BRANCHED COVER HOMOLOGY AND Q EVALUATIONS

This is a draft. I would be grateful for any comments and corrections!

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Abstract. We show that the known evaluations of the Q polynomial are the only ones determined by the homology of the double branched cover of S^3 over a link.

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1. Introduction

The Alexander polynomial [Al] is a classical invariant in the theory of knots and links in S^3, which is closely related to the topology of the knot (or link) complement [Ro].

After the appearance of the relatives of the Alexander polynomial [J, H, BLM, Ka], it has been tried to find a topological understanding of these invariants, too. This succeeded only for special evaluations of the above polynomials [LM, J2]. A summary on this can be found in [L]. In all cases these evaluations have been related to the homology modules of branched coverings of S^3 over the link with values in some finite field. Of particular interest is the Q polynomial of Brandt-Lickorish-Millett-Ho [BLM, Ho], a polynomial invariant with values in $\mathbb{Z}[z,z^{-1}]$, from which the rank of the homology of the double branched cover with values in $\mathbb{Z}_3$ can be obtained from the evaluation at $z=-1$ (see Theorem 8.4.8 (2) of [Kw]) or the one with values in $\mathbb{Z}_5$, recoverable from the evaluation at $z=(\sqrt{5}-1)/2$ [J2]. Thus, by [We, theorem p. 690] or by direct skein/Kauffman relation arguments as in [Tr], these evaluations give lower bounds for the unknotting number.

In this note we attempt to show that these are the only evaluations of Q, which are, in some reasonable sense, determined by the homology of the double branched cover with values in some finite field. A similar reasoning for the Jones polynomial we leave to an interested reader.

2. Special Q evaluations determined completely by $\dim H_1(D_L,\mathbb{Z}_n)$

We start with the following

Theorem 2.1 Let $z\in\mathbb{C}\setminus\{0\}$ be so that $Q_L(z)$ is determined by the rank of the homology $H_1(D_L,\mathbb{Z}_n)$ of the double branched cover D_L of L with values in some finite field $\mathbb{Z}_n$, n prime. Then $z=-2,1$, or $z=-1$ and $n=3$.

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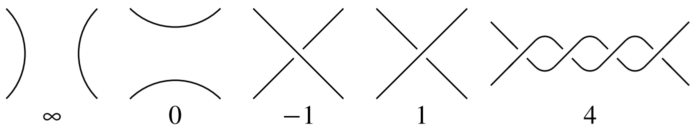
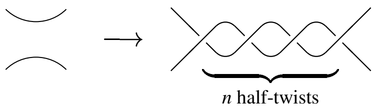


Figure 1: The Conway tangles.

Remark 2.1 The case $z=(\sqrt{5}-1)/2$ does not occur here, because the evaluation is determined *just up to a sign* by $\dim H_1(D_L,\mathbb{Z}_5)$. Thus we will attempt to prove a stronger version of theorem 2.1, where $Q_L(z)$ is replaced by $|Q_L(z)|$, but, unfortunately, our method will work with much more effort.

Proof. We use the result of Przytycki [Pr], that $H_1(D_L,\mathbb{Z}_n)$ is unchanged by a n-move. A n-move is a move on a knot or link diagram, replacing a 0 tangle (in the Conway [Co] sense) by an n or $-n$ tangle (where a $-n$ tangle is the obverse of an n tangle):



Two links are called n-equivalent if there is a sequence of Reidemeister and n-moves transforming a diagram of the one link into one of the other link.

Recall, that the Q polynomial is a Laurent polynomial in one variable z for links without orientation, defined by being 1 on the unknot and the relation

$$A_1+A_{-1}=z(A_0+A_\infty)\,, \tag{1}$$

where A_i are the Q polynomials of links K_i and K_i ($i\in\mathbb{Z}\cup\{\infty\}$) possess diagrams equal except in one room, where an i-tangle (in the Conway sense) is inserted, see figure 1.

Now take all K_i, except K_∞, to be knots, and consider the series

$$f(x,z):=\sum_{i=0}^{\infty}A_i(z)x^i\,.$$

As the coefficients of Q are exponentially bounded in the crossing number, the series has a positive convergence radius around $(0,0)$ and converges absolutely within this radius and defines an analytic function.

More precisely, the series converges absolutely for $|x|<1$ and $|z|<1$, and for $|z|>1$ and $|xz|<1$. Therefore, for any given z we can choose x small enough so as to perform the following calculations.

From (1) we obtain

$$f(x,z)\;=\;\frac{-f(x,z)+A_0(z)+A_1(z)x}{x^2}+z\frac{f(x,z)-A_0(z)}{x}+\frac{z}{1-x}A_\infty(z)\,,$$

whence

$$f\;=\;\frac{(1-zx)(1-x)A_0+x(1-x)A_1+zx^2A_\infty}{(1-zx+x^2)(1-x)}\,. \tag{2}$$

The dependency of f on $A_{0,1,\infty}$ we will not mark explicitly, but should implicitly keep in mind.

