcal{A}$ have kernels of pairs of arrows. If (V, W), (W, V), (V, V) are admissible pairs, then the presheaf $G_{W,V}$ is representable.

(b) Suppose, in addition, that the category $\mathcal{A}$ has fiber products and the pair (W, W) are admissible. Then the presheaf $Iso_{V,W}$ is representable.

Proof. (a) Suppose that the pairs (V, W), (W, V), and (V, V) are admissible. Then

$$\mathcal{H}_{V,V} \simeq \mathbb{V}(V^\wedge V)) \quad \text{and} \quad \mathcal{H}_{V,W} \times \mathcal{H}_{W,V} \simeq \mathbb{V}(W^\wedge V) \times \mathbb{V}(V^\wedge W) \simeq \mathbb{V}(W^\wedge V \coprod V^\wedge W).$$

Let

$$\mathbb{V}(W^\wedge V \coprod V^\wedge W) \simeq \mathbb{V}(W^\wedge V) \times \mathbb{V}(V^\wedge W) \overset{\alpha'}{\underset{\beta'}{\rightrightarrows}} \mathbb{V}(V^\wedge V) \tag{3}$$

be the pair of morphisms of vector fibers corresponding to (2). It follows that the presheaf $G_{V,W}$ is representable by the kernel of the pair (3), which, by hypothesis, exists.

(b) The set $Iso_{V,W}(\mathcal{S}) = Iso(\mathfrak{s}^*(V), \mathfrak{s}^*(W))$ is naturally isomorphic to the set of pairs of morphisms $\mathfrak{s}^*(V) \xrightarrow{u} \mathfrak{s}^*(W)$, $\mathfrak{s}^*(W) \xrightarrow{v} \mathfrak{s}^*(V)$ such that $u \circ v = id$ and $v \circ u = id$. So $Iso(\mathfrak{s}^*(V), \mathfrak{s}^*(W))$ is identified with the fiber product of the morphisms

$$G_{V,W}(\mathcal{S}) \xrightarrow{\mathfrak{j}(\mathcal{S})} \mathcal{H}_{V,W}(\mathcal{S}) \times \mathcal{H}_{W,V}(\mathcal{S}) \xleftarrow{\psi(\mathcal{S})} G_{W,V}(\mathcal{S}),$$

where $\mathfrak{j} = \mathfrak{j}_{V,W}$ is the natural embedding, ψ is the composition of the natural embedding $G_{W,V} \xrightarrow{\mathfrak{j}_{W,V}} \mathcal{H}_{W,V} \times \mathcal{H}_{V,W}$ and the functorial isomorphism

$$\mathcal{H}_{W,V} \times \mathcal{H}_{V,W} \xrightarrow{\sim} \mathcal{H}_{V,W} \times \mathcal{H}_{W,V}, \quad (u, v) \longmapsto (v, u)$$

Thus the presheaf $Iso_{V,W}$ is determined via a cartesian square

$$\begin{array}{ccc} Iso_{V,W} & \longrightarrow & G_{W,V} \\ \Big\downarrow & cart & \Big\downarrow \psi_{W,V} \\ G_{V,W} & \xrightarrow{\mathfrak{j}_{V,W}} & \mathcal{H}_{V,W} \times \mathcal{H}_{W,V} \end{array}$$

Suppose that (V, W), (W, V), (V, V), and (W, W) are admissible pairs of objects of $\mathcal{C}_X$. Then both the source and the target of the arrows $\mathfrak{j}, \psi$, the presheaves $G_{V,W}$ and $\mathcal{H}_{V,W} \times \mathcal{H}_{W,V}$, are representable. Since, by hypothesis, the category $\mathcal{A}$ has fiber products, the presheaf $Iso_{V,W}$ is representable too. ■

2.5.2. Corollary. *Let V be an object of the category $\mathcal{C}_X$ such that the pair (V, V) is admissible. Then the presheaf $GL_V \overset{\text{def}}{=} Iso_{V,V}$ is representable by an affine $\mathcal{A}^{\sim}$-scheme in groups.*

3. Morphisms, isomorphisms and automorphisms of functors.

3.1. The presheaf of natural transformations. Let $\mathcal{B}$ be a svelte category and $\mathcal{B} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightrightarrows}} \mathcal{C}_X$ a pair of functors. We denote by $\mathfrak{H}_{\mathcal{F},\mathcal{G}}$ the presheaf of sets on $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ which assigns to every associative unital algebra $\mathcal{S}$ in the monoidal category $\mathcal{A}^{\sim}$ the set $Hom(\mathfrak{s}^* \circ \mathcal{F}, \mathfrak{s}^* \circ \mathcal{G})$ of functor morphisms from $\mathfrak{s}^* \circ \mathcal{F}$ to $\mathfrak{s}^* \circ \mathcal{G}$.

Here $\mathfrak{s}^*$ denotes the canonical left adjoint to the forgetful functor $\mathcal{S} - mod_X \xrightarrow{\mathfrak{s}_*} \mathcal{C}_X$.

3.2. Isomorphisms. We denote by $\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}$ the presheaf of sets on $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ which maps every associative unital algebra $\mathcal{S}$ in $\mathcal{A}^{\sim}$ the set $Iso(\mathfrak{s}^* \circ \mathcal{F}, \mathfrak{s}^* \circ \mathcal{G})$ of isomorphisms from $\mathfrak{s}^* \circ \mathcal{F}$ to $\mathfrak{s}^* \circ \mathcal{G}$.

3.3. The group $GL_{\mathcal{F}}$. We shall write $GL_{\mathcal{F}}$ instead of $\mathfrak{Iso}_{\mathcal{F},\mathcal{F}}$. The presheaf $GL_{\mathcal{F}}$ has a natural structure of a group in the category of presheaves of sets on $\mathbf{Aff}_{\mathcal{A}^{\sim}}$.

3.4. Proposition. *Let $\mathcal{B}$ be a svelte category and $\mathcal{B} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightrightarrows}} \mathcal{C}_X$ a pair of functors.*

Suppose that the category $\mathcal{A}$ has colimits of diagrams of the cardinality less or equal to the cardinality of $Hom\mathcal{B}$.

(a) Let the pair of objects $(\mathcal{F}(L), \mathcal{G}(M))$ of the category $\mathcal{C}_X$ be admissible for every pair of objects L, M of the category $\mathcal{B}$ such that $\mathcal{B}(L, M)$ is not empty. Then the presheaf $\mathfrak{H}_{\mathcal{F},\mathcal{G}}$ is representable by a vector fiber.

(b) Let the pairs of objects $(\mathcal{F}(L), \mathcal{G}(M))$ and $(\mathcal{G}(L), \mathcal{F}(M))$ be admissible for every pair of objects L, M of the category $\mathcal{B}$ such that $\mathcal{B}(L, M)$ is not empty.

Then the presheaf $\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}$ is representable.

Proof. (a1) If the category $\mathcal{B}$ is discrete, then

$$\mathfrak{H}_{\mathcal{F},\mathcal{G}}(\mathcal{S}) \stackrel{\mathrm{def}}{=} Hom(\mathfrak{s}^* \circ \mathcal{F}, \mathfrak{s}^* \circ \mathcal{G}) = \prod_{L \in Ob\mathcal{B}} \mathcal{S} - mod_X(\mathfrak{s}^*(\mathcal{F}(L)), \mathfrak{s}^*(\mathcal{G}(L))) =$$
$$\prod_{L \in Ob\mathcal{B}} \mathcal{C}_X(\mathcal{F}(L), \mathfrak{s}_*\mathfrak{s}^*(\mathcal{G}(L))) = \prod_{L \in Ob\mathcal{B}} \mathcal{C}_X(\mathcal{F}(L), \Phi(\eta_{\mathcal{S}*}(\mathcal{S}))(\mathcal{G}(L))) \simeq$$
$$\prod_{L \in Ob\mathcal{B}} \mathcal{A}(\mathcal{G}(L)^\wedge \mathcal{F}(L), \eta_{\mathcal{S}*}(\mathcal{S})) \simeq \mathcal{A}(\coprod_{L \in Ob\mathcal{B}} \mathcal{G}(L)^\wedge \mathcal{F}(L), \eta_{\mathcal{S}*}(\mathcal{S})) \simeq$$
$$Alg_{\mathcal{A}^\sim}(\mathbb{V}(\coprod_{L \in Ob\mathcal{B}} \mathcal{G}(L)^\wedge \mathcal{F}(L)), \mathcal{S})$$

which shows that, in this case, the presheaf $\mathfrak{H}_{\mathcal{F},\mathcal{G}}$ is representable by the vector fiber $\mathbb{V}(\coprod_{L \in Ob\mathcal{B}} \mathcal{G}(L)^\wedge \mathcal{F}(L))$ of the object $\coprod_{L \in Ob\mathcal{B}} \mathcal{G}(L)^\wedge \mathcal{F}(L)$.

(a2) In the general case, every morphism $L \xrightarrow{f} M$ of the category $\mathcal{B}$ gives rise to the pair of maps:

$$\begin{array}{ccc} \mathcal{S} - mod_X(\mathfrak{s}^*(\mathcal{F}(L)), \mathfrak{s}^*(\mathcal{G}(L))) & & \mathcal{S} - mod_X(\mathfrak{s}^*(\mathcal{F}(M)), \mathfrak{s}^*(\mathcal{G}(M))) \\ \mathfrak{s}^*(f) \circ \big\downarrow & \text{and} & \big\downarrow \circ \mathfrak{s}^*(f) \\ \mathcal{S} - mod_X(\mathfrak{s}^*(\mathcal{F}(L)), \mathfrak{s}^*(\mathcal{G}(M))) & & \mathcal{S} - mod_X(\mathfrak{s}^*(\mathcal{F}(L)), \mathfrak{s}^*(\mathcal{G}(M))) \\ g \longmapsto \mathfrak{s}^*(f) \circ g & & h \longmapsto h \circ \mathfrak{s}^*(f). \end{array}$$

It follows that the set

$$\mathfrak{H}_{\mathcal{F},\mathcal{G}}(\mathcal{S}) \stackrel{\mathrm{def}}{=} Hom(\mathfrak{s}^* \circ \mathcal{F}, \mathfrak{s}^* \circ \mathcal{G})$$

of functor morphisms is the limit of the diagram

$$\begin{array}{ccc} & & \mathcal{S} - mod_X(\mathfrak{s}^*(\mathcal{F}(M)), \mathfrak{s}^*(\mathcal{G}(M))) \\ & & \big\downarrow \circ \mathfrak{s}^*(f) \\ \mathcal{S} - mod_X(\mathfrak{s}^*(\mathcal{F}(L)), \mathfrak{s}^*(\mathcal{G}(L))) & \xrightarrow{\mathfrak{s}^*(f)\circ} & \mathcal{S} - mod_X(\mathfrak{s}^*(\mathcal{F}(L)), \mathfrak{s}^*(\mathcal{G}(M))), \\ & & (L \xrightarrow{f} M) \in Hom\mathcal{B}. \end{array} \tag{1}$$

Under the assumptions of (a), the diagram (1) is isomorphic to the limit of the diagram

$$
\begin{array}{ccc}
 & & \mathcal{A}(\mathcal{G}(M)^\wedge\mathcal{F}(M), S) \\
 & & \big\downarrow \mathcal{A}(\mathcal{G}(M)^\wedge\mathcal{F}(f), S) \\
\mathcal{A}(\mathcal{G}(L)^\wedge\mathcal{F}(L), S) & \xrightarrow{\mathcal{A}(\mathcal{G}(f)^\wedge\mathcal{F}(L),S)} & \mathcal{A}(\mathcal{G}(M)^\wedge\mathcal{F}(L), S), \\
 & & (L \xrightarrow{f} M) \in Hom\mathcal{B}.
\end{array}
\qquad (2)
$$

This shows that, under the conditions of (a), the presheaf $\mathfrak{H}_{\mathcal{F},\mathcal{G}}$ is representable by the vector fiber $\mathbb{V}(\mathcal{M}_{\mathcal{F},\mathcal{G}})$, where the object $\mathcal{M}_{\mathcal{F},\mathcal{G}}$ is the colimit of the diagram

$$
\begin{array}{ccc}
\mathcal{G}(M)^\wedge\mathcal{F}(L) & \xrightarrow{\mathcal{G}(f)^\wedge\mathcal{F}(L)} & \mathcal{G}(L)^\wedge\mathcal{F}(L) \\
\mathcal{G}(M)^\wedge\mathcal{F}(f) \big\downarrow & & \\
\mathcal{G}(M)^\wedge\mathcal{F}(M) & & \\
 & & (L \xrightarrow{f} M) \in Hom\mathcal{B}.
\end{array}
$$

(b) For every associative unital algebra $\mathcal{S}$ in $\mathcal{A}^\sim$, the set $\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}(\mathcal{S})$ is the limit of the diagram

$$
\begin{array}{ccc}
\mathfrak{H}_{\mathcal{F},\mathcal{G}}(\mathcal{S}) \times \mathfrak{H}_{\mathcal{G},\mathcal{F}}(\mathcal{S}) & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{G}}}{\rightrightarrows}} & \mathfrak{H}_{\mathcal{G},\mathcal{G}}(\mathcal{S}) \\
\mathfrak{S}(\mathcal{S}) \big\downarrow \wr & & \\
\mathfrak{H}_{\mathcal{G},\mathcal{F}}(\mathcal{S}) \times \mathfrak{H}_{\mathcal{F},\mathcal{G}}(\mathcal{S}) & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{F}}}{\rightrightarrows}} & \mathfrak{H}_{\mathcal{F},\mathcal{G}}(\mathcal{S})
\end{array}
\qquad (3)
$$

where $\mathfrak{S}$ is the standard symmetry, $(x,y) \longmapsto (y,x)$, $\mathfrak{m}$ is the composition; and $\mathfrak{p}_{\mathcal{G}}(\mathcal{S})$ maps $\mathfrak{H}_{\mathcal{F},\mathcal{G}}(\mathcal{S}) \times \mathfrak{H}_{\mathcal{G},\mathcal{F}}(\mathcal{S}) = Hom(\mathfrak{s}^* \circ \mathcal{F}, \mathfrak{s}^* \circ \mathcal{G}) \times Hom(\mathfrak{s}^* \circ \mathcal{G}, \mathfrak{s}^* \circ \mathcal{F})$ to the identical endomorphism of $\mathfrak{s}^* \circ \mathcal{G}$. The diagram (3) is functorial in $\mathcal{S}$; so that the presheaf $\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}$ of isomorphisms from $\mathcal{F}$ to $\mathcal{G}$ is the limit of the diagram

$$
\begin{array}{ccc}
\mathfrak{H}_{\mathcal{F},\mathcal{G}} \times \mathfrak{H}_{\mathcal{G},\mathcal{F}} & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{G}}}{\rightrightarrows}} & \mathfrak{H}_{\mathcal{G},\mathcal{G}} \\
\mathfrak{S} \big\downarrow \wr & & \\
\mathfrak{H}_{\mathcal{G},\mathcal{F}} \times \mathfrak{H}_{\mathcal{F},\mathcal{G}} & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{F}}}{\rightrightarrows}} & \mathfrak{H}_{\mathcal{F},\mathcal{G}}
\end{array}
\qquad (3')
$$

of presheaves of sets. It follows from (a2) that, if the conditions of (b) hold, each of the functors in the diagram (3') is representable. Therefore, $\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}$ is representable by the limit of the diagram $\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}$ is the limit of the diagram

$$\begin{array}{ccccc} \mathbb{V}((\mathcal{M}_{\mathcal{F},\mathcal{G}}\coprod\mathcal{M}_{\mathcal{G},\mathcal{F}}) & \xrightarrow{\sim} & \mathbb{V}(\mathcal{M}_{\mathcal{F},\mathcal{G}})\times\mathbb{V}(\mathcal{M}_{\mathcal{G},\mathcal{F}}) & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{G}}}{\rightrightarrows}} & \mathbb{V}(\mathcal{M}_{\mathcal{G},\mathcal{G}}) \\ \sigma \big\downarrow \wr & & \mathfrak{S} \big\downarrow \wr & & \\ \mathbb{V}((\mathcal{M}_{\mathcal{G},\mathcal{F}}\coprod\mathcal{M}_{\mathcal{F},\mathcal{G}}) & \xrightarrow{\sim} & \mathbb{V}(\mathcal{M}_{\mathcal{G},\mathcal{F}})\times\mathbb{V}(\mathcal{M}_{\mathcal{F},\mathcal{G}}) & \overset{\mathfrak{m}}{\underset{\mathfrak{p}_{\mathcal{F}}}{\rightrightarrows}} & \mathbb{V}(\mathcal{M}_{\mathcal{F},\mathcal{F}}) \end{array} \qquad (4)$$

which exists, because, by hypothesis, $\mathcal{A}$ has colimits. ■

3.5. Corollary. *Let $\mathcal{B}$ be a svelte category and $\mathcal{B} \xrightarrow{\mathcal{F}} \mathcal{C}_X$ a functor. Suppose that the category $\mathcal{A}$ has colimits of diagrams of the cardinality less or equal to the cardinality of $Hom\mathcal{B}$, and the pair of objects $(\mathcal{F}(L), \mathcal{F}(M))$ of the category $\mathcal{C}_X$ be admissible whenever $\mathcal{B}(L, M)$ is not empty. Then the presheaf of groups $GL_{\mathcal{F}}$ is representable.*

In other words, the group $GL_{\mathcal{F}}$ is affine.

Proof. The fact follows from 3.4(b). ■

3.6. Finiteness conditions.

3.6.0. Finitely presentable functors and actions. We call a functor *finitely presentable*, if it preserves colimits of inductive systems (that is filtered diagrams).

We call an action $\Phi^{\sim} = (\Phi, \phi, \phi_0)$ of a monoidal category $\mathcal{A}^{\sim}$ on a category $\mathcal{C}_X$ *finitely presentable*, if the functor $\mathcal{A} \xrightarrow{\Phi} End(\mathcal{C}_X)$ is finitely presentable.

In this subsection, we assume that the action $\Phi^{\sim}$ of $\mathcal{A}^{\sim}$ on $\mathcal{C}_X$ is finitely presentable.

3.6.1. Proposition. *Let $\mathcal{B}$ be a finite category and $\mathcal{B} \overset{\mathcal{F}}{\underset{\mathcal{G}}{\rightrightarrows}} \mathcal{C}_X$ a pair of functors.*

Suppose that the pair of objects $(\mathcal{F}(L), \mathcal{G}(M))$ of the category $\mathcal{C}_X$ is admissible for every pair of objects L, M of the category $\mathcal{B}$ such that $\mathcal{B}(L, M)$ is not empty.

(a) If $\mathcal{F}(L)$ is a finitely presentable object of $\mathcal{C}_X$ for every $L \in Ob\mathcal{B}$, then the presheaf $\mathfrak{H}_{\mathcal{F},\mathcal{G}}$ of morphisms from $\mathcal{F}$ to $\mathcal{G}$ (see 3.1) is locally finitely copresentable.

(b) If the category $\mathcal{A}$ has finite colimits, then the presheaf $\mathfrak{H}_{\mathcal{F},\mathcal{G}}$ is representable by a vector fiber of a finitely presentable object of the category $\mathcal{A}$.

Proof. (a) It follows from the argument of 3.4(a) that the presheaf $\mathfrak{H}_{\mathcal{F},\mathcal{G}}$ is the limit of the finite diagram of presheaves

$$\begin{array}{ccc} & & \mathcal{A}(\mathcal{G}(M)^\wedge\mathcal{F}(M), \mathfrak{f}_*(-)) \\ & & \big\downarrow \mathcal{A}(\mathcal{G}(M)^\wedge\mathcal{F}(\xi), \mathfrak{f}_*(-)) \\ \mathcal{A}(\mathcal{G}(L)^\wedge\mathcal{F}(L), \mathfrak{f}_*(-)) & \xrightarrow{\mathcal{A}(\mathcal{G}(\xi)^\wedge\mathcal{F}(L), \mathfrak{f}_*(-))} & \mathcal{A}(\mathcal{G}(M)^\wedge\mathcal{F}(L), \mathfrak{f}_*(-)), \end{array} \tag{1}$$

$$(L \xrightarrow{\xi} M) \in Hom\mathcal{B}.$$

Here $\mathfrak{f}_*$ is the forgetful functor $Alg\mathcal{A}^\sim \longrightarrow \mathcal{A}$. The functor $\mathfrak{f}_*$ preserves colimits of filtered systems and, by 2.3.5(b), the objects $\mathcal{G}(M)^\wedge\mathcal{F}(L)$, $\mathcal{G}(M)^\wedge\mathcal{F}(M)$ and $\mathcal{G}(L)^\wedge\mathcal{F}(L)$ are finitely presentable. Therefore, all the presheaves of the diagram (1) are locally finitely copresentable. By (the dual version of) II.1.4, the presheaf $\mathfrak{H}_{\mathcal{F},\mathcal{G}}$, being a limit of a finite diagram of locally finitely copresentable presheaves, is locally finitely copresentable.

(b) By the argument of 3.3(a), the presheaf $\mathfrak{H}_{\mathcal{F},\mathcal{G}}$ is representable by the vector fiber $\mathbb{V}(\mathcal{M}_{\mathcal{F},\mathcal{G}})$, where the object $\mathcal{M}_{\mathcal{F},\mathcal{G}}$ is the colimit of the diagram

$$\begin{array}{ccc} \mathcal{G}(M)^\wedge\mathcal{F}(L) & \xrightarrow{\mathcal{G}(\xi)^\wedge\mathcal{F}(L)} & \mathcal{G}(L)^\wedge\mathcal{F}(L) \\ {\scriptstyle \mathcal{G}(M)^\wedge\mathcal{F}(\xi)}\big\downarrow & & \\ \mathcal{G}(M)^\wedge\mathcal{F}(M) & & \end{array} \tag{2}$$

$$(L \xrightarrow{\xi} M) \in Hom\mathcal{B}.$$

Since the category $\mathcal{B}$ is finite, the diagram (2) is finite. It follows from 2.3.5 that, under the assumptions, each of the objects of the diagram (2) is finitely presentable. Therefore, the colimit of the diagram (2) is a finitely presentable object. ∎

3.6.2. Proposition. *Let $\mathcal{B}$ be a finite category and $\mathcal{B} \underset{\mathcal{G}}{\overset{\mathcal{F}}{\rightrightarrows}} C_X$ a pair of functors.*

Suppose that the pairs of objects $(\mathcal{F}(L), \mathcal{G}(M))$ and $(\mathcal{F}(M), \mathcal{G}(L))$ of C_X are admissible for every pair of objects L, M of the category $\mathcal{B}$ such that $\mathcal{B}(L, M)$ is not empty.

(a) If $\mathcal{F}(L)$ is a finitely presentable object of C_X for every $L \in Ob\mathcal{B}$, then the presheaf $\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}$ of isomorphisms from $\mathcal{F}$ to $\mathcal{G}$ (see 3.2) is locally finitely copresentable.

(b) If the category $\mathcal{A}$ has finite colimits, then the presheaf $\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}$ is representable.

Proof. (a) It follows from the argument of 3.4(b) that, under the assumptions, the presheaf $\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}$ is the limit a finite diagram of locally finitely copresentable presheaves. Therefore, by II.1.4, it is locally finitely copresentable.

(b) This is a consequence of 3.4(b). ∎

4. A construction of affine schemes. The following proposition is a generalization of Theorem 4.3 in [B].

4.1. Proposition. *Let B and D be small categories and G a functor $B \longrightarrow D$ which is a bijection on objects. Let $B \xrightarrow{E} \mathcal{C}_X$ be a functor having the following property:*

(†) *the pair of objects $(E(M), E(L))$ is admissible if $D(G(M), G(L)) \neq \emptyset$.*

Suppose that the category $\mathcal{A}$ has small colimits. Then there exist an associative unital algebra $R_{E,G}$ in $\mathcal{A}^\sim$ and a functor $D \xrightarrow{H_{E,G}} R_{E,G} - mod_X$ which make the diagram

$$\begin{array}{ccc} B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \eta^*_{E,G}} \\ D & \xrightarrow{H_{E,G}} & R_{E,G} - mod_X \end{array} \qquad (1)$$

commute, and which are universal for this property.

Proof. (a) Suppose first that D coincides with the image of G. Then applying 2.4.2(b), we obtain an algebra R (which is a quotient of $\mathbb{I}$) and a functor $D \xrightarrow{H} R - mod_X$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle \eta^*_R} \\ D & \xrightarrow{H} & R - mod_X \end{array}$$

commutes and the pair (H, R) is universal for this property.

(b) If $HomB \xrightarrow{G} HomD$ is not surjective, (a) gives a reduction to the case when B is a subcategory of D with the same set of objects and G is the inclusion functor. We apply 2.4.1(b) to obtain all morphisms needed, and then apply 2.4.2(b) for relations. Details are left to the reader. ■

4.2. Localizations and universal localizations. We have the following corollary of Proposition 4.1:

4.2.1. Proposition. *Let $B \xrightarrow{E} \mathcal{C}_X$ be a functor satisfying the condition*

(†*) *The pair $(E(M), E(L))$ is admissible if $B(M, L)$ is non-empty.*

Let Σ be a class of morphisms of the category B and $G = \mathfrak{q}^*_\Sigma$ the localization functor $B \longrightarrow \Sigma^{-1}B$. Then there exists a unique (up to isomorphism) algebra R_Σ and a unique *functor* $\Sigma^{-1}B \xrightarrow{E_\Sigma} R_\Sigma - mod_X$ *such that the diagram*

$$\begin{array}{ccc} B & \xrightarrow{E} & \mathcal{C}_X \\ \mathfrak{q}^*_\Sigma \downarrow & & \downarrow \eta^*_{R_\Sigma} \\ \Sigma^{-1}B & \xrightarrow{E_\Sigma} & R_\Sigma - mod_X \end{array} \qquad (3)$$

commutes and which are universal for this property.

4.2.2. Note. Suppose the conditions of 4.1 on functors $D \xleftarrow{G} B \xrightarrow{E} \mathcal{C}_X$ and the category $\mathcal{A}$ hold. Consider a map, which assigns to any algebra S in $\mathcal{A}^\sim$ the set $\mathcal{H}_{E,G}(S)$ of all functors $D \xrightarrow{H} S - mod_X$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{E} & \mathcal{C}_X \\ G \downarrow & & \downarrow \eta^*_S \\ D & \xrightarrow{H} & S - mod_X \end{array}$$

commutes. It is easy to see that the map is functorial. The assertion 4.1 means that this functor is (representable by) an affine scheme in the monoidal category $\mathcal{A}^\sim$.

In the case when the functor $B \xrightarrow{G} D$ is a localization, the set $\mathcal{H}_{E,G}(S)$ is either empty, or has only one element.

The bijectivity condition in 4.1 can be replaced by a weaker requirement:

4.3. Proposition. *Let B and D be small categories, and let $B \xrightarrow{G} D$ be a functor injective on objects and such that every object of D is isomorphic to an object of the image of the functor G. Suppose that the category $\mathcal{A}$ has small colimits. Then, for any functor $B \xrightarrow{E} \mathcal{C}_X$ satisfying the condition (†) of 4.1, there exist an algebra $R_{E,G}$ and a functor*

$$D \xrightarrow{H_{E,G}} R_{E,G} - mod_X$$

which make the following diagram commute

$$\begin{array}{ccc} B & \xrightarrow{E} & \mathcal{C}_X \\ G \downarrow & & \downarrow \eta^*_{E,G} \\ D & \xrightarrow{H_{E,G}} & R_{E,G} - mod_X \end{array} \qquad (1)$$

and which are universal for this property. In particular, the functor $H_{E,G}$ is defined uniquely up to isomorphism.

Proof. Let D' be the full subcategory of D defined by $ObD' = G(ObB)$, and let $B \xrightarrow{G'} D'$ be the corestriction of the functor G to D'. By 4.1, there exist an algebra $R_{E,G'}$ and a functor $D' \xrightarrow{H_{E,G'}} \mathcal{C}_{E,G'}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G'}\downarrow & & \downarrow{\scriptstyle \eta^*_{E,G'}} \\ D' & \xrightarrow{H_{E,G'}} & R_{E,G'} - mod_X \end{array} \qquad (2)$$

commutes and which are universal for this property. The embedding $D' \xrightarrow{J_*} D$ is an equivalence of categories. Let $D \xrightarrow{J^*} D'$ denote a left adjoint (a quasi-inverse) to J_* such that $J^* \circ J_* = Id_{D'}$. Then $H_{E,G} = H_{E,G'} \circ J^*$ is the universal functor. ■

4.4. The pseudo-functor $\mathfrak{F}^{\sim}_{E,G}$. Passing to *essentially bijective*, but, not necessarily injective functors requires a relaxation of the setting, which is as follows.

Fix svelte categories D and B and functors $D \xleftarrow{G} B \xrightarrow{E} \mathcal{C}_X$.

For any associative unital algebra S in the monoidal category $\mathcal{A}^{\sim}$, let $\mathfrak{F}^{\sim}_{E,G}(S)$ denote the category whose objects are pairs $(\mathcal{L}, \gamma_{\mathcal{L}})$, where $\mathcal{L}$ is a functor $D \longrightarrow S - mod_{\mathcal{C}_X}$ and $\gamma_{\mathcal{L}}$ a functor isomorphism $\mathcal{L} \circ G \xrightarrow{\sim} s^* \circ E$.

A morphism $(\mathcal{L}, \gamma_{\mathcal{L}}) \longrightarrow (\mathcal{L}', \gamma_{\mathcal{L}'})$ is given by a functor morphism $\mathcal{L} \xrightarrow{\psi} \mathcal{L}'$ such that $\gamma_{\mathcal{L}'} \circ \psi G = \gamma_{\mathcal{L}}$. In particular, ψG is a functor isomorphism.

The correspondence $S \longmapsto \mathfrak{F}^{\sim}_{E,G}(S)$ extends to a pseudo-functor

$$Alg\mathcal{A}^{\sim} = \mathbf{Aff}^{op}_{\mathcal{A}^{\sim}} \xrightarrow{\mathfrak{F}^{\sim}_{E,G}} Cat$$

which maps every algebra morphism $S \xrightarrow{\mathsf{t}} T$ to the functor

$$\mathfrak{F}^{\sim}_{E,G}(S) \xrightarrow{\mathfrak{F}^{\sim}_{E,G}(\mathsf{t})} \mathfrak{F}^{\sim}_{E,G}(T)$$

assigning to each object $(\mathcal{L}, \gamma_{\mathcal{L}})$ of the category $\mathfrak{F}^{\sim}_{E,G}(S)$ the object $(\mathsf{t}^* \circ \mathcal{L}, \alpha_{s,\mathsf{t}}E \circ \mathsf{t}^*\gamma_{\mathcal{L}})$ of the category $\mathfrak{F}^{\sim}_{E,G}(T)$ and to every morphism $(\mathcal{L}, \gamma_{\mathcal{L}}) \xrightarrow{\psi} (\mathcal{L}', \gamma_{\mathcal{L}'})$ the morphism

$$(\mathsf{t}^* \circ \mathcal{L}, \alpha_{s,\mathsf{t}}E \circ \mathsf{t}^*\gamma_{\mathcal{L}}) \xrightarrow{\mathsf{t}^*\psi} (\mathsf{t}^* \circ \mathcal{L}', \alpha_{s,\mathsf{t}}E \circ \mathsf{t}^*\gamma_{\mathcal{L}'}).$$

Here $\alpha_{s,\mathfrak{t}}$ is the unique isomorphism $\mathfrak{t}^* \circ s^* \xrightarrow{\sim} (\mathfrak{t} \circ s)^*$.

4.4.1. The presheaf $\mathfrak{F}_{E,G}$. The map $\mathfrak{F}_{E,G}$, which assigns to every associative unital algebra S in $\mathcal{A}^\sim$ the set $\mathfrak{F}_{E,G}(S)$ of isomorphism classes of objects of the category $\mathfrak{F}^\sim_{E,G}(S)$ extends to a presheaf of sets

$$Alg\mathcal{A}^\sim = \mathbf{Aff}^{op}_{\mathcal{A}^\sim} \xrightarrow{\mathfrak{F}_{E,G}} Sets$$

on the category $\mathbf{Aff}_{\mathcal{A}^\sim}$ of affine $\mathcal{A}^\sim$-schemes.

4.4.2. Proposition. *Suppose that the functor $B \xrightarrow{G} D$ in the data $\mathfrak{E}$ is essentially bijective on objects (that is it induces bijective map of isomorphism classes of objects) and the functor $B \xrightarrow{E} \mathcal{C}_X$ satisfies the condition*

(†) *the pair of objects $(E(M), E(L))$ is admissible, if $D(G(M), G(L)) \neq \emptyset$.*

Then the presheaf $\mathfrak{F}_{E,G}$ is representable, provided the category $\mathcal{A}$ has colimits.

Proof. Since the category B is svelte and the functor G is essentially bijective on objects (that is it induces a bijection of the sets of isomorphism classes of objects), there is a small full subcategory $\mathfrak{B}$ of the category B such that the inclusion functor $\mathfrak{B} \xrightarrow{\mathfrak{J}_*} B$ is a category equivalence and the restriction of the functor $B \xrightarrow{G} D$ to $\mathfrak{B}$ is injective on objects. By 4.3, there exists a universal commutative diagram

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{E\circ\mathfrak{J}_*} & \mathcal{C}_X \\ {\scriptstyle G\circ\mathfrak{J}_*}\big\downarrow & & \big\downarrow {\scriptstyle \eta^*_{E,G}} \\ D & \xrightarrow{H_{E,G}} & R_{E,G} - mod_X \end{array} \qquad (1)$$

Since the composition $\mathfrak{J}_* \circ \mathfrak{J}^*$ of the inclusion functor $\mathfrak{B} \xrightarrow{\mathfrak{J}_*} B$ with its quasi-inverse is isomorphic to the identical functor $B \longrightarrow B$, it follows that the diagram

$$\begin{array}{ccc} B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G}\big\downarrow & & \big\downarrow {\scriptstyle \eta^*_{E,G}} \\ D & \xrightarrow{H_{E,G}} & R_{E,G} - mod_X \end{array} \qquad (2)$$

quasi-commutes. It follows from the argument that the diagram (2) is universal for this property. ∎

4.5. Proposition. *Let* $\mathfrak{D} \stackrel{\mathfrak{G}}{\longleftarrow} D \stackrel{G}{\longleftarrow} B \stackrel{E}{\longrightarrow} \mathcal{C}_X$ *be a diagram such that the presheaf* $\mathfrak{F}_{E,G}$ *is representable by* $\mathcal{R}^{\vee}_{E,G}$. *Then the presheaf* $\mathfrak{F}_{\mathfrak{G}\circ G,E}$ *is naturally isomorphic to the presheaf* $\mathfrak{F}_{H_{E,G},\mathfrak{G}}$, *where* $H_{E.G}$ *is the unique up to isomorphism functor*

$$D \longrightarrow R_{E,G} - mod_X$$

defined by the quasi-commutativity of the diagram 4.4.2(2)).

Proof. The argument is left to the reader. ■

4.5.1. An application. Let B be a svelte category and $B \stackrel{E}{\longrightarrow} \mathcal{C}_X$ a functor satisfying the condition

($\dagger^*$) The pair $(E(M), E(L))$ is admissible if $B(M, L)$ is non-empty.

Suppose that the category $\mathcal{A}$ has colimits. Then, for any functor $B \stackrel{G}{\longrightarrow} D$, the presheaf $\mathfrak{F}_{E,G}$ is isomorphic to the presheaf $\mathfrak{F}_{E_{\Sigma_G},G_{\mathfrak{s}}}$, where Σ_G is the class of all arrows of B, which the functor G maps to isomorphisms, and $G_{\mathfrak{s}}$ is the "conservative" component of the functor G (– the functor G is the composition of the localization at Σ_G and $G_{\mathfrak{s}}$); and E_{Σ_G} is the canonical functor

$$\Sigma_G^{-1}B \xrightarrow{E_{\Sigma_G}} R_{\Sigma_G} - mod_X$$

whose existence follows from 4.4.1.

4.6. Proposition (base change). *Let the conditions of 4.1 hold. Let* $\mathcal{S}$ *be an algebra in* $\mathcal{A}^{\sim}$ *and* $E_{\mathcal{S}}$ *the composition of* $B \stackrel{E}{\longrightarrow} \mathcal{C}_X$ *and* $\mathcal{C}_X \stackrel{\mathfrak{s}^*}{\longrightarrow} \mathcal{S} - mod_X$.

Then the universal algebra $R_{E_{\mathcal{S}},G}$ *is naturally identified with* $R_{E,G} \coprod \mathcal{S}$ *and the canonical functor* $H_{E_{\mathcal{S}},G}$ *with the composition of the functor*

$$D \xrightarrow{H_{E,G}} R_{E,G} - mod_X$$

and the functor

$$R_{E,G} - mod_X \xrightarrow{\widetilde{\mathfrak{s}}^*_X} R_{E,G} \coprod \mathcal{S} - mod_X$$

corresponding to the coprojection $R_{E,G} \xrightarrow{\widetilde{\mathfrak{s}}} R_{E,G} \coprod \mathcal{S}$.

Proof. It follows from the commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{E} & \mathcal{C}_X & \xrightarrow{\mathfrak{s}^*} & \mathcal{S} - mod_X \\ G \big\downarrow & & \big\downarrow & & \big\downarrow \\ D & \xrightarrow{H_{E,G}} & R_{E,G} - mod_X & \xrightarrow{\widetilde{\mathfrak{s}}^*_X} & R_{E,G} \coprod \mathcal{S} - mod_X \end{array} \qquad (4)$$

and the universal property of the algebra $R_{E,G}$ that there exists a unique algebra morphism $R_{E_{\mathcal{S}},G} \xrightarrow{\psi} R_{E,G} \coprod \mathcal{S}$ such that $\widetilde{\mathfrak{s}}^* \circ H_{E,G}$ is the composition of $H_{E_{\mathcal{S}},G}$ and the functor $R_{E_{\mathcal{S}},G} - mod_X \xrightarrow{\psi^*} R_{E,G} \coprod \mathcal{S} - mod_X$.

On the other hand, the universal property of the algebra $R_{E,G}$ gives rise to a commutative square

$$\begin{array}{ccc} \mathbb{I} & \xrightarrow{\eta_{\mathcal{S}}} & \mathcal{S} \\ \downarrow & & \downarrow \\ R_{E,G} & \longrightarrow & R_{E_{\mathcal{S}},G} \end{array}$$

of algebra morphisms; and, by the universal property of a coproduct, this square determines a unique unital algebra morphism $R_{E,G} \coprod \mathcal{S} \xrightarrow{\varphi} R_{E_{\mathcal{S}},G}$. By a standard argument, the algebra morphisms ψ and φ are inverse to each other. ■

4.7. Finiteness conditions.

4.7.1. Proposition. *Suppose that the action $\Phi^\sim$ of the monoidal category $\mathcal{A}^\sim$ on the category $\mathcal{C}_X$ is finitely presentable (see 3.6.0). Let $\mathcal{B}$ and $\mathcal{D}$ be finite categories and $\mathcal{B} \xrightarrow{G} \mathcal{D}$ a functor, which is essentially bijective on objects. Let $\mathcal{B} \xrightarrow{E} \mathcal{C}_X$ be a functor, which maps all objects of $\mathcal{B}$ to finitely presentable objects of the category $\mathcal{C}_X$ and satisfies the property*

(†) *the pair of objects $(E(M), E(L))$ is admissible, if $D(G(M), G(L)) \neq \emptyset$.*

Then the presheaf of sets $\mathfrak{F}_{E,G}$ is locally finitely copresentable.

Proof. The presheaf $\mathfrak{F}_{E,G}$ is the limit of the presheaves

$$\mathbf{Aff}^{op}_{\mathcal{A}^\sim} = Alg\mathcal{A}^\sim \xrightarrow{\mathcal{H}_{E(M),E(L)}} Sets, \quad \mathcal{S} \longmapsto \mathcal{S} - mod_X(\mathfrak{s}^*(E(M)), \mathfrak{s}^*(E(L))),$$

where $D(G(M), G(L)) \neq \emptyset$. If the condition (†) holds, then it follows from 2.4.1 that $\mathfrak{F}_{E,G}$ is the limit of the presheaves representable by the vector bundles $\mathbb{V}(E(L)^\wedge E(M))$ with $D(G(M), G(L)) \neq \emptyset$. Since each $E(M)$, $M \in Ob\mathcal{B}$, is a finitely presentable object of the category $\mathcal{C}_X$, it follows from 2.3.5 that $E(L)^\wedge E(M)$ is a finitely presentable object of the category $\mathcal{A}$; hence $\mathbb{V}(E(L)^\wedge E(M))$ is a finitely corepresentable affine scheme.

Therefore, since the categories $\mathcal{B}$ and $\mathcal{D}$ are finite, the presheaf $\mathfrak{F}_{E,G}$ is the limit of a finite diagram of locally finitely corepresentable presheaves. By II.1.4, this implies that the presheaf $\mathfrak{F}_{E,G}$ is locally finitely copresentable. ■

4.7.2. Note. Suppose that the conditions of 4.7.1 hold. Then it follows from 4.4.2 that, if the category $\mathcal{A}$ has colimits of finite diagrams, then the presheaf $\mathfrak{F}_{E,G}$ is representable.

