Construction of $N = 2$ Superconformal Algebra from Affine Algebras with Extended Symmetry

Shun-Jen Cheng

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-53225 Bonn

Germany

MPI / 94-94

Construction of N=2 Superconformal Algebra from Affine Algebras with Extended Symmetry

by

Shun-Jen Cheng

Max-Planck-Institut für Mathematik

Gottfried-Claren-Straße 26, Bonn 53225, Germany

Abstract. The purpose of this paper is to study the connections between affine algebras with extended symmetry and superconformal algebras. We investigate which of the super-symmetries have super-affine backgrounds and in the course derive simple free field realizations of the $N = 2$ superconformal algebra with the underlying representation spaces being Fock spaces related to affine algebras with $2 + k$ extended symmetry, where $k \geq 1$.

§1. Introduction

Construction of representations of the $N = 2$ superconformal algebra has received some attention recently, mainly because of its connection to Mirror symmetry. Kazama and Suzuki [KZ1] [KZ2] investigated under what conditions the GKO coset method for the $N = 1$ symmetry can be applied to obtain an $N = 2$ symmetry. Roughly speaking an $N = 1$ coset model possesses an $N = 2$ symmetry if and only if the coset G/H can be polarized into 2 isotropic subalgebras. This construction links the $N = 2$ symmetry to Kähler manifolds. More recently Parkhomenko [P] pointed out a connection between Manin triples and the $N = 2$ symmetry. This model is similar to the Kazama-Suzuki model, since in this case a Lie algebra with a polarization into 2 isotropic subalgebras (a Manin triple) gives rise to an $N = 2$ symmetry [G]. In this paper we will use the Sugawara construction to give yet another approach to constructing representations of the $N = 2$ superconformal algebra. Although the representations we get are known, we believe that it is worth mentioning, because constructing superconformal algebras from affine algebras with extended symmetry is very natural. It turns out that one of our models coincides with a special case of the so-called (b, c, β, γ)-model [FGLS]. This model is currently of interest in the Landau-Ginzburg description of the $N = 2$ theories [W]. Our other model is equivalent to the Manin triple model with an even dimensional abelian Lie algebra as the underlying Manin triple [G].

In this letter we will only deal with the Neveu-Schwarz algebra. The connection between the Neveu-Schwarz algebra and the Ramon algebra was pointed out by Seiberg and Schwimmer [SS].

Computations involving operator product expansions are bound to be cumbersome and error-prone. We would like to thank Thielemans for making his Mathematica package available to the public [T].

§2. Preliminaries

Our field will always be the field of complex numbers $\mathbb{C}$. We let $\mathbb{C}[t, t^{-1}]$ stand for the Laurent polynomials. The Grassmann superalgebra in the n variables $\theta_1, \theta_2, \cdots, \theta_n$ will be denoted by $\Lambda(n)$. We will write $W(1, n)$ for the derivation superalgebra of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. The degree of a homogenous element $g \in \mathfrak{g}$ will be denoted by $\deg g$. Suppose that α is a 2-cocycle of $\mathfrak{g}$. α defines a central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ with the new commutation relation given by

$$[x, y] = [x, y]_0 + \alpha(x, y)z,$$

where $x, y \in \mathfrak{g}$, z is a central element and $[\cdot, \cdot]$ and $[\cdot, \cdot]_0$ denote the new and the old Lie brackets, respectively.

Suppose that $\mathfrak{g}$ possesses a nondegenerate supersymmetric invariant bilinear form. Then there exits a 1-1 correspondence between outer superskewsymmetric derivations of $\mathfrak{g}$ and nontrivial 2-cocycles [C]. If D is such a derivation and $(\cdot, \cdot)$ is such a form, then $\alpha_D(x, y) := (D(x), y)$ is a 2-cocycle. And conversely every 2-cocycle is necessarily of this form. We will call such derivations *central derivations*.

Suppose $\mathfrak{g} = S \otimes \mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$, where S is a simple finite dimensional Lie algebra. Let $(\cdot,\cdot)_S$ be a symmetric nondegenerate invariant bilinear form on S. We can define a bilinear form on $\mathfrak{g}$ by

$$(x \otimes \lambda, y \otimes \mu) := (x, y)_S \oint \lambda \wedge \mu,$$

where $x, y \in S$; $\lambda, \mu \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$ and $\oint$ denotes the coefficient of $t^{-1} \otimes \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n$. $(\cdot,\cdot)$ is supersymmetric, invariant and nondegenerate. Since S is simple, one can prove that the outer derivations of $\mathfrak{g}$ is isomorphic to $W(1, n)$. It is then not hard to see that $D = a_0 \frac{\partial}{\partial t} + \sum_{i=1}^n a_i \frac{\partial}{\partial \theta_i}$ with $a_i \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(n)$ is central if and only if D is divergence free, i.e. $\operatorname{div} D := \frac{\partial}{\partial t} a_0 + \sum_{i=1}^n (-1)^{\deg a_i} \frac{\partial}{\partial \theta_i} a_i = 0$.

Let $\operatorname{der}_{\mathbb{C}} \mathfrak{g}$ be the derivation superalgebra of $\mathfrak{g}$ and $D \in \operatorname{der}_{\mathbb{C}} \mathfrak{g}$. We say that D is *compatible* with α if D is superskewsymmetric with respect to α, i.e.

$$\alpha(D(x), y) = (-1)^{\deg D \deg x} \alpha(x, D(y)), \quad x, y \in \mathfrak{g}$$

The set of all α-compatible derivations form a subalgebra of $\operatorname{der}_{\mathbb{C}} \mathfrak{g}$ and we will denote this subalgebra by $(\operatorname{der}_{\mathbb{C}} \mathfrak{g})^\alpha$. We like to point out that $(\operatorname{der}_{\mathbb{C}} \mathfrak{g})^\alpha$ consists precisely of those $D \in \operatorname{der}_{\mathbb{C}} \mathfrak{g}$ such that $D \in \operatorname{der}_{\mathbb{C}} \hat{\mathfrak{g}}$, where $\hat{\mathfrak{g}}$ is the central extension of $\mathfrak{g}$ corresponding to the cocycle α.

Given a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ equipped with a consistent nondegenerate supersymmetric bilinear form $(\cdot,\cdot)$. By *consistent* we mean that $(V_{\bar{0}}, V_{\bar{1}}) = 0$. On the space $\mathrm{L} := \Lambda^2(V_{\bar{0}}) \oplus S^2(V_{\bar{1}}) \oplus (V_{\bar{0}} \otimes V_{\bar{1}})$ with obvious $\mathbb{Z}_2$-grading one can define a Lie bracket using $(\cdot,\cdot)$. For homogeneous elements $x, y \in V$ one defines $[x, y] := (x, y) \cdot 1$. One extends this bracket operation, using the Leibniz rule, to L. This gives a realization of the Lie superalgebra $\operatorname{osp}(V_{\bar{0}}|V_{\bar{1}})$, the orthosymplectic Lie superalgebra of $\dim_{\mathbb{C}} V_{\bar{0}}$ even dimension and $\dim_{\mathbb{C}} V_{\bar{1}}$ odd dimension [K].