Assume now that for some concrete value of z, $Q(z)$ depends just on $H_1(D_L,\mathbb{Z}_n)$. Then $f(x,z)$ would have a n-periodic Taylor expansion in x around 0 (for any z_0, $f(x,z_0)$ converges absolutely for $|x|<\varepsilon_{z_0}$). So

$$\big((1-zx)(1-x)A_0+x(1-x)A_1+zx^2A_\infty\big)(1-x^n)\;=\;(1-x)(x^2-zx+1)P(x,z)\,,$$

for some polynomial $P\in\mathbb{Z}[z,x]$ of degree at most $n-1$ in x.

Now, first we show that $z=2\cos 2\pi k/n$ for some natural number $0\le k\le n-1$. Assume, that it is not the case. Then x^2-zx+1 has zeros, which are not zeros of $1-x^n$. Then x^2-zx+1 must divide the first factor on the left. But, already making the simplest non-trivial choice $A_\infty = A_1 = 1$, $A_0 = 2/z-1$, the rest is $x\big(2z^2-2\big)+(2-2z)\neq 0$, unless $z=-1$, which is uninteresting.

Therefore, $z = 2\cos 2\pi k/n$. Now, connected sum shows that the map $\operatorname{rank} H_1 \mapsto Q(z)$, should be multiplicative under addition, and hence must be an exponential. The only candidate for a base is found by examining the 2 component unlink. So we must have

$$Q(2\cos 2\pi k/n) = \left(\frac{1}{\cos 2\pi k/n}-1\right)^{\dim H_1(D_L,\mathbb{Z}_n)}.$$

Consider the trefoil with polynomial $Q=-3+2z+2z^2$ and $n>3$. The above equality specializes to

$$-3+4c+8c^2 = 1\,,$$

with $c := \cos 2\pi k/n$, whence $c = 1/2$ or $c=-1$, which implies $z=1$ or $z=-2$. So we are done checking the cases $n\leq 3$ directly. □

3. The evaluations determined by norm

Now replace $Q(z)$ by $|Q(z)|$. This makes life somewhat more complicated. The generating series is now

$$\tilde f(x,z) := \sum_{i=0}^{\infty} |A_i(z)|^2 x^i\,.$$

This series converges absolutely for $|x|<1$ and $|z|<1$, and for $|z|>1$ and $|xz|<1$. Therefore, for any given z we can choose x small enough so as to perform the following calculations.

$\tilde f$ still can be expressed in terms of f by Fourier calculus, but rather complicatedly. One way is

$$\tilde f(x,z) = \int_0^1 f(\sqrt{x}e^{2\pi i k},z)\,\overline{f(\sqrt{\overline{x}}e^{2\pi i k},z)}\,dk = \int_0^1 f(\sqrt{x}e^{2\pi i k},z)\,f(\sqrt{x}e^{-2\pi i k},\overline{z})\,dk\,. \tag{3}$$

Now we need to examine for which z there is an n such that

$$\frac{\partial^n}{\partial x^n}\bigg|_{|x|<\varepsilon}\int_0^1 f(\sqrt{x}e^{2\pi i k},z)\,\overline{f(\sqrt{\overline{x}}e^{2\pi i k},z)}\,(1-x^n)\,dk \equiv 0$$

for *any* choice of $A_{0,1,\infty}$.

The result is the one we like.

Theorem 3.1 Let $z\in\mathbb{C}\setminus\{0\}$ be so that $|Q_L(z)|$ is determined by $\dim H_1(D_L,\mathbb{Z}_n)$, n prime. Then $z=-2,1$, or $z=-1$ and $n=3$, or $z=\frac{-1\pm\sqrt{5}}{2}$ and $n=5$.

Proof. Again we are interested in a periodic Taylor expansion with regard to x around $x=0$, this time of the integral in (3). This integral can be expressed as a curve integral

$$\begin{aligned}
&\frac{1}{2\pi i}\oint\limits_{|k|=\sqrt{x}} f(k,z)\,f\Big(\frac{x}{k},\overline{z}\Big)\,\frac{1}{k}\,dk \\
&= \frac{1}{2\pi i}\oint\limits_{|k|=\sqrt{x}} \frac{\big[(1-zk)(1-k)A_0(z)+k(1-k)A_1(z)+zk^2A_\infty(z)\big]}{(1-kz+k^2)(1-k)} \times \frac{\big[(1-\overline{z}\frac{x}{k})(1-\frac{x}{k})A_0(\overline{z})+\frac{x}{k}(1-\frac{x}{k})A_1(\overline{z})+\overline{z}\frac{x^2}{k^2}A_\infty(\overline{z})\big]\,dk}{\big(1-\frac{x}{k}\overline{z}+\frac{x^2}{k^2}\big)\big(1-\frac{x}{k}\big)\,k} \\
&= \frac{1}{2\pi i}\oint\limits_{|k|=\sqrt{x}} \frac{\big[(1-zk)(1-k)A_0(z)+k(1-k)A_1(z)+zk^2A_\infty(z)\big]\ \big[(1-\overline{z}\frac{x}{k})(1-\frac{x}{k})A_0(\overline{z})+\frac{x}{k}(1-\frac{x}{k})A_1(\overline{z})+\overline{z}\frac{x^2}{k^2}A_\infty(\overline{z})\big]\ k^2\,dk}{(1-kz+k^2)(1-k)(k^2-xk\overline{z}+x^2)(k-x)}\,,
\end{aligned}$$