5. Grassmannians.

5.1. The presheaf $Gr_{M,V}$. For a pair (M,V) of objects of the category $\mathcal{C}_X$, consider the presheaf of sets

$$\mathbf{Aff}^{op}_{\mathcal{A}^\sim} = Alg\mathcal{A}^\sim \xrightarrow{Gr_{M,V}} Sets$$

which assigns to every associative unital algebra $\mathcal{R}$ in $\mathcal{A}^\sim$ the set of isomorphism classes of split epimorphisms $\eta^*_{\mathcal{R}}(M) \longrightarrow \eta^*_{\mathcal{R}}(V)$ and to any unital algebra morphism $\mathcal{R} \xrightarrow{\varphi} \mathcal{S}$ the map

$$Gr_{M,V}(\mathcal{R}) \xrightarrow{Gr_{M,V}(\varphi)} Gr_{M,V}(\mathcal{S})$$

induced by the inverse image functor $\mathcal{R} - mod_X \xrightarrow{\varphi^*_X} \mathcal{S} - mod_X$.

5.1.1. The presheaf $G_{M,V}$. We denote by $G_{M,V}$ the presheaf of sets

$$\mathbf{Aff}^{op}_{\mathcal{A}^\sim} = Alg\mathcal{A}^\sim \longrightarrow Sets$$

which assigns to every associative unital algebra $\mathcal{R}$ in $\mathcal{A}^\sim$ the set of pairs of morphisms $\eta^*_{\mathcal{R}}(V) \xrightarrow{v} \eta^*_{\mathcal{R}}(M) \xrightarrow{u} \eta^*_{\mathcal{R}}(V)$ such that $u \circ v = id_{\eta^*_{\mathcal{R}}(V)}$ and acts naturally on morphisms. The map

$$G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V}, \quad (v,u) \longmapsto [u], \tag{1}$$

is an epimorphism of presheaves.

5.1.2. Relations. Let $\mathfrak{R}_{M,V}$ denote the "presheaf of relations", which is, by definition, the kernel pair $Ker_2(\pi_{M,V}) = G_{M,V} \prod_{Gr_{M,V}} G_{M,V}$ of the morphism (1). Explicitly, $\mathfrak{R}_{M,V}$ is a subpresheaf of the product $G_{M,V} \times G_{M,V}$ which assigns to each associative unital algebra $\mathcal{R}$ in $\mathcal{A}^\sim$ the set of all 4-tuples $(u_1, v_1; u_2, v_2) \in G_{M,V}(\mathcal{R}) \times G_{M,V}(\mathcal{R})$ such that the split epimorphisms $\eta^*_{\mathcal{R}}(M) \overset{u_1}{\underset{u_2}{\rightrightarrows}} \eta^*_{\mathcal{R}}(V)$ are equivalent. The latter means that there exists an isomorphism $\eta^*_{\mathcal{R}}(V) \xrightarrow{\psi} \eta^*_{\mathcal{R}}(V)$ such that $u_2 = \psi \circ u_1$, or, equivalently, $\psi^{-1} \circ u_2 = u_1$. Since $u_i \circ v_i = id$, $i = 1,2$, these equalities imply that $\psi = u_2 \circ v_1$ and $\psi^{-1} = u_1 \circ v_2$. Thus $\mathfrak{R}_{M,V}(\mathcal{R})$ is a subset of all $(u_1, v_1; u_2, v_2) \in G_{M,V}(\mathcal{R}) \times G_{M,V}(\mathcal{R})$ satisfying the following relations:

$$u_2 = (u_2 \circ v_1) \circ u_1, \quad u_1 = (u_1 \circ v_2) \circ u_2 \tag{2}$$

in addition to the relations describing $G_{M,V}(\mathcal{R}) \times G_{M,V}(\mathcal{R})$:

$$u_1 \circ v_1 = id = u_2 \circ v_2 \tag{3}$$

Let $\mathfrak{R}_{M,V} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{M,V}$ denote the canonical projections. It follows from the surjectivity of the projection $G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V}$ that the diagram

$$\mathfrak{R}_{M,V} \overset{p_1}{\underset{p_2}{\rightrightarrows}} G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V} \tag{4}$$

is exact.

5.1.3. Remark. Let $\mathcal{A}^\sim = \mathcal{R}^e - mod^\sim$ acting on the category $\mathcal{R} - mod$ of left modules over an associative unital k-algebra $\mathcal{R}$. This is the setting of Chapter III (starting from Section 5). Notice, however, that the presheaves $Gr_{M,V}$ and $G_{M,V}$ are defined for arbitrary pairs of $\mathcal{R}$-modules (M, V), while in Chapter III, we required that V is a projective $\mathcal{R}$-module (cf. III.6.0).

5.1.4. Proposition. *Suppose that the category $\mathcal{A}$ has limits of finite diagrams.*

If the pairs of objects (M, V), (V, M) and (V, V) are admissible, then the presheaves $G_{M,V}$ and $\mathfrak{R}_{M,V}$ are representable.

Proof. The argument follows the lines of the argument of III.6.3.

The presheaf $G_{M,V}$ is the kernel of the pair

$$\mathcal{H}_{M,V} \times \mathcal{H}_{V,M} \overset{\mathfrak{c}}{\underset{\wp}{\rightrightarrows}} \mathcal{H}_{V,V} \tag{5}$$

of presheaf morphisms, where, for every unital associative algebra S in $\mathcal{A}^\sim$, the map $\mathfrak{c}(S)$ is the composition of morphisms and the map $\wp(S)$ sends the whole set to the identical morphism of $\eta_S^*(V)$. If the pairs of objects (M, V), (V, M) and (V, V) are admissible, then the presheaves in (5) are representable. Therefore, since $\mathcal{A}$ has limits of finite diagrams, the presheaf $G_{M,V}$ is representable.

The representability of the presheaf $\mathfrak{R}_{M,V}$ follows from the representability of $G_{M,V}$, the description of $\mathfrak{R}_{M,V}$ in terms of relations (cf. 5.1.2), and Proposition 2.4.2. Details are left to the reader. ∎

5.1.4.1. Note. The representability of the presheaf $G_{M,V}$ is a consequence of 4.1 applied to the following setting: D is a category with two objects, x and y, and arrows

$x \xrightarrow{f} y,\ y \xrightarrow{g} x$ such that $f \circ g = id_y$; B is the discrete subcategory of D (i.e. it has only identical morphisms) with objects x and y; and the functor $B \xrightarrow{F} \mathcal{C}$ maps the object x to M and the object y to V.

5.1.5. Finiteness conditions. There is the following generalization of III.6.3.1:

5.1.5.1. Proposition. *Let the action $\Phi^\sim$ of the monoidal category $\mathcal{A}^\sim$ on the category $\mathcal{C}_X$ be finitely presentable. Suppose that the pairs (M,V), (V,M) and (V,V) are admissible and M is a finitely presentable object of the category $\mathcal{C}_X$. Then the presheaves $\mathfrak{R}_{M,V}$, $G_{M,V}$ and $Gr_{M,V}$ are locally finitely copresentable.*

Proof. Notice that if M is a finitely presentable object of $\mathcal{C}_X$, then $\eta_S^*(M)$ is a finitely presentable object of the category $S - mod_X$; and $\eta_S^*(V)$ is finitely presentable object of $S - mod_X$, if $G_{M,V}(S) \neq \emptyset$. This implies that $G_{M,V}$ is the kernel of a pair of morphisms between locally finitely copresentable presheaves. Therefore, the presheaf $G_{M,V}$ is locally finitely copresentable. One of the consequences of the latter fact is that the presheaf of relations $\mathfrak{R}_{M,V}$ is the limit of a finite diagram of locally finitely copresentable presheaves, hence it is locally finitely copresentable. The presheaf $Gr_{M,V}$ is locally finitely copresentable, because it is the cokernel of a pair of morphisms between locally finitely copresentable presheaves. ■

5.2. Generic Grassmannians. Fix an object E of the category $\mathcal{C}_X$. For any associative unital algebra $\mathcal{S}$ in $\mathcal{A}^\sim$, we denote by $Gr_E(S)$ the set of isomorphism classes of split epimorphisms $S(E) \longrightarrow L$. The map $\mathcal{S} \longmapsto Gr_E(\mathcal{S})$ extends naturally to a presheaf of sets

$$\mathbf{Aff}_{\mathcal{A}^\sim}^{op} = Alg\mathcal{A}^\sim \xrightarrow{Gr_E} Sets.$$

For any $L \in Ob\mathcal{A}$, there is a natural presheaf morphism $G_{E,L} \xrightarrow{\rho_L} Gr_E$.

5.2.1. The presheaf $\mathfrak{Pr}_E$. Let

$$\mathbf{Aff}_{\mathcal{A}^\sim}^{op} = Alg\mathcal{A}^\sim \xrightarrow{\mathfrak{Pr}_E} Sets$$

be presheaf of sets which assigns to any associative unital algebra $\mathcal{S}$ in $\mathcal{A}^\sim$ the set of projectors of $\eta_{\mathcal{S}}^*(E)$, i.e. morphisms $\eta_{\mathcal{S}}^*(E) \xrightarrow{p} \eta_{\mathcal{S}}^*(E)$ such that $p^2 = p$.

5.2.1.1. Lemma. *Suppose that the category $\mathcal{C}_X$ is Karoubian; i.e. every idempotent splits. Then, for any monad $\mathcal{F} = (F, \mu_{\mathcal{F}})$ on $\mathcal{C}_X$, the category $(\mathcal{F}/X) - mod$ of $\mathcal{F}$-modules is Karoubian too.*

Proof. Let $\mathcal{M} = (M, \xi_{\mathcal{M}})$ be an $\mathcal{F}$-module and $\mathcal{M} \xrightarrow{\mathfrak{p}} \mathcal{M}$ an idempotent such that the idempotent $M \xrightarrow{\mathfrak{p}} M$ of the object M is the composition of an epimorphism

$M \xrightarrow{\mathfrak{e}} L$ and a monomorphism $L \xrightarrow{\mathfrak{j}} M$. The fact that $\mathfrak{p} = \mathfrak{p}^2$ can be expressed as $\mathfrak{j} \circ (\mathfrak{e} \circ \mathfrak{j}) \circ \mathfrak{e} = \mathfrak{j} \circ \mathfrak{e}$, which implies that $\mathfrak{j} \circ (\mathfrak{e} \circ \mathfrak{j}) = \mathfrak{j}$ (because $\mathfrak{e}$ is an epimorphism) and $\mathfrak{e} \circ \mathfrak{j} = id_L$, because $\mathfrak{j}$ is a monomorphism. The fact that $\mathfrak{p} = \mathfrak{j} \circ \mathfrak{e}$ is an $\mathcal{F}$-module morphism is expressed by

$$\mathfrak{j} \circ \mathfrak{e} \circ \xi_{\mathcal{M}} = \xi_{\mathcal{M}} \circ F(\mathfrak{j}) \circ F(\mathfrak{e}).$$

Set $\xi_{\mathcal{L}} = \mathfrak{e} \circ \xi_{\mathcal{M}} \circ F(\mathfrak{j})$. It follows that

$$\mathfrak{e} \circ \xi_{\mathcal{M}} = \mathfrak{e} \circ \mathfrak{j} \circ \mathfrak{e} \circ \xi_{\mathcal{M}} = (\mathfrak{e} \circ \xi_{\mathcal{M}} \circ F(\mathfrak{j})) \circ F(\mathfrak{e}) = \xi_{\mathcal{L}} \circ F(\mathfrak{e}) \quad \text{and}$$
$$\mathfrak{j} \circ \xi_{\mathcal{L}} = \mathfrak{j} \circ (\mathfrak{e} \circ \xi_{\mathcal{M}} \circ F(\mathfrak{j})) = \xi_{\mathcal{M}} \circ F(\mathfrak{j}) \circ F(\mathfrak{e}) \circ F(\mathfrak{j}) = \xi_{\mathcal{M}} \circ F(\mathfrak{j}),$$

which shows that $\mathfrak{j}$ is a morphism from the action $\mathcal{L} = (L, \xi_{\mathcal{L}})$ to $\mathcal{M} = (M, \xi_{\mathcal{M}})$ and $\mathfrak{e}$ is a morphism from $\mathcal{M}$ to the action $\mathcal{L}$. It remains to observe that

$$\xi_{\mathcal{L}} \circ F(\xi_{\mathcal{L}}) = \mathfrak{e} \circ \xi_{\mathcal{M}} \circ F(\mathfrak{j}) \circ F(\mathfrak{e}) \circ F(\xi_{\mathcal{M}}) \circ F^2(\mathfrak{j}) =$$
$$\mathfrak{e} \circ \xi_{\mathcal{M}} \circ F(\mathfrak{j} \circ \mathfrak{e}) \circ F(\xi_{\mathcal{M}}) \circ F^2(\mathfrak{j}) = \mathfrak{e} \circ (\mathfrak{j} \circ \mathfrak{e}) \circ \xi_{\mathcal{M}} \circ F(\xi_{\mathcal{M}}) \circ F^2(\mathfrak{j}) =$$
$$\mathfrak{e} \circ \xi_{\mathcal{M}} \circ \mu(M) \circ F^2(\mathfrak{j}) = \mathfrak{e} \circ \xi_{\mathcal{M}} \circ F(\mathfrak{j}) \circ \mu(L) = \xi_{\mathcal{L}} \circ \mu(L) \qquad \text{and}$$
$$\xi_{\mathcal{L}} \circ \eta_{\mathcal{F}}(L) = \mathfrak{e} \circ \xi_{\mathcal{M}} \circ F(\mathfrak{j}) \circ \eta_{\mathcal{F}}(L) = \mathfrak{e} \circ \xi_{\mathcal{M}} \circ \eta_{\mathcal{F}}(M) \circ \mathfrak{j} = \mathfrak{e} \circ \mathfrak{j} = id_L,$$

where $Id_{\mathcal{C}_X} \xrightarrow{\eta_{\mathcal{F}}} F$ is the unit of the monad $\mathcal{F}$. This shows that $F(L) \xrightarrow{\xi_{\mathcal{L}}} L$ is an $\mathcal{F}$-module structure. ∎

5.2.1.2. Remark. The object L in the decomposition $M \xrightarrow{\mathfrak{e}} L \xrightarrow{\mathfrak{j}} M$ of the idempotent $\mathfrak{p}$ is isomorphic to the kernel and the cokernel of the pair of morphisms $M \overset{\mathfrak{p}}{\underset{id}{\rightrightarrows}} M$. Since the both arrows are endomorphisms of the $\mathcal{F}$-module $\mathcal{M}$, the action $\mathcal{L} = (L, \xi_{\mathcal{L}})$ is the kernel of the pair $\mathcal{M} \overset{\mathfrak{p}}{\underset{id}{\rightrightarrows}} \mathcal{M}$ of $\mathcal{F}$-module morphisms. This also shows, by passing, that $F(L) \xrightarrow{\xi_{\mathcal{L}}} L$ is an $\mathcal{F}$-module structure.

5.2.1.3. Corollary. *Suppose that the category $\mathcal{C}_X$ is Karoubian. Then there is a natural presheaf epimorphism*

$$\mathfrak{Pr}_E \xrightarrow{\pi_E} Gr_E \qquad (1)$$

Proof. It follows from 5.2.1.1 that, for any associative unital algebra $\mathcal{S}$ in $\mathcal{A}^\sim$, there is a well defined map

$$\mathfrak{Pr}_E(\mathcal{S}) \xrightarrow{\pi_E(\mathcal{S})} Gr_E(\mathcal{S}),$$

which assigns to every idempotent $\eta_{\mathcal{S}}^*(\mathcal{M}) \xrightarrow{\mathfrak{p}} \eta_{\mathcal{S}}^*(\mathcal{M})$ of $\Phi_{Alg}^{\sim}(\mathcal{S})$-module $\eta_{\mathcal{S}}^*(\mathcal{M})$ the isomorphism class of the cokernel $\eta_{\mathcal{S}}^*(\mathcal{M}) \longrightarrow \mathcal{L}$ of the pair $\eta_{\mathcal{S}}^*(\mathcal{M}) \underset{id}{\overset{\mathfrak{p}}{\rightrightarrows}} \eta_{\mathcal{S}}^*(\mathcal{M})$. ■

5.2.2. Relations. Two projectors, $\eta_{\mathcal{S}}^*(E) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \eta_{\mathcal{S}}^*(E)$ are equivalent, if their images by (1) are isomorphic. The latter can be expressed by the equalities

$$p_1p_2p_1 = p_1 \quad \text{and} \quad p_2p_1p_2 = p_2. \tag{2}$$

Thus, the functor of relations $\mathfrak{R}_E \stackrel{\text{def}}{=} Ker_2(\pi_E) = \mathfrak{Pr}_E \prod_{Gr_E} \mathfrak{Pr}_E$ of the morphism (1) assigns to each associative unital algebra $\mathcal{S}$ in $\mathcal{A}^{\sim}$ the subset of all elements (p_1, p_2) of the set $\mathfrak{Pr}_E(\mathcal{S}) \times \mathfrak{Pr}_E(\mathcal{S})$ satisfying the relations (2) above (in addition to the relations $p_i^2 = p_i,\ i = 1, 2$ describing $\mathfrak{Pr}_E(\mathcal{S})$).

We have an exact diagram

$$\mathfrak{R}_E \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathfrak{Pr}_E \xrightarrow{\pi_E} Gr_E \tag{3}$$

5.2.3. Proposition. *Suppose that the category $\mathcal{C}_X$ is Karoubian and the category $\mathcal{A}$ has limits of finite diagrams. Let M be an object of $\mathcal{C}_X$ such that the pair (M, M) is admissible. Then the presheaves $\mathfrak{Pr}_M$ and $\mathfrak{R}_M$ are representable.*

Proof. The argument below mimics the corresponding part of the proof of III.8.5.

(a) For any associative unital algebra S in $\mathcal{A}^{\sim}$, the set $\mathfrak{Pr}_{\mathcal{M}}(S)$ is the kernel of the pair of maps

$$S - mod_X(\eta_S^*(M), \eta_S^*(M)) = \mathcal{H}_{M,M}(S) \underset{id}{\overset{\mathfrak{s}(S)}{\rightrightarrows}} \mathcal{H}_{M,M}(S), \tag{1}$$

where $\mathfrak{s}$ is the "taking square" – the composition of the diagonal map

$$\mathcal{H}_{M,M}(S) \xrightarrow{\Delta} \mathcal{H}_{M,M}(S) \times \mathcal{H}_{M,M}(S)$$

and the composition of endomorphisms. The diagram (1) is functorial in S; i.e. it is the value at S of the pair

$$\mathcal{H}_{M,M} \underset{id}{\overset{\mathfrak{s}}{\rightrightarrows}} \mathcal{H}_{M,M} \tag{2}$$

of morphisms of presheaves. If (M, M) is an admissible pair, then the presheaf $\mathcal{H}_{M,M}$ is representable by the vector fiber of $M^\wedge M$. Therefore, the presheaf $\mathfrak{Pr}_{\mathcal{M}}$ is representable by the cokernel of the pair of unital algebra morphisms

$$T(M^\wedge M) \underset{id}{\overset{\sigma}{\rightrightarrows}} T(M^\wedge M),$$

where σ is the morphism of algebras corresponding to the presheaf morphism $\mathfrak{s}$ in (2).

(b) For any associative unital algebra S in $\mathcal{A}^\sim$, the set $\mathfrak{R}_{\mathcal{M}}(S)$ is the limit of the diagram

$$\begin{array}{ccc} \mathfrak{Pr}_M(S) \times \mathfrak{Pr}_M(S) & \underset{\pi_1}{\overset{\mathfrak{m}}{\rightrightarrows}} & \mathfrak{Pr}_M(S) \\ \mathfrak{S} \downarrow \wr & & \downarrow id \\ \mathfrak{Pr}_M(S) \times \mathfrak{Pr}_M(S) & \underset{\pi_1}{\overset{\mathfrak{m}}{\rightrightarrows}} & \mathfrak{Pr}_M(S) \end{array} \qquad (3)$$

where $\mathfrak{m}$ is the composition map, π_2 is the projection on the first component, $(x, y) \longmapsto x$; and the left vertical map is the standard symmetry: $(x, y) \longmapsto (y, x)$.

If the pair (M, M) is admissible, then, by (a) above, the presheaf $\mathfrak{Pr}_M$ is representable. Therefore, since, by hypothesis, the category $\mathcal{A}$ has finite limits, the presheaf $\mathfrak{R}_M$ is representable. ∎

5.2.4. Finiteness conditions.

5.2.4.1. Proposition. *Suppose that the category $\mathcal{C}_X$ is Karoubian and the action $\Phi^\sim$ of the monoidal category $\mathcal{A}^\sim$ on $\mathcal{C}_X$ is finitely presentable. Let M be a finitely presentable object of the category $\mathcal{C}_X$ such that the pair (M, M) is admissible. Then the presheaves $\mathfrak{R}_M$, $\mathfrak{Pr}_M$ and Gr_M are locally finitely copresentable.*

Proof. By the argument of 5.2.3, the presheaf $\mathfrak{Pr}_M$ is the kernel of the pair $\mathcal{H}_{M,M} \underset{id}{\overset{\mathfrak{s}}{\rightrightarrows}} \mathcal{H}_{M,M}$ of presheaf morphisms. If (M, M) is an admissible pair, then the presheaf $\mathcal{H}_{M,M}$ is presentable by the vector fiber of the object $M^\wedge M$ of the category $\mathcal{A}$. If, in addition, M is a finitely presentable object of the category $\mathcal{C}_X$, then the object $M^\wedge M$ of $\mathcal{A}$ is finitely presentable, hence the presheaf $\mathcal{H}_{M,M}$ is a locally finitely copresentable presheaf of sets. Therefore the presheaf $\mathfrak{Pr}_M$ is locally finitely copresentable.

Therefore, the presheaf of relations $\mathfrak{R}_M$ is the limit of a finite diagram (– the diagram (3) in the argument of 5.2.3) formed by finitely corepresentable presheaves. Therefore, $\mathfrak{R}_M$ is locally finitely corepresentable.

The presheaf Gr_M is locally finitely corepresentable, because it is the cokernel of a pair of morphisms between locally finitely corepresentable presheaves. ■

5.3. Some properties of Grassmannians.

5.3.1. Functoriality. One can see that the maps $E \longmapsto Gr_E$ and $E \longmapsto Gr_{E,L}$ are functorial for split epimorphisms. Moreover, we have the following

5.3.1.1. Proposition. *For any split epimorphism $E' \longrightarrow E$, the corresponding morphisms $Gr_E \longrightarrow Gr_{E'}$ and $Gr_{E,L} \longrightarrow Gr_{E',L}$ are closed immersions.*

Proof. An adaptation of the argument of III.8.6, which is left to the reader. ■

5.3.1.2. Proposition. *The canonical morphism $Gr_{E,L} \xrightarrow{\rho_E} Gr_E$ is functorial in E. Moreover, for any locally split epimorphism $E' \longrightarrow E$, the square*

$$\begin{array}{ccc} Gr_{E,L} & \xrightarrow{\rho_E} & Gr_E \\ \downarrow & cart & \downarrow \\ Gr_{E',L} & \xrightarrow{\rho_{E'}} & Gr_{E'} \end{array} \qquad (1)$$

is cartesian.

Proof. See III.8.6. ■

5.3.2. Proposition. *Grassmannians are separated.*

Proof. Let $\mathcal{R}^\vee$ be an arbitrary affine scheme, and let $\mathcal{R}^\vee \overset{u_1}{\underset{u_2}{\rightrightarrows}} Gr_{E,L}$ be a pair of morphisms. The claim is that the kernel of the pair (u_1, u_2) is representable.

Let $\eta^*_{\mathcal{R}}(E) \xrightarrow{\xi_i} L_i$ be a split morphism corresponding to u_i, $i = 1, 2$. Let (v^i_1, v^i_2) be a pair of arrows $M_i \overset{v^i_1}{\underset{v^i_2}{\rightrightarrows}} \eta^*_{\mathcal{R}}(E)$ such that u_i is a cokernel of (v^i_1, v^i_2), $i = 1, 2$. Consider the compositions

$$M_2 \overset{\xi_1 v^2_1}{\underset{\xi_1 v^2_2}{\rightrightarrows}} L_1 \quad \text{and} \quad M_1 \overset{\xi_2 v^1_1}{\underset{\xi_2 v^1_2}{\rightrightarrows}} L_2 \qquad (1)$$

By 2.4.2, there exists a universal affine scheme morphism $\mathcal{S}^\vee \xrightarrow{\psi} \mathcal{R}^\vee$ such that the image of each of the pairs (1) by ψ^* belongs to the diagonal. This morphism ψ is a closed immersion. ■

5.4. Vector bundles and Grassmannians. Fix a morphism $L \xrightarrow{\phi} E$. For any algebra $\mathcal{S}$, consider the set $F_{\phi;E,L}(\mathcal{S})$ of all morphisms $\eta_{\mathcal{S}}^*(E) \xrightarrow{v} L'$ such that $v \circ \eta_{\mathcal{S}}^*(\phi)$ is an isomorphism.

5.4.1. Proposition. *(a) The map* $\mathcal{S} \longmapsto F_{\phi;E,L}(\mathcal{S})$ *is naturally extended to a subpresheaf* $F_{\phi;E,L} : \mathbf{Aff}_{\mathcal{A}^\sim}^{op} = Alg\mathcal{A}^\sim \longrightarrow Sets$ *of the presheaf* $Gr_{E,L}$.

(b) Suppose that the pairs (E, L) *and* (L, L) *are admissible. Then the presheaf* $F_{\phi;E,L}$ *is representable.*

(c) The canonical morphism $F_{\phi;E,L} \longrightarrow Gr_{E,L}$ *is an affine localization.*

Proof. (a) (i) In fact, if $\eta_{\mathcal{S}}^*(E) \xrightarrow{v} L'$ belongs to $F_{\phi;E,L}(\mathcal{S})$, i.e. $v \circ \eta_{\mathcal{S}}^*(\phi)$ is an isomorphism, then for any morphism $\mathcal{S} \xrightarrow{h} T$, the composition $h^*(v) \circ h^*\eta_{\mathcal{S}}^*(\phi)$ is an isomorphism. There is a natural morphism $F_{\phi;E,L} \longrightarrow Gr_{E,L}$.

(ii) One can identify $F_{\phi;E,L}(\mathcal{S})$ with the set of morphisms $\eta_{\mathcal{S}}^*(E) \xrightarrow{v} \eta_{\mathcal{S}}^*(L)$ such that $v \circ \eta_{\mathcal{S}}^*(\phi) = id_{\eta_{\mathcal{S}}^*(L)}$. In fact, if $\eta_{\mathcal{S}}^*(E) \xrightarrow{v'} L'$ is such that

$$w = v' \circ \eta_{\mathcal{S}}^*(\phi) : \eta_{\mathcal{S}}^*(L) \longrightarrow L'$$

is an isomorphism, then $v = w^{-1} \circ v'$ has the required property.

(iii) One of the consequences of the observation (ii) is that the canonical morphism $F_{\phi;E,L} \longrightarrow Gr_{E,L}$ is a monomorphism.

(b) There are two maps,

$$\mathcal{S} - mod_X(\eta_{\mathcal{S}}^*(E), \eta_{\mathcal{S}}^*(L)) \overset{\alpha_{\mathcal{S}}}{\underset{\beta_{\mathcal{S}}}{\rightrightarrows}} \mathcal{S} - mod_X(\eta_{\mathcal{S}}^*(L), \eta_{\mathcal{S}}^*(L)),$$

defined by $v \overset{\alpha_{\mathcal{S}}}{\longmapsto} v \circ \eta_{\mathcal{S}}^*(\phi)$, $v \overset{\beta_{\mathcal{S}}}{\longmapsto} id_{\eta_{\mathcal{S}}^*(L)}$. The maps $\alpha_{\mathcal{S}}$ and $\beta_{\mathcal{S}}$ are functorial in $\mathcal{S}$, hence they define morphisms, resp. α and β, from the presheaf

$$\mathcal{S}^\vee \longmapsto \mathcal{S} - mod_X(\eta_{\mathcal{S}}^*(E), \eta_{\mathcal{S}}^*(L)) \simeq \mathcal{A}(E, \eta_{\mathcal{S}}^*(L))$$

to the presheaf

$$\mathcal{S}^\vee \longmapsto \mathcal{S} - mod_X(\eta_{\mathcal{S}}^*(L), \eta_{\mathcal{S}}^*(L)) \simeq \mathcal{A}(L, \eta_{\mathcal{S}}^*(L)).$$

The first presheaf is representable by $\mathbb{V}(L^\wedge E)$, the second one is representable by $\mathbb{V}(L^\wedge L)$. Let α' and β' be morphisms from $\mathbb{V}(L^\wedge E)$ to $\mathbb{V}(L^\wedge L)$ corresponding to resp. α and β. The presheaf $F_{\phi;E,L}$ is the kernel of the pair (α, β), hence it is representable by the kernel, $\mathbf{F}_{\phi;E,L}$, of the pair (α', β').

(c) *The presheaf morphism* $F_{\phi;E,L} \longrightarrow Gr_{E,L}$ *is representable by an affine morphism*; i.e. for any affine $\mathcal{A}^{\sim}$-scheme $\mathcal{S}^{\vee}$ and any morphism $\mathcal{S}^{\vee} \longrightarrow Gr_{E,L}$, the presheaf

$$\mathbf{Aff}^{op}_{\mathcal{A}^{\sim}} \longrightarrow Sets, \quad \mathcal{T} \longmapsto F_{\phi;E,L}(\mathcal{T}) \prod_{Gr_{E,L}(\mathcal{T})} \mathcal{S}^{\vee}(\mathcal{T})$$

is representable by an affine subscheme of $\mathcal{S}^{\vee}$.

In fact, any morphism $\mathcal{S}^{\vee} \longrightarrow Gr_{E,L}$ is uniquely determined by an element of $Gr_{E,L}(\mathcal{S})$, i.e. by the equivalence class, $[v]$, of a split epimorphism $\eta^*_{\mathcal{S}}(E) \xrightarrow{v} L'$. The corresponding map $\mathcal{S}^{\vee}(Z) \longrightarrow Gr_{E,L}(Z)$ sends any morphism $\mathcal{S} \xrightarrow{t} \mathcal{T}$ to $[t^*(v)]$. The fiber product $F_{\phi;E,L}(\mathcal{T}) \prod_{Gr_{E,L}(\mathcal{T})} \mathcal{S}^{\vee}(\mathcal{T})$ consists of all pairs (w,t), where $t \in \mathcal{S}^{\vee}(\mathcal{T})$ and $[T(E) \xrightarrow{w} T(L)]$ are such that $w \circ T(\phi) = id_{T(L)}$ and $w = t^*(v)$. Since v and ϕ here are fixed, the fiber product $F_{\phi;E,L}(\mathcal{T}) \prod_{Gr_{E,L}(\mathcal{T})} \mathcal{S}^{\vee}(\mathcal{T})$ can be identified with the set of all morphisms $\mathcal{S} \xrightarrow{t^{\vee}} \mathcal{T}$ such that $t^*(v \circ T(\phi)) = id_{\eta^*_{\mathcal{T}}(L)}$. In other words, the fiber product $F_{\phi;E,L}(\mathcal{T}) \prod_{Gr_{E,L}(\mathcal{T})} \mathcal{S}^{\vee}(\mathcal{T})$ is identified with the kernel of the pair of morphisms

$$\mathcal{S}^{\vee}(\mathcal{T}) \underset{\beta_{\mathcal{T}}}{\overset{\alpha_{\mathcal{T}}}{\rightrightarrows}} \mathcal{T} - mod_X(T(L), T(L))$$

defined by

$$\beta_{\mathcal{T}} : t \longmapsto id_{T(L)}, \ \alpha_{\mathcal{T}} : t \longmapsto t^*(v \circ T(\phi)).$$

The morphisms $\beta_{\mathcal{T}}$, $\alpha_{\mathcal{T}}$ are functorial in $\mathcal{T}$, and the presheaf

$$\mathcal{T}^{\vee} \longmapsto \mathcal{T} - mod_X(\eta^*_{\mathcal{T}}(L), \eta^*_{\mathcal{T}}(L))$$

is representable by the vector fiber $\mathbb{V}(L^{\wedge}L)$. Therefore, the morphisms $\beta = (\beta_{\mathcal{T}})$, $\alpha = (\alpha_{\mathcal{T}})$ define a pair of morphisms

$$\mathcal{S}^{\vee} \underset{\beta'}{\overset{\alpha'}{\rightrightarrows}} \mathbb{V}(L^{\wedge}L),$$

and the presheaf of sets $\mathcal{T} \longmapsto F_{\phi;E,L}(\mathcal{T}) \prod_{Gr_{E,L}(\mathcal{T})} \mathcal{S}^{\vee}(\mathcal{T})$ is representable by the kernel of the pair (α', β'). ∎

5.4.2. Projective completion of a vector bundle. Let $E'' = E \coprod L$; and let $L \xrightarrow{j_L} E''$ be the coprojection. The presheaf $F_{j_L;E'',L}$ is isomorphic to the presheaf, which assigns to an affine $\mathcal{A}^{\sim}$-scheme $\mathcal{S}^{\vee}$ the set $\mathcal{S} - mod_X(\eta_{\mathcal{S}}^*(E), \eta_{\mathcal{S}}^*(L)) = \mathcal{H}_{E,L}(S)$ (see (ii) and (b) in the argument of 5.4.1). If the pair (E, L) is admissible, then the presheaf $\mathcal{H}_{E,L}$ is representable by the vector fiber $\mathbb{V}(L^{\wedge}E)$ of the object $L^{\wedge}L$ of the category $\mathcal{A}$. By 5.4.1 we have an affine embedding (an open immersion) $\mathbb{V}(L^{\wedge}E) \longrightarrow Gr_{E,L}$. In particular, taking $L = \mathbb{I}$, we obtain a canonical immersion $\mathbb{V}(E) \longrightarrow \mathbf{P}_E$. The projective space $\mathbb{P}_{E\sqcup\mathbb{I}}$ can be regarded (as in the commutative case) as the *projective completion* of the vector bundle $\mathbb{V}(E)$.

6. Generic flags and generalized Stiefel varieties.

6.1. Generic flags. Let $\mathfrak{I}$ be a preordered set with initial object $*$. Fix an object, E, of the category $\mathcal{C}_X$. For any associative unital algebra S in $\mathcal{A}^{\sim}$, we denote by $\mathfrak{Fl}_E^{\mathfrak{I}}(S)$ the set of isomorphism classes of functors $\mathfrak{I} \longrightarrow \mathcal{C}_X$ which map all arrows $* \longrightarrow i$ to split epimorphisms $\eta_{\mathcal{S}}^*(E) \longrightarrow L_i$. The map $S \longmapsto \mathfrak{Fl}_E^{\mathfrak{I}}(S)$ is functorial in S; i.e. it defines a presheaf of sets

$$\mathbf{Aff}_{\mathcal{A}^{\sim}}^{op} = Alg\mathcal{A}^{\sim} \xrightarrow{\mathfrak{Fl}_E^{\mathfrak{I}}} Sets, \tag{1}$$

which we call the *variety of generic flags.*

6.2. Generalized Stiefel varieties. Let E be an object of the category $\mathcal{C}_X$. Consider the presheaf

$$\mathbf{Aff}_{\mathcal{A}^{\sim}}^{op} = Alg\mathcal{A}^{\sim} \xrightarrow{\mathfrak{F}_E^{\mathfrak{I}}} Sets, \tag{2}$$

which assigns to every associative unital algebra S in the monoidal category $\mathcal{A}^{\sim}$ the set of projectors $\eta_{\mathcal{S}}^*(E) \xrightarrow{p_i} \eta_{\mathcal{S}}^*(E)$ such that $p_i p_j = p_i$, if $i \leq j$.

6.2.1. Definition. We call the presheaf $\mathbf{Aff}_{\mathcal{A}^{\sim}}^{op} \xrightarrow{\mathfrak{F}_E^{\mathfrak{I}}} Sets$ the *generalized Stiefel variety* of the object E, or, more precisely, the *Stiefel $\mathfrak{I}$-variety* of the object E.

The reason for this terminology is provided by III.11.1.

6.3. Note. If $\mathcal{A}^{\sim}$ is the monoidal category of R^e-modules acting on the category $R-mod$ of left R-modules for some associative unital k-algebra R and E is a projective R-module of finite type, then the definition 6.2.1 gives the same notion as III.11.3. Otherwise, these two notions are different.

6.4. The canonical projection and the relations. Suppose that all projectors of the object E split (say, $\mathcal{C}_X$ is a Karoubian category). Then there is a natural presheaf morphism

$$\mathfrak{F}_E^{\mathfrak{I}} \xrightarrow{\pi_E^{\mathfrak{I}}} \mathfrak{Fl}_E^{\mathfrak{I}} \tag{2}$$

and its kernel pair $\mathfrak{R}_E^{\mathfrak{I}} \overset{{}^1\pi_E^{\mathfrak{I}}}{\underset{{}^2\pi_E^{\mathfrak{I}}}{\rightrightarrows}} \mathfrak{F}_E^{\mathfrak{I}}$. The presheaf of relations $\mathfrak{R}_E^{\mathfrak{I}} = \mathfrak{F}_E^{\mathfrak{I}} \prod_{\mathfrak{Fl}_E^{\mathfrak{I}}} \mathfrak{F}_E^{\mathfrak{I}}$ consists of all $(\{p_i | i \in \mathfrak{I}\}, \{p'_i | i \in \mathfrak{I}\}) \in \mathfrak{F}_E^{\mathfrak{I}} \times \mathfrak{F}_E^{\mathfrak{I}}$ satisfying the relations

$$p_i p'_i p_i = p_i \quad \text{and} \quad p'_i p[_i p'_i = p'_i \tag{3}$$

(see the argument of 5.2.1).

If $\mathfrak{I} = \{0, 1, 2, ..., r\}$ with the natural order, we shall write $\mathfrak{Fl}_E^r$ instead of $\mathfrak{Fl}_E^{\mathfrak{I}}$ and $\mathfrak{F}_E^r$ instead of $\mathfrak{F}_E^{\mathfrak{I}}$.

6.4.1. Proposition. *Suppose that* (E, E) *is an admissible pair and the category* $\mathcal{A}$ *has coproducts of sets of copies of the object* $E^\wedge E$ *of the cardinality* $\leq 4|\mathfrak{I}|$. *Then the presheaves* $\mathfrak{F}_E^{\mathfrak{I}}$ *and* $\mathfrak{R}_E^{\mathfrak{I}}$ *are representable.*

Proof. The argument (mimicking that of III.9.2(c)) is as follows.

(a) The presheaf $\mathfrak{F}_E^{\mathfrak{I}}$ is the limit of the diagram

$$\begin{array}{c} \mathcal{H}_{E,E}^{\times \mathfrak{I}} \xrightarrow{\mathfrak{p}_i \times \mathfrak{p}_j} \mathcal{H}_{E,E} \prod \mathcal{H}_{E,E} \overset{\mathfrak{m}}{\underset{\mathfrak{p}_1}{\rightrightarrows}} \mathcal{H}_{E,E} \\ \text{for all} \quad i, j \in \mathfrak{I} - \{\bullet\} \quad \text{such that} \quad i \leq j, \end{array} \tag{4}$$

where the left arrow is the projection of the product of $\mathfrak{I}$ copies of $\mathcal{H}_{E,E}$ on the product of i^{th} and j^{th} components, the upper right arrow is the composition and the lower one is the projection on the first component. The limit of the diagram (4) is the kernel of the corresponding pair of arrows

$$\mathcal{H}_{E,E}^{\times \mathfrak{I}} \overset{\mathfrak{m}^{\mathfrak{I}}}{\underset{\mathfrak{p}_1^{\mathfrak{I}}}{\rightrightarrows}} \mathcal{H}_{E,E}^{\times Hom(\mathfrak{I} - \{\bullet\})}.$$

Here the preorder $\mathfrak{I}$ is regarded as a category; so that $Hom(\mathfrak{I} - \{\bullet\})$ can be identified with the set of all pairs (i, j) such that $i \leq j$ and $i \neq \bullet$.

If the pair (E, E) is admissible, then $\mathcal{H}_{E,E}^{\times \mathfrak{I}}$ is representable by the vector fiber of the coproduct of $\mathfrak{I}$ copies of $E^\wedge E$. Similarly, $\mathcal{H}_{E,E}^{\times Hom(\mathfrak{I} - \{\bullet\})}$ is representable by the vector fiber of the coproduct of $Hom(\mathfrak{I} - \{\bullet\})$ copies of $E^\wedge E$. Both coproducts exist by hypothesis. Therefore, the presheaf $\mathfrak{F}_E^{\mathfrak{I}}$ is representable.