Now let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite dimensional Lie superalgebra eqiupped with a nondegenerate superskewsymmetric consistent bilinear form $\alpha(\cdot,\cdot)$. α for example could be a 2-cocycle. We obtain an embedding

$$(\operatorname{der}_{\mathbb{C}} \mathfrak{g})^\alpha \xrightarrow{\kappa} \operatorname{osp}(\mathfrak{g}_{\bar{1}}|\mathfrak{g}_{\bar{0}}),$$

where κ is defined as follows: Let $\{x_1, x_2, \cdots, x_n\}$ be a homogeneous basis of $\mathfrak{g}$. Let $\{x_1^*, x_2^*, \cdots, x_n^*\}$ be elements in $\mathfrak{g}$ such that $\alpha(x_i^*, x_j) = \delta_{ij}$. Let $D \in (\mathrm{der}_{\mathbb{C}}\mathfrak{g})^\alpha$. Suppose $[D, x_i] = \sum_{j=1}^n a_{ij}x_j$. Then

$$\kappa(D) := -\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}x_i^* x_j. \qquad (*)$$

We note that if α is invariant, then $\mathfrak{g} \subset (\mathrm{der}_{\mathbb{C}}\mathfrak{g})^\alpha$. Therefore if we have a representation of $\mathrm{osp}(\mathfrak{g}_{\bar{1}}|\mathfrak{g}_{\bar{0}})$ it will pull back to a representation of $(\mathrm{der}_{\mathbb{C}}\mathfrak{g})^\alpha$. Now let $\mathfrak{a}$ be the Lie superalgebra generated by $\{x_1, x_2, \cdots, x_n; z\}$ with commutation relations given by α, i.e. $[x_i, x_j] := \alpha(x_i, x_j) \cdot 1$. If $\mathfrak{F}$ is a representation of $\mathfrak{a}$, then $\mathrm{osp}(\mathfrak{g}_{\bar{1}}|\mathfrak{g}_{\bar{0}})$ acts on $\mathfrak{F}$, and hence $(\mathrm{der}_{\mathbb{C}}\mathfrak{g})^\alpha$. If $\mathfrak{g}$ is infinite dimensional, this idea can be used to construct representations of some central extension of $(\mathrm{der}_{\mathbb{C}}\mathfrak{g})^\alpha$. However, we have to introduce "normal ordering" in order to give an expression like $(*)$ meaning. This simple idea is what we will use in the next 2 sections.

Let us conclude this section with some remarks on fields. We let $\psi(z) = \sum_j \psi_j z^{-j-\Delta}$ be a field of conformal dimension Δ. We can then define

$$\psi_+(z) = \sum_{j>-\Delta} \psi_j z^{-j-\Delta}$$

and

$$\psi_-(z) = \sum_{j\leq-\Delta} \psi_j z^{-j-\Delta}.$$

We call this field even if all the ψ_j's are even elements of some Lie superalgebra, and odd if they are all odd elements. Given another field $\phi(z) = \sum_j \phi_j z^{-j-\Delta'}$ of conformal dimension Δ'. We define the normally ordered product of the fields $\psi(z)$ and $\phi(z)$ to be the following field of conformal dimension $\Delta + \Delta'$:

$$:\psi(z)\phi(z): := \psi_-(z)\phi(z) + \phi(z)\psi_+(z),$$

if one of the two fields is even and

$$:\psi(z)\phi(z): := \psi_-(z)\phi(z) - \phi(z)\psi_+(z),$$

if both fields are odd. So the normally ordered product is the field

$$:\psi(z)\phi(z): = \sum_j\Big(\sum_{n\leq-\Delta}\psi_n\phi_{j-n} \pm \sum_{n>-\Delta}\phi_{j-n}\psi_n\Big)z^{-j-\Delta-\Delta'}.$$

Furthermore the derivative of the field $\psi(z)$ is the following field of conformal dimension $\Delta+1$

$$\partial\psi(z) := \sum_j(-j-\Delta)\psi_j z^{-j-\Delta-1}.$$

Finally all fields we consider in this paper will be local with respect to one another. The short distance operator product expansion of two fields $\psi(z)$ and $\phi(z)$ will be denoted by $\psi(z)\cdot\phi(w)$. We will only write down the singular terms, which are known to determine the commutation relations between mutually local fields completely.

§3. Construction in $N=3$

The fields of the $N=2$ superconformal algebra are

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

$$T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n-1}$$

$$G^+(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} G_r^+ z^{-r-\frac{3}{2}}$$

$$G^-(z) = \sum_{r\in\frac{1}{2}+\mathbb{Z}} G_r^- z^{-r-\frac{3}{2}}.$$

They satisfy the following nontrivial operator product expansions:

$$L(z)\cdot L(w) \sim \frac{\frac{3}{2}c}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)}$$

$$T(z)\cdot T(w) \sim \frac{c}{(z-w)^2}$$

$$L(z)\cdot T(w) \sim \frac{T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)}$$

$$L(z)\cdot G^+(w) \sim \frac{\frac{3}{2}G^+(w)}{(z-w)^2}+\frac{\partial G^+(w)}{(z-w)}$$

$$L(z)\cdot G^-(w) \sim \frac{\frac{3}{2}G^-(w)}{(z-w)^2}+\frac{\partial G^-(w)}{(z-w)}$$

$$T(z)\cdot G^+(w) \sim \frac{G^+(w)}{(z-w)}$$

$$T(z)\cdot G^-(w) \sim \frac{-G^-(w)}{(z-w)}$$

$$G^+(z)\cdot G^-(w) \sim \frac{c}{(z-w)^3}+\frac{T(w)}{(z-w)^2}+\frac{L(w)+\frac{1}{2}\partial T(w)}{(z-w)}.$$

The Sugawara construction [S] of the Virasoro algebra relies on the fact that the unique nontrivial (up to a scalar) 2-cocycle of the loop algebra $S\otimes\mathbb{C}[t,t^{-1}]$ is compatible with the centerless Virasoro algebra, realized as a subalgebra of the derivation algebra of the loop algebra. It is important that this 2-cocycle establishes a nondegenerate pairing for $\mathbb{C}[t,t^{-1}]$, considered as an abelian Lie algebra. This enables one to define κ as in §2 and hence, after introduction of normal ordering, to extend a highest weight representation of the central extension of $S\otimes\mathbb{C}[t,t^{-1}]$ to include the Virasoro algebra in a natural way. Let us consider two simple examples to illustrate this: Take S to be the 1-dimensional Lie algebra with a nodegenerate symmetric invariant bilinear form. In this case κ gives us the usual bosonic construction of the Virasoro algebra; i.e. $L(z)=\frac{1}{2}:\alpha(z)^2:$, where $\alpha(z)\cdot\alpha(w)\sim\frac{1}{(z-w)^2}$ and $L(z)$ denotes the Virasoro field. As the second example we take S to be the 2-dimensional abelian odd Lie superalgebra with nondegenerate skewsymmetric invariant bilinear form. In this case we obtain the usual fermionic construction of the Virasoro algebra. That is $L(z)=\frac{1}{2}(:\partial\psi^+(z)\psi^-(z):+:\partial\psi^-(z)\psi^+(z):)$, where $\psi^+(z)\cdot\psi^-(w)\sim\frac{1}{z-w}$.

