IMAGES OF LAGRANGIAN SUBMANIFOLDS, GENERATING FAMILIES FOR OPEN SWALLOWTAILS AND APPLICATIONS

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Summary. In this paper we study the symplectic relations appearing as the generalized cotangent bundle liftings of smooth stable mappings. Using this class of symplectic relations the classification theorem for generic (pre)images of lagrangian submanifolds is proved. The normal forms for the respective classified pullbacks and pushforwards are provided and the connections between the singularity types of symplectic relation, mapped lagrangian submanifold and singular image, are established. The notion of special symplectic triplet is introduced and the generic local models of such triplets are studied. We show that the open swallowtails are canonically represented as pushforwards of the appropriate regular lagrangian submanifolds. Using the $Sl_2(\mathbb{R})$ invariant symplectic structure on the space of binary forms of an appropriate dimension we derive the generating families for the open swallowtails and the respective generating functions for its regular resolutions. The regularly intersecting pairs of holonomic components are resolved using an appropriate reduction relation. Examples of singular images encountered in physics are given.

1. Introduction.

The classification of singular lagrangian submanifolds as the sets of rays tangent to the geodesic flows on a hypersurface is carried out in the previous papers [15],[3]. This is connected to the theory of nested hypersurfaces in a symplectic manifold describing the geodesics on a Riemannian manifold with boundary [14]. In particular it is closely related to the problem of the shortest bypassing of the obstacle represented by a smooth hypersurface [3],[2], which we can briefly formulate as follows: let $\mathbb{R}^{2n} = \{(x,y)\}$ be a phase space of particle in a classical mechanics [1], let $h(x,p) = \frac{1}{2}(p^2 - 1)$ be a Hamilton function for this particle. Then the space of bicharacteristics in $H = \{h=0\}$, say M, which forms a manifold of all oriented lines in $\mathbb{R}^n$ has a canonical symplectic structure. Let K be a hypersurface in $\mathbb{R}^n$ (an obstacle) and γ a geodesic flow on K (e.g. this one defined on K by the variational problem of shortest bypassing of K). It is proved in [2] that the set of oriented lines tangent to γ on K forms a lagrangian submanifold in M which is not necessary smooth. The appropriate local classification of these singular lagrangian submanifolds is carried out in the cited paper. It turned out that the generic singularities of this classification so-called open swallowtails can be conveniently described in the $Sl_2(\mathbb{R})$ invariant symplectic space of binary forms of an appropriate degree. We find that the open swallowtails can be obtained as images from the regular lagrangian submanifolds by means of a canonical symplectic procedure. This observation suggests the further generalization of the problem and classification of images of lagrangian submanifolds by means of (widely used in physical applications[5]) symplectic relations[16]. It appeared that the begining of the geometrical classification presented in this paper provides the new types of singular lagrangian submanifolds (cf.[12]).

The another motivation, for investigations presented here, comes from thermodynamics of phase transitions [12] and independently from statics of controlled mechanical systems[18], Let us consider the simple one-component thermodynamical system (cf.[12]) and admit the class of deformations onto two isolated subsystems of the same sample. The phase space for such deformations is following (cf.[12],[17]),

$$(T^*Y_1 \times T^*Y_2,\ -S_1 dT_1 - p_1 dV_1 + \mu_1 dN_1 - S_2 dT_2 - p_2 dV_2 + \mu_2 dN_2),$$

where $T^*Y_1, \{V_1, T_1, N_1, -p_1, -S_1, \mu_1\}$; $T^*Y_2, \{V_2, T_2, N_2, -p_2, -S_2, \mu_2\}$ are the phase spaces of the respective subsystems and V_i, T_i, N_i, p_i, S_i, μ_i are the standard thermodynamical coordinates. Let a lagrangian submanifold $L_1 \times L_2 \subset T^*Y_1 \times T^*Y_2$ be a space of equilibrium states of a composite isolated system. After removing of all (chemical, thermical, mechanical) constraints the virtual states of the system are defined by the coisotropic submanifold $C \subseteq T^*Y_1 \times T^*Y_2$ (cf.[5]),

$$C = \{T_1 = T_2,\ p_1 = p_2, \mu_1 = \mu_2,\ N_1 + N_2 = N = \text{const.}, N_1 > 0,\ N_2 > 0\}.$$

C provides the canonical characteristic submersion, say ρ, onto the phase space of composite system $(T^*Y, -SdT - pdV)$, $: C \longrightarrow T^*Y$, $\rho(V_1, T_1, N_1, p_1, S_1, \mu_1, V_2, T_2, N_2, p_2, \mu_2) = (V_1 + V_2, T_1, p_1, S_1 + S_2)$.
Hence the space of equilibrium states of composite system is an image $\rho(L_1 \times L_2)$, which for the Van der Waals gas forms a singular lagrangian submanifold in T^*Y well known in thermodynamics of coexistence states[12].

The aim of this paper is to set up a method of formalizing and generalizing these examples and derive the first results for further applications. We now outline the organization of the paper. In Section 2, in the begining, we introduce some known but perhaps unfamiliar results of symplectic geometry, which we shall need later on. Then we formulate the problem of classification of images of lagrangian submanifolds by means of the special classes of symplectic relations, namely these ones generated by modified pushforwards and pullbacks of smooth mappings. This classification forced us to introduce a notion of singular lagrangian submanifold and to prove some results concerning the generating families (useful physical potentials) of these classified images. Restricting considerations to the dimensions of symplectic manifolds not greater than four we prove the classification theorem for the normal forms of generic, generating families of the respective images of stable lagrangian submanifolds with respect to the stable mapping. This classification substantially depends on the results of [13] but provides the more exact description of singular images and their maximally reduced generating families. Section 3 is devoted to the investigation of local properties of general symplectic triplets. We show here the classification theorem for the so-called special symplectic triplets and derive the respective generating families for the respective lagrangian sets which its provide. In Section 4 we introduce the basis of Arnold's theory of open swallowtails represented in the symplectic space of binary forms. We show that the open swallowtails are provided by the appropriate special symplectic triplets. Using the methods of symplectic relations developed before we prove that the open swallowtails are images from the regular lagrangian submanifolds by the canonical symplectic reduction relation. This fact allow us to conduct the precise calculations for generating families of the open swallowtails and compare them to these ones for the respective special symplectic triplets.

2. Symplectic relations and images of lagrangian submanifolds.

Let (P_1,ω_1), (P_2,ω_2) be two symplectic manifolds (see[1]). We define the product $(P_1,\omega_1)\times(P_2,\omega_2)$ as the symplectic manifold $(P_1\times P_2, pr_1^*\omega_1 + pr_2^*\omega_2)$, where $pr_i\colon P_1\times P_2 \longrightarrow P_i$ $(i=1,2)$ are the cartesian projections. We define a symplectic relation from (P_1,ω_1) to (P_2,ω_2) as an immersed lagrangian submanifold of $(P_1,-\omega_1)\times(P_2,\omega_2)$ and denote it by R (see [16],[5]).

We recall a notion of symplectic relation of particular kind namely the symplectic reduction relation. Such relations are morphisms in the category of symplectic manifolds and are very widely used in mathematical physics (cf.[19],[17],[18],[5],[1],[14]). A submanifold $C\subseteq(P,\omega)$ is called coisotropic if at each $x\in C$

(1) $$(T_xC)^{\S} = \{v\in T_xP;\ \forall_{u\in T_xC}\ \langle v\wedge u,\omega\rangle = 0\} \subset T_xC.$$

Let $D = \{v\in TC;\ v \rfloor(\omega|_C) = 0\}$, we call D the characteristic distribution of C. Let B be the set of characteristics. We consider the following relation from P to B:

(2) $$R = \{(x,b)\in P\times B;\ x\in C,\ b=\rho(x)\},$$

where $\rho\colon C \longrightarrow B$ is the canonical projection. If B admits a differentiable structure and the map ρ is a submersion (cf. [19]) then there is a unique symplectic structure β on B such that

(3) $$\rho^*\beta = \omega|_C .$$

In this case (B,β) is called the reduced symplectic manifold, and R is a symplectic relation from (P,ω) to (B,β). R is called the symplectic reduction relation of the symplectic manifold (P,ω) with respect to a coisotropic submanifold C (see[5],[16]).

Let $R \subset (P_1 \times P_2, pr_2^*\omega_2 - pr_1^*\omega_1)$ be a symplectic relation and $L \subset P_1$ be a lagrangian submanifold of (P_1,ω_1). The set

(4) $R(L) = \{p_2 \in P_2;$ there exists $p_1 \in L$ such that $(p_1,p_2) \subset R\}$

is called an image of L under a symplectic relation R. Using the transpose relation ${}^tR$ (cf.[5]) we analogously define the counter-image of $N \subset (P_2,\omega_2)$, namely ${}^tR(N)$.

According to the purpose of this paper we confine our attention to the typical example of symplectic manifold, namely to cotangent bundle (i.e. the symplectic manifolds found in most applications are isomorphic to cotangent bundles [16],[12],[1]) (T^*X,ω_X) where $\omega_X = d\Theta_X$, and Θ_X is the Liouville form in the cotangent bundle T^*X (over a smooth manifold X).