(b) The presheaf $\mathfrak{R}_E^{\mathfrak{I}}$ is the limit of the diagram

$$
\begin{array}{ccccc}
\mathcal{H}_{E,E}^{\times\mathfrak{I}} & \xrightarrow{\mathfrak{p}_i\times\mathfrak{p}_j} & \mathcal{H}_{E,E}\times\mathcal{H}_{E,E} & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathcal{H}_{E,E} \\
\pi_1^{\mathfrak{I}}\big\uparrow & & & & \\
\mathcal{H}_{E,E}^{\times\mathfrak{I}}\coprod\mathcal{H}_{E,E}^{\times\mathfrak{I}} & \xrightarrow{\mathfrak{p}_i\times\mathfrak{p}_i} & \mathcal{H}_{E,E}\times\mathcal{H}_{E,E} & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathcal{H}_{E,E} \\
\mathfrak{S}\big\downarrow\wr & & \sigma\big\downarrow\wr & & \\
\mathcal{H}_{E,E}^{\times\mathfrak{I}}\coprod\mathcal{H}_{E,E}^{\times\mathfrak{I}} & \xrightarrow{\mathfrak{p}_i\times\mathfrak{p}_i} & \mathcal{H}_{E,E}\times\mathcal{H}_{E,E} & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathcal{H}_{E,E} \\
\pi_1^{\mathfrak{I}}\big\downarrow & & & & \\
\mathcal{H}_{E,E}^{\times\mathfrak{I}} & \xrightarrow{\mathfrak{p}_i\times\mathfrak{p}_j} & \mathcal{H}_{E,E}\times\mathcal{H}_{E,E} & \overset{\mathfrak{m}}{\underset{\pi_1}{\rightrightarrows}} & \mathcal{H}_{E,E} \\
 & & i\in\mathfrak{I}-\{\bullet\}\ni j & \text{and} & i<j.
\end{array}
\qquad (5)
$$

Here $\mathfrak{S}$ denotes the standard symmetry; the vertical arrow $\pi_1^{\mathfrak{I}}$ and the (four times repeated) horizontal arrow π_1 in the right column are projections to the first component, and the (four times repeated) arrow $\mathcal{H}_{E,E}\times\mathcal{H}_{E,E}\xrightarrow{m}\mathcal{H}_{E,E}$ is the composition map.

The limit of the diagram (5) is isomorphic to the kernel of the pair of arrows between presheaves representable by the vector fibers of coproducts of sets of copies of the object $E^\wedge E$ associated with the diagram (5). Hence the representability of the presheaf $\mathfrak{R}_E^{\mathfrak{I}}$. ∎

6.5. Proposition. *Suppose that the category $\mathcal{A}$ has limits of finite diagrams, $\mathfrak{I}$ is finite, and the pair (E,E) is admissible. Then the presheaf of sets $\prod_{i\in\mathfrak{I}} Gr_E$ is locally representable and the natural embedding*

$$\mathfrak{Fl}_E^{\mathfrak{I}} \longrightarrow \prod_{i\in\mathfrak{I}} Gr_E \qquad (3)$$

is a closed immersion.

Proof. The argument is an adaptation of the proof of III.9.3 left to the reader. ∎

6.6. An action of GL_E on generic flag varieties. The presheaf of groups GL_E acts naturally on the presheaf $\mathfrak{F}_E^{\mathfrak{I}}$ and on $\mathfrak{Fl}_E^{\mathfrak{I}}$, and the canonical morphism 6.1(1) is compatible with these actions.In particular, the induced action of GL_E on $\mathfrak{F}_E^{\mathfrak{I}}\times\mathfrak{F}_E^{\mathfrak{I}}$ preserves the subpresheaf of relations $\mathfrak{R}_E^{\mathfrak{I}}$.

7. General Grassmannian type spaces.

7.1. The combinatorial data. We fix an action of the monoidal category $\mathcal{A}^\sim$ on a category $\mathcal{C}_X$. For any unital associative algebra $\mathcal{S}$ in $\mathcal{A}^\sim$, we denote by $\mathcal{C}_X \xrightarrow{s^*} S{-}mod_X$ the canonical left adjoint to the restriction of scalars functor $S - mod_X \xrightarrow{s_*} \mathcal{C}_X$.

We denote by $\mathfrak{E}$ the diagram

$$\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G_1}\big\downarrow & & \big\downarrow{\scriptstyle G} & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array}$$

formed by small categories and functors and by $\mathfrak{E}^S$ the data

$$\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G_1}\big\downarrow & & \big\downarrow{\scriptstyle G} & & \big\downarrow{\scriptstyle s^*} \\ D_1 & \xrightarrow{G_2} & D_2 & & S - mod_X \end{array}$$

obtained from $\mathfrak{E}$ by 'extension of scalars'.

7.2. The presheaf $\mathfrak{Fl}_{\mathfrak{E}}$. Let $(\mathcal{L}, \gamma_{\mathcal{L}})$, $(\mathcal{L}', \gamma_{\mathcal{L}'})$ be objects of the category $\mathfrak{F}^\sim_{E,G}(S)$ introduced in 4.4. Recall $\mathcal{L}$ is a functor from D_2 to $\mathcal{S} - mod_X$ and $\gamma_{\mathcal{L}}$ is a functor isomorphism $\mathcal{L} \xrightarrow{\sim} s^* \circ E$. We say that the objects $(\mathcal{L}, \gamma_{\mathcal{L}})$, $(\mathcal{L}', \gamma_{\mathcal{L}'})$ are *equivalent*, if there is a functor isomorphism

$$\mathcal{L} \circ G_2 \xrightarrow{\lambda} \mathcal{L}' \circ G_2$$

satisfying the equality

$$\gamma_{\mathcal{L}'} G_0 \circ \lambda G_1 = \gamma_{\mathcal{L}} G_0. \tag{1}$$

This is, indeed, an equivalence relation such that isomorphic objects belong to the same equivalence class. We denote by $\mathfrak{Fl}_{\mathfrak{E}}(S)$ the set of equivalence classes of objects of the category $\mathfrak{F}_{E,G}(S)^\sim$ and by $\pi_{\mathfrak{E}}(S)$ the natural surjection $\mathfrak{F}_{E,G}(S) \longrightarrow \mathfrak{Fl}_{\mathfrak{E}}(S)$.

The equivalence relation is functorial in S. So that $S \longmapsto \mathfrak{Fl}_{\mathfrak{E}}(S)$ is a presheaf of sets and $S \longmapsto \pi_{\mathfrak{E}}(S)$ is a presheaf epimorphism $\mathfrak{F}_{E,G} \xrightarrow{\pi_{\mathfrak{E}}} \mathfrak{Fl}_{\mathfrak{E}}$.

7.3. Relations. We complement the projection $\mathfrak{F}_{E,G} \xrightarrow{\pi_{\mathfrak{E}}} \mathfrak{Fl}_{\mathfrak{E}}$ with its kernel pair – the relations:

$$\mathfrak{R}_{\mathfrak{E}} \underset{\mathfrak{p}^2_{\mathfrak{E}}}{\overset{\mathfrak{p}^1_{\mathfrak{E}}}{\rightrightarrows}} \mathfrak{F}_{E,G} \xrightarrow{\pi_{\mathfrak{E}}} \mathfrak{Fl}_{\mathfrak{E}}. \tag{1}$$

By definition, $\mathfrak{R}_{\mathfrak{E}}(S)$ consists of all pairs $([(\mathcal{L},\gamma_{\mathcal{L}})],[(\mathcal{L}',\gamma_{\mathcal{L}'})])\in\mathfrak{F}_{E,G}(S)\times\mathfrak{F}_{E,G}(S)$ such that there exists an isomorphism $\mathcal{L}\circ G_2\xrightarrow{\ \lambda\ }\mathcal{L}'\circ G_2$ satisfying $\gamma_{\mathcal{L}'}G_0\circ\lambda G_1=\gamma_{\mathcal{L}}G_0$.

7.4. Remark. Suppose that the functor $B\xrightarrow{G}D_2$ in the data $\mathfrak{E}$ is injective and *essentially surjective* on objects. The latter means that G induces a surjection (hence bijection) of isomorphism classes of objects. Then it follows from 4.3 that each object $(\mathcal{L},\gamma_{\mathcal{L}})$ of the category $\mathfrak{F}_{\widetilde{E,G}}(\mathcal{S})$ is isomorphic to an object of the form $(\mathfrak{L},id)$, that is $\mathfrak{L}\circ G=\mathfrak{s}^*\circ E$. (Evidently, such representative is unique, if G is bijective on objects.)

Therefore, in this case, elements of the set $\mathfrak{Fl}_{\widetilde{\mathfrak{E}}}(\mathcal{S})$ are equivalence classes of the objects of the form $(\mathcal{L},id)$, where $(\mathcal{L},id)$ and $(\mathcal{L}',id)$ are equivalent, if there exists an isomorphism $\mathcal{L}\circ G_2\xrightarrow{\ \lambda\ }\mathcal{L}'\circ G_2$ such that $\lambda G_1=id$.

7.5. Representability and finiteness conditions.

7.5.1. Proposition. *(a) Suppose that the functor* $B\xrightarrow{G}D_2$ *in the data* $\mathfrak{E}$ *induces bijective map on isomorphism classes of objects and the pair of functors* $D_2\xleftarrow{G}B\xrightarrow{E}\mathcal{C}_X$ *satisfies the condition*

(†) *the pair of objects* $(E(M),E(L))$ *is admissible, if* $D_2(G(M),G(L))\neq\emptyset$.

Then the presheaf $\mathfrak{F}_{E,G}$ *is representable, provided the category* $\mathcal{A}$ *has colimits.*

(b) If, in addition, the functor $D_1\xrightarrow{G_2}D_2$ *in the data* $\mathfrak{E}$ *induces a bijective map on isomorphism classes of objects, then the presheaf* $\mathfrak{R}_{\mathfrak{E}}$ *of relations is representable too.*

Proof. (a) The assertion follows from Proposition 4.4 applied to the pair of functors $D_2\xleftarrow{G}B\xrightarrow{E}\mathcal{C}_X$.

(b1) We consider the data

$$\widetilde{\mathfrak{E}}=\left(\begin{array}{ccccc}\widetilde{B}_0 & \xrightarrow{\widetilde{G}_0} & B & \xrightarrow{E} & \mathcal{C}_X\\ \widetilde{G}_1\big\downarrow & cart & \big\downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array}\right)$$

obtained via replacing the square in $\mathfrak{E}$ with the canonical cartesian (in pseudo-sense) square. The fact that the functors $B\xrightarrow{G}D_2$ and $D_1\xrightarrow{G_2}D_2$ induce bijections on objects implies that the functors $\widetilde{B}_0\xrightarrow{\widetilde{G}_0}B$ and $\widetilde{B}_0\xrightarrow{\widetilde{G}_1}D_1$ have the same property. Besides, they are surjective on objects. It follows from the surjectivity of the functor $\widetilde{B}_0\xrightarrow{\widetilde{G}_0}B$ on objects that the condition (†) for the pair $D_2\xleftarrow{G}B\xrightarrow{E}\mathcal{C}_X$ implies the condition (†) for the pair $D_1\xleftarrow{\widetilde{G}_1}\widetilde{B}_0\xrightarrow{E\circ\widetilde{G}_0}\mathcal{C}_X$. Set $\widetilde{E}=E\circ\widetilde{G}_0$.

(b2) For every associative unital algebra $\mathcal{S}$ is $\mathcal{A}^\sim$, there is a natural functor

$$\widetilde{\mathfrak{F}}_{E,G}(\mathcal{S}) \longrightarrow \widetilde{\mathfrak{F}}_{\widetilde{E}.\widetilde{G}_1}(\mathcal{S})$$

which maps every object $(\mathcal{L}, \gamma_{\mathcal{L}})$ of the category $\widetilde{\mathfrak{F}}_{E,G}(\mathcal{S})$ to $(\mathcal{L} \circ G_2, \gamma_{\mathcal{L}}\widetilde{G}_0)$ and acts accordingly on morphisms. This functor induces a map

$$\mathfrak{F}_{E,G}(\mathcal{S}) \longrightarrow \mathfrak{F}_{\widetilde{E}.\widetilde{G}_1}(\mathcal{S})$$

between the sets of isomorphism classes of objects, which is functorial in $\mathcal{S}$. So that we have a presheaf morphism

$$\mathfrak{F}_{E,G} \longrightarrow \mathfrak{F}_{\widetilde{E},\widetilde{G}_1}.$$

(b3) If the functor $B \xrightarrow{G} D_2$ is essentially bijective on objects, then the functor $\widetilde{B}_0 \xrightarrow{\widetilde{G}_1} D_1$ is surjective on objects, which implies that the natural map from $\mathfrak{R}_{\mathfrak{E}}(\mathcal{S})$ to the kernel pair of the map $\mathfrak{F}_{E,G}(\mathcal{S}) \longrightarrow \mathfrak{F}_{\widetilde{E}.\widetilde{G}_1}(\mathcal{S})$ defined in (b2) is an isomorphism for any associative unital algebra $\mathcal{S}$ in $\mathcal{A}^\sim$.

(b4) Suppose now that the functor $D_1 \xrightarrow{G_2} D_2$ is essentially bijective on objects. Then the pair of functors $D_1 \xleftarrow{\widetilde{G}_1} \widetilde{B}_0 \xrightarrow{E\circ\widetilde{G}_0} \mathcal{C}_X$ satisfies the conditions of (a); hence the presheaf $\mathfrak{F}_{\widetilde{E},\widetilde{G}_1}$ is representable. Therefore, it follows from (b3) that the presheaf of relations $\mathfrak{R}_{\mathfrak{E}}$ is the kernel pair of a morphism between representable presheaves. Therefore, since, by hypothesis, the category $\mathcal{A}^\sim$ has colimits (in particular, the cokernel pairs of morphisms), the presheaf of relations $\mathfrak{R}_{\mathfrak{E}}$ is representable. ∎

7.5.2. Proposition. *Suppose that the categories $\mathcal{B}$, D_1, D_2 in the data*

$$\mathfrak{E} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1 \big\downarrow & & \big\downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array} \right)$$

are finite, the functors $B \xrightarrow{G} D_2$ and $D_1 \xrightarrow{G_2} D_2$ are essentially bijective on objects, and the pair of functors $D_2 \xleftarrow{G} B \xrightarrow{E} \mathcal{C}_X$ satisfies the condition

(†) *the pair of objects $(E(M), E(L))$ is admissible, if $D_2(G(M), G(L)) \neq \emptyset$.*

Then the presheaves $\mathfrak{F}_{E,G}$, $\mathfrak{R}_{\mathfrak{E}}$, and $\mathfrak{Fl}_{\mathfrak{E}}$ are locally finitely corepresentable.

Proof. It follows from 4.7.1 that the presheaf $\mathfrak{F}_{E,G}$ is locally finitely copresentable. By the argument of 7.5.1, the presheaf $\mathfrak{R}_{\mathfrak{E}}$ is the limit of a square formed by presheaves, which, by the same 4.7.1, are locally finitely copresentable. Therefore, $\mathfrak{R}_{\mathfrak{E}}$ is locally finitely copresentable. Finally, $\mathfrak{Fl}_{\mathfrak{E}}$ is locally finitely copresentable, because it is the cokernel of a pair of arrows between locally finitely copresentable presheaves. ■

7.5.3. Note. It follows from 7.5.1 that, under the conditions of 7.5.2, the presheaves $\mathfrak{F}_{E,G}$ and $\mathfrak{F}_{\mathfrak{E}}$ are representable, if the category $\mathcal{A}$ has colimits of finite diagrams.

7.6. Functorialities. A morphism from the data

$$\mathfrak{E} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1 \downarrow & & \downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array} \right)$$

to the data

$$\mathfrak{E}' = \left(\begin{array}{ccccc} B_0' & \xrightarrow{G_0'} & B' & \xrightarrow{E'} & \mathcal{C}_X \\ G_1' \downarrow & & \downarrow G & & \\ D_1' & \xrightarrow{G_2'} & D_2' & & \end{array} \right)$$

is a 1-morphism of the diagram $\mathfrak{E}$ to the diagram $\mathfrak{E}'$ identical on $\mathcal{C}_X$. Explicitly, it is given by the commutative diagram

$$\begin{array}{ccccccc} & & B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ & & \Psi_0 \downarrow & & \downarrow \Psi^{\dagger} & & \downarrow Id_{\mathcal{C}_X} \\ B_0 & \xrightarrow{\Psi_0} & B_0' & \xrightarrow{G_0'} & B' & \xrightarrow{E'} & \mathcal{C}_X \\ G_1 \downarrow & & G_1' \downarrow & & \downarrow G & & \\ D_1 & \xrightarrow{\Psi_1} & D_1' & \xrightarrow{G_2'} & D_2' & & \\ & & \Psi_1 \uparrow & & \uparrow \Psi_2 & & \\ & & D_1 & \xrightarrow{G_2} & D_2 & & \end{array} \qquad (1)$$

The composition of morphisms is defined in a natural way.

To each morphism $\mathfrak{E} \xrightarrow{\bar{\Psi}} \mathfrak{E}'$ (that is the diagram (1) above) and every algebra S in $\mathcal{A}^{\sim}$, there corresponds a functor

$$\mathfrak{F}^{\sim}_{E',G'}(S) \xrightarrow{\mathfrak{F}^{\sim}_{\bar{\Psi}}(S)} \mathfrak{F}^{\sim}_{E,G}(S), \quad (L, \gamma_L) \longmapsto (L \circ \Psi_2, \gamma_L \Psi_2), \qquad (2^{\sim})$$

which induces maps

$$\mathfrak{F}_{E',G'}(S) \xrightarrow{\mathfrak{F}_{\bar{\Psi}}(S)} \mathfrak{F}_{E,G}(S) \quad \text{and} \quad \mathfrak{Fl}_{\mathfrak{E}'}(S) \xrightarrow{\mathfrak{Fl}_{\bar{\Psi}}(S)} \mathfrak{Fl}_{\mathfrak{E}}(S)$$

functorially depending on S and such that the diagram

$$\begin{array}{ccc} \mathfrak{F}_{E',G'}(S) & \xrightarrow{\pi_{\mathfrak{E}'}(S)} & \mathfrak{Fl}_{\mathfrak{E}'}(S) \\ \mathfrak{F}_{\bar{\Psi}}(S) \big\downarrow & & \big\downarrow \mathfrak{Fl}_{\bar{\Psi}}(S) \\ \mathfrak{F}_{E,G}(S) & \xrightarrow{\pi_{\mathfrak{E}}(S)} & \mathfrak{Fl}_{\mathfrak{E}}(S) \end{array}$$

commutes. The latter implies the existence of a unique morphism

$$\mathfrak{R}_{\mathfrak{E}'}(S) \xrightarrow{\mathfrak{R}_{\bar{\Psi}}(S)} \mathfrak{R}_{\mathfrak{E}}(S)$$

which makes the diagram

$$\begin{array}{ccccc} \mathfrak{R}_{\mathfrak{E}'}(S) & \overset{\mathfrak{p}^1_{\mathfrak{E}'}(S)}{\underset{\mathfrak{p}^2_{\mathfrak{E}'}(S)}{\rightrightarrows}} & \mathfrak{F}_{E',G'}(S) & \xrightarrow{\pi_{\mathfrak{E}'}(S)} & \mathfrak{Fl}_{\mathfrak{E}'}(S) \\ \mathfrak{R}_{\bar{\Psi}}(S) \big\downarrow & & \mathfrak{F}_{\bar{\Psi}}(S) \big\downarrow & & \big\downarrow \mathfrak{Fl}_{\bar{\Psi}}(S) \\ \mathfrak{R}_{\mathfrak{E}}(S) & \overset{\mathfrak{p}^1_{\mathfrak{E}}(S)}{\underset{\mathfrak{p}^2_{\mathfrak{E}}(S)}{\rightrightarrows}} & \mathfrak{F}_{E,G}(S) & \xrightarrow{\pi_{\mathfrak{E}}(S)} & \mathfrak{Fl}_{\mathfrak{E}}(S) \end{array}$$

commute. This diagram depends functorially on S; i.e. it is the value at S of the commutative diagram

$$\begin{array}{ccccc} \mathfrak{R}_{\mathfrak{E}'} & \overset{\mathfrak{p}^1_{\mathfrak{E}'}}{\underset{\mathfrak{p}^2_{\mathfrak{E}'}}{\rightrightarrows}} & \mathfrak{F}_{E',G'} & \xrightarrow{\pi_{\mathfrak{E}'}} & \mathfrak{Fl}_{\mathfrak{E}'} \\ \mathfrak{R}_{\bar{\Psi}} \big\downarrow & & \mathfrak{F}_{\bar{\Psi}} \big\downarrow & & \big\downarrow \mathfrak{Fl}_{\bar{\Psi}} \\ \mathfrak{R}_{\mathfrak{E}} & \overset{\mathfrak{p}^1_{\mathfrak{E}}}{\underset{\mathfrak{p}^2_{\mathfrak{E}}}{\rightrightarrows}} & \mathfrak{F}_{E,G} & \xrightarrow{\pi_{\mathfrak{E}}} & \mathfrak{Fl}_{\mathfrak{E}} \end{array}$$

of presheaves and presheaf morphisms.

7.6.1. Reduction. We call a data

$$\mathfrak{E} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1 \downarrow & & \downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array}\right)$$

reduced if the functors G and G_2 are injective – that is they are faithful and injective on objects. Let

$$\mathfrak{E} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1 \downarrow & & \downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array}\right)$$

be arbitrary and S an associative unital algebra in $\mathcal{A}^\sim$ such that the category $\mathfrak{F}_{E,G}(S)^\sim$ is non-empty. Then we have an obvious morphism

$$\begin{array}{ccccccc} & & B_0 & \xrightarrow{G_0} & B & \xrightarrow{s^* \circ E} & S - mod_X \\ & & Id_{B_0} \downarrow & & \downarrow \Psi^\dagger & & \downarrow Id_{\mathcal{C}_X} \\ B_0 & \xrightarrow{Id_{B_0}} & B_0 & \xrightarrow{G_0'} & G(B) & \xrightarrow{E_S'} & S - mod_X \\ G_1 \downarrow & & G_1' \downarrow & & \downarrow G & & \\ D_1 & \xrightarrow{\Psi_1} & G_2(D_1) & \xrightarrow{G_2'} & D_2 & & \\ & & \Psi_1 \uparrow & & \uparrow Id_{D_2} & & \\ & & D_1 & \xrightarrow{G_2} & D_2 & & \end{array} \qquad (2)$$

from the data

$$\mathfrak{E}^S = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{s^* \circ E} & S - mod_X \\ G_1 \downarrow & & \downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array}\right)$$

to the associated with it *reduced* data

$$\mathfrak{E}^S_{red} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0'} & G(B) & \xrightarrow{E_S'} & S - mod_X \\ G_1' \downarrow & & \downarrow G' & & \\ G_2(D_1) & \xrightarrow{G_2'} & D_2 & & \end{array}\right) \qquad (3)$$

It follows that the morphism (3) induces isomorphisms

$$\mathfrak{F}_{\mathfrak{E}^S,G} \xrightarrow{\ \sim\ } \mathfrak{F}_{E,G_{\mathfrak{r}}} \quad \text{and} \quad \mathfrak{Fl}_{\mathfrak{E}^S} \xrightarrow{\ \sim\ } \mathfrak{Fl}_{\mathfrak{E}^S_{red}}$$

of presheaves of sets on $\mathbf{Aff}_{\mathcal{A}_S^\sim}$.

7.6.2. Cutting off objects. Let

$$\mathfrak{E} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle G} & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array} \right)$$

be a reduced data; that is G and G_2 are injective functors. Then we have a morphism

$$\begin{array}{ccccccc} & & B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ & & {\scriptstyle \Psi_0}\downarrow & & \downarrow{\scriptstyle \Psi^\dagger} & & \downarrow{\scriptstyle Id_{\mathcal{C}_X}} \\ B_0 & \xrightarrow{Id_{B_0}} & B_0 & \xrightarrow{G_0'} & B' & \xrightarrow{E'} & \mathcal{C}_X \\ {\scriptstyle G_1^{\mathfrak{r}}}\downarrow & & {\scriptstyle G_1'}\downarrow & & \downarrow{\scriptstyle G} & & \\ D_1^{\mathfrak{r}} & \xrightarrow{\Psi_1} & D_1 & \xrightarrow{G_2} & D_2^{\mathfrak{r}} & & \\ & & {\scriptstyle \Psi_1}\uparrow & & \uparrow{\scriptstyle \Psi_2} & & \\ & & D_1^{\mathfrak{r}} & \xrightarrow{G_2^{\mathfrak{r}}} & D_2^{\mathfrak{r}} & & \end{array} \qquad (4)$$

where $D_2^{\mathfrak{r}}$ is the full subcategory of D_2 generated by the image of ObB and $D_1^{\mathfrak{r}} = G_2^{-1}(D_2^{\mathfrak{r}})$. Therefore, we have canonical morphisms

$$\mathfrak{R}_{\mathfrak{E}} \xrightarrow{\mathfrak{R}_{\bar{\mathfrak{J}}}} \mathfrak{R}_{\mathfrak{E}^{\mathfrak{r}}}, \quad \mathfrak{F}_{E,G} \xrightarrow{\mathfrak{F}_{\bar{\mathfrak{J}}}} \mathfrak{F}_{E,G^{\mathfrak{r}}} \quad \text{and} \quad \mathfrak{Fl}_{\mathfrak{E}} \xrightarrow{\mathfrak{Fl}_{\bar{\mathfrak{J}}}} \mathfrak{Fl}_{\mathfrak{E}^{\mathfrak{r}}}. \qquad (5)$$

making the diagram

$$\begin{array}{ccccc} \mathfrak{R}_{\mathfrak{E}} & \overset{\mathfrak{p}^1_{\mathfrak{E}}}{\underset{\mathfrak{p}^2_{\mathfrak{E}}}{\rightrightarrows}} & \mathfrak{F}_{E,G} & \xrightarrow{\pi_{\mathfrak{E}}} & \mathfrak{Fl}_{\mathfrak{E}} \\ {\scriptstyle \mathfrak{R}_{\bar{\mathfrak{J}}}}\downarrow & & {\scriptstyle \mathfrak{F}_{\bar{\mathfrak{J}}}}\downarrow & & \downarrow{\scriptstyle \mathfrak{Fl}_{\bar{\mathfrak{J}}}} \\ \mathfrak{R}_{\mathfrak{E}^{\mathfrak{r}}} & \overset{\mathfrak{p}^1_{\mathfrak{E}^{\mathfrak{r}}}}{\underset{\mathfrak{p}^2_{\mathfrak{E}^{\mathfrak{r}}}}{\rightrightarrows}} & \mathfrak{F}_{E,G^{\mathfrak{r}}} & \xrightarrow{\pi_{\mathfrak{E}^{\mathfrak{r}}}} & \mathfrak{Fl}_{\mathfrak{E}^{\mathfrak{r}}} \end{array}$$

commute.

7.7. Proposition. *(a) Let the functor G in the data*

$$\mathfrak{E}_i = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E_i} & \mathcal{C}_X \\ G_1 \big\downarrow & & \big\downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array}\right), \qquad i = 1,\ 2,$$

be faithful and bijective on objects; and let $E_1 \xrightarrow{\beta} E_2 \xrightarrow{\varphi} E_1$ be functor morphisms such that

$$\varphi \circ \beta = id_{E_1} \quad \text{and} \quad E_2(\xi) = \beta(b) \circ E_1(\xi) \circ \varphi(a)$$

for every arrow $a \xrightarrow{\xi} b$ of $G(HomB)$ such that $G(\xi)$ is a composition of arrows from $HomD_2 - G(HomB)$.

Then the pair (β, φ) determines a presheaf morphism

$$\mathfrak{F}_{E_1,G_1} \xrightarrow{\mathfrak{F}_{\beta,\varphi}} \mathfrak{F}_{E_2,G_2}. \tag{6}$$

(b) Suppose, in addition, that $\varphi(a)$ (hence $\beta(a)$) is an isomorphism for every $a \in G_2(ObD_1) - G_2 \circ G_1(ObB_0)$. then the pair (β, φ) determines a presheaf morphism

$$\mathfrak{Fl}_{\mathfrak{E}_1} \xrightarrow{\mathfrak{Fl}_{\beta,\varphi}} \mathfrak{Fl}_{\mathfrak{E}_2}$$

making the diagram

$$\begin{array}{ccc} \mathfrak{F}_{\mathfrak{E}_1} & \xrightarrow{\pi_{\mathfrak{E}_1}} & \mathfrak{Fl}_{\mathfrak{E}_1} \\ \mathfrak{F}_{\beta,\varphi} \big\downarrow & & \big\downarrow \mathfrak{Fl}_{\beta,\varphi} \\ \mathfrak{F}_{E_2,G_2} & \xrightarrow{\pi_{\mathfrak{E}_2}} & \mathfrak{Fl}_{\mathfrak{E}_2} \end{array} \tag{7}$$

commute.

Proof. (a) Let the conditions of (a) hold.

(i) Let $(\mathcal{L}, \gamma_{\mathcal{L}})$ be any object of the category $\mathfrak{F}^{\sim}_{E_1,G_1}(S)$; that is $\mathcal{L}$ is a functor from D_2 to $S - mod_X$ and $\gamma_{\mathcal{L}}$ a functor isomorphism $\mathcal{L} \circ G \xrightarrow{\sim} s^* \circ E_1$.

The bijectivity of the functor G on objects implies that there exists a unique functor $D_2 \xrightarrow{\widetilde{\mathcal{L}}} S - mod_X$ such that $\widetilde{\mathcal{L}}(G(a) = s^* \circ E_1(a)$ for all $a \in ObB$ and the object $(\widetilde{\mathcal{L}}, id)$ is isomorphic to $(\mathcal{L}, \gamma_{\mathcal{L}})$. The functor $\widetilde{\mathcal{L}}$ maps a morphism $a \xrightarrow{\xi} b$ to the composition of

$$\widetilde{\mathcal{L}}(a) = s^* \circ E_1(G^{-1}(a)) \xrightarrow{\gamma_{\mathcal{L}}^{-1}(G^{-1}(a))} \mathcal{L}(a) \xrightarrow{\mathcal{L}(\xi)} \mathcal{L}(b) \xrightarrow{\gamma_{\mathcal{L}}(G^{-1}(b))} s^* \circ E_1(G^{-1}(b)) = \widetilde{\mathcal{L}}(b).$$

(ii) It follows from (i) that the category $\mathfrak{F}_{\mathfrak{E}_1}(S)^\sim$ is equivalent to its discrete subcategory consisting of the objects of the form $(\mathcal{L}, id)$. In other words, the set $\mathfrak{F}_{\mathfrak{E}_1}(S)$ is realized as the set of all functors $D_2 \xrightarrow{\mathcal{L}} S - mod_X$ such that $\mathcal{L} \circ G = s^* \circ E_1$.

Since this fact is based on bijectivity of the functor G on objects, same holds for the category $\mathfrak{F}_{\mathfrak{E}_2}(S)^\sim$.

(iii) For any object a of D_2, we set (using the condition that G is identical on objects) $\mathcal{L}_{\beta,\varphi}(a) = s^* \circ E_2(G^{-1}(a))$. For a morphism $a \xrightarrow{\xi} b$, of D_2, set

$$\begin{aligned}
&\mathcal{L}_{\beta,\varphi}(\xi) = s^* \circ E_2(\xi'), \quad \text{if} \quad \xi = G(\xi') \quad \text{for some} \quad \xi' \in HomB;\\
&\mathcal{L}_{\beta,\varphi}(\xi) = s^*(\beta(b)) \circ \mathcal{L}(\xi) \circ s^*(\varphi(a)) \quad \text{otherwise.}
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathcal{L}_{\beta,\varphi}(\xi_1) \circ \mathcal{L}_{\beta,\varphi}(\xi_2) = s^*(\beta(b)) \circ \mathcal{L}(\xi_1) \circ s^*(\varphi(a)) \circ s^*(\beta(b)) \circ \mathcal{L}(\xi_2) \circ s^*(\varphi(a)) =\\
&s^*(\beta(b)) \circ \mathcal{L}(\xi_1) \circ \mathcal{L}(\xi_2) \circ s^*(\varphi(a)) = s^*(\beta(b)) \circ \mathcal{L}(\xi_1 \circ \xi_2) \circ s^*(\varphi(a)) = \mathcal{L}_{\beta,\varphi}(\xi_1 \circ \xi_2)
\end{aligned}$$

for any pair $a \xrightarrow{\xi_1} b \xrightarrow{\xi_2} c$ of arrows from $HomD_2 - (G(HomB)$.

Suppose that $\xi_1 \circ \xi_2 = G(\zeta)$ for some $\zeta \in G(HomB)$. Then, by hypothesis,

$$\begin{aligned}
&\mathcal{L}_{\beta,\varphi} \circ G(\zeta) = \mathcal{L}_{\beta,\varphi}(\xi_1 \circ \xi_2) = s^*(\beta(b)) \circ \mathcal{L}(\xi_1 \circ \xi_2) \circ s^*(\varphi(a)) =\\
&s^*(\beta(b)) \circ \mathcal{L} \circ G(\zeta) \circ s^*(\varphi(a)) = s^*(\beta(b)) \circ s^* \circ E_1(\zeta) \circ s^*(\varphi(a)) =\\
&s^*(\beta(b) \circ E_1(\zeta) \circ \varphi(a)) = s^* \circ E_2(\zeta).
\end{aligned}$$

Altogether shows that $D_2 \xrightarrow{\mathcal{L}_{\beta,\varphi}} S - mod_X$ is a well defined functor and the pair $(\mathcal{L}_{\beta,\varphi}, \mathcal{L}_{\beta,\varphi} \circ G \xrightarrow{id} s^* \circ E)$ is an object of the category $\mathfrak{F}^{\sim}_{\mathfrak{E}_2}(S)$; or, what is the same, the functor $\mathcal{L}_{\beta,\varphi}$ is an element of the set $\mathfrak{F}^{\sim}_{\mathfrak{E}_2}(S)$.

(b) Let $\mathcal{L}$, $\mathcal{L}'$ be two elements of $\mathfrak{F}_{\mathfrak{E}_1}(S)$. By definition, they are equivalent iff there is a functor isomorphism $\mathcal{L} \circ G_2 \xrightarrow{\lambda} \mathcal{L}' \circ G_2$ such that $\lambda G_1 = id_{E_1 G_2 G_1}$.

For $a \in ObD_1$, we set

$$\begin{aligned}
&\lambda_{\beta,\varphi}(a) = id_{G_2(a)}, \quad \text{if} \quad a \in G_0(ObB_0);\\
&\lambda_{\beta,\varphi}(a) = s^*(\beta(a)) \circ \lambda(a) \circ s^*(\varphi(a)) \quad \text{otherwise.}
\end{aligned}$$

One can see that $\lambda_{\beta,\varphi} = \{\lambda_{\beta,\varphi}(a) \mid a \in ObD_2\}$ is a functor morphism

$$\mathcal{L}_{\beta,\varphi} \circ G_2 \longrightarrow \mathcal{L}'_{\beta,\varphi} \circ G_2$$

such that $\lambda_{\beta,\varphi} G_1 = id_{E_2 G_2 G_1}$.

If the conditions (b) hold, than the morphism $\lambda_{\beta,\varphi}$ is an isomorphism; i.e. it establishes equivalence between $\mathcal{L}_{\beta,\varphi}$ and $\mathcal{L}'_{\beta,\varphi}$. ■

7.7.1. Remarks. (a) In the condition (a) of Proposition 7.7, it suffices to require the *essential* bijectivity of the functor G on objects: that is G induces a bijective map between the sets of isomorphism classes of objects of the categories B and D_2.

(b) The commutativity of the diagram (7) implies that there exists a unique morphism $\mathfrak{R}_{\mathfrak{E}_1} \xrightarrow{\mathfrak{R}_{\beta,\varphi}} \mathfrak{R}_{\mathfrak{E}_2}$ making the diagram

$$\begin{array}{ccccc}
\mathfrak{R}_{\mathfrak{E}_1} & \overset{\mathfrak{p}^1_{\mathfrak{E}_1}}{\underset{\mathfrak{p}^2_{\mathfrak{E}_1}}{\rightrightarrows}} & \mathfrak{F}_{\mathfrak{E}_1} & \xrightarrow{\pi_{\mathfrak{E}_1}} & \mathfrak{Fl}_{\mathfrak{E}_1} \\
{\scriptstyle \mathfrak{R}_{\beta,\varphi}}\downarrow & & {\scriptstyle \mathfrak{F}_{\beta,\varphi}}\downarrow & & \downarrow{\scriptstyle \mathfrak{Fl}_{\beta,\varphi}} \\
\mathfrak{R}_{\mathfrak{E}_2} & \overset{\mathfrak{p}^1_{\mathfrak{E}_2}}{\underset{\mathfrak{p}^2_{\mathfrak{E}_2}}{\rightrightarrows}} & \mathfrak{F}_{\mathfrak{E}_2} & \xrightarrow{\pi_{\mathfrak{E}_2}} & \mathfrak{Fl}_{\mathfrak{E}_2}
\end{array} \qquad (8)$$

commute.

(c) Let the functors $B \xrightarrow{G} D_2 \xleftarrow{G_2} D_1$ in the assumptions of 7.7 be injective. Then, by 7.6.1, there are canonical morphisms

$$\begin{array}{l}
\mathfrak{R}_{\mathfrak{E}_2} \xrightarrow{\mathfrak{R}_{\bar{\mathfrak{J}}}} \mathfrak{R}_{\mathfrak{E}_2^{\mathfrak{r}}}, \quad \mathfrak{F}_{\mathfrak{E}_2} \xrightarrow{\mathfrak{F}_{\bar{\mathfrak{J}}}} \mathfrak{F}_{\mathfrak{E}^{\mathfrak{r}}} \quad \text{and} \quad \mathfrak{Fl}_{\mathfrak{E}_2} \xrightarrow{\mathfrak{Fl}_{\bar{\mathfrak{J}}}} \mathfrak{Fl}_{\mathfrak{E}^{\mathfrak{r}}} \\
\mathfrak{R}_{\mathfrak{E}_1} \xrightarrow{\mathfrak{R}_{\bar{\mathfrak{J}}'}} \mathfrak{R}_{\mathfrak{E}_1^{\mathfrak{r}}}, \quad \mathfrak{F}_{\mathfrak{E}_1} \xrightarrow{\mathfrak{F}_{\bar{\mathfrak{J}}'}} \mathfrak{F}_{\mathfrak{E}_1^{\mathfrak{r}}} \quad \text{and} \quad \mathfrak{Fl}_{\mathfrak{E}_1} \xrightarrow{\mathfrak{Fl}_{\bar{\mathfrak{J}}'}} \mathfrak{Fl}_{\mathfrak{E}_1^{\mathfrak{r}}}
\end{array} \qquad (9)$$

corresponding to the morphism $\mathfrak{E}^{\mathfrak{r}} \xrightarrow{\bar{\mathfrak{J}}} \mathfrak{E}$ of the data. The data $\mathfrak{E}^{\mathfrak{r}}$ and $\mathfrak{E}_1^{\mathfrak{r}}$ satisfy the assumptions (a) of 7.7. Therefore, under assumptions (b), we have a commutative diagram

$$\begin{array}{ccccccc}
\mathfrak{R}_{\mathfrak{E}_1} & \xrightarrow{\mathfrak{R}_{\bar{\mathfrak{J}}'}} & \mathfrak{R}_{\mathfrak{E}_1^{\mathfrak{r}}} & \xrightarrow{\mathfrak{R}_{\beta,\varphi}} & \mathfrak{R}_{\mathfrak{E}_2^{\mathfrak{r}}} & \xleftarrow{\mathfrak{R}_{\mathfrak{J}}} & \mathfrak{R}_{\mathfrak{E}_2} \\
{\scriptstyle \mathfrak{p}^1_{\mathfrak{E}_1}}\downarrow\downarrow{\scriptstyle \mathfrak{p}^2_{\mathfrak{E}_1}} & & {\scriptstyle \mathfrak{p}^1_{\mathfrak{E}_1^{\mathfrak{r}}}}\downarrow\downarrow{\scriptstyle \mathfrak{p}^2_{\mathfrak{E}_1^{\mathfrak{r}}}} & & {\scriptstyle \mathfrak{p}^1_{\mathfrak{E}^{\mathfrak{r}}}}\downarrow\downarrow{\scriptstyle \mathfrak{p}^2_{\mathfrak{E}^{\mathfrak{r}}}} & & {\scriptstyle \mathfrak{p}^1_{\mathfrak{E}_2}}\downarrow\downarrow{\scriptstyle \mathfrak{p}^2_{\mathfrak{E}_2}} \\
\mathfrak{F}_{\mathfrak{E}_1} & \xrightarrow{\mathfrak{F}_{\bar{\mathfrak{J}}'}} & \mathfrak{F}_{\mathfrak{E}_1^{\mathfrak{r}}} & \xrightarrow{\mathfrak{F}_{\beta,\varphi}} & \mathfrak{F}_{\mathfrak{E}^{\mathfrak{r}}} & \xleftarrow{\mathfrak{F}_{\mathfrak{J}}} & \mathfrak{F}_{\mathfrak{E}_2} \\
{\scriptstyle \pi_{\mathfrak{E}_1}}\downarrow & & {\scriptstyle \pi_{\mathfrak{E}_1^{\mathfrak{r}}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathfrak{E}^{\mathfrak{r}}}} & & \downarrow{\scriptstyle \pi_{\mathfrak{E}_2}} \\
\mathfrak{F}_{\mathfrak{E}_1} & \xrightarrow{\mathfrak{F}_{\bar{\mathfrak{J}}'}} & \mathfrak{Fl}_{\mathfrak{E}_1^{\mathfrak{r}}} & \xrightarrow{\mathfrak{Fl}_{\beta,\varphi}} & \mathfrak{Fl}_{\mathfrak{E}^{\mathfrak{r}}} & \xleftarrow{\mathfrak{Fl}_{\mathfrak{J}}} & \mathfrak{Fl}_{\mathfrak{E}_2}
\end{array}$$

of presheaves of sets on $\mathbf{Aff}_{\mathcal{A}^{\sim}}$.

