ALMOST REGULAR QUATERNARY QUADRATIC FORMS

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Abstract. In this paper we investigate the almost regular positive definite integral quaternary quadratic forms. In particular, we show that every such form is p-anisotropic for at most one prime number p. Moreover, for a prime p there is an almost regular p-anisotropic quaternary quadratic form if and only if $p \leq 37$. We also study the genera containing some almost regular p-anisotropic quaternary form. We show several finiteness results concerning the families of these genera and give effective criteria for almost regularity.

1. Introduction

In this paper we investigate properties of the almost regular positive definite integral quaternary quadratic forms. An integral quadratic form f is called *almost regular* (resp. *regular*) if it represents every large (resp. every) rational positive integer which is represented by f over the ring $\mathbb{Z}_p$ of p-adic integers, for every prime p. By an *integral form* we shall always mean a positive definite nonsingular quadratic form having an integer matrix. Quite often we shall make no distinction between a form and its equivalence class. If a form f represents zero nontrivially over $\mathbb{Z}_p$, we say that f is *p-isotropic*, otherwise f is called *p-anisotropic*.

The following result is contained in the works of Kloosterman, Tartakowsky, Pall and Ross (cf. [10], [16], [14], [4]).

Theorem 1.1. *Every positive definite integral quadratic form which is p-isotropic for every prime p is almost regular.*

In particular, every integral form in more than 4 variables is almost regular. An almost regular quadratic form which is p-anisotropic for some prime p is called *exceptional*. Since there are simple and effective criteria of p-isotropy (cf. [15], p.37), Theorem 1.1 reduces the problem of deciding almost regularity of forms to the case of exceptional ones. There are no exceptional almost regular forms in more than 4 variables. All almost regular forms in less than 4 variables are exceptional. In the case of 2 variables there is only a finite number of primitive almost regular forms. They are, in fact, all regular and coincide with primitive binary forms representing one-class genera. In the case of 3 variables there are at most 913 primitive regular ternaries (cf. [9]), and the structure of almost regular ternaries is well understood from works [5], [6], [7]. Primitive almost regular ternaries constitute an infinite family.

In dimension 4, both families of exceptional and nonexceptional primitive regular forms are infinite (an example of the former is provided by $x^2 + y^2 +$

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$z^2+4^n t^2, n \geq 0$). We shall focus our attention on the case of the exceptional almost regular quaternaries. Strangely enough, they never seemed to be the object of any investigation. We shall show that the set of all exceptional almost regular quaternaries, although infinite, is nevertheless very "small" in the sense which is explained by several results stated below in this section.

Theorem 1.2. *Any integral positive definite almost regular quaternary quadratic form is p-anisotropic for at most one prime p. Moreover, for a prime p there exists an almost regular p-anisotropic quaternary form if and only if $p \leq 37$.*

This theorem shows already that the exceptional quaternaries do not occur frequently. But even *within* the family of p-anisotropic quaternaries, $p \leq 37$, they are very rare. To make this claim precise we need some preparation.

Recall that a quadratic form g over $\mathbb{Z}_p$ is said to be *p-universal* if g represents all p-adic integers. A quaternary quadratic form g over $\mathbb{Z}_2$ is called *$2\mathbb{Z}_2$-universal* if g is *even* and either g is 2-isotropic and represents all numbers in $2\mathbb{Z}_2$, or $g \simeq h \perp 2h$ over $\mathbb{Z}_2$, where $h = 2x^2+2xy+2y^2$; in either case g represents all numbers in $2\mathbb{Z}_2$. An integral quadratic form f is said to be *p-universal* (resp. *$2\mathbb{Z}_2$-universal*) if $f_p = f \otimes \mathbb{Z}_p$ is p-universal (resp. $f_2 = f \otimes \mathbb{Z}_2$ is $2\mathbb{Z}_2$-universal). If an integral quadratic form f is *even* and represents all even positive integers, f is said to be *even-universal*.

Let $\mathcal{E}_p$ be the family of all primitive p-anisotropic almost regular integral quaternary quadratic forms. Since $\mathcal{E}_p = \emptyset$ for $p > 37$, we only need to investigate twelve families $\mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_5, \ldots, \mathcal{E}_{37}$. We shall see that the study of $\mathcal{E}_p$ is often reduced to the study of its subfamily $\mathcal{R}_p$ consisting of these forms in $\mathcal{E}_p$ which are regular, and p-universal (if $p > 2$) or $2\mathbb{Z}_2$-universal (if $p = 2$). The families $\mathcal{R}_p$ are *finite* (cf. Theorem 5.4). This allows to prove several finiteness results concerning $\mathcal{E}_p$. The families $\mathcal{R}_p$ play the crucial role in our investigations. One can say that $\mathcal{R}_p$ is the key for understanding the structure and properties of forms in $\mathcal{E}_p$. In turn, $\mathcal{R}_p$ contains an important subfamily $\mathcal{U}_p$ consisting of these forms in $\mathcal{R}_p$ which are even-universal. In Section 3 we shall find explicitly all forms in $\mathcal{U}_p$, and in Section 6 we shall prove that for $p \geq 17$ one has $\mathcal{R}_p = \mathcal{U}_p$. On the other hand, if $p \leq 13$ then $\mathcal{R}_p$ contains an even form which is not in $\mathcal{U}_p$ (cf. Theorem 8.1). The forms in $\mathcal{E}_p$ for the top six primes p, $17 \leq p \leq 37$, often behave differently than the forms in the remaining six families (corresponding to $p \leq 13$). For example, for $p \geq 17$ a primitive p-anisotropic quaternary form f is in $\mathcal{E}_p$ if and only if f represents all positive integers divisible by $2p^s$, where s depends only on f_p. The analogous property is not valid anymore for $p \leq 13$ (cf. Section 7). This difference in behavior of elements of $\mathcal{E}_p$, depending whether $p \geq 17$ or $p \leq 13$, is explained precisely by the fact that $\mathcal{R}_p = \mathcal{U}_p$ only for $p \geq 17$. It also explains why we can say much more about the six families $\mathcal{E}_{17}$, $\mathcal{E}_{19}$, $\ldots$, $\mathcal{E}_{37}$, than about the remaining six $\mathcal{E}_2$, $\mathcal{E}_3$, $\ldots$, $\mathcal{E}_{13}$. The families $\mathcal{R}_p$, for $17 \leq p \leq 37$, are very small, they are just reduced to one or two forms (cf. Theorem 6.1). The families $\mathcal{R}_p$, for $p \leq 13$, are not known explicitly but they are certainly much larger. For example, $\mathcal{R}_2$ contains at least 81 forms but most likely a few hundreds.

The families $\mathcal{R}_p$ are also used to prove several *effective* criteria of almost regularity (cf. Theorems 1.3, 1.8, 1.9, 4.2, Section 7). For example, $\mathcal{R}_p$ intervenes in an explicit description of all genera containing some almost regular p-anisotropic quaternary quadratic form.

Given a genus Γ containing an integral quaternary form f, let Γ_p be the class of equivalence over $\mathbb{Z}_p$ of $\mathbb{Z}_p$-forms represented by $f_p = f \otimes \mathbb{Z}_p$. Clearly, the sequence $\{\Gamma_p\}_{p\in P}$, where P is the set of all primes, completely determines Γ, and conversely. By abuse of notation, we shall identify Γ and the sequence $\{\Gamma_p\}_{p\in P}$, $\Gamma = \{\Gamma_p\}_{p\in P}$. Given a quadratic form H over $\mathbb{Z}_p$, we say that a genus Γ of an integral quadratic form is of *type* H if $\Gamma_p \simeq H$, where $\simeq$ denotes the equivalence over $\mathbb{Z}_p$.

The next theorem shows that for an *arbitrary* p-anisotropic quaternary quadratic form H over $\mathbb{Z}_p$, $p \leq 37$, there exists a genus Γ of type H containing some almost regular p-anisotropic integral quaternary form. The theorem also provides a characterization of all such genera.

Theorem 1.3. *Let H be a primitive p-anisotropic quaternary quadratic form over $\mathbb{Z}_p$. For any genus Γ of type H the following conditions are equivalent:*

(i) Γ *contains a primitive almost regular integral quaternary quadratic form (which is necessarily p-anisotropic).*

(ii) *There is a form g in $\mathcal{R}_p$ such that $g_q \simeq \Gamma_q$ for all primes $q \neq p$.*

Theorems 1.1, 1.2 and 1.3 imply

Corollary 1.4. *Let H be a quaternary quadratic form over $\mathbb{Z}_p$. Then the following properties are equivalent:*

(i) *There is an integral almost regular quaternary quadratic form f such that $f_p \simeq H$.*

(ii) *H is p-isotropic or $p \leq 37$.*

Let α_p be the number of all genera represented by the forms of family $\mathcal{R}_p$. In Section 6 we shall prove that $\alpha_{17} = \alpha_{19} = \alpha_{29} = \alpha_{37} = 1$ and $\alpha_{23} = \alpha_{31} = 2$. Of course, $\alpha_p = 0$ for $p > 37$. The finiteness of $\mathcal{R}_p$, together with Theorem 1.3, implies the following result.

Corollary 1.5. *Let H be a primitive p-anisotropic quaternary quadratic form over $\mathbb{Z}_p$. Then the number of all genera of type H, containing some primitive almost regular quaternary, is finite and equal to α_p. In particular, this number depends only on p, but not on H.*

For H as in Corollary 1.5 the number of all genera of type H, containing some primitive integral quaternary form anisotropic only at p, is infinite. Therefore Corollary 1.5 shows the scarcity of almost regular quadratic forms even *within* the family of all integral quaternaries which are anisotropic only at p. It also shows a remarkable equidistribution of the genera containing a primitive almost regular p-anisotropic quaternary: their number is finite and *identical* within each p-type.

The fact that $\mathcal{E}_p$ constitutes a very "small" part of the family of all primitive quaternaries which are anisotropic only at p can also be seen from the shape of the discriminants of the forms in $\mathcal{E}_p$. This is well illustrated by the next two theorems.

Theorem 1.6. *For each prime p there is a finite set A_p of integers relatively prime to p, such that a number d is the discriminant of a form in $\mathcal{E}_p$ if and only if $d = rp^{2n}$, where $r \in A_p$ and n is an integer, $n \geq 1$ for $p > 2$, and $n \geq 0$ for $p = 2$.*

We shall see in Section 6 that $A_p = \{1\}$ for $p = 17, 19, 29, 31$, and $A_p = \{1, 4\}$ for $p = 23, 31$. For $p \leq 13$ the sets A_p are not known explicitly, but the elements of A_p are bounded by an effective constant (cf. Theorem 9.1).

We are able to determine with more accuracy the sets

$$D_p = \{q \in P \mid q \text{ divides } d(f),\ f \in \mathcal{E}_p\},$$

where $d(f)$ is the discriminant of f.

Theorem 1.7. *One has*

$$\begin{aligned} &D_2 = \{2,3,5,7,11,13,17,19,23,29,31,37,41,47,59,61,67,73,89,97,\\ &\qquad 107,113,137,193,233,241,257,281,353\}\\ &D_3 = \{2,3,5,7,37\}, \quad D_5 = \{2,3,5,11\},\\ &\{2,p\} \subset D_p \subset \{2,3,5,p\} \textit{ for } p = 7, 11, \quad \{2,3,13\} \subset D_{13} \subset \{2,3,5,13\},\\ &D_p = \{2,p\} \textit{ for } p = 23, 31, \quad D_p = \{p\} \textit{ for } p = 17, 19, 29, 37. \end{aligned}$$

Letting D denote the set of all primes dividing the discriminant of some primitive exceptional almost regular quaternary quadratic form, one has $D = D_2$.

Define

$$P^* = \{17, 19, 23, 29, 31, 37\},$$
$$\Omega_{17} = \{2,4,6,34,68\}, \quad \Omega_{19} = \{2,4,6,20\}, \quad \Omega_{23} = \{2,4,6,10,46\}$$
$$\Omega_{23} = \{2,4,10,58,290\}, \quad \Omega_{31} = \{2,4,6,10,62\}, \quad \Omega_{37} = \{2,4,10,58\}.$$

In Section 4 we shall define, for every primitive integral quaternary quadratic form f and every prime p, an invariant $s(f,p)$. This invariant is a nonnegative integer which can be effectively computed and which depends only on f_p.

The explicit knowledge of all elements of $\mathcal{R}_p$ for $p \in P^*$ leads to the following *effective* criterion of almost regularity of p-anisotropic integral quaternaries.

Theorem 1.8. *Let $p \in P^*$. For any primitive integral p-anisotropic quaternary quadratic form f the following properties are equivalent:*

(i) *f is almost regular.*
(ii) *f is even and represents all numbers in the set $p^{s(f,p)}\Omega_p$.*

For primes p in P^* we can formulate a particularly simple and explicit description of all genera containing some p-anisotropic primitive almost regular quaternary quadratic form.

Theorem 1.9. *Let $p \in \{17, 19, 29, 37\}$ (resp. $p \in \{23, 31\}$). For any primitive p-anisotropic integral quaternary quadratic form f the following properties are equivalent:*

(i) *The genus of f contains an almost regular form.*
(ii) *The form f is even and the discriminant $d(f) = p^{2n}$ (resp. $d(f) = p^{2n}$, or $d(f) = 4p^{2n}$ and $f_2 \simeq 2(xy + z^2 - t^2)$ over $\mathbb{Z}_2$), where $n \geq 1$ is an integer.*

The proofs of Theorems 1.2 and 1.3 are given, respectively, in Section 3 and 4. Theorems 1.6, 1.7 (resp. 1.8 and 1.9) are proved in Section 9 (resp. 7). These proofs rely heavily on the properties of a descent method known as the "Watson transformation" presented in Section 2, and on the knowledge of the list of all even-universal quaternaries. Such a list was compiled recently with the help of "The 290 Theorem" (cf. [8], [12]).

2. The Watson transformation

First we shall fix the notation and terminology which will be used throughout this paper. As is well known, there is a natural bijection between classes of integral quadratic forms and lattices having integral inner product: the Gram matrix of a quadratic form f can be regarded as the matrix of the corresponding lattice L_f. Although the results in the Introduction were stated in the language of forms, it is more convenient to use the language of lattices in the proofs. We shall therefore oscillate between these two languages. A $\mathbb{Z}$-lattice (by definition always equipped with an integral inner product) is called *regular*, *almost regular*, *p-universal* etc., if the corresponding form is regular, almost regular, p-universal, etc. We shall also work with $\mathbb{Z}_p$-lattices, always equipped with a $\mathbb{Z}_p$-valued inner product. The notation $\langle a_1, a_2, \ldots, a_n \rangle$ is used to indicate the diagonal form $\sum_{i=1}^n a_i x_i^2$, or the corresponding lattice.

Given a $\mathbb{Z}$-lattice L, we denote by $L_p = L \otimes \mathbb{Z}_p$ the lattice L regarded as a lattice over $\mathbb{Z}_p$. For two lattices L and M over a ring R, where $R = \mathbb{Z}$ or $\mathbb{Z}_p$, we use the notation $M \simeq L$ to indicate that they are isometric over R. For an R-lattice L we denote by $B_L(x, y)$ the inner product of elements x, y in L, and we use the notation $L(x) = B_L(x, x)$.

For a rational number s we denote by sL the lattice L equipped with a new inner product B_{sL} defined by

$$B_{sL}(x, y) = sB_L(x, y).$$

In particular, $(sL)(x) = s(L(x))$. The lattice sL is thus obtained from L by *scaling* s. Such a lattice is sometime denoted by L^s (cf. [13]), but the notation L^s would not be practical in this paper. We should stress that in the case when s is in $\mathbb{Z}$, the lattice sL should *not* be confused with a sublattice of L defined by

$$\{x \in L \mid x = sy \text{ for some } y \in L\},$$

the latter being isometric to $s^2 L$.

A $\mathbb{Z}_p$-lattice L is called *unimodular* if its discriminant $d(L)$ is a unit in $\mathbb{Z}_p$. Clearly, L is a primitive $\mathbb{Z}_p$-lattice if and only if L splits off a unimodular sublattice.

Often we shall make no distinction between a lattice L and its Gram matrix, and we shall use the same symbol for both objects. In particular, throughout this paper we shall always denote by $\mathbb{A}$ (resp. $\mathbb{H}$) the $\mathbb{Z}$- or $\mathbb{Z}_2$-lattice corresponding to the matrix

$$\mathbb{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \left(\text{resp. } \mathbb{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right).$$

All $\mathbb{Z}$-lattices are assumed to be positive definite.

The Watson transformation. We shall now recall the definition of the Watson transformation δ_p and present its properties needed in this paper. (Watson introduced this transformation in [17]; see also [18], [5], [6], [7]).

Given a $\mathbb{Z}$- or $\mathbb{Z}_p$-lattice L define, for an integer m, a sublattice $\Lambda_m(L)$ by

$$\Lambda_m(L) = \{x \in L \mid L(x+y) - L(y) \equiv 0 \pmod{m} \ \text{ for all } y \in L\}.$$

The defining condition can also be expressed equivalently as

$$L(x) \equiv 2B_L(x,y) \equiv 0 \pmod{m}.$$

One sees easily that mx is in $\Lambda_m(L)$ for all x in L. The operation Λ_m was introduced essentially in order to handle the subsets

$$\{x \in L \mid L(x) \equiv 0 \pmod{m}\},$$

which are not lattices, in general (see however Proposition 2.2). We shall be interested only in the case when m is a prime or $m = 4$.

If p is an odd prime, let $\delta_p(L)$ be a primitive lattice obtained from $\Lambda_p(L)$ by scaling by a suitable rational number.

If $p = 2$ and L is an odd (resp. even) $\mathbb{Z}$- or $\mathbb{Z}_2$-lattice, let $\delta_2(L)$ be a primitive lattice obtained from $\Lambda_2(L)$ (resp. $\Lambda_4(L)$) by a suitable scaling.