Let (T^*X,ω_X), (T^*Y,ω_Y) be two cotangent bundles. The product

$$\Omega = (T^*X \times T^*Y,\ pr_2^*\omega_Y - pr_1^*\omega_X)$$

is a symplectic manifold which, for further purposes, will be identified with $T^*(X \times Y)$. Let $f\colon X \to Y$ be a smooth mapping. By $\Gamma f \subset X \times Y$ we denote the graph of f, Γf is a submanifold of $X \times Y$. Any function on Γf can be pulled-back onto X, so the smooth structure on Γf is equivalent to the smooth structure on X. As we know (see[17], Prop. 3.1) the set

(5) $\{p \in T^*(X \in Y);\ \pi_{X \in Y}(p) \in \Gamma f$ and $<u,p> = <u,d\bar{g}>$ for each $u \in T(\Gamma f) \subset T(X \times Y)$ such that $\tau_{X\times Y}(u) = \pi_{X\times Y}(p)\}$.

is a symplectic relation in Ω. Here $\bar{g}$ is a smooth function on Γf; $\pi_{X\times Y}$, $\tau_{X\times Y}$ are the projections for cotangent and tangent bundles respectively. Let g denotes the function $\bar{g}$ pulled-back to X.

Definition 2.1. Let $f\colon X \to Y$, $g\colon X \to \mathbb{R}$ be smooth functions. A symplectic relation, defined in (5) and denoted by (f,g), is called f-constrained symplectic relation. By F we denote the set of all f-constrained symplectic relations in Ω .

In the present paper we are interested only in local properties of symplectic relations as well as in local properties of images of lagrangian submanifolds. Hence X, Y will be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and instead of lagrangian submanifolds or mappings we shall consider in fact their germs (see[6]). Further on, to avoid an inessential rigour, we speak about mappings submanifolds etc. as representants of germs.

Let us introduce in F an action of a subgroup of the group of symplectomorphisms (an equivalence relation) such that for the images of lagrangian submanifolds this action reduces to the standard action (see[4],[22]) of the group of symplectomorphisms preserving the fibre structure of cotangent bundle. Hence we introduce in Ω the canonical action of the group $\mathcal{G} = G_X \times G_Y$, where by G_X (resp. G_Y) we denote the group of symplectomorphisms preserving the fibre structure of T^*X (resp. T^*Y). It is evident that $\mathcal{G}$ acts on Ω transforming a symplectic relation $R \subset \Omega$ onto $(\Phi,\Psi)(R)$, where $(\Phi,\Psi) \in \mathcal{G}$. As we know (see[7],[22]) a symplectomorphism $(\Phi,\Psi) \in \mathcal{G}$, locally, has the following form,

$$
(6) \qquad \begin{aligned} \Phi(x,\xi) &= (\varphi(x), {}^tD\varphi(x)^{-1}(\xi + d\alpha(x))): T^*X \longrightarrow T^*X, \\ \Psi(y,\eta) &= (\psi(y), {}^tD\psi(y)^{-1}(\eta + d\beta(y))): T^*Y \longrightarrow T^*Y, \end{aligned}
$$

where φ, ψ are diffeomorphisms, $\varphi: X \to X$, $\psi: Y \to Y$ and α, β are smooth functions on X and Y respectively. Thus the group $\mathcal{G}$ is defined as a system of functions and diffeomorphisms : $(\varphi,\alpha,\psi,\beta)$ with an appropriate composition formula.

By straightforward calculations, using[6]and definition of symplectic relation belonging to F , we obtain

<u>Proposition 2.2.</u> For the pairs (f,g) determining the respective

symplectic relations belonging to F we have the following transformation law

(7) $\quad (f,g) \longrightarrow (\Phi,\Psi)(f,g) = (\psi\circ f\circ\psi^{-1}, g\circ\varphi^{-1}+\beta\circ f\circ\varphi^{-1}-\alpha\circ\varphi^{-1})$.

Taking $\beta = 0$ and $\alpha = g$ we see that the second component in the right hand side of (7) vanishes. Thus we have

Corollary 2.3. For any orbit of action (7) there exists a representative of the form $(f,0)$, i.e. a pure lifting of f to cotangent bundle Ω ; in the sequel denoted by T^*f (cf.[19],[12]).

If we take the subgroup of G, say G', elements of which are determined by triplets (φ,α,Ψ) and act on a relation R by means of symplectomorphism $(\varphi,\alpha,\Psi,\alpha\circ f)$ then immediately we obtain,

Corollary 2.4. The action (7) restricted to the subgroup $G'\subset G$ is well defined action on the space, say F', of canonical liftings T^*f of smooth mappings $f: X \longrightarrow Y$ to Ω. An element of F' is represented by a pair $(f,0)$.

Let $R \in F$ and L, N be lagrangian submanifolds in (T^*X,ω_X) and (T^*Y,ω_Y) respectively.

Definition 2.5. Let $R = (f,g)$. The subset $R(L) \subset T^*Y$ $({}^tR(N)\subset T^*X)$ is called the pushforward of L (pullback of N respectively) with respect to R.

If f is an immersion (see[10]) then the pushforwards are always smooth lagrangian submanifolds of T^*Y. Analogously if f is a submersion then the respective pullbacks are smooth lagrangian submanifolds of T^*X. Moreover if $\pi_{Y|N}: N \longrightarrow Y$ and $f:X \longrightarrow Y$ are transversal mappings then the respective pullback ${}^tR(N)$ is a lagrangian submanifold of T^*X. Analogously for pushforwards if f has a constant rank and L is transversal to ${}^tR(T^*Y)$ then R(L) is a lagrangian submanifold of T^*Y.

In this paper we study the more general situation when the mentioned above, transversality conditions are not assumed and f is a stable mapping. The possible approach to classification of singular images (pullbacks and pushforwards) by specifying the various types of respective nontransversalities we shall study in forthcoming paper.

Let us denote a pushforward of L with respect to R by a pair (R,L) and similarly for a pullback of N with respect to R we use notation: (N,R).

Definition 2.6. The pushforwards (R_1,L_1), (R_2,L_2), (pullbacks: (N_1,R_1), (N_2,R_2)) are equivalent if there exists $g \in G$, $g = (\Phi,\Psi)$ such that

$$(R_2,L_2) = (g(R_1),\Phi(L_1)) \tag{8}$$

$((N_2,R_2) = (\Psi(N_1),g(R_1))$ respectively).

Before we prceed to classification of images we recall the very convenient notion of Morse family. It is well known (see [19], [5]) that any lagrangian submanifold L of T^*X, can be locally generated by a family of functions, so-called Morse family, F: $X \times \mathbb{R}^k \longrightarrow \mathbb{R}$ (for some $k \in \mathbb{N}$, $k \leq \dim X$) so that

$$L = \{(x,\xi);\ \xi = \frac{\partial F}{\partial x}(x,\lambda),\ 0 = \frac{\partial F}{\partial \lambda}(x,\lambda)\}, \tag{9}$$

where $\operatorname{rank}(\frac{\partial^2 F}{\partial \lambda^2}, \frac{\partial^2 F}{\partial \lambda \partial x}) = k$ in an appropriate source point of the germ F. Between Morse families, with a minimal number of parameters (see [22]), there is a following equivalence: two Morse families (or generating families as below) F, F : $X \times \mathbb{R}^k \longrightarrow \mathbb{R}$ are equivalent if there exists a diffeomorphism $\Xi: X \times \mathbb{R}^k \longrightarrow X \times \mathbb{R}^k$, $\rho_X \circ \Xi = \rho_X$, such that $F = F' \circ \Xi$, where $\rho_X: X \times \mathbb{R}^k \longrightarrow X$ is the projection. Let us notice that the equivalent Morse families represent the same lagrangian submanifold of T^*X. For the proof of the inverse statement see e.g. [22]. In this paper, most frequently, we use rather the following notion.

Definition 2.7. A family of functions on X, which describes a lagrangian subset in T^*X (it can be nondifferentiable but endowed with the Whitney stratification [11], the maximal strata of which are lagrangian) by the formula (9), not necessary with the rank assumption, is called a generating family for the considered lagrangian subset.

Proposition 2.8. Let $L\subset T^*X$, $N\subset T^*Y$ be lagrangian submanifolds generated by Morse families, say $G: X\times\mathbb{R}^k \longrightarrow \mathbb{R}$ and $F: Y\times\mathbb{R}^l \longrightarrow \mathbb{R}$ respectively. Let $R = (f,g)\in F$, then the image of L and N with respect to R have the following generating families:

i) for pushforward (R,L); $P: Y\times\mathbb{R}^M \longrightarrow \mathbb{R}$,

$$P(y;\lambda,\mu,\nu) = \sum_{i=1}^{m} \lambda_i(y_i-f_i(\mu)) + g(\mu) + G(\mu,\nu),$$

where $\mu = (\mu_1,\dots,\mu_n)$, $\nu = (\nu_1,\dots,\nu_k)$, $M \le m+n+k$,

ii) for pullback (N,R); $H: X\times\mathbb{R}^l \longrightarrow \mathbb{R}$,

$$H(x;\lambda) = F(f(x),\lambda) - g(x),$$

in respective local Darboux coordinates on T^*X and T^*Y.

