Zakharov-Shabat technique with quantized spectral parameter in the theory of integrable models

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Abstract.

Two new integrable hierarchies are constructed: $MILW_n$ and the hierarchy of nonlocal two-dimensional Toda lattices. A new version of the Zakharov-Shabat technique with quantized (non-commutative) spectral parameter is proposed. We apply also our technique to investigate the ILW_n hierarchy. It is also worth mentioning that the zero-curvature equations we obtain are very similar to those which appear in application of non-commutative geometry techniques to the gauge theory of non-commutative two-tori.

1. Introduction.

Recently the Zakharov-Shabat technique with quantized spectral parameter was introduced [1]. This technique was applied to the construction of three types of integrable nonlocal hierarchies: ILW_n with $n = 1, 2$ [2 – 3], $MILW_n$ with $n = 2$ and nonlocal partners of two-dimensional Toda lattices [4].

In this note we will generalize this construction to the case of arbitrary n. The three hierarchies are deeply connected with each other: the $MILW_n$ hierarchy reduces to the ILW_n hierarchy via the generalized Miura transformation and the hierarchy of nonlocal two-dimensional Toda lattices is the hierarchy of equations for which $MILW_n$ hierarchy are the symmetry equations [5].

It is worth mentioning that matrix elements of $U\left(\hat{\Lambda}\right)$ and $V\left(\hat{\Lambda}\right)$ in eq. (9) below have the form $\sum_n f_n(x)\hat{\lambda}^n$, $\hat{\lambda} = \lambda e^{-2ih\partial_x}$ and therefore belong to the irrational rotation $C^* - algebra$ A_h which describes a non-commutative (quantum) 2–tori [6] (if the function $f_n(x)$ are defined on the circle). Therefore the nonlocal effects which appear in our theory are very similar to those which appear, for example, in the theory of two-dimensional Yang-Mills equations on noncommutative two-tori [7]. We would like to stress that it is very important to find the right point of switching on the non-commutative effects. In the theory of integrable equations the successful point is the Zakharov-Shabat dressing technique. For this reason we hope that our results might give additional intuition on the possible physical applications of A. Connes' philosophy of quantization of space-time manifolds [8].

We do not give any proofs in this paper. These can be obtained by straightforward generalization of earlier results given in [1], [3], [4]. In the same papers one can find also the simplest examples of our general equations.

2. ILW_n Hierarchy in Zero-Curvature Representation with Quantized Spectral Parameter.

The simplest representative of the ILW_n hierarchy is the ILW equation [2]:

$$u_t + 2uu_x + iT[u_{xx}] = 0 \ , \tag{1}$$

where

$$Tu(x) = \frac{1}{2h} P.V. \int_{-\infty}^{\infty} coth[\pi(y-x)/2h]u(y)dy$$

and $u_x = \partial u/\partial x$, $u_t = \partial u/\partial t$. Here, the real parameter h corresponds to the depth of stratified fluid.

To formulate our results concerning the ILW_n hierarchy it is useful to introduce some further notation.

Let

$$K(z, \partial_z) = 1 + \sum_{i \geq 1} K_i(z,t)\partial_z^{-i}, \ (\partial_z = \partial/\partial z)$$

where the functions $K_i(z,t)$ are holomorphic, bounded and continuous up to the boundary in the strip $\prod_{2h} = \{z \mid -2h < Im\, z < 0\}$ and t denotes the set of time-parameters $t_2, t_3 \cdots$. Define $K_i^+(z) = K_i(x)$, $K_i^-(z) = K_i(x - 2ih)$ where $x = Re\, z$ and $K^{\pm}(\partial_x) = 1 + \sum_{i \geq 1} K_i^{\pm}(z,t)\partial_x^{-i}$. (We will use also the notation $K^+(\partial_x) \equiv K(\partial_x))$.