Proposition 2.1. *Let L be a primitive $\mathbb{Z}$-lattice and let p be a prime number. The operations Λ_p and δ_p satisfy the following properties*

(i) $(\delta_p(L))_p \simeq \delta_p(L_p)$.

(ii) *If $p > 2$ and $L_p \simeq M \perp pN$ for some $\mathbb{Z}_p$-lattices M and N, M unimodular, then*

$$\Lambda_p(L_p) \simeq p^2 M \perp pN.$$

(iii) *For any prime $q \neq p$*

$$(\delta_p(L))_q \simeq \eta L_q,$$

where $\eta = 1$ or $1/p$. In particular, L is q-universal if and only if $\delta_p(L)$ is q-universal.

(iv) *For any prime q, L is q-anisotropic if and only if $\delta_p(L)$ is q-anisotropic.*

(v) *For any prime q*

$$\delta_p(\delta_q(L)) \simeq \delta_q(\delta_p(L)).$$

(vi) *If the lattices L and L' are in the same genus, then so are $\delta_p(L)$ and $\delta_p(L')$.*

(vii) *For each $\mathbb{Z}$-lattice M' in the genus of $\delta_p(L)$ there is a lattice M in the genus of L such that $\delta_p(M) = M'$.*

Proof. (i) is obvious from the definition of δ_p.

(ii) Let $x \in \Lambda_p(L_p)$ and $x = y + z$, where $y \in M$ and $z \in pN$. It suffices to show that y is in the sublattice of M consisting of all nonprimitive elements. If y were not of the form $y = pv$, $v \in M$, then $B_L(y, u) = 1$ for some $u \in M$ and one would have

$$L(x+u) - L(u) = L(x) + 2B_L(x, u) = L(x) + 2.$$

But the left side of this equality is 0 (mod p), while the right one is 2 (mod p), which is impossible.

(iii)-(vi) These properties are immediate consequences of the fact that $\delta_p(L)$ is obtained from $\Lambda_p(L)$ (resp. $\Lambda_4(L)$, if L is even and $p = 2$) by scaling $1, 1/p$ or $1/p^2$, and that $(\Lambda_p(L))_q \simeq L_q$ for $q \neq p$ (resp. $(\Lambda_4(L))_q \simeq L_q$ for $q > 2$).

(vii) is proved in [18]. □

Clearly,

$$\Lambda_2(L) = \{x \in L \mid L(x) \equiv 0 \pmod 2\}.$$

The analogous equality for $p > 2$ is not always valid. However it is valid in a few important cases.

Proposition 2.2. *Let L be a primitive quaternary $\mathbb{Z}$-lattice and let p be a prime.*

(i) *If $p > 2$ and L_p is either p-anisotropic or is not p-universal, then*

$$\Lambda_p(L) = \{x \in L \mid L(x) \equiv 0 \pmod p\}.$$

(ii) *If $p = 2$, L_2 is even and either L_2 is 2-anisotropic or L_2 does not represent all elements in $2\mathbb{Z}_2$, then*

$$\Lambda_4(L) = \{x \in L \mid L(x) \equiv 0 \pmod 4\}.$$

Proof. Since

$$\Lambda_m(L) = \{x \in L \mid L(x) \equiv 2B_L(x, y) \equiv 0 \pmod m \ \text{ for all } y \in L\},$$

in order to prove (i) (resp. (ii)), it suffices to prove the implication

$$L_p(x) \equiv 0 \ (\mathrm{mod}\ p) \Longrightarrow B_{L_p}(x, y) \equiv 0 \ (\mathrm{mod}\ p) \ \text{ for all } y \in L_p \qquad (*)$$

$$(\text{resp. } L_2(x) \equiv 0 \ (\mathrm{mod}\ 4) \Longrightarrow B_{L_2}(x, y) \equiv 0 \ (\mathrm{mod}\ 2) \text{ for all } y \in L_2). \quad (**)$$

(i) Since L_p is either p-anisotropic or is not p-universal, it suffices to prove the implication $(*)$ for L_p of the form

$$L_p \simeq \langle \epsilon, p\alpha, p\beta, p\gamma \rangle \quad \text{or} \quad L_p \simeq \langle 1, -\eta, p\alpha, p\beta \rangle,$$

where ϵ is a unit in $\mathbb{Z}_p$, α, β, γ are in $\mathbb{Z}_p$ and η is a nonsquare unit in $\mathbb{Z}_p$. But in either of these cases the implication $(*)$ is then trivial to check.

(ii) Since L_2 is even, primitive and either 2-anisotropic or does not represent all elements in $2\mathbb{Z}_2$, one has

$$L_2 \simeq \mathbb{A} \perp \ell,$$

for some even binary $\mathbb{Z}_2$-lattice ℓ. Moreover, $\ell(u) \equiv 0 \pmod 4$ for all u in ℓ. Indeed, if we would have $\ell(u) = 2\epsilon$ for some u in ℓ and some unit ϵ in $\mathbb{Z}_2$, then L_2 would contain a sublattice $\mathbb{A} \perp \langle 2\epsilon \rangle \simeq \mathbb{H} \perp \langle 10\epsilon \rangle$, which is 2-isotropic and represents all elements in $2\mathbb{Z}_2$.

If e_1, e_2 (resp. e_3, e_4) is a $\mathbb{Z}_2$-basis of $\mathbb{A}$ (resp. ℓ), and if for a vector $x = \sum_{i=1}^4 x_i e_i$ in L_2, $x_i \in \mathbb{Z}_2$, one has

$$L(x) \equiv 0 \pmod 4,$$

then

$$L_2(x_1 e_1 + x_2 e_2) = 2x_1^2 + 2x_1 x_2 + 2x_2^2 \equiv 0 \pmod 4,$$

which implies $x_1 \equiv x_2 \equiv 0 \pmod 2$. But then necessarily $B_{L_2}(x, y) \equiv 0 \pmod 2$ for all y in L_2, which shows $(**)$. □

Proposition 2.3. *Let L be a primitive almost regular (resp. regular) quaternary $\mathbb{Z}$-lattice. Then the lattice $\delta_p(L)$ is also almost regular (resp. regular) if any of the following conditions is satisfied:*

(i) *$p > 2$ and L is not p-universal.*
(ii) *L is p-anisotropic.*
(iii) *$p = 2$ and either L is odd or L is even and does not represent all elements in $2\mathbb{Z}_2$.*

Proof. Let $E(L)$ be the set of all positive integers represented by the genus of L but not by L itself. In particular, L is almost regular (resp. regular) if and only if $E(L)$ is finite (resp. empty).

To prove the proposition it suffices to show that, assuming either (i) or (ii) or (iii), the set $E(\Lambda_p(L))$ (or $E(\Lambda_4(L))$ if $p = 2$ and L is even) is contained in $E(L)$.

Case $p > 2$. Assume that L_p is either p-anisotropic or is not p-universal. Let $b \in E(\Lambda_p(L))$. Then b is represented by the $\mathbb{Z}_p$-lattice $\Lambda_p(L_p)$. In particular, b is divisible by p. If b were represented by L, then, by Proposition 2.2 (i), b would be represented by $\Lambda_p(L)$, contrary to the assumption that $b \in E(\Lambda_p(L))$. Hence b is not represented by L. But b is represented by $(\Lambda_p(L))_q \simeq L_q$ for every $q \neq p$, and clearly by L_p itself. Hence b is represented by the genus of L and thus b is in $E(L)$.

Case $p = 2$. If L is odd, then $E(L) \cap 2\mathbb{Z} = E(\Lambda_2(L))$.

Assume therefore that L is even and either L_2 is 2-anisotropic or L_2 is not representing all elements in $2\mathbb{Z}_2$. Let $b \in E(\Lambda_4(L))$. Then b is represented by $\Lambda_4(L_2)$. In particular, b is divisible by 4 and is represented by L_2. If b were represented by L, then, by Proposition 2.2 (ii), it would also be represented by $\Lambda_4(L)$, contrary to the assumption that $b \in E(\Lambda_4(L))$. Since b is represented by $(\Lambda_4(L))_q \simeq L_q$ for every $q > 2$, it follows that b is in $E(L)$. □

Now we shall investigate the effect of the iteration of the operation δ_p. For $k \geq 0$ define inductively $\delta_p^k(L)$ as follows:

$$\delta_p^0(L) = L \quad \text{and} \quad \delta_p^{k+1}(L) = \delta_p(\delta_p^k(L)).$$

Recall that for $p > 2$, a quaternary $\mathbb{Z}_p$-lattice M is p-universal if and only if M contains the hyperbolic plane $\langle 1, -1 \rangle$, or

$$M \simeq \langle 1, -\tau_p, p, -\tau_p p \rangle,$$

where τ_p is a nonsquare unit in $\mathbb{Z}_p$.

Proposition 2.4. *Let $p > 2$ be a prime number. For any primitive quaternary $\mathbb{Z}$-lattice L there is an integer $k \geq 0$ such that the $\mathbb{Z}$-lattice $\delta_p^k(L)$ is p-universal.*

Proof. Assume that L is not p-universal, otherwise $k = 0$. Let

$$L_p \simeq \langle \epsilon_1, \epsilon_2 p^{\alpha_2}, \epsilon_3 p^{\alpha_3}, \epsilon_4 p^{\alpha_4} \rangle,$$

where ϵ_i are units in $\mathbb{Z}_p$ and α_i are integers satisfying $\alpha_2 \leq \alpha_3 \leq \alpha_4$.

Given a p-adic integer $\gamma = p^v \epsilon$, ϵ a unit in $\mathbb{Z}_p$, let denote $v = \mathrm{ord}_p(\gamma)$. Since L_p is not p-universal, one has necessarily $\alpha_3 \geq 1$ and $\mathrm{ord}_p(d(L_p)) \geq 3$, where $d(L_p)$ is the discriminant of L_p.

If $\mathrm{ord}_p(d(L_p)) = 3$, then

$$L_p \simeq \langle 1, -\tau_p, \epsilon_3 p, \epsilon_4 p^2 \rangle \quad \text{or} \quad L_p \simeq \langle \epsilon_1, \epsilon_2 p, \epsilon_3 p, \epsilon_4 p \rangle.$$

Applying Proposition 2.1 (ii) one sees immediately that

$$\delta_p(L_p) \simeq \langle \epsilon_3, p, -\tau_p p, \epsilon_4 p \rangle \quad \text{or} \quad \delta_p(L_p) \simeq \langle \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_1 p \rangle.$$

In the second case, $\delta_p(L_p)$ is p-universal, and one takes $k = 1$. In the first case, applying again δ_p one gets $\delta_p^2(L_p)$ p-universal.

Assume now that $\mathrm{ord}_p(d(L_p)) \geq 4$. Since

$$\delta_p(L_p) \simeq \begin{cases} \langle \epsilon_1, \epsilon_2 p^{\alpha_2 - 2}, \epsilon_3 p^{\alpha_3 - 2}, \epsilon_4 p^{\alpha_4 - 2} \rangle & \text{if } \alpha_2 \geq 2, \\ \langle \epsilon_2, \epsilon_1 p, \epsilon_3 p^{\alpha_3 - 1}, \epsilon_4 p^{\alpha_4 - 1} \rangle & \text{if } \alpha_2 = 1, \\ \langle \epsilon_1, \epsilon_2, \epsilon_3 p^{\alpha_3 - 2}, \epsilon_4 p^{\alpha_4 - 2} \rangle & \text{if } \alpha_2 = 0 \text{ and } \alpha_3 \geq 2, \\ \langle \epsilon_3, \epsilon_1 p, \epsilon_2 p, \epsilon_4 p^{\alpha_4 - 1} \rangle & \text{if } \alpha_2 = 0 \text{ and } \alpha_3 = 1, \end{cases}$$

we have

$$\mathrm{ord}_p(d(\delta_p(L_p))) < \mathrm{ord}_p(d(L_p)).$$

Hence, arguing by induction, we deduce the existence of an integer k as requested in the proposition. $\square$

A quaternary $\mathbb{Z}_2$-lattice M is said to be $2\mathbb{Z}_2$-*universal* if M is *even* and either M is 2-isotropic and represents all numbers in $2\mathbb{Z}_2$, or $M \simeq \mathbb{A} \perp 2\mathbb{A}$. A quaternary $\mathbb{Z}$-lattice L is $2\mathbb{Z}_2$-universal if L_2 is $2\mathbb{Z}_2$-universal.

Proposition 2.5. *Let L be a primitive quaternary $\mathbb{Z}$-lattice. Then there is a $k \geq 0$ such that the $\mathbb{Z}$-lattice $\delta_2^k(L)$ is $2\mathbb{Z}_2$-universal.*

Before giving the proof we need five lemmas.

Lemma 2.6. *Let M be an odd unimodular quaternary $\mathbb{Z}_2$-lattice. Then $\delta_2(M)$ is $2\mathbb{Z}_2$-universal.*

Proof. There are exactly 8 odd unimodular quaternary $\mathbb{Z}_2$-lattices M. They are listed below, together with the corresponding expressions for $\delta_2(M)$.

M	$\delta_2(M)$	M	$\delta_2(M)$
$\mathbb{A} \perp \langle 1, 3 \rangle$	$\mathbb{A} \perp 2\mathbb{A}$	$\mathbb{A} \perp \langle 1, 7 \rangle$	$\mathbb{A} \perp 2\mathbb{H}$
$\mathbb{H} \perp \langle 1, 5 \rangle$	$\mathbb{H} \perp \langle 6, 14 \rangle$	$\mathbb{H} \perp \langle 1, 3 \rangle$	$\mathbb{H} \perp 2\mathbb{A}$
$\mathbb{A} \perp \langle 1, 1 \rangle$	$\mathbb{A} \perp \langle 2, 2 \rangle$	$\mathbb{A} \perp \langle 1, 5 \rangle$	$\mathbb{A} \perp \langle 6, 14 \rangle$
$\mathbb{A} \perp \langle 3, 3 \rangle$	$\mathbb{A} \perp \langle 6, 6 \rangle$	$\mathbb{A} \perp \langle 3, 7 \rangle$	$\mathbb{A} \perp \langle 2, 10 \rangle$

One sees easily that all these $\delta_2(M)$ are even $\mathbb{Z}_2$-lattices representing all elements in $2\mathbb{Z}_2$. Only the first lattice $M \simeq \mathbb{A} \perp \langle 1, 3\rangle$ is 2-anisotropic. The lemma follows. $\square$

Lemma 2.7. *Let $N = \langle \epsilon_1, \epsilon_2\rangle$ be a binary $\mathbb{Z}_2$-lattice, where ϵ_i are units in $\mathbb{Z}_2$. Let M be a quaternary $\mathbb{Z}_2$-lattice of the form*

$$M \simeq N \perp 2^\alpha \ell,$$

where $\alpha \geq 1$ and ℓ is a primitive binary $\mathbb{Z}_2$-lattice. Then

$$\delta_2(M) \simeq N' \perp 2^{\alpha-1}\ell$$

for some unimodular binary $\mathbb{Z}_2$-lattice N'.

Proof. First observe that N is isometric over $\mathbb{Z}_2$ to one of the following $\mathbb{Z}_2$-lattices:

$$\langle 1,1\rangle, \quad \langle 3,3\rangle, \quad \langle 1,3\rangle, \quad \langle 1,5\rangle, \quad \langle 3,7\rangle, \quad \langle 1,7\rangle.$$

Also observe that if $\{e_1, e_2\}$ is a basis of N satisfying $B_N(e_i, e_i) = \epsilon_i$, $B_N(e_1, e_2) = 0$, then

$$\Lambda_2(N) \simeq \mathbb{Z}_2(2e_1) + \mathbb{Z}_2(e_1 + e_2).$$

This allows to compute $\Lambda_2(N \perp 2^\alpha \ell)$, and thus also $\delta_2(N \perp 2^\alpha \ell)$, for each of the 6 cases of N. One has

$$\delta_2(N \perp 2^\alpha \ell) \simeq N' \perp 2^{\alpha-1}\ell,$$

where the N' corresponding to N are listed in the table below:

N	$\langle 1,1\rangle$	$\langle 3,3\rangle$	$\langle 1,3\rangle$	$\langle 1,5\rangle$	$\langle 3,7\rangle$	$\langle 1,7\rangle$
N'	$\langle 1,1\rangle$	$\langle 3,3\rangle$	$\mathbb{A}$	$\langle 3,7\rangle$	$\langle 1,5\rangle$	$\mathbb{H}$

Since every N' is unimodular, the proof is complete. $\square$

Lemma 2.8. *Let M be an odd quaternary $\mathbb{Z}_2$-lattice. Then there is a $k \geq 0$ such that $\delta_2^k(M)$ is either even or splits off a unimodular sublattice of rank ≥ 3.*

Proof. We may assume that either

$$M \simeq \langle \epsilon_1\rangle \perp 2^\alpha U \perp \langle \epsilon_2 2^\beta\rangle, \quad 1 \leq \alpha < \beta,$$

or

$$M \simeq \langle \epsilon_1, \epsilon_2 2^\beta\rangle \perp 2^\alpha \ell, \quad 0 \leq \beta \leq \alpha,$$

where ϵ_i are units in $\mathbb{Z}_2$, $U \simeq \mathbb{A}$ or $\mathbb{H}$, and ℓ is a primitive binary $\mathbb{Z}_2$-lattice.

In the first case, one has

$$\delta_2^k(M) \simeq \begin{cases} \langle 2\epsilon_1\rangle \perp U \perp \langle \epsilon_2 2^{\beta-\alpha}\rangle, & \text{if } \alpha = 2k-1,\ k > 0 \\ \langle \epsilon_1\rangle \perp U \perp \langle \epsilon_2 2^{\beta-\alpha}\rangle, & \text{if } \alpha = 2k,\ k > 0. \end{cases}$$

Therefore in this case $\delta_2^k(M)$ is either even, or the rank of its unimodular component is 3.