Proof. On the basis of (5) and Lecture 6 in [19](see also[17]) a Morse family for the relation R is following

$$A(x,y;\lambda) = \sum_{i=1}^{m} \lambda_i(y_i-f_i(x)) + g(x),$$

i.e. R locally can be expressed by the following equations

$$-\xi_j = -\sum_{i=1}^{m}\lambda_i \frac{\partial f}{\partial x_j}(x) + \frac{\partial g}{\partial x_j}(x), \qquad 1\le j\le n$$

$$\eta_r = \lambda_r, \quad 1\le r\le m.$$

L is described by the equations $\xi_j = \frac{\partial G}{\partial x_j}(x,\nu)$, $0 = \frac{\partial G}{\partial \nu_i}(x,\nu)$, $1\le j\le n$, $1\le i\le k$. Hence using (4) for (R,L) we obtain i). By the same way, reducing only an appropriate part of parameters (as for the stable equivalence in[22]) we obtain ii).

On the basis of (7), (8) and Proposition 2.8 we obtain immediately.

<u>Corollary 2.9.</u> Let $P(y;\lambda,\mu,\nu)$, $H(x;\lambda)$ be generating families for pushforward (R,L) and pullback (N,R) respectively as in Proposition 2.8, then the respective generating families, for equivalent pushforward and pullback, are following

$$\tilde{P}(y,\lambda,\mu,\nu) = \sum_{i=1}^{m} \lambda_i(y_i-(\psi\circ f\circ\varphi^{-1})_i(\mu))+g\circ\varphi^{-1}(\mu)+\beta\circ f\circ\varphi^{-1}(\mu)+G(\varphi^{-1}(\mu),\nu),$$

$$\tilde{H}(x,\lambda) = F(f\circ\varphi^{-1}(x),\lambda) - g\circ\varphi^{-1}(x)+\alpha\circ\varphi^{-1}(x),$$

where the equivalent symplectic relation R has a form (7).

Now we provide the begining of classification of normal forms for the appropriate pushforwards and pullbacks. Let us denote by (Σ^{ijk},A_r) for pushforward and (A_r,Σ^{ijk}) for pullback, the types of the respective equivalence classes, where Σ^{ijk} is a Boardman symbol of $f\colon (X,x_0)\to Y$ (cf.[11]) and A_r is a singularity type of L (or N), (cf.[4]) at a source or target point of the germ of symplectic relation R.

<u>Proposition 2.10.</u> Let $\dim X$, $\dim Y < 3$, then the normal forms for the generating families of generic pushforwards and pullbacks of the appropriate types are given in the following table

n,m	type	$P: Y\times\mathbb{R}^N \to \mathbb{R}$	type	$H: X\times\mathbb{R}^1 \to \mathbb{R}$
1,1	(Σ^0,A_1)	0	(A_1,Σ^0)	0
	(Σ^0,A_2)	$\lambda y+\lambda^3$	(A_2,Σ^0)	$\lambda^3+\lambda x$
	(Σ^{10},A_1)	λy	(A_1,Σ^{10})	0
	(Σ^{10},A_2)	λy	(A_2,Σ^{10})	$\lambda^3 \pm \lambda x^2$
1,2	(Σ^0,A_1)	λy_2	(A_1,Σ^0)	0
	(Σ^0,A_2)	$\lambda_1^3+\lambda_1y_1+\lambda_2y_2$	(A_2,Σ^0)	$\lambda^3+\lambda x$
			(A_3,Σ^0)	$\lambda^4+\lambda^2x+\lambda\varphi(x)$, $\varphi(0)=0$

	(Σ^1, A_1)	$0;\ \lambda^3 + y\lambda$	(A_1, Σ^1)	0
	(Σ^1, A_2)	$-y^3 + 0(y^4);\ \lambda^3 + y\varphi(\lambda), \varphi'(0) \neq 0$	(A_2, Σ^1)	$\lambda^3 + x_1\lambda$
2,1	(Σ^1, A_3)	$y^2\varphi(y)$		
	(Σ^{20}, A_1)	λy	(A_1, Σ^{20})	0
	(Σ^{20}, A_2)	λy	(A_2, Σ^{20})	$\lambda^3 + \lambda(\pm x_1^2 \pm x_2^2)$
	(Σ^{20}, A_3)	λy		
	(Σ^0, A_1)	0	(A_1, Σ^0)	0
	(Σ^0, A_2)	$\lambda^3 + y_1\lambda$	(A_2, Σ^0)	$\lambda^3 + x_1\lambda$
	(Σ^0, A_3)	$\lambda^4 + \lambda^2 y_1 + y_2\lambda$	(A_3, Σ^0)	$\lambda^4 + \lambda^2 x_1 + x_2\lambda$
	(Σ^{10}, A_1)	$-\lambda_2\lambda_1^2 \pm \lambda_1^2 + \lambda_2 y_2 + \lambda_1 y_1$	(A_1, Σ^{10})	0
	(Σ^{10}, A_2)	$\lambda_1 y_1 + \lambda_2 y_2 - \mu_1\lambda_1 - \lambda_2\mu_2^2 + \nu^3 + \nu(\mu_1 + \mu_2) + a\mu_2^2 + \mu_2\varphi(\mu_1, \mu_2^2)$	(A_2, Σ^{10})	$\lambda^3 + x_1\lambda$
2,2	(Σ^{10}, A_3)	$\lambda_1 y_1 + \lambda_2 y_2 - \lambda_1\mu_1 - \lambda_2\mu_2^2 + \nu^4 + \nu^2(\mu_1 + \mu_2) + \nu\varphi_2(\mu_1, \mu_2) + a\mu_2^2 + \mu_2\varphi_1(\mu_1, \mu_2^2)$	(A_3, Σ^{10})	$\lambda^4 + \lambda^2 x_1 + \lambda(x_2^2 + \varphi(x_1))$
	(Σ^{110}, A_1)	$\lambda_1(y_1 - \mu_1) + \lambda_2(y_2 - \mu_1\mu_2 - \mu_2^3) \pm \mu_2^2 + \mu_2\varphi(\mu_1\mu_2 + \mu_2^3)$	(A_1, Σ^{110})	0
	(Σ^{110}, A_2)	$\lambda_1(y_1 - \mu_1) + \lambda_2(y_2 - \mu_1\mu_2 - \mu_2^3) + \nu^3 + \nu(\mu_2 + \varphi_3(\mu_1, \mu_1\mu_2 + \mu_2^3)) + \mu_2\varphi_1(\mu_1, \mu_1\mu_2 + \mu_2^3) + \mu_2^2\varphi_2(\mu_1, \mu_1\mu_2 + \mu_2^3) + a\mu_2^3$	(A_2, Σ^{110})	$\lambda^3 \pm x_1\lambda$
	(Σ^{110}, A_3)	$\lambda_1(y_1 - \mu_1) + \lambda_2(y_2 - \mu_1\mu_2 - \mu_2^3) +$	(A_3, Σ^{110})	$\lambda^4 \pm \lambda^2 x_1 + \lambda(\varphi_1(x_1) +$

	$+\mu^4+\mu^2(\mu_2+\varphi_3(\mu_1\mu_2+\mu_2^3,\mu_1))+$ $+\nu\varphi_4(\mu_1,\mu_2)+\mu_2\varphi_1(\mu_1,\mu_1\mu_2+\mu_2^3)+$ $+ {}_2^2\varphi_2(\mu_1,\mu_1\mu_2+\mu_2^3)+ a\mu_2^3$		$+\varphi_2(x_1)(x_2^3+x_1x_2)),$ $\varphi_2(0) > 0$

Proof. By assumption $\dim X$, $\dim Y \leqq 2$. The types (cf.[4],[11]) for f, L, N are these ones listed in the table. Hence for the respective germs, say for $L \subset T^*X$, we use the following Morse families (cf.[22]),

$$
(10)\qquad
\begin{aligned}
A_1&: G(x,\lambda) = g_o(x),\\
A_2&: G(x,\lambda) = \lambda^3+g_1(x)\lambda + g_o(x),\\
A_3&: G(x,\lambda) = \lambda^4+g_1(x)\lambda^2+g_2(x)\lambda+g_o(x)\ldots
\end{aligned}
$$

Following [12] the classification of germs of pushforwards (R,L), restricting R to that one belonging to $\mathcal{F}$ (see Corollary 2.3), reduces to classification of germs of mapping diagrams

$$(\mathbb{R}^1\times\mathbb{R}^s,0) \xleftarrow{(g_o,g)} (\mathbb{R}^n,0) \xrightarrow{f} (\mathbb{R}^k,0),$$

where $s = 0,1,2$, $g = g_1$ or $g = (g_1,g_2)$ for $s=1$ or 2 respectively, with respect to the equivalence relation represented by the following commuting diagram

$$
\begin{tikzcd}
(\mathbb{R}^1\times\mathbb{R}^s,0) \arrow[d, "id\times(id+\alpha\circ f)"'] & (\mathbb{R}^n,0) \arrow[l, "(g_o,g)"'] \arrow[r, "f"] \arrow[d, "h"] & (\mathbb{R}^k,0) \arrow[d, "h"] \\
(\mathbb{R}^1\times\mathbb{R}^s,0) & (\mathbb{R}^n,0) \arrow[l, "(g_o,g)"'] \arrow[r, "f"] & (\mathbb{R}^k,0)
\end{tikzcd}
$$

Analogous but a bit more simple situation there is for pullbacks. Here any pullback of type (A_1,Σ^{ijk}) can be reduced to normal form with the trivial Morse family. Hence the problem of classification of pullbacks can be reduced to the problem of finding the normal forms for mapping diagrams (cf.[4],[12]):

$$(R^n,0) \xrightarrow{\ f\ } (R^k,0) \xrightarrow{\ g\ } (R^l,0), \ (l=1,2)$$

with a given type of singularity of the germ f and endowed with the following equivalence relation (cf. [4]),

$$\begin{tikzcd}
(\mathbb{R}^n,0) \arrow[r, "f"] \arrow[d, "h"] & (\mathbb{R}^k,0) \arrow[d, "\tilde{h}"] \arrow[dr, "g"] & \\
(R^n,0) \arrow[r, "f'"] & (R^k,0) \arrow[r, "g'"'] & (R^l,0)
\end{tikzcd}$$

where $g_0 = 0$, $g = g_1$ or $g = (g_1, g_2)$.