The Kač-Todorov construction [KT] is similar. In this case one looks for 2-cocycles of $S\otimes\mathbb{C}[t,t^{-1}]\otimes\Lambda(1)$ such that the corresponding compatible subalgebras of the derivation superalgebra $W(1,1)$ contain the $N=1$ superconformal algebra.

Remark. Although the S in our two examples above are not simple, our argument still applies. The only reason we have insisted on S being simple in $\mathfrak{g}=S\otimes\mathbb{C}[t,t^{-1}]\otimes$

$\Lambda(n)$ is to assure that the outer derivations coincides with $W(1,n)$. However, this is by no means required, because $W(1,n)$ will always be a subalgebra of the outer derivation of $\mathfrak{g}$ for arbitrary S.

To apply the above construction to the $N=2$ superconformal algebra one is therefore led naturally to consider $S\otimes\mathbb{C}[t,t^{-1}]\otimes\Lambda(2)$ and look for central derivations in $W(1,2)$ which are compatible with the (centerless) $N=2$ superconformal algebra. Such central derivations could enable one to construct representations of the $N=2$ symmetry from the corresponding central extensions of $S\otimes\mathbb{C}[t,t^{-1}]\otimes\Lambda(2)$. However, the only such central derivation is the trivial one [CFRS], which does not give rise to a nondegenerate pairing in $\mathbb{C}[t,t^{-1}]\otimes\Lambda(2)$. But the (centerless) $N=2$ superconformal algebra is obviously embedded in the derivation superalgebra of $W(1,3)$. One could ask if there is a nontrivial cocycle α of $S\otimes\mathbb{C}[t,t^{-1}]\otimes\Lambda(3)$ such that $(W(1,3))^{\alpha}$ contains the $N=2$ superconformal algebra. It turns out that there exists such a 2-cocycle. It is unique up to a scalar and this cocycle corresponds to the central derivation $\frac{\partial}{\partial\theta_3}$ [CFRS].

From now on we will assume that S is the 1-dimensional Lie algebra equipped with a symmetric nondegenerate bilinear form, unless otherwise stated. Furthermore we will drop S in $S\otimes\mathbb{C}[t,t^{-1}]\otimes\Lambda(n)$ when we make this assumption. Consider the central extension of $\mathbb{C}[t,t^{-1}]\otimes\Lambda(3)$ corresponding to $\frac{\partial}{\partial\theta_3}$. This cocycle is defined by

$$\alpha(x,y)=\oint\frac{\partial}{\partial\theta_3}(x)\wedge y,$$

where $x,y\in\mathbb{C}[t,t^{-1}]\otimes\Lambda(3)$. Now only nontrivial pairings of α can be used to construct the $N=2$ symmetry. One checks easily that the only nontrival pairings of α are

$$\alpha(t^n\otimes\theta_1\wedge\theta_3,t^{-n-1}\otimes\theta_2\wedge\theta_3)=-1,$$

$$\alpha(t^n\otimes\theta_3,t^{-n-1}\otimes\theta_1\wedge\theta_2\wedge\theta_3)=1.$$

The centerless $N = 2$ superconformal algebra in $W(1,3)$ can be realized as

$$
\begin{aligned}
L_n &= -t^{n+1}\frac{\partial}{\partial t} - (\frac{n+1}{2})t^n \otimes (\theta_1 \frac{\partial}{\partial \theta_1} + \theta_2 \frac{\partial}{\partial \theta_2}) \\
T_n &= t^n \otimes (\theta_1 \frac{\partial}{\partial \theta_2} - \theta_2 \frac{\partial}{\partial \theta_1}) \\
G_r^1 &= t^{r+\frac{1}{2}} \otimes (\theta_1 \frac{\partial}{\partial t} - \frac{\partial}{\partial \theta_1}) + (r+\frac{1}{2})t^{r-\frac{1}{2}} \otimes \theta_1 \wedge \theta_2 \frac{\partial}{\partial \theta_2} \\
G_r^2 &= t^{r+\frac{1}{2}} \otimes (\theta_2 \frac{\partial}{\partial t} - \frac{\partial}{\partial \theta_2}) - (r+\frac{1}{2})t^{r-\frac{1}{2}} \otimes \theta_1 \wedge \theta_2 \frac{\partial}{\partial \theta_1} \\
G_r^+ &= \frac{1}{\sqrt{2}}(G_r^1 + iG_r^2) \\
G_r^- &= \frac{1}{\sqrt{2}}(G_r^1 - iG_r^2)
\end{aligned}
$$

Let $\mathfrak{a}$ denote the Lie superalgebra generated by $\{t^n \otimes \theta_1 \wedge \theta_3, t^n \otimes \theta_2 \wedge \theta_3, t^n \otimes \theta_3, t^n \otimes \theta_1 \wedge \theta_2 \wedge \theta_3, z; n \in \mathbb{Z}\}$ with nontrivial commutation relations

$$[t^n \otimes \theta_1 \wedge \theta_3, t^m \otimes \theta_2 \wedge \theta_3] = -\delta_{n+m,-1} z$$

$$[t^n \otimes \theta_3, t^m \otimes \theta_1 \wedge \theta_2 \wedge \theta_3] = \delta_{n+m,-1} z.$$

One checks that $\mathfrak{a}$ is invariant under L_n, T_n, G_r^1, G_r^2. We want to write the generators of $\mathfrak{a}$ in field notation to make computations easier. For this we need to determine the conformal dimensions and the modes of these fields. We use the Virasoro field $L(z)$ to force these fields to be primary. This immediately implies that the fields of $\mathfrak{a}$ are of the following form:

$$
\begin{aligned}
\alpha^+(z) &= \sum_{r \in \frac{1}{2}+\mathbb{Z}} \alpha_r^+ z^{-r-\frac{1}{2}} \\
\alpha^-(z) &= \sum_{r \in \frac{1}{2}+\mathbb{Z}} \alpha_r^- z^{-r-\frac{1}{2}} \\
\psi^+(z) &= \sum_{j \in \mathbb{Z}} \psi_j^+ z^{-j-1} \\
\psi^-(z) &= \sum_{j \in \mathbb{Z}} \psi_j^- z^{-j-1}
\end{aligned}
$$

where

$$\alpha_r^+ = t^{r-\frac{1}{2}} \otimes \theta_3 \wedge \theta_1$$
$$\alpha_r^- = t^{r-\frac{1}{2}} \otimes \theta_2 \wedge \theta_3$$
$$\psi_j^+ = t^j \otimes \theta_3$$
$$\psi_j^- = -jt^{j-1} \otimes \theta_1 \wedge \theta_2 \wedge \theta_3$$

Here $\alpha^+(z)$ and $\alpha^-(z)$ are bosonic fields and $\psi^+(z)$ and $\psi^-(z)$ are fermionic fields. They satisfy the following operator product expansions:

$$\alpha^+(z) \cdot \alpha^-(w) \sim \frac{k}{(z-w)},$$
$$\psi^+(z) \cdot \psi^-(w) \sim \frac{k}{(z-w)^2},$$

where $k \in \mathbb{C}^*$ is the scalar with which z acts. To construct an $N = 2$ symmetry from these four fields one needs to compute the actions of L_n, T_n, G_r^1, G_r^2 on these fields. The computation is straightforward and we will leave the verification to the reader. However, because we need these formulas for §4, let us write them down explicitly:

PROPOSITION 3.1. *Let $\lambda \in \Lambda(\theta_3, \cdots, \theta_{2+k})$. Consider the $N = 2$ superconformal algebra as a subalgebra of $\mathrm{der}_{\mathbb{C}}\Lambda(\theta_1, \theta_2, \theta_3, \cdots, \theta_{2+k})$. Set*

$$\alpha_r^+ = t^{r-\frac{1}{2}} \otimes \lambda \wedge \theta_1 \quad \alpha_r^- = t^{r-\frac{1}{2}} \otimes \theta_2 \wedge \lambda$$
$$\psi_j^+ = t^j \otimes \lambda \qquad \psi_j^- = -jt^{j-1} \otimes \theta_1 \wedge \theta_2 \wedge \lambda$$

Then one has

$$L_n\alpha_r^+ = (-r - \frac{n}{2})\alpha_{n+r}^+ \quad L_n\alpha_r^- = (-r - \frac{n}{2})\alpha_{n+r}^-$$
$$L_n\psi_j^+ = -j\psi_{n+j}^+ \qquad L_n\psi_j^- = -j\psi_{n+j}^-$$
$$T_n\alpha_r^+ = \alpha_{n+r}^- \qquad T_n\alpha_r^- = -\alpha_{n+r}^+$$
$$T_n\psi_j^+ = T_n\psi_j^- = 0 \qquad G_n^1\alpha_r^+ = \psi_{r+n}^+$$
$$G_n^1\alpha_r^- = -\psi_{r+n}^- \qquad G_n^1\psi_j^+ = -j\alpha_{n+j}^+$$
$$G_n^1\psi_j^- = j\alpha_{n+j}^- \qquad G_n^2\alpha_r^+ = -\psi_{r+n}^-$$
$$G_n^2\alpha_r^- = -\psi_{r+n}^+ \qquad G_n^2\psi_j^+ = j\alpha_{n+j}^-$$
$$G_n^2\psi_j^- = j\alpha_{n+j}^+.$$

Using the above proposition it is not hard to derive the following formula for the $N = 2$ symmetry. We summarize this in the following

PROPOSITION 3.2. *On the Fock space of the central extension of the Lie superalgebra $C[t, t^{-1}] \otimes \Lambda(3)$ with corresponding central derivation $\frac{\partial}{\partial \theta_3}$ one can define a representation of the $N = 2$ superconformal algebra of central charge $c = -1$. It is equivalent to the following free field realization:*

$$
\begin{aligned}
L(z) &= \frac{1}{2k}(:\partial\alpha^-(z)\alpha^+(z): - :\partial\alpha^+(z)\alpha^-(z):) - \frac{1}{k}:\psi^+(z)\psi^-(z): \\
T(z) &= \frac{1}{2k}(:\alpha^-(z)\alpha^-(z): - :\alpha^+(z)\alpha^+(z):) \\
G^+(z) &= \frac{1}{2k}(:\alpha^-(z)\psi^+(z): + :\alpha^+(z)\psi^-(z): - :\alpha^-(z)\psi^-(z): - :\alpha^+(z)\psi^+(z):) \\
G^-(z) &= \frac{1}{2k}(:\alpha^-(z)\psi^+(z): + :\alpha^+(z)\psi^-(z): + :\alpha^-(z)\psi^-(z): + :\alpha^+(z)\psi^+(z):)
\end{aligned}
$$

Proof. One can check the OPE's using Wick's theorem.

We will conclude this section with a few remarks.

Remarks.

(i) Let $\mathfrak{a}^+$ be the subalgebra of $\mathfrak{a}$ generated by $\{\psi_j^+, \psi_j^-, \alpha_r^+, \alpha_r^-, z \mid j \geq 0, r > 0\}$. Let $|0>$ be the one dimensional representation of $\mathfrak{a}^+$ such that $z|0> = k|0>$ and $\psi_j^+|0> = \psi_j^-|0> = \alpha_r^+|0> = \alpha_r^-|0> = 0$, for $j \geq 0$ and $r > 0$. Let $\mathfrak{F} = \mathrm{Ind}_{\mathfrak{a}^+}^{\mathfrak{a}}|0>$, the induced $\mathfrak{a}$-module. Then $\mathfrak{F}$ is clearly an irreducible $\mathfrak{a}$-module. The $N = 2$ superconformal algebra acts on $\mathfrak{F}$ and $|0>$ is a highest weight vector with $L_0|0> = T_0|0> = 0$. $\mathfrak{F}$ however is not irreducible as a representation of the $N = 2$ superconformal algebra. One checks that $\alpha^+_{-\frac{1}{2}}|0> + \alpha^-_{-\frac{1}{2}}|0>$ is a singular vector on which L_0 and T_0 act as the scalars $\frac{1}{2}$ and -1, respectively. $\alpha^+_{-\frac{1}{2}}|0> - \alpha^-_{-\frac{1}{2}}|0>$ is also singular with

$$L_0(\alpha^+_{-\frac{1}{2}}|0> - \alpha^-_{-\frac{1}{2}})|0> = \frac{1}{2}(\alpha^+_{-\frac{1}{2}}|0> - \alpha^-_{-\frac{1}{2}}|0>)$$

and

$$T_0(\alpha^+_{-\frac{1}{2}}|0> - \alpha^-_{-\frac{1}{2}}|0>) = (\alpha^+_{-\frac{1}{2}}|0> - \alpha^-_{-\frac{1}{2}}|0>).$$

(ii) Just as in the case of the Sugawara construction one can take S to be a simple Lie algebra. In the Sugawara case one can get representations of different central charges this way. In our case the situation is different. The $N = 2$ symmetry constructed from $S \otimes \mathbb{C}[t, t^{-1}] \otimes \Lambda(3)$, is equivalent to the one constructed by using $A \otimes \mathbb{C}[t, t^{-1}] \otimes \Lambda(3)$, where A is an abelian Lie algebra of the same dimension as S. Therefore the $N = 2$ symmetry obtained from $S \otimes \mathbb{C}[t, t^{-1}] \otimes \Lambda(3)$ is a direct sum of $\dim_{\mathbb{C}} S$ copies of the one we have constructed here.

(iii) This model is indeed a special case of the (b, c, β, γ)-model [FGLS] as we have pointed out in the beginning as follows: Let $j \in \mathbb{Z}$ and $\omega = \frac{1}{j+2}$. We let $b(z)$ and $c(z)$ be two fermionic fields of conformal dimensions $\frac{1+\omega}{2}$ and $\frac{1-\omega}{2}$, respectively. Furthermore suppose $\beta(z)$ and $\gamma(z)$ are two bosonic fields of dimensions $1 - \frac{\omega}{2}$ and $\frac{\omega}{2}$, respectively. The above fields satisfy the following nontrivial OPE:

$$b(z) \cdot c(w) \sim \frac{1}{(z-w)} \qquad \beta(z) \cdot \gamma(w) \sim \frac{1}{(z-w)}.$$