Consider now the second possibility, that is, $M \simeq \langle \epsilon_1, \epsilon_2 2^\beta\rangle \perp 2^\alpha \ell$, where $0 \leq \beta \leq \alpha$. If $\beta = 2m+1$ for some $m \geq 0$, then

$$\delta_2^m(M) \simeq \langle \epsilon_1, 2\epsilon_2\rangle \perp 2^{\alpha-2m}\ell$$

and

$$\delta_2^{m+1}(M) \simeq \langle \epsilon_2, 2\epsilon_1 \rangle \perp 2^{\alpha-\beta}\ell.$$

It follows that

$$\delta_2^{m+1+(\alpha-\beta)}(M) \simeq \langle \epsilon_i, 2\epsilon_j \rangle \perp \ell,$$

where $i, j = 1, 2$, $i \neq j$, either has a unimodular component of rank 3 (if ℓ is unimodular), or

$$\langle \epsilon_i, 2\epsilon_j \rangle \perp \ell \simeq \langle \epsilon_i, \epsilon_3 \rangle \perp 2\langle \epsilon_j, 2^\gamma \epsilon_4 \rangle$$

for some units ϵ_3, ϵ_4 in $\mathbb{Z}_2$ and some $\gamma \geq 0$. By Lemma 2.7, applying δ_2 to the latest lattice, one obtains a $\mathbb{Z}_2$-lattice which is either unimodular (if $\gamma = 0$), or has a unimodular component of rank 3 (if $\gamma > 0$).

Finally, if $\beta = 2m$ for some $m \geq 0$, then

$$\delta_2^m(M) \simeq \langle \epsilon_1, \epsilon_2 \rangle \perp 2^{\alpha-\beta}\ell.$$

By Lemma 2.7, applying to this lattice the operation δ_2 at most $\alpha - \beta$ times, one obtains either an even lattice, or a lattice having a unimodular component of rank ≥ 3. This completes the proof. $\square$

Lemma 2.9. *Let M be a quaternary $\mathbb{Z}_2$-lattice which the unimodular component is of rank 3. Then $\delta_2(M)$ is an even $\mathbb{Z}_2$-lattice.*

Proof. Since any unimodular ternary $\mathbb{Z}_2$-lattice splits off either the lattice $\mathbb{A}$ or $\mathbb{H}$, we may assume that

$$M \simeq U \perp \langle \epsilon_1 \rangle \perp \langle \epsilon_2 2^\alpha \rangle,$$

where $U \simeq \mathbb{A}$ or $\mathbb{H}$, the ϵ_i are units in $\mathbb{Z}_2$, and $\alpha \geq 1$. Then

$$\Lambda_2(M) \simeq U \perp \langle 4\epsilon_1 \rangle \perp \langle \epsilon_2 2^\alpha \rangle \quad \text{and} \quad \delta_2(M) = \Lambda_2(M),$$

which is an even lattice. $\square$

Lemma 2.10. *Let M be a primitive even quaternary $\mathbb{Z}_2$-lattice. Then, for some $k \geq 0$, the $\mathbb{Z}_2$-lattice $\delta_2^k(M)$ is $2\mathbb{Z}_2$-universal.*

Proof. The lattice M is either $2\mathbb{Z}_2$-universal or is of the form $M \simeq \mathbb{A} \perp \ell$ for some binary even $\mathbb{Z}_2$-lattice ℓ. Assume first that $\ell = 2^\alpha U$, where $U \simeq \mathbb{A}$ or $\mathbb{H}$. If $\alpha = 0$ or 1, the lattice M is $2\mathbb{Z}_2$-universal, and we take $k = 0$. For $\alpha \geq 2$ one has

$$\Lambda_4(M) \simeq 4\mathbb{A} \perp 2^\alpha U \quad \text{and} \quad \delta_2(M) \simeq \mathbb{A} \perp 2^{\alpha-2}U.$$

It follows that for $k = \left[\frac{\alpha}{2}\right]$

$$\delta_2^k(M) \simeq \mathbb{A} \perp U \quad \text{or} \quad \mathbb{A} \perp 2U,$$

and either of these lattices is $2\mathbb{Z}_2$-universal.

Assume now that $\ell \simeq \langle \epsilon_1 2^{\alpha_1}, \epsilon_2 2^{\alpha_2} \rangle$, where ϵ_i are units in $\mathbb{Z}_2$ and $1 \leq \alpha_1 \leq \alpha_2$. Observe that for any unit ϵ in $\mathbb{Z}_2$ the lattice $\mathbb{A} \perp \langle 2\epsilon \rangle \simeq \mathbb{H} \perp \langle 10\epsilon \rangle$ represents all elements in $2\mathbb{Z}_2$.

If $\alpha_1 = 1$ then $M \simeq \mathbb{A} \perp \langle 2\epsilon_1 \rangle \perp \langle \epsilon_2 2^{\alpha_2} \rangle$ is $2\mathbb{Z}_2$-universal. For $\alpha_1 \geq 2$ one has

$$\Lambda_4(M) \simeq 4\mathbb{A} \perp \langle \epsilon_1 2^{\alpha_1}, \epsilon_2 2^{\alpha_2} \rangle \ \text{ and } \ \delta_2(M) \simeq \mathbb{A} \perp \langle \epsilon_1 2^{\alpha_1-2}, \epsilon_2 2^{\alpha_2-2} \rangle.$$

If $\alpha_1 = 2m+1$, then

$$\delta_2^m(M) \simeq \mathbb{A} \perp \langle 2\epsilon_1 \rangle \perp \langle \epsilon_2 2^{\alpha_2-2m} \rangle$$

is $2\mathbb{Z}_2$-universal.

If $\alpha_1 = 2m$, then

$$\delta_2^m(M) \simeq \mathbb{A} \perp \langle \epsilon_1, \epsilon_2 2^{\alpha_2 - 2m} \rangle$$

is an *odd* $\mathbb{Z}_2$-lattice. If $\alpha_2 - 2m = 2t$ for some $t \geq 0$, then

$$\delta_2^{m+2t}(M) \simeq \mathbb{A} \perp \langle \epsilon_1, \epsilon_2 \rangle$$

is an odd unimodular $\mathbb{Z}_2$-lattice. By Lemma 2.6, the lattice $\delta_2^{m+2t+1}(M)$ is then $2\mathbb{Z}_2$-universal. Finally, if $\alpha_2 - 2m = 2t+1$, then

$$\delta_2^{m+2t+1}(M) \simeq \mathbb{A} \perp \langle 2\epsilon_2 \rangle \perp \langle 4\epsilon_1 \rangle$$

is $2\mathbb{Z}_2$-universal. This completes the proof. □

Now we are ready to prove Proposition 2.5.

Proof of Proposition 2.5. If L is an odd primitive quaternary $\mathbb{Z}$-lattice then, by Lemmas 2.6, 2.8 and 2.9, the lattice $\delta_2^m(L)$ is even for some $m > 0$. We can therefore assume L even and apply Lemma 2.10 to complete the proof. □

Operation Δ_p . Fix a prime number p. Let L be a primitive quaternary $\mathbb{Z}$- or $\mathbb{Z}_p$-lattice. Let $k \geq 0$ be the smallest integer such that $\delta_p^k(L)$ is p-universal (if $p > 2$), or $2\mathbb{Z}_2$-universal (if $p = 2$). The existence of such a k follows from Propositions 2.4 and 2.5. For such a k define

$$\Delta_p(L) = \delta_p^k(L).$$

Operation Δ. Let L be a primitive quaternary $\mathbb{Z}$-lattice and let $\{p_1, \ldots, p_m\}$ be the set of all primes p such that L_p is not p-universal (if $p > 2$), or not $2\mathbb{Z}_2$-universal (if $p = 2$). Define $\Delta(L)$ to be a quaternary $\mathbb{Z}$-lattice obtained from L by applying successively the operations Δ_{p_i}, $i = 1, \ldots, m$,

$$\Delta(L) = \Delta_{p_m}(\cdots(\Delta_{p_1}(L))\cdots).$$

The lattice $\Delta(L)$ is therefore even, p-universal for every $p > 2$ and $2\mathbb{Z}_2$-universal.

The next theorem is the main result of this section and one of the main tools needed in the proof of the results stated in the Introduction. Recall that a $\mathbb{Z}$-lattice L is called *even-universal* if L is even and represents all even positive integers. If L represents all positive integers then L is said to be *universal*. Let $S(L)$ be the set of all primes p such that L is p-anisotropic.

Theorem 2.11. *Let L be a primitive almost regular quaternary $\mathbb{Z}$-lattice. Then*

(i) *$\Delta_p(L)$ is an almost regular $\mathbb{Z}$-lattice which is p-universal if $p > 2$, and $2\mathbb{Z}_2$-universal if $p = 2$. If L is regular or p-anisotropic then $\Delta_p(L)$ is regular.*

(ii) *$\Delta(L)$ is an even $\mathbb{Z}$-lattice representing all large even integers. If $S(L) \neq \emptyset$, then $\Delta(L)$ is even-universal.*

(iii) *$S(L) = S(\Delta_p(L)) = S(\Delta(L))$.*

Proof. (i) The first part of (i) follows from Proposition 2.3 and the definition of $\Delta_p(L)$. Proposition 2.3 also implies that $\Delta_p(L)$ is regular if L is regular. Assume therefore that L is p-anisotropic. To show that $\Delta_p(L)$ is regular let consider an integer m represented by the genus of $\Delta_p(L)$. Then $\Delta_p(L)$ represents all mp^{2n} for n large. Since $\Delta_p(L_p)$ is p-anisotropic and p-universal (if $p > 2$), or $\Delta_2(L_2) \simeq \mathbb{A} \perp 2\mathbb{A}$ (if $p = 2$), if $\Delta_p(L)(x) = mp^{2n}$ for some $n \geq 1$, then $x = py$ for some $y \in L$ and $\Delta_p(L)(y) = mp^{2(n-1)}$. Arguing by induction one deduces that $\Delta_p(L)$ represents m. Hence $\Delta_p(L)$ is regular.

(ii) The lattice $\Delta(L)$ represents every even positive integer over every ring $\mathbb{Z}_q$. Since $\Delta(L)$ is almost regular, it represents all large even integers. If $S(L) \neq \emptyset$, then by (i), $\Delta(L)$ is regular and thus represents all even positive integers.

(iii) Follows from Proposition 2.1 (iv) and the definition of $\Delta_p(L)$ and $\Delta(L)$. □

3. Even-universal p-anisotropic lattices

Recall that $\mathcal{E}_p$ denotes the family of all primitive almost regular p-anisotropic quaternary $\mathbb{Z}$-lattices. In this section we shall prove Theorem 1.2 after first investigating the subfamily $\mathcal{U}_p$ of $\mathcal{E}_p$ of lattices in $\mathcal{E}_p$ which are even-universal (if $p = 2$ we further require that for L in $\mathcal{U}_2$ one has $L_2 \simeq \mathbb{A} \perp 2\mathbb{A}$). This family is easier to handle than the (often) larger family $\mathcal{R}_p$, and yet $\mathcal{U}_p$ already provides important information about $\mathcal{E}_p$. Our interest in $\mathcal{U}_p$ comes essentially from two facts: for $L \in \mathcal{E}_p$ one has $\Delta(L) \in \mathcal{U}_p$ (cf. Theorem 2.11), and, for $p \geq 17$, $\mathcal{R}_p = \mathcal{U}_p$ (cf. Theorem 6.1). Both these properties have for reaching consequences.

The list $\mathcal{F}$ of all even-universal quaternary $\mathbb{Z}$-lattices was recently obtained with the help of "The 290 Theorem", which says that an even lattice L is in $\mathcal{F}$ if and only if L represents every positive integer not exceeding 290 (in fact, only 29 specific integers, the largest being 290, should be represented) (cf. [8], [12]). The family $\mathcal{F}$ is quite large, it contains exactly 6436 elements. The largest discriminant of a lattice in $\mathcal{F}$ is 4292. We shall see that the subset $\bigcup_p \mathcal{U}_p$ of $\mathcal{F}$ is relatively small and contains only 123 lattices, most of them belonging to $\mathcal{U}_2$.

Family $\mathcal{U}_2$. The elements of $\mathcal{U}_2$ are precisely the even-universal quaternary $\mathbb{Z}$-lattices L with $L_2 \simeq \mathbb{A} \perp 2\mathbb{A}$. The family $\mathcal{U}_2$ was already investigated in [3] where it played a crucial role in obtaining a complete description of all almost universal quaternaries. It contains exactly 79 lattices which are listed, with their discriminants, in [3] Table 4. All of them are q-isotropic for $q > 2$ and they represent 65 genera. The set U_2 of all primes dividing the discriminant of some lattice in $\mathcal{U}_2$ contains exactly 29 primes which are listed below:

$$U_2 = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 47, 59, 61, 67, 73, 89, 97, 107,\\ 113, 137, 193, 233, 241, 257, 281, 353\}.$$

It should be mentioned that there are even-universal 2-anisotropic $\mathbb{Z}$-lattices which are not in $\mathcal{U}_2$. They are of two kinds: the lattices of the form $2M$, where M is universal and 2-anisotropic (the most famous example is $M =$

$\langle 1,1,1,1\rangle$), and the lattices L which are primitive, 2-anisotropic and even-universal, but with $L_2 \not\cong \mathbb{A} \perp 2\mathbb{A}$. Since for any lattice L in $\mathcal{E}_2$ one has $\Delta(L)$ in $\mathcal{U}_2$, it suffices for our purposes to deal only with those 2-anisotropic even-universal lattices which are in $\mathcal{U}_2$. □

We shall now study the families $\mathcal{U}_p$ with $p > 2$.

Proposition 3.1. *If $p > 37$ the family $\mathcal{U}_p$ is empty.*

Proof. Let L be a lattice in $\mathcal{U}_p$. Then L contains at least one of the following binary $\mathbb{Z}$-lattices defined below by their Gram matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

The smallest even positive integers *not* represented by these lattices are, respectively, 6, 4, 10, 6. It follows that L contains a ternary sublattice M having the successive minima $\mu_i(M)$ satisfying

$$\mu_1(M) = 2, \quad \mu_2(M) \leq 4, \quad \mu_3(M) \leq 10,$$

and therefore

$$d(M) \leq \prod_{i=1}^{3} \mu_i(M) \leq 80.$$

Since the discriminant of any even p-anisotropic ternary $\mathbb{Z}$-lattice is divisible by $2p$, it follows that $2p \leq 80$, which shows that $p \leq 37$. □

Next we shall study the families $\mathcal{U}_p$, where $17 \leq p \leq 37$.

Let

$$P^* = \{17, 19, 23, 29, 31, 37\}.$$

For each $p \in P^*$ (resp. $p \in \{23, 31\}$) let $G(p)$ (resp. $G'(p)$) be a quaternary $\mathbb{Z}$-lattice defined by the following matrices:

$$G(17) = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & 2 & -1 \\ 1 & 2 & 6 & 2 \\ 0 & -1 & 2 & 10 \end{bmatrix}, \quad G(19) = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 6 & 3 \\ 0 & 2 & 3 & 12 \end{bmatrix},$$

$$G(23) = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ 1 & 0 & 0 & 12 \end{bmatrix}, \quad G(29) = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & 1 & 8 & 1 \\ 1 & 2 & 1 & 16 \end{bmatrix},$$

$$G(31) = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 8 & 0 \\ 1 & 0 & 0 & 16 \end{bmatrix}, \quad G(37) = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 4 & 1 & 2 \\ 1 & 1 & 10 & 1 \\ 1 & 2 & 1 & 20 \end{bmatrix},$$

$$G'(23) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 46 \end{bmatrix}, \quad G'(31) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 62 \end{bmatrix}.$$

Observe that $d(G(p)) = p^2$ and $d(G'(p)) = 4p^2$.

In the next theorem we shall refer to the sets Ω_p, $p \in P^*$, defined in the Introduction.

Theorem 3.2. *Let $p \in P^*$. For any even p-anisotropic quaternary $\mathbb{Z}$-lattice L the following properties are equivalent:*

(i) *L is in $\mathfrak{U}_p$.*

(ii) *L represents all elements of the set Ω_p.*

(iii) $L \simeq \begin{cases} G(p) & \text{if } p \in \{17, 19, 29, 37\}, \\ G(p) \text{ or } G'(p) & \text{if } p \in \{23, 31\}. \end{cases}$

Before giving the proof we need some preparation. A sublattice M of a lattice L, $M \neq L$, is said to be *primitive* if M has the following property: for each x in L, if kx is in M for some k in $\mathbb{Z} \setminus \{0\}$, then x is in M. Observe that a primitive sublattice need not be primitive as a lattice.

Lemma 3.3. *Let M be a primitive sublattice of L and let $x \in L \setminus M$. Then*

$$d(M + \mathbb{Z}x) \leq d(M)L(x).$$

Furthermore, the equality implies that $M + \mathbb{Z}x = M \perp \langle L(x) \rangle$.

For the proof of this result see [1] p. 330. $\square$

It is convenient to introduce the following notation: if A is a lattice isometric to a sublattice of L, we shall write $A \to L$. In the contrary case, we put $A \not\to L$.

Let denote

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Lemma 3.4. *Let $p \in P^*$ and let L be an even p-anisotropic quaternary $\mathbb{Z}$-lattice which represents all elements in Ω_p. If $p \in \{17, 19\}$ (resp. $p \in \{23, 29, 31, 37\}$), then $A_3 \to L$ and $A_i \not\to L$ for $i \neq 3$ (resp. $A_4 \to L$ and $A_j \not\to L$ for $j \neq 4$).*

Proof. Since 2 and 4 are in Ω_p, $A_i \to L$ for some i. We shall prove the rest of the lemma in 3 steps.