7.8. Non-generic, generic and partly generic flag varieties.

7.8.1. Grassmannians. Let D_2 be the category with two objects, x_0, x_1, and three non-identical morphisms: $x_0 \xrightarrow{e} x_1 \xrightarrow{m} x_0$, and $m \circ e$ such that $e \circ m = id_{x_1}$. Let B be the discrete subcategory of D_2 with objects x_0, x_1, B_0 the discrete subcategory of D_2 generated by object x_0, and D_1 the subcategory of D_2 generated by the arrow $x_1 \xrightarrow{e} x_0$. The functors

$$\begin{array}{ccc} B_0 & \xrightarrow{G_0} & B \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle G} \\ D_1 & \xrightarrow{G_2} & D_2 \end{array}$$

are natural embeddings. Fix a functor

$$B \xrightarrow{E} \mathcal{C}_X \quad x_i \longmapsto E_i, \ i = 0, 1.$$

The presheaf of sets $\mathfrak{F}_{E,G}$ corresponding to the data

$$\mathfrak{E} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle G} & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array} \right)$$

coincides with the Grassmannian Gr_{E_0,E_1}.

7.8.2. Flag varieties. Let $\mathfrak{I} = (I, \leq)$ be a preorder with an initial object $\bullet$, B the discrete subcategory of $\mathfrak{I}$ with the set of objects I and B_0 the subcategory of B generated by $\bullet$. Let D_1 coincide with $(I, \leq)$. Finally, let D_2 be the category with $ObD_2 = I$ and the set of morphisms generated by morphisms $y \xrightarrow{e_{xy}} x$ and $x \xrightarrow{m_{yx}} y$ defined for all $x, y \in I$ such that $x \leq y$, which satisfy the following relations:

$$e_{xy} m_{yx} = id_x, \quad \text{and, for any} \quad x \leq y \leq z, \quad e_{xy} e_{yz} = e_{xz}, \ m_{zy} m_{yx} = m_{zx}.$$

In particular, there are projections $m_{yx} e_{xy} : y \longrightarrow y$. The functors

$$\begin{array}{ccc} B_0 & \xrightarrow{G_0} & B \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle G} \\ D_1 & \xrightarrow{G_2} & D_2 \end{array}$$

are natural embeddings. Fix a functor $B \xrightarrow{E} \mathcal{C}_X, \ x \longmapsto E_x$. The presheaf of sets corresponding to the data

$$\mathfrak{E} = \begin{pmatrix} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1 \Big\downarrow & & \Big\downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{pmatrix}$$

will be denoted by $\mathfrak{Fl}_{E,\mathcal{I}}$ and called the *flag variety* corresponding to the preorder $\mathcal{I} = (I, \leq)$ and the functor $I \xrightarrow{E} \mathcal{C}_X$.

Taking $\mathcal{I} = (x_1 \leq x_0)$, we recover Grassmannians.

7.8.3. From flag varieties to the varieties of generic flags. Let $\mathcal{I} = (I, \leq)$ be a preorder with an initial object $\bullet$. Let the categories $B_0, \ D_1$, and D_2 be like in 7.8.3; and let $B = B_0$. The functors $G_i, \ i = 0, 1, 2,$ are natural embeddings. The functor $B \xrightarrow{E} \mathcal{C}_X$ maps the initial object $\bullet$ to an object E of the category $\mathcal{C}_X$.

Applying the procedures of 7.6.1 ("reduction") and 7.6.2 ("cutting objects off"), we obtain the data consisting of the full subcategory D_2' of the category D_2 generated by the object $\bullet$ and its trivial subcategories $B_0 = B = D_1'$. One can see that $HomD_2'$ is the set $\{p_x | \ x \in I\}$ of endomorphisms of $\bullet$ satisfying the conditions: $p_x p_y = p_x$ if $x \leq y$. The corresponding presheaf of sets $\mathfrak{F}_{\mathfrak{E}'}$ is the generic flag variety $\mathfrak{Fl}_E$ of E. We recover generic Grassmannians taking $\mathcal{I} = \{0, 1\}$.

7.8.3.1. Proposition. *The canonical presheaf morphism* $\mathfrak{Fl}_{\mathfrak{E}} \longrightarrow \mathfrak{F}_{\mathfrak{E}'} = \mathfrak{Fl}_E$ *is an isomorphism.*

Proof. The argument is left to the reader. ∎

7.8.4. Partly generic flags. Let $\mathcal{I} = (I, \leq)$ be a preorder, and let I_0 be a *cofinal* subset of I; that is, for any $x \in I$, there exists $y \in I_0$ such that $y \leq x$. Let $B_0 = B$ be the discrete category with the set of objects I_0; and let the categories $D_1, \ D_2$ be as in 7.8.2. The functors G and $G_i, \ i = 1, 2,$ are natural embeddings. Fix a functor $B \xrightarrow{E} \mathcal{C}_X, \ x \longmapsto E_x, \ x \in I_0$. Applying the procedure of 7.6.1 and 7.6.2, we obtain the data

$$\mathfrak{E}' = \begin{pmatrix} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1' \Big\downarrow & & \Big\downarrow G' & & \\ D_1' & \xrightarrow{G_2'} & D_2' & & \end{pmatrix},$$

where D_i' is the full subcategory of $D_i, \ i = 1, 2,$ such that $ObD_1' = ObD_2' = I_0$. Clearly the flag variety of 7.8.2 and the generic flag variety of 7.8.3 are particular extreme cases of this example. By an obvious reason, we call $\mathfrak{F}_{\mathfrak{C}'}$ *variety of partly generic flags.* We denote it by $\mathfrak{Fl}_{E,I_0,\mathcal{I}}$. As in the particular case 7.8.3, the canonical morphism $\mathfrak{Fl}_{\mathfrak{C}} \longrightarrow \mathfrak{Fl}_{E,I_0,\mathcal{I}}$ is an isomorphism.

7.8.5. Example. Consider the category D_2 formed by three objects, x, y, z, and generating arrows

$$\begin{array}{ccc} x & \xrightarrow{\;a\;} & y \\ i \nwarrow \searrow b & & \nearrow c \\ & z & \end{array}$$

subject to the relations $b \circ i = id_z$, $c \circ b = a$, which imply that $c = a \circ i$ and $e = i \circ b$ is an idempotent. Let D_1 be the subcategory of D_2 generated by arrows

$$\begin{array}{ccc} x & \xrightarrow{\;a\;} & y \\ b \searrow & & \nearrow c \\ & z & \end{array}$$

and B the subcategory of D_1 generated by $x \xrightarrow{a} y$. Finally, the category B_0 is trivial; i.e. it has only one arrow – id_x. The functors G and G_i, $i = 0, 1, 2$, are natural embeddings.

Fix a functor $B \xrightarrow{E} \mathcal{C}_X$. Applying the procedure of 7.6.1, we obtain the functors

$$D_2' \xleftarrow{G_2'} D_1' = B \xleftarrow{G_1'} B \xrightarrow{E} \mathcal{C}_X,$$

where G_1' is the identical functor and D_2' is the category generated by $x \xrightarrow{e} x \xrightarrow{a} y$ subject to the relations $e^2 = e$, $a \circ e = a$.

7.8.5.1. Proposition. *The canonical morphism* $\mathfrak{Fl}_{\mathfrak{C}} \longrightarrow \mathfrak{Fl}_{\mathfrak{C}'}$ *is an isomorphism.*

Proof. The argument is left to the reader. ∎

7.9. Base change. Fix a data

$$\mathfrak{C} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1 \downarrow & & \downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array} \right)$$

Due to the universality of our constructions, the diagram $\mathfrak{R}_{\mathfrak{C}} \rightrightarrows \mathfrak{F}_{E,G}$ is compatible with the base change. That is for any affine scheme $\mathcal{S}^{\vee}$, we have a canonical commutative diagram with isomorphic horizontal arrows

$$\begin{array}{ccc} \mathcal{S}^\vee \times \mathfrak{R}_{\mathfrak{E}} & \xrightarrow{\sim} & \mathfrak{R}_{\mathfrak{E}^S} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathcal{S}^\vee \times \mathfrak{F}_{E,G} & \xrightarrow{\sim} & \mathfrak{F}_{\mathfrak{E}^S} \end{array} \qquad (1)$$

Here

$$\mathfrak{E}^S = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1 \downarrow & & \downarrow G & & \downarrow s^* \\ D_1 & \xrightarrow{G_2} & D_2 & & \mathcal{S} - mod_X \end{array} \right)$$

This implies that the diagram

$$\mathfrak{R}_{\mathfrak{E}} \rightrightarrows \mathfrak{F}_{E,G} \longrightarrow \mathfrak{Fl}_{\mathfrak{E}}$$

is compatible with the base change. In particular, we have a unique isomorphism

$$\mathcal{S}^\vee \times \mathfrak{Fl}_{\mathfrak{E}} \xrightarrow{\sim} \mathfrak{Fl}_{\mathfrak{E}^S}$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{S}^\vee \times \mathfrak{R}_{\mathfrak{E}} & \xrightarrow{\sim} & \mathfrak{R}_{\mathfrak{E}^S} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathcal{S}^\vee \times \mathfrak{F}_{E,G} & \xrightarrow{\sim} & \mathfrak{F}_{\mathfrak{E}^S} \\ \downarrow & & \downarrow \\ \mathcal{S}^\vee \times \mathfrak{Fl}_{\mathfrak{E}} & \xrightarrow{\sim} & \mathfrak{Fl}_{\mathfrak{E}^S} \end{array} \qquad (2)$$

commute.

8. Formal smoothness and smoothness.

8.1. Relatively projective and relatively conservative objects. Fix a functor $C_{\mathfrak{X}} \xrightarrow{\mathfrak{f}^*} C_{\mathfrak{Y}}$. We call an object $\mathcal{L}$ of the category $C_{\mathfrak{X}}$

– $\mathfrak{f}^*$-*projective,* if the map

$$C_{\mathfrak{X}}(\mathcal{L}, \mathcal{M}) \xrightarrow{\mathfrak{f}^*_{\mathcal{L},\mathcal{M}}} C_{\mathfrak{Y}}(\mathfrak{f}^*(\mathcal{L}), \mathfrak{f}^*(\mathcal{M})), \quad \alpha \longmapsto \mathfrak{f}^*(\alpha),$$

is surjective for all $\mathcal{M} \in ObC_{\mathfrak{X}}$,

– *conservative* for $\mathfrak{f}^*$, if the map

$$C_{\mathfrak{X}}(\mathcal{L},\mathcal{L}) \xrightarrow{\mathfrak{f}^*_{\mathcal{L},\mathcal{L}}} C_{\mathfrak{Y}}(\mathfrak{f}^*(\mathcal{L}),\mathfrak{f}^*(\mathcal{L}))$$

reflects isomorphisms.

8.2. Proposition. *Let* $D_2 \xleftarrow{G} B \xrightarrow{E} \mathcal{C}_X$ *be the pair of functors between svelte categories such that* G *is essentially bijective on objects and, in addition, the following condition holds:*

(i) $HomD_2 - G(HomB)$ *is generated by a set* Ξ *of arrows with the relations of the form* $\xi\circ\zeta = id_V$, *where* $\xi\in\Xi$, ζ *is the composition of arrows of* $\Xi - \{\xi\}$ *and* $G(HomB)$; *and for each* $\xi\in\Xi$, *there is at most one such relation.*

Let $S \xrightarrow{\varphi} T$ *be a morphism of algebras in* $\mathcal{A}^\sim$ *such that*

(ii) for every relation $\xi\circ\zeta = id_V$, $\xi\in\Xi$, *from (i), the object* $s^*(E(V))$ *is conservative for* $S - mod_X \xrightarrow{\varphi^*} T - mod_X$.

(iii) the functor $B \xrightarrow{s^*\circ E} S - mod_X$ *takes values in* φ^**-projective objects;*

Then the map $\mathfrak{F}_{E,G}(S) \xrightarrow{\mathfrak{F}_{E,G}(\varphi)} \mathfrak{F}_{E,G}(T)$ *is surjective.*

Proof. Let $D_2 \xrightarrow{\mathcal{L}} T - mod_X$ be an object of the category $\mathfrak{F}_{E,G}(T)$; that is

$$t^*\circ E = \varphi^*\circ s^*\circ E = \mathcal{L}\circ G.$$

The claim is that the functor $\mathcal{L}$ is isomorphic to the composition $\varphi^*\circ\mathcal{L}'$ for some object $D_2 \xrightarrow{\mathcal{L}'} S - mod_X$ of the category $\mathfrak{F}^{\sim}_{E,G}(S)$.

We start with constructing a diagram $D_2 \xrightarrow{\widetilde{\mathcal{L}}'} S - mod_X$ such that $\varphi^*\circ\widetilde{\mathcal{L}}' = \mathcal{L}$ and then transform it to a functor with the same property.

The restriction of the diagram $\widetilde{\mathcal{L}}'$ to the subcategory $G(HomB)$ of D_2 is determined by the equality $\widetilde{\mathcal{L}}'\circ G = s^*\circ E$. By Zorn Lemma, there exists a maximal subcategory D_2' of D_2 containing $G(B)$ such that $\widetilde{\mathcal{L}}'$ extends to a functor $\mathcal{L}'$ on D_2' with the property that $\varphi^*\circ\mathcal{L}''$ is the restriction of the functor $\mathcal{L}$ to the subcategory D_2'. If $D_2'\neq D_2$, then there exists an arrow $V_1 \xrightarrow{\xi} V_2$ which belongs to $\Xi - HomD_2'$.

By hypothesis, the functor G is bijective on objects. Therefore, it follows from (iii) that there is a morphism

$$s^*\circ E(V_1) \xrightarrow{\widetilde{\mathcal{L}}(\xi)} s^*\circ E(V_2)$$

such that $\varphi^*(\widetilde{\mathcal{L}}'(\xi)) = L(\xi)$.

If ξ enters into the relation $\xi \circ \zeta = id_V$ of (i) with $\zeta \in HomD_2'$, then, by (ii), the object $s^*(E(V))$ is conservative for φ^*. Therefore, the fact that $\varphi^*(\widetilde{\mathcal{L}}'(\xi\circ\zeta))$ is an (identical) isomorphism implies that $\widetilde{\mathcal{L}}'(\xi\circ\zeta)$ is an isomorphism. We set $\mathcal{L}'(\xi) = \widetilde{\mathcal{L}}'(\xi\circ\zeta)^{-1}\circ\widetilde{\mathcal{L}}'(\xi)$.

If ξ does not enter into any relation $\xi\circ\zeta = id_V$ of (i) with $\zeta \in HomD_2'$, then we set $\mathcal{L}'(\xi) = \widetilde{\mathcal{L}}'(\xi)$. It follows from (i) that this produces a well defined extension of $\mathcal{L}'$ to the subcategory of D_2 generated by $HomD_2'$ and ξ, which is strictly larger than D_2' – a contradiction with the maximality of D_2'. ∎

8.3. Infinitesimal morphisms. Let $\mathcal{S}$ and $\mathcal{T}$ be associative unital algebras in the monoidal category $\mathcal{A}^\sim$. We call a unital algebra morphism $\mathcal{S} \xrightarrow{\varphi} \mathcal{T}$ *infinitesimal,* if all finite objects of the category $\mathcal{S} - mod_X$ (with respect to the natural action of the monoidal category $\mathcal{S} - bim^\sim$ of $\mathcal{S}$-bimodules) are φ^*-projective and φ^*-conservative.

8.3.1. Strongly infinitesimal morphisms. We say that an infinitesimal morphism $\mathcal{S} \xrightarrow{\varphi} \mathcal{T}$ of associative unital algebras in $\mathcal{A}^\sim$ is *strongly infinitesimal,* if every finite object of the category $\mathcal{T} - mod_X$ is isomorphic to $\varphi^*(\mathcal{V})$ for some finite object $\mathcal{V}$ of the category $\mathcal{S} - mod_X$.

8.3.2. Formal smoothness. In what follows, *formal smoothness* is the formal smoothness with respect to strongly infinitesimal morphisms.

8.4. Proposition. *Let the functor G in the data*

$$\mathfrak{E} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle G} & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array}\right)$$

be essentially bijective on objects and the following conditions hold:

(i) $HomD_2 - G(HomB)$ is generated by a set Ξ of arrows with the relations of the form $\xi\circ\zeta = id_V$, where $\xi \in \Xi$, ζ is the composition of arrows of $\Xi - \{\xi\}$ and $G(HomB)$; and for each $\xi \in \Xi$, there is at most one such relation.

(ii) for every relation $\xi\circ\zeta = id_V$, $\xi \in \Xi$, from (i), the object $E(V)$ is finite;

(iii) the functor $B \xrightarrow{E} \mathcal{C}_X$ maps those objects of B which do not belong to the essential image of B_0 to finite objects.

Then all presheaves and presheaf morphisms of the diagram

$$\mathfrak{R}_{\mathfrak{E}} \underset{\mathfrak{p}^2_{\mathfrak{E}}}{\overset{\mathfrak{p}^1_{\mathfrak{E}}}{\rightrightarrows}} \mathfrak{F}_{E,G} \xrightarrow{\pi_{\mathfrak{E}}} \mathfrak{Fl}_{\mathfrak{E}}$$

are formally smooth.

Proof. (a) It follows from 8.2 that the presheaf $\mathfrak{F}_{E,G}$ is formally smooth.

(b) Since $\mathfrak{F}_{E,G} \xrightarrow{\pi_{\mathfrak{E}}} \mathfrak{Fl}_{\mathfrak{E}}$ is a presheaf epimorphism, it follows from the formal smoothness of $\mathfrak{F}_{E,G}$ that the presheaf $\mathfrak{Fl}_{\mathfrak{E}}$ is also formally smooth.

(c) *The presheaf morphism* $\mathfrak{F}_{E,G} \xrightarrow{\pi_{\mathfrak{E}}} \mathfrak{Fl}_{\mathfrak{E}}$ *is formally smooth.*

Let $S \xrightarrow{\varphi} T$ be an infinitesimal algebra morphism and

$$\begin{array}{ccc} T^\vee & \xrightarrow{\zeta} & \mathfrak{F}_{E,G} \\ {\scriptstyle \varphi^\vee}\big\downarrow & & \big\downarrow {\scriptstyle \pi_{\mathfrak{E}}} \\ S^\vee & \xrightarrow{\xi} & \mathfrak{Fl}_{\mathfrak{E}} \end{array} \qquad (1)$$

a commutative diagram of presheaf morphisms. Let $D_2 \xrightarrow{\mathcal{L}_\zeta} T - mod_X$ be the functor corresponding to the morphism ζ and $D_2 \xrightarrow{\mathcal{L}_\xi} S - mod_X$ a representative of the element of $\mathfrak{Fl}_{\mathfrak{E}}$ corresponding to ξ. The commutativity of the diagram (1) means that there is a functor isomorphism $\varphi^* \circ \mathcal{L}_\xi \circ G_2 \xrightarrow{\lambda} \mathcal{L}_\zeta \circ G_2$ such that $\lambda G_1 = id_{G \circ G_0}$.

By (a), the presheaf $\mathfrak{F}_{E,G}$ is formally smooth. Therefore, there exists a functor

$$D_2 \xrightarrow{\widetilde{\mathcal{L}}_\zeta} S - mod_X$$

which is an element of $\mathfrak{F}_{E,G}(S)$ satisfying the equality $\varphi^* \circ \widetilde{\mathcal{L}}_\zeta = \mathcal{L}_\zeta$.

(d) Since formally smooth morphisms are stable under pull-backs, it follows from (c) that the morphisms $\mathfrak{R}_{\mathfrak{E}} \underset{\mathfrak{p}^2_{\mathfrak{E}}}{\overset{\mathfrak{p}^1_{\mathfrak{E}}}{\rightrightarrows}} \mathfrak{F}_{E,G}$ are formally smooth. This and the fact that the presheaf $\mathfrak{F}_{E,G}$ is formally smooth imply that the presheaf of relations $\mathfrak{R}_{\mathfrak{E}}$ is formally smooth too. ∎

8.5. Smoothness. Smoothness is understood as above, as the smoothness with respect to *strongly infinitesimal* morphisms (see 8.3.1).

8.5.1. Proposition. *Suppose that the categories* $\mathcal{B}$, D_1, D_2 *in the data*

$$\mathfrak{E} = \left(\begin{array}{ccccc} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ {\scriptstyle G_1}\big\downarrow & & \big\downarrow{\scriptstyle G} & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{array} \right)$$

are finite, the functors $B \xrightarrow{G} D_2$ *and* $D_1 \xrightarrow{G_2} D_2$ *are essentially bijective on objects, and the pair of functors* $D_2 \xleftarrow{G} B \xrightarrow{E} \mathcal{C}_X$ *satisfies the condition*

(†) *the pair of objects* $(E(M), E(L))$ *is admissible, if* $D_2(G(M), G(L)) \neq \emptyset$.

Suppose, in addition, that the conditions (i), (ii) and (iii) of 8.4 hold. Then all presheaves and presheaf morphisms of the diagram

$$\mathfrak{R}_{\mathfrak{E}} \underset{\mathfrak{p}^2_{\mathfrak{E}}}{\overset{\mathfrak{p}^1_{\mathfrak{E}}}{\rightrightarrows}} \mathfrak{F}_{E,G} \xrightarrow{\pi_{\mathfrak{E}}} \mathfrak{Fl}_{\mathfrak{E}} \qquad (1)$$

are smooth.

Proof. It follows from 8.4 that all presheaves and presheaf morphisms of the diagram (1) are formally smooth. By 7.5.2, all presheaves of the diagram (1) are locally finitely copresentable. Therefore they are smooth. ■

9. Quasi-coherent modules and bimodules.

9.0. The $\mathcal{A}^{\sim}$-ringed categories. We call this way pairs $(\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})$, where $\mathfrak{B}$ is a category and $\mathcal{O}_{\mathfrak{B}}$ is a presheaf of associative unital algebras in $\mathcal{A}^{\sim}$ on the category $\mathfrak{B}$.

9.1. The cofibred categories associated with an action. Fix an $\mathcal{A}^{\sim}$-ringed category $(\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})$. Given an action $\Phi^{\sim}$ of $\mathcal{A}^{\sim}$ on a svelte category $\mathcal{C}_X$, we associate with the pair $(\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})$ a pseudo-functor from the category $\mathfrak{B}$ to the bicategory of actions of the monoidal categories, which assigns to every object $\mathcal{Z}$ of $\mathfrak{B}$ the category $\mathcal{O}_{\mathfrak{B}}(\mathcal{Z}) - mod_X$ endowed with the action of the monoidal category $\mathcal{O}_{\mathfrak{B}}(\mathcal{Z}) - bim^{\sim}$ of $\mathcal{O}_{\mathfrak{B}}(\mathcal{Z})$-bimodules and to every morphism $\mathcal{Y} \longrightarrow \mathcal{Z}$ the restriction of scalars functors

$$\begin{aligned} \mathcal{O}_{\mathfrak{B}}(\mathcal{Y}) - mod_X &\xrightarrow{\mathcal{O}_{\mathfrak{B}}(\varphi)_{X*}} \mathcal{O}_{\mathfrak{B}}(\mathcal{Z}) - mod_X \quad \text{and} \\ \mathcal{O}_{\mathfrak{B}}(\mathcal{Y}) - bim^{\sim} &\xrightarrow{\mathcal{O}_{\mathfrak{B}}(\varphi)^{\sim}_{*}} \mathcal{O}_{\mathfrak{B}}(\mathcal{Z}) - bim^{\sim}. \end{aligned} \qquad (1)$$

(see A2.10.1). This defines a pair of pseudo-functor: the first one from the category $\mathfrak{B}$ to *Cat* and the second one to the category $\mathfrak{MCat}$ of monoidal categories and monoidal functors. In addition, the second one *acts* on the first one.

These pseudo-functors define a cofibred category and, respectively, a cofibred monoidal category. Therefore, we have a monoidal category $\mathcal{O}_{\mathfrak{B}} - Mod^{\sim}$ of modules on the latter, which acts on the category $\mathcal{O}_{\mathfrak{B}} - Mod_X$ of modules on the former.

Under our standard hypothesis on the monoidal category $\mathcal{A}^{\sim}$ and its action $\Phi^{\sim}$, both the cofibred category and the cofibred monoidal category are *bifibred.* In this case, the category $Qcoh(\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})^{\sim}$ of quasi-coherent $\mathcal{O}_{\mathfrak{B}}$-bimodules is a monoidal subcategory of $\mathcal{O}_{\mathfrak{B}} - Mod^{\sim}$, which acts on the subcategory $Qcoh(\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})_X$ of quasi-coherent $\mathcal{O}_{\mathfrak{B}}$-modules in $\mathcal{C}_X$.

9.2. Remark. A presheaf of associative unital algebras in $\mathcal{A}^{\sim}$ on a category $\mathfrak{B}$ is, by definition, the dual to a functor the category $\mathfrak{B} \xrightarrow{\mathcal{O}^{\vee}} \mathbf{Aff}_{\mathcal{A}^{\sim}}$ from $\mathfrak{B}$ to the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ of affine schemes in $\mathcal{A}^{\sim}$. We have canonical bifibred category of modules in $\mathcal{C}_X$ and the monoidal fibred category of bimodules with the base $\mathbf{Aff}_{\mathcal{A}^{\sim}}$, which acts on the bifibred category of modules. The actions described above are pull-backs along the functor $\mathfrak{B} \xrightarrow{\mathcal{O}^{\vee}_{\mathfrak{B}}} \mathbf{Aff}_{\mathcal{A}^{\sim}}$ of the similar actions over $\mathbf{Aff}_{\mathcal{A}^{\sim}}$.

9.3. Quasi-coherent modules and bimodules over a presheaf of sets. For every presheaf of sets $\mathfrak{X}$ on the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$, we take as $\mathfrak{B}$ the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}/\mathfrak{X}$ and apply the generalities above to the forgetful functor

$$\mathbf{Aff}_{\mathcal{A}^{\sim}}/\mathfrak{X} \xrightarrow{\mathcal{O}^{\vee}_{\mathfrak{X}}} \mathbf{Aff}_{\mathcal{A}^{\sim}}$$

corresponding to a presheaf $\mathcal{O}_{\mathfrak{X}}$ of associative unital algebras in $\mathcal{A}^{\sim}$.

We denote by $\mathcal{O}_{\mathfrak{X}} - Mod^{\sim}$ the corresponding monoidal category of $\mathcal{O}_{\mathfrak{X}}$-bimodules on $\mathbf{Aff}_{\mathcal{A}^{\sim}}/\mathfrak{X}$; and we denote by $Qcoh(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})^{\sim}$ its monoidal subcategory formed by quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-bimodules. The monoidal category $Qcoh(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})^{\sim}$ acts on the category $Qcoh(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})_X$ of quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules in $\mathcal{C}_X$.

If the presheaf $\mathfrak{X}$ is representable by an affine scheme $\mathcal{R}^{\vee}$, then it follows from IV.1.3 that the category $Qcoh(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})_X$ is naturally equivalent to the category $\mathcal{R} - mod_X$ of $\mathcal{R}$-modules in $\mathcal{C}_X$, the monoidal category $Qcoh(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})^{\sim}$ is equivalent to the monoidal category $\mathcal{R} - bim^{\sim}$ of $\mathcal{R}$-bimodules, amd the action of $Qcoh(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})^{\sim}$ on $Qcoh(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})_X$ corresponds, via these equivalences, to the standard action of $\mathcal{R} - bim^{\sim}$ on the category $\mathcal{R} - mod_X$ (see A2.10.1).

9.4. Quasi-coherent sheaves of modules and bimodules. Let τ be a quasi-topology on the category $\mathfrak{B}$ and $\mathcal{O}_{\mathfrak{B}}$ a sheaf of associative unital algebras on $(\mathfrak{B}, \tau)$.

Thanks to the fact that the forgetful functor $Alg\mathcal{A}^{\sim} \xrightarrow{\mathfrak{f}_*} \mathcal{A}$ is conservative and preserves limits, this means, precisely, that the composition of the presheaf

$$\mathfrak{B}^{op} \xrightarrow{\mathcal{O}_{\mathfrak{B}}} Alg\mathcal{A}^{\sim}$$

with the forgetful functor $Alg\mathcal{A}^{\sim} \xrightarrow{\mathfrak{f}_*} \mathcal{A}$ is a presheaf on the quasi-site $(\mathfrak{B}, \tau)$ with the values in the category $\mathcal{A}$.

Applying the generalities of Sections 3 and 4 of Chapter IV, we obtain the monoidal category $\mathcal{O}_{\mathfrak{B}}^{\sim} - Sh_{\tau}Mod^{\sim}$, which acts on the category $\mathcal{O}_{\mathfrak{B}} - Sh_{\tau}Mod_X$ of sheaves of $\mathcal{O}_{\mathfrak{B}}$-modules in $\mathcal{C}_X$, and its monoidal subcategory $Qcoh((\mathfrak{B},\tau),\mathcal{O}_{\mathfrak{B}}^{\sim})^{\sim}$ acting on the category $Qcoh((\mathfrak{B},\tau),\mathcal{O}_{\mathfrak{B}})_X$ of quasi-coherent *sheaves* of modules in $\mathcal{C}_X$.

9.4.1. Quasi-coherent sheaves of modules and bimodules on a presheaf of sets. Let τ be a quasi-topology on the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ of affine schemes in $\mathcal{A}^{\sim}$ such that the canonical presheaf of algebras on $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ (– the identical functor $\mathbf{Aff}_{\mathcal{A}^{\sim}}^{op} \longrightarrow Alg\mathcal{A}^{\sim}$) is a sheaf. For any presheaf of sets $\mathfrak{X}$ on the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ and the forgetful functor

$$\mathbf{Aff}_{\mathcal{A}^{\sim}}/\mathfrak{X} \xrightarrow{\mathcal{O}_{\mathfrak{X}}^{\vee}} \mathbf{Aff}_{\mathcal{A}^{\sim}}$$

the corresponding to a presheaf $\mathcal{O}_{\mathfrak{X}}$ of associative unital algebras in $\mathcal{A}^{\sim}$ is a sheaf with respect to the induced quasi-topology $\tau_{\mathfrak{X}}$ on the category $\mathbf{Aff}_{\mathcal{A}^{\sim}}/\mathfrak{X}$. Thus, we obtain the monoidal category $Qcoh((\mathfrak{X},\tau),\mathcal{O}_{\mathfrak{X}})^{\sim}$ of quasi-coherent sheaves of $\mathcal{O}_{\mathfrak{X}}$-bimodules on the quasi-site $(\mathbf{Aff}_{\mathcal{A}^{\sim}}/\mathfrak{X},\tau_{\mathfrak{X}})$ acting on the category $Qcoh((\mathfrak{B},\tau),\mathcal{O}_{\mathfrak{B}})_X$ of quasi-coherent *sheaves* of modules in $\mathcal{C}_X$.

If the τ is a quasi-topology of 1-descent (which holds for all canonical pretopologies, starting from the most important smooth pretopology, which is, even, of effective descent) the categories of quasi-coherent sheaves of bimodules and modules defined above, coincide with the corresponding categories of presheaves; so that the choice of the quasi-topology τ does not matter, as long as the quasi-topology is of effective descent.

We leave to the reader the description of these categories and the action for the varieties described in this Chapter.

Appendix 1: Fibred Categories.

Main references are Exposé VI in [SGA1] and Exposé VI in [SGA4]. The purpose of this appendix is to recall basic notions and fix notations. All categories we consider here belong to a fixed universum, $\mathfrak{U}$.

A1.1. Categories over a category. Fix a category $\mathcal{E}$. Let $(A, A \stackrel{F}{\to} \mathcal{E})$ and $(B, B \stackrel{G}{\to} \mathcal{E})$ be objects of the category $Cat/\mathcal{E}$. For any two morphisms, Φ, Ψ from (A, F) to (B, G) (called *$\mathcal{E}$-functors*), an *$\mathcal{E}$-morphism* $\Phi \to \Psi$ is defined as any functor morphism $\phi : \Phi \longrightarrow \Psi$ such that $G(\phi(x)) = id_{F(x)}$ for all $x \in ObA$. This defines a subcategory, $Hom_{\mathcal{E}}((A, F), (B, G))$, of the category $Hom(A, B)$ of all functors from A to B. The composition

$$Hom(A, B) \times Hom(B, C) \longrightarrow Hom(A, C)$$

induces a composition

$$Hom_{\mathcal{E}}((A, F), (B, G)) \times Hom_{\mathcal{E}}((B, G), (C, H)) \longrightarrow Hom_{\mathcal{E}}((A, F), (C, H)).$$

The map $((A, F), (B, G)) \longmapsto Hom_{\mathcal{E}}((A, F), (B, G))$ defines a functor

$$(Cat/\mathcal{E})^{op} \times Cat/\mathcal{E} \longrightarrow Cat.$$

A1.2. Inner hom. For any two categories $\mathcal{F}$, $\mathcal{G}$ over $\mathcal{E}$ and any category $\mathcal{H}$, there is an isomorphism

$$Hom(\mathcal{H}, Hom_{\mathcal{E}}(\mathcal{F}, \mathcal{G})) \xrightarrow{\sim} Hom_{\mathcal{E}}(\mathcal{F} \times \mathcal{H}, \mathcal{G})$$

functorial in all three arguments).

A1.3. Base change. If $\mathcal{F}$ and $\mathcal{E}'$ are two categories over $\mathcal{E}$, $\mathcal{F} \times_{\mathcal{E}} \mathcal{E}'$ denotes their product in $Cat/\mathcal{E}$. Recall that $Ob(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}') = Ob\mathcal{F} \times_{Ob\mathcal{E}} Ob\mathcal{E}'$ and $Hom(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}') = Hom\mathcal{F} \times_{Hom\mathcal{E}} Hom\mathcal{E}'$. Fixing $\gamma : \mathcal{E}' \longrightarrow \mathcal{E}$, we obtain the *base change functor*

$$Cat/\mathcal{E} \longrightarrow Cat/\mathcal{E}', \quad (\mathcal{F} \longmapsto \mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \pi_{\mathcal{E}'}),$$

where $\pi_{\mathcal{E}'}$ is the canonical projection.

For any two categories, $\mathcal{F}$, $\mathcal{G}$, over $\mathcal{E}$, the projection $\mathcal{G} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{G}$ induces a category isomorphism

$$Hom_{\mathcal{E}'}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{G} \times_{\mathcal{E}} \mathcal{E}') \longrightarrow Hom_{\mathcal{E}}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{G}).$$

The inverse morphism sends any $\mathcal{E}$-functor Φ to the $\mathcal{E}'$-functor $\Phi \times_{\mathcal{E}} \mathcal{E}'$.

A1.3.1. Proposition. *If an $\mathcal{E}$-functor $\Phi : \mathcal{F} \longrightarrow \mathcal{G}$ is fully faithful, then for any base change $\mathcal{E}' \longrightarrow \mathcal{E}$, the corresponding $\mathcal{E}'$-functor $\Phi \times_{\mathcal{E}} \mathcal{E}' : \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{G} \times_{\mathcal{E}} \mathcal{E}'$ is fully faithful too.*

A1.3.2. Definition. An $\mathcal{E}$-functor $\Phi : \mathcal{F} \longrightarrow \mathcal{G}$ is called an *$\mathcal{E}$-equivalence* if there exists $\mathcal{E}$-functor $\Psi : \mathcal{G} \longrightarrow \mathcal{F}$ and $\mathcal{E}$-isomorphisms $\Phi \circ \Psi \xrightarrow{\sim} Id_{\mathcal{G}}$, $\Psi \circ \Phi \xrightarrow{\sim} Id_{\mathcal{F}}$.

A1.3.3. Proposition. *The following conditions on an $\mathcal{E}$-functor $\Phi : \mathcal{F} \longrightarrow \mathcal{G}$ are equivalent:*

(i) Φ is an $\mathcal{E}$-equivalence.

(ii) For any category $\mathcal{E}'$ over $\mathcal{E}$, the functor $\Phi \times_{\mathcal{E}} \mathcal{E}' : \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{G} \times_{\mathcal{E}} \mathcal{E}'$ is an equivalence of categories.

(iii) Φ is an equivalence of categories, and for any $X \in Ob\mathcal{E}$, the functor $\Phi_X : \mathcal{F}_X \longrightarrow \mathcal{G}_X$ induced by Φ is an equivalence of categories.

A1.4. Cartesian morphisms. Inverse image functors. Fix a category $\mathcal{E}$ and an object $\mathcal{A} = (A, A \xrightarrow{F} \mathcal{E})$ of the category $Cat/\mathcal{E}$. For any $X \in Ob\mathcal{E}$, we denote by $\mathcal{A}_X$ the *fiber of F in X* which is the subcategory $F^{-1}(id_X)$ of A. For any $f : X \to Y$ of $\mathcal{E}$ and $x, y \in ObA$ such that $F(x) = X$, $F(y) = Y$, we set $\mathcal{A}_f(x, y) := \{\xi : x \to y |\ F(\xi) = f\}$.

A1.4.1. Cartesian morphisms. A morphism $\xi \in A(x, y)$ is called *cartesian* if for any $x' \in Ob\mathcal{A}_X$ and any $\xi' : x' \to y$ such that $F(\xi') = f := F(\xi)$, there exists a unique *X-morphism* $u : x' \to x$ (that is $Fu = id_X$) such that $\xi' = \xi \circ u$. In other words, for any $y \in \mathcal{A}_{F(x)}$, the map

$$\mathcal{A}_X(x', x) \longrightarrow \mathcal{A}_f(x', y), \quad v \longmapsto \xi \circ v,$$

is bijective. This means also that the pair (x, ξ) represents the functor

$$\mathcal{A}_X^{op} \longrightarrow \mathbf{Sets}, \quad x' \longmapsto \mathcal{A}_f(x', y).$$

If for a morphism $f \in \mathcal{E}(X, Y)$, there exists a cartesian morphism $\xi : x \to y$ such that $F(\xi) = f$, then the object x is defined uniquely up to isomorphism and is called *inverse image of y by f*. The standard notation: $x = f^*(y)$. The morphism $\xi : f^*(y) \to y$ is then denoted by ξ_f, or by $\xi_f(y)$.

A1.4.2. Inverse image functor. Suppose an inverse image exists for all $y \in \mathcal{A}_Y$. Then the map $y \longmapsto (f^*(y), \xi_f(y))$ defines a functor $f^* : \mathcal{A}_Y \longrightarrow \mathcal{A}_X$.

In fact, fix objects y, y' of $\mathcal{A}_Y$ and a cartesian morphisms $\xi_f(y) : f^*(y) \longrightarrow y$ and $\xi_f(y') : f^*(y') \longrightarrow y'$. For any morphism $\phi : y \to y'$ of $\mathcal{A}_Y$, there exists a unique morphism,

$f^*(\phi) : f^*(y) \longrightarrow f^*(y')$, such that the diagram

$$\begin{array}{ccc} f^*(y) & \xrightarrow{\xi_f(y)} & y \\ {\scriptstyle f^*(\phi)}\downarrow & & \downarrow{\scriptstyle \phi} \\ f^*(y') & \xrightarrow{\xi_f(y')} & y' \end{array} \qquad (1)$$

commutes.

A1.4.3. Note. Let

$$\begin{array}{ccc} x & \xrightarrow{\xi} & y \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \phi} \\ x' & \xrightarrow{\xi'} & y' \end{array}$$

be a commutative diagram in A such that $\psi \in Iso\mathcal{A}_X$ and $\phi \in Iso\mathcal{A}_Y$. Then ξ is cartesian iff ξ' is cartesian.

A1.5. Cartesian functors. Let $\mathcal{A} = (A, F)$, $\mathcal{B} = (B, G)$ be $\mathcal{E}$-categories. An $\mathcal{E}$-functor $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ is called a *cartesian functor* if it transforms cartesian morphisms to cartesian morphisms. The full subcategory of $Hom_{\mathcal{E}}(\mathcal{A}, \mathcal{B})$ formed by cartesian functors is denoted by $Cart_{\mathcal{E}}(\mathcal{A}, \mathcal{B})$.

A1.5.1. Proposition. *(a) Any $\mathcal{E}$-equivalence is a cartesian functor.*

(a') Given an $\mathcal{E}$-equivalence $\Phi : \mathcal{A} = (A, F) \longrightarrow \mathcal{B}$, a morphism ξ of A is cartesian iff $\Phi(\xi)$ is cartesian.

(b) Any $\mathcal{E}$-functor which is isomorphic to a cartesian functor is cartesian.

(c) Composition of cartesian functors is a cartesian functor.