Let $\hat{\lambda} = \lambda e^{-2ih\partial_x}$ and introduce the $n \times n$ matrices

$$\hat{\Lambda} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & \hat{\lambda} \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \cdots & \cdots & 1 & 0 \end{pmatrix} \equiv I + \hat{\lambda} e_{1,n} , \quad q^{can} = \begin{pmatrix} u_{n-1}, & \cdots, & u_0 \\ 0 & \cdots & 0 \\ & & \\ 0 & \cdots & 0 \end{pmatrix} . \tag{2}$$

One can define the following formal series with matrix coefficients

$$G\left(\hat{\Lambda}\right) = 1 + \sum_{j \geq 0} G_j(x,t)\hat{\Lambda}^{-(j+1)} , \tag{3}$$

where

$$(G_j(x,t))_{ab} = \begin{Bmatrix} C_{b-a}^{n-a}\partial_x^{b-a}K_j(x,t) , & b \geq a \\ & \\ 0 , & b < 0 \end{Bmatrix} \tag{4}$$

and $C_b^a = \frac{b!}{(b-a)!a!}$.

Now we are in position to state our results concerning the ILW_n case [3]. We will formulate them in a series of theorems.

Theorem 1 *(i) Fix* $G\left(\hat{\Lambda}\right)$ *by the equation*

$$G\left(\hat{\Lambda}\right)\left(\partial_x - \hat{\Lambda}\right)G^{-1}\left(\hat{\Lambda}\right) = \partial_x - \hat{\Lambda} + q^{can} \equiv \partial_x - U\left(\hat{\Lambda}\right) . \tag{5}$$

Then, the recurrence equations in the coefficients K_i *are the same as the equations which can be obtained from the equation*

$$K^-(\partial_x)\partial_x^n K(\partial_x)^{-1} = \partial_x^n + u_{n-1}\partial_x^{n-1} + \cdots + u_0 \equiv L \tag{6}$$

and therefore can be solved [3].

(ii) Let us require in addition that

$$G\left(\hat{\Lambda}\right)\left(\partial_{t_s} - \hat{\Lambda}^s\right)G^{-1}\left(\hat{\Lambda}\right) = \partial_{t_s} - V_s\left(\hat{\Lambda}\right) , \tag{7}$$

where $V_s\left(\hat{\Lambda}\right) = \sum_{j=0}^{s} V_{s,j}\hat{\lambda}^j$.

Equation (7) is equivalent to the equation

$$\left[K_{t_s}(\partial_x)K^{-1}(\partial_x) + K(\partial_x)\partial_x^s K^{-1}(\partial_x)\right]_- = 0 , \tag{8}$$

and therefore can be solved [3]. In the last formula the lower index "-" defines the projection onto the negative powers in ∂_x *.*

Comment. Appearance of additional field $u_{n-1}(x) = K_1(x-2ih) - K_1(x)$ in (5) and (6) is the consequence of the switching on "non-commutative tension" in x, λ direction in (5).

Using the Theorem 1 one can construct the zero-curvature equation

$$\left[\partial_x - U\left(\hat{\Lambda}\right),\ \partial_{t_s} - V_s\left(\hat{\Lambda}\right)\right] = 0 . \tag{9}$$

Following the usual arguments, the last equation can be rewritten to the form

$$q^{can}_{t_s} = \left[V_{s,0},\ \partial_x + q^{can} - I\right] , \tag{10}$$

where q^{can} and I are defined by eq. (2) and the matrix $V_{s,0}$ is the zero-order coefficient in the polynomial $V_s\left(\hat{\Lambda}\right) = \sum_{i\geq 0} V_{s,i}\hat{\lambda}^i$. Due to eq. (7) one would expect that the entries of $V_{s,0}$ consist of functions shifted in their argument by different multiples of h . Now, the nontrivial result is that the entries of $V_{s,0}$ either depend on x or on $(x-2ih)$ only. More exactly