(1) Since A_i should be p-anisotropic and $d(A_i)$ is not divisible by p, it follows that if $A_i \to L$ then

$$\left(\frac{d(A_i)}{p}\right) = -\left(\frac{-1}{p}\right).$$

This excludes the following cases

$$\begin{array}{rll} A_2,\ A_4 \not\to L & \text{if} & p = 17 \\ A_1,\ A_4 \not\to L & \text{if} & p = 19 \\ A_3 \not\to L & \text{if} & p = 23 \end{array} \qquad \begin{array}{rll} A_2,\ A_3 \not\to L & \text{if} & p = 29 \\ A_1 \not\to L & \text{if} & p = 31 \\ A_1,\ A_2,\ A_3 \not\to L & \text{if} & p = 37. \end{array}$$

(2) Now we shall prove that $A_1 \not\to L$ and $A_2 \not\to L$. Recall that the discriminant of any ternary sublattice of an even p-anisotropic quaternary $\mathbb{Z}$-lattice is divisible by $2p$. If $A_1 \to L$, then L contains a ternary sublattice T of the form

$$T = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_1}} & a \\ & & b \\ \hline a & b & c \end{array}\right],$$

where $0 \le a \le 1$, $0 \le |b| \le 1$, c even and $2 \le c \le 4$. Since $2p$ should divide $d(T)$ and, by Lemma 3.3, $d(T) \le cd(A) \le 12$, we get a contradiction. If $A_2 \to L$, then by (1), $p = 19$, 23 or 31. Since $6 \not\to A_2$ but $6 \in \Omega_p$ for $p = 19$, 23, 31, it follows as above that for some ternary sublattice T of L one has $d(T) \le 6d(A_2) = 24$. This is not possible, as $d(T)$ should be divisible by $2p$.

(3) If $p = 31$, then $A_3 \not\to L$. Indeed, since $6 \not\to A_3$ but $6 \in \Omega_{31}$, if $A_3 \to L$ then L contains a ternary sublattice T with $d(T) \le 6d(A_3) = 42$. But then $d(T)$ would not be divisible by $2 \cdot 31$.

Now the lemma follows from (1), (2) and (3). □

Observe that for L as in Lemma 3.4 one has $\mu_1(L) = 2$ and $\mu_2(L) = 4$. Define the following ternary $\mathbb{Z}$-lattices by their Gram matrices:

$$T(17) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_3}} & 1 \\ & & 2 \\ \hline 1 & 2 & 6 \end{array}\right], \quad T(19) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_3}} & 1 \\ & & 1 \\ \hline 1 & 1 & 6 \end{array}\right], \quad T(23) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_4}} & 0 \\ & & 1 \\ \hline 0 & 1 & 6 \end{array}\right],$$

$$T(29) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_4}} & 1 \\ & & 1 \\ \hline 1 & 1 & 8 \end{array}\right], \quad T(31) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_4}} & 0 \\ & & 1 \\ \hline 0 & 1 & 8 \end{array}\right], \quad T(37) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_4}} & 1 \\ & & 1 \\ \hline 1 & 1 & 10 \end{array}\right].$$

Observe that $d(T(p)) = 2p$ and that $\mu_3(T(p))$ is the integer in the lower right corner of the matrix defining $T(p)$.

Lemma 3.5. *Let $p \in P^*$ and let L be an even p-anisotropic quaternary $\mathbb{Z}$-lattice representing all elements in the set Ω_p. Let*

$$T(a,b,c) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & a \\ & & b \\ \hline a & b & c \end{array}\right],$$

where $0 \le a \le 1$, $0 \le |b| \le 2$, c even and

$$\begin{aligned} A &= A_3, \quad 4 \le c \le 6 \quad \text{if} \quad p = 17, 19, \\ A &= A_4, \quad 4 \le c \le 10 \quad \text{if} \quad p > 19. \end{aligned}$$

Then $T(a,b,c) \to L$ for some a, b, c as above, and any such $T(a,b,c)$ is isometric to $T(p)$. Furthermore, $\mu_i(T(p)) = \mu_i(L)$ for $i = 1, 2, 3$.

Proof. By assumption, $6 \to L$ for $p = 17, 19$ (resp. $10 \to L$ for $p > 19$). Since $6 \not\to A_3$ and $10 \not\to A_4$, it follows from Lemma 3.4 that $T(a,b,c) \to L$ for some a, b, c. The direct computation of all discriminants $d(T(a,b,c))$ shows that either $d(T(a,b,c)) = 2p$, or $d(T(a,b,c))$ is not divisible by $2p$. Then one checks that every $T(a,b,c)$ with $d(T(a,b,c)) = 2p$ is isometric to $T(p)$. Since the discriminant of any ternary sublattice of L is divisible by $2p$, the lemma follows. □

Proof of Theorem 3.2.

(i) $\Longrightarrow$ (ii) is obvious.

(iii) $\Longrightarrow$ (i) Each of the lattices $G(p)$ or $G'(p)$ is p-anisotropic primitive and on the list $\mathcal{F}$ of all even-universal $\mathbb{Z}$-lattices (cf. [8], [12]). Hence if $L \simeq G(p)$ or $G'(p)$, then L is in $\mathcal{U}_p$.

(ii) $\Longrightarrow$ (iii) Let L be an even p-anisotropic quaternary $\mathbb{Z}$-lattice representing all elements in Ω_p.

Define $w_{17} = 34$, $w_{19} = 20$, $w_{23} = 46$, $w_{29} = 58$, $w_{31} = 62$, $w_{37} = 58$. Note that $w_p \in \Omega_p$ and $w_p \not\to T(p)$. We know from Lemma 3.5 that $T(p) \to L$. We can therefore assume that $T(p)$ is a primitive sublattice of L and $\mu_i(T(p)) = \mu_i(L)$ for $i = 1, 2, 3$. Choose a basis $e_1, \ldots, e_4$ such that $\mu_i(L) = L(e_i)$, $i = 1, \ldots, 4$. Such a basis always exists for a quaternary $\mathbb{Z}$-lattice (cf. [11] Corollary 6.2.3). We can therefore deduce that the Gram matrix of L with respect to this basis is of the form

$$L \simeq L(a, b, c, f) = \left[\begin{array}{ccc|c} & & & a \\ & T(p) & & b \\ & & & c \\ \hline a & b & c & f \end{array}\right],$$

for some integers a, b, c, f satisfying the following conditions: $0 \le a \le 1$, $0 \le |b| \le 2$, $0 \le |c| \le \frac{1}{2}\mu_3(T(p))$, and $\mu_3(T(p)) \le f = \mu_4(L) \le w_p$. Since $L \simeq L(a, b, c, f)$ is p-anisotropic, the discriminant $d(L(a, b, c, f))$ is divisible by p^2. The direct computation of all $d(L(a, b, c, f))$ and checking the isometry class of these $L(a, b, c, f)$ whose discriminant is divisible by p^2, imply the following conclusions which prove the implication (ii)$\Longrightarrow$(iii).

For $p = 17$, if 17^2 divides $d = d(L(a, b, c, f))$, then $d = 17^2$ and

$$L \simeq L(a, b, c, f) \simeq G(17) \quad \text{or} \quad L \simeq T(17) \perp \langle 34 \rangle.$$

But $68 \not\to T(17) \perp \langle 34 \rangle$ and $68 \in \Omega_{17}$. Hence $L \simeq G(17)$.

For $p = 19$, if 19^2 divides $d = d(L(a, b, c, f))$, then $d = 19^2$ and

$$L \simeq L(a, b, c, f) \simeq G(19).$$

For $p = 23$, if 23^2 divides $d = d(L(a, b, c, f))$, then $d = 23^2$ or $d = 4 \cdot 23^2$, and

$$L \simeq L(a, b, c, f) \simeq G(23) \quad \text{or} \quad L \simeq G'(23).$$

For $p = 29$, if 29^2 divides $d = d(L(a, b, c, f))$, then $d = 29^2$ and

$$L \simeq L(a, b, c, f) \simeq G(29) \quad \text{or} \quad L \simeq T(29) \perp \langle 58 \rangle.$$

But $290 \not\to T(29) \perp \langle 58 \rangle$ and $290 \in \Omega_{29}$. Hence $L \simeq G(29)$.

For $p = 31$, if 31^2 divides $d = d(L(a, b, c, f))$, then $d = 31^2$ or $d = 4 \cdot 31^2$, and

$$L \simeq L(a, b, c, f) \simeq G(31) \quad \text{or} \quad L \simeq G'(31).$$

For $p = 37$, if 37^2 divides $d = d(L(a, b, c, f))$, then $d = 37^2$ and

$$L \simeq L(a, b, c, f) \simeq G(37).$$

This completes the proof of Theorem 3.2. $\square$

We shall now deal with the remaining families $\mathcal{U}_p$ for $p = 3, 5, 7, 11, 13$. For each $p \in \{3, 5, 7, 11, 13\}$ define first a Table (p) containing some quaternary $\mathbb{Z}$-lattices G and their discriminants $d(G)$. A lattice G in Table (p) will be identified by its Gram matrix $(a_{ij})_{ij}$, and we shall represent a symmetric 4×4 matrix $(a_{ij})_{ij}$ by a sequence of 10 numbers a_{11}, a_{22}, a_{33}, a_{44}, a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}.

G	$d(G)/3^2$	G	$d(G)/3^2$
2 2 2 2 1 0 0 0 0 1	1	2 4 6 6 1 0 0 0 0 0	$2^2 \cdot 7$
2 2 4 4 0 1 1 1 1 1	2^2	2 2 6 12 0 0 0 0 0 3	$2^2 \cdot 7$
2 2 2 6 0 0 0 1 0 0	2^2	2 4 6 8 1 0 0 0 2 3	5^2
2 2 6 8 0 0 0 0 1 3	2^4	2 4 4 10 1 1 1 1 0 0	5^2
2 4 4 6 1 1 0 1 0 0	2^4	2 4 6 12 1 0 0 0 0 3	7^2
2 4 4 8 1 1 0 1 2 2	2^4	2 4 6 10 1 0 1 0 0 3	37

Table (3)

G	$d(G)/5^2$	G	$d(G)/5^2$
2 2 4 4 1 1 0 1 1 2	1	2 4 4 14 0 0 0 1 1 −1	2^4
2 2 4 10 1 1 0 1 0 0	2^2	2 4 8 8 0 1 1 0 0 −2	2^4
2 4 4 4 0 0 0 1 1 −1	2^2	2 4 4 20 1 1 0 2 0 0	2^4
2 4 4 6 0 1 0 2 2 1	2^2	2 4 6 10 1 0 0 1 0 0	2^4
2 4 10 14 0 0 0 0 1 0	$2^2 \cdot 11$	2 4 4 8 0 0 1 1 0 0	3^2
2 4 4 20 0 0 0 1 0 0	$2^3 \cdot 3$	2 4 4 12 0 1 1 2 1 1	3^2
2 4 8 10 0 1 0 0 0 0	$2^3 \cdot 3$	2 4 6 6 1 0 0 1 1 1	3^2

Table (5)

G	$d(G)/7^2$	G	$d(G)/7^2$
2 2 4 4 0 0 1 1 0 0	1	2 4 4 8 1 0 0 0 0 2	2^2
2 4 6 6 0 1 1 1 1 −1	2^2	2 2 4 14 0 0 0 1 0 0	2^2

Table (7)

G	$d(G)/11^2$	G	$d(G)/11^2$
2 2 6 6 0 0 1 1 0 0	1	2 2 6 22 0 0 0 1 0 0	2^2

Table (11)

G	$d(G)/13^2$	G	$d(G)/13^2$
2 4 4 8 0 1 1 1 2 1	1	2 4 10 10 0 0 0 1 1 −3	2^2
2 4 4 26 0 1 0 1 0 0	2^2	2 4 10 26 0 0 0 1 0 0	$2^2 \cdot 3$

Table (13)

Define also the sets

$$\Omega_3 = \{2, 4, 6, 10, 12, 14, 20, 26\}, \quad \Omega_5 = \{2, 4, 6, 10, 20, 30\}$$
$$\Omega_7 = \{2, 4, 6, 10, 12, 14, 42, 70\}, \quad \Omega_{11} = \{2, 4, 6, 10, 20, 44\},$$
$$\Omega_{13} = \{2, 4, 10, 26\}$$

Theorem 3.6. *Let $p \in \{3, 5, 7, 11, 13\}$. For any even primitive p-anisotropic quaternary $\mathbb{Z}$-lattice L the following properties are equivalent:*

(i) *L is $\mathcal{U}_p$.*
(ii) *L represents all elements of the set Ω_p.*
(iii) *$L \simeq G$, where G is a lattice listed in Table (p).*

Sketch of the proof.
(i) $\Longrightarrow$ (ii) is obvious.

(iii) $\Longrightarrow$ (i) Each of the lattices G listed in Table (p) is primitive p-anisotropic and on the list $\mathcal{F}$ of all even-universal quaternary $\mathbb{Z}$-lattices (cf. [8], [12]). It follows that G, and therefore L, is in $\mathcal{U}_p$.

(ii) $\Longrightarrow$ (iii) The proof of this implication goes along similar lines as the proof of the analogous implication in Theorem 3.2. Let us briefly indicate how the arguments used in the proof of Theorem 3.2 should be modified. Lemma 3.4 should be replaced by

Lemma 3.4′. *Let $p \in \{3,5,7,11,13\}$ and let L be an even p-anisotropic quaternary $\mathbb{Z}$-lattice which represents all elements in Ω_p. Then*

$$\begin{array}{lll}
A_2 \text{ or } A_3 \to L, \quad \text{and } A_1,\ A_4 \nrightarrow L, & \text{if} & p=3,\\
A_1 \text{ or } A_3 \text{ or } A_4 \to L, \quad \text{and } A_2 \nrightarrow L, & \text{if} & p=5,\\
A_2 \text{ or } A_3 \text{ or } A_4 \to L, \quad \text{and } A_1 \nrightarrow L, & \text{if} & p=7,\\
A_2 \to L, \quad \text{and } A_1,\ A_3,\ A_4 \nrightarrow L, & \text{if} & p=11,\\
A_4 \to L, \quad \text{and } A_1,\ A_2,\ A_3 \nrightarrow L, & \text{if} & p=13.
\end{array}$$

The proof of this lemma is analogous to that of Lemma 3.4.

Lemma 3.5 should be modified as follows. In Lemma 3.5 intervenes, for each $p \in P^*$, a single ternary lattice $T(p)$. Now, for $p \le 13$, we have usually several ternary lattices $T_i(p)$ playing this role. These lattices are defined below, by their Gram matrices, for each $p = 3,5,7,11,13$.
$p=3$

$$T_1(3) = \begin{bmatrix} 2&0&0\\0&2&0\\0&0&6 \end{bmatrix}^{12}_{24},\ T_2(3) = \begin{bmatrix} 2&0&0\\0&2&1\\0&1&2 \end{bmatrix}^{12}_{6},\ T_3(3) = \begin{bmatrix} 2&0&1\\0&2&1\\1&1&4 \end{bmatrix}^{6}_{12},$$
$$T_4(3) = \begin{bmatrix} 2&1&0\\1&4&0\\0&0&6 \end{bmatrix}^{12}_{42},\ T_5(3) = \begin{bmatrix} 2&1&1\\1&4&0\\1&0&4 \end{bmatrix}^{12}_{24}$$

$p=5$

$$T_1(5) = \begin{bmatrix} 2&1&1\\1&2&1\\1&1&4 \end{bmatrix}^{10}_{10},\ T_2(5) = \begin{bmatrix} 2&0&0\\0&4&0\\0&0&10 \end{bmatrix}^{20}_{80},\ T_3(5) = \begin{bmatrix} 2&0&0\\0&4&1\\0&1&4 \end{bmatrix}^{20}_{30},$$
$$T_4(5) = \begin{bmatrix} 2&0&0\\0&4&2\\0&2&6 \end{bmatrix}^{10}_{40},\ T_5(5) = \begin{bmatrix} 2&0&1\\0&4&0\\1&0&8 \end{bmatrix}^{10}_{60},\ T_6(5) = \begin{bmatrix} 2&1&1\\1&4&2\\1&2&4 \end{bmatrix}^{20}_{20},$$
$$T_7(5) = \begin{bmatrix} 2&1&0\\1&4&1\\0&1&6 \end{bmatrix}^{10}_{40}$$

$p=7$

$$T_1(7) = \begin{bmatrix} 2&0&0\\0&2&1\\0&1&4 \end{bmatrix}^{42}_{14},\ T_2(7) = \begin{bmatrix} 2&0&0\\0&4&2\\0&2&8 \end{bmatrix}^{14}_{56},\ T_3(7) = \begin{bmatrix} 2&0&1\\0&4&1\\1&1&6 \end{bmatrix}^{14}_{42},$$
$$T_4(7) = \begin{bmatrix} 2&1&0\\1&4&0\\0&0&4 \end{bmatrix}^{10}_{28}$$

$p = 11$

$$T(11) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 6 \end{bmatrix}^{44}_{22}$$

$p = 13$

$$T_1(13) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 10 \end{bmatrix}^{26}_{78}, \; T_2(13) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}^{26}_{26}, \; T_3(13) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 4 & 2 \\ 1 & 2 & 8 \end{bmatrix}^{10}_{52}$$

The numbers given at the right side of each of the matrices above indicate the smallest positive integer not represented by the corresponding lattice (the upper number), and its discriminant (the lower number). These data are used in the proof of Theorem 3.6 in the way as the numbers w_p and $d(T(p))$ were used in the proof of Theorem 3.2.

The version of Lemma 3.5 for, say, $p = 3$ is now as follows.

Lemma 3.5′. ($p = 3$) *Let L be a primitive even 3-anisotropic quaternary $\mathbb{Z}$-lattice which represents all numbers in the set Ω_3. Let*

$$T(a,b,c,A) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{A}} & a \\ & & b \\ \hline a & b & c \end{array}\right],$$

where $0 \le a \le 1$, c even, and either $0 \le |b| \le 1$, $2 \le c \le 6$, $A = A_2$, or $0 \le |b| \le 2$, $4 \le c \le 10$, $A = A_3$. Then $T(a,b,c,A) \to L$ for some a, b, c, A as above, and any such $T(a,b,c,A)$ is isometric to some $T_i(3)$, $i = 1, \ldots, 5$.

The proof of this lemma is an easy adaptation of the proof of Lemma 3.5. The analogous statements for $p = 5$, 7, 11, 13 are now obvious to formulate (using the lattices $T_i(p)$, $p = 5$, 7, 11, 13).