where '$|k|=\sqrt{x}$' means the sphere in $\mathbb{C}$ with origin zero and radius $\sqrt{x}$, positively oriented. As $x\to 0$, the relevant zeros of the denominator are x, $x\cdot\overline{k_0}$ and $x/\overline{k_0}$, where $k=k_0$ is one zero of k^2-zk+1, so that

$$z=k_0+\frac{1}{k_0}\,. \tag{4}$$

To express ourselves more briefly, set

$$\Phi(k,x,z)\,:=\,\Big[(1-zk)(1-k)A_0(z)+k(1-k)A_1(z)+zk^2A_\infty(z)\Big]\Big[(k-\overline{z}x)(k-x)A_0(\overline{z})+x(k-x)A_1(\overline{z})+\overline{z}x^2A_\infty(\overline{z})\Big]\,.$$

The denominators of the residues are (in the order of appearance)

$$\begin{gathered}(1-xz+x^2)(1-x)x^2(2-\overline{z})\,,\\ (1-x\overline{k_0}z+x^2\overline{k_0}^2)(1-x\overline{k_0})(-x+x\overline{k_0})\big(x\overline{k_0}-\frac{x}{\overline{k_0}}\big)\,,\\ \Big(1-\frac{x}{\overline{k_0}}z+\frac{x^2}{\overline{k_0}^2}\Big)\Big(1-\frac{x}{\overline{k_0}}\Big)\Big(-x+\frac{x}{\overline{k_0}}\Big)\Big(\frac{x}{\overline{k_0}}-x\overline{k_0}\Big)\,.\end{gathered}$$

We may assume that $k_0\neq 1$ (else $z=2$, which is clearly not of the kind we want). Then, regarded as functions in x, they have the following zeros (with multiplicities)

$$\begin{array}{ccccc}\overline{k_0}, & \frac{1}{\overline{k_0}}, & 1, & 0, & 0\,;\\ \frac{1}{\overline{k_0}^2}, & 1, & \frac{1}{\overline{k_0}}, & 0, & 0\,;\\ \overline{k_0}^2, & 1, & \overline{k_0}, & 0, & 0\,.\end{array}$$

If k_0 is an n-th unity root we would have $z=2\cos 2\pi l/n$. Now assume, that k_0 is not an n-th unity root. Then, because $\overline{k_0}^{\pm 2}$ appear as zeros in only one of the denominators above, we must have that they divide the corresponding numerators. Therefore, $x-\overline{k_0}^{-2}$ divides the numerator of the first residue, and $x-\overline{k_0}^2$ divides the numerator of the second residue for *any* choice of $A_{0,1,\infty}$. These numerators are $\Phi(x\overline{k_0},x,z)$ and $\Phi(x/\overline{k_0},x,z)$, respectively.

Therefore, $\Phi(\overline{k_0},\overline{k_0}^2,z)=\Phi(\frac{1}{\overline{k_0}},\frac{1}{\overline{k_0}^2},z)=0$, for the given choice of z. Set again $A_\infty=A_1=1$ and $A_0=2/z-1$. The equality $\Phi(\overline{k_0},\overline{k_0}^2,z)=0$ yields

$$\hat{z}^2k_0+\hat{z}(k_0^2-1)+(2-2k_0)=0\,, \tag{5}$$

where '$\hat{z}$' means that one of both identities with $\hat{z}$ replaced by z or $\overline{z}$ is to be satisfied. But because of (4), the substitution $k_0\to 1/k_0$ does not change z (and $\overline{z}$), and hence from (5) we obtain under this substitution

$$\hat{z}^2k_0+\hat{z}(1-k_0^2)+(2k_0^2-2k_0)=0\,,$$

with '$\hat{z}$' meaning the *same* choice between z and $\overline{z}$ as in (5).

Taking difference with (5), we find $k_0=\pm 1$, hence $z=\pm 2$, or $z=-1$, which are evaluations where the picture is clear by [BLM].

Hence, assume that $z=2\cos 2\pi k/n$. As in the previous proof, we get

$$|Q(2\cos 2\pi k/n)|\,=\,\left|\frac{1}{\cos 2\pi k/n}-1\right|^{\dim H_1(D_L,\mathbb{Z}_n)}\,.$$

Again with the trefoil we find for $n>3$ this implies

$$|-3+4c+8c^2|=1\,,$$

with c as before. But $-3+4c+8c^2$ is real, so $-3+4c+8c^2=\pm 1$. In former case we get where we did in the previous proof, and in latter case we obtain $c=\frac{-1\pm\sqrt{5}}{4}$, giving the case $n=5$ of Jones. □

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