Applying the Malgrange Preparation Theorem (cf.[6]), the generalized Morse Lemma and method of liftable and lowerable vector fields (cf.[4]) we obtain the following list of normal forms for generic pairs (R,L) and (N,R) defining the pushforwards and pullbacks respectively (see also [13]).

Pushforwards:

<u>(n,k)=(1,1)</u>

$f(x) = x$: A_1: $g_0 = 0$; A_2: $(g_0,g)(x) = (0,x)$,

$f(x) = x^2$: A_1: $g_0(x) = \pm x^2 + ax^3$; A_2: $(g_0,g)(x) = (ax^2 + x^3\varphi_1(x^2), x + \varphi_2(x^2))$

<u>(n,k)=(1,2)</u>

$f(x) = (x,0)$: $g_0 = 0$; A_2: $(g_0,g)(x) = (0,x)$

<u>(n,k)=(2,1)</u>

$f(x) = x_1$: A_1: $g_0(x) = \pm x_2^2$ or $g_0(x) = x_1x_2 + x_2^3$; A_2: $(g_0,g)(x) = (x_1x_2 + x_2^2\varphi(x), x_2)$ or $(g_0,g)(x) = (x_2\varphi(x), x_1 \pm x_2^2)$, A_3: $(g_0,g)(x) = (x_2\varphi_1(x), x_2, x_1 + x_2\varphi_2(x))$,

$f(x) = x_1^2 \pm x_2^2$: A_1: $g_0(x) = \pm x_1^2 + ax_2^2 + x_1\varphi_1(f(x)) + x_2\varphi_2(f(x))$, $a \neq 1$;

A_2: $(g_0,g)(x) = (ax_1^2 + x_1\varphi_1(f(x)) + x_2\varphi_2(x), x_1 + \varphi_3(f(x)))$;

A_3: $(g_0,g)(x) = (ax_1^2 + x_1\varphi_1(f(x)) + x_2\varphi_2(x), x_1 + \varphi_3(f(x)), \varphi_4(x))$,

$\underline{(n,k)=(2,2)}$

$f(x)=x$: A_1: $g_o=0$; A_2: $(g_o,g)(x)=(0,x_1)$; A_3: $(g_o,g)=(0,x_1,x_2)$,

$f(x)=(x_1,x_2^2)$: A_1: $g_o(x)=x_1x_2\pm x_2^2$; A_2:$(g_o,g)(x)=(ax_2^2+x_2\varphi(f(x)),$ $x_1+x_2)$; A_3:$(g_o,g) = (ax_2^2+x_2\varphi_1(f(x)),x_1+x_2,\varphi_2(x))$.

$f(x)=(x_1,x_1x_2+x_2^3)$: A_1: $g_o(x)=\pm x_2^2+x_2\varphi(x_1x_2+x_2^3)+ax_2^3$; A_2:$(g_o,g) =$ $(x_2\varphi_1(f(x))+x_2^2\varphi_2(f(x))+ax_2^3,x_2+\varphi_3(f(x)))$; A_3: $(g_o,g)(x)=(x_2\varphi_1(f(x))+x_2^2\varphi_2(f(x))+ax_2^3,$ $x_2+\varphi_3(f(x)),\varphi_4(x))$.

Pullbacks:

$\underline{(n,k)=(1,1)}$

$f(x)=x$: A_2: $g(x)=x$,

$f(x)=x^2$: A_2: $g(x)= \pm x^2$,

$\underline{(n,k)=(1,2)}$

$f(x)=(x,0)$: A_2: $g(x)=x$; A_3: $g(x)=(x,\varphi(x))$,

$\underline{(n,k)=(2,1)}$

$f(x)=x_1$: A_2: $g(x)=x_1$,

$f(x)= x_1^2\pm x_2^2$: A_2: $g(x)= x_1^2\pm x_2^2$.

$\underline{(n,k)=(2,2)}$

$f(x)=(x_1,x_2)$: A_2: $g(x)=x_1$; A_3: $g(x)=(x_1,x_2)$,

$f(x)=(x_1,x_2^2)$: A_2: $g(x)=x_1$; A_3: $g(x)=(x_1,\varphi(x_1)+x_2^2)$,

$f(x)=(x_1,x_2^3+x_1x_2)$: A_2: $g(x)= \pm x_1$; A_3: $g(x)=(\pm x_1,\varphi_1(x_1)+$ $+(x_2^3+x_1x_2)\varphi_2(x_1))$.

Using these normal forms and Proposition 2.8 we can write down the generating families for the respective images of $L\subset T^*X$ and $N\subset T^*Y$. It is easy to check that $R(L)$, for the listed above types, are germs of smooth lagrangian submanifolds in T^*Y. Hence on the basis of [22] Theorem 4 and providing some calculations, we can conduct further reduction of number of parameters for generating families of pushforwards. Thus the proof of Proposition 2.10

is completed.

Let us abbreviate the notation for pullbacks and pushforwards writting $(A_r, \Sigma^{ijk})_{(n,m)}$ and $(\Sigma^{ijk}, A_r)_{(n,m)}$ respectively. Thus on the basis of Proposition 2.10 almost immediately we obtain

Corollary 2.11. For the generic pullbacks and pushforwards listed in Proposition 2.10 we have the following relations

$(\Sigma^0, A_i)_{(1,1)} = A_i$,

$(\Sigma^{10}, A_i)_{(1,1)} = \begin{pmatrix}\text{constrained lagrangian submanifold, unstable} \\ \text{in the standard sense (cf.[22],[12],[16])}\end{pmatrix}$

$(\Sigma^0, A_i)_{(1,2)} =$ (constrained lagrangian submanifold), $(i=1,2)$,

$(\Sigma^1, A_1)_{(2,1)} = \{A_1, A_2\}$,

$(\Sigma^1, A_2)_{(2,1)} = \{A_1, A_2\}$,

$(\Sigma^1, A_3)_{(2,1)} = A_1$,

$(\Sigma^{20}, A_i)_{(2,1)} =$ (constrained lagrangian submanifold), $(i=1,2,3)$,

$(\Sigma^0, A_i)_{(2,2)} = A_i$, $(i=1,2,3)$,

$(\Sigma^{10}, A_i)_{(2,2)} =$ (unstable), $(i=1,2,3)$,

$(\Sigma^{110}, A_i)_{(2,2)} =$ (unstable), $(i=1,2,3)$,

$(A_i, \Sigma^0)_{(1,1)} = A_i$, $(i=1,2)$,

$(A_1, \Sigma^{10})_{(1,1)} = A_1$,

$(A_2, \Sigma^{10})_{(1,1)} =$ (singular),

$(A_i, \Sigma^0)_{(1,2)} = A_i$, $(i=1,2)$,

$(A_3, \Sigma^0)_{(1,2)} =$ (unstable smooth if $\varphi'(0) \neq 0$),

$(A_i, \Sigma^1)_{(2,1)} = A_i$, $(i= 1,2,)$,

$(A_1, \Sigma^{20})_{(2,1)} = A_1$,

$(A_2, \Sigma^{20})_{(2,1)} =$ (singular),

$(A_i, \Sigma^0)_{(2,2)} = A_i$, $(i=1,2,3)$,

$(A_i, \Sigma^{10})_{(2,2)} = A_i$, $(i=1,2)$,

$(A_3, \Sigma^{10})_{(2,2)} =$ (singular),

$(A_i, \Sigma^{110})_{(2,2)} = A_i$, $(i=1,2)$,

$(A_3, \Sigma^{110})_{(2,2)} =$ (singular).

Example 2.12. The analogous phenomenon as for the unstable pushforwards in Proposition 2.10 appears in many mechanical and thermodynamical systems (see e.g. [18],[12]). Let Y be the Euclidean plane. Equations

$$\eta_1 = r\cos\theta\,,\quad \eta_2 = r\sin\theta\,,$$
$$y_1 = -k(r-a)\cos\theta\,,\quad y_2 = -k(r-a)\sin\theta$$

describe a lagrangian submanifold N of T^*Y with coordinates (θ, r), $0 \leq \theta < 2\pi$, $-\infty < r < \infty$. (N can be obtained as a canonical pushforward, see [18]). N represents the position-force relation for a point subject to a simple restoring force whose centre of attraction is allowed to move freely on the circle $\eta_1^2 + \eta_2^2 = a^2$. We see that for $r=a$, $T_0^*Y \cap N$ is the circle $\eta_1 = a\cos\theta$, $\eta_2 = a\sin\theta$ and for $(y_1, y_2) \neq 0$, N is transversal to the fibers $T^*_{(y_1,y_2)}Y$. Hence N is unstable lagrangian submanifold likes these ones listed in Proposition 2.10. The respective, physically realisable reduction relation R and lagrangian submanifold L is constructed in [18].