Once Lemmas 3.4 and 3.5 are modified as just explained, the rest of the proof of Theorem 3.6 (ii) $\Rightarrow$ (iii) goes along the same line as that of Theorem 3.2, and is left to the reader. □

Let $\mathcal{U}'_p$ be the set of all even-universal p-anisotropic $\mathbb{Z}$-lattices. If M is a p-anisotropic universal $\mathbb{Z}$-lattice then obviously $2M$ is in $\mathcal{U}'_p$. No such a lattice exists for $p > 7$ (cf. [2]). For $p = 3$ and 7 there is just one universal lattice (namely $M(3)$ and $M(7)$, corresponding to the forms $x^2 + y^2 + 3z^2 + 3t^2$ and $x^2 + 2y^2 + 4z^2 + 7t^2 + 2yz$), and for $p = 5$ there are exactly two universal lattices $M(5)$ and $M'(5)$ (corresponding to the form $x^2 + 2y^2 + 5z^2 + 10t^2$ and $x^2 + 2y^2 + 3z^2 + 5t^2 + 2yz$), (cf. [2]). One has therefore $\mathcal{U}'_3 = \mathcal{U}_3 \cup \{2M(3)\}$, $\mathcal{U}'_5 = \mathcal{U}_5 \cup \{2M(5), 2M'(5)\}$, $\mathcal{U}'_7 = \mathcal{U}_7 \cup \{2M(7)\}$ and $\mathcal{U}'_p = \mathcal{U}_p$ for $p > 7$.

Let U_p be the set of all primes dividing the discriminant of some lattice in $\mathcal{U}_p$. Theorems 3.2 and 3.6 imply the following corollary.

Corollary 3.7. *One has*

$$U_3 = \{2,3,5,7,37\}, \quad U_5 = \{2,3,5,11\}, \quad U_{13} = \{2,3,13\},$$
$$U_p = \{2,p\} \;\; \textit{for} \;\; p = 7,11,23,31, \quad U_p = \{p\} \;\; \textit{for} \;\; p = 17,19,29,37.$$

We are now ready to prove Theorem 1.2 stated in the Introduction. Although all results in the Introduction were formulated in the language of quadratic forms, we shall give their proofs in the language of lattices.

Proof of Theorem 1.2. We have to show that $\mathcal{E}_p \neq \emptyset$ if and only if $p \leq 37$, and that $\mathcal{E}_p \cap \mathcal{E}_q = \emptyset$ for every pair of primes $p < q$. By Theorem 2.11, if $L \in \mathcal{E}_p$ then $\Delta(L) \in \mathcal{U}_p$ and $S(L) = S(\Delta(L))$. It suffices therefore to show the analogous properties only for $\mathcal{U}_p$. We have shown already in this section that $\mathcal{U}_p \neq \emptyset$ if and only if $p \leq 37$.

Consider now $\mathcal{U}_p \cap \mathcal{U}_q$, $p < q$. By Corollary 3.4 in [3] one has $S(L) = \{2\}$ for all L in $\mathcal{U}_2$. Hence $\mathcal{U}_2 \cap \mathcal{U}_q = \emptyset$ for all $q \geq 3$. Assume now that $\mathcal{U}_p \cap \mathcal{U}_q \neq \emptyset$ for some primes $3 \leq p < q$. Then Corollary 3.7 implies that $p = 3$ and $q = 5$. But no lattice in $\mathcal{U}_3$ is 5-anisotropic, as one can check by the direct inspection of the lattices in Table (3) (it suffices to check two lattices G with $d(G) = 3^2 \cdot 5^2$). If follows that $\mathcal{U}_p \cap \mathcal{U}_q = \emptyset$ for all primes $p < q$. This completes the proof of Theorem 1.2. □

4. Genera containing almost regular quaternary $\mathbb{Z}$-lattices

In this section we shall study the genera containing some p-anisotropic almost regular quaternary $\mathbb{Z}$-lattice. In particular, we shall prove Theorem 1.3.

Lemma 4.1. *Let L be a quaternary $\mathbb{Z}$-lattice with $S(L) = \{p\}$. Then L is almost regular if and only if $\delta_p(L)$ is almost regular.*

Proof. We can assume L to be primitive. By Proposition 2.3, if L is almost regular, then so is $\delta_p(L)$.

Conversely, suppose $\delta_p(L)$ is almost regular. Let $\alpha = 2$ if L is even and $p = 2$; otherwise let $\alpha = 1$. Then the lattice $\Lambda_{\alpha p}(L)$ is also almost regular and, by Proposition 2.2,

$$\Lambda_{\alpha p}(L) = \{x \in L \mid L(x) \equiv 0 \pmod{\alpha p}\}.$$

Thus the lattice $\Lambda_{\alpha p}(L)$ represents exactly all integers represented by L and divisible by αp. In particular, the set

$$E(L) \cap \alpha p\mathbb{Z} = E(\Lambda_{\alpha p}(L))$$

is finite (recall that $E(L)$ is the set of all integers represented by the genus of L but not by L itself). Since $S(L) = \{p\}$, all large integers not divisible by αp and represented by the genus of L are represented by L (cf. [14], Theorem 1). This, together with the finiteness of $E(L) \cap \alpha p\mathbb{Z}$, implies that L is almost regular. □

Remark. In general, Lemma 4.1 is not valid if we drop the assumption $S(L) = \{p\}$. For example, for $L = \langle 2, 2, 3, 3\rangle$ one has $S(L) = \{3\}$ and $\delta_2(L) = \langle 1, 1, 3, 3\rangle$ is regular (in fact, universal), but L itself is not almost regular. The numbers 3^{2n} are represented by the genus of L but not by L itself. □

Recall that $\mathcal{R}_p$ is the family of all primitive regular p-universal (or $2\mathbb{Z}_2$-universal, if $p = 2$) p-anisotropic quaternary $\mathbb{Z}$-lattices.

Theorem 4.2. *For a primitive quaternary $\mathbb{Z}$-lattice L the following properties are equivalent:*

(i) *L is almost regular.*

(ii) *$S(L) = \{p\}$ and $\Delta_p(L) \in \mathcal{R}_p$ for some prime $p \leq 37$, or $S(L) = \emptyset$.*

Proof. (i)$\Longrightarrow$(ii) follows from Theorems 1.2 and 2.11.
(ii)$\Longrightarrow$(i) follows from Theorem 1.1 and Lemma 4.1. □

Theorem 4.2 provides a strong criterion for almost regularity. Since for $p \geq 17$ the families $\mathcal{R}_p$ are known explicitly (cf. Theorem 6.1), this criterion is effective for these values of p. It will become unconditionally effective, once the finite families $\mathcal{R}_p$ are enumerated for the remaining 6 primes $p \leq 13$.

Lemma 4.3. *For each lattice G in $\mathcal{R}_p$ there is a lattice G' in $\mathcal{R}_p$ such that $G_q \simeq pG'_q$ for all primes $q \neq p$.*

Proof. Define $\alpha(p) = p$ if $p > 2$, and $\alpha(2) = 4$. For each G in $\mathcal{R}_p$ one has

$$\delta_p(G) = \frac{1}{p}\Lambda_{\alpha(p)}(G).$$

Moreover, $G' = \delta_p(G)$ is in $\mathcal{R}_p$ and

$$G'_q \simeq \frac{1}{p}(\Lambda_{\alpha(p)}(G))_q \simeq \frac{1}{p}G_q$$

for all $q \neq p$. The lemma follows. □

Invariants $b_{H,p}$ and $s(H,p)$. Now we shall define, for each prime p, two invariants $b_{H,p} \in \{1,p\}$ and $s(H,p) \in \mathbb{Z}$, of a primitive quaternary $\mathbb{Z}$- or $\mathbb{Z}_p$-lattice H. As defined in Section 2, $\Delta_p(H) = \delta_p^k(H)$ for some minimal $k \geq 0$. If $k = 0$, we put $b_{H,p} = 1$ and $s(H,p) = 0$. For $k > 0$, each operation δ_p used in the definition of $\delta_p^k(H)$ consists of taking a lattice defined by the transformation Λ_p (or Λ_4) and scaling it by 1, $1/p$ or $1/p^2$ (cf. Proposition 2.1 (ii) for $p > 2$, and Lemmas 2.6–2.10 for $p = 2$). The scaling 1 can occur only if $p = 2$. Let $\beta_{H,p}$ be the number of all operations δ_p, with scaling $1/p$, involved in the definition of $\Delta_p(H)$. Define

$$b_{H,p} = \begin{cases} p & \text{if } \beta_{H,p} \text{ is odd,} \\ 1 & \text{if } \beta_{H,p} \text{ is even,} \end{cases}$$

and

$$s(H,p) = \begin{cases} 2k - \beta_{H,p} & \text{if } p > 2, \\ 2k - \beta_{H,p} - 2\gamma_H & \text{if } p = 2, \end{cases}$$

where γ_H is the number of all operations δ_2, with scaling 1, involved in the definition of $\Delta_2(H)$. Both invariants can be computed effectively and, for a $\mathbb{Z}$-lattice H, $b_{H,p} = b_{H_p,p}$ and $s(H,p) = s(H_p,p)$.

Corollary 4.4. *Let L be a primitive quaternary $\mathbb{Z}$-lattice.*

(i) *One has*

$$b_{L,p}(\Delta_p(L))_q \simeq L_q$$

for all primes $q \neq p$.

(ii) *If L is p-anisotropic, then an integer n (n even if $p = 2$) is represented by $\Delta_p(L)$ if and only if $p^{s(L,p)}n$ is represented by L.*

Proof. (i) Follows from the definition of $b_{L,p}$ and $\Delta_p(L)$, and the fact that $(\Lambda_p(L))_q \simeq L_q$ for all $q \neq p$, $(\Lambda_4(L))_q \simeq L_q$ for all $q > 2$.

(ii) Follows from Proposition 2.2 and the definition of $s(L,p)$ and $\Delta_p(L)$. □

We can now prove Theorem 1.3 stated in the Introduction. The proof is given in the language of lattices.

Proof of Theorem 1.3. Let H be an arbitrary primitive p-anisotropic quaternary $\mathbb{Z}_p$-lattice and let Γ be a genus of type H.

(i) $\Longrightarrow$ (ii) Let L be an almost regular primitive $\mathbb{Z}$-lattice representing genus Γ. The quaternary $\mathbb{Z}$-lattice $G' = \Delta_p(L)$ is in $\mathcal{R}_p$ and

$$b_{L,p} G'_q \simeq L_q$$

for all primes $q \neq p$ (cf. Theorem 2.11 and Corollary 4.4 (i)). Let $G = G'$ if $b_{L,p} = 1$. If $b_{L,p} = p$, choose G in $\mathcal{R}_p$ such that $G_q \simeq pG'_q$ for all $q \neq p$ (cf. Lemma 4.3). Then in both cases $\Gamma_q \simeq L_q \simeq G_q$ for all $q \neq p$, and $G \in \mathcal{R}_p$.

(ii) $\Longrightarrow$ (i) Let G be a lattice in $\mathcal{R}_p$ and let Γ satisfy $\Gamma_q \simeq G_q$ for all $q \neq p$. We shall show that Γ contains an almost regular $\mathbb{Z}$-lattice N (which is necessarily p-anisotropic and primitive). By Theorem 81.14 in [13] there exists a $\mathbb{Z}$-lattice L such that

$$L_q \simeq \begin{cases} H & \text{if } q = p, \\ G_q & \text{if } q \neq p. \end{cases}$$

In particular, L is in Γ. If $b_{L,p} = 1$, let $G' = G$. If $b_{L,p} = p$, let G' be a lattice in $\mathcal{R}_p$ satisfying $G_q \simeq pG'_q$ for all primes $q \neq p$ (cf. Lemma 4.3). For arbitrary p-anisotropic quaternary $\mathbb{Z}$-lattices M and M' one has $\Delta_p(M) \simeq \Delta_p(M')$. In particular,

$$\Delta_p(L_p) \simeq \Delta_p(H) \simeq \Delta_p(G'_p) \simeq G'_p.$$

By Corollary 4.4 (i),

$$b_{L,p}(\Delta_p(L))_q \simeq L_q \simeq b_{L,p} G'_q$$

for all $q \neq p$. It follows that

$$(\Delta_p(L))_q \simeq G'_q$$

for *all* primes q. In other words, $\Delta_p(L)$ and G' are the $\mathbb{Z}$-lattices representing the same genus. By Proposition 2.1 (vii), there is a $\mathbb{Z}$-lattice N in the genus of L such that $\Delta_p(N) = G'$. By Theorem 4.2, the lattice N is almost regular. Since, by construction, N is in Γ, the theorem follows. □

Corollary 4.5. *Let H be a primitive 2-anisotropic quaternary $\mathbb{Z}_2$-lattice. Let $\Gamma = \{\Gamma_p\}_{p \in P}$ be a genus of an integral quaternary $\mathbb{Z}$-lattice. Then the following properties are equivalent:*

(i) Γ *contains an almost regular $\mathbb{Z}$-lattice which is q-universal for all odd primes q, and $\Gamma_2 \simeq H$.*

(ii) *There is a $\mathbb{Z}$-lattice G in $\mathcal{U}_2$ such that $G_q \simeq \Gamma_q$ for all $q > 2$.*

Proof. The corollary follows from Theorem 1.3. □

Corollary 4.6. *Let H be a primitive p-anisotropic quaternary $\mathbb{Z}_p$-lattice. The numbers η_p of all genera Γ containing some almost regular $\mathbb{Z}$-lattice such that Γ_q is q-universal for all $q \neq p$ and $\Gamma_p \simeq H$ are as follows:*

$$\eta_p = \begin{cases} 65 & \text{if } p = 2, \\ 1 & \text{if } p = 3 \text{ or } 7, \\ 2 & \text{if } p = 5, \\ 0 & \text{if } p > 7. \end{cases}$$

Proof. For $p = 2$ the corollary follows from Corollary 4.5 and the fact that the lattices in $\mathcal{U}_2$ represent exactly 65 genera. For $p > 2$ the corollary is a consequence of Theorem 1.3 and the facts mentioned in Section 3 about p-anisotropic universal $\mathbb{Z}$-lattices, $p > 2$. □

5. Regular p-anisotropic p-universal $\mathbb{Z}$-lattices

In this section we shall prove that the families $\mathcal{R}_p$ are finite.

Lemma 5.1. *Let L be a primitive regular quaternary $\mathbb{Z}$-lattice. Assume that L is not p-universal for some prime $p > 5$, but is q-universal for all odd primes $q \neq p$ and is $2\mathbb{Z}_2$-universal (in particular L is even). Then*

$$L_p \simeq L_0 \perp pL_1 \perp p^2 M$$

for some unimodular $\mathbb{Z}_p$-lattices L_0 and L_1, and some $\mathbb{Z}_p$-lattice M.

Proof. Let

$$L_p \simeq L_0 \perp pL_1 \perp p^2 M,$$

where L_0 and L_1 are either unimodular or null. Since L is primitive and is not p-universal, the rank of L_0 is either 1 or 2. It suffices to show that $L_1 \neq 0$. Assume to the contrary that $L_1 = 0$ and that $L_0 \simeq \langle \epsilon \rangle$ for some ϵ which is a p-adic unit. Let

$$T = \left\{ 2t \in 2\mathbb{Z} \;\middle|\; 1 \leq t \leq p-1, \left(\frac{2t}{p}\right) = \left(\frac{\epsilon}{p}\right) \right\}.$$

Clearly, T contains exactly $(p-1)/2$ elements, and every element of T is represented by the genus of L. Since L is regular, L represents every element of T. Let K be a sublattice of L generated by all elements x in L with $L(x)$ in T. Let k be the rank of K; necessarily $k \geq 1$. Since in the present situation the rank of M is 3, the discriminant of K satisfies $p^{2(k-1)} \leq d(K)$. Let $\mu_i(K)$, $i = 1, \ldots, k$, be the successive minima of K. In particular, $\mu_1(K) \leq p - 1$.

If $k \geq 3$, then

$$p^{2(k-1)} \leq d(K) \leq \mu_1(K) \cdots \mu_k(K) \leq (p-1)(2p-2)^{k-1},$$

which is impossible.

If $k = 2$, then K corresponds to a reduced even binary quadratic form

$$K = ax^2 + 2bxy + cy^2,$$

where a, b, c are integers satisfying $0 \leq 2b \leq a \leq c$, $0 \leq a \leq p-1$, a, c even. Assume first that $p \geq 31$. For any $2t$ in T the equation

$$aK(x, y) = (ax + by)^2 + d(K)y^2 = a2t \tag{1}$$

should have an integer solution x, y. Since $d(K) \geq p^2$, one has necessarily $y = 0$ or $y = \pm 1$. It follows that the number of t's, $1 \leq t \leq p-1$, for which (1) has an integer solution x (where $y = 0$ or ± 1) is not exceeding $(\sqrt{2}+1)\sqrt{p-1}+1$, which is less than $(p-1)/2$ if $p \geq 31$. This contradicts the fact that K represents $(p-1)/2$ elements in T. If $7 \leq p < 31$ we have to use a different argument. Since by assumption $L_p \simeq \langle \epsilon \rangle \perp p^2 M$, it follows that the discriminant of every binary sublattice of L is divisible by p^2. We shall show the contradiction with the assumption $L_1 = 0$ by constructing for each prime p, $7 \leq p < 31$, a binary sublattice $B(p)$ of L such that $d(B(p))$ is not divisible by p^2. Let $p = 7$. Consider first the case where ϵ is a square unit in $\mathbb{Z}_7$. Then 2 and 4, being square units in $\mathbb{Z}_7$, are represented by L_7, and therefore by any L_q, q prime. Since L is regular, 2 and 4 are represented by L. It follows that one of the following binary $\mathbb{Z}$-lattices

$$\begin{bmatrix} 2 & a \\ a & b \end{bmatrix},$$

where $a = 0$, 1 and $b = 2$, 4, is contained in L, which is impossible, as their discriminants are smaller than 7^2. Let now ϵ be a nonsquare unit in $\mathbb{Z}_7$. Then 6 and 10, being nonsquare units in $\mathbb{Z}_7$, are both represented by L_7, and therefore by L. Hence L contains at least one of the following binaries

$$\begin{bmatrix} 6 & a \\ a & b \end{bmatrix},$$

where $a = 0, 1, 2, 3$ and $b = 6, 8, 10$, which is impossible because none of them has the discriminant divisible by 7^2.