An $N = 2$ symmetry with central charge $\frac{j}{j+2}$ using the fields $b(z)$, $c(z)$, $\beta(z)$ and $\gamma(z)$ can be obtained as follows [FGLS]:

$$\begin{aligned} L(z) &= -\frac{1+\omega}{2} : b(z)\partial c(z) : + \frac{1-\omega}{2} : \partial b(z) c(z) : \\ &\quad + (1 - \frac{\omega}{2}) : \beta(z)\partial\gamma(z) : - \frac{\omega}{2} : \partial\beta(z)\gamma(z) : \\ T(z) &= -(1-\omega) : b(z)c(z) : + \omega : \beta(z)\gamma(z) : \\ G^+(z) &= -\omega : \gamma(z)\partial c(z) : + (1-\omega) : \partial\gamma(z) c(z) : \\ G^-(z) &=: \beta(z) b(z) : . \end{aligned}$$

The (b, c, β, γ)-model with $j = -1$ is equivalent to our affine model via the following substitutions:

$$\begin{aligned} b(z) &= \frac{1}{\sqrt{2k}}(\psi^-(z) + \psi^+(z)) \\ \partial c(z) &= \frac{1}{\sqrt{2k}}(\psi^-(z) - \psi^+(z)) \\ \beta(z) &= \frac{1}{\sqrt{2k}}(\alpha^-(z) + \alpha^+(z)) \\ \gamma &= \frac{1}{\sqrt{2k}}(\alpha^-(z) - \alpha^+(z)) \end{aligned}$$

So our model provides a purely algebraic background for the (b, c, β, γ)-model in the special case when $j = -1$.

§4. The construction in $N > 3$

From the previous section one is led naturally to consider the following question: Can one find central derivations in $W(1, N+k)$ that are compatible with the N-superconformal algebra for $k > 1$ and $N \geq 2$. §3 settles this question in the case when $k = 1$. To answer this question we will need to study the "compatibility condition" more carefully. For this we introduce some more notation. Let $D_i = \theta_i \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta_i} \in W(1, N+k)$ for $i = 1, \cdots, N+k$. We have $[D_i, D_j] = 2\delta_{ij} \frac{\partial}{\partial t}$. The N-superconformal algebra can be realized as the subalgebra of $W(1, N)$ (and therefore as the subalgebra of $W(1, N+k)$) consisting of elements of the form [CFRS]

$$L(u) = u\frac{\partial}{\partial t} + \frac{1}{2}\sum_{j=1}^{N}(-1)^{\deg u} D_j(u) D_j, \quad u \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N).$$

Let $D \in W(1, N+k)$ be a central derivation and α_D the corresponding 2-cocycle. The compatibility condition means that $L(u)$ is superskew with respect to α_D. We prove the following simple lemma, which was stated in [CFRS].

Lemma 4.1. *Let $D \in W(1, n)$ be central and α_D be the corresponding cocycle. Then $A \in W(1, n)$ is superskew with respect to α_D if and only if $[A, D] = -\mathrm{div}(A) \wedge D$.*

Proof. Write $A = a_0 \frac{\partial}{\partial t} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial \theta_i}$, $a_i \in \Lambda(n) \otimes \mathbb{C}[t, t^{-1}]$. We have for $x, y \in \Lambda(n) \otimes \mathbb{C}[t, t^{-1}]$

$$\begin{aligned} 0 &= \int \frac{\partial}{\partial t}(a_0 \wedge D(x) \wedge y) + \sum_{i=1}^{n}(-1)^{\deg a_i} \frac{\partial}{\partial \theta_i}(a_i \wedge D(x) \wedge y) \\ &= \int \frac{\partial}{\partial t}(a_0) \wedge D(x) \wedge y + \sum_{i=1}^{n}(-1)^{\deg a_i} \frac{\partial}{\partial \theta_i}(a_i) \wedge D(x) \wedge y \\ &+ \int a_0 \wedge \frac{\partial}{\partial t}(D(x) \wedge y) + \sum_{i=1}^{n} a_i \wedge \frac{\partial}{\partial \theta_i}(D(x) \wedge y). \end{aligned}$$

This implies that

$$\begin{aligned} \int -(\mathrm{div}(A) \wedge D)(x) \wedge y &= \int A(D(x) \wedge y) \\ &= \int A(D(x)) \wedge y + (-1)^{\deg A \deg D + \deg A \deg x} D(x) \wedge A(y). \end{aligned}$$

Now A is superskew if and only if

$$\int DA(x) \wedge y = -(-1)^{\deg A \deg x} \int D(x) \wedge A(y).$$

Thus

$$\int -(\operatorname{div}(A) \wedge D)(x) \wedge y = \int AD(x) \wedge y - (-1)^{\deg A \deg D} DA(x) \wedge y$$
$$= \int [A, D](x) \wedge y.$$

Therefore $-\operatorname{div}(A) \wedge D = [A, D]$. □

Thus our task of finding compatible cocycles reduces to finding derivations $D \in W(1, N+k)$ such that $[L(u), D] = -\operatorname{div}(L(u)) \wedge D$ for all $u \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$. We let $D = \alpha_0 \frac{\partial}{\partial t} + \sum_{i=1}^{N+k} \alpha_i D_i,\ \alpha_i \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N+k)$. One computes $[L(u), D]$ and set it equal to $-\operatorname{div}(L(u)) \wedge D$. It is easy to see that $-\operatorname{div}(L(u)) = (\frac{N}{2} - 1)\frac{\partial}{\partial t}(u)$. From the above equality one derives the following set of equations:

$$u\frac{\partial}{\partial t}(\alpha_0) - (-1)^{\deg u \deg \alpha_0}\alpha_0 \frac{\partial}{\partial t}(u) + \frac{1}{2}\sum_{j=1}^{N}(-1)^{\deg u} D_j(u) D_j(\alpha_0)$$
$$- \sum_{j=1}^{N+k}(-1)^{\deg u(\deg \alpha_j + 1)}\alpha_j D_j(u) + \sum_{j=1}^{N}(-1)^{\deg u + \deg \alpha_j} D_j(u)\alpha_j$$
$$= (\frac{N}{2} - 1)\frac{\partial}{\partial t}(u)\alpha_0. \tag{1}$$
$$-\frac{1}{2}(-1)^{\deg u(1+\deg \alpha_0)}\alpha_0 \frac{\partial}{\partial t}(D_j(u)) + \frac{1}{2}(-1)^{\deg u}\sum_{i=1}^{N} D_i(u) D_i(\alpha_j)$$
$$-\frac{1}{2}\sum_{i=1}^{N+k}(-1)^{\deg u + \deg u(1+\deg \alpha_i)}\alpha_i D_i(D_j(u)) + u\frac{\partial}{\partial t}(\alpha_j)$$
$$= (\frac{N}{2} - 1)\frac{\partial}{\partial t}(u)\alpha_j, \quad j = 1, \cdots, N. \tag{2}$$
$$u\frac{\partial}{\partial t}(\alpha_j) + \frac{1}{2}(-1)^{\deg u}\sum_{i=1}^{N} D_i(u) D_i(\alpha_j)$$
$$= (\frac{N}{2} - 1)\frac{\partial}{\partial t}(u)\alpha_j, \quad j = N+1, \cdots, N+k. \tag{3}$$

From equations (1), (2) and (3) we obtain by putting $u = 1$:

$$\frac{\partial}{\partial t}(\alpha_0) = 0 \tag{1'}$$

$$\frac{\partial}{\partial t}(\alpha_j) = 0, \quad j = 1, \cdots, N \tag{2'}$$

$$\frac{\partial}{\partial t}(\alpha_j) = 0, \quad j = N+1, \cdots, N+k \tag{3'}$$