Remark 2.13. Let us notice that an every symplectic relation R, in general, is locally generated by the Morse family $(x,y,\lambda) \longrightarrow G(x,y,\lambda)$, ($\lambda$-paremeter). The classification of images (preimages) for more arbitrary (than these ones considered in this paper) symplectic relations R can be conducted using the following symplectic equivalence: let R_X, R_Y be two symplectic relations in $(T^*X \times T^*X, \pi_2^*\Theta_X - \pi_1^*\Theta_X)$ and in $(T^*Y \times T^*Y, \pi_2^*\Theta_Y - \pi_1^*\Theta_Y)$ respectively, representing the appropriate elements of the group of symplectomorphisms of T^*X and T^*Y respectively(cf.[17]). We say that the symplectic relations R, $R' \subset (T^*X \times T^*Y, \pi_2^*\Theta_Y - \pi_1^*\Theta_X)$ are equivalent if there exist relations R_X, R_Y such that

$$R' = R_X \circ R \circ R_Y, \text{ (cf. 10).}$$

If $G_X: X \times X \longrightarrow \mathbb{R}$, $G_Y: Y \times Y \longrightarrow \mathbb{R}$ are Morse families for R_X and R_Y respectively then R' has a following Morse family

(11) $$G'(x,y;\mu,\nu,\lambda) = G_X(x,\mu) + G(\mu,\nu,\lambda) + G_Y(\nu,y).$$

It is easily seen that if $X = \mathbb{R}^1$, $Y = \mathbb{R}^1$ and R is transversal to the fibres of $T^*(X \times Y)$ then we can reduce G to the normal form:

$$G(x,y;\mu,\nu) = x\nu + y\mu + \nu\mu f(\nu,\mu).$$

It seems to be interesting the classification problem for images of lagrangian submanifolds with respect to more general class (than this one considered here) of symplectic relations. The more consequent analysis of this problem we leave to the forthcoming paper.

3. Special symplectic triplets.

Now we pass to the images of lagrangian submanifolds provided by symplectic reduction relation defined by hypersurface H in a symplectic manifold (P,ω). The first, nontrivial step in the study of mutual intersection of lagrangian submanifold $X \subseteq P$ and hypersurface $H \subseteq P$ was done in [15] and[2]. It turned out that the nontransversal positions of X and H, i.e. a mutual tangency of the first order along the hypersurface $H \cap X$ of X, so-called symplectic triplets $(H, X, H \cap X)$ provide the singular images $\rho(X)$ (see [2]), which are encountered in variational calculus of physical systems [2],[3] and in boundary value problems for differential operators [14] ,[15].

It is easy to establish that at any point, say $p \in H \cap X$, (for a symplectic triplet $(H, X, H \cap X)$ one can choose a local special symplectic structure on P (see [17] and [20], Theorem 4.1.) so-called Weinstein symplectic structure $T^*X \cong P$, such that

$$(12) \qquad \begin{aligned} &H = \{(x,\xi)\in T^*X;\ h(x,\xi) = \sum_{i=1}^{n} a_i(x,\xi)\xi_i + \chi(x) = 0\}, \\ &l = H\cap X = \{x\in X;\ \chi(x)=0\} \text{ is a submanifold of codim. 1,} \end{aligned}$$

where a_i, χ, $1 \leq i \leq n$ are smooth functions and in addition, graph $\chi \subseteq X\times \mathbb{R}$ has a first order tangency to X along l.

Definition 3.1. Let $(H, X, l=H\cap X)$ be a symplectic triplet in (P,ω). We say that it is a special symplectic triplet if there exists Weinstein symplectic structure, say T^*X, such that h generates a hamiltonian flow preserving this structure.

Locally a special symplectic triplet is described by (12) with an additional assumption that $h(x,\xi) = \sum_{i=1}^{n} a_i(x)\xi_i + \chi(x)$. We see that the characteristics (provided by h) on X are defined by the vector field $V = \sum_i a_i(x)\frac{\partial}{\partial x_i}$. Using the symplectomorphisms preserving an affine form of h and zero section X (i.e. a class of special symplectic triplets, see [9] for contact equivalence) as well as the standard equivalence for Hamiltonians (i.e. $h \sim h'$ iff $h = ah'$ for some smooth function a, such that $a(0)\neq 0$) we obtain the following result,

Proposition 3.2. Let (H,X,l) be a special symplectic triplet, then generically in a neighbourhood of any point of l, (H,X,l) can be reduced to one from the following normal forms

$$(13) \qquad \begin{aligned} &H_k = \{(x,\xi)\in T^*X;\ h = \xi_1 \pm a(x_1^{k+1}+x_2x_1^{k-1}+\dots+x_{k+1})^2 = 0 \quad , \\ &l_k = \{x\in X;\ x_1^{k+1}+x_2x_1^{k-1}+\dots+x_{k+1} = 0\}, \end{aligned}$$

where $k \leq \dim X-1$, $a:(T^*X,0) \longrightarrow \mathbb{R}$ and $a(0)\neq 0$.

Proof. We show, at first, that a germ $\chi:(X,0)\longrightarrow \mathbb{R}$, defining a symplectic triplet as in (12) can be brought to the form g^2 for some smooth function-germ $g:(X,0) \longrightarrow \mathbb{R}$ defining hypersurface l. Let us take the coordinates on X, and simultaneously the symplectic coordinates on T^*X by cotangent bundle lifting (see[19]), such that $l = \{x \in X;\ x_1=0\}$. So $\chi(x)=x_1g_1(x)$ and, since l is a hypersurface of nonisolated critical points for f, i.e. $\nabla\chi|_l=(g_1(x)+x_1\frac{\partial g_1}{\partial x_1}(x), x_1\frac{\partial g_1}{\partial x_2}(x),\ldots,x_1\frac{\partial g_1}{\partial x_n}(x)) = 0$, thus $g_1(x)=x_1g_2(x)$. By assumption of the first order tangency of graph χ to X we have $g_2(0)\neq 0$. Hence we can write $\chi = \pm g^2$, where $g(x)=x_1\sqrt{\pm g_2(x)}$.

The vector field $V=\sum_i a_i(x)\frac{\partial}{\partial x_i}$ can be straightened in a neighbourhood of the considered point, so that $\varphi^*V=\frac{\partial}{\partial x_1}$ for some diffeomorphism $\varphi:(\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$. Taking the canonical lifting of φ to T^*X we obtain the following normal form for h (for an equivalent special symplectic triplet), namely

$$h(x,\xi) = \xi_1 \pm g^2(x). \tag{14}$$

Now we have the natural group of equivalences for integral curves of $\frac{\partial}{\partial x_1}$, i.e. diffeomorphisms germs preserving the fibre structure $(x_1,x_2,\ldots,x_n) \xrightarrow{\pi} (x_2,\ldots,x_n)$. Using these equivalences we reduce the problem of description of mutual generic positions of characteristics of h and the submanifold $l(\{g(x)=0\})$ to the classification problem for Whitney's projections (see[2],[11],[6])

$$\pi|_l = l \longrightarrow \mathbb{R}^{n-1}.$$

Hence (14) can be brought into the following normal form

$$h(x,\xi)=\xi_1 b(x) \pm (x_1^{k+1}+x_2x_1^{k-1}+\ldots+x_{k+1})^2,$$

where $b(0)\neq 0$. Taking an equivalent Hamiltonian for H we obtain (13). Thus the proof of Proposition 3.2 is completed.

For any special symplectic triplet (H,X,l) there exists a canonical special symplectic structure on the space (B,β) of characteristics (bicharacteristics) on H (cf. (3)), say T^*Y such that for the reduction relation

(15) $$R = \{(p_1,p_2) \in T^*X \times T^*Y;\ p_2 = \rho(p_1),\ p_1 \in H\},$$

where $\rho : H \longrightarrow T\,Y$ is the canonical projection onto bicharacteristics, we have a following commuting diagram

(16) $$\begin{array}{ccc} T^*X \supseteq H & \xrightarrow{\ \rho\ } & T^*Y \\ \Big\downarrow \pi_{X|H} & & \Big\downarrow \pi_Y \\ X & \xrightarrow{\ \pi\ } & Y \end{array}$$

where π is a submersion (along characteristics).

<u>Corollary 3.3.</u> Let (H,X,1) be special symplectic triplet, then a stationary lagrangian submanifold R(X) (cf.[2]) is a canonical pushforward, i.e.

(17) $$R(X) = T^*\pi(L),$$

and for the respective types of triplets described in Proposition 3.2, we have:

$$\pi : X \longrightarrow Y, \qquad \pi : (x_1,x_2,\ldots,x_n) \longrightarrow (x_2,\ldots,x_n).$$

Moreover L is generated (for the respective type of the triplet) by the following generating function

(18) $$F(x_1,\ldots,x_n) = \pm \int_0^{x_1} a(s,x_2,\ldots,x_n)(s^{k+1}+x_2 s^{k-1}+\ldots+x_{k+1})^2 ds.$$

Proof. We see that the space of characteristics of H, described by the dynamical system

$$\dot{x}_1 = 1,\ \dot{x}_2 = 0,\ \ldots\ ,\ \dot{x}_n = 0,$$
$$\dot{\xi}_1 = -\frac{\partial h}{\partial x_1}\ ,\ \ldots\ ,\ \dot{\xi}_n = -\frac{\partial h}{\partial x_n}\ ,$$

can be easily obtained. As the canonical variables $(\bar{x}_1,\ldots\bar{x}_{n-1},\ \bar{\xi}_1,\ldots,\bar{\xi}_{n-1})$, parametrizing the symplectic space of characteristics, we can take the initial values for $x_2,\ldots,x_n,\xi_2,\ldots,\xi_n$, where the initial value for x_1 is equal to zero. The respective symplectic form β, such that $\rho^*\beta = \sum_{i=1}^{n} d\xi_i \wedge dx_i|_H$, for this space may be choosen in the Darboux form. Now it is easy to check that (17) holds for each type of symplectic triplet of Proposition 3.2, with generating function (18) for L.