For p satisfying $7 < p \leq 29$ the argument is identical. Assume that for any such p we can find four even positive integers s_p, s'_p, r_p, r'_p such that s_p, s'_p (resp. r_p, r'_p) are square (resp. nonsquare) units in $\mathbb{Z}_p$, $s_p s'_p < p^2$, $r_p r'_p < p^2$, and s'_p/s_p, r'_p/r_p are not squares in $\mathbb{Q}$. Then L represents s_p and s'_p (resp. r_p and r'_p), if ϵ is a square (resp. nonsquare) in $\mathbb{Z}_p$. The existence of these numbers implies the existence of a binary sublattice $B(p)$ of L, such that

$$d(B(p)) \leq \max(s_p s'_p, r_p r'_p) < p^2,$$

which is impossible. It suffices therefore to exhibit s_p, s'_p, r_p, r'_p with these properties, which is done in the table below.

p	11	13	17	19	23	29
r_p, r'_p	2, 6	2, 6	6, 12	2, 20	10, 14	2, 10
s_p, s'_p	4, 12	4, 12	2, 4	4, 6	2, 6	4, 6

If $k = 1$, then $K = \langle 2m \rangle$ for some $2m$ in T, and the number of t's, $1 \leq t \leq p-1$, for which the equation

$$K(x) = 2mx^2 = 2t$$

has an integer solution x is not exceeding $\sqrt{p-1} < (p-1)/2$. This again contradicts the fact that K represents $(p-1)/2$ elements of T. Hence $L_1 \neq 0$ if L_0 is of rank 1.

Assume now that $L_1 = 0$ and L_0 has rank 2. It follows immediately that the discriminant of any ternary (resp. quaternary) sublattice of L is divisible by $2p^2$ (resp. p^4). Also it follows that the genus of L represents all elements of the set

$$\{2t \in 2\mathbb{Z} \mid 1 \le t \le p-1\}.$$

Since L is regular, L represents all these elements.

Let N be a sublattice of L generated by the vectors x satisfying $L(x) \le 2(p-1)$. Since $p \ge 7$, N represents therefore all even positive integers not exceeding 12. If N was of rank 1, then necessarily $N = \langle 2 \rangle$ contradicting the fact that N represents 4. Assume that the rank of N is $k \ge 2$. If $k = 2$ (resp. $k \ge 3$), then N is isometric to (resp. contains) one of the following binary lattices, given below by their matrices:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Since the smallest even integers not represented by these lattices are, respectively, 4, 6, 6, 10, the lattice N cannot be of rank 2. If $k = 3$, then N would be a ternary lattice of discriminant $d(N) \le 80$, contradicting that $2p^2$ divides $d(N)$ and $2p^2 \ge 2 \cdot 7^2$. Finally, if $k = 4$, one would have

$$d(N) \le \prod_{i=1}^{4} \mu_i(N) \le 2 \cdot 4 \cdot 10 \cdot 2(p-1) \le 160(p-1),$$

contradicting the fact that p^4 divides $d(N)$ and $p^4 \ge 7^4$. Hence $L_1 \ne 0$ if L_0 is of rank 2. This completes the proof of Lemma 5.1. $\square$

For a $\mathbb{Z}$-lattice L let denote by $A(L)$ the set

$$A(L) = \{p \in P \mid p > 5 \text{ and } p \text{ divides } d(L)\}.$$

Lemma 5.2. *Let L be a primitive regular quaternary $\mathbb{Z}$-lattice. Then*

$$A(L) = A(\Delta(L)).$$

Proof. Fix a prime $p > 5$. We shall show that $p \in A(L)$ if and only if $p \in A(\Delta(L))$. The property $p \in A(L)$ depends only on the p-adic structure of L and is therefore not affected if L is replaced by $\Delta_q(L)$, $q \ne p$. The lattice $\Delta_q(L)$ is regular if L is regular (cf. Theorem 2.11). Hence we can assume without loss of generality that L_q is q-universal for $q > 2$, $q \ne p$, and L_2 is $2\mathbb{Z}_2$-universal. In particular, $\Delta_p(L) = \Delta(L)$.

If L is p-universal, then $L = \Delta(L)$ and there is nothing further to prove. Assume therefore that L is not p-universal. Then, by Lemma 5.1,

$$L_p \simeq L_0 \perp pL_1 \perp p^2 M,$$

where L_0 and L_1 are unimodular $\mathbb{Z}_p$-lattices.

By Proposition 2.1 (ii)

$$\Lambda_p(L_p) \simeq p^2 L_0 \perp pL_1 \perp p^2 M \quad \text{and} \quad \delta_p(L_p) \simeq pL_0 \perp L_1 \perp pM.$$

The $\mathbb{Z}$-lattice $\delta_p(L)$ is regular and

$$\mathrm{ord}_p(d(L)) > \mathrm{ord}_p(d(\delta_p(L))) \ge 1.$$

It follows that applying, if necessary, to $\delta_p(L)$ the operations δ_p, finally one finds the smallest $k \geq 1$ such that the $\mathbb{Z}$-lattice $\delta_p^k(L)$ will satisfy

$$1 \leq \mathrm{ord}_p(d(\delta_p^k(L))) \leq 2.$$

Such a lattice is necessarily p-universal. Thus, $\Delta(L) = \Delta_p(L) = \delta_p^k(L)$, and p divides both $d(L)$ and $d(\Delta(L))$. $\square$

Recall that

$$U_p = \{q \in P \mid q \text{ divides } d(L),\ L \in \mathcal{U}_p\},$$
$$D_p = \{q \in P \mid q \text{ divides } d(L),\ L \in \mathcal{E}_p\}.$$

Also recall that the sets U_p are known explicitly (cf. Corollary 3.7 for $p > 2$, and the subsection on "Family $\mathcal{U}_2$" in Section 3 for $p = 2$).

Corollary 5.3. *For every prime number p one has*

$$U_p \subset D_p \subset U_p \cup \{2, 3, 5\}.$$

Proof. Since $\mathcal{U}_p \subset \mathcal{E}_p$, clearly $U_p \subset D_p$. Let $\pi(L)$ be the set of all primes dividing $d(L)$. By Proposition 2.1 (iii), for $L \in \mathcal{E}_p$ one has

$$\pi(L) = \pi(\Delta_p(L)) \quad \text{for } p > 2, \text{ and } \quad \pi(L) \cup \{2\} = \pi(\Delta_2(L)).$$

Since $\Delta(L) = \Delta(\Delta_p(L))$, by Lemma 5.2 one has that

$$\pi(\Delta_p(L)) \cup \{2, 3, 5\} = \pi(\Delta(L)) \cup \{2, 3, 5\}.$$

By Theorem 2.11,

$$\mathcal{U}_p = \{\Delta(L) \mid L \in \mathcal{E}_p\}.$$

The corollary follows. $\square$

Later on we shall describe the sets D_p with more accuracy (cf. Theorem 9.2).

Theorem 5.4. *There is a constant c, which can be computed effectively, such that for every lattice L in $\mathcal{R}_p$ one has $d(L) \leq c$. In particular, the family $\mathcal{R}_p$ is finite.*

Proof. Case $p > 2$. Clearly, $\mathcal{R}_p = \emptyset$ for $p > 37$. Define

$$\Sigma_3 = \{2, 5, 7, 37\}, \quad \Sigma_5 = \{2, 3, 11\}, \quad \Sigma_p = \{2, 3, 5\} \quad \text{for } p \geq 7.$$

It follows from Corollaries 3.7 and 5.3 that if $L \in \mathcal{R}_p$ then L_q is q-universal for all $q \in P \setminus \Sigma_p$ (simply observe that $D_p \subset \Sigma_p \cup \{p\}$).

Fix $p > 2$, and let $\Sigma_p = \{q_0 = 2, q_1, \dots, q_n\}$. Let $C_i = \{1, \tau_i\}$, where τ_i is the smallest positive integer which is a nonsquare unit in $\mathbb{Z}_{q_i}$, $i = 1, \dots, n$, and let

$$C_0 = \{1, 2, 3, 5, 6, 7, 10, 14\}.$$

Observe that the elements of C_0 represent all different 8 square classes in $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$. Let

$$S_p = \prod_{i=0}^{n} C_i.$$

For a, b in $\mathbb{Z}_q$ we shall write $a \sim b$ to indicate that $a = u^2 b$ for some unit u in $\mathbb{Z}_q$. For every $\alpha = (\alpha_0, \dots, \alpha_n)$ in S_p let a_α, b_α, c_α and d_α be positive integers such that:

(i) $a_\alpha \sim b_\alpha \sim pc_\alpha \sim pd_\alpha \sim \alpha_j$ in $\mathbb{Z}_{q_j}$ for $j = 0, \ldots, n$.
(ii) a_α, b_α, pc_α, pd_α represent different square classes in $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$.

Observe that the integers a_α, b_α, c_α, d_α depend only on p, and can be computed effectively.

Let $L \in \mathcal{R}_p$ and choose an element $\alpha = (\alpha_0, \ldots, \alpha_n)$ in S_p such that α_i is represented by L over $\mathbb{Z}_{q_i}$, $i = 0, \ldots, n$. Condition (i) implies that a_α, b_α, pc_α, pd_α are represented by the genus of L. Since L is regular, L represents these integers over $\mathbb{Z}$. Condition (ii) implies that if $L(x_1) = a_\alpha$, $L(x_2) = b_\alpha$, $L(x_3) = pc_\alpha$ and $L(x_4) = pd_\alpha$, then the vectors $x_1, \ldots, x_4$ are linearly independent. Hence the successive minima $\mu_i(L)$ of L satisfy

$$\prod_{i=1}^{4} \mu_i(L) \le a_\alpha b_\alpha c_\alpha d_\alpha p^2 = s_\alpha,$$

and the discriminant

$$d(L) \le \max\{s_\alpha \mid \alpha \in S_p\}$$

is bounded by a constant which depends only on p and can be computed effectively. (Observe that $S_p = C_0 \times \{1,2\} \times \{1,2\}$ for $p \ge 5$, and $S_3 = C_0 \times \{1,2\} \times \{1,3\} \times \{1,2\}$).

Case $p = 2$. The set U_2 is known explicitly, and $U_2 = \{2, 3, 5, \ldots, 281, 353\}$ contains exactly 29 primes listed in Section 3. If follows from Corollary 5.3 that $D_2 = U_2$, and thus the $m = 28$ primes constituting the set

$$D_2 \setminus \{2\} = \{q_1, \ldots, q_m\} = \{3, \ldots, 353\}$$

are also known explicitly. Let $C_i = \{1, \tau_i\}$, where τ_i is the smallest positive integer which is a nonsquare unit in $\mathbb{Z}_{q_i}$, $i = 1, \ldots, m$. Let

$$S = \prod_{i=1}^{m} C_i,$$

and let $A = \{2, 4, 6, 10, 12, 14, 20, 28\}$.

For every $\alpha = (\alpha_1, \ldots, \alpha_m)$ in S fix a sequence

$$A_\alpha = \{a_\alpha(t) \mid t \in A\}$$

of 8 even positive integers which satisfy the following properties:

(i) $a_\alpha(t) \sim \alpha_i$ in $\mathbb{Z}_{q_i}$, for all $i = 1, \ldots, m$ and all $t \in A$.
(ii) $a_\alpha(t) \sim t$ in $\mathbb{Z}_2$, for all $t \in A$.

Observe that for each α the elements of A_α represent all different 8 square classes in $\mathbb{Q}_2^*/\mathbb{Q}_2^{*2}$. Let $L \in \mathcal{R}_2$ and choose $\alpha = (\alpha_1, \ldots, \alpha_m)$ in S such that α_i is represented by L over $\mathbb{Z}_{q_i}$, $i = 1, \ldots, m$. Condition (i) implies that every element in A_α is represented by the genus of L, and thus by L itself, L being regular. If $\{x_t \mid t \in A\}$ are the vectors in L such that $L(x_t) = a_\alpha(t)$, then condition (ii) implies that the sublattice of L generated by these 8 vectors is of rank 4. As above, it follows that

$$d(L) \le \max\left\{ \prod_{t \in A} a_\alpha(t) \,\middle|\, \alpha \in S \right\}.$$

Thus $d(L)$ is bounded by a constant which is independent of L and can be computed effectively. □

6. Families $\mathcal{R}_p$, $17 \le p \le 37$

In this section we shall study in more detail the families $\mathcal{R}_p$ of all primitive regular p-universal p-anisotropic quaternary $\mathbb{Z}$-lattices for $p \in P^* = \{17, 19, 23, 29, 31, 37\}$, and we shall determine explicitly all their elements. As proved earlier, $\mathcal{R}_p = \emptyset$ for $p > 37$. We shall prove below that for $p \in P^*$ one has $\mathcal{R}_p = \mathcal{U}_p$. Recall that $\mathcal{U}_p$ is the family of all p-anisotropic even-universal quaternary $\mathbb{Z}$-lattices. These families were studied in Section 3. In particular it was shown that

$$\mathcal{U}_p = \begin{cases} \{G(p)\} & \text{if } p \in \{17, 19, 29, 37\}, \\ \{G(p), G'(p)\} & \text{if } p \in \{23, 31\}, \end{cases}$$

where the lattices $G(p)$ and $G'(p)$ are defined in Section 3 (cf. Theorem 3.2). One has $d(G(p)) = p^2$ and $d(G'(p)) = 4p^2$. In Section 7 we shall give some important applications of Theorem 6.1. The equality $\mathcal{R}_p = \mathcal{U}_p$ is no longer valid for $p < 17$ (cf. Section 8).

Theorem 6.1. *For $p \in P^*$ one has $\mathcal{R}_p = \mathcal{U}_p$.*

It follows immediately from this theorem that for $p \in P^*$ the invariants α_p and the sets A_p, defined in the Introduction, are as follows:

$$\alpha_{17} = \alpha_{19} = \alpha_{29} = \alpha_{37} = 1, \quad \alpha_{23} = \alpha_{31} = 2,$$
$$A_{17} = A_{19} = A_{29} = A_{37} = \{1\}, \quad A_{23} = A_{31} = \{1, 4\}.$$

Before proving Theorem 6.1 we need some preparation.

Lemma 6.2. *Let $L \in \mathcal{R}_p$ for some $p \in P^*$. Then*

$$\Delta(L) = \Delta_5(\Delta_3(\Delta_2(L))).$$

Proof. By Theorem 2.11, $\Delta(L)$ is in $\mathcal{U}_p$. Since $p \in P^*$, one has $d(\Delta(L)) = p^2$ or $4p^2$. Hence, by Lemma 5.2, all primes dividing $d(L)$ are contained in $\{2, 3, 5, p\}$. In particular, L is q-universal for all $q \ne 2$, 3, 5 and $\Delta(L) = \Delta_5(\Delta_3(\Delta_2(L)))$. □

Lemma 6.3. *Let $L \in \mathcal{R}_p$ for some $p \in P^*$. Then $\Delta(L) = \Delta_3(\Delta_2(L))$.*

Proof. Let denote $\Delta_3(\Delta_2(L)) = L(2, 3)$. By Lemma 6.2 we only have to show that $L(2, 3)$ is 5-universal. The lattice $L(2, 3)$ is regular, p-anisotropic, $2\mathbb{Z}_2$-universal and q-universal for every odd prime $q \ne 5$. Moreover, since $d(\Delta(L))$ is not divisible by 5, $L(2, 3)_5$ is of the form $H \perp 5^2M$, where H and M are the $\mathbb{Z}_5$-lattices and H is unimodular.

Suppose that $L(2, 3)$ is not 5-universal. Then we may assume that

$$L(2, 3)_5 \simeq \langle \epsilon \rangle \perp 5^2M \quad \text{or} \quad L(2, 3)_5 \simeq \langle 1, 2 \rangle \perp 5^2N,$$

where ϵ is a unit in $\mathbb{Z}_5$, and M and N are the $\mathbb{Z}_5$-lattices.

Consider first $L(2, 3)_5 \simeq \langle \epsilon \rangle \perp 5^2M$. If ϵ is a square (resp. nonsquare) unit in $\mathbb{Z}_5$, then $L(2, 3)$ represents the numbers 4 and 6 (resp. 2 and 12). Then one has for the successive minima μ_i

$$\mu_1(L(2, 3)) \le 4 \text{ (resp. 2)}, \quad \mu_2(L(2, 3)) \le 6 \text{ (resp. 12)}.$$

Hence $L(2, 3)$ contains a binary sublattice of discriminant at most 24, contradicting the fact that, in the case under consideration, the discriminant of any binary sublattice of $L(2, 3)$ is divisible by 25.

In the second case, i.e., if $L(2,3)_5 \simeq \langle 1,2\rangle \perp 5^2N$, the lattice $L(2,3)$ represents 2 and 4. Hence $\mathrm{L}(2,3)$ contains at least one of the following binary $\mathbb{Z}$-lattices

$$\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}, \quad \begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}, \quad \begin{bmatrix}2 & 1\\ 1 & 4\end{bmatrix}, \quad \begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}.$$

The smallest even positive integers not represented by the first three lattices are, respectively, 6 , 4, 6. The number 6 is represented by $L(2,3)_5$ and thus by $L(2,3)$ itself. The smallest even positive integer not represented by $\langle 2,4\rangle$ is 10, but 10 is not necessarily represented by $L(2,3)_5$. On the other hand, 14 is represented by $L(2,3)_5$, and thus by $L(2,3)$. Hence

$$\mu_i(L(2,3)) \le n_i,$$

where $n_1 = 2$, $n_2 = 4$, $n_3 = 14$. Therefore $L(2,3)$ contains a ternary sublattice whose discriminant is not exceeding $2 \cdot 4 \cdot 14 = 112$, contrary to the property that, in the case under consideration, the discriminant of any ternary sublattice of $L(2,3)$ is divisible by $25p$.