A1.5.2. Corollary. *Let $\Phi : \mathcal{A} = (A, F) \longrightarrow \mathcal{B}$ be an $\mathcal{E}$-equivalence. Then for any $\mathcal{E}$-category $\mathcal{C}$, the functors $\Psi \longmapsto \Psi \circ \Phi$ and $\Psi \longmapsto \Phi \circ \Psi$ induce equivalence of categories:*

$$Cart_{\mathcal{E}}(\mathcal{B}, \mathcal{C}) \xrightarrow{\sim} Cart_{\mathcal{E}}(\mathcal{A}, \mathcal{C})$$

$$Cart_{\mathcal{E}}(\mathcal{C}, \mathcal{A}) \xrightarrow{\sim} Cart_{\mathcal{E}}(\mathcal{C}, \mathcal{B}).$$

A1.5.3. The category $Cart_{\mathcal{E}}$. We denote by $Cart_{\mathcal{E}}$ the category objects of which are same as objects of $Cat/\mathcal{E}$ and morphisms are cartesian functors.

A1.5.3.1. The category $Cart$**.** For a given universum $\mathfrak{U}$, we denote by $Cart_{\mathfrak{U}}$, or by $Cart$, the subcategory of the category $Cat^2_{\mathfrak{U}}$, whose objects are same as objects of $Cat^2_{\mathfrak{U}}$ – functors, and morphisms from $\mathcal{A}' \xrightarrow{F'} \mathcal{E}'$ to $A \xrightarrow{F} \mathcal{E}$ are commutative diagrams

$$\begin{array}{ccc} A' & \xrightarrow{G} & A \\ {\scriptstyle F'}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{E}' & \xrightarrow{G'} & \mathcal{E} \end{array}$$

such that G is a cartesian functor from $(A', G' \circ F')$ to (A, F).

A1.5.4. Colimits. Let B, C be categories and $\mathcal{S}$ a family of morphisms of B. Denote by $Hom_{\mathcal{S}}(B, C)$ the category of functors $B \to C$ which transform morphisms of $\mathcal{S}$ into isomorphisms.

Let $\mathcal{A} = (A, F)$ be a category over $\mathcal{E}$ and $\mathcal{S}_{\mathcal{A}}$ the family of cartesian morphisms of A. The $\mathcal{E}$-category $\mathcal{A}$ defines two functors $Cat \longrightarrow$ **Sets**:

$$C \longmapsto Hom_{\mathcal{S}_{\mathcal{A}}}(A, C), \tag{1}$$

$$C \longmapsto Cart_{\mathcal{E}}(\mathcal{A}, (C \times \mathcal{E}, P_{\mathcal{E}})). \tag{2}$$

Here $P_{\mathcal{E}}$ is the natural projection $C \times \mathcal{E} \longrightarrow \mathcal{E}$.

A1.5.4.1. Lemma. *The functors (1) and (2) are canonically isomorphic.*

Proof. Cartesian morphisms of $C \times \mathcal{E}$ are all morphisms of the form (m, f), where m is an isomorphism of A. ■

A1.5.4.2. Corollary. *For any $\mathcal{E}$-category $\mathcal{A} = (A, F)$, the functor*

$$Cat \longrightarrow \textbf{Sets}, \quad C \longmapsto Cart_{\mathcal{E}}(\mathcal{A}, (C \times \mathcal{E}, P_{\mathcal{E}})), \tag{2}$$

is representable by the category $(\mathcal{S}_{\mathcal{A}})^{-1}A$.

A1.5.4.3. Definition. The functor (2) and the category $(\mathcal{S}_{\mathcal{A}})^{-1}A$ representing it are denoted by $Colim\mathcal{A}/\mathcal{E}$ and are called a *colimit of A over $\mathcal{E}$.*

A1.5.5. Limits.

A1.5.5.1. Proposition. *Let $\mathcal{A} = (A, F)$ be a category over $\mathcal{E}$. The functor*

$$Cat \longrightarrow \textbf{Sets}, \quad C \longmapsto Cart_{\mathcal{E}}((C \times \mathcal{E}, P_{\mathcal{E}}), \mathcal{A}),$$

is representable by the category $Cart_{\mathcal{E}}(\mathcal{E}, \mathcal{A})$ of $\mathcal{E}$-cartesian functors $\mathcal{E} \to \mathcal{A}$.

Proof. Let A be a category and $G : (C \times \mathcal{E}) \longrightarrow \mathcal{A}$ a cartesian functor. For any $z \in ObC$, the functor

$$\mathcal{E} \longrightarrow \mathcal{A}, \quad X \longmapsto G(z, X),$$

is cartesian. This gives a map $Cart_{\mathcal{E}}((C \times \mathcal{E}, P_{\mathcal{E}}) \longrightarrow Cart_{\mathcal{E}}(\mathcal{E}, \mathcal{A})$ functorial in A. This map is a bijection. ∎

A1.5.5.2. Definition. The category $Cart_{\mathcal{E}}(\mathcal{E}, \mathcal{F})$ is called the category of *cartesian sections of $\mathcal{A}$ over $\mathcal{E}$*. It is also called a *limit of $\mathcal{A}$ over $\mathcal{E}$* and is denoted by $Lim\mathcal{A}/\mathcal{E}$.

A1.6. Fibred and prefibred categories.

A1.6.1. Definitions. (a) A category $\mathcal{A} = (A, A \stackrel{F}{\to} \mathcal{E})$ over $\mathcal{E}$ is called *prefibred* if for any morphism $f : X \to Y$ of $\mathcal{E}$, an inverse image functor exists.

(b) A prefibred category $\mathcal{A}$ over $\mathcal{E}$ is called *fibred* if the composition of cartesian morphisms is a cartesian morphism.

A1.6.1.1. The 2-category of fibred categories over $\mathcal{E}$. We denote by $Fib_{/\mathcal{E}}$ the 2-category of fibred categories over $\mathcal{E}$. Its 1-morphisms are cartesian functors and 2-morphisms natural transformation of (cartesian) functors.

A1.6.2. Fibred and prefibred subcategories. Let $\mathcal{A} = (A, F)$ be a fibred (resp. prefibred) category over $\mathcal{E}$. A subcategory B of A is called a *fibred subcategory of $\mathcal{A}$* (resp. a *prefibred subcategory of $\mathcal{A}$*) if $\mathcal{B} = (B, F|_{\mathcal{B}})$ is a fibred (resp. prefibred) category and the inclusion functor is a cartesian functor $\mathcal{B} \longrightarrow \mathcal{A}$.

A1.6.2.1. Lemma. *Let $\mathcal{A} = (A, F)$ be a fibred (resp. prefibred) category over $\mathcal{E}$. If B is a full subcategory of A, then B is a fibred (resp. prefibred) subcategory of $\mathcal{A}$ iff for any morphism $f : X \to Y$ of $\mathcal{E}$ and for any $y \in Ob\mathcal{B}_Y$, the inverse image, $f^*_{\mathcal{A}}(y)$ is $\mathcal{A}_X$-isomorphic to an object of $\mathcal{B}_X$.*

A1.6.2.2. Example. Let $\mathcal{A} = (A, F)$ be a fibred category over $\mathcal{E}$. And let B be a subcategory of A having same objects; morphisms of B are cartesian morphisms of $\mathcal{A}$. In particular, for any $X \in Ob\mathcal{E}$, morphisms of $\mathcal{B}_X$ are all isomorphisms of $\mathcal{A}_X$. The subcategory B is a fibred subcategory of $\mathcal{A}$.

A1.6.3. Proposition. *Let $F : A \longrightarrow \mathcal{E}$ be a functor. The following conditions are equivalent:*

(a) All morphisms of A are cartesian.

(b) $\mathcal{A} = (A, F)$ is a fibred category and all fibers are groupoids.

If the equivalent conditions (a), (b) hold, (A, F) is called *fibred category of groupoids*.

If $\mathcal{E}$ is a groupoid, then the conditions (a), (b) are equivalent to the condition

(c) The category A is a groupoid, and the functor $F : A \longrightarrow \mathcal{E}$ is *transportable*. The latter means that for any isomorphism $f : X \to Y$ of $\mathcal{E}$ and any object x of $\mathcal{A}_X$, there exists an isomorphism $\xi : x \to y$ such that $F(\xi) = f$.

A1.6.4. Proposition. *Let $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ be an $\mathcal{E}$-equivalence. Then $\mathcal{A}$ is a fibred (resp. prefibred) category over $\mathcal{E}$ iff $\mathcal{B}$ is such.*

Proof. The assertion follows from the fact that a morphism ξ of A is cartesian iff $\Phi(\xi)$ is cartesian. ∎

A1.6.5. Proposition. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be categories over $\mathcal{E}$ and let $\xi = (\xi_1, \xi_2)$ be a morphism of $\mathcal{A} = \mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2$. The morphism ξ is cartesian iff ξ_1, ξ_2 are cartesian.*

A1.6.6. Proposition. *Let $\mathcal{A} = \mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2$, and let $\Psi = (\Psi_1, \Psi_2)$ be an $\mathcal{E}$-functor $\mathcal{B} \to \mathcal{A}$. The functor Ψ is cartesian iff Ψ_1 and Ψ_2 are cartesian. Thus one has a category isomorphism*

$$Cart_{\mathcal{E}}(\mathcal{B}, \mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2) \xrightarrow{\sim} Cart_{\mathcal{E}}(\mathcal{B}, \mathcal{A}_1) \times Cart_{\mathcal{E}}(\mathcal{B}, \mathcal{A}_2)$$

In particular, there is a natural isomorphism of categories

$$Lim(\mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2/\mathcal{E}) \xrightarrow{\sim} Lim(\mathcal{A}_1/\mathcal{E}) \times Lim(\mathcal{A}_2/\mathcal{E})$$

Proof. The assertion follows from A1.6.5. ∎

A1.6.6.1. Corollary. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be fibred (resp. prefibred) categories over $\mathcal{E}$. Then $\mathcal{A}_1 \times_{\mathcal{E}} \mathcal{A}_2$ is a fibred (resp. prefibred) category over $\mathcal{E}$.*

A1.6.6.2. Remark. The results above hold for fibred products of any (small) set of categories over $\mathcal{E}$.

A1.6.7. Proposition. *Let $\mathcal{A} = (A, F)$ be a category over $\mathcal{E}$ and $G : \mathcal{E}' \longrightarrow \mathcal{E}$ a functor. Let $\mathcal{A}' = (A', F')$, where $A' = A \times_{\mathcal{E}} \mathcal{E}'$and F' is the projection $A' \to \mathcal{E}'$. A morphism, ξ', of A' is cartesian iff its image, ξ, in A is cartesian.*

A1.6.7.1. Corollary. *For any cartesian functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ of categories over $\mathcal{E}$ and any functor $G : \mathcal{E}' \longrightarrow \mathcal{E}$, the functor $\mathcal{F}' = \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' : \mathcal{A}' = \mathcal{A} \times_{\mathcal{E}} \mathcal{E}' \longrightarrow \mathcal{B}' = \mathcal{B} \times_{\mathcal{E}} \mathcal{E}'$ is cartesian.*

Thus the functor $Hom_{\mathcal{E}}(\mathcal{A}, \mathcal{B}) \longrightarrow Hom_{\mathcal{E}'}(\mathcal{A}', \mathcal{B}')$ induces a functor

$$Cart_{\mathcal{E}}(\mathcal{A}, \mathcal{B}) \longrightarrow Cart_{\mathcal{E}'}(\mathcal{A}', \mathcal{B}').$$

Taking into consideration the canonical isomorphism

$$Hom_{\mathcal{E}'}(\mathcal{A}', \mathcal{B}') \xrightarrow{\sim} Hom_{\mathcal{E}}(\mathcal{A} \times_{\mathcal{E}} \mathcal{E}', \mathcal{B}),$$

one can see that cartesian $\mathcal{E}'$-functors correspond to $\mathcal{E}$-functors $\mathcal{A}\times_{\mathcal{E}}\mathcal{E}' \longrightarrow \mathcal{B}$ transforming any morphism the first projection of which is cartesian into a cartesian morphism of $\mathcal{B}$.

A1.6.7.2. Corollary. *The category $Lim(\mathcal{A}'/\mathcal{E}')$ is isomorphic to the full subcategory of $Hom_{\mathcal{E}}(\mathcal{E}', \mathcal{A})$ formed by $\mathcal{E}$-functors which transform any morphism into a cartesian morphism.*

In particular, if $\mathcal{A}$ is a fibred category over $\mathcal{E}$ and $\mathcal{A}_c$ is the subcategory of $\mathcal{A}$ morphisms of which are all cartesian morphisms of $\mathcal{A}$, then there is a bijection

$$ObLim(\mathcal{A}'/\mathcal{E}') \xrightarrow{\sim} ObHom_{\mathcal{E}}(\mathcal{E}', \mathcal{A}_c).$$

A1.6.8. Proposition. *Let $\mathcal{A}$ be a fibred (resp. prefibred) category over $\mathcal{E}$. Then for any functor $\mathcal{E}' \longrightarrow \mathcal{E}$, $\mathcal{A}' := \mathcal{A} \times_{\mathcal{E}} \mathcal{E}'$ is a fibred (resp. prefibred) category over $\mathcal{E}'$.*

A1.6.9. Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be prefibred categories over $\mathcal{E}$, Φ a cartesian functor $\mathcal{A} \longrightarrow \mathcal{B}$. The functor Φ is faithful (resp. fully faithful, resp. $\mathcal{E}$-equivalence) iff for any $X \in Ob\mathcal{E}$, the induced functor $\Phi_X : \mathcal{A}_X \longrightarrow \mathcal{B}_X$ is faithful (resp. fully faithful, resp. an equivalence).*

Proof. The fact follows from definitions. ■

A1.6.10. Proposition. *Let $\mathcal{A} = (A, F)$ be a prefibred category over $\mathcal{E}$. It is fibred iff the following condition holds:*

(Fib) Let $\xi : x \to y$ be a cartesian morphism over $f : X \to Y$ (i.e. $F(\xi) = f$). For any morphism $g : V \to X$ and any $v \in Ob\mathcal{A}_V$, the map

$$Hom_g(v, x) \longrightarrow Hom_{fg}(v, x), \quad u \longmapsto \xi \circ u,$$

is bijective.

A1.6.10.1. Corollary. *Let $\mathcal{A} = (A, F)$ be a category over $\mathcal{E}$, ξ a morphism of A.*

(a) If ξ an isomorphism, then ξ is cartesian and $F(\xi)$ is an isomorphism.

(b) If $\mathcal{A}$ is fibred, then the inverse is true.

A1.6.10.2. Corollary. *Let $\xi : x \to y$ and $\alpha : v \to x$ be morphisms of a fibred category $\mathcal{A}$ over $\mathcal{E}$. Suppose ξ is cartesian. Then α is cartesian iff $\xi \circ \alpha$ is cartesian.*

A1.7. Fibred categories and pseudo-functors. A pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$ is given by the following data:

(i) A map $Ob\mathcal{E} \longrightarrow ObCat,\ X \longmapsto \mathcal{A}_X$.

(ii) A map $Hom\mathcal{E} \longrightarrow HomCat$ which associates to any $f : X \to Y$ a functor $f^* : \mathcal{A}_Y \longrightarrow \mathcal{A}_X$.

(iii) A map which associates to any pair of composable morphisms, $S \xrightarrow{f} T \xrightarrow{g} U$ a functor morphism $c_{g,f} : f^*g^* \longrightarrow (gf)^*$.

This data should satisfy the following conditions:

(a) $c_{f,id_S} = id_{f^*} = c_{id_T,f}$ for any morphism $f : S \to T$ of $\mathcal{E}$,

(b) For any composable morphisms, $f : S \to T,\ g : T \to U,\ h : U \to V$ of $\mathcal{E}$, the diagram

$$\begin{array}{ccc} f^*g^*h^* & \xrightarrow{c_{f,g}h^*} & (gf)^*h^* \\ {\scriptstyle f^*c_{h,g}}\downarrow & & \downarrow{\scriptstyle c_{h,gf}} \\ f^*(hg)^* & \xrightarrow{c_{hg,f}} & (hgf)^* \end{array}$$

commutes.

Pseudo-functors $\mathcal{E}^{op} \longrightarrow Cat$ form a category defined in a natural way.

A1.7.1. Prefibred categories and pseudo-functors. Let $\mathcal{A} = (A, F)$ be a prefibred category over $\mathcal{E}$. Then there is a function which assigns to any morphism f of $\mathcal{E}$ its inverse image functor, f^* in such a way that $(id_X)^* = Id_{\mathcal{A}_X}$ for all $X \in Ob\mathcal{E}$.

Let $f : X \to Y$ and $g : V \to X$ be morphisms of $\mathcal{E}$ and y an object of $\mathcal{A}_Y$. There exists a unique V-morphism

$$c_{f,g}(y) : g^*f^*(y) \longrightarrow (fg)^*(y)$$

such that the diagram

$$\begin{array}{ccc} g^*f^*(y) & \xrightarrow{\xi_g(f^*(y))} & f^*(y) \\ {\scriptstyle c_{f,g}(y)}\downarrow & & \downarrow{\scriptstyle \xi_f(y)} \\ (fg)^*(y) & \xrightarrow{\xi_{fg}(y)} & y \end{array}$$

is commutative. These morphisms are functorial in y, i.e. $c_{f,g} = \{c_{f,g}(y) | y \in Ob\mathcal{A}_Y\}$ is a morphism $g^*f^* \longrightarrow (fg)^*$ of functors $\mathcal{A}_Y \longrightarrow \mathcal{A}_V$. One can check that they satisfy the conditions (a), (b) of A1.7.1.

Conversely, let $\mathcal{E}^{op} \longrightarrow Cat,\ X \longmapsto \mathcal{A}_X,\ f \longmapsto f^*$ be a pseudo-functor. Set $Ob\mathcal{A} = \coprod_{X \in Ob\mathcal{E}} \mathcal{A}_X = \{(X, x) |\ X \in Ob\mathcal{E},\ x \in Ob\mathcal{A}_X\}$. A morphism from $\bar{x} = (X, x)$ to $\bar{y} = (Y, y)$ is a pair (f, ξ), where f is a morphism $X \to Y$, ξ a morphism $x \to f^*(y)$. A composition is defined by

$$(f, \xi) \circ (g, \mu) := c_{f,g}(y) \circ g^*(\xi) \circ \mu. \qquad (1)$$

Set $h_f(\bar{x},\bar{y}) := Hom_{\mathcal{A}_X}(x, f^*(y))$ and $\mathcal{A}(\bar{x},\bar{y}) = \coprod_{f\in\mathcal{E}(X,Y)} h_f(\bar{x},\bar{y})$. The composition (1) defines the composition on $\mathcal{A}$. The projection functor, $F : \mathcal{A} \longrightarrow \mathcal{E}$, is given by the maps $(X, x) \longmapsto X,\ (f, \xi) \longmapsto f$.

The $\mathcal{E}$-category $F : \mathcal{A} \longrightarrow \mathcal{E}$ is fibred iff all morphisms $c_{f,g}$ are isomorphisms.

A1.8. Limits, Colimits, and pseudo-functors. Let $\mathcal{A} = (A, F)$ be a prefibred category corresponding to a pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$,

$$Ob\mathcal{E} \ni X \longmapsto \mathcal{A}_X,\ Hom\mathcal{E} \ni f \longmapsto f^*,\ Hom\mathcal{E} \times_{Ob\mathcal{E}} Hom\mathcal{E} \ni (f,g) \longmapsto c_{f,g}. \qquad (1)$$

A1.8.1. Colimits of pseudo-functors. The composition of inclusion functors $\mathcal{A}_X \hookrightarrow A$ and the canonical functor $A \longrightarrow Colim\mathcal{A}/\mathcal{E}$ provide for any $X \in Ob\mathcal{E}$ a functor $q_X : \mathcal{A}_X \longrightarrow Colim\mathcal{A}/\mathcal{E}$, and for any morphism $f : X \to Y$ of $\mathcal{E}$ a diagram commutative up to isomorphism:

$$\begin{array}{ccc} \mathcal{A}_Y & \xrightarrow{f^*} & \mathcal{A}_X \\ q_Y \searrow & & \swarrow q_X \\ & Colim\mathcal{A}/\mathcal{E} & \end{array}$$

Thus $Colim\mathcal{A}/\mathcal{E}$ is a colimit in the sense of pseudo-functors of the pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$.

Note that if $X \longmapsto \mathcal{A}_X,\ f \longmapsto f^*$ is a functor, the category $Colim\mathcal{A}/\mathcal{E}$ is not, in general, the colimit of this functor.

A1.8.2. Limits of pseudo-functors. Fix a pseudo-functor

$$\mathcal{E}^{op} \longrightarrow Cat,\ X \longmapsto \mathcal{A}_X,\ f \longmapsto f^*,\ (f,g) \longmapsto c_{f,g}.$$

For any $X \in Ob\mathcal{E}$, denote by p_X the functor $Lim\mathcal{A}/\mathcal{E} \longrightarrow \mathcal{A}_X$ of evaluation at X. For any morphism $f : X \to Y$ of $\mathcal{E}$, there is a diagram

$$\begin{array}{ccc} \mathcal{A}_Y & \xrightarrow{f^*} & \mathcal{A}_X \\ p_Y \nwarrow & & \nearrow p_X \\ & Lim\mathcal{A}/\mathcal{E} & \end{array}$$

commutative up to isomorphism. This means that $Lim\mathcal{A}/\mathcal{E}$ is a limit in the sense of pseudo-functors of the pseudo-functor $\mathcal{E}^{op} \longrightarrow Cat$.

If $X \longmapsto \mathcal{A}_X,\ f \longmapsto f^*$ is a functor, the category $Lim\mathcal{A}/\mathcal{E}$ is not, in general, the limit of this functor.

A1.9. Cofibred and bifibred categories. Fix a category $\mathcal{A} = (A, A \xrightarrow{F} \mathcal{E})$, over $\mathcal{E}$. A morphism $\xi : x \to y$ of A is called *cocartesian* if it is a cartesian morphism of the category $\mathcal{A}^{op} := (A^{op}, F^{op})$ over $\mathcal{E}^{op}$. This means that for any $x' \in Ob\mathcal{A}_Y$, the map $\mathcal{A}_X(y, y') \longrightarrow Hom_f(x, y')$, $u \longmapsto u \circ \xi$, is bijective. In this case, (y, ξ) is called a *direct image of* x *by* f. If it exists for any $x \in \mathcal{A}_X$, then there exists a *direct image functor* $f_* : \mathcal{A}_X \longrightarrow \mathcal{A}_Y$. It is defined (uniquely up to isomorphism) by an isomorphism of bifunctors

$$\mathcal{A}_Y(f_*(x), y) \xrightarrow{\sim} Hom_f(x, y).$$

A1.9.1. Suppose f_* exists. Then f^* exists iff f_* has a right adjoint.

In fact, the functor f^* is defined (uniquely up to isomorphism) by a functorial isomorphism $\mathcal{A}_X(x, f^*(y)) \xrightarrow{\sim} Hom_f(x, y)$. Therefore we have a functorial isomorphism $\mathcal{A}_X(x, f^*(y)) \simeq \mathcal{A}_Y(f_*(x), y)$.

Appendix 2: Monoidal categories and their actions.

The main practical purpose here is to fix notations and give an overview of the subject in the form convenient for its use in Chapter V. The reader might look shortly at this text and then return to its specific parts when needed.

A2.1. Categories with multiplication and morphisms between them. A category with multiplication is a pair $(\mathcal{A}, \odot)$, where $\mathcal{A}$ is a category and $\odot$ is a functor $\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$. Categories with multiplications form a category: a morphism from $(\mathcal{A}, \odot)$ to $(\mathcal{A}', \odot')$ is a pair (Φ, ϕ), where Φ is a functor $\mathcal{A} \longrightarrow \mathcal{A}'$ and ϕ is a functor morphism $\odot' \circ (\Phi \times \Phi) \longrightarrow \Phi \circ \odot$. The composition of

$$(\mathcal{A}, \odot) \xrightarrow{(\Phi,\phi)} (\mathcal{A}', \odot') \quad \text{and} \quad (\mathcal{A}', \odot') \xrightarrow{(\Psi,\psi)} (\mathcal{A}'', \odot'')$$

is $(\Psi \circ \Phi, \Psi\phi \circ \psi(\Phi \times \Phi))$.

A2.1.1. Strict morphisms. A morphism

$$(\mathcal{A}, \odot) \xrightarrow{(\Phi,\phi)} (\mathcal{A}', \odot')$$

of categories with multiplication is called *strict* if ϕ is the identical morphism; that is $\odot' \circ (\Phi \times \Phi) = \Phi \circ \odot$. It follows that strict morphisms are closed under composition.

A2.1.2. Associative multiplications. The category with multiplication $(\mathcal{A}, \odot)$ is called *strict* if the functor

$$\mathcal{A} \xrightarrow{\mathfrak{L}_\odot} End(\mathcal{A}), \quad a \longmapsto a \odot -, \tag{1}$$

is a strict morphism from $(\mathcal{A}, \odot)$ to $(End(\mathcal{A}), \circ)$. The latter means precisely that the multiplication $\odot$ is associative: $\odot \circ (\odot \times Id_{\mathcal{A}}) = \odot \circ (Id_{\mathcal{A}} \times \odot)$.

A2.2. Strict monoidal categories. A strict category with multiplication $(\mathcal{A}, \odot)$ is called a *strict monoidal* category, if there exists an object $\mathbb{I}$ of $\mathcal{A}$ such that

$$\mathbb{I} \odot - = Id_{\mathcal{A}} = - \odot \mathbb{I}.$$

Notice that the *unit* object $\mathbb{I}$ is uniquely determined by these equalities.

A standard example of a strict monoidal category is the category $End_{\mathcal{C}}$ of endofunctors of a category $\mathcal{C}$ with the composition as multiplication.

A *strict monoidal* functor from a strict monoidal category $(\mathcal{A}, \odot, \mathbb{I})$ to a strict monoidal category $(\mathcal{A}', \odot', \mathbb{I}')$ is a functor $\mathcal{A} \xrightarrow{\Phi} \mathcal{A}'$ such that

$$\odot' \circ (\Phi \times \Phi) = \Phi \circ \odot \quad \text{and} \quad \Phi(\mathbb{I}) = \mathbb{I}'.$$

A2.2.1. If $(\mathcal{A}, \odot, \mathbb{I})$ is a strict monoidal category, then the canonical functors

$$\mathcal{A} \xrightarrow{\mathfrak{L}_\odot} End(\mathcal{A}), \quad a \longmapsto a \odot -,$$
$$\mathcal{A} \xrightarrow{\mathfrak{R}_\odot} End(\mathcal{A}), \quad a \longmapsto - \odot a,$$

are strict monoidal functors from $(\mathcal{A}, \odot, \mathbb{I})$ to $(End(\mathcal{A}), \circ, Id_{\mathcal{A}})$.

A2.3. Actions. An action of a category with multiplication $(\mathcal{A}, \odot)$ on a category C_X is a morphism $(\mathcal{A}, \odot) \xrightarrow{(\Phi.\phi)} (End(C_X), \circ)$.

A2.3.1. Associativity constraints. Every extension $(\mathfrak{L}_\odot, \mathfrak{a})$ of the canonical functor

$$\mathcal{A} \xrightarrow{\mathfrak{L}_\odot} End(\mathcal{A}), \quad x \longmapsto x \odot -, \tag{1}$$

to an action of $(\mathcal{A}, \odot)$ on the category $\mathcal{A}$ is a choice of a morphism

$$\odot \circ (Id_{\mathcal{A}} \times \odot) \xrightarrow{\mathfrak{a}} \odot \circ (\odot \times Id_{\mathcal{A}}). \tag{2}$$

of functors from $\mathcal{A} \times \mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. If the morphism $\mathfrak{a}$ in (2) is an isomorphism, it is called an *associativity constraint.* Every associativity constraint $\mathfrak{a}$ on $(\mathcal{A}, \odot)$ gives an extension $(\mathfrak{R}_\odot, \mathfrak{a}^{-1})$ of the functor

$$\mathcal{A} \xrightarrow{\mathfrak{R}_\odot} End(\mathcal{A}), \quad x \longmapsto - \odot x,$$

to an action of $(\mathcal{A}, \odot)$ on the category $\mathcal{A}$.

Triples $(\mathcal{A}, \odot; \mathfrak{a})$, where $\mathfrak{a}$ is an associativity constraint, are objects of a category whose morphisms $(\mathcal{A}, \odot; \mathfrak{a}) \longrightarrow (\mathcal{A}', \odot'; \mathfrak{a}')$ are morphisms $(\mathcal{A}, \odot) \xrightarrow{(\Phi.\phi)} (\mathcal{A}', \odot')$, which are *compatible* with associativity constraints in the sense that the diagram

$$\begin{array}{ccccc}
\odot' \circ (\Phi \times \odot' \circ (\Phi \times \Phi)) & \xrightarrow{\odot'(id_\Phi \times \phi)} & \odot' \circ (\Phi \times \Phi \circ \odot) & \xrightarrow{\phi(Id_{\mathcal{A}} \times \odot)} & \Phi \circ \odot \circ (Id_{\mathcal{A}} \times \odot) \\
{\scriptstyle \mathfrak{a}'\Phi} \downarrow & & & & \downarrow {\scriptstyle \Phi\mathfrak{a}} \\
\odot' \circ (\odot' \circ (\Phi \times \Phi) \times \Phi) & \xrightarrow{\odot'(\phi \times id_\Phi)} & \odot' \circ (\Phi \circ \odot \times \Phi) & \xrightarrow{\phi(\odot \times Id_{\mathcal{A}})} & \Phi \circ \odot \circ (\odot \times Id_{\mathcal{A}})
\end{array} \tag{3}$$

commutes. It is immediate that the compatibility with associativity constraints – that is the commutativity of the diagram (3), survives composition.

A2.3.2. Associative actions. An action of $(\mathcal{A}, \odot; \mathfrak{a})$ on a category $\mathcal{C}_X$ (sometimes called an *associative* action) is a morphism $(\mathcal{A}, \odot; \mathfrak{a}) \xrightarrow{(\Phi,\phi)} (End(\mathcal{C}_X), \circ; id)$.

In other words, the diagram

$$\begin{array}{ccccc}
\Phi(x) \circ (\Phi(y) \circ \Phi(z)) & \xrightarrow{\Phi(x)\phi_{y,z}} & \Phi(x) \circ \Phi(y \odot z) & \xrightarrow{\phi_{x,y\odot z}} & \Phi(x \odot (y \odot z)) \\
{\scriptstyle id}\downarrow & & & & \downarrow{\scriptstyle \Phi\mathfrak{a}} \\
(\Phi(x) \circ \Phi(y)) \circ \Phi(z)) & \xrightarrow{\phi_{x,y}\Phi(z)} & \Phi(x \odot y) \circ \Phi(z)) & \xrightarrow{\phi_{x\odot y,z}} & \Phi((x \odot y) \odot z)
\end{array} \qquad (4)$$

commutes for all objects x, y, z of the category $\mathcal{A}$.

A2.3.3. Unital actions.

A2.3.3.1. Categories with multiplication and 'unit' objects. Those are triples $(\mathcal{A}, \odot; \mathbb{I})$, where $\mathbb{I}$ is an object of the category $\mathcal{A}$. A morphism from $(\mathcal{A}, \odot; \mathbb{I})$ to $(\mathcal{A}', \odot'; \mathbb{I}')$ is a triple (Φ, ϕ, ϕ_0), where (Φ, ϕ) is a morphism $(\mathcal{A}, \odot) \longrightarrow (\mathcal{A}', \odot')$ and ϕ_0 is a morphism $\mathbb{I}' \longrightarrow \Phi(\mathbb{I})$. The composition is defined naturally.

A2.3.3.2. Unital actions. Suppose that (Φ, ϕ, ϕ_0) is a morphism from $(\mathcal{A}, \odot; \mathbb{I})$ to $(End(\mathcal{C}_X), \circ; Id_{\mathcal{C}_X})$, i.e. (Φ, ϕ) is an action of $(\mathcal{A}, \odot)$ on the category $\mathcal{C}_X$ and ϕ_0 is a morphism $Id_{\mathcal{C}_X} \longrightarrow \Phi(\mathbb{I})$. We denote by $\Phi(-) \xrightarrow{\phi^\ell} \Phi(\mathbb{I} \odot -)$ the composition of

$$\Phi(-) \xrightarrow{\phi_0\Phi} \Phi(\mathbb{I}) \circ \Phi(-) \quad \text{and} \quad \Phi(\mathbb{I}) \circ \Phi(-) \xrightarrow{\phi_{\mathbb{I},-}} \Phi(\mathbb{I} \odot -)$$

and by $\Phi(-) \xrightarrow{\phi^{\mathfrak{r}}} \Phi(- \odot \mathbb{I})$ the composition of

$$\Phi(-) \xrightarrow{\phi_0\Phi} \Phi(\mathbb{I}) \circ \Phi(-) \quad \text{and} \quad \Phi(\mathbb{I}) \circ \Phi(-) \xrightarrow{\phi_{-,\mathbb{I}}} \Phi(- \odot \mathbb{I}).$$

We call the action (Φ, ϕ, ϕ_0) *'unital'* if $\phi^\ell(\mathbb{I}) = \phi^{\mathfrak{r}}(\mathbb{I})$.

A2.3.3.3. Associative unital actions. Consider now categories with multiplication, associativity constraint and a 'unit' object. Let $(\mathcal{A}, \odot; \mathfrak{a}, \mathbb{I})$ be one of them and (Φ, ϕ, ϕ_0) a morphism of $(\mathcal{A}, \odot; \mathbb{I})$ to $(End(\mathcal{C}_X), \circ; Id_{\mathcal{C}_X})$. We call this morphism an *associative unital* action if (Φ, ϕ) is an associative action of $(\mathcal{A}, \odot; \mathfrak{a})$ on the category $\mathcal{C}_X$ (i.e. the diagram 3.2.2(4) commutes for all x. y, z in $Ob\mathcal{A}$) and (Φ, ϕ, ϕ_0) is unital: $\phi^\ell(\mathbb{I}) = \phi^{\mathfrak{r}}(\mathbb{I})$.

A2.4. Monoidal categories and their actions. A *monoidal category* is a data $(\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$, where $(\mathfrak{L}_\odot, \mathfrak{a}, \mathfrak{l})$ and $(\mathfrak{R}_\odot, \mathfrak{a}^{-1}, \mathfrak{r})$ are associative unital actions of $(\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I})$ on the category $\mathcal{A}$ such that

$$\mathbb{I} \odot Id_{\mathcal{A}} \xleftarrow{\ \mathfrak{l}\ } Id_{\mathcal{A}} \xrightarrow{\ \mathfrak{r}\ } Id_{\mathcal{A}} \odot \mathbb{I}$$

are functor isomorphisms making the diagram

$$\begin{array}{ccccc}
\mathbb{I} \odot Id_{\mathcal{A}} & \xleftarrow{\ \mathfrak{l}\ } & Id_{\mathcal{A}} & \xrightarrow{\ \mathfrak{r}\ } & Id_{\mathcal{A}} \odot \mathbb{I} \\
{\scriptstyle \mathbb{I} \odot \mathfrak{r}} \downarrow & & & & \downarrow {\scriptstyle \mathfrak{l} \odot \mathbb{I}} \\
\mathbb{I} \odot (Id_{\mathcal{A}} \odot \mathbb{I}) & & \xrightarrow{\ \mathfrak{a}\ } & & (\mathbb{I} \odot Id_{\mathcal{A}}) \odot \mathbb{I}
\end{array} \qquad (1)$$

commute.

A2.4.1. Monoidal functors. A morphism from a monoidal category $(\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ to a monoidal category $(\mathcal{A}', \odot'; \mathfrak{a}'; \mathbb{I}', \mathfrak{l}', \mathfrak{r}')$ (otherwise called a *monoidal functor*) is a morphism (Φ, ϕ, ϕ_0) from $(\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I})$ to $(\mathcal{A}', \odot'; \mathfrak{a}'; \mathbb{I}')$ such that the diagram

$$\begin{array}{ccccc}
\mathbb{I}' \odot' \Phi(-) & \xleftarrow{\ \mathfrak{l}'\Phi\ } & \Phi & \xrightarrow{\ \mathfrak{r}'\Phi\ } & \Phi(-) \odot' \mathbb{I}' \\
{\scriptstyle \phi_{\mathbb{I},-}} \downarrow & & \downarrow {\scriptstyle id_\Phi} & & \downarrow {\scriptstyle \phi_{-,\mathbb{I}}} \\
\Phi(\mathbb{I} \odot -) & \xleftarrow{\ \Phi\mathfrak{l}\ } & \Phi & \xrightarrow{\ \Phi\mathfrak{r}\ } & \Phi(- \odot \mathbb{I})
\end{array} \qquad (2)$$

commutes. The composition of monoidal functors is a monoidal functor.

A2.4.2. Morphisms of monoidal functors. Let $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ be a svelte monoidal category; and let (Φ, ϕ, ϕ_0) and (Ψ, ψ, ψ_0) be monoidal functors from $\mathcal{A}^\sim$ to a monoidal category $\widetilde{\mathcal{A}}^\sim = (\widetilde{\mathcal{A}}, \widetilde{\odot}; \widetilde{\mathfrak{a}}; \widetilde{\mathbb{I}}, \widetilde{\mathfrak{l}}, \widetilde{\mathfrak{r}})$. A morphism from (Φ, ϕ, ϕ_0) to (Ψ, ψ, ψ_0) is a functor morphism $\Phi \xrightarrow{\ \zeta\ } \Psi$ such that the diagrams

$$\begin{array}{ccc}
\Phi(x)\widetilde{\odot}\Phi(y) & \xrightarrow{\ \phi_{x,y}\ } & \Phi(x \odot y) \\
{\scriptstyle \zeta(x)\widetilde{\odot}\zeta(y)} \downarrow & & \downarrow {\scriptstyle \zeta(x \odot y)} \\
\Psi(x)\widetilde{\odot}\Psi(y) & \xrightarrow{\ \psi_{x,y}\ } & \Psi(x \odot y)
\end{array} \quad \text{and} \quad
\begin{array}{ccc}
\Phi(\mathbb{I}) & \xrightarrow{\ \zeta(\mathbb{I})\ } & \Psi(\mathbb{I}) \\
{\scriptstyle \phi_0} \nwarrow & & \nearrow {\scriptstyle \psi_0} \\
& \widetilde{\mathbb{I}} &
\end{array}$$

commute for all $x, y \in Ob\mathcal{A}$.

A2.4.3. The 2-category of monoidal categories. We denote by $\mathfrak{MCat}$ the 2-category whose objects are svelte monoidal categories, 1-morphisms are monoidal functors and 2-morphisms are morphisms of monoidal functors.

A2.4.4. Actions of monoidal categories. An action of a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ on a category $\mathcal{C}_X$ is a morphism (Φ, ϕ, ϕ_0) of $\mathcal{A}^\sim$ to the the strict monoidal category $(End(\mathcal{C}_X), \circ, Id_{\mathcal{C}_X})$. It follows that (Φ, ϕ, ϕ_0) is an assoicative unital action of $(\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I})$ on $\mathcal{C}_X$ such that the morphism $\Phi \xrightarrow{\phi^{\mathfrak{l}}} \Phi(\mathbb{I} \odot -)$ (defined in A2.3.3.2) coincides with $\Phi(\mathfrak{l})$ and $\Phi \xrightarrow{\phi^{\mathfrak{r}}} \Phi(- \odot \mathbb{I})$ coincides with $\Phi(\mathfrak{r})$.

A2.4.5. Morphisms between actions of monoidal categories. Let $\Phi^\sim = (\Phi, \phi, \phi_0)$ be an action of a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ on the category $\mathcal{C}_X$ and $\widetilde{\Phi}^\sim = (\widetilde{\Phi}, \widetilde{\phi}, \widetilde{\phi}_0)$ be an action of a monoidal category $\widetilde{\mathcal{A}}^\sim = (\widetilde{\mathcal{A}}, \widetilde{\odot}; \widetilde{\mathfrak{a}}; \widetilde{\mathbb{I}}, \widetilde{\mathfrak{l}}, \widetilde{\mathfrak{r}})$ on the category $\mathcal{C}_Y$. A morphism from $\Phi^\sim = (\Phi, \phi, \phi_0)$ to $\widetilde{\Phi}^\sim = (\widetilde{\Phi}, \widetilde{\phi}, \widetilde{\phi}_0)$ is a triple $(\Psi^\sim; g_*, \gamma)$, where $\Psi^\sim = (\Psi, \psi, \psi_0)$ is a monoidal functor from $\mathcal{A}^\sim$ to $\widetilde{\mathcal{A}}^\sim$, g_* is a functor $\mathcal{C}_X \longrightarrow \mathcal{C}_Y$, γ is a morphism $\widetilde{\Phi} \circ \Psi(-) \circ g_* \longrightarrow g_*\Phi(-)$ of functors from $\mathcal{A}$ to $Hom(\mathcal{C}_X, \mathcal{C}_Y)$ satisfying natural compatibility conditions.

A2.4.6. Examples of actions.

A2.4.6.1. Bimodules and modules. A standard noncommutative example is the monoidal category $R^e - mod^\sim = (R^e - mod, \otimes_R, R)$ of left R^e-modules acting on the category of left modules over an associative k-algebra R.