4. Generating families for the open swallowtails.

Let us consider now the most representative example of the Arnold's theory[3] for singular lagrangian submanifolds.

We are given the space of binary forms of degree d=2k+3, dimension of which is equal to 2k+4 (cf.[2]). This space can be endowed with the unique $Sl_2(\mathbb{R})$ - invariant symplectic form. In the appropriate Darboux coordinates $(q_o,\ldots,q_{k+2},p_o,\ldots,p_{k+2})$ a binary form, say $\varphi(x,y)$, can be written as follows

$$\varphi(x,y)=q_o\frac{x^{2k+3}}{(2k+3)!}+q_1\frac{x^{2k+2}y}{(2k+2)!}+\ldots+q_{k+1}\frac{x^{k+2}y^{k+1}}{(k+2)!}+(-1)^1 p_{k+1}\frac{x^{k+1}y^{k+1}}{(k+1)!}+\ldots$$
$$\ldots+(-1)^{k+1}p_o y^{2k+3}.$$

We see that the space of characteristics of the coisotropic hypersurface $\{q_o=1\}$ identifies with the space of polynomials of degree 2k+2 (derivatives of the respective polynomials $\varphi(x,1)$), i.e.

$$T^*Q = \{\frac{x^{2k+2}}{(2k+2)!}+ q_1\frac{x^{2k+1}}{(2k+1)!}+\ldots+q_{k+1}\frac{x^{k+1}}{(k+1)!}-p_{k+1}\frac{x^k}{k!}+\ldots+(-1)^k p_1\} ,$$

with the reduced symplectic form $\omega = \sum_{i=1}^{k+1} dp_i \wedge dq_i$, where $(q_1,\ldots,q_{k+1})$ are coordinates on Q.

<u>Proposition 4.1.</u> (cf.[2]), The triplet (H,Q,l), where $H=\{h(q,p)= = p_1+q_1p_2+\ldots+q_kp_{k+1}+q_{k+1}^2/2 = 0\}$, is a special symplectic triplet in (T^*Q,ω_Q) such that $\rho(l) \subseteq T^*Y$ is an open k-dimensional swallowtail.

Proof. The space of characteristics of the hamiltonian system

$$\begin{aligned}&\dot q_1= 1,\ \dot q_2= q_1,\ldots,\ \dot q_{k+1}= q_k,\\ &\dot p_1=-p_2,\ldots,\ \dot p_k=-p_{k+1},\ \dot p_{k+1}=-q_{k+1}\end{aligned} \tag{19}$$

can be identified to a space of polynomials of degree 2k+2 in $H \subseteq T^*Q$, such that $q_1=0$. We see that the zero section $Q \subseteq T^*Q$ intersected with H (i.e. 1) forms the space of polynomials divisible by x^{k+2}, so the canonical projection $\rho(1)$ onto the space of characteristics T^*Y (endowed with the Darboux coordinates $(q_2,\ldots,q_{k+1},p_2,\ldots,p_{k+1})$ can be identified with the polynomials of degree 2k+1 of the form:

$$\varphi(x,1)=\frac{x^{2k+1}}{(2k+1)!}+q_2\frac{x^{2k-1}}{(2k-1)!}+\ldots+q_{k+1}\frac{x^k}{k!}-p_{k+1}\frac{x^{k-1}}{(k-1)!}+\ldots+(-1)^{k-1}p_2$$

such that $\varphi(x,1)=(x-\xi)^{k+1}(x^k+\ldots)$ for some $\xi\in\mathbb{R}$. But this is nothing else than the definition of open swallowtail introduced in [2].

We can also use the initial values of $(q_1,\ldots q_{k+1},p_1,\ldots,p_{k+1})$ on characteristics to parametrize the space T^*Y. Remembering that h is a Hamiltonian of translations along the variable x, for the polynomial parametrization of characteristics we can write the following identification

$$\frac{(x-t)^{2k+2}}{(2k+2)!}+q_1\frac{(x-t)^{2k+1}}{(2k+1)!}+\ldots+q_{k+1}\frac{(x-t)^{k+1}}{(k+1)!}-p_{k+1}\frac{(x-t)^k}{k!}+\ldots+$$
$$+(-1)^kp_1=\frac{x^{2k+2}}{(2k+2)!}+\bar{q}_2\frac{x^{2k}}{(2k)!}+\ldots+\bar{q}_{k+1}\frac{x^{k+1}}{(k+1)!}-\bar{p}_{k+1}\frac{x^k}{k!}+\ldots+(-1)^k\bar{p}_1, \tag{20}$$

where $h(\bar{q}_1,\ldots,\bar{q}_{k+1},\bar{p}_1,\ldots,\bar{p}_{k+1})=0$ and $\bar{q}_1=0$ implies $q_1=t$. Hence we can take $(\bar{q}_2,\ldots,\bar{q}_{k+1},\bar{p}_2,\ldots,\bar{p}_{k+1})$ as a Darboux coordinates on T^*Y where Y is parametrized by $(\bar{q}_2,\ldots,\bar{q}_{k+1})$ and $\omega_Q|_H=\rho^*\omega_Y$. Likewise, in (16) $\pi:(q_1,\ldots,q_{k+1})\longrightarrow(\bar{q}_2,\ldots,\bar{q}_{k+1})$ has a following form

$$\bar{q}_j=\sum_{r=0}^{j-2}(-1)^r\frac{1}{r!}q_1^rq_{j-r}-(-1)^j\frac{j(j-1)}{j!}q_1^j,\quad j=2,\ldots,k+1. \tag{21}$$

Let R be a canonical symplectic reduction relation connected with H, i.e. R is a graph of ρ in $(T^*Q\times T^*Y,\pi_2^*\omega_Y-\pi_1^*\omega_Q)$.

Proposition 4.2. An open, k-dimensional swallowtail, can be represented as a canonical pushforward of a regular lagrangian submanifold, i.e.

$$R(Q) = T^*\pi(L_k),\ \dim Q = k+1,$$

where L_k is a lagrangian submanifold of (T^*Q,ω_Q) with the following generating function:

$$F_k(q_1,\dots,q_{k+1}) = \sum_{i=-1}^{k-2}\sum_{s=2}^{k-i-1} D^{(k)}_{k-i,s} q_1^{k+i-s+3} q_s q_{k-i} +$$

$$(22)\quad +\frac{1}{2}\sum_{i=0}^{k-2} D^{(k)}_{k-i,k-i} q_1^{2i+3} q_{k-i}^2 + \sum_{i=0}^{k-2} E^{(k)}_{k-i} q_1^{k+i+3} q_{k-i} + \frac{1}{2} D^{(k)}_{k+1,k+1} q_1 q_{k+1}^2 +$$

$$+E^{(k)}_{k+1} q_1^{k+2} q_{k+1} - \frac{E^{(k)}_2}{2k+3} q_1^{2k+3},$$

where

$$D^{(k)}_{r,s} = (-1)^{k-r}\sum_{j=s}^{k+1}\frac{(-1)^{j-s}}{(j-s)!(2k+3-j-r)!},$$

(23)

$$E^{(k)}_r = (-1)^{k-r}\left(\frac{1}{(2k+3-r)!} - \sum_{j=2}^{k+1}\frac{(-1)^j(j-1)}{j!(2k+3-j-r)!}\right),$$

$1 \le r,s \le k+1$.

Proof. On the basis of (21), $T^*\pi$ can be written as follows

$$p_1 = \sum_{j=1}^{k}\bar{p}_{j+1}\sum_{l=1}^{j}(-1)^l\frac{1}{(l-1)!}q_1^{l-1}q_{j-l+1},$$

$(T^*\pi)$

$$p_r = \bar{p}_r + \sum_{j=r}^{k}\bar{p}_{j+1}\frac{(-1)^{j-r+1}}{(j-r+1)!}q_1^{j-r+1},\quad 1 < r \le k+1.$$

On the other hand, providing further calculations, for R we obtain

$$p_1 = \sum_{j=1}^{k}\sum_{i=0}^{k-j}\frac{(-1)^{k-j-i+1}}{(k-i-j)!}q_j\bar{p}_{k-i+1}q_1^{k-j-i} - \sum_{j=1}^{k}\sum_{s=2}^{k+1}q_1^{2k+2-s-j}q_j\cdot$$

$$(R)\qquad \cdot q_s D^{(k)}_{j+1,s} - \sum_{j=1}^{k}E^{(k)}_{j+1}q_j q_1^{2k+2-j} - \frac{1}{2}q_{k+1}^2,$$

$$p_r = \sum_{j=0}^{k+1-r}(-1)^{k+1-r-j}\frac{1}{(k+1-r-j)!}\bar{p}_{k+1-j}q_1^{k+1-r-j} +$$

$$+\sum_{s=2}^{k+1}D^{(k)}_{r,s}q_s q_1^{2k+3-s-r} + E^{(k)}_r q_1^{2k+3-r},$$

where $2 \leq r \leq k+1$ and $D^{(k)}_{s,r}$, $E^{(k)}_{r}$ are defined in (23). Comparing both sets of equations for R and for $T^*\pi$, and remembering that Q is described by equations $p_1=p_2=\dots=p_{k+1}=0$ after simple but long calculations we obtain (22).