Theorem 2. *(i) Let*

$$Grad\,\tilde{l}_X = \left\{ \begin{array}{ll} res\,\partial_x^{n-1} X\left(L\,\partial_x^{-(n+1)+j}\right)_+ , & j \leq i ; \\ -res\,\partial_x^{n-i} X\left(L\,\partial_x^{-(n+1)+j}\right)_- & \\ +\sum_{a=0}^{n-1} C^{a+n-i}_{a+j-i}\partial_x^{a+j-i} res\left(\partial_x^{-(a+1)} X L\right) , & j > i \end{array} \right\} \tag{11}$$

where

$$X = \sum_{i=1}^{\infty} \partial_x^{-i} \circ X_i = \left[K(\partial_x)\partial_x^{s-n} K^{-}(\partial_x)^{-1}\right]_- \tag{12}$$

and $res\left(\Sigma a_i \partial_x^i\right) = a_{-1}$ *. Then* $V_{s,0}$ *defined by eq. (7) is equal to*

$$V_{s,0} = Grad\,\tilde{l}_X \tag{13}$$

(ii) Equation (9) or (10) are equivalent to

$$L_{t_s} = (LX)_+ L - L(XL)_+ \ , \tag{14}$$

where X is defined by eq. (12).

Summarizing, we have proved that eq. (9) is equivalent to eqs. (14), (12), which define the ILW_n hierarchy [3].

Furthermore, there is a Hamiltonian formulation of eq. (10). Following [5] let $\mathcal{M}_I = \{\mathcal{L} = \partial_x - I + q\}$, where $q(x) \in b_+$ and $b_+ = n_+ + h$. Here n_+ is the algebra of upper triangular matrices with zeros on the diagonal and h is the algebra of diagonal matrices. The group $\tilde{N} = e^{\tilde{n}_+}$ acts on the space $\mathcal{M}_I$ by the gauge transformations. (Here $\tilde{n}_+$ denotes the Lie algebra of linear maps $R \to n_+$) . Let $\bar{\mathcal{M}}_I$ be the quotient space $\mathcal{M}_I/\tilde{N}$. One can realize $\bar{\mathcal{M}}_I$ as the linear space of differential operators $\mathcal{L}^{can} = \partial_x - I + q^{can}$, where q^{can} is defined by (2). The Poisson bracket on the space of functionals on $\bar{\mathcal{M}}_I$ is given by the Poisson bracket on the space $\mathcal{F}$ of gauge-invariant functionals on $\mathcal{M}_I$. This means that $\mathcal{F}$ consists of the functionals satisfying the condition $f(q) = f(\tilde{q})$, where $\tilde{q}$ is defined by $s^{-1}(\partial_x - I + q)s = \partial_x - I + \tilde{q}$. There is a natural Poisson bracket on $\mathcal{F}$:

$$\{f, g\} = \int tr\ grad_q f[grad_q g,\ \partial_x + q - I]\ dx \ , \tag{15}$$

where the $n \times n$ matrix $grad_q f$ is determined by the following equation

$$\frac{d}{d\epsilon} f(q + \epsilon h) \mid_{\epsilon=0} = \int tr(grad_q f\ h) dx \ , \tag{16}$$

where $h(x) \in \tilde{b}_+$. Of course, the matrix $grad_q f$ is defined by eq. (16) up to addition of an arbitrary matrix $\theta(x) \in \tilde{n}_+$. But it is easy to check that the Poisson bracket (15) on the gauge invariant functionals does not depend on the choice of $grad_q f$, i.e. is correctly defined on the functionals from the space $\mathcal{F}$. It is also easy to see that $\{f, g\}(\tilde{q}) = \{f, g\}(q)$, i.e. the Poisson bracket (15) is gauge invariant. (cf. for the proofs [5]).

Now, we can state the following properties of eqs. (10–13):

Theorem 3. *Let $\tilde{l}_X = \int\ res\ LX\ dx$, where L and X are defined by the eqs. (6) and (12). Then*

(i) If

$$H_s = \int res_\lambda \Big[\Big(i\lambda^n + \frac{n}{2h} \lambda^{n-1} \Big) \lambda^s ln \frac{K^-(\lambda)}{K^+(\lambda)} \Big] dx \ , \tag{17}$$

where $K(\partial_x)$ satisfies the condition (6) and $K^{\pm}(\lambda) = 1 + \sum_{i \geq 1} K_i^{\pm}(x) \lambda^{-i}$ with commutative parameter λ . Then

$$\delta\ H_{s-n} = \int tr\Big(Grad\ \tilde{l}_X\ \delta q^{can} \Big) dx \tag{18}$$

(ii) The functionals $H_s \in \mathcal{F}$ are in involution under the Poisson bracket (15).