It follows that $L(2,3)$ is 5-universal and $\Delta(L) = \Delta_3(\Delta_2(L))$. $\square$

Lemma 6.4. *Let $L \in \mathcal{R}_p$ for some $p \in P^*$. Then $\Delta(L) = \Delta_2(L)$.*

Proof. It follows from Lemma 6.3 that we only have to show that $\Delta_2(L)$ is 3-universal. Let denote $\Delta_2(L) = L(2)$. The lattice $L(2)$ is regular p-anisotropic $2\mathbb{Z}$-universal and q-universal for every odd $q \neq 3$. Since $d(\Delta(L))$ is not divisible by 3, $L(2)_3$ is of the form $H \perp 3^2M$, where H and M are the $\mathbb{Z}_3$-lattices and H is unimodular.

Suppose that $L(2)$ is not 3-universal. Then we may assume that

$$L(2)_3 \simeq \langle \epsilon\rangle \perp 3^2M \quad \text{or} \quad L(2)_3 \simeq \langle 1,1\rangle \perp 3^2N,$$

where $\epsilon = 1$ or 2, and M, N are the $\mathbb{Z}_3$-lattices.

Consider first $L(2)_3 \simeq \langle \epsilon\rangle \perp 3^2M$. If $\epsilon = 1$ (resp. 2), then 4 and 10 (resp. 2 and 14) are represented by $L(2)$. Since in the case under consideration the discriminant of any binary sublattice of $L(2)$ is divisible by 9, it follows easily that $L(2)$ contains a primitive sublattice isometric to

$$B = \begin{bmatrix}4 & 2\\ 2 & 10\end{bmatrix} \quad \left(\text{resp. } \begin{bmatrix}2 & 1\\ 1 & 14\end{bmatrix}\right).$$

For example, to show that $L(2)$ contains the first of these lattices, one observes that 2, 6, 8 are not represented by $L(2)$ if $\epsilon = 1$. Then L contains $\left[\begin{smallmatrix}4 & a\\ a & b\end{smallmatrix}\right]$, for some $a = 0, 1, 2$, and $b = 4$ or 10. But the cases $b = 10$ and $a = 0$, 1 or $b = 4$, are excluded by the discriminant condition. The number 22 (resp. 20) is represented by $L(2)$, but not by B, which implies by Lemma 3.3 that $L(2)$ contains a ternary sublattice whose discriminant is not exceeding $36 \cdot 22$ (resp. $27 \cdot 20$). This however is not possible, because the discriminant of any ternary sublattice of $L(2)$ is, in the present case, divisible by 3^4p. Hence $L(2)$ is necessarily 3-universal in this case.

If $L(2)_3 \simeq \langle 1,1\rangle \perp 3^2N$, the numbers 2 and 4 are represented by $L(2)$. Hence $L(2)$ contains a binary sublattice isometric to one of the following lattices

$$B_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Since 14 (resp. 10) is not represented by B_1 (resp. B_2), but 14 and 10 are both represented by $L(2)$, it follows by Lemma 3.3 that if $L(2)$ contains B_1 or B_2 then $L(2)$ contains a ternary sublattice T with $d(T) \leq 70$, in contradiction to the property that $d(T)$ is divisible by $9p$. On the other hand, the lattices C and D are isometric over $\mathbb{Z}_3$ to, respectively, $\langle 1, 3\rangle$ and $\langle 1, -1\rangle$. But neither $\langle 1, -1\rangle$, which is 3-universal, nor $\langle 1, 3\rangle$ embeds in $L(2)_3$. It follows that none of these 4 binary lattices can be contained in $L(2)$.

Therefore $L(2)$ is 3-universal and thus $\Delta(L) = \Delta_2(L)$. $\square$

In the next results of this section we shall use the following six quaternary $\mathbb{Z}_2$-lattices:

$$K(1) = \langle 1, 3\rangle \perp 2\mathbb{A}, \quad K(2) = \mathbb{A} \perp 4\mathbb{A}, \quad K(3) = \langle 1, 3\rangle \perp \langle 4, 12\rangle,$$
$$K(4) = \langle 1, 28\rangle \perp 2\mathbb{H}, \quad K(5) = \langle 3, 20\rangle \perp 2\mathbb{H}, \quad K(6) = \langle 8, 24\rangle \perp \mathbb{A}.$$

Lemma 6.5. *Let K be a primitive quaternary $\mathbb{Z}_2$-lattice such that $\delta_2(K) \simeq \mathbb{A} \perp \mathbb{A}$. Then K is either 2-universal or $2\mathbb{Z}_2$-universal or $K \simeq K(i)$ for some $i \in \{1, 2\}$.*

Proof. Assume that K is neither 2-universal nor $2\mathbb{Z}_2$-universal. Consider first the case when K is an odd lattice. Let r_0 be the rank of the unimodular component of K. Clearly, $r_0 \leq 3$. Let ϵ and ϵ_i denote a unit in $\mathbb{Z}_2$.

If r_0 was 1, then $K \simeq \langle \epsilon\rangle \perp 2\ell$ for some ternary $\mathbb{Z}_2$-lattice ℓ and

$$\Lambda_2(K) \simeq \langle 4\epsilon\rangle \perp 2\ell,$$

contradicting the assumption that $\delta_2(K) \simeq \mathbb{A} \perp \mathbb{A}$.

If r_0 was 3, then $K \simeq N \perp \langle \epsilon, 2a\rangle$, where $N \simeq \mathbb{A}$ or $\mathbb{H}$ and $a \in \mathbb{Z}_2$. But then $\delta_2(K) \simeq N \perp \langle 4\epsilon, 2a\rangle$, contradicting again the assumption $\delta_2(K) \simeq \mathbb{A} \perp \mathbb{A}$.

Therefore $r_0 = 2$ and $K \simeq \langle \epsilon_1, \epsilon_2\rangle \perp 2\ell$, for some binary $\mathbb{Z}_2$-lattice ℓ. It follows that

$$\delta_2(K) \simeq \delta_2(\langle \epsilon_1, \epsilon_2\rangle) \perp \ell,$$

which implies that either

$$\langle \epsilon_1, \epsilon_2\rangle \simeq \langle 1, 3\rangle \quad \text{and} \quad \ell \simeq \mathbb{A},$$

or

$$\langle \epsilon_1, \epsilon_2\rangle \simeq \langle 1, -1\rangle \quad \text{and} \quad \ell \simeq \mathbb{H}.$$

Since $\langle 1, 3\rangle \perp 2\mathbb{A} \simeq \langle 1, -1\rangle \perp 2\mathbb{H}$, the lemma follows for K odd.

If K is even, then $K \simeq \mathbb{A} \perp \ell$ for some binary even $\mathbb{Z}_2$-lattice ℓ. Since K is not $2\mathbb{Z}_2$-universal, one has $\ell(x) \equiv 0 \pmod 4$ for every $x \in \ell$. Hence $\Lambda_4(K) \simeq 4\mathbb{A} \perp \ell$, which implies $\ell = 4\mathbb{A}$. The lemma follows. $\square$

Lemma 6.6. *Let K be a primitive quaternary $\mathbb{Z}_2$-lattice such that $\delta_2(K) \simeq \langle 2, 6\rangle \perp \mathbb{A}$. Then K is either 2-universal or $2\mathbb{Z}_2$-universal or $K \simeq K(i)$ for some $i \in \{3, 4, 5, 6\}$.*

The proof of this lemma is similar to that of Lemma 6.5 and is left to the reader.

Definition. Let p be a prime, k, s positive integers, and B a binary $\mathbb{Z}$-lattice. A quaternary $\mathbb{Z}$-lattice L is said to have *property* $\pi_p(k,s,B)$ if the following conditions are satisfied:

(1) The discriminant of any ternary sublattice of L is divisible by kp.
(2) L contains a primitive sublattice isometric to B.
(3) s is represented by L but not by B.

Lemma 6.7. *If $sd(B) < kp$, then there is no quaternary $\mathbb{Z}$-lattice L satisfying property $\pi_p(k,s,B)$.*

Proof. By (2), (3) and Lemma 3.3, such a lattice L contains a ternary sublattice T with $d(T) \le sd(B)$. But then (1) contradicts the assumption $sd(B) < kp$. □

Lemma 6.8. *Let M be a primitive regular quaternary $\mathbb{Z}$-lattice which is q-universal for every odd prime q, and p-anisotropic for some $p \in P^*$ (resp. $p \in \{23, 31\}$). Then M_2 is not isometric to any $K(i)$, $i = 1,2,3,4$ (resp. $K(i)$, $i = 5$, 6).*

Proof. The proof is divided into several cases. In each case, the assumption $M_2 \simeq K(i)$ leads to a contradiction.

Case 1. Assume $p \in P^*$ and $M_2 \simeq K(1)$ or $K(3)$. Then M satisfies property $\pi_p(4,5,\langle 1,3\rangle)$, which is impossible by Lemma 6.7.

Case 2. Assume $p \in P^*$ and $M_2 \simeq K(2)$. In particular, every ternary sublattice of M has the discriminant divisible by $8p$, and $d(M)$ is divisible by $16p^2$.

(a) Since 2 and 6, but not 4, are represented by M, M contains one of the following primitive sublattices

$$B_1 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}.$$

The lattices B_1 and B_2 are excluded, otherwise M would satisfy property $\pi_p(8,10,B_i)$, $i = 1$, 2, which is impossible by Lemma 6.7. As for B_3, M would have property $\pi_p(8,14,B_3)$ which is excluded by Lemma 6.7, if $p \ge 23$.

(b) In the remaining two cases $p = 17$ and 19 we argue as follows. If $p = 17$ (resp. 19), then M contains a ternary sublattice of the form

$$T(p) = \begin{bmatrix} 2 & 1 & a \\ 1 & 6 & b \\ a & b & 14 \end{bmatrix},$$

where $0 \le a \le 5$ and $-9 \le b \le 9$ depend on p, $d(T(p)) \le 14d(B_3) = 154$, and $d(T(17))$ (resp. $d(T(19))$) is divisible by $8 \cdot 17 = 136$ (resp. $8 \cdot 19 = 152$). The bounds on a and b are simply dictated by the requirement $d(T(p)) > 0$. It follows that $a = 0$, $b = \pm 1$ for $p = 17$, and $a = 1$, $b = 3$ for $p = 19$. In turn, this implies that $d(T(p)) = 8p$ and that $T(p)$ is a primitive sublattice of M. If $p = 17$, the number 34 is represented by M but not by $T(17)$. Therefore M contains a quaternary sublattice M' such that

$$16 \cdot 17^2 \le d(M) \le d(M') \le d(T(17)) \cdot 34 = 16 \cdot 17^2.$$

Hence $M = M'$ and, by Lemma 3.3,

$$M = M' \simeq T(17) \perp \langle 34 \rangle,$$

contradicting the assumption $M_2 \simeq K(2) = \mathbb{A} \perp 4\mathbb{A}$.

If $p = 19$, then the argument above is valid replacing 17 by 19, and 34 by 38.

Case 3. Assume $M_2 \simeq K(4)$ and $p \in P^*$. Here every ternary sublattice T of M has $d(T)$ divisible by $4p$. Since 1 and 5, but not 2 and 3, are represented by M, M contains a sublattice isometric to $\langle 1, 4 \rangle$. It follows that M has property $\pi_p(4, 12, \langle 1, 4 \rangle)$, which is impossible by Lemma 6.7.

Case 4. Assume $M_2 \simeq K(5)$ and $p \in \{23, 31\}$. Here every ternary sublattice of M has the discriminant divisible by $4p$. Since 3 and 4, but not 1 and 2, are represented by M, the lattice M contains one of the following binary sublattices

$$B_1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{or} \quad B_2 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}.$$

In case of B_1 the lattice M would have property $\pi_p(4, 7, B_1)$, which is impossible. In case of B_2, the number 8 is represented by M but not by B_2. Then, for $p = 31$, M would have property $\pi_{31}(4, 8, B_2)$ which is impossible. For $p = 23$ the lattice M would have a ternary sublattice of the form

$$T(a, b) = \begin{bmatrix} 3 & 0 & a \\ 0 & 4 & b \\ a & b & 8 \end{bmatrix}$$

where $0 \le a \le 4$, $-5 \le b \le 5$. For each $(a, b) \neq (1, 0)$ the discriminant of $T(a, b)$ is not divisible by $4 \cdot 23$. On the other hand, $d(T(1, 0)) = 4 \cdot 23$ but the rank of the unimodular component of $T(1, 0)_2$ is 2 in contradiction with the assumption that $M_2 \simeq K(5) = \langle 3, 20 \rangle \perp 2\mathbb{H}$.

Case 5. Assume $M_2 \simeq K(6)$ and $p \in \{23, 31\}$. The argument used in Case 2 (a) applies to this case without modification. $\square$

Lemma 6.9. *Let $L \in \mathcal{R}_p$ for some $p \in P^*$. Then $\Delta(L) = L$.*

Proof. By Lemma 6.4, $\Delta(L) = \Delta_2(L)$. If $\Delta(L) \neq L$, then $\Delta(L) = \Delta_2(L) = \delta_2^k(L)$ for some minimal $k > 0$. Define $M = \delta_2^{k-1}(L)$. Then M is a primitive regular p-anisotropic $\mathbb{Z}$-lattice which is q-universal for all primes $q > 2$. By Lemma 6.8, for $p \in P^*$ and $i \in \{1, 2\}$ (resp. $p \in \{23, 31\}$ and $i \in \{1, \dots, 6\}$) the lattice M_2 is not isometric to $K(i)$, where $K(1)$, $\dots$, $K(6)$ are the $\mathbb{Z}_2$-lattices listed above Lemma 6.5.

Since $\Delta(L)$ is in $\mathcal{U}_p$, by Theorem 3.2 one has

$$\Delta(L) \simeq \begin{cases} G(p) & \text{if } p \in P^* \setminus \{23, 31\}, \\ G(p) \text{ or } G'(p) & \text{if } p \in \{23, 31\}. \end{cases}$$

One can check easily that $(G(p))_2 \simeq \mathbb{A} \perp \mathbb{A}$ and $(G'(p))_2 \simeq \langle 2, 6 \rangle \perp \mathbb{A}$. It follows that

$$\delta_2(M_2) = \Delta_2(L_2) \simeq \Delta(L)_2 \simeq \begin{cases} \mathbb{A} \perp \mathbb{A} & \text{if } p \in P^* \setminus \{23, 31\}, \\ \mathbb{A} \perp \mathbb{A} \text{ or } \langle 2, 6 \rangle \perp \mathbb{A} & \text{if } p \in \{23, 31\}. \end{cases}$$

Lemmas 6.5 and 6.6 imply that M_2 is either 2-universal or $2\mathbb{Z}_2$-universal. However M_2 cannot be 2-universal because then M would be an universal p-anisotropic quaternary $\mathbb{Z}$-lattice, and no such a lattice exists for $p > 7$ (cf. [2]). Thus M is $2\mathbb{Z}_2$-universal and $\Delta(L) = \delta_2^{k-1}(L)$, contradicting the assumption of the minimality of k. Hence $\Delta(L) = L$, □

Finally we can complete the proof of Theorem 6.1.

Proof of Theorem 6.1. Let $L \in \mathcal{R}_p$. By Lemma 6.9, $L = \Delta(L)$. By Theorem 2.11, $\Delta(L)$ is in $\mathcal{U}_p$. The theorem follows. □

7. Criteria for almost regularity

The explicit knowledge of all lattices in $\mathcal{R}_p$ for $p \in P^*$ allows us to formulate a very simple and effective description of all genera containing some primitive p-anisotropic almost regular quaternary $\mathbb{Z}$-lattice.

Theorem 7.1. *Let $p \in \{17, 19, 29, 37\}$ (resp. $p \in \{23, 31\}$). For any primitive p-anisotropic quaternary $\mathbb{Z}$-lattice L the following properties are equivalent:*

(i) *The genus of L contains some almost regular lattice.*
(ii) *The lattice L is even and $d(L) = p^{2n}$ (resp. $d(L) = p^{2n}$, or $d(L) = 4p^{2n}$ and $L_2 \simeq \mathbb{A} \perp \langle 2, 6\rangle$) for some integer $n \geq 1$.*

Proof. (i) $\Longrightarrow$ (ii) Without loss of generality we may assume that $L \in \mathcal{E}_p$. One has $d(L) = kp^{2n}$, where k and p are relatively prime, and $n \geq 1$. By Theorem 1.3, $L_q \simeq G_q$ for some $G \in \mathcal{R}_p$ and all $q \neq p$. It follows that $k = d(G)/p^2$. If $p \in \{17, 19, 29, 37\}$ (resp. $p \in \{23, 31\}$), then by Theorem 6.1, $d(G) = p^2$ (resp. $d(G) = p^2$ or $4p^2$). Furthermore, if $d(G) = 4p^2$ then $L_2 \simeq G_2 \simeq \mathbb{A} \perp \langle 2, 6\rangle$. Finally, since for $p \in P^*$ all lattices in $\mathcal{R}_p = \mathcal{U}_p$ are even, and $L_2 \simeq G_2$, the lattice L is even.