Using Proposition 2.8 and function (22) we obtain a generating family (not necessary Morse family) for the singular lagrangian submanifold in T^*Y called the open swallowtail (see[2]).

Corollary 4.3. A generating family for an open, k-dimensional swallowtail can be written in the following form

$$P_k(\bar{q}_2,\dots,\bar{q}_{k+1};\mu_1,\dots,\mu_{k+1},\lambda_1,\dots,\lambda_k)= F_k(\mu_1,\dots,\mu_{k+1})+$$
$$+\sum_{i=1}^{k}\pi_i(\bar{q}_{i+1}-\sum_{l=0}^{i-1}(-1)^l\frac{1}{l!}\mu_1^l\mu_{i-l+1}+(-1)^i\frac{i}{(i+1)!}\mu_1^{i+1}),$$

where F_k is defined in (22), $\mu_1,\dots,\mu_{k+1}$, $\lambda_1,\dots,\lambda_k$ are paremeters of the family.

Example 4.4. Let k= 1,2, then the respective generating functions for smooth (resolvents) lagrangian submanifolds L_1, L_2 are following

$$L_1: \quad F_1(q_1,q_2)=-\frac{1}{15}q_1^5+\frac{1}{3}q_1^3q_2-\frac{1}{2}q_1q_2^2,$$
$$L_2: \quad F_2(q_1,q_2,q_3)=-\frac{11}{840}q_1^7+\frac{11}{120}q_1^5q_2-\frac{1}{8}q_1^4q_3-\frac{1}{6}q_1^3q_2^2+\frac{1}{2}q_1^2q_2q_3-$$
$$-\frac{1}{2}q_1q_3^2.$$

Now by Corollary 4.3 and the standard method for reduction of parameters in generating family we obtain the generating, one-parameter, families for the cusp singularity (see[6]) and the two-dimensional open swallowtail singularity (see[3]) of lagrangian submanifold.

cusp:
$$P_1(\bar{q}_2,\lambda) = -\frac{1}{40}\lambda^5-\frac{1}{6}\lambda^3\bar{q}_2-\frac{1}{2}\lambda\bar{q}_2^2,$$

open swallowtail:
$$P_2(\bar{q}_2,\bar{q}_3,\lambda)=-\frac{1}{576}\lambda^7-\frac{1}{30}\lambda^5\bar{q}_2-\frac{1}{24}\lambda^4\bar{q}_3-\frac{1}{6}\lambda^3\bar{q}_2^2-\frac{1}{2}\lambda^2\bar{q}_2\bar{q}_3-\frac{1}{2}\lambda\bar{q}_3^2 .$$

Remark 4.5. Taking a new coordinates on T^*Q, defined by (21) and formulae $\bar{q}_1=q_1$ we have $\pi:Q \longrightarrow Y$, $\pi(q_1,\dots,q_{k+1})=(q_2,\dots,q_{k+1})$. So after straightforward calculations (cf.[13]) we derive the following generating families for the respective open swallowtails $R(Q)$, namely,

$$P_k(\bar{q}_2,\dots,\bar{q}_{k+1},\lambda)=\frac{1}{2}\int_0^\lambda\Big(\frac{k+2}{(k+1)!}x^{k+1}+\sum_{i=2}^{k+1}\frac{1}{(k-i+1)!}\bar{q}_i x^{k-i+1}\Big)^2dx.$$

Comparing this formula with Corollary 3.3 (formula (18)) we see that the special symplectic triplets with $k=n-1$ are diffeomorphic to these ones providing the open swallowtails.

Remark 4.6. One of the most interesting appearance of the open swallowtail ($k=2$) is that one proposed by V.I. Arnold (and coworkers)[2],[3] in variational calculus, which is frequently called "shortest bypassing of the obstacle". It has some precisely indefinite connection to geometrical optics (see[3]). Let us consider a piece D of a hypersurface (obstacle) in $\mathbb{R}^3$, and we define the geodesic flow on D by the time function $\tau:D \longrightarrow \mathbb{R}$. Hence $(\nabla\tau)^2=1$. An appropriate symplectic triplet connected with this situation is defined by $\phi:T^*R^3 \longrightarrow \mathbb{R}$ (defining H), $\varphi=p^2-1$ (all directions in the fibres) and the lagrangian submanifold L as all extensions to $T_q^*\mathbb{R}^3$ of the 1-forms $p=d\tau|_q$ defined on the tangent space to D. It turns out that $(H,L,H\cap L)$ is a symplectic triplet diffeomorphic to this one considered in §§ 3,4 of the present paper.

5. Final remarks and applications.

(5.1) As a simple mechanical example of singular image with respect to the symplectic reduction we consider the finite element analogue of the Euler beam problem. This system consisting of two rigid rods of unit length connected by frictionless pins, is subjected to a compressive force $-p_q$ which is resisted by a torsion

where $2 \leq r \leq k+1$ and $D^{(k)}_{s,r}$, $E^{(k)}_{r}$ are defined in (23).
Comparing both sets of equations for R and for $T^*\pi$, and remembering that Q is described by equations $p_1=p_2=\ldots=p_{k+1}=0$ after simple but long calculations we obtain (22).

Using Proposition 2.8 and function (22) we obtain a generating family (not necessary Morse family) for the singular lagrangian submanifold in T^*Y called the open swallowtail (see[2]).

Corollary 4.3. A generating family for an open, k-dimensional swallowtail can be written in the following form

$$P_k(\bar{q}_2,\ldots,\bar{q}_{k+1};\mu_1,\ldots,\mu_{k+1},\lambda_1,\ldots,\lambda_k)= F_k(\mu_1,\ldots,\mu_{k+1})+$$
$$+\sum_{i=1}^{k}\pi_i(\bar{q}_{i+1}-\sum_{l=0}^{i-1}(-1)^l\frac{1}{l!}\mu_1^l\mu_{i-l+1}+(-1)^i\frac{i}{(i+1)!}\mu_1^{i+1}),$$

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5. Final remarks and applications.

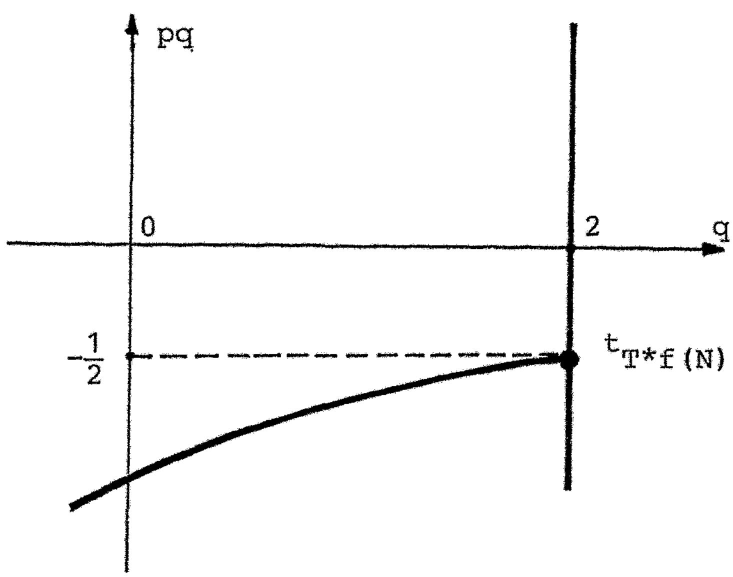
(5.1) As a simple mechanical example of singular image with respect to the symplectic reduction we consider the finite element analogue of the Euler beam problem. This system consisting of two rigid rods of unit length connected by frictionless pins, is subjected to a compressive force $-p_q$ which is resisted by a torsion spring of unit strength. The angle φ and the force p_q are considered coordinates of a manifold X. Together with the torque p_φ and the position q they form a canonical coordinate system $(\varphi, p_q, p_\varphi, -q)$ of T^*X. The potential energy of this system (generating function of the lagrangian submanifold $N \subset T^*X$) has the form

$$V(\varphi, p_q) = \tfrac{1}{2}\varphi^2 - 2p_q \cos\varphi.$$

If we take the reduced phase space T^*Y with the localcoordinate system $(p_q, -q)$ and the mapping $f: X \longrightarrow Y$, $f(\varphi, p_q) = p_q$ then we obtain for the image of N the following formula

$${}^{t}T^*f(N) = \{(p_q, -q) \in T^*Y;\ 0 = \frac{\partial V}{\partial\varphi}(\varphi, p_q) = \varphi + 2p_q \sin\varphi,\ -q = \frac{\partial V}{\partial p_q}(\varphi, p_q) = -2\cos\varphi\},$$

which is a space of equilibrium states (constitutive set) in the control phase space T^*Y. A simple calculation shows that if $p_q = -\frac{1}{2}$ $\varphi = 0$; V is not Morse family and the set ${}^{t}T^*f(N)$ has a standard singularity well known in the imperfect bifurcation theory (see fig. below)



Unfortunately that singularity is not stable, it disappears after small deformation of V because the respective transversality condition (cf. §2) is not fulfilled. However for examples of this type we can construct the space of deformations and treat the unstable singular lagrangian submanifold as an element of a family of deformations (a kind of unfolding[9]or more precisely Wassermann's (r,s)-unfolding [26]). Number of parameters of this family is connected to the codimension according to the above classified singularity types. This approach leads to the classification of stable images according to the composition of two reduction relations.