Now set $u = \theta_l$, $l = 1, \cdots, N$ and we get

$$D_l(\alpha_0) = 0 \tag{1''}$$

$$D_l(\alpha_j) = 0, \quad j = 1, \cdots, N \tag{2''}$$

$$D_l(\alpha_j) = 0, \quad j = N+1, \cdots, N+k \tag{3''}$$

Finally if we let $u = t$ we conclude that

$$-\sum_{j=N+1}^{N+k} \alpha_j \theta_j = \frac{N}{2}\alpha_0 \tag{1'''}$$

$$(N-1)\alpha_j = 0, \quad j = 1, \cdots, N \tag{2'''}$$

$$(N-2)\alpha_j = o, \quad j = N+1, \cdots, N+k \tag{3'''}$$

Using the above nine identities we may rewrite (1), (2) and (3) as

$$\frac{N}{2}(\frac{\partial}{\partial t}(u)\alpha_0) = -\sum_{j=N+1}^{N+k} (-1)^{\deg u(\deg \alpha_j + 1)} \alpha_j D_j(u) \tag{1*}$$

$$-\frac{1}{2}\{(-1)^{\deg u(1+\deg\alpha_0)}\alpha_0 \frac{\partial}{\partial t}(D_j(u)) + \sum_{i=1}^{N+k}(-1)^{\deg u + \deg u(1+\deg\alpha_i)}\alpha_i D_i(D_j(u))\}$$

$$= (\frac{N}{2} - 1)\frac{\partial}{\partial t}(u)\alpha_j, \quad j = 1, \cdots, N \tag{2*}$$

$$(\frac{N}{2} - 1)\frac{\partial}{\partial t}(u)\alpha_j = 0, \quad j = N+1, \cdots, N+k \tag{3*}$$

We obtain the following corollaries.

COROLLARY 4.1. *If D is compatible with the N-superconformal algebra for $N \geq 3$, then $D = 0$.*

Proof. By (2''') and (3''') $\alpha_j = 0$ for $j = 1, \cdots, N+k$. Therefore $D = \alpha_0 \frac{\partial}{\partial t}$. Now we have by (1''') $\alpha_0 = 0$. □

Remark. Corollary 4.1 in the special case when $k = 1$ was stated in [CFRS]. From this Corollary we can also deduce that one can only construct the $N = 0, 1, 2$ superconformal algebras from affine algebras with extended symmetry using the approach in this paper.

COROLLARY 4.2. *D is compatible with the $N = 2$ superconformal algebra if and only if $D = \sum_{i=3}^{2+k} \alpha_i \frac{\partial}{\partial\theta_i}$, with $\mathrm{div}(D) = 0$ and $\alpha_i \in \Lambda(\theta_3, \cdots, \theta_{2+k})$.*

Proof. We have by $(2''')$ $\alpha_1 = \alpha_2 = 0$. Also by $(1''')$ one has

$$D = -\sum_{j=3}^{2+k} \alpha_j \theta_j \frac{\partial}{\partial t} + \sum_{j=3}^{2+k} \alpha_j (\theta_j \frac{\partial}{\partial t} + \frac{\partial}{\partial \theta_j})$$
$$= \sum_{j=3}^{2+k} \alpha_j \frac{\partial}{\partial \theta_j}.$$

By $(3')$ and $(3'')$ $\alpha_j \in \Lambda(\theta_3, \cdots, \theta_{2+k})$. Therefore the condition is necessary. One checks that if D has the above form, then D satisfies (1^*), (2^*) and (3^*) with $u = t^n, \theta_1 \otimes t^n, \theta_2 \otimes t^n, \theta_1 \wedge \theta_2 \otimes t^n$ for $n \in \mathbb{Z}$. □

Therefore $D \in W(1, 2+k)$ is $N = 2$ compatible if $D \in \mathrm{der}_{\mathbb{C}} \Lambda(\theta_3, \cdots, \theta_{2+k})$ and divergence-free. Let D be such a derivation. The $N = 2$ will be constructed on the Fock space of the central extension $\hat{\mathrm{L}}_D$ of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(\theta_1, \cdots, \theta_{2+k})/\mathrm{ker}\alpha_D$. This we will do now.

When we consider D as a derivation of $\Lambda(\theta_3, \cdots, \theta_{2+k})$ we will denote it by $\bar{D}$ to avoid confusion. We let r be the rank of the linear map $\bar{D}$. For parity reason it is necessary to assume that the 2-cocycle α_D is a consistent bilinear form. Let $\alpha_{\bar{D}}(x, y) = \int \bar{D}(x) \wedge y$, where $x, y \in \Lambda(\theta_3, \cdots, \theta_{2+k})$ and $\int$ here denotes the coefficient of $\theta_3 \wedge \cdots \wedge \theta_{2+k}$. The bilinear form $\alpha_{\bar{D}}$ is symmetric on $\Lambda(\theta_3, \cdots, \theta_{2+k})_{\bar{1}}$ and skewsymmetric on $\Lambda(\theta_3, \cdots, \theta_{2+k})_{\bar{0}}$ and furthermore it determines a 2-cocycle on $\Lambda(\theta_3, \cdots, \theta_{2+k})/\mathrm{ker}\alpha_{\bar{D}}$. Let $\hat{\mathrm{L}}_{\bar{D}}$ be the corresponding central extension. $\alpha_{\bar{D}}$ is nondegenerate on $\Lambda(\theta_3, \cdots, \theta_{2+k})/\mathrm{ker}\alpha_{\bar{D}}$, thus we can choose an orthonormal basis $\{\lambda_i | i = 1, \ldots, r_{\bar{1}}\}$ of $(\Lambda(\theta_3, \cdots, \theta_{2+k})/\mathrm{ker}\alpha_{\bar{D}})_{\bar{1}}$ and a basis $\{\mu_i, \mu_i^* | \alpha_{\bar{D}}(\mu_i, \mu_j^*) = \delta_{ij}, i = 1, \ldots, \frac{r_{\bar{0}}}{2}\}$ of $(\Lambda(\theta_3, \cdots, \theta_{2+k})/\mathrm{ker}\alpha_{\bar{D}})_{\bar{0}}$. Here one has of course $r_{\bar{0}} + r_{\bar{1}} = r$. Denote by $\mathfrak{a}_i$ the Lie superalgebra generated by $\{t^n \otimes \theta_1 \wedge \lambda_i, t^n \otimes \theta_2 \wedge \lambda_i, t^n \otimes \theta_1 \wedge$

$\theta_2 \wedge \lambda_i, t^n \otimes \lambda_i, z_i;\ n \in \mathbb{Z}\}$ with commutation relations determined by α_D, i.e.