(iii) The equation (10) is Hamiltonian under the Poisson bracket (15) with the Hamiltonian H_{s-n}.

3. $MILW_n$ Hierarchy in Zero-Curvature Representation with Quantized Spectral Parameter

To state the second block of theorems concerning $MILW_n$ hierarchy we introduce some further notation. Let

$$q^{diag} = diag(v_1(x,t), \cdots, v_n(x,t)) .$$

Define the matrix $S = ((S_{ij}))$ by

$$S_{ij} = res\left[\prod_{k=i+1}^{n} (\partial_x + v_k(x,t))\partial_x^{-(n+1)+j}\right] . \tag{19}$$

Theorem 4. *(i)*

$$S\left(\partial_x - \hat{\Lambda} + q^{can}\right)S^{-1} = \partial_x - \hat{\Lambda} + q^{diag} . \tag{20}$$

(ii) Let $\tilde{G}\left(\hat{\Lambda}\right) = SG\left(\hat{\Lambda}\right)$, where $G\left(\hat{\Lambda}\right)$ is the same as in Theorem 1, then

$$\tilde{G}\left(\hat{\Lambda}\right)\left(\partial_x - \hat{\Lambda}\right)\tilde{G}^{-1}\left(\hat{\Lambda}\right) = \partial_x - \hat{\Lambda} + q^{diag} \equiv \partial_x - \tilde{U}\left(\hat{\Lambda}\right) , \tag{21}$$

$$\tilde{G}\left(\hat{\Lambda}\right)\left(\partial_{t_s} - \hat{\Lambda}^s\right)\tilde{G}^{-1}\left(\hat{\Lambda}\right) = \partial_{t_s} - \tilde{V}_s\left(\hat{\Lambda}\right) , \tag{22}$$

where $\tilde{V}_s\left(\hat{\Lambda}\right) = \sum_{\alpha=0}^{s} \tilde{V}_{s,\alpha}\hat{\lambda}^\alpha$.

(iii) The equations (21)-(22) are equivalent to the following equations on $K(\partial_x)$:

$$K^-(\partial_x)\partial_x^n K(\partial_x)^{-1} = \prod_{i=1}^{n} (\partial_x + v_i) \tag{23}$$

$$\left[K_{t_s}(\partial_x)K(\partial_x)^{-1} + K(\partial_x)\partial_x^s K(\partial_x)^{-1}\right]_- = 0 . \tag{24}$$

Using (21)-(22) one can construct the zero-curvature equation

$$\left[\partial_x - \tilde{U}\left(\hat{\Lambda}\right),\ \partial_{t_s} - \tilde{V}_s\left(\hat{\Lambda}\right)\right] = 0 . \tag{25}$$

As in the previous, the last equation can be rewritten to the form

$$q^{diag}_{t_s} = \left[\tilde{V}_{s,0},\ \partial_x + q^{diag} - I\right] , \tag{26}$$

where $\tilde{V}_{s,0} = SV_{s,0}S^{-1} + S_{t_s}S^{-1}$ and $V_{s,o}$, S are defined by the equations (13) and (19). We will call eq. (26) as $MILW_n$ hierarchy. This equation satisfies the following properties:

Theorem 5. *(i) The equation (25) (or(26)) is equivalent to the equations on the functions* v_i :

$$v_{i,t_s} = -\partial_x\left[SV_{s,0}S^{-1}\right]_{i,i}, \quad i = 1,\cdots,n\,. \tag{27}$$

(ii) The Miura map defined by

$$\prod_{i=1}^{n}(\partial_x + v_i) = \partial_x^n + u_{n-1}\partial_x^{n-1} + \cdots + u_0 \tag{28}$$

transforms the $MILW_n$ *hierarchy (25)-(27) into the* ILW_n *hierarchy (9)-(10).*

(iii) Due to the Hamiltonian property of the Miura map, the conservation laws of $MILW_n$ *hierarchy are given by eq. (17) where we changed from the variables* u_i *to* v_i *using eq. (28).*