A2.4.6.2. Continuous endofunctors. Let $\mathcal{C}_X$ be a svelte k-linear category and $End_k^{\mathfrak{c}}(\mathcal{C}_X)$ the category of continuous k-linear endofunctors of the category $\mathcal{C}_X$. It is the monoidal subcategory of the strict monoidal category $End(\mathcal{C}_X)^\sim = (End(\mathcal{C}_X), \circ, Id_{\mathcal{C}_X})$ naturally acting on the category $\mathcal{C}_X$. It follows from Eilenberg-Moore theorem that, it $\mathcal{C}_X = R - mod$, this example is naturally equivalent to A2.4.6.1.

A2.4.6.3. Weakly continuous functors. Let $\mathcal{C}_X$ be a svelte category with cokernels of reflexive pairs of arrows, and let $End^w(\mathcal{C}_X)$ denote the category of *weakly continuous* (that is preserving cokernels of reflexive pairs of arrows) endofunctors of the category $\mathcal{C}_X$. The monoidal category $End^w(\mathcal{C}_X)^\sim = (End^w(\mathcal{C}_X), \circ, Id_{\mathcal{C}_X})$ acts on the category $\mathcal{C}_X$.

A2.4.6.4. One of the main examples. Let $\mathcal{C}_X$ be a svelte category with cokernels of reflexive pairs of arrows and countable coproducts; and let $\mathfrak{End}^{\mathfrak{w}}(\mathcal{C}_X)$ denote the full subcategory of $End^w(\mathcal{C}_X)$ generated by all endofinctors of $\mathcal{C}_X$ which preserve countable coproducts. The monoidal subcategory $\mathfrak{End}^{\mathfrak{w}}(\mathcal{C}_X)^\sim = (\mathfrak{End}^{\mathfrak{w}}(\mathcal{C}_X), \circ, Id_{\mathcal{C}_X})$ is one of our main examples: the actions of other monoidal categories on $\mathcal{C}_X$ are usually required to be monoidal functors to the monoidal $\mathfrak{End}^{\mathfrak{w}}(\mathcal{C}_X)^\sim$.

A2.5. Algebras in monoidal categories. An *algebra* in a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ is a pair $(\mathcal{R}, \mu)$, where $\mathcal{R}$ is an object of the category $\mathcal{A}$ and μ is a morphism $\mathcal{R} \odot \mathcal{R} \longrightarrow \mathcal{R}$. A morphism of algebras $(\mathcal{R}, \mu_{\mathcal{R}}) \longrightarrow (\mathcal{S}, \mu_{\mathcal{S}})$ is given by a morphism $\mathcal{R} \xrightarrow{\varphi} \mathcal{S}$ of the category $\mathcal{A}$ such that the diagram

$$\begin{array}{ccc} \mathcal{R} \odot \mathcal{R} & \xrightarrow{\varphi \odot \varphi} & \mathcal{S} \odot \mathcal{S} \\ {\scriptstyle \mu_{\mathcal{R}}} \downarrow & & \downarrow {\scriptstyle \mu_{\mathcal{S}}} \\ \mathcal{R} & \xrightarrow{\varphi} & \mathcal{S} \end{array}$$

commutes. If $(\mathcal{R}, \mu_{\mathcal{R}}) \xrightarrow{\varphi} (\mathcal{S}, \mu_{\mathcal{S}})$ and $(\mathcal{S}, \mu_{\mathcal{S}}) \xrightarrow{\psi} (\mathcal{T}, \mu_{\mathcal{T}})$ are algebra morphisms, then the composition $\mathcal{R} \xrightarrow{\psi \circ \varphi} \mathcal{T}$ is an algebra morphism $(\mathcal{R}, \mu_{\mathcal{R}}) \xrightarrow{\psi \circ \varphi} (\mathcal{T}, \mu_{\mathcal{T}})$.

This defines the category $\mathfrak{Alg}\mathcal{A}^{\sim}$ of *algebras in the monoidal category* $\mathcal{A}^{\sim}$.

A2.5.1. Unital algebras. An algebra $(\mathcal{R}, \mu)$ is called *unital* if there is a morphism – the unit element, $\mathbb{I} \xrightarrow{\eta_{\mathcal{R}}} \mathcal{R}$ such that the diagram

$$\begin{array}{ccccc} \mathbb{I} \odot \mathcal{R} & \xleftarrow{\mathfrak{l}_{\mathcal{R}}} & \mathcal{R} & \xrightarrow{\mathfrak{r}_{\mathcal{R}}} & \mathcal{R} \odot \mathbb{I} \\ {\scriptstyle \eta_{\mathcal{R}} \odot id_{\mathcal{R}}} \downarrow & & \downarrow {\scriptstyle id_{\mathcal{R}}} & & \downarrow {\scriptstyle id_{\mathcal{R}} \odot \eta_{\mathcal{R}}} \\ \mathcal{R} \odot \mathcal{R} & \xrightarrow{\mu} & \mathcal{R} & \xleftarrow{\mu} & \mathcal{R} \odot \mathcal{R} \end{array} \qquad (2)$$

commutes. There is at most one unit element, because if $\mathbb{I} \xrightarrow{\eta} \mathcal{R} \xleftarrow{\widetilde{\eta}} \mathbb{I}$ are unit elements, then it follows from the commutativity of the diagram (2) that

$$\eta = (\mu \circ (\widetilde{\eta} \odot id_{\mathcal{R}}) \circ \mathfrak{l}_{\mathcal{R}}) \circ \eta = \mu \circ (\eta \odot \widetilde{\eta}) \circ \mathfrak{l}_{\mathbb{I}} =$$
$$\mu \circ (\eta \odot \widetilde{\eta}) \circ \mathfrak{r}_{\mathbb{I}} = (\mu \circ (id_{\mathcal{R}} \odot \widetilde{\eta}) \circ \mathfrak{r}_{\mathcal{R}}) \circ \widetilde{\eta} = \widetilde{\eta}.$$

A2.5.1.1. Unital algebra morphisms. Let $(\mathcal{R}, \mu_{\mathcal{R}})$ and $(\mathcal{S}, \mu_{\mathcal{S}})$ be unital algebras with the unit elements $\mathcal{R} \xleftarrow{\eta_{\mathcal{R}}} \mathbb{I} \xrightarrow{\eta_{\mathcal{S}}} \mathcal{S}$. An algebra morphism

$$(\mathcal{R}, \mu_{\mathcal{R}}) \xrightarrow{\varphi} (\mathcal{S}, \mu_{\mathcal{S}})$$

is called *unital* if $\varphi \circ \eta_{\mathcal{R}} = \eta_{\mathcal{S}}$. The composition of unital morphisms is a unital morphism.

Unital algebras and unital algebra morphisms form a subcategory of the category of $\mathfrak{Alg}\mathcal{A}^{\sim}$ of algebras in $\mathcal{A}^{\sim}$, which we denote by $\mathfrak{Alg}^{\mathfrak{u}}\mathcal{A}^{\sim}$.

A2.5.2. The category of associative unital algebras. An algebra $(\mathcal{R}, \mu)$ in the monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ is called *associative* if the diagram

$$\begin{array}{ccccc} \mathcal{R} \odot (\mathcal{R} \odot \mathcal{R}) & & \xrightarrow{\mathfrak{a}} & & (\mathcal{R} \odot \mathcal{R}) \odot \mathcal{R} \\ {\scriptstyle id_{\mathcal{R}} \odot \mu} \downarrow & & & & \downarrow {\scriptstyle \mu \odot id_{\mathcal{R}}} \\ \mathcal{R} \odot \mathcal{R} & \xrightarrow{\mu} & \mathcal{R} & \xleftarrow{\mu} & \mathcal{R} \odot \mathcal{R} \end{array} \qquad (1)$$

commutes.

We denote by $Alg\mathcal{A}^{\sim}$ the full subcategory of the category $\mathfrak{Alg}^{\mathfrak{u}}\mathcal{A}^{\sim}$ whose objects are associative unital algebras in the monoidal category $\mathcal{A}^{\sim}$.

A2.5.2.1. Note. Associative unital algebras in a monoidal category $\mathcal{A}^{\sim}$ are oftenly called *monoids* in $\mathcal{A}^{\sim}$. We prefer to use 'monoids' in the traditional meaning – as algebras in the monoidal category of sets.

A2.5.3. Monoidal functors and algebras.

A2.5.3.1. Functors between algebras induced by monoidal functors. Every morphism $\Phi^{\sim} = (\Phi, \phi, \phi_0)$ from a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ to a monoidal category $\mathcal{A}^{'\sim} = (\mathcal{A}', \odot'; \mathfrak{a}'; \mathbb{I}', \mathfrak{l}', \mathfrak{r}')$ induces a functor

$$\mathfrak{Alg}\mathcal{A}^{\sim} \xrightarrow{\Phi^{\sim}_{\mathfrak{Alg}}} \mathfrak{Alg}\mathcal{A}^{'\sim}, \quad (\mathcal{R}, \mu) \longmapsto (\Phi(\mathcal{R}), \Phi(\mu) \circ \phi_{\mathcal{R},\mathcal{R}}), \tag{1}$$

which maps an algebra morphism $(\mathcal{R}, \mu_{\mathcal{R}}) \xrightarrow{\varphi} (\mathcal{S}, \mu_{\mathcal{S}})$ to

$$(\Phi(\mathcal{R}), \mu_{\mathcal{R}} \circ \phi_{\mathcal{R},\mathcal{R}}) \xrightarrow{\Phi(\varphi)} (\Phi(\mathcal{S}), \mu_{\mathcal{S}} \circ \phi_{\mathcal{R},\mathcal{R}}).$$

It follows that the functor $\mathfrak{Alg}\mathcal{A}^{\sim} \xrightarrow{\Phi^{\sim}_{\mathfrak{Alg}}} \mathfrak{Alg}\mathcal{A}^{'\sim}$ maps associative algebras to associative algebras and unital algebra morphisms to unital algebra morphisms.

In particular, the functor $\Phi^{\sim}_{\mathfrak{Alg}}$ induces a functor

$$Alg\mathcal{A}^{\sim} \xrightarrow{\Phi^{\sim}_{Alg}} Alg\mathcal{A}^{'\sim} \tag{2}$$

from the category of associative unital algebras in $\mathcal{A}^{\sim}$ to the category of associative unital algebras in $\mathcal{A}^{'\sim}$.

A2.5.3.2. Actions and monads. The category of associative unital algebras in the strict monoidal category $(End(\mathcal{C}_X), \circ, Id_{\mathcal{C}_X})$ of endofunctors of a svelte category $\mathcal{C}_X$ coincides with the category $\mathfrak{Mon}(X)$ of monads on $\mathcal{C}_X$. So that every action $\Phi^{\sim} = (\Phi, \phi, \phi_0)$ of a monoidal category $\mathcal{A}^{\sim}$ on the category $\mathcal{C}_X$ induces a functor

$$Alg\mathcal{A}^{\sim} \xrightarrow{\Phi^{\sim}_{Alg}} \mathfrak{Mon}(X). \tag{3}$$

A2.5.3.3. Elements of algebras. Let $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ be a monoidal category. For every object $\mathcal{V}$ of the category $\mathcal{A}$, we denote by $|\mathcal{V}|$ the set $\mathcal{A}(\mathbb{I}, \mathcal{V})$. The elements of $|\mathcal{V}|$ are called *elements* of the object $\mathcal{V}$. The functor

$$\mathcal{A} \xrightarrow{\mathcal{A}(\mathbb{I},-)} Sets, \quad \mathcal{V} \longmapsto |\mathcal{V}|,$$

assigning to each object the set of its elements has a natural structure of a monoidal functor $|-|^\sim = (|-|, \phi, \phi_0)$ from $\mathcal{A}^\sim$ to the monoidal category $Sets^\sim = (Sets, \times, \bullet)$ of sets. Here $\bullet$ is a one element set – the unit object of $Sets^\sim$. The morphism $\bullet \xrightarrow{\phi_0} |\mathbb{I}| = \mathcal{A}(\mathbb{I}, \mathbb{I})$ assigns to the unique element of $\bullet$ the identical morphism $\mathbb{I} \to \mathbb{I}$.

For any pair of objects $\mathcal{V}$, $\mathcal{W}$ of the category $\mathcal{A}$, the morphism

$$|\mathcal{V}| \times |\mathcal{W}| \xrightarrow{\phi_{\mathcal{V},\mathcal{W}}} |\mathcal{V} \odot \mathcal{W}|$$

is defined by $\phi_{\mathcal{V},\mathcal{W}}(\mathfrak{v}, \mathfrak{w}) = (\mathfrak{v} \odot \mathfrak{w}) \circ \mathfrak{l}_{\mathbb{I}}$ – the composition of $\mathbb{I} \odot \mathbb{I} \xrightarrow{\mathfrak{v}\odot\mathfrak{w}} \mathcal{V} \odot \mathcal{W}$ and the canonical isomorphism $\mathbb{I} \xrightarrow{\mathfrak{l}_{\mathbb{I}}} \mathbb{I} \odot \mathbb{I}$.

By the generality A2.5.3.1, the monoidal functor $|-|^\sim = (|-|, \phi, \phi_0)$ from $\mathcal{A}^\sim$ to $Sets^\sim$ induces a functor $|-|^\sim_{Alg}$ from the category $Alg\mathcal{A}^\sim$ of associative unital algebras in $\mathcal{A}^\sim$ to the category of monoids, which is precisely the category of associative unital algebras in the monoidal category $Sets^\sim$.

A2.5.3.3.1. Invertible elements of algebras. Let $\mathcal{R}$ be an associative unital algebra in a monoidal category $\mathcal{A}^\sim$ and $|\mathcal{R}|$ the monoid of its elements. We denote by $|\mathcal{R}|^*$ the group of all invertible elements of the monoid $|\mathcal{R}|$ and call it the *group of invertible elements* of the algebra $\mathcal{R}$.

A2.5.4. Reflection of monoidal categories and opposite algebras.

A2.5.4.1. Reflection of monoidal categories. Let $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ be a monoidal category. We denote by $\odot^\sigma$ the composition of the functor $\mathcal{A} \times \mathcal{A} \xrightarrow{\odot} \mathcal{A}$ with the standard symmetry

$$\mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A} \times \mathcal{A}, \quad (x, y) \longmapsto (y, x),$$

and set

$$\mathfrak{l}^\sigma_x = \mathfrak{r}_x, \quad \mathfrak{r}^\sigma_y = \mathfrak{l}_y, \quad \mathfrak{a}^\sigma_{x,y,z} = \mathfrak{a}^{-1}_{z,y,x} \quad \text{for all} \quad x, y, z \in Ob\mathcal{A}. \tag{1}$$

One can see that $\mathcal{A}^\sim_\sigma = (\mathcal{A}, \odot^\sigma; \mathfrak{a}^\sigma; \mathbb{I}, \mathfrak{l}^\sigma, \mathfrak{r}^\sigma)$ is a monoidal category, which we call the *reflection* of $\mathcal{A}^\sim$. Evidently, the map $\mathcal{A}^\sim \longmapsto \mathcal{A}^\sim_\sigma$ is involutive: $(\mathcal{A}^\sim_\sigma)^\sim_\sigma = \mathcal{A}^\sim$.

A2.5.4.2. Reflection of monoidal functors. If $\Phi^\sim = (\Phi, \phi, \phi_0)$ is a monoidal functor from $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ to $\widetilde{\mathcal{A}}^\sim = (\widetilde{\mathcal{A}}, \widetilde{\odot}; \widetilde{\mathfrak{a}}; \widetilde{\mathbb{I}}, \widetilde{\mathfrak{l}}, \widetilde{\mathfrak{r}})$, then $\Phi^\sim_\sigma = (\Phi, \phi^\sigma, \phi_0)$, where $\phi^\sigma_{x,y} = \phi_{x,y}$ for all $x, y \in Ob\mathcal{A}$, is a monoidal functor from $\mathcal{A}^\sim_\sigma$ to $\widetilde{\mathcal{A}}^\sim_\sigma$.

The map which assigns to monoidal categories and monoidal functors their respective reflections is an involutive automorphism of the category (actually the 2-category) $\mathfrak{MCat}$ of svelte monoidal categories and monoidal functors.

A2.5.4.3. Opposite algebras. Every algebra $\mathcal{R} = (R, \mu_{\mathcal{R}})$ in $\mathcal{A}^{\sim}$ defines an algebra $\mathcal{R}^o = (R, \mu_{\mathcal{R}^o})$ in the monoidal category $\mathcal{A}^{\sim}_{\sigma}$, which we call the *algebra opposite* to $\mathcal{R}$.

If $\mathcal{R} = (R, \mu_{\mathcal{R}}) \xrightarrow{\ \varphi\ } \mathcal{S} = (S, \mu_{\mathcal{S}})$ is a morphism of algebras in $\mathcal{A}^{\sim}$, then the same $R \xrightarrow{\varphi} S$ defines a morphism $\mathcal{R}^o = (R, \mu_{\mathcal{R}^o}) \xrightarrow{\ \varphi\ } \mathcal{S}^o = (S, \mu_{\mathcal{S}^o})$ of opposite algebras.

This produces an isomorphism between the category $\mathfrak{Alg}\mathcal{A}^{\sim}$ of algebras in the monoidal category $\mathcal{A}^{\sim}$ and the category $\mathfrak{Alg}\mathcal{A}^{\sim}_{\sigma}$ of algebras in the monoidal category $\mathcal{A}^{\sim}_{\sigma}$ – the reflection of $\mathcal{A}^{\sim}$. Since $\mathcal{R} \longmapsto \mathcal{R}^o$ maps unital (resp. associative) algebras to unital (resp. associative) algebras, the isomorphism

$$\mathfrak{Alg}\mathcal{A}^{\sim} \xrightarrow{\sim} \mathfrak{Alg}\mathcal{A}^{\sim}_{\sigma}$$

induces an isomorphism

$$Alg\mathcal{A}^{\sim} \xrightarrow{\sim} Alg\mathcal{A}^{\sim}_{\sigma}$$

between the categories of associative unital algebras.

A2.5.5. Digression: braidings, symmetries and commutative algebras.

A2.5.5.1. Morphisms to the reflection. Let $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ be a monoidal category and $(Id_{\mathcal{A}}, \beta, id_{\mathbb{I}})$ a monoidal functor from the monoidal category $\mathcal{A}^{\sim}$ to its reflection $\mathcal{A}^{\sim}_{\sigma} = (\mathcal{A}, \odot^{\sigma}; \mathfrak{a}^{\sigma}; \mathbb{I}, \mathfrak{l}^{\sigma}, \mathfrak{r}^{\sigma})$ (see A2.5.4.1). This means that β is a functor morphism $\odot^{\sigma} \longrightarrow \odot$ making commute the diagram

$$\begin{array}{ccccc}
\odot^{\sigma} \circ (Id_{\mathcal{A}} \times \odot^{\sigma}) & \xrightarrow{\odot^{\sigma}(Id_{\mathcal{A}} \times \beta)} & \odot^{\sigma} \circ (Id_{\mathcal{A}} \times \odot) & \xrightarrow{\beta(Id_{\mathcal{A}} \times \odot)} & \odot \circ (Id_{\mathcal{A}} \times \odot) \\
{\scriptstyle \mathfrak{a}^{\sigma}} \big\downarrow \wr & & & & \wr \big\downarrow {\scriptstyle \mathfrak{a}} \\
\odot^{\sigma} \circ (\odot^{\sigma} \times Id_{\mathcal{A}}) & \xrightarrow{\odot^{\sigma}(\beta \times Id_{\mathcal{A}})} & \odot^{\sigma} \circ (\odot \times Id_{\mathcal{A}}) & \xrightarrow{\beta(\odot \times Id_{\mathcal{A}})} & \odot \circ (\odot \times Id_{\mathcal{A}})
\end{array} \qquad (1)$$

obtained via specialization of the diagram A2.3.1(3) and the diagram

$$\begin{array}{ccccc}
\mathbb{I} \odot^{\sigma} Id_{\mathcal{A}} & \xleftarrow{\ \mathfrak{l}^{\sigma}\ } & Id_{\mathcal{A}} & \xrightarrow{\ \mathfrak{r}^{\sigma}\ } & Id_{\mathcal{A}} \odot^{\sigma} \mathbb{I} \\
{\scriptstyle \beta_{\mathbb{I},-}} \big\downarrow & & \big\downarrow {\scriptstyle id} & & \big\downarrow {\scriptstyle \beta_{-,\mathbb{I}}} \\
\mathbb{I} \odot Id_{\mathcal{A}} & \xleftarrow{\ \mathfrak{l}\ } & Id_{\mathcal{A}} & \xrightarrow{\ \mathfrak{r}\ } & Id_{\mathcal{A}} \odot \mathbb{I})
\end{array} \qquad (2)$$

which is a special case of A2.4.1(2). Or, what is the same, the diagrams

$$\begin{array}{ccccc}
(z \odot y) \odot x) & \xrightarrow{\beta_{y,z} \odot id_x} & (y \odot z) \odot x & \xrightarrow{\beta_{x,y \odot z}} & x \odot (y \odot z) \\
{\scriptstyle \mathfrak{a}_{z,y,x}} \big\uparrow \wr & & & & \wr \big\downarrow {\scriptstyle \mathfrak{a}_{x,y,z}} \\
z \odot (y \odot x) & \xrightarrow{id_z \odot \beta_{x,y}} & z \odot (x \odot y) & \xrightarrow{\beta_{x \odot y, z}} & (x \odot y) \odot z
\end{array} \qquad (3)$$

and

$$\begin{array}{ccccc} x\odot\mathbb{I} & \xleftarrow{\ \mathfrak{r}_x\ } & x & \xrightarrow{\ \mathfrak{l}_x\ } & \mathbb{I}\odot x \\ {\scriptstyle\beta_{\mathbb{I},x}}\big\downarrow & & \big\downarrow{\scriptstyle id_x} & & \big\downarrow{\scriptstyle\beta_{x,\mathbb{I}}} \\ \mathbb{I}\odot x & \xleftarrow{\ \mathfrak{l}_x\ } & x & \xrightarrow{\ \mathfrak{r}_x\ } & x\odot\mathbb{I} \end{array} \qquad (4)$$

commute for all $x, y, z \in Ob\mathcal{A}$.

The commutativity of the diagram (4) means that

$$\beta_{\mathbb{I},x} = \mathfrak{l}_x \circ \mathfrak{r}_x^{-1} \quad \text{and} \quad \beta_{x,\mathbb{I}} = \mathfrak{r}_x \circ \mathfrak{l}_x^{-1} \quad \text{for all} \quad x \in Ob\mathcal{A}. \qquad (5)$$

A2.5.5.2. Note. Applying A2.5.5.1 to a monoidal functor $(Id_{\mathcal{A}}, \beta', id_{\mathbb{I}})$ from the reflection $\mathcal{A}^{\sim}_{\sigma} = (\mathcal{A}, \odot^{\sigma}; \mathfrak{a}^{\sigma}; \mathbb{I}, \mathfrak{l}^{\sigma}, \mathfrak{r}^{\sigma})$ of the monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ to $\mathcal{A}^{\sim}$, we obtain that $(Id_{\mathcal{A}}, \beta', id_{\mathbb{I}})$ being a monoidal functor is equivalent to the commutativity of the diagram

$$\begin{array}{ccccc} (z\odot y)\odot x & \xleftarrow{\ \beta'_{z\odot y,z}\ } & x\odot(z\odot y) & \xleftarrow{\ id_x\odot\beta'_{y,z}\ } & x\odot(y\odot z) \\ {\scriptstyle\mathfrak{a}_{z,y,x}}\big\uparrow\wr & & & & \wr\big\downarrow{\scriptstyle\mathfrak{a}_{x,y,z}} \\ z\odot(y\odot x) & \xleftarrow{\ \beta'_{y\odot x,z}\ } & (y\odot x)\odot z & \xleftarrow{\ \beta'_{x,y}\odot id_z\ } & (x\odot y)\odot z \end{array} \qquad (6)$$

for all $x, y, z \in Ob\mathcal{A}$ together with the equalities

$$\beta'_{x,\mathbb{I}} = \mathfrak{l}_x \circ \mathfrak{r}_x^{-1} \quad \text{and} \quad \beta'_{\mathbb{I},x} = \mathfrak{r}_x \circ \mathfrak{l}_x^{-1} \quad \text{for all} \quad x \in Ob\mathcal{A}. \qquad (7)$$

A2.5.5.3. The action on algebras. As any monoidal functor, a monoidal functor $\beta^{\sim} = (Id_{\mathcal{A}}, \beta, id_{\mathbb{I}})$ from a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ to its reflection $\mathcal{A}^{\sim}_{\sigma} = (\mathcal{A}, \odot^{\sigma}; \mathfrak{a}^{\sigma}; \mathbb{I}, \mathfrak{l}^{\sigma}, \mathfrak{r}^{\sigma})$ gives rise to a functor

$$\mathfrak{Alg}\mathcal{A}^{\sim} \xrightarrow{\ \beta^{\sim}_{\mathfrak{Alg}}\ } \mathfrak{Alg}\mathcal{A}^{\sim}_{\sigma}$$

which maps an algebra $\mathcal{R} = (R, \mu_{\mathcal{R}})$ in $\mathcal{A}^{\sim}$ to the algebra $\mathcal{R}^{\beta} = (R, \mu^{\beta}_{\mathcal{R}})$, where $\mu^{\beta}_{\mathcal{R}} = \mu_{\mathcal{R}} \circ \beta_{R,R}$.

A2.5.5.4. Definition. Let $\beta^{\sim} = (Id_{\mathcal{A}}, \beta, id_{\mathbb{I}})$ be a monoidal functor from a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ to its reflection $\mathcal{A}^{\sim}_{\sigma} = (\mathcal{A}, \odot^{\sigma}; \mathfrak{a}^{\sigma}; \mathbb{I}, \mathfrak{l}^{\sigma}, \mathfrak{r}^{\sigma})$.

We say that an algebra $\mathcal{R} = (R, \mu_{\mathcal{R}})$ in $\mathcal{A}^{\sim}$ is *β-commutative* if $\mathcal{R}^{\beta} = (R, \mu^{\beta}_{\mathcal{R}}) = (R, \mu_{\mathcal{R}} \circ \beta_{R,R})$ coincides with the algebra $\mathcal{R}^{o}$ (see A2.5.4.3).

A2.5.5.5. Braidings. A monoidal functor $\beta^\sim = (Id_\mathcal{A}, \beta, id_\mathbb{I})$ from a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ to its reflection $\mathcal{A}^\sim_\sigma = (\mathcal{A}, \odot^\sigma; \mathfrak{a}^\sigma; \mathbb{I}, \mathfrak{l}^\sigma, \mathfrak{r}^\sigma)$ is called a *braiding* if it has the inverse monoidal functor. Equivalently, $(Id_\mathcal{A}, \beta^{-1}, id_\mathbb{I})$ is a monoidal functor from $\mathcal{A}^\sim_\sigma$ to $\mathcal{A}^\sim$. Here $\beta^{-1}_{x,y} \stackrel{\text{def}}{=} (\beta_{y,x})^{-1}$.

Applying A2.5.5.2(6) to $\beta' = \beta^{-1}$, we obtain the diagram

$$\begin{array}{ccccc} (z\odot y)\odot x & \xrightarrow{\beta_{x,z\odot y}} & x\odot(z\odot y) & \xrightarrow{id_x\odot\beta_{y,z}} & x\odot(y\odot z) \\ \mathfrak{a}_{z,y,x}\big\uparrow\wr & & & & \wr\big\downarrow \mathfrak{a}_{x,y,z} \\ z\odot(y\odot x) & \xrightarrow{\beta_{y\odot x,z}} & (y\odot x)\odot z & \xrightarrow{\beta_{x,y}\odot id_z} & (x\odot y)\odot z \end{array} \qquad (8)$$

which commutes for all $x, y, z \in Ob\mathcal{A}$.

Thus, a braiding $\beta^\sim$ is identified with a morphism $\odot \xrightarrow{\beta} \odot^\sigma$ such that the diagrams (8) and

$$\begin{array}{ccccc} (z\odot y)\odot x & \xrightarrow{\beta_{y,z}\odot id_x} & (y\odot z)\odot x & \xrightarrow{\beta_{x,y\odot z}} & x\odot(y\odot z) \\ \mathfrak{a}_{z,y,x}\big\uparrow\wr & & & & \wr\big\downarrow \mathfrak{a}_{x,y,z} \\ z\odot(y\odot x) & \xrightarrow{id_z\odot\beta_{x,y}} & z\odot(x\odot y) & \xrightarrow{\beta_{x\odot y,z}} & (x\odot y)\odot z \end{array} \qquad (3)$$

commute for all $x, y, z \in Ob\mathcal{A}$ and

$$\beta_{\mathbb{I},x} = \mathfrak{l}_x \circ \mathfrak{r}_x^{-1} \quad \text{and} \quad \beta_{x,\mathbb{I}} = \mathfrak{r}_x \circ \mathfrak{l}_x^{-1} \quad \text{for all} \quad x \in Ob\mathcal{A}. \qquad (5)$$

A2.5.5.6. Symmetries. We call a monoidal functor $\beta^\sim = (Id_\mathcal{A}, \beta, id_\mathbb{I})$ from a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ to its reflection $\mathcal{A}^\sim_\sigma = (\mathcal{A}, \odot^\sigma; \mathfrak{a}^\sigma; \mathbb{I}, \mathfrak{l}^\sigma, \mathfrak{r}^\sigma)$ a *symmetry* if the *reflection* $\beta^{\sigma^\sim} = (Id_\mathcal{A}, \beta^\sigma, id_\mathbb{I})$ of $\beta^\sim$ (cf. A2.4.3) is the inverse of $\beta^\sim$.

In other words, $\beta_{x,y} \circ \beta_{y,x} = id_{x\odot y}$ for any $x, y \in Ob\mathcal{A}$.

A2.5.5.7. Note. Evidently, every symmetry is a braiding. Moreover, if $\beta^\sim = (Id_\mathcal{A}, \beta, id_\mathbb{I})$ is a symmetry, then the commutativity of the diagram (8) is equivalent to the commutativity of the diagram (3). So that symmetries are identified with morphisms $\odot \xrightarrow{\beta} \odot^\sigma$ such that $\beta_{x,y} \circ \beta_{y,x} = id_{x\odot y}$ and $\beta_{\mathbb{I},x} = \mathfrak{l}_x \circ \mathfrak{r}_x^{-1}$ for all $x, y \in Ob\mathcal{A}$ and the diagram (8) (or (3)) commutes for all $x, y, z \in Ob\mathcal{A}$.

A2.5.6. Digression: operads and algebras over operads. Fix a symmetric monoidal k-linear category $\mathcal{C}^\sim = (\mathcal{C}, \otimes, \mathbb{I}, \mathfrak{a}, \mathfrak{l}, \mathfrak{r}; \beta)$; here $\beta = \big(X \otimes Y \xrightarrow{\beta_{X,Y}} Y \otimes X\big)$ is a symmetry. Let **S** denote the category whose objects are sets $[n] = \{1, ..., n\}$, $n \geq 1$, and

$[0] = \emptyset$ and morphisms are bijections. Denote by $\mathcal{C}^{\mathbf{S}}$ the category of functors $\mathbf{S}^{op} \longrightarrow \mathcal{C}$. In other words, objects of $\mathcal{C}^{\mathbf{S}}$ are collections $\mathcal{M} = (M(n)|\ n \geq 0)$, where M_n is an object of $\mathcal{C}$ with an action of the symmetric group S_n.

The category $\mathcal{C}^{\mathbf{S}}$ acts on the category $\mathcal{C}$ by *polynomial functors*:

$$M : V \longmapsto M(V) = \bigoplus_{n\geq 0} M(n) \otimes_{S_n} V^{\otimes n} \tag{1}$$

The composition of polynomial functors is again a polynomial functor. This defines a tensor product, $\odot$, on $\mathcal{C}^{\mathbf{S}}$ called the *plethism product*. We denote the corresponding monoidal category $(\mathcal{C}^{\mathbf{S}}, \odot, \mathbb{I}_{\mathbf{S}})$ by $\mathcal{C}^{\sim\mathbf{S}}$. Here $\mathbb{I}_{\mathbf{S}}$ is the unit object $\mathbb{I}_{\mathbf{S}}$. One can see that $\mathbb{I}_{\mathbf{S}}(n) = 0$ if $n \neq 1$ and $\mathbb{I}_{\mathbf{S}}(1)$ is the unit object of the category $\mathcal{C}^{\sim}$.

Thus, we have a natural action of the monoidal category $\mathcal{C}^{\mathbf{S}}$ on the category $\mathcal{C}$.

Associative unital algebras in the monoidal category $\mathcal{C}^{\sim\mathbf{S}}$ are called *operads* in the monoidal category $\mathcal{C}^{\sim}$, or $\mathcal{C}^{\sim}$-*operads*. Modules in $\mathcal{C}$ over an operad $\mathcal{P}$ are traditionally called *algebras*.

A2.6. Actions and modules. Fix an action $\Phi^{\sim} = (\Phi, \phi, \phi_0)$ of a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ on a category $\mathcal{C}_X$ such that the functor $\mathcal{A} \xrightarrow{\Phi} End(\mathcal{C}_X)$ takes values in the full monoidal subcategory $End^w(\mathcal{C}_X)$ of the category $End(\mathcal{C}_X)$ whose objects are *weakly continuous* (that is preserving cokernels of reflexive pairs of arrows) endofunctors. For any unital associative algebra $\mathcal{R}$ in $\mathcal{A}^{\sim}$, we denote by $\mathcal{R} - mod_X$ the category of modules over the monad $\Phi^{\sim}_{Alg}(\mathcal{R})$.

A2.6.1. The left standard action and left modules. Consider the standard left action of the monoidal category $\mathcal{A}^{\sim}$ on the category $\mathcal{A}$:

$$\mathcal{A}^{\sim} \xrightarrow{(\mathfrak{L}_{\odot}, \mathfrak{a}, \mathfrak{l})} (End(\mathcal{A}), \circ, Id_{\mathcal{A}}), \quad x \longmapsto x \odot -.$$

For any associative unital algebra $\mathcal{R} = (R, \mu_{\mathcal{R}})$, we denote the corresponding category of $\mathcal{R}$-modules by $\mathcal{R} - mod$ and call its objects *left* $\mathcal{R}$-*modules*. It follows that objects of $\mathcal{R} - mod$ are pairs $(M, \xi_{\mathcal{M}})$, where M is an object of the category $\mathcal{A}$ and $\xi_{\mathcal{M}}$ a left $\mathcal{R}$-module structure – a morphism $R \odot M \longrightarrow M$ such that the diagrams

$$\begin{array}{ccccc} R \odot (R \odot M) & \xrightarrow{\mathfrak{a}_{R,R,M}} & & & (R \odot R) \odot M \\ R \odot \xi_{\mathcal{M}} \downarrow & & & & \downarrow \mu_{\mathcal{R}} \odot M \\ R \odot M & \xrightarrow{\xi_{\mathcal{M}}} & M & \xleftarrow{\xi_{\mathcal{M}}} & R \odot M \end{array} \quad \text{and} \quad \begin{array}{ccc} R \odot M & \xrightarrow{\xi_{\mathcal{M}}} & M \\ {\scriptstyle \eta_{\mathcal{R}} \odot M} \nwarrow & & \nearrow {\scriptstyle \mathfrak{l}_M} \\ & \mathbb{I} \odot M & \end{array}$$

commute. Here $\mathbb{I} \xrightarrow{\eta_{\mathcal{R}}} R$ is the unit of the algebra $\mathcal{R} = (R, \mu_{\mathcal{R}})$.

A morphism from a left $\mathcal{R}$-module $\mathcal{L} = (L, \xi_{\mathcal{L}})$ to a left $\mathcal{R}$-module $\mathcal{M} = (M, \xi_{\mathcal{M}})$ is given by a morphism $L \xrightarrow{g} M$ such that the diagram

$$\begin{array}{ccc} R \odot L & \xrightarrow{R \odot g} & R \odot M \\ {\scriptstyle \xi_{\mathcal{L}}} \downarrow & & \downarrow {\scriptstyle \xi_{\mathcal{M}}} \\ L & \xrightarrow{g} & M \end{array}$$

commutes.

A2.6.2. The right standard action and right modules. The *right standard action* of the monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ on the category $\mathcal{A}$ is given by

$$\mathcal{A}^{\sim} \xrightarrow{(\mathfrak{R}_{\odot}, \mathfrak{a}^{-1}, \mathfrak{r})} (End(\mathcal{A}), \circ, Id_{\mathcal{A}}), \quad x \longmapsto - \odot x.$$

For any associative unital algebra $\mathcal{R} = (R, \mu_{\mathcal{R}})$, we denote the corresponding category of $\mathcal{R}$-modules by $mod - \mathcal{R}$ and call its objects *right $\mathcal{R}$-modules.* It follows that objects of $\mathcal{R} - mod$ are pairs $(M, \zeta_{\mathcal{M}})$, where M is an object of the category $\mathcal{A}$ and $\zeta_{\mathcal{M}}$ a morphism $M \odot R \longrightarrow M$ making the diagrams

$$\begin{array}{ccccc} (M \odot R) \odot R & & \xleftarrow{\mathfrak{a}_{M,R,R}} & & M \odot (R \odot R) \\ {\scriptstyle \zeta_{\mathcal{M}} \odot R} \downarrow & & & & \downarrow {\scriptstyle M \odot \mu_{\mathcal{R}}} \\ M \odot R & \xrightarrow{\zeta_{\mathcal{M}}} & M & \xleftarrow{\zeta_{\mathcal{M}}} & M \odot R \end{array} \quad \text{and} \quad \begin{array}{ccc} M \odot R & \xrightarrow{\zeta_{\mathcal{M}}} & M \\ {\scriptstyle M \odot \eta_{\mathcal{R}}} \nwarrow & & \nearrow {\scriptstyle \mathfrak{r}_M} \\ & M \odot \mathbb{I} & \end{array}$$

commute.

A2.6.3. Note. The right action of the monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ on $\mathcal{A}$ is the same as the left action of the monoidal category $\mathcal{A}^{\sim}_{\sigma} = (\mathcal{A}, \odot^{\sigma}; \mathfrak{a}^{\sigma}; \mathbb{I}, \mathfrak{l}^{\sigma}, \mathfrak{r}^{\sigma})$ – the *reflection* of $\mathcal{A}^{\sim}$.

A2.6.4. Monoidal functors and left and right modules. Let $\mathcal{R} = (R, \mu_{\mathcal{R}})$ be an associative unital algebra in a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$. Any monoidal functor $\Phi^{\sim} = (\Phi, \phi, \phi_0)$ from $\mathcal{A}^{\sim}$ to $\widetilde{\mathcal{A}}^{\sim} = (\widetilde{\mathcal{A}}, \widetilde{\odot}; \widetilde{\mathfrak{a}}; \widetilde{\mathbb{I}}, \widetilde{\mathfrak{l}}, \widetilde{\mathfrak{r}})$ induces a functor

$$\mathcal{R} - mod \xrightarrow{\Phi^{\sim}_{\mathcal{R}}} \Phi^{\sim}_{Alg}(\mathcal{R}) - mod$$

which maps an $\mathcal{R}$-module $\mathcal{L} = (L, \xi_{\mathcal{L}})$ to the $\Phi^{\sim}_{Alg}(\mathcal{R})$-module $(\Phi(L), \xi_{\Phi(\mathcal{L})})$, where the action $\xi_{\Phi(\mathcal{L})}$ is the composition of

$$\Phi(R) \widetilde{\odot} \Phi(L) \xrightarrow{\phi_{R,L}} \Phi(R \odot L) \quad \text{and} \quad \Phi(R \odot L) \xrightarrow{\Phi(\xi_{\mathcal{L}})} \Phi(L).$$

Simmetrically, the monoidal functor $\Phi^{\sim}$ induces a functor

$$mod - \mathcal{R} \xrightarrow{\ {}^{\mathcal{R}}\Phi^{\sim}\ } mod - \Phi^{\sim}_{Alg}(\mathcal{R})$$

from the category of right $\mathcal{R}$-modules to the category of right $\Phi^{\sim}_{Alg}(\mathcal{R})$-modules.

A2.6.5. Digression: inner homs related to an action. Let $\Phi^{\sim} = (\Phi, \phi, \phi_0)$ be an action of the monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ on a category $\mathcal{C}_X$. So that, for any associative unital algebra $\mathcal{R}$ in $\mathcal{A}^{\sim}$, we have a functor

$$\mathcal{R} - mod \xrightarrow{\ \Phi^{\sim}_{\mathcal{R}}\ } \Phi^{\sim}_{Alg}(\mathcal{R}) - mod$$

from the category of left $\mathcal{R}$-modules to the category of left $\Phi^{\sim}_{Alg}(\mathcal{R})$-modules.