$$[t^n \otimes \nu, t^m \otimes \nu'] = \alpha_D(t^n \otimes \nu, t^m \otimes \nu')z_i,$$

where z_i is a central element. Also let $\mathfrak{b}_i$ be the Lie superalgebra generated by the elements $\{t^n \otimes \theta_1 \wedge \mu_i, t^n \otimes \theta_1 \wedge \mu_i^*, t^n \otimes \theta_2 \wedge \mu_i, t^n \otimes \theta_2 \wedge \mu_i^*, t^n \otimes \theta_1 \wedge \theta_2 \wedge \mu_i, t^n \otimes \theta_1 \wedge \theta_2 \wedge \mu_i^*, t^n \otimes \mu_i, t^n \otimes \mu_i^*, \tilde{z}_i;\ n \in \mathbb{Z}\}$ with $\tilde{z}_i$ central and commutation relations defined by α_D. We have

$$\hat{\mathrm{L}}_D \cong (\oplus_{i=1}^{r_{\bar{1}}} \mathfrak{a}_i) \oplus (\oplus_{j=1}^{\frac{r_{\bar{0}}}{2}} \mathfrak{b}_i)/(z = z_1 = \cdots = z_{r_{\bar{1}}} = \tilde{z}_1 = \cdots = \tilde{z}_{\frac{r_{\bar{0}}}{2}}).$$

So to study the action of our $N = 2$ on the Fock space of $\hat{\mathrm{L}}_D$ it suffices to see what happens on each $\mathfrak{a}_i$ and $\mathfrak{b}_j$.

Let's first look at $\mathfrak{a}_i$, $i = 1, \ldots, r_{\bar{1}}$. In this case λ_i and θ_3 have the same parity. It follows that we have a Lie superalgebra isomorphism $\mathfrak{a} \cong \mathfrak{a}_i$, where $\mathfrak{a}$ is the Lie superalgebra in §3. Furthermore $\mathfrak{a} \cong \mathfrak{a}_i$ as modules of the $N = 2$ superconformal algebra by Proposition 3.1. Thus the $N = 2$ symmetry one obtains on the Fock space of $\mathfrak{a}_i$ is isomorphic to the one we have constructed on $\mathfrak{a}$ in §3.

Next consider the action of the $N = 2$ superconformal algebra on $\mathfrak{b}_j$, $j = 1, \cdots, \frac{r_{\bar{0}}}{2}$. The situation here is slightly different, because μ_j and μ_j^* are even. So even though by Proposition 3.1 we have an isomorphism of $N = 2$-modules $\mathfrak{b}_i \cong (\mathfrak{a} \oplus \mathfrak{a})$ after identification of the central elements on the right hand side, this isomorphism is not degree preserving. Furthermore these 2 Lie superalgebras are not isomorphic. So in this case we have to proceed as in §3 and use the actions of the $N = 2$ on $\mathfrak{b}_i$ to construct the map κ of §2. $\mathfrak{b}_i$ defines 8 free fields. The odd fields are

$$a^+(z) = \sum_{r \in \frac{1}{2}+\mathbb{Z}} a_r^+ z^{-r-\frac{1}{2}}, a_r^+ = -t^{r-\frac{1}{2}} \otimes \theta_1 \wedge \mu_i$$

$$a^-(z) = \sum_{r \in \frac{1}{2}+\mathbb{Z}} a_r^- z^{-r-\frac{1}{2}}, a_r^- = t^{r-\frac{1}{2}} \otimes \theta_2 \wedge \mu_i^*$$

$$b^+(z) = \sum_{r \in \frac{1}{2}+\mathbb{Z}} b_r^+ z^{-r-\frac{1}{2}}, b_r^+ = t^{r-\frac{1}{2}} \otimes \theta_2 \wedge \mu_i$$

$$b^-(z) = \sum_{r \in \frac{1}{2}+\mathbb{Z}} b_r^- z^{-r-\frac{1}{2}}, b_r^- = -t^{r-\frac{1}{2}} \otimes \theta_1 \wedge \mu_i^*$$

The even fields are

$$
\begin{aligned}
A^{+}(z) &= \sum_{n\in\mathbf{Z}} A_n^{+} z^{-n-1}, A_n^{+} = t^n \otimes \mu_i \\
A^{-}(z) &= \sum_{n\in\mathbf{Z}} A_n^{-} z^{-n-1}, A_n^{-} = -nt^{n-1} \otimes \theta_1 \wedge \theta_2 \wedge \mu_i^* \\
B^{+}(z) &= \sum_{n\in\mathbf{Z}} B_n^{+} z^{-n-1}, B_n^{+} = -nt^{n-1} \otimes \theta_1 \wedge \theta_2 \wedge \mu_i \\
B^{-}(z) &= \sum_{n\in\mathbf{Z}} B_n^{-} z^{-n-1}, B_n^{-} = t^n \otimes \mu_i^*.
\end{aligned}
$$

They satisfy the following nontrivial operator product expansions:

$$
\begin{aligned}
a^{+}(z)\cdot a^{-}(w) &\sim \frac{-k}{(z-w)} & \quad A^{+}(z)\cdot A^{-}(w) &\sim \frac{k}{(z-w)^2} \\
b^{+}(z)\cdot b^{-}(w) &\sim \frac{k}{(z-w)} & \quad B^{+}(z)\cdot B^{-}(w) &\sim \frac{-k}{(z-w)^2}
\end{aligned}
$$

Here $k \in \mathbf{C}^*$ is the scalar with which z acts on the Fock space of $\mathfrak{b}_i$. The fields of the $N = 2$ can be then written as

$$
\begin{aligned}
L(z) &= \frac{1}{2k}(: a^{+}(z)\partial a^{-}(z) : + : a^{-}(z)\partial a^{+}(z) : + : \partial b^{+}(z) b^{-}(z) : \\
&\quad + : \partial b^{-}(z) b^{+}(z) : +2 : A^{+}(z)A^{-}(z) : +2 : B^{+}(z)B^{-}(z) :) \\
T(z) &= \frac{-i}{k}(: a^{-}(z)b^{+}(z) : - : a^{+}(z)b^{-}(z) :) \\
G^{+}(z) &= \frac{1}{2k}(: A^{+}(z)a^{-}(z) : - : A^{-}(z)a^{+}(z) : + B^{+}(z)b^{-}(z) : - : B^{-}(z)b^{+}(z) :) \\
&\quad + \frac{i}{2k}(: A^{+}(z)b^{-}(z) : + : A^{-}(z)b^{+}(z) : - B^{+}(z)a^{-}(z) : - B^{-}(z)a^{+}(z) :) \\
G^{-}(z) &= \frac{1}{2k}(: A^{+}(z)a^{-}(z) : - : A^{-}(z)a^{+}(z) : + B^{+}(z)b^{-}(z) : - : B^{-}(z)b^{+}(z) :) \\
&\quad - \frac{i}{2k}(: A^{+}(z)b^{-}(z) : + : A^{-}(z)b^{+}(z) : - B^{+}(z)a^{-}(z) : - B^{-}(z)a^{+}(z) :)
\end{aligned}
$$

These fields satisfy an $N = 2$ symmetry with central charge $c = 2$. We can write these formulas in a more familiar form. To do so, let us introduce the following new fields:

$$\alpha^+(z) = \frac{A^+(z) - iB^+(z)}{\sqrt{2k}} \qquad \alpha^-(z) = \frac{A^-(z) - iB^-(z)}{\sqrt{2k}}$$
$$\beta^+(z) = \frac{A^-(z) + iB^-(z)}{\sqrt{2k}} \qquad \beta^+(z) = \frac{A^+(z) + iB^+(z)}{\sqrt{2k}}$$
$$\psi^+(z) = \frac{-a^+(z) - ib^+(z)}{\sqrt{2k}} \qquad \psi^-(z) = \frac{a^-(z) + ib^-(z)}{\sqrt{2k}}$$
$$\phi^+(z) = \frac{a^-(z) - ib^-(z)}{\sqrt{2k}} \qquad \phi^-(z) = \frac{-a^+(z) + ib^+(z)}{\sqrt{2k}}$$