(ii) $\Longrightarrow$ (i) By Theorem 6.1, if $G \in \mathcal{R}_p$ then $G_q \simeq x_1^2 + \cdots + x_4^2$ over $\mathbb{Z}_q$ for all $q \neq 2$, p, and $G_2 \simeq \mathbb{A} \perp \mathbb{A}$ (if $d(G) = p^2$), or $G_2 \simeq \mathbb{A} \perp \langle 2, 6\rangle$ (if $d(G) = 4p^2$). If $d(L) = p^{2n}$ or $4p^{2n}$, then clearly $L_2 \simeq x_1^2 + \cdots + x_4^2$ over $\mathbb{Z}_q$ for all $q \neq 2$, p. Since L is even and p-anisotropic, if $d(L) = p^{2n}$ then necessarily $L_2 \simeq \mathbb{A} \perp \mathbb{A}$. If $d(L) = 4p^{2n}$ then $L_2 \simeq \mathbb{A} \perp \langle 2, 6\rangle$ by assumption. In either case, $L_q \simeq G_q$ for some G in $\mathcal{R}_p$ and all $q \neq p$. It follows from Theorem 1.3 that the genus of L contains some almost regular lattice. □

The sets $\Omega_p \subset 2\mathbb{Z}$, $p \in P^*$, intervening in the next theorem are defined in the Introduction. We shall also use the lattices $G(p)$ and $G'(p)$ defined in Section 3 (cf. Theorem 3.2), and the invariant $s(L, p)$ defined in Section 4. Recall that $s(L, p)$ is a nonnegative integer which depends only on L_p and can be effectively computed. The next result gives effective criteria for almost regularity of p-anisotropic quaternaries for $p \in P^*$.

Theorem 7.2. *Let $p \in P^*$. For any primitive p-anisotropic quaternary $\mathbb{Z}$-lattice L the following properties are equivalent:*

(i) *L is almost regular.*

(ii) *One has*

$$\Delta_p(L) \simeq \begin{cases} G(p) & \text{if } p \in \{17, 19, 29, 37\}, \\ G(p) \text{ or } G'(p) & \text{if } p \in \{23, 31\}. \end{cases}$$

(iii) *L is even and represents all positive integers divisible by $2p^{s(L,p)}$.*
(iv) *L is even and represents all elements of the set $p^{s(L,p)}\Omega_p$.*

Proof. (i) $\Longleftrightarrow$ (ii) follows from Theorems 4.2, 6.1 and 3.2.

(i) $\Longrightarrow$ (iii) By Theorem 4.2, $\Delta_p(L)$ is in $\mathcal{R}_p$, and by Theorem 6.1, $\Delta_p(L)$ is even-universal. Hence L is even and, by Corollary 4.4 (ii), L represents all positive integers divisible by $2p^{s(L,p)}$.

(iii) $\Longrightarrow$ (iv) is obvious.

(iv) $\Longrightarrow$ (i) Since L is even, $\Delta_p(L)$ is also even and, by Corollary 4.4 (ii), $\Delta_p(L)$ represents all elements in Ω_p. By Theorem 3.2, $\Delta_p(L)$ is even-universal. In particular $\Delta_p(L)$ is in $\mathcal{R}_p$ and L is almost regular by Theorem 4.2. □

Theorem 7.2 does not extend to primes $p \leq 13$. For $p \leq 7$ there exist odd p-anisotropic almost regular quaternaries. But the reason for failure is more deeper. Even if we disregard odd lattices, the implication (i) $\Rightarrow$ (iii) is not valid for all even lattices in $\mathcal{E}_p$ if $p \leq 13$. Indeed, (iii) implies that every even lattice in $\mathcal{R}_p$ is even-universal, which is not the case if $p \leq 13$ (cf. Theorem 8.1). We conjecture that the equivalence (i) $\Leftrightarrow$ (iv) is not valid either for $p \leq 13$ and any set S_p in place of Ω_p (even if we consider only even lattices). A counterexample for $p = 2$ is given below.

Counterexample 7.3. There is no set $S \subset \mathbb{Z}$ (finite or infinite) such that the following statement holds true:

Statement A. For any even primitive 2-anisotropic quaternary $\mathbb{Z}$-lattice L the following properties are equivalent:

(i) L is almost regular.
(ii) L represents all integers in the set $2^{s(L,2)}S$.

Since $\Delta_2(L) \simeq \mathbb{A} \perp 2\mathbb{A}$, the statement above is equivalent, by Corollary 4.4 (ii), to the following one:

Statement A′. Any primitive quaternary $\mathbb{Z}$-lattice L, with $L_2 \simeq \mathbb{A} \perp 2\mathbb{A}$, is regular if and only if L represents all elements in S.

Assume that A′ holds true for some S necessarily contained in $2\mathbb{Z}$. We claim that $m \in S$, for some integers m with $m \not\equiv 0 \pmod{18}$. Indeed, consider the $\mathbb{Z}$-lattice $H = \mathbb{A} \perp 18\mathbb{A}$. Clearly $H_2 \simeq \mathbb{A} \perp 2\mathbb{A}$. Since H contains $9(\mathbb{A} \perp 2\mathbb{A})$, H represents all positive integers divisible by 18. The lattice H is not regular because 20 which is represented by the genus of H is not represented by H. Hence S must contain an integer not divisible by 18. We shall show that this is not possible. Let M and M' be the $\mathbb{Z}$-lattices corresponding to the quadratic forms

$$M = 4x^2 + 10y^2 + 10z^2 + 10t^2 + 4xy + 4xz + 4xt + 2yz + 2yt + 2zt$$

$$M' = 2x^2 + 14y^2 + 14z^2 + 14t^2 + 2xy + 2xz + 2xt + 10yz + 10yt - 8zt.$$

Observe that $d(M) = d(M') = 2^2 \cdot 3^6$, M and M' are primitive, $M_2 \simeq M'_2 \simeq \mathbb{A} \perp 2\mathbb{A}$. One can check that the class number of M and M' is

1. In particular, M and M' are regular. Since $M_3 \simeq \langle 1, 3^2, 3^2, 3^2 \rangle$ and $M'_3 \simeq \langle 2, 3^2, 3^2, 2\cdot 3^2 \rangle$, it follows that any integer represented simultaneously by M_3 and M'_3 is divisible by 9. Hence any integer represented by M and M' is divisible by 18, and $S \subset 18\mathbb{Z}$. Therefore S does not exist. $\square$

However we conjecture that for each $p \leq 13$ there is a finite collection $\Omega_{1,p}, \ldots, \Omega_{n_p,p}$ of finite sets of positive integers such that a primitive p-anisotropic quaternary $\mathbb{Z}$-lattice L is almost regular if and only if L represents all elements in $p^{s(L,p)}\Omega_{i,p}$, for *some* i.

Here we shall prove only a weaker result. The sets $\Omega_3, \ldots, \Omega_{13}$ intervening in the next proposition are defined in Section 3 (cf. Theorem 3.6).

Proposition 7.4. *Let L be a primitive p-anisotropic quaternary $\mathbb{Z}$-lattice, $3 \leq p \leq 13$. Then the following properties are equivalent:*

(i) *L is almost regular, $2\mathbb{Z}_2$-universal and q-universal for all $q \neq 2$, p.*
(ii) *$\Delta_p(L)$ is in $\mathfrak{U}_p$.*
(iii) *L is even and represents all elements in $p^{s(L,p)}\Omega_p$.*

Proof. (i)$\Longleftrightarrow$(ii) follows from Proposition 2.1 (iii) and Theorem 4.2.
(ii)$\Longleftrightarrow$(iii) follows from Theorem 3.6 and Corollary 4.4 (ii). $\square$

Proposition 7.5. *For any primitive 2-anisotropic quaternary $\mathbb{Z}$-lattice L the following properties are equivalent:*

(i) *L is almost regular and q-universal for all $q > 2$.*
(ii) *$\Delta_2(L) \in \mathfrak{U}_2$.*
(iii) *L represents all elements of the set $2^{s(L,2)}\Omega_2$, where*

$$\Omega_2 = \{2, 4, 6, 10, 12, 14, 20, 28\}.$$

Proof. (i) $\Longleftrightarrow$ (ii) follows from Proposition 2.1 (iii) and Theorem 4.2.
(ii) $\Longleftrightarrow$ (iii) follows from Corollary 4.4 (ii) and Theorem 3.1 in [3] which says that a $\mathbb{Z}$-lattice M with $M_2 \simeq \mathbb{A} \perp 2\mathbb{A}$ is even-universal if and only if M represents all elements in Ω_2. $\square$

Example 7.6. For every $n \geq 0$ the form $f_n = x^2 + y^2 + z^2 + 4^n t^2$ is regular, 2-anisotropic and q-universal for all $q > 2$. The last two properties are obvious. Let $n \geq 1$. Let A_n be the set of all integers represented by the genus of f_n, let B be the set of all integers represented by $x^2 + y^2 + z^2$, and let

$$C_n = \{4^s(8k+7) \mid s \geq n-1, k \geq 0\}.$$

Clearly, $A_n = B \cup C_n$. To show that f_n is regular it suffices to show that f_n represents every number of the type $4^{n-1}(8k+7)$, $k \geq 0$. The equation

$$4^{n-1}(8k+7) = x^2 + y^2 + z^2 + 4^n$$

is equivalent to

$$4^{n-1}(8k+3) = x^2 + y^2 + z^2,$$

which is solvable in integers. The regularity of f_n follows. $\square$

8. Families $\mathcal{R}_p$, $2 \le p \le 13$

We have seen in Section 6 that for all $p \ge 17$ one has $\mathcal{R}_p = \mathcal{U}_p$. This is no longer true for $p < 17$.

Theorem 8.1. *For each prime $p \le 13$ there is an even lattice $H(p)$ in $\mathcal{R}_p$ which is not even-universal.*

The lattices $H(p)$ will by defined explicitly below.

Family $\mathcal{R}_{13}$. This family contains at least 5 elements, namely the 4 lattices of $\mathcal{U}_{13}$ and an another lattice $H(13)$ defined by

$$H(13) = \begin{bmatrix} 4 & 1 \\ 1 & 10 \end{bmatrix} \perp \langle 6, 78 \rangle,$$

which is not even-universal.

Proposition 8.2. *The lattice $H(13)$ is in $\mathcal{R}_{13}$, but is not 3-universal.*

Proof. One sees easily that for $H = H(13)$ one has $d(H) = 2^2 \cdot 3^3 \cdot 13^2$ and

$$H_2 \simeq \mathbb{H} \perp \langle 6, 14 \rangle, \quad H_3 \simeq \langle 1, 3, 3, 3 \rangle \quad \text{and} \quad H_{13} \simeq \langle 1, -2, 13, -26 \rangle.$$

It follows that H is 13-anisotropic, q-universal for every prime $q \ge 5$ and $2\mathbb{Z}_2$-universal. Clearly, H is not 3-universal.

Checking that H is regular is much more delicate. It consists in showing that every positive integer *not* of the form $6k + 2$ is represented by H.

Since

$$H \simeq \begin{bmatrix} 4 & 3 \\ 3 & 12 \end{bmatrix} \perp \langle 6, 78 \rangle,$$

it follows that

$$\Lambda_3(H) \simeq \begin{bmatrix} 36 & 9 \\ 9 & 12 \end{bmatrix} \perp \langle 6, 78 \rangle,$$

which implies that

$$\delta_3(H) = \frac{1}{3}\Lambda_3(H) \simeq \begin{bmatrix} 4 & 1 \\ 1 & 10 \end{bmatrix} \perp \langle 2, 26 \rangle$$

is even-universal (cf. Section 3, Table (13)). Then any even positive integer $2m$ is represented by $\delta_3(H)$, which implies that any $6m$ is represented by $\Lambda_3(H) \subset H$. It suffices therefore to show that every positive integer of the form $6m + 4$ is represented by H. One easily shows that

$$\delta_{13}(H) \simeq \frac{1}{13}\Lambda_{13}(H) \simeq H.$$

Hence, if a number k is represented by H, then $13k$ is represented by $\Lambda_{13}(H)$, and thus by H. Consequently, it suffices to prove that H represents every positive integer of the form $6m + 4$ which is not divisible by 13. Now the regularity of H follows from the next lemma. $\square$

Lemma 8.3. *The ternary $\mathbb{Z}$-lattice*

$$\ell = \begin{bmatrix} 4 & 1 \\ 1 & 10 \end{bmatrix} \perp \langle 6 \rangle$$

represents every integer in the set

$$U = \{n \in \mathbb{Z} \mid n \ge 0,\ n \equiv 4 \pmod 6,\ n \not\equiv 0 \pmod{13}\}.$$

Sketch of the proof. Let ℓ' be a ternary $\mathbb{Z}$-lattice defined by its Gram matrix

$$\ell' = \begin{bmatrix} 4 & 3 & 3 \\ 3 & 6 & 3 \\ 3 & 3 & 18 \end{bmatrix}.$$

The lattices ℓ and ℓ' are not isometric and the genus of ℓ is $\{\ell, \ell'\}$. The lattice ℓ is not regular (ℓ does not represent 138 which is represented by its genus). It is easy to see that any number in U is represented by the genus of ℓ, that is, either by ℓ or ℓ'. To prove the lemma it suffices therefore to show that if $\xi \in U$ is represented by ℓ' then ξ is also represented by ℓ. However this part of the proof is quite long and will be omitted. □

Family $\mathcal{R}_{11}$. This family contains at least 3 elements. There are exactly two lattices in $\mathcal{R}_{11}$ which are even-universal. But there is also a lattice $H(11) = B \perp 4B$, where

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix},$$

which is not even-universal.

Lemma 8.4. *The lattice $H(11)$ is in $\mathcal{R}_{11}$, but is not $2\mathbb{Z}_2$-universal.*

Proof. Let denote $H(11) = H$. Since $H_{11} \simeq \langle 1, 1, 11, 11\rangle$ and $H_2 \simeq \mathbb{A} \perp 4\mathbb{A}$, the lattice H is 11-anisotropic, 11-universal, but is not $2\mathbb{Z}_2$-universal. We have to show that H is regular. Consider that lattice $M = B \perp B$ which is even-universal (cf. Theorem 3.6). The ternary sublattice $T = B \perp \langle 8\rangle$ of M is regular, though its class number is 2 (cf. [9]). The lattice T_2 represents every element $b \in 2\mathbb{Z}_2$ such that $\mathrm{ord}_2(b) \neq 2$, and T_q is q-universal for every odd prime $q \neq 11$. Since $T_{11} \simeq \langle 1, 1, -11\rangle$, T represents every even positive integer $b \not\equiv 4 \pmod 8$, except integers of the form $11^{2n+1}u$, where n and u are nonnegative integers, and u is a square unit in $\mathbb{Z}_{11}$.

Let $e_1, \ldots, e_4$ be a basis of H corresponding to the matrix $B \perp 4B$. Since the sublattice $\mathbb{Z}2e_1 + \mathbb{Z}2e_2 + \mathbb{Z}e_3 + \mathbb{Z}e_4$ of H is isometric to $4B \perp 4B = 4M$, H represents every positive integer divisible by 8. Hence if we show that every positive integer of the form $2(2s+1)$ is represented by H, then H is regular. Furthermore, since T is also a sublattice of H, it suffices to show that every integer of the form

$$v(n, m, k) = 11^{2n+1}(44m + k),$$

where n, m are nonnegative integers and $k = 14, 26, 34, 38, 42$, is represented by H.

Let $N = \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3 + \mathbb{Z}(e_3 - 2e_4)$ be a sublattice of H. Then $N \simeq T \perp \langle 88\rangle$. Hence if $v(n, m, k) - 88c^2$ is represented by T for some integer c, then $v(n, m, k)$ is represented by H.

Let $r_k = 8$ (resp. 32) if $k = 14, 26, 38$ (resp. $k = 34, 42$). Then $11^{2n+1}r_k$ is of the form $88c^2$ and

$$\beta = v(n, m, k) - 11^{2n+1}r_k = 11^{2n+1}(44m + k - r_k)$$

is of the form $\beta = 11^{2n+1}t$, where t is a nonsquare unit in $\mathbb{Z}_{11}$, and $\beta/2$ is odd. It follows that β is represented by T and therefore $v(n, m, k)$ is represented by H, which completes the proof. □

Families $\mathcal{R}_3$, $\mathcal{R}_5$ and $\mathcal{R}_7$. These families contain, respectively, 12, 14 and 4 even-universal lattices (cf. Theorem 3.6). Furthermore, there is a single universal lattice in each of $\mathcal{R}_3$ and $\mathcal{R}_7$, and there are two in $\mathcal{R}_5$ (cf. [2]). Each $\mathcal{R}_p$, $p = 3,\ 5,\ 7$, also has at least one even lattice $H(p)$ which is not even-universal. They are

$$H(3) = \mathbb{A} \perp 4\mathbb{A},\ H(5) = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} \perp 3\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix},\ H(7) = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 6 & -1 & 2 \\ 1 & -1 & 6 & 2 \\ 0 & 2 & 2 & 16 \end{bmatrix}.$$

One can easily verify that $H(p)$ is p-universal and p-anisotropic. If $p = 3$ or 7 then $H(p)_2 \simeq \mathbb{A} \perp 4\mathbb{A}$, so neither $H(3)$ nor $H(7)$ is $2\mathbb{Z}_2$-universal. The regularity of $H(3)$ and $H(7)$ follows from the fact that each of them is single in its genus. Since $H(5)_3 \simeq \langle 1, 3, 6, 18\rangle$, the lattice $H(5)$ is not 3-universal. The question of its regularity is more delicate, because the genus of $H(5)$ contains two lattices. It requires a computation similar to that given above for $H(11)$ and will be omitted. □

Family $\mathcal{R}_2$. This family consists of all primitive regular quaternaries L such that $L_2 \simeq \mathbb{A} \perp 2\mathbb{A}$. Among them there are 79 lattices of the family $\mathcal{U}_2$ mentioned in Section 3 and listed in [3]. The set $\mathcal{R}_2 \setminus \mathcal{U}_2$ is certainly quite large. Two examples of the lattices M and M' which are in $\mathcal{R}_2 \setminus \mathcal{U}_2$ were given in Counterexample 7.3. The invariant α_2 is at least 67, but probably much bigger. □

Solving the following problem would be of great interest.

Open problem. Determine all elements of $\mathcal{R}_p$, $p \leq 13$.

Since the discriminants of the lattices in $\mathcal{R}_p$ are bounded by an effective constant (cf. Theorem 5.4), the obstacles to solve this problem are essentially