Notice that every $\Phi^{\sim}_{Alg}(\mathcal{R})$-module can be viewed as a functor from $\mathcal{C}_X$ to the category $\mathcal{R} - mod_X$ of modules over the monad $\Phi^{\sim}_{Alg}(\mathcal{R})$. In particular, for every left $\mathcal{R}$-module $\mathcal{L}$ and any $\mathcal{M} \in Ob\mathcal{R} - mod_X$, we have a functor $\mathcal{R} - mod_X(\Phi^{\sim}_{\mathcal{R}}(\mathcal{L})(-), \mathcal{M})$ from $\mathcal{C}_X^{op}$ to *Sets*. If this functor is representable, we call an object of $\mathcal{C}_X$ representing it *inner hom* from the left $\mathcal{R}$-module $\mathcal{L}$ to the module $\mathcal{M}$ over the monad $\Phi^{\sim}_{Alg}(\mathcal{R})$ and denote it by $\mathfrak{Hom}^{\Phi^{\sim}}_{\mathcal{R}}(\mathcal{L}, \mathcal{M})$.

A2.6.5.1. The standard inner hom. If $\Phi^{\sim}$ is the standard left action of the monoidal category $\mathcal{A}^{\sim}$ on the category $\mathcal{A}$, then both $\mathcal{L}$ and $\mathcal{M}$ are left $\mathcal{R}$-modules and $\mathfrak{Hom}^{\Phi^{\sim}}_{\mathcal{R}}(\mathcal{L}, \mathcal{M}) = \mathfrak{Hom}_{\mathcal{R}}(\mathcal{L}, \mathcal{M})$ coincides with the standard notion of inner hom – the object representing the presheaf $Hom_{\mathcal{R}}(\mathcal{L} \odot -, \mathcal{M}) \stackrel{\mathrm{def}}{=} \mathcal{R} - mod(\mathcal{L} \odot -, \mathcal{M})$.

A2.6.5.2. Finite objects of a monoidal category. An object $\mathcal{L}$ of the category $\mathcal{A}$ is called *finite* if the functor $\mathcal{A} \xrightarrow{\mathcal{L}\odot -} \mathcal{A}$ has a right adjoint of the form $\mathcal{L}^! \odot -$ for some object $\mathcal{L}^!$ of the category $\mathcal{A}$. Notice that the object $\mathcal{L}^!$ is unique up to isomorphism, because $\mathcal{L}^! \odot \mathbb{I} \simeq \mathcal{L}^!$. It follows from the definitions that the inner hom $\mathfrak{Hom}(\mathcal{L}, \mathcal{M})$ exists for all $\mathcal{M} \in Ob\mathcal{A}$ and is isomorphic to $\mathcal{L}^! \odot \mathcal{M}$. In particular, we have an isomorphism $\mathcal{L}^! \xrightarrow{\sim} \mathfrak{Hom}(\mathcal{L}, \mathbb{I})$; that is $\mathcal{L}^!$ is isomorphic to the object *dual* to $\mathcal{L}$.

A2.6.5.2.1. Note. Let $\mathcal{L}$ and $\mathcal{M}$ be objects of the category $\mathcal{A}$ such that there exist $\mathcal{L}^! \stackrel{\mathrm{def}}{=} \mathfrak{Hom}(\mathcal{L}, \mathbb{I})$ and $\mathfrak{Hom}(\mathcal{L}, \mathcal{M})$. Then there is a natural morphism

$$\mathcal{L}^! \odot \mathcal{M} \longrightarrow \mathfrak{Hom}(\mathcal{L}, \mathcal{M}) \tag{1}$$

constructed as follows. Let

$$\mathcal{L} \odot \mathcal{L}^! \xrightarrow{\ \mathfrak{e}_{\mathcal{L}}\ } \mathbb{I} \tag{2}$$

denote the *evaluation* morphism, which is, by definition, the image of the identical arrow $\mathcal{L}^! \longrightarrow \mathcal{L}^!$ by the adjunction isomorphism $\mathcal{A}(\mathcal{L}^!, \mathcal{L}^!) \xrightarrow{\sim} \mathcal{A}(\mathcal{L} \odot \mathcal{L}^!, \mathbb{I})$.

The composition $\mathcal{L} \odot (\mathcal{L}^! \odot \mathcal{M}) \longrightarrow \mathcal{M}$ of the morphisms

$$\mathcal{L} \odot (\mathcal{L}^! \odot \mathcal{M}) \xrightarrow{\mathfrak{a}} (\mathcal{L} \odot \mathcal{L}^!) \odot \mathcal{M} \xrightarrow{\mathfrak{e}_{\mathcal{L}} \odot \mathcal{M}} \mathbb{I} \odot \mathcal{M} \xrightarrow{\mathfrak{l}^{-1}_{\mathcal{M}}} \mathcal{M}$$

determines, by the adjunction $\mathcal{A}(\mathcal{L} \odot -, \mathcal{M}) \xrightarrow{\sim} \mathcal{A}(-, \mathfrak{Hom}(\mathcal{L}, \mathcal{M}))$, the morphism (1).

A2.6.5.2.1. Digression: traces. Let $\beta^\sim = (Id_{\mathcal{A}}, \beta, id_{\mathbb{I}})$ be a monoidal functor from the monoidal category $\mathcal{A}^\sim$ to its reflection $\mathcal{A}^\sim_\sigma$. Let $\mathcal{L}$ be an object of the category $\mathcal{A}$ such that $\mathcal{L}^! = \mathfrak{Hom}(\mathcal{L}, \mathbb{I})$ exists. The composition

$$\mathcal{L}^! \odot \mathcal{L} \xrightarrow{\mathfrak{tr}_\beta} \mathbb{I}$$

of the morphism

$$\mathcal{L}^! \odot \mathcal{L} \xrightarrow{\beta_{\mathcal{L}^!, \mathcal{L}}} \mathcal{L} \odot \mathcal{L}^!$$

and the evaluation morphism $\mathcal{L} \odot \mathcal{L}^! \xrightarrow{\mathfrak{e}_{\mathcal{L}}} \mathbb{I}$ is called the β-*trace* on $\mathcal{L}$.

If $\mathcal{L}$ is a finite object, then the canonical morphism $\mathcal{L}^! \odot \mathcal{L} \longrightarrow \mathfrak{Hom}(\mathcal{L}, \mathcal{L})$ is an isomorphism. So, in this case, the β-trace is defined (for any β) on the object of inner endomorphisms of the object $\mathcal{L}$.

A2.6.6. Bimodules. Let $\mathcal{R} = (R, \mu_{\mathcal{R}})$ and $\mathcal{S} = (S, \mu_{\mathcal{S}})$ be associative unital algebras in the monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$.

An $(\mathcal{R}, \mathcal{S})$-*bimodule* is a triple $(\xi_{\mathcal{M}}, M, \zeta_{\mathcal{M}})$, where $(M, \xi_{\mathcal{M}})$ is a left $\mathcal{R}$-module and $(M, \zeta_{\mathcal{M}})$ is a right $\mathcal{S}$-module and the diagram

$$\begin{array}{ccccc} R \odot (M \odot S) & & \xrightarrow{\mathfrak{a}_{R,M,S}} & & (R \odot M) \odot S \\ R \odot \zeta_{\mathcal{M}} \big\downarrow & & & & \big\downarrow \xi_{\mathcal{M}} \odot S \\ R \odot M & \xrightarrow{\xi_{\mathcal{M}}} & M & \xleftarrow{\zeta_{\mathcal{M}}} & M \odot S \end{array}$$

commutes. A morphism $(\xi_{\mathcal{L}}, L, \zeta_{\mathcal{L}}) \longrightarrow (\xi_{\mathcal{M}}, M, \zeta_{\mathcal{M}})$ is given by a morphism $L \xrightarrow{f} M$ such that the diagram

$$\begin{array}{ccccc} R \odot L & \xrightarrow{\xi_{\mathcal{L}}} & L & \xleftarrow{\zeta_{\mathcal{L}}} & L \odot S \\ R \odot f \big\downarrow & & \big\downarrow f & & \big\downarrow f \odot S \\ R \odot M & \xrightarrow{\xi_{\mathcal{M}}} & M & \xleftarrow{\zeta_{\mathcal{M}}} & M \odot S \end{array}$$

commutes. In other words, $L \xrightarrow{f} M$ gives a morphism $(L, \xi_{\mathcal{L}}) \longrightarrow (M, \xi_{\mathcal{M}})$ of left $\mathcal{R}$-modules and a morphism $(L, \zeta_{\mathcal{L}}) \longrightarrow (M, \zeta_{\mathcal{M}})$ of right $\mathcal{S}$-modules.

A2.6.6.1. Notations. We denote the category of $(\mathcal{R}, \mathcal{S})$-bimodules by $(\mathcal{R}, \mathcal{S}) - bim$ and will write $\mathcal{R} - bim$ instead of $(\mathcal{R}, \mathcal{R}) - bim$.

Notice that the category $\mathcal{R} - mod$ is naturally isomorphic to the category of $(\mathcal{R}, \mathbb{I})$-bimodules and the category $mod - \mathcal{S}$ is isomorphic to the category of $(\mathbb{I}, \mathcal{S})$-bimodules.

A2.7. The bicategory of algebras. Recall that a pair of arrows $M \underset{g_2}{\overset{g_1}{\rightrightarrows}} L$ of a category is called *reflexive*, if there exists a morphism $L \xrightarrow{h} M$ splitting both of them, that is $g_1 \circ h = id_M = g_2 \circ h$. A functor is called *weakly continuous*, if it preserves cokernels of reflexive pairs of arrows (see I.4.2.1).

Fix a monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$. We assume that the category $\mathcal{A}$ has cokernels of reflexive pairs of arrows and, for every object M of $\mathcal{A}$, the functors

$$\mathcal{A} \xleftarrow{M \odot -} \mathcal{A} \xrightarrow{- \odot M} \mathcal{A}$$

are weakly continuous.

Notice that, for any pair $\mathcal{R}$, $\mathcal{S}$ of associative unital algebras in $\mathcal{A}^{\sim}$, the category $(\mathcal{S}, \mathcal{R}) - bim$ of $(\mathcal{S}, \mathcal{R})$-bimodules has cokernels of reflexive pairs of arrows.

Let $\mathcal{L} = (\xi_{\mathcal{L}}, L, \zeta_{\mathcal{L}})$ be an $(\mathcal{S}, \mathcal{R})$-bimodule and $\mathcal{M} = (\xi_{\mathcal{M}}, M, \zeta_{\mathcal{M}})$ a $(\mathcal{T}, \mathcal{S})$-bimodule. Then $M \odot S \odot L$ and $M \odot L$ have natural structures of $(\mathcal{T}, \mathcal{S})$-bimodules and

$$M \odot S \odot L \underset{M \odot \xi_{\mathcal{L}}}{\overset{\zeta_{\mathcal{M}} \odot L}{\rightrightarrows}} M \odot L$$

is a reflexive pair of morphisms of $(\mathcal{T}, \mathcal{S})$-bimodules. Its cokernel is a $(\mathcal{T}, \mathcal{R})$-bimodule, which is denoted by $\mathcal{M} \odot_{\mathcal{S}} \mathcal{L}$ (cf. I.4.2.2).

We denote by $\mathfrak{BAlg}\mathcal{A}^{\sim}$ the bicategory whose objects are associative unital algebras in the monoidal category $\mathcal{A}^{\sim}$ and, for any pair $\mathcal{R}$, $\mathcal{S}$ of such algebras, the category $\mathfrak{Hom}_{\mathcal{A}^{\sim}}(\mathcal{R}, \mathcal{S})$ of morphisms from $\mathcal{R}$ to $\mathcal{S}$ is the category $(\mathcal{S}, \mathcal{R}) - bim$ of $(\mathcal{S}, \mathcal{R})$-bimodules.

The composition is given by the 'tensor' product:

$$\mathfrak{Hom}_{\mathcal{A}^{\sim}}(\mathcal{S}, \mathcal{T}) \times \mathfrak{Hom}_{\mathcal{A}^{\sim}}(\mathcal{R}, \mathcal{S}) \xrightarrow{\odot_{\mathcal{S}}} \mathfrak{Hom}_{\mathcal{A}^{\sim}}(\mathcal{R}, \mathcal{T}), \quad (\mathcal{M}, \mathcal{L}) \longmapsto \mathcal{M} \odot_{\mathcal{S}} \mathcal{L}. \quad (1)$$

In particular, for every assoicative unital algebra $\mathcal{R}$ in $\mathcal{A}^{\sim}$, we have the monoidal category $\mathfrak{End}_{\mathcal{A}^{\sim}}(\mathcal{R}) = (\mathcal{R} - bim, \odot_{\mathcal{R}}, \mathfrak{a}; \mathcal{R}, \mathfrak{l}, \mathfrak{r})$ of "endomorphisms" of $\mathcal{R}$, whose unit object is the algebra $\mathcal{R}$.

A2.8. The bicategory Cat^w of weakly cocomplete categories and weakly continuous functors. We call a category $\mathcal{C}_X$ *weakly cocomplete* it has cokernels of reflexive pairs of arrows. The objects of Cat^w are svelte weakly cocomplete categories. Given two such categories, $\mathcal{C}_X$ and $\mathcal{C}_Y$, the category of morphisms from $\mathcal{C}_X$ and $\mathcal{C}_Y$ is the category $\mathfrak{Hom}^w(\mathcal{C}_X, \mathcal{C}_Y)$ of weakly continuous functors $\mathcal{C}_X \longrightarrow \mathcal{C}_Y$. The composition of 1-morphisms is the composition of functors. Two-morphisms are, as usual, morphisms of functors.

A2.9. Bimodules and actions.

A2.9.1. Assumptions. Fix a monoidal category $\mathcal{A}^\sim = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ and its action $\Phi^\sim = (\Phi, \phi, \phi_0)$ on a category $\mathcal{C}_X$. We assume that the categories $\mathcal{A}$ and $\mathcal{C}_X$ have cokernels of reflexive pairs of arrows (see I.4.2.1) and, for every $M \in Ob\mathcal{A}$, the functors

$$\mathcal{A} \xrightarrow{M\odot -} \mathcal{A} \xleftarrow{-\odot M} \mathcal{A} \quad \text{and} \quad \mathcal{C}_X \xrightarrow{\Phi(M)} \mathcal{C}_X$$

are *weakly* continuous, i.e. they preserve cokernels of reflexive pairs.

A2.9.2. The action of bimodules. Let $\mathcal{R}$ and $\mathcal{S}$ be assoicative unital algebras in $\mathcal{A}^\sim$. The action $\Phi^\sim$ induces a functor from the category of $(\mathcal{S}, \mathcal{R})$-bimodules to the category of weakly continuous functors from $\mathcal{R} - mod_X$ to $\mathcal{S} - mod_X$ which assigns to every $(\mathcal{S}, \mathcal{R})$-bimodule $\mathcal{M} = (\xi_{\mathcal{M}}, M, \zeta_{\mathcal{M}})$ the endofunctor

$$\mathcal{R} - mod_X \xrightarrow{\mathcal{M}\odot^{\Phi}_{\mathcal{R}}} \mathcal{S} - mod_X$$

mapping each $\mathcal{R}$-module $\mathcal{L} = (L, \xi_{\mathcal{L}})$ to the cokernel $\mathcal{M} \odot^{\Phi}_{\mathcal{R}} \mathcal{L}$ of the reflexive pair

$$\begin{array}{ccc} \Phi(M)\Phi(R)(L) & \overset{\Phi(M)\xi_{\mathcal{L}}}{\underset{\zeta'_{\mathcal{L}}}{\rightrightarrows}} & \Phi(M)(L) \\ {\scriptstyle \phi_{M,R}(L)} \searrow & & \nearrow {\scriptstyle \Phi(\zeta_{\mathcal{M}})(L)} \\ & \Phi(M \odot R)(L) & \end{array}$$

of $\Phi^\sim_{Alg}(\mathcal{S})$-module morphisms (see the argument of I.4.2.2).

A2.9.3. The canonical functor. The map which assigns to every associative algebra $\mathcal{R}$ in $\mathcal{A}^\sim$ the category $\mathcal{R} - mod_X$ and to every $(\mathcal{S}, \mathcal{R})$-bimodule $\mathcal{M}$ the functor

$$\mathcal{R} - mod_X \xrightarrow{\mathcal{M}\odot^{\Phi}_{\mathcal{R}}} \mathcal{S} - mod_X$$

is a 2-functor from the bicategory $\mathfrak{BAlg}\mathcal{A}^{\sim}$ of algebras in $\mathcal{A}^{\sim}$ (defined in A2.7) to the bicategory Cat^w of svelte categories with cokernels of reflexive pairs of arrows and weakly continuous functors (see A2.8).

A2.10. Bimodules and restriction of scalars. It follows that, for every associative unital algebra $\mathcal{R}$ in the monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$, an action $\Phi^{\sim} = (\Phi, \phi, \phi_0)$ of $\mathcal{A}^{\sim}$ on a category $\mathcal{C}_X$, which satisfies the conditions A2.9.1 gives rise to an action $\Phi^{\sim}_{\mathcal{R}} = (\Phi^{\mathcal{R}}, \phi^{\mathcal{R}}, \phi^{\mathcal{R}}_0)$ of the monoidal category

$$\mathfrak{End}_{\mathcal{A}^{\sim}}(\mathcal{R}) = \mathcal{R} - bim^{\sim} = (\mathcal{R} - bim, \odot_{\mathcal{R}}, \mathfrak{a}; \mathcal{R}, \mathfrak{l}^{\mathcal{R}}, \mathfrak{r}^{\mathcal{R}})$$

of $\mathcal{R}$-bimodules on the category $\mathcal{R} - mod_X$ of $\mathcal{R}$-modules in $\mathcal{C}_X$.

Notice that any morphism $\mathcal{R} \longrightarrow \mathcal{S}$ of unital associative algebras makes the algebra $\mathcal{S}$ a unital associative algebra in the monoidal category $\mathfrak{End}_{\mathcal{A}^{\sim}}(\mathcal{R})$ of $\mathcal{R}$-bimodules.

A2.10.1. Restrictions of scalars. We assume that the monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$, the category $\mathcal{C}_X$ and the action $\Phi^{\sim} = (\Phi, \phi, \phi_0)$ of $\mathcal{A}^{\sim}$ on $\mathcal{C}_X$ satisfy the conditions A2.9.1; that is $\mathcal{A}$ and $\mathcal{C}_X$ have cokernels of reflexive pairs of arrows and the monoidal structure $\odot$ and the functor Φ preserve them.

Let $\mathcal{R}$ and $\mathcal{S}$ be associative unital algebras in the monoidal category $\mathcal{A}^{\sim}$ and $\mathcal{R} \xrightarrow{\varphi} \mathcal{S}$ a unital algebra morphism. Then we have restriction of scalars functors

$$\mathcal{S} - mod_X \xrightarrow{\varphi_{X*}} \mathcal{R} - mod_X \quad \text{and}$$
$$\mathcal{S} - bim \xrightarrow{\widetilde{\varphi}_*} \mathcal{R} - bim.$$

together with the canonical strict epimorphisms

$$\widetilde{\varphi}_*(\mathcal{M}) \odot_{\mathcal{R}} \varphi_{X*}(\mathcal{V}) \xrightarrow{\phi^{\varphi}_{\mathcal{M},\mathcal{V}}} \varphi_{X*}(\mathcal{M} \odot_{\mathcal{S}} \mathcal{V}) \quad \text{and}$$
$$\widetilde{\varphi}_*(\mathcal{M}) \odot_{\mathcal{R}} \widetilde{\varphi}_*(\mathcal{N}) \xrightarrow{\lambda^{\varphi}_{\mathcal{M},\mathcal{N}}} \widetilde{\varphi}_*(\mathcal{M} \odot_{\mathcal{S}} \mathcal{N})$$

for any $\mathcal{S}$-bimodules $\mathcal{M}$, $\mathcal{N}$ and any $\mathcal{R}$-module $\mathcal{V}$ in $\mathcal{C}_X$, which depend functorially on respectively $(\mathcal{M}, \mathcal{V})$ and $(\mathcal{M}, \mathcal{N})$. The triple $(\widetilde{\varphi}_*, \lambda^{\phi}, \phi)$ is a monoidal functor

$$\mathcal{S} - bim^{\sim} \xrightarrow{\varphi^{\sim}_*} \mathcal{R} - bim^{\sim}$$

which we call, naturally, the monoidal functor of *restrictions of scalars* along $\mathcal{R} \xrightarrow{\varphi} \mathcal{S}$.

A2.10.1.1. Note. The triple $(\varphi_*^\sim; \varphi_{X*}, \phi^\varphi)$ is a morphism from the action $\Phi_{\mathcal{S}}^\sim = (\Phi^{\mathcal{S}}, \phi^{\mathcal{S}}, \phi_0^{\mathcal{S}})$ of the monoidal category $\mathcal{S} - bim^\sim$ on the category $\mathcal{S} - mod_X$ to the action $\Phi_{\mathcal{R}}^\sim$ of the monoidal category $\mathcal{R} - bim^\sim$ on the category $\mathcal{R} - mod_X$ (see A2.4.5).

A2.10.2. Digression: inner hom and restriction of scalars. Let $\mathcal{R} = (R, \mu_{\mathcal{R}})$ and $\mathcal{S} = (S, \mu_{\mathcal{S}})$ be associative unital algebras in the monoidal category $\mathcal{A}^\sim$ and $\mathcal{R} \xrightarrow{\varphi} \mathcal{S}$ a unital algebra morphism. Under the assumptions of A2.9.1, the restriction of scalars functor

$$\mathcal{S} - mod_X \xrightarrow{\varphi_{X*}} \mathcal{R} - mod_X$$

is weakly continuous and has a left adjoint, φ_X^*. That is φ_{X*} and φ_X^* can be regarded as respectively direct and inverse images of a *weakly affine* morphism

$$\mathbf{Sp}(\Phi_{Alg}^\sim(\mathcal{S})/X) \xrightarrow{\varphi_X} \mathbf{Sp}(\Phi_{Alg}^\sim(\mathcal{R})/X). \tag{1}$$

A2.10.2.1. Proposition. *The direct image functor*

$$\mathcal{S} - mod_X \xrightarrow{\varphi_{X*}} \mathcal{R} - mod_X$$

of the morphism (1) has a right adjoint, $\varphi_X^!$ (that is the morphism (1) is affine) iff the inner hom $\mathfrak{Hom}_{\mathcal{R}}^{\Phi^\sim}(\varphi_*(\mathcal{S}), \mathcal{M})$ exists for every $\mathcal{M} \in Ob\mathcal{R} - mod_X$.

Proof. (i) Suppose that the inner hom $\mathfrak{Hom}_{\mathcal{R}}^{\Phi^\sim}(\varphi_*(\mathcal{S}), \mathcal{M})$ exists for every object $\mathcal{M}$ of the category $\mathcal{R} - mod_X$. That is, for every $\mathcal{V} \in Ob\mathcal{C}_X$, there is an isomorphism

$$\begin{aligned} &\mathcal{C}_X(\mathcal{V}, \mathfrak{Hom}_{\mathcal{R}}^{\Phi^\sim}(\varphi_*(\mathcal{S}), \mathcal{M})) \xrightarrow{\sim} \mathcal{R} - mod_X(\Phi_{\mathcal{R}}^\sim(\varphi_*(\mathcal{S}))(\mathcal{V}), \mathcal{M}) = \\ &\mathcal{R} - mod_X(\varphi_{X*}(\Phi_{\mathcal{S}}^\sim(\mathcal{S})(\mathcal{V})), \mathcal{M}) = \mathcal{R} - mod_X((\varphi_{X*}(\eta_{\mathcal{S}})_X^*(\mathcal{V})), \mathcal{M}), \end{aligned} \tag{2}$$

where $\eta_{\mathcal{S}}$ is the unit $\mathbb{I} \longrightarrow \mathcal{S}$ of the algebra $\mathcal{S}$ regarded as an algebra morphism.

(ii) We have natural morphisms

$$\begin{array}{ccc} \mathcal{C}_X(\mathcal{V}, \mathfrak{Hom}_{\mathcal{R}}^{\Phi^\sim}(\varphi_*(\mathcal{S}), \mathcal{M})) & \xrightarrow{\sim} & \mathcal{R} - mod_X(\Phi_{\mathcal{R}}^\sim(\varphi_*(\mathcal{S}))(\mathcal{V}), \mathcal{M}) \\ & & \big\downarrow \\ \mathcal{C}_X(\Phi(S)(\mathcal{V}), \mathfrak{Hom}_{\mathcal{R}}^{\Phi^\sim}(\varphi_*(\mathcal{S}), \mathcal{M})) & \xleftarrow{\sim} & \mathcal{R} - mod_X(\Phi_{\mathcal{R}}^\sim(\varphi_*(\mathcal{S}))(\Phi(S)(\mathcal{V})), \mathcal{M}) \end{array} \tag{3}$$

Here the vertical arrow is induced by the multiplication morphism

$$\Phi(S)^2(\mathcal{V}) \xrightarrow{\mu(\mathcal{V})} \Phi(S)(\mathcal{V}).$$

Taking $\mathcal{V} = \mathfrak{Hom}_{\mathcal{R}}^{\Phi^{\sim}}(\varphi_*(\mathcal{S}), \mathcal{M})$, we obtain a canonical morphism

$$\Phi(S)\big(\mathfrak{Hom}_{\mathcal{R}}^{\Phi^{\sim}}(\varphi_*(\mathcal{S}), \mathcal{M})\big) \xrightarrow{\ \zeta\ } \mathfrak{Hom}_{\mathcal{R}}^{\Phi^{\sim}}(\varphi_*(\mathcal{S}), \mathcal{M}), \tag{4}$$

which is the image of the identical morphism

$$\mathfrak{Hom}_{\mathcal{R}}^{\Phi^{\sim}}(\varphi_*(\mathcal{S}), \mathcal{M})\big) \xrightarrow{\ id\ } \mathfrak{Hom}_{\mathcal{R}}^{\Phi^{\sim}}(\varphi_*(\mathcal{S}), \mathcal{M}).$$

The morphism (4) is a canonical $\Phi^{\sim}_{Alg}(\mathcal{S})$-module structure on $\mathfrak{Hom}_{\mathcal{R}}^{\Phi^{\sim}}(\varphi_*(\mathcal{S}), \mathcal{M})$. We denote the $\Phi^{\sim}_{Alg}(\mathcal{S})$-module $(\mathfrak{Hom}_{\mathcal{R}}^{\Phi^{\sim}}(\varphi_*(\mathcal{S}), \mathcal{M}), \zeta)$ by $\varphi^!_X(\mathcal{M})$.

Therefore, for an arbitrary object $\mathcal{V}$ of $\mathcal{C}_X$, there is an adjunction isomorphism

$$\mathcal{C}_X(\mathcal{V}, \mathfrak{Hom}_{\mathcal{R}}^{\Phi^{\sim}}(\varphi_*(\mathcal{S}), \mathcal{M})) \xrightarrow{\ \sim\ } \mathcal{S} - mod_X(\Phi^{\sim}_{\mathcal{S}}(\mathcal{V}), \varphi^!_X(\mathcal{M})).$$

(iii) For any $\mathcal{L} = (L, \xi_{\mathcal{L}}) \in Ob\mathcal{S} - mod_X$, there is a canonical exact diagram

$$((\eta_{\mathcal{S}})^*_X(\eta_{\mathcal{S}})_{X*})^2(\mathcal{L}) = \Phi^{\sim}_{\mathcal{S}}(\mathcal{S})^2(L) \ \underset{\Phi(S)(\xi_{\mathcal{L}})}{\overset{\mu(L)}{\rightrightarrows}}\ \Phi^{\sim}_{\mathcal{S}}(\mathcal{S})(L) = (\eta_{\mathcal{S}})^*_X(\eta_{\mathcal{S}})_{X*}(\mathcal{L}) \xrightarrow{\ \xi_{\mathcal{L}}\ } \mathcal{L} \tag{5}$$

of $\Phi^{\sim}_{Alg}(\mathcal{S})$-modules. Since the pair of arrows in (3) is reflexive and the functor

$$\mathcal{S} - mod_X \xrightarrow{\ \varphi_{X*}\ } \mathcal{R} - mod_X$$

preserves cokernels of reflexive pairs of morphisms, the diagram

$$\varphi_{X*}\big(\Phi^{\sim}_{\mathcal{S}}(\mathcal{S})^2(L) \ \underset{\Phi(S)(\xi_{\mathcal{L}})}{\overset{\mu(L)}{\rightrightarrows}}\ \Phi^{\sim}_{\mathcal{S}}(\mathcal{S})(L) \xrightarrow{\ \xi_{\mathcal{L}}\ } \mathcal{L}\big) \tag{6}$$

is exact. It follows from (i) and (ii) that we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{S} - mod_X(\mathcal{L}, \varphi^!_X(\mathcal{M})) & & \mathcal{R} - mod_X(\varphi_{X*}(\mathcal{L}), \mathcal{M}) \\
\downarrow & & \downarrow \\
\mathcal{S} - mod_X(\Phi^{\sim}_{\mathcal{S}}(\mathcal{S})(L), \varphi^!_X(\mathcal{M})) & \xrightarrow{\ \sim\ } & \mathcal{R} - mod_X(\varphi_{X*}(\Phi^{\sim}_{\mathcal{S}}(\mathcal{S})(L)), \mathcal{M}) \\
\downarrow\downarrow & & \downarrow\downarrow \\
\mathcal{S} - mod_X(\Phi^{\sim}_{\mathcal{S}}(\mathcal{S})^2(L), \varphi^!_X(\mathcal{M})) & \xrightarrow{\ \sim\ } & \mathcal{R} - mod_X(\varphi_{X*}(\Phi^{\sim}_{\mathcal{S}}(\mathcal{S})^2(L)), \mathcal{M})
\end{array} \tag{7}$$

functorial in $\mathcal{L}$ and $\mathcal{M}$, whose horizontal arrows are isomorphisms and columns are exact diagrams. Therefore, there exists a unique (hence functorial in $\mathcal{L}$ and $\mathcal{M}$) isomorphism

$$\mathcal{S} - mod_X(\mathcal{L}, \varphi^!_X(\mathcal{M})) \longrightarrow \mathcal{R} - mod_X(\varphi_{X*}(\mathcal{L}), \mathcal{M})$$

making the diagram

$$\begin{array}{ccc}
\mathcal{S} - mod_X(\mathcal{L}, \varphi^!_X(\mathcal{M})) & \longrightarrow & \mathcal{R} - mod_X(\varphi_{X*}(\mathcal{L}), \mathcal{M}) \\
\downarrow & & \downarrow \\
\mathcal{S} - mod_X(\Phi^{\sim}_{\mathcal{S}}(\mathcal{S})(L), \varphi^!_X(\mathcal{M})) & \xrightarrow{\ \sim\ } & \mathcal{R} - mod_X(\varphi_{X*}(\Phi^{\sim}_{\mathcal{S}}(\mathcal{S})(L)), \mathcal{M})
\end{array}$$

commute. ∎

References

[A] M. Artin, Geometry of quantum planes, Contemporary Mathematics, v.124, 1992

[AB] M. Artin, M. Van den Bergh, Twisted homogenious coordinate rings, J. Algebra 133, 1990

[AZ] M. Artin, J.J. Zhang, Noncommutative projective schemes, Adv. in Math. 109 (1994), 228-287

[B] N. Bourbaki, Algebre commutative, Masson, Paris, New York, Barcelone, 1965

[BD] Bucur, I. and Deleanu, A, Introduction to the theory of categories and functors, Pure and Appl. Math., VXIX, John Wiley & Sons LTD, London – New York – Sidney, (1969)

[BeDr] A. Beilinson, V. Drinfeld, Hitchin's integrable system, preprint, 1999

[BN] M. Böckstedt, A. Neeman, Homotopy limits in triangulated categories, Compositio Math. 86 (1993), 209-234

[BO1] A. Bondall, D. Orlov, Semiorthogonal decompositions for algebraic varieties, alg-geom/9506012

[BO2] A. Bondall, D. Orlov, Reconstruction of a variety from the derived category and groups of autoequivalences, preprint MPI, 1995

[BOVdB] A. Bondal, M. Van den Bergh, Generates and representability of functors in commutative and noncommutative geometry, math.AG/0204218 v2

[D] J. Dixmier, Algèbres Enveloppantes, Gauthier-Villars, Paris/Bruxelles/Montreal, 1974.

[Gab] P.Gabriel, Des catégories abéliennes, Bull. Soc. Math. France, 90 (1962), 323-449

[GZ] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, Springer Verlag, Berlin-Heidelberg-New York, 1967

[GrD] A. Grothendieck, J.A. Dieudonné, Eléments de Géométrie Algébrique, SpringeVerlag, New York - Heidelberg - Berlin, 1971

[Ha] T. Hayashi, Q-analogues of Clifford and Weyl algebras – spinor and oscillator representations of quantum enveloping algebras, Commun. Math. Phys., 127, 129–144 (1990)

[K] M. Kapranov, Noncommutative geometry based on commutator expansions, math.AG/9802041 (1998), 48 pp.

[KR1] M. Kontsevich, A.L. Rosenberg, Noncommutative smooth spaces, in "The Gelfand Mathematical Seminar 1996–1999" (2000), 87–109.

[KR2] M. Kontsevich, A. Rosenberg, Noncommutative spaces, preprint MPI, 2004(35), 79 pp.

[KR3] M. Kontsevich, A. Rosenberg, Noncommutative spaces and flat descent, preprint MPI, 2004(36), 108 pp.

[KR4] M. Kontsevich, A. Rosenberg, Noncommutative stacks, preprint MPI, 2004(37), 55 pp.

[KR5] M. Kontsevich, A. Rosenberg, Noncommutative Grassmannians and related constructions, preprint MPI, 2008(89), 31 pp.

[KR6] M. Kontsevich, A. Rosenberg, Orthogonals, in preparation.
[LR1] V. Lunts, A.L. Rosenberg, Differential calculus in noncommutative algebraic geometry I. D-calculus on noncommutative rings (1996), MPI 96-53
[LR2] V. Lunts, A.L. Rosenberg, Differential calculus in noncommutative algebraic geometry II. D-calculus in the braided case. The localization of quantized enveloping algebras. (1996), MPI 96-76
[LR3] V. Lunts, A.L. Rosenberg, Differential operators on noncommutative rings, Selecta Mathematica , v. 3, 1997, 335–359 pp.
[LR4] V. Lunts, A.L. Rosenberg, Localization for quantum groups, Selecta Mathematica, New Series, 5 (1999), 123–159.
[M1] Yu.I. Manin, Quantum Groups and Noncommutative Geometry, Publ. du C.R.M.; Univ. de Montreal, 1988
[M2] Yu.I. Manin, Topics in Noncommutative Geometry, Princeton University Press, Princeton New Jersey (1991)
[ML] S. Mac-Lane, Categories for the working mathematicians, Springer - Verlag; New York - Heidelberg - Berlin (1971)
[MLM] S. Mac-Lane, L. Moerdijk, Sheaves in Geometry and Logic, Springer - Verlag; New York - Heidelberg - Berlin (1992)
[Or] D. Orlov, Quasi-coherent sheaves in commutative and noncommutative geometry, preprint MPIM (1999)
[R] A.L. Rosenberg, Noncommutative algebraic geometry and representations of quantized algebras, Kluwer Academic Publishers, Mathematics and Its Applications, v.330 (1995), 328 pages.
[R1] A.L. Rosenberg, Noncommutative local algebra, Geometric and Functional Analysis (GAFA), v.4, no.5 (1994), 545-585
[R2] A.L. Rosenberg, Noncommutative schemes, Compositio Mathematica 112 (1998), 93-125
[R3] A.L. Rosenberg, Noncommutative spaces and schemes, preprint MPIM, 1999
[R4] A.L. Rosenberg, Underlying spaces of noncommutative schemes, preprint, 2002
[R5] A.L. Rosenberg, Spectra of noncommutative spaces, preprint, 2003
[R6] A.L. Rosenberg, Spectra related with localizations, preprint MPIM, 2003(112), 77 pp
[R7] A. L. Rosenberg, Spectra of 'spaces' represented by abelian categories, preprint MPIM, 2004, 73 pp
[R8] A.L. Rosenberg, The spectrum of the algebra of skew differential operators and the irreducible representations of quantum Heisenberg algebra, Commun. Math. Phys., 142, 567-588 (1991).
[R9] A.L. Rosenberg, Spectra, associated points, and representations, preprint MPIM, 2007(10), 71 pp.