They satisfy

$$\alpha^+(z) \cdot \alpha_-(w) \sim \frac{1}{(z-w)^2} \qquad \beta^+(z) \cdot \beta_-(w) \sim \frac{1}{(z-w)^2}$$
$$\psi^+(z) \cdot \psi_-(w) \sim \frac{1}{z-w} \qquad \phi^+(z) \cdot \phi_-(w) \sim \frac{1}{z-w}$$

In terms of these fields we can rewrite our symmetry as:

$$\begin{aligned} L(z) &= \frac{1}{2}(- : \psi^+(z)\partial\psi^-(z) : +2 : \alpha^+(z)\alpha^-(z) : - : \phi^+(z)\partial\phi^-(z) : \\ &\quad + 2 : \beta^+(z)\beta^-(z) : + : \partial\psi^+(z)\psi^-(z) : + : \partial\phi^+(z)\phi^-(z) :) \\ J(z) &= - : \psi^+(z)\psi^-(z) : - : \phi^+(z)\phi^-(z) : \\ G^+(z) &=: \alpha^+(z)\psi^-(z) : + : \beta^+(z)\phi^-(z) : \\ G^-(z) &=: \alpha^-(z)\psi^+(z) : + : \beta^-(z)\phi^+(z) : \end{aligned}$$

This symmetry can also be obtained via Getzler's Manin triple construction [G], where one takes the 4-dimensional abelian Lie algebra as the underlying Manin triple. It clearly decomposes into a direct sum of two copies of the $N = 2$ symmetry with underlying Manin triple being the 2-dimensional abelian Lie algebra. It is a unitary representation. We summarize our results in §4.

THEOREM 4.1. *Let $D \in W(1, 2+k)$ be a central derivation of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(2+k)$ and let α_D be the corresponding 2-cocycle Then D is compatible with the $N = 2$ superconformal algebra if and only if $D \in \mathrm{der}_{\mathbb{C}}(\Lambda(\theta_3, \cdots, \theta_{2+k}))$. Let $r = r_{\bar{0}} + r_{\bar{1}}$ be the rank of D as a map of $\Lambda(\theta_3, \cdots, \theta_{2+k})$. Let $\hat{L}_D$ be the central extension $\mathbb{C}[t, t^{-1}] \otimes \Lambda(2+k)/\ker\alpha_D$ corresponding to the 2-cocycle α_D, restricted to $\mathbb{C}[t, t^{-1}] \otimes \Lambda(2+k)/\ker\alpha_D$. We have $\hat{L}_D \cong (\oplus_{i=1}^{r_{\bar{1}}} \mathfrak{a}_i) \oplus (\oplus_{j=1}^{\frac{r_{\bar{0}}}{2}} \mathfrak{b}_i)/(z = z_1 = \cdots = z_{r_{\bar{1}}} =$*

$\tilde{z}_1 = \cdots = \tilde{z}_{\frac{r_{\bar{0}}}{2}}$) *as Lie superalgebras. Furthermore the* $N = 2$ *symmetry on* $\mathfrak{a}_i$ *is equivalent to the one in Proposition 3.2 and on* $\mathfrak{b}_i$ *it is equivalent to the Manin triple model with the 4-dimensional abelian Lie algebra as the underlying Manin triple. The total central charge of the* $N = 2$ *symmetry on the Fock space of* $\hat{L}_D$ *is* $r_{\bar{0}} - r_{\bar{1}}$.

Remark. We have taken S to be abelian for the following reason. Suppose that S is a Lie algebra equipped with a symmetric invariant nondegenerate bilinear form $(\cdot,\cdot)_S$. Let D define a cocycle α_D. We need to construct an $N = 2$ symmetry on a representation space of $S \otimes \hat{\mathrm{L}}_D$, where $\hat{\mathrm{L}}_D$ is as in Theorem 4.1 with the Lie bracket defined by $[s_1 \otimes l_1, s_2 \otimes l_2] = [s_1, s_2] \otimes l_1 l_2 + (s_1, s_2)_S \alpha_D(l_1, l_2) z$. However, although $\ker\alpha_D$ is an associative superalgebra, $S \otimes \hat{\mathrm{L}}_D$ is not for nonabelian S in general. This causes problem as to what space the $N = 2$ superconformal algebra should act on.

Acknowledgments. I would like to thank Victor G. Kač and Weiqiang Wang for discussions. Furthermore I want to thank the anonymous referee of Letters of Mathematical Physics who has pointed out the connection between our $c = -1$ model and the Fre et al model. Finally I like to express my gratitude to the Max-Planck-Institut für Mathematik for its hospitality and financial support.

REFERENCES

[C] Cheng, S.-J.: *Representations of Central Extensions of Differentiably Simple Lie Superalgebras,* Comm. Math. Phys. 154, 555 – 568 (1993).

[K] Kač, V.: *Lie Superalgebras,* Adv. in Math. 26 (1977).

[KT] Kač, V.;Todorov, I.: *Superconformal Current Algebras and Their Unitary Representations,* Comm. Math. Phys. 102, 337–347 (1985).

[T] Thielemans, K.: *A Mathematica Package for Computing Operator Product Expansions,* Int. Journal Mod. Phys. C 2, No.3, 787–798 (1991).

[KZ1] Kazama, Y.;Suzuki, H.: *Characterization of $N = 2$ Superconformal Models Generated by the Coset Space Method,* Phys. Letters B 216, 112–116 (1989).

[KZ2] Kazama, Y.;Suzuki, H.: *New $N = 2$ Superconformal Field Theories and Superstring Compactification,* Nuclear Phys. B 321, 232–268 (1989).

[P] Parkhomenko, S.: *Extended Superconformal Current Algebras and Finite Dimensional Manin Triples,* Sov. Phys. JETP 75, 1–3 (1992).

[CFRS] Coquereuax, R.;Frappat, L.;Ragoucy, E.; Sorba, P.: *Extended Super Kač-Moody Algebras and Their Super-Derivations Algebras,* Comm. Math. Phys. 133, 1–35 (1990).

[G] Getzler, E.:*Manin Pairs and Toplogical Field Theory,* hep-th. 9309057 (1993).

[S] Sugawara, H.: *A Field Theory of Currents,* Phys. Rev. 170, 1659–1662 (1968).

[SS] Schwimmer, A.; Seiberg, N.: *Comments on the $N = 2, 3, 4$ Superconformal Algebras in Two Dimensions,* Phys. Letters B 184, 191–196 (1987).

[FGLS] Fre, P.; Girardello, L.; Lerda, A.; Soriani, P.: *Topological First-order systems with Landau-Ginzburg interactions,* Nuclear Phys. B387, 333–372 (1992).

[W] Witten, E.: *On the Landau-Ginzburg description of N=2 minimal models,* IAS preprint IASSNS-HEP-93/10 (hep-th. 9304026) (1993).