(5.2) Now we formulate the resolution problem for singular lagrangian submanifolds. Let $L \subset (P,\omega)$ be a germ of singular lagr. submanifold. The question is: do there exist

i) special symplectic structure (X,π,Θ,α) on (P,ω),
ii) a submersion $\rho : \Lambda \longrightarrow X$,
iii) a regular lagrangian submanifold $N \subset (T^*\Lambda,\omega_\Lambda)$

such that

$$L = T^*\rho(N).$$

Now we show that the regular geometric interaction between holonomic components (in the sense of Kashiwara [15],[19]) can be resolved in this way.

Let V_1, V_2 be lagrangian submanifolds of a symplectic manifold (P,ω) (cf.[18]).

<u>Definition.</u> The lagrangian subset $V_1 \cup V_2$ (or pair (V_1,V_2)) of (P,ω) is called a regular geometric interaction if the following conditions are fulfilled

a) $V_1 \cap V_2$ is a submanifold of P, $\dim V_1 \cap V_2 = \dim V_1 - 1$,
b) for every point $p \in V_1 \cap V_2$ we have

$$T_p(V_1 \cap V_2) = T_pV_1 \cap T_pV_2.$$

Let $(V_1 \cup V_2, p)$ be a germ of a regular geometric interaction in (P,ω).

Proposition. There exist a symplectic manifold $(\widetilde{P},\widetilde{\omega})$ and a symplectic reduction relation $R \subset (\widetilde{P}\times P, \widetilde{\omega} \ominus \omega)$ such that for a germ of regular geometric interaction, say $(V_1 \cup V_2, p) \subset (P,\omega)$ we have a canonical resolution formulae

$$V_1 \cup V_2 = R(L),$$

for some regular lagrangian submanifold $L \subset (\widetilde{P},\widetilde{\omega})$.

Proof. On the basis of the Kostant-Weinstein theorem (see e.g. [10], [24]) we can isomorphically represent (P,ω) by means of (T^*V_1, ω_{V_1}), where V_1 is a zero-section of the bundle. Hence $V_1 = \{p_1 =, \dots, p_n = 0\}$ and a generating function for V_2, in T^*V_1, can be written as $H(q) = q_1^2\varphi(q)$, where $\varphi(0) \neq 0$ (because of the point b) of the definition). So we can choose local Darboux coordinates on T^*V_1, near p, preserving the zero section V_1 and such that the respective germ of generating function for V_2 is following

$$H(q) = q_1^2.$$

Taking the new Darboux coordinates in T^*V_1 preserving V_1, namely

$$\Phi(q_1,\dots,q_n,p_1,\dots,p_n) = (q_1 - \tfrac{1}{2}p_1, q_2,\dots,q_n,p_1,\dots,p_n)$$

we obtain the following local equations for V_1 and V_2 respectively

$$V_1: \quad p_1 = ,\dots, = p_n = 0,$$

$$V_2: \quad p_2 = 0,\dots, p_n = 0, \ q_1 = 0.$$

But for this germ of geometric interaction we can easily write the respective generating family:

$$F(q_1,\dots,q_n,\lambda) = q_1\lambda^3.$$

If T^*X is any initial, special symplectic structure on (P,ω) then using the Morse family, say $G\colon X \times Q \times \mathbb{R}^N \longrightarrow \mathbb{R}$, for the respective symplectomorphisms in the above procedure (according to [21], [22]) we can write down the desired generating family for $V_1 \cup V_2$:

$$\widetilde{F}(x;\nu,\mu,\lambda) = G(x_1,\dots,x_n,\nu_1,\dots,\nu_n,\mu_1,\dots,\mu_N) + \nu_1\lambda^3$$

This completes the proof.

Having an analytical description of regular holonomic interaction we can formulate the appropriate stability problem and use it to determine the respective Gauss-Manin systems[18],[19].

(5.3), (Landau singularities). Let us consider the motion of a free particle of mass m in space-time $\mathbb{R}^4$ endowed with the Minkowski metric tensor. The phase space is the cotangent bundle $P \cong T^*\mathbb{R}^4$. A mass surface or a first class constraint submanifold $\tilde{M} \subset P$ is defined by

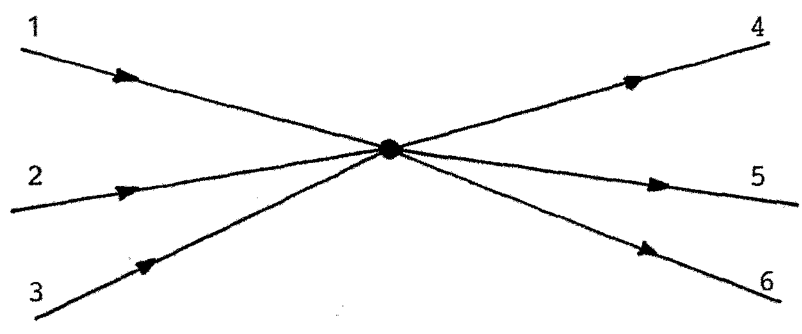
$$\tilde{M} = \{(x,p) \in P;\ p^2 = p_0^2 - \vec{p}^2 = m^2,\ p_0 > 0\},$$

where the respective Hamiltonian is defined as a zero function on $\tilde{M}$.

In the elementary particle physics the collision processes constitute a one of the main subjects of interest (for the basis of the theory of multiple collisins processes see e.g.[20]).
Let us consider a collision process $I \longrightarrow J$ described by the coisotropic submanifold $\tilde{M}_{(I,J)}$ in $\prod_{i \in I \cup J} P_i$, namely

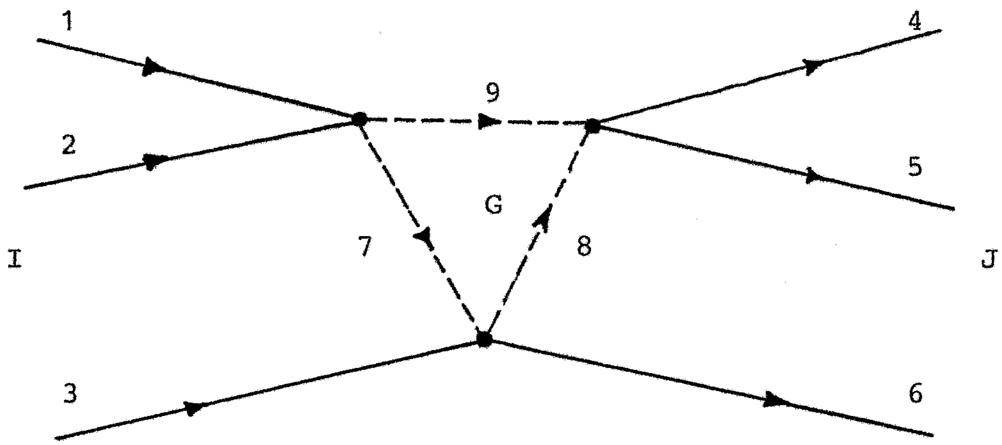
(*) $\tilde{M}_{(I,J)} = \{(\tilde{x},\tilde{p}) \in \prod_{i \in I \cup J} P_i;\ (\tilde{x},\tilde{p}) \in \prod_{i \in I \cup J} M_i,\ \sum_{i \in I} p_i = \sum_{j \in J} p_j\}$,

where I, J are the numbering sets for the respective particles (as in Fig. below) in the collision process (I,J).



$I = \{1,2,3\}$ $J = \{4,5,6\}$

Let us consider an associate causal configuration for (I,J) corresponding to the graph G of an appropriate multiple diffusion process (see fig. below).



Let I resp. J denote the set of external lines incoming and resp. outgoing from G. Let $\widetilde{M}_G$ be the coisotropic submanifold defined analogously as in (*) using the conservation laws. It is easy to check that the symplectic spaces $\widetilde{M}_{(I,J)}/\sim$ and $\widetilde{M}_G/\sim$ associated canonically to $\widetilde{M}_{(I,J)}$ and $\widetilde{M}_G$ resp. are isomorphic to $T^*M_{(I,J)}$ and T^*M_G respectively, where

$\mathbb{R}^{4N} \supset M_{(I,J)} = \{(p_i) \in \mathbb{R}^{4N};\ p_{oi}^2 - \vec{p}_i^{\,2} = m_i^2,\ p_{oi} > 0,\ \sum_{i\in I} p_i = \sum_{j\in J} p_j\}$

and analogously for M_G.

We have here the canonical projection

$$f:\ M_G \longrightarrow M_{(I,J)},$$

which defines the respective symplectic relation

$$T^*f \subset \widetilde{M}_{(I,J)}/\sim \ \times\ \widetilde{M}_G/\sim$$

responsible for the geometrical properties of the collision process. The set of critical values of f, say $\Gamma f \subset M_{(I,J)}$ (an apparent contour of f) is called a Landau set corresponding to the graph G. The singularity type of f is responsible for singularity type of the Landau set and is frequently called the Landau singularity.

Corollary. The geometrical properties of a multiple diffusion process with a graph G are described by the following pair:

$$(L_{\Gamma f}, T^{*}f),$$

where $L_{\Gamma f}$ is a constrained lagrangian submanifold over constraint Γf (cf.[14]). Hence the classification of normal forms, as in the Pham approach to the Landau singularities can be easily derived using our classification theorem for pullbacks.

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