[R10] A.L. Rosenberg, Trieste Lectures, in "Some Recent Developments in Algebraic K-Theory, ICTP Lecture Notes, v. 23, 2008, 307–428
[R11] A.L. Rosenberg, Topics in Noncommutative Algebraic Geometry, Homological Algebra and K-Theory, preprint MPIM, 2008(57), 105 pp.
[R12] A.L. Rosenberg, Geometry of right exact 'spaces', preprint MPIM, 2008(85), 66 pp.
[R13] A.L. Rosenberg, Homological Algebra of Noncommutative 'spaces' I, preprint MPIM, 2008(91), 198 pp.
[R14] A.L. Rosenberg, Homological Algebra of Noncommutative 'spaces' I, 2009, 234 pp, a monograph in preparation
[R15] A.L. Rosenberg, Geometry of Noncommutative 'Spaces', 2009, 298 pp, a monograph
[S] J.-P. Serre, Faisceaux algébriques cohérents, Annals of Math.62, 1955
[ThT] R. W. Thomason and T. Trobaugh, Higher Algebraic K-Theory of Schemes and of Derived Categories, Grothendieck Festsschrift, v. 3, 247 – 435, Progr. Math, 88, Birkhäuser Boston, MA, 1990
[Ve1] J.–L. Verdier, Catégories dérivées, Séminaire de Géométrie Algébrique 4 1/2, Cohomologie Étale, LNM 569, pp. 262–311, Springer, 1977
[Ve2] J.–L. Verdier, Des catégories dérivées des catégories abéliennes, Astérisque, v.239, 1996
[V1] A.B. Verevkin, On a noncommutative analogue of the category of coherent sheaves on a projective scheme, Amer. Math. Soc. Transl. (2) v. 151, 1992
[V2] A.B. Verevkin, Serre injective sheaves, Math. Zametki 52 (1992) 35-41
[XT] Xin Tang, Applications of noncommutative algebraic geometry to representation theory, PhD Thesis, Kansas State University, 2006

Glossary of notations

Chapter I

$|Cat|^o$ the category of 'spaces', I.1.0
C_X the category associated with the 'space' X, I.1.0
$|\mathcal{A}|$ the *underlying 'space'* of $\mathcal{A}$ defined by $C_{|\mathcal{A}|} = \mathcal{A}$, I.1.0
$C_X \xrightarrow{f^*} C_Y$ an inverse image functor of a morphism of 'spaces' $X \xrightarrow{f} Y$, 1.0
$f = [F]$ the morphism of 'spaces' with an inverse image functor F, I.1.0
$C_X \xrightarrow{f_*} C_Y$ a direct image functor of a continuous morphism $X \xrightarrow{f} Y$, 1.0.1
$C_Y \xrightarrow{f^!} C_X$ a right adjoint to direct image functor, I.1.0.1
$\mathbf{Sp}(R)$ the *categoric spectrum* of a unital ring R defined by $C_{\mathbf{Sp}(R)} = R - mod$, I.1.1
$\mathbf{Sp}_{\mathcal{G}}(R)$ the 'space' represented by the category of $\mathcal{G}$-graded R-modules, I.1.2
$\mathcal{T}_{R_+}$ the full subcategory $R - mod$ generated by R-modules annihilated by R_+, I.1.3
$\mathcal{T}^-_{R_+}$ the smallest Serre subcategory of $R - mod$ containing $\mathcal{T}_{R_+}$, I.1.3
$\mathbf{Cone}(R_+)$ the cone of a non-unital ring R^+ defined by $C_{\mathbf{Cone}(R_+)} = R - mod/\mathcal{T}^-_{R_+}$, I.1.3
$gr_{\mathcal{G}}\mathcal{T}_{R_+} = \mathcal{T}_{R_+} \cap gr_{\mathcal{G}}R - mod$ I.1.4
$gr_{\mathcal{G}}\mathcal{T}^-_{R_+} = gr_{\mathcal{G}}R - mod \cap \mathcal{T}^-_{R_+}$ the Serre envelope of $gr_{\mathcal{G}}\mathcal{T}_{R_+}$, I.1.4
$\mathbf{Proj}_{\mathcal{G}}$ the **Proj** of $\mathcal{G}$-graded algebras: $C_{\mathbf{Proj}_{\mathcal{G}}(R)} = gr_{\mathcal{G}}R - mod/gr_{\mathcal{G}}\mathcal{T}^-_{R_+}$, I.1.4
$\mathbf{P}^r_{\mathbf{q}}$ projective **q**-'space', I.1.5.1
$U_q(\mathfrak{g})$ the quantized enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$, I.1.7
$R = \bigoplus_{\lambda \in \mathcal{G}_+} R_\lambda$ the algebra of regular functions on the quantum base affine 'space', I.1.7
$W = W(\mathfrak{g})$ the Weyl group of a semisimple Lie algebra $\mathfrak{g}$, I.1.7.1
$S_w = \{k^* e^\lambda_{w\lambda} \mid \lambda \in \mathcal{G}_+\}$ the multiplicative set of w-exremal vectors, $w \in W$, I.1.7.1
$\mathbf{Sp}(S_w^{-1}R) \xrightarrow{\widetilde{\mathfrak{u}}_w} \mathbf{Cone}(R)$, $w \in W$, the canonical open cover of the quantum base affine 'space' of $\mathfrak{g}$, I.1.7.1
$(S_w^{-1}R)_0$ the zero component of the $\mathcal{G}$-graded algebra $S_w^{-1}R$, $w \in W$, I.1.7.1
$\mathbf{Sp}((S_w^{-1}R)_0) \simeq \mathbf{Sp}_{\mathcal{G}}(S_w^{-1}R) \xrightarrow{\mathfrak{u}_w} \mathbf{Proj}_{\mathcal{G}}(R)$, $w \in W$, the canonical open cover of the quantum flag variety of $\mathfrak{g}$, I.1.7.1
$\mathcal{O} = f^*(R)$ for $X \xrightarrow{f} \mathbf{Sp}(R)$ I.3.1
$\Gamma_X\mathcal{O}$ the algebra $C_X(\mathcal{O}, \mathcal{O})^o$, I.3.2
$C_X \xrightarrow{f_{\mathcal{O}*}} \Gamma_X\mathcal{O} - mod$, $M \longmapsto C_X(\mathcal{O}, M)$, the global sections functor on $(X, \mathcal{O})$, I.3.2
$X \xrightarrow{f_{\mathcal{O}}} \mathbf{Sp}(\Gamma_X\mathcal{O})$ 'global sections' morphism, I.3.3.1

$|Cat|^o_{\mathbb{Z}}$ the category of $\mathbb{Z}$-'spaces', I.3.3.2
$\mathfrak{Ass}$ the category of associative rings and conjugation classes of ring morphisms, I.3.3.5
$\mathfrak{Mon}_X$ the category of monads on a 'space' X, I.4.1
$\mathbf{Sp}(\mathcal{F}_f/Y)$ the categoric spectrum of the monad $\mathcal{F}_f$ on a 'space' Y, I.4.1
$\mathfrak{Mon}^w_X$ the category of weakly continuous monads on a 'space' X, I.4.2.1
$\mathfrak{CMon}_X$ the category of comonads on a 'space' X, I.4.3
$\mathbf{Sp^o}(Y\backslash\mathcal{G})$ the cospectrum of the comonad $\mathcal{G}$ on a 'space' Y, I.4.3
$\mathfrak{CMon}^w_X$ the category of weakly flat comonads of a 'space' X, I.4.5
$\mathbf{Aff}^w_Y$ the category of weakly affine 'spaces' over Y, I.4.5
$Flat^w_X$ the category of weakly flat 'spaces' under X, I.4.5
$|V|$ the set of elements of an object V of a monoidal category, I.4.6.1
$Alg\mathcal{A}^\sim$ the category of associative unital algebras in a monoidal category $\mathcal{A}^\sim$, I.4.6.1
$\mathfrak{Ass}\mathcal{A}^\sim$ the category of assoicative unital algebras in $\mathcal{A}^\sim$ and conjugation classes of unital algebra morphisms, I.4.6.1
$\mathfrak{Ass}^w_X$ the category of weakly continuous monads on X and conjugation classes of monad morphisms, I.4.6.2.2
$End_w(C_X)$ the category of weakly continuous endofunctors, I.4.6.1
$\mathbf{Aff}_X$ the category of affine schemes over X, I.5.4
$\mathfrak{Mon}^{\mathfrak{c}}_X$ the category of continuous monads of a 'space' X, I.5.7
$\mathfrak{Ass}_X$ the category of continuous monads of a 'space' X and conjugation classes of monad morphisms, I.5.7
Σ_F the class of arrows, which the functor F maps to isomorphisms, I.7.1
$\mathcal{T}^-$ Serre envelope of a subcategory $\mathcal{T}$, I.7.2.2
$\mathbf{Cone}(F_+/X)$ the cone of a non-unital monad F_+ on X, I.7.2.3
$\mathbf{Proj}_{\mathcal{G}}(\mathbb{F}_+/X)$ the Proj of a $\mathcal{G}$-graded non-unital monad on X, I.7.4.1

Chapter II

$\mathfrak{A} \xrightarrow{\mathfrak{F}} \mathfrak{B}$ 'local data', II.1.0
$CAlg_k$ the category of commutative unital k-algebras, II.1.0.1
$\mathcal{M} \longmapsto \widehat{\mathcal{M}} = \mathfrak{A}(-,\mathcal{M})$ the Yoneda embedding, II.1.0.2
$(\mathfrak{A},\tau)^\wedge$ the category of sheaves of sets on the presite $(\mathfrak{A},\tau)$, II.1.0.2
$\mathbf{Aff}_k = Alg^{op}_k$ the category of noncommutative affine k-schemes, II.1.0.3
$\mathcal{R}^\vee = Alg_k(\mathcal{R},-)$ II.1.0.3
$(\mathfrak{A},\mathfrak{F})$-cofinite locally cofinite, II.1.1
$(\mathfrak{A},\mathfrak{F})$-finitely copresentable locally finitely copresentable, II.1.1
$\mathfrak{B}^{\mathfrak{f}}_{\mathfrak{A},\mathfrak{F}}$ the subcategory of $\mathfrak{B}$ generated by locally cofinite objects, II.1.1
$\mathfrak{B}^{\mathfrak{fp}}_{\mathfrak{A},\mathfrak{F}}$ the subcategory of $\mathfrak{B}$ generated by locally finitely copresentable objects, II.1.1
$\mathfrak{B}^{\mathfrak{f}}$ the subcategory of cofinite objects, II.1.1.1

$\mathfrak{B}^{\mathfrak{fp}}$ the subcategory of finitely copresentable objects, II.1.1.1
$\mathfrak{B}_{\mathfrak{f}}^{\mathfrak{A},\mathfrak{F}}$ the subcategory of $\mathfrak{B}$ generated by locally finite objects, II.1.2
$\mathfrak{B}_{\mathfrak{fp}}^{\mathfrak{A},\mathfrak{F}}$ the subcategory of $\mathfrak{B}$ generated by locally finitely presentable objects, II.1.2
$\mathfrak{B}_{\mathfrak{f}}$ the subcategory of finite objects, II.1.2
$\mathfrak{B}_{\mathfrak{fp}}$ the subcategory of finitely presentable objects, II.1.2
$\mathfrak{E}_{\mathfrak{B}}^{\mathfrak{s}}$ the class of strict epimorphisms of $\mathfrak{B}$, II.1.5
$T_k(\mathcal{M})$ the tensor algebra of the k-module $\mathcal{M}$, II.1.6.2
$k - mod \xrightarrow{\phi_k} Alg_k$ maps $\mathcal{N}$ to $k \oplus \mathcal{N}$ with zero multiplication on $\mathcal{N}$, II.1.6.2.1
$\mathbb{V}_k(\mathcal{M}) \stackrel{\text{def}}{=} T_k(\mathcal{M})^\vee \stackrel{\text{def}}{=} Alg_k(T_k(\mathcal{M}), -)$ vector fiber of a k-module $\mathcal{M}$, II.1.8.1
$\Sigma_{\mathfrak{A}}^1$ the class of $(\mathfrak{A}, \mathfrak{F})$-finitely copresentable morphisms, II.1.11.1
$\Sigma_{\mathfrak{A}}^0$ the class of morphisms of $(\mathfrak{A}, \mathfrak{F})$-cofinite type, II.1.11.1
$\widehat{\mathcal{P}}_{\mathfrak{F}}$ the class of morphisms of $\mathfrak{B}$ representable by morphisms of $\mathcal{P} \subset Hom\mathfrak{A}$, II.2.0
$\mathcal{P}^\infty$ all compositions of arrows from the class $\mathcal{P}$, II.2.2
$\widehat{\mathfrak{A}}_{\mathfrak{F}}$ the class of representable morphisms of $\mathfrak{B}$, II.2.5
Λ_f the class of all pairs of arrows $X \underset{u_2}{\overset{u_1}{\rightrightarrows}} V$ equalizing $Y \xrightarrow{f} X$, II.2.8.1
$\mathfrak{M}_s = \mathfrak{M}_s(A)$ the class of strict monomorphisms of a category A, II.2.8.1
$\mathcal{X} \xrightarrow{\Delta_{\mathfrak{f}}} K_2(\mathfrak{f}) = \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ the diagonal morphism for $\mathcal{X} \xrightarrow{\mathfrak{f}} \mathcal{Y}$, II.2.6.4
$\mathfrak{B}_{fsm}^{\mathfrak{M}}$ the full subcategory of $\mathfrak{B}$ generated by formally $\mathfrak{M}$-smooth objects, II.3.0.2
$\mathfrak{M}_{\mathfrak{I}}$ the largest class of arrows of $\mathfrak{B}$ such that all objects of $\mathfrak{I}$ are $\mathfrak{M}_{\mathfrak{I}}$-formally smooth, II.3.0.2
$\mathfrak{M}_{\mathfrak{n}}$ the class of nilpotent closed immersions of affine schemes, II.3.0.3.2
$R^e \stackrel{\text{def}}{=} R \otimes_k R^o$, where R^o is the algebra opposite to R, II.3.0.3.3
$\Omega_{R|k}^1 \stackrel{\text{def}}{=} Ker(R^e \longrightarrow R)$ the R^e-module of Kähler differentials of R, II.3.0.3.3
$\mathfrak{Ass}_k$ the category of associative unital k-algebras and conjugation classes of algebra morphisms, II.3.0.4
$\mathfrak{Aff}_k \stackrel{\text{def}}{=} \mathfrak{Ass}_k^{op}$ II.3.0.4
$\bar{\mathfrak{M}}_{\mathfrak{n}}$ the image in $\mathfrak{Aff}_k^\wedge$ of the class $\mathfrak{M}_{\mathfrak{n}}$, II.3.0.4
$\mathfrak{M}_{fsm}$ formally $\mathfrak{M}$-smooth morphisms, 3.1
$\mathfrak{M}_{fnr}$ formally $\mathfrak{M}$-unramified morphisms, II.3.1
$\mathfrak{M}_{fet}$ formally $\mathfrak{M}$-étale morphisms, II.3.1
$\mathfrak{N}_{inf}$ $\mathfrak{N}$-infinitesimal morphisms, II.3.2
$\Omega_{S|R}^1 \stackrel{\text{def}}{=} Ker(S \otimes_R S^o \longrightarrow S)$ II.3.6.1
$\mathfrak{M}_{\mathfrak{J}}$ the class of radical closed immersions of affine schemes, II.3.8.1.
$\mathfrak{M}_{sm}$ $\mathfrak{M}$-smooth morphisms, II.4.1.1
$\mathfrak{M}_{nr}$ $\mathfrak{M}$-unramified morphisms, II.4.1.1
$\mathfrak{M}_{et}$ $\mathfrak{M}$-étale morphisms, II.4.1.1

$\mathfrak{M}_{zar}$ the class of $\mathfrak{M}$-open immersions, II.4.1.2
$\mathbf{CAff}_k$ the category of commutative affine k-schemes, II.4.4.1
$\mathfrak{E}^{\mathfrak{c}}_{\mathfrak{B}}$ the canonical (the finest) right exact structure on $\mathfrak{B}$, II.5.1.3
$\mathfrak{P}^{\tau}$ the class of arrows of a presite, which locally belong to a class $\mathfrak{P}$, II.5.2
$\tau^{\mathcal{P}}$ the family of all covers of a pretopology τ formed by arrows from $\mathcal{P}$, II.6.1
$\tau^{\mathfrak{M}}_{et}$ étale pretopology, II.6.3.1
$\tau^{\mathfrak{M}}_{\mathfrak{Z}}$ Zariski pretopology, II.6.3.2
$\tau^{\mathfrak{M}}_{sm}$ smooth pretopology, II.6.3.3
$\mathcal{E}sp$ the category of sheaves of sets on $CRings^{op}$ for the **fpqc** topology, II.7.4.3
$\mathfrak{T}_{\mathfrak{Z}}$ Zariski pretopology on $\mathbf{CAff}_k = CAlg_k^{op}$, II.7.4.3
$\tau_{\mathbf{fpqc}}$ **fpqc** quasi-pretopology on $\mathbf{Aff}_k$, II.7.5.1
$\tau_{\mathbf{fppf}}$ **fppf** quasi-pretopology, II.7.5.2
$\tau_{\mathfrak{Z}}$ Zariski pretopology, II.7.5.2
$\mathcal{N}\mathcal{E}sp_k$ the category of sheaves of sets on $\mathbf{Aff}_k$ for **fpqc** quasi-pretopology, II.7.5.3
$\mathbf{Proj}_{\mathcal{G}}(R\#U_{\mathfrak{q}}(\mathfrak{g}))$ quantum D-scheme associated with the Lie algebra $\mathfrak{g}$, II.8.5
$\mathbf{Sp}(S_w^{-1}R) \longrightarrow \mathbf{Cone}(R_+),\ w \in W,$ canonical affine cover of a base affine 'space', II.8.5

Chapter III

$\mathcal{P}^{\wedge}$ the class of morphisms representable by morphisms of $\mathcal{P}$, III.1.1
$\mathfrak{M}_s^{\wedge} = \mathfrak{M}_s(A)^{\wedge}$ the class of closed immersions of presheaves of sets on A, III.2.1
$\mathcal{C}_2(f)$ the cokernel pair of a morphism f, III.2.4
$\mathfrak{M}_{\mathfrak{st}}(\mathcal{A})$ the class of universally strict monomorphisms the category $\mathcal{A}$, III.2.5
$\mathfrak{M}_{\mathfrak{sm}}(\mathcal{A})$ the class of strict monomorphisms stable under push-forwards along strict monomorphisms, III.2.5
$T_R(\mathcal{M}) = \bigoplus_{n\geq 0} \mathcal{M}^{\otimes n}$ the tensor algebra of the R^e-module $\mathcal{M}$, III.5.1
$\mathbb{V}_R(\mathcal{M})$ the fiber of an R^e-module $\mathcal{M}$, III.5.1
$\mathcal{H}_R(M,V)$ inner hom, III.5.3
$V_R^* = Hom_R(V,R)$ dual module, III.5.3.2
$\mathfrak{Iso}_R(\mathcal{M},\mathcal{N})$ the presheaf of isomorphisms from $\mathcal{M}$ to $\mathcal{N}$, III.5.4
$GL_{\mathcal{V}}$ the presheaf of automorphisms of the R-module $\mathcal{V}$, III.5.4.2
$Gr_{M,V}$ the Grassmannian, III.6.0
$G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V}$ a natural cover of the Grassmannian, III.6.1
$\mathfrak{R}_{M,V} = G_{M,V} \prod_{Gr_{M,V}} G_{M,V}$ the "functor of relations", III.6.2
$F_{\phi;M,V} \longrightarrow Gr_{M,V}$ a natural subscheme of the Grassmannian, III.6.7.1

$\mathbb{V}(M) \longrightarrow \mathbb{P}_{M\oplus R^1}$ the projective completion of the vector fiber $\mathbb{V}(M)$, III.6.7.4
$\mathfrak{F}\ell_{M,\bar{V}}$ a non-commutative flag variety, III.7.1
$\mathfrak{F}\ell_{M,\bar{V}} \longrightarrow \prod_{1\le i\le n} Gr_{M,V_i}$ the natural embedding, III.7.2
$\mathfrak{Gr}_{\mathcal{M}}$ generic Grassmannian of an R-module $\mathcal{M}$, III.8.1
$Gr_{\mathcal{M},V} \xrightarrow{\rho_V} \mathfrak{Gr}_{\mathcal{M}}$ a canonical embedding, III.8.1.1
$\mathfrak{Pr}_{\mathcal{M}}(S,s)$ the set of projectors $s^*(\mathcal{M}) \xrightarrow{p} s^*(\mathcal{M})$, III.8.2
$\mathfrak{R}_{\mathcal{M}} = \mathfrak{Pr}_{\mathcal{M}} \prod_{\mathfrak{Gr}_{\mathcal{M}}} \mathfrak{Pr}_{\mathcal{M}}$ relations, 8.3
$\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ generic flags, III.9.1
$\mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\pi^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}$ the canonical cover (for a projective $\mathcal{M}$), III.9.2.1
$\mathfrak{R}^{\mathfrak{I}}_{\mathcal{M}} = \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}} \prod_{\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}}} \mathfrak{F}^{\mathfrak{I}}_{\mathcal{M}}$ the relations, III.9.2.1
$\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \longrightarrow \prod_{i\in\mathfrak{I}} \mathfrak{Gr}_{\mathcal{M}}$ a canonical embedding, III.9.4
$\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\mathfrak{p}^I_{\mathfrak{I}}} \mathfrak{Fl}^{I}_{\mathcal{M}}$ the restriction morphism, III.9.8.1
$\mathfrak{Fl}^{\mathfrak{I}}_{\mathcal{M}} \xrightarrow{\sim} \lim_{I\in\mathfrak{S}^o_{\mathfrak{f}}(\mathfrak{I})} (\mathfrak{Fl}^{I}_{\mathcal{M}} \mid \mathfrak{p}^J_L)$ a natural isomorphism, III.9.8.2
$\mathfrak{F}\ell_{\mathcal{M},\bar{V}} \xrightarrow{\mathfrak{j}_{\bar{V}}} \mathfrak{Fl}^{[n]}_{\mathcal{M}}$ the embedding of a flag variety into the generic flag variety, III.10
$Stief^{n+1}_{\mathcal{M}}$ Stiefel scheme, II.11

Chapter IV and Appendix 1

$\mathcal{M}od(\mathfrak{F})$ the category of modules on $\mathfrak{F} = (\mathcal{F}, \mathcal{F} \xrightarrow{\pi} \mathcal{E})$, IV.1.1
$\mathrm{Lim}\mathfrak{F} \stackrel{\mathrm{def}}{=} Cart_{\mathcal{E}}(\mathcal{E},\mathfrak{F})$ the category of cartesian sections of $\mathfrak{F} = (\mathcal{F}, \mathcal{F} \xrightarrow{\pi} \mathcal{E})$, IV.1.1.1
$Qcoh(\mathfrak{F}) = (Lim\mathfrak{F})^{op}$ the category of quasi-coherent modules on $\mathfrak{F}$, IV.1.1
$Cart_{\mathcal{E}}$ the category of cartesian functors over $\mathcal{E}$, IV.1.1.1
$Cart_{\mathfrak{V}}$ IV.1.4.2, A1.1.5.3.1
$Cart_{\mathfrak{U},\mathfrak{V}}$ 2-subcategory of $Cart_{\mathfrak{V}}$ generated by $\mathfrak{F} = (\mathcal{F} \xrightarrow{\pi} \mathcal{E})$ with $\mathcal{E}$ in the universum $\mathfrak{V}$ and each fiber in $\mathfrak{U} \in \mathfrak{V}$, IV.1.4.2
$\mathfrak{M}Cart_{\mathfrak{U},\mathfrak{V}}$ 2-subcategory of $Cart_{\mathfrak{U},\mathfrak{V}}$ formed by 1-morphisms inducing equivalences on fibers, IV.1.4.2
$Cart_{\mathcal{E}}(\mathcal{E}/X,\mathfrak{F})$ the category of cartesian functors $\mathcal{E}/X \longrightarrow \mathfrak{F}$, IV.1.5
$Qcoh(\mathfrak{F}/X) = Cart_{\mathcal{E}}(\mathcal{E}/X,\mathfrak{F})^{op}$ IV.1.5
$\mathfrak{F}^+(X) \stackrel{\mathrm{def}}{=} Cart_{\mathcal{E}}(\mathcal{E}/X,\mathfrak{F}) = Qcoh(\mathfrak{F}/X)^{op}$ IV.1.5.1
$\mathfrak{I}_{\mathfrak{X}}$ the subpreshef of X assocated with a family $\mathfrak{X} = \{X_i \to X \mid i \in I\}$, IV.2.1.1

$\mathcal{M}od(\mathfrak{F},\mathfrak{T})$ the category of sheaves of modules on a cofibred category $\mathfrak{F}$, IV.3.1
$\mathcal{M}od(\mathfrak{F},\mathfrak{T})$ the category of sheaves of modules on a fibred category $\mathfrak{F}$, IV.3.2.1
$Qcoh(\mathfrak{F},\mathfrak{T}) \stackrel{\mathrm{def}}{=} Qcoh(\mathfrak{F})\bigcap\mathcal{M}od(\mathfrak{F},\mathfrak{T})$ the category of quasi-coherent sheaves of modules on $(\mathfrak{F},\mathfrak{T})$, 3.2.2
$\mathfrak{M}od^+(\mathfrak{F},\mathfrak{T})$ fibred category of sheaves of modules over presheaves of sets, IV.3.4
$\mathfrak{M}od(\mathfrak{F},\mathfrak{T})$ the restriction of the fibred category $\mathfrak{M}od^+(\mathfrak{F},\mathfrak{T})$ to $\mathcal{E}$, IV.3.4.1
$(\mathcal{A},\mathcal{O})$ a ringed category, IV.4.0
$\mathfrak{M}(\mathcal{A},\mathcal{O})$ a (bi)fibred category associated with a ringed category $(\mathcal{A},\mathcal{O})$, IV.4.0
$\mathcal{O}-mod \stackrel{\mathrm{def}}{=} \mathcal{M}od(\mathfrak{M}(\mathcal{A},\mathcal{O}))$ the category of modules on $\mathfrak{M}(\mathcal{A},\mathcal{O})$, IV.4.2
$Qcoh(\mathcal{A},\mathcal{O}) \stackrel{\mathrm{def}}{=} Qcoh(\mathfrak{M}(\mathcal{A},\mathcal{O}))$ the category of quasi-coherent modules on $(\mathcal{A},\mathcal{O})$, IV.4.3
$\mathcal{M}od_X = \mathcal{O}_X - mod$ the category of presheaves of $\mathcal{O}_X$-modules, IV.4.6
$Qcoh_X \stackrel{\mathrm{def}}{=} Qcoh(\mathcal{A}/X,\mathcal{O}_X)$ category of quasi-coherent $\mathcal{O}_X$-modules, IV.4.6
$\mathfrak{Mod}(\mathcal{A},\mathcal{O})$ fibred category of modules over $\mathcal{A}^\wedge$, IV.4.6.1
$\mathfrak{Qcoh}(\mathcal{A},\mathcal{O})$ fibred category of quasi-coherent modules on $\mathcal{A}^\wedge$, IV.4.6.2
$\mathcal{M}od(\mathcal{A},\mathfrak{T};\mathcal{O})$ the category of sheaves of left $\mathcal{O}$-modules on the site $(\mathcal{A},\mathfrak{T})$, IV.4.8
$\mathfrak{Bmod}(\mathcal{A},\mathcal{O})$ the fibred category of $(\mathcal{A},\mathcal{O})$-bimodules, IV.6.3.2

Chapter V and Appendix 2

$(\mathcal{A},\odot) \xrightarrow{(\Phi,\phi)} (\mathcal{A}',\odot')$ morphism of categories with multiplication, A2.1
$\mathcal{A} \xrightarrow{\mathfrak{L}_\odot} End(\mathcal{A})$ the standard left action, $a \longmapsto a\odot -$, A2.1.2
$(\mathcal{A},\odot) \xrightarrow{(\Phi.\phi)} (End(C_X),\circ)$ an action of $(\mathcal{A},\odot)$ on a category C_X, A2.3
$(\mathcal{A},\odot;\mathbb{I})$ a category with multiplication and a 'unit' object, A2.3.3.1
$(\mathcal{A},\odot,\mathbb{I}) \xrightarrow{(\Phi,\phi,\phi_0)} (\mathcal{A}',\odot',\mathbb{I}')$ unital morphism, A2.3.3.1
$\mathcal{A}^\sim = (\mathcal{A},\odot;\mathfrak{a};\mathbb{I},\mathfrak{l},\mathfrak{r})$ a monoidal category, A2.4
$\Phi^\sim = (\Phi,\phi,\phi_0)$ a monoidal functor, A2.4.1
$\mathfrak{MCat}$ the 2-category of monoidal categories, A2.4.3
$\mathcal{A}^\sim \xrightarrow{(\Phi,\phi,\phi_0)} (End(C_X),\circ,Id_{C_X})$ an action of a monoidal category, A2.4.4
$R^e-mod^\sim = (R^e-mod,\otimes_R,R)$ the monoidal category of left R^e-modules, A2.4.6.1
$End_k^{\mathfrak{c}}(\mathcal{C}_X)$ the category of continuous k-linear endofunctors of the category $\mathcal{C}_X$, A2.4.6.2
$End^w(\mathcal{C}_X)^\sim = (End^w(\mathcal{C}_X),\circ,Id_{\mathcal{C}_X})$ the monoidal category of weakly continuous endofunctors of $\mathcal{C}_X$, A2.4.6.3
$\mathfrak{End}^{\mathfrak{w}}(\mathcal{C}_X)^\sim = (\mathfrak{End}^{\mathfrak{w}}(\mathcal{C}_X),\circ,Id_{\mathcal{C}_X})$ the monoidal category of weakly continuous functors preserving countable coproducts, A2.4.6.4
$\mathfrak{Alg}\mathcal{A}^\sim$ the category of algebras in the monoidal category $\mathcal{A}^\sim$, A2.5

$\mathfrak{Alg}^{\mathfrak{u}}\mathcal{A}^{\sim}$ the category of unital algebras and unital morphisms in $\mathcal{A}^{\sim}$, A2.5.1.1
$Alg\mathcal{A}^{\sim}$ the category of associative unital algebras in a monoidal category $\mathcal{A}^{\sim}$, A2.5.2
$Alg\mathcal{A}^{\sim} \xrightarrow{\Phi^{\sim}_{Alg}} Alg\mathcal{A}'^{\sim}$ the functor induced by a monoidal functor $\mathcal{A}^{\sim} \xrightarrow{\Phi^{\sim}} \mathcal{A}'^{\sim}$, A2.5.3.2
$|\mathcal{V}| \stackrel{\text{def}}{=} \mathcal{A}(\mathbb{I}, \mathcal{V})$ the elements of the object $\mathcal{V}$, A2.5.3.3
$|\mathcal{R}|^*$ the group of invertible elements of a unital associative algebra $\mathcal{R}$ in $\mathcal{A}^{\sim}$, A2.5.3.3.1
$\mathcal{A}^{\sim}_{\sigma} = (\mathcal{A}, \odot^{\sigma}; \mathfrak{a}^{\sigma}; \mathbb{I}, \mathfrak{l}^{\sigma}, \mathfrak{r}^{\sigma})$ the reflection of a monoidal category $\mathcal{A}^{\sim}$, A2.5.4.1
$\Phi^{\sim}_{\sigma} = (\Phi, \phi^{\sigma}, \phi_0)$ the reflection of a monoidal functor $\Phi^{\sim} = (\Phi, \phi, \phi_0)$, A2.5.4.2
$\mathcal{R}^o = (R, \mu_{\mathcal{R}^o})$ the algebra in $\mathcal{A}^{\sim}_{\sigma}$ opposite to an algebra $\mathcal{R}$ in $\mathcal{A}^{\sim}$, A2.5.4.3
$\beta^{\sim} = (Id_{\mathcal{A}}, \beta, id_{\mathbb{I}})$ a monoidal functor from $\mathcal{A}^{\sim}$ to its reflection $\mathcal{A}^{\sim}_{\sigma}$, A2.5.5.3
$\mathfrak{Alg}\mathcal{A}^{\sim} \xrightarrow{\beta^{\sim}_{\mathfrak{Alg}}} \mathfrak{Alg}\mathcal{A}^{\sim}_{\sigma}, \quad \mathcal{R} \longmapsto \mathcal{R}^{\beta} \stackrel{\text{def}}{=} (R, \mu^{\beta}_{\mathcal{R}}) \stackrel{\text{def}}{=} (R, \mu_{\mathcal{R}} \circ \beta_{R,R}),$ A2.5.5.3
$\mathcal{C}^{\sim\mathbf{S}} = (\mathcal{C}^{\mathbf{S}}, \odot, \mathbb{I}_{\mathbf{S}})$ A2.5.6
$Alg\mathcal{C}^{\sim\mathbf{S}}$ operads in the monoidal category $\mathcal{C}^{\sim}$, A2.5.6
$\mathcal{R}-mod_X$ the category of modules over the monad $\Phi^{\sim}_{Alg}(\mathcal{R})$ in the category $\mathcal{C}_X$, A2.6
$\mathcal{A}^{\sim} \xrightarrow{(\mathfrak{L}_{\odot}, \mathfrak{a}, \mathfrak{l})} (End(\mathcal{A}), \circ, Id_{\mathcal{A}}), \quad x \longmapsto x \odot -,$ the standard left action of the monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ on the category $\mathcal{A}$, A2.6.1
$\mathcal{R}-mod$ the category of left modules over an algebra $\mathcal{R}$ in $\mathcal{A}^{\sim}$, A2.6.1
$\mathcal{A}^{\sim} \xrightarrow{(\mathfrak{R}_{\odot}, \mathfrak{a}^{-1}, \mathfrak{r})} (End(\mathcal{A}), \circ, Id_{\mathcal{A}}), \quad x \longmapsto - \odot x,$ right standard action of the monoidal category $\mathcal{A}^{\sim} = (\mathcal{A}, \odot; \mathfrak{a}; \mathbb{I}, \mathfrak{l}, \mathfrak{r})$ on the category $\mathcal{A}$, A2.6.2
$mod-\mathcal{R}$ the category of right modules over an algebra $\mathcal{R}$ in $\mathcal{A}^{\sim}$, A2.6.2
$\mathcal{R}-mod \xrightarrow{\Phi^{\sim}_{\mathcal{R}}} \Phi^{\sim}_{Alg}(\mathcal{R})-mod$ the functor induced by a monoidal functor $\Phi^{\sim}$, A2.6.4
$\mathfrak{Hom}^{\Phi^{\sim}}_{\mathcal{R}}(\mathcal{L}, \mathcal{M})$ inner hom from $\mathcal{L} \in Ob\mathcal{R}-mod$ to $\mathcal{M} \in Ob\mathcal{R}-mod_X$, A2.6.5
$(\mathcal{R}, \mathcal{S})-bim$ the category of $(\mathcal{R}, \mathcal{S})$-bimodules in $\mathcal{A}^{\sim}$, A2.6.6.1
$\mathcal{R}-bim \stackrel{\text{def}}{=} (\mathcal{R}, \mathcal{R})-bim$ A2.6.6.1
$\mathfrak{BAlg}\mathcal{A}^{\sim}$ the bicategory of associative unital algebras in $\mathcal{A}^{\sim}$, A2.7
$\mathfrak{End}_{\mathcal{A}^{\sim}}(\mathcal{R}) = \mathcal{R}-bim^{\sim} = (\mathcal{R}-bim, \odot_{\mathcal{R}}, \mathfrak{a}; \mathcal{R}, \mathfrak{l}^{\mathcal{R}}, \mathfrak{r}^{\mathcal{R}})$ the monoidal category of "endomorphisms" of an algebra $\mathcal{R}$ in $\mathcal{A}^{\sim}$, A2.7
$\mathcal{R}-mod_X \xrightarrow{\mathcal{M}\odot^{\Phi}_{\mathcal{R}}} \mathcal{S}-mod_X$ the action of an $(\mathcal{S}, \mathcal{R})$-bimodule $\mathcal{M}$ induced by an action of $\mathcal{A}^{\sim}$ on $\mathcal{C}_X$, A2.9.2
$\Phi^{\sim}_{\mathcal{R}} = (\Phi^{\mathcal{R}}, \phi^{\mathcal{R}}, \phi^{\mathcal{R}}_0)$ the induced by $\Phi^{\sim}$ action of the monoidal category of $\mathcal{R}$-bimodules $\mathcal{R}-bim^{\sim}$ on the category $\mathcal{R}-mod_X$ of $\mathcal{R}$-modules in $\mathcal{C}_X$, A2.10
$\mathcal{S}-mod_X \xrightarrow{\varphi_{X*}} \mathcal{R}-mod_X$ restriction of scalars along $\mathcal{R} \xrightarrow{\varphi} \mathcal{S}$, A2.10.1
$\mathcal{S}-bim \xrightarrow{\widetilde{\varphi_*}} \mathcal{R}-bim$ restriction of scalars along $\mathcal{R} \xrightarrow{\varphi} \mathcal{S}$, A2.10.1

$\mathcal{C}_X \xrightarrow{(\eta_{\mathcal{S}})^*_X} \mathcal{S}-mod_X$ the inverse image functor of the unit $\mathbb{I} \xrightarrow{\eta_{\mathcal{S}}} \mathcal{S}$ of algebra $\mathcal{S}$, A2.10.2.1

$T(E) = (\coprod_{n\geq 0} E^{\odot n}, \mu_E)$ the tensor algebra of an object E of $\mathcal{A}^{\sim}$, V.1.2.1

$\mathbf{Aff}_{\mathcal{A}^{\sim}} \stackrel{\text{def}}{=} (Alg\mathcal{A}^{\sim})^{op}$ affine schemes in $\mathcal{A}^{\sim}$, V.2.1

$\mathbb{V}(\mathcal{L}) \stackrel{\text{def}}{=} T(\mathcal{L})^{\vee}$ vector fibre of an object $\mathcal{L}$ of $\mathcal{A}^{\sim}$, V.2.2

$L^{\wedge}M$ an object of $\mathcal{A}$ corepresenting the functor $\mathcal{C}_X(M, \Phi(-)(L))$, V.2.3

$Iso_{V,W}$ the presheaf of sets on $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ which maps every associative unital algebra $\mathcal{S}^{\vee}$ to the set $Iso_{\mathcal{S}}(\mathfrak{s}^*(V), \mathfrak{s}^*(W))$ of isomorphism from $\mathfrak{s}^*(V)$ to $\mathfrak{s}^*(W)$, 2.5

$GL_E \stackrel{\text{def}}{=} Iso_{E,E}$ the group "scheme" of automorphisms of an object E, V.2.5.2

$\mathfrak{H}_{\mathcal{F},\mathcal{G}}$ the presheaf of functor morphisms from $\mathcal{B} \xrightarrow{\mathcal{F}} \mathcal{C}_X$ to $\mathcal{B} \xrightarrow{\mathcal{G}} \mathcal{C}_X$, V.3.1

$\mathfrak{Iso}_{\mathcal{F},\mathcal{G}}$ the presheaf of isomorphisms from $\mathcal{B} \xrightarrow{\mathcal{F}} \mathcal{C}_X$ to $\mathcal{B} \xrightarrow{\mathcal{G}} \mathcal{C}_X$, V.3.2

$GL_{\mathcal{F}}$ the presheaf of groups $\mathfrak{Iso}_{\mathcal{F},\mathcal{F}}$, V.3.3

$D \xrightarrow{H_{F,G}} R_{F,G} - mod_X$ universal functor corresponding to $D \xleftarrow{G} B \xrightarrow{F} \mathcal{C}_X$ and an action of $\mathcal{A}^{\sim}$ on $\mathcal{C}_X$, V.4.1

$\Sigma^{-1}B \xrightarrow{F_{\Sigma}} R_{\Sigma} - mod_X$ the universal localization at $\Sigma \subset HomB$ corresponding to a functor $B \xrightarrow{F} \mathcal{C}_X$ and an action of $\mathcal{A}^{\sim}$ on $\mathcal{C}_X$, V.4.2.1

$\mathfrak{F}^{\sim}_{E,G}$ the pseudo-functor associated with a diagram $D \xleftarrow{G} B \xrightarrow{E} \mathcal{C}_X$, V.4.4

$\mathfrak{F}_{E,G}$ the presheaf of sets on $\mathbf{Aff}_{\mathcal{A}^{\sim}}$ associated with the pseudo-functor $\mathfrak{F}^{\sim}_{E,G}$, V.4.4.1

$\mathbf{Aff}^{op}_{\mathcal{A}^{\sim}} = Alg\mathcal{A}^{\sim} \xrightarrow{Gr_{M,V}} Sets$ the Grassmannian related to an action of $\mathcal{A}^{\sim}$ on $\mathcal{C}_X$ and a pair of objects (M, V) of the category $\mathcal{C}_X$, V.5.1

$G_{M,V} \xrightarrow{\pi_{M,V}} Gr_{M,V}$ the canonical cover of the Grassmannian, V.5.1.1

$\mathbf{Aff}^{op}_{\mathcal{A}^{\sim}} = Alg\mathcal{A}^{\sim} \xrightarrow{Gr_E} Sets$ the generic Grassmannian of an object E of $\mathcal{C}_X$, V.5.2

$\mathbf{Aff}^{op}_{\mathcal{A}^{\sim}} \xrightarrow{\mathfrak{Pr}_E} Sets$ the presheaf of projecttors of an object E, V.5.2.1

$\mathbf{Aff}^{op}_{\mathcal{A}^{\sim}} = Alg\mathcal{A}^{\sim} \xrightarrow{\mathfrak{Fl}^{\mathfrak{I}}_E} Sets$ the variety of generic flags, V.6.1

$\mathbf{Aff}^{op}_{\mathcal{A}^{\sim}} \xrightarrow{\mathfrak{F}^{\mathfrak{I}}_E} Sets$ the generalized Stiefel variety of an object E, V.6.2

$\mathfrak{F}^{\mathfrak{I}}_E \xrightarrow{\pi^{\mathfrak{I}}_E} \mathfrak{Fl}^{\mathfrak{I}}_E$ the canonical cover, V.6.2.1

$\mathfrak{R}^{\mathfrak{I}}_E \underset{{}^2\pi^{\mathfrak{I}}_E}{\overset{{}^1\pi^{\mathfrak{I}}_E}{\rightrightarrows}} \mathfrak{F}^{\mathfrak{I}}_E.$ the relations, V.6.2.1

$\mathfrak{Fl}^{\mathfrak{I}}_E \longrightarrow \prod_{i\in\mathfrak{I}} Gr_E$ a natural embedding, V.6.3

$$\mathfrak{E} = \begin{pmatrix} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1 \downarrow & & \downarrow G & & \\ D_1 & \xrightarrow{G_2} & D_2 & & \end{pmatrix}$$ the combinatorial data, V.7.1

$\mathcal{C}_X \xrightarrow{s^*} S - mod_X$ the left adjoint to the restriction of scalars $S - mod_X \xrightarrow{s_*} \mathcal{C}_X$, V.7.1

$\mathfrak{E}^S$ the combinatorial data obtained by extension of scalars $\mathcal{C}_X \xrightarrow{s^*} S - mod_X$, V.7.1

$\mathbf{Aff}^{op}_{\mathcal{A}^\sim} \xrightarrow{\mathfrak{Fl}^\sim_{\mathfrak{E}}} Sets$ the generalized flag variety associated with the data $\mathfrak{E}$, V.7.2

$\mathfrak{R}_{\mathfrak{E}} \overset{\mathfrak{p}^1_{\mathfrak{E}}}{\underset{\mathfrak{p}^2_{\mathfrak{E}}}{\rightrightarrows}} \mathfrak{F}_{\mathfrak{E}} \xrightarrow{\pi_{\mathfrak{E}}} \mathfrak{Fl}_{\mathfrak{E}}$ the canonical exact diagram, V.7.3

$$\mathfrak{E}^S_{red} = \begin{pmatrix} B_0 & \xrightarrow{G'_0} & G(B) & \xrightarrow{E'_S} & S - mod_X \\ G'_1 \downarrow & & \downarrow G' & & \\ G_2(D_1) & \xrightarrow{G'_2} & D_2 & & \end{pmatrix}$$ the reduced data corresponding to the data $$\mathfrak{E}^S = \begin{pmatrix} B_0 & \xrightarrow{G_0} & B & \xrightarrow{E} & \mathcal{C}_X \\ G_1 \downarrow & & \downarrow G & & \downarrow s^* \\ D_1 & \xrightarrow{G_2} & D_2 & & S - mod_X \end{pmatrix},$$ V.7.6.1

$(\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})$ an $\mathcal{A}^\sim$-ringed category, V.9.0

$\mathcal{O}_{\mathfrak{B}} - Mod_X$ the category of $\mathcal{O}_{\mathfrak{B}}$-modules in $\mathcal{C}_X$, V.9.1

$Qcoh(\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})_X$ the category of quasi-coherent $\mathcal{O}_{\mathfrak{B}}$-modules in $\mathcal{C}_X$, V.9.1

$\mathcal{O}_{\mathfrak{B}} - Mod^\sim$ the monoidal category of $\mathcal{O}_{\mathfrak{B}}$-bimodules, V.9.1

$Qcoh(\mathfrak{B}, \mathcal{O}_{\mathfrak{B}})^\sim$ the monoidal category of quasi-coherent $\mathcal{O}_{\mathfrak{B}}$-bimodules, V.9.1

$\mathbf{Aff}_{\mathcal{A}^\sim}/\mathfrak{X} \xrightarrow{\mathcal{O}^\vee_{\mathfrak{X}}} \mathbf{Aff}_{\mathcal{A}^\sim}$ the forgetful functor for a presheaf of sets $\mathfrak{X}$ on $\mathbf{Aff}_{\mathcal{A}^\sim}$, V.9.3

$\mathcal{O}_{\mathfrak{X}}$ the presheaf of associative unital algebras in $\mathcal{A}^\sim$ corresponding to the forgetful functor $\mathbf{Aff}_{\mathcal{A}^\sim}/\mathfrak{X} \xrightarrow{\mathcal{O}^\vee_{\mathfrak{X}}} \mathbf{Aff}_{\mathcal{A}^\sim}$, V.9.3

Index