4. Nonlocal Partners for the Generalized Two-Dimensional Toda Lattice.

Let us define two first order differential operators

$$\begin{aligned} \mathcal{L} &= \partial_x - \hat{\Lambda} + \phi'\,, \\ \bar{\mathcal{L}} &= \partial_\tau - e^{-\phi}\hat{\Lambda}^{-1}e^{\phi}\,, \end{aligned} \tag{29}$$

where $\phi = diag(\phi_1,\cdots,\phi_n)$, $\phi_i = \phi_i(x,\tau,t)$ and $\hat{\Lambda}$ is defined by the eq. (2). Consider the zero-curvature equation

$$\left[\mathcal{L},\bar{\mathcal{L}}\right] = 0\,. \tag{30}$$

It is easy to check that eq. (30) is equivalent to the following equations on the fields ϕ_i :

$$\left\{\begin{array}{lcll} \phi_{1,x\tau} & = & e^{\phi_1(x,\tau,t)-\phi_n(x-2ih,\tau,t)} & - \quad e^{\phi_2(x,t)-\phi_1(x,t)} \\ \cdots\cdots\cdots & & & \\ \phi_{i,x\tau} & = & e^{\phi_i(x,\tau,t)-\phi_{i-1}(x,\tau,t)} & - \quad e^{\phi_{i+1}(x,\tau,t)-\phi_i(x,\tau,t)} \\ \cdots\cdots\cdots & & & \\ \phi_{n,x\tau} & = & e^{\phi_n(x,\tau,t)-\phi_{n-1}(x,\tau,t)} & - \quad e^{\phi_1(x+2ih,\tau,t)-\phi_n(x,\tau,t)} \end{array}\right\} \tag{31}$$

One important property of the eqs. (31) is

Theorem 6. *Let the* t_s- *evolution of* $\mathcal{L}$ *coincide with the* $MILW_n$ *eq. (25) where we have set* $q^{diag} = \phi'$. *Then*

$$\partial_{t_s}\left[\mathcal{L},\bar{\mathcal{L}}\right] = 0$$

i.e. the $MILW_n$ *equations are the symmetry equations to the* nonlocal *generalized 2–dimensional Toda lattice. This property is very similar to that one which appeared in the theory of the generalized* local *two-dimensional Toda lattice [5].*

Remark 1. It is impossible to reduce eqs. (29)-(31) putting $\phi_n = -\sum_{i=1}^{n-1} \phi_i$ i.e. to reduce the gl_n case to the sl_n one.

Remark 2. It is worth mentioning that eqs. (31) have a very similar structure to those in [9] but are different.

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References

1. A. Degasperis et al. Generalized Internal Long-Waves Hierarchy in Zero-Curvature Representation with Noncommutative Spectral Parameter: Preprint Roma University, 1990.

2. M. Ablowits, Y. Kodama, J. Satsuma: Phys. Rev. Lett. **46** (1981) 677; Phys. Lett. A **73** (1979) 283; J. Math. Phys. **23** (4) (1982) 564.

3. D. Lebedev, A. Radul: Comm. Math. Phys. **91** (1983) 543.

4. A. Degasperis et al. Nonlocal Integrable Partners to Generalized MKdV and Two-Dimensional Toda Lattice Equations in Formalism of Dressing Method with Quantized Spectral Parameter: Preprint Bonn-HE-90–14. Recent Developments for Integrable Integro-Differential Equations. Preprint Bonn-HE-90–13.

5. V. Drinfeld, V. Sokolov: Journal of Soviet Math. **30** (1985) 1975.

6. M. Pimsner, D. Voiculescu: J. of operator theory 4 (1980), 201–211.

A. Connes: C.R. Acad. Sc. Paris, t. 290 (1980) Serie A-599.

M. Rieffel: Pac. J. of Math. **95** (2) (1981), 415–429.

7. A. Connes and M. Rieffel: Contemp. Math. **62** (1987), 237.

M. Rieffel: J. of Diff. Geometry **31** (1990), 535.

8. A. Connes: Essay on Physics and Non-commutative Geometry, in The Interface of Mathematics and Particle Physics, Edited by D.G. Quillen, G.B. Segal, and Tsou S.T., Clarendon Press, Oxford, 1990.

9. M. Saveliev, A. Vershik: Comm. Math. Phys. **126** (1989) 367; Phys. Lett. **A 143** (1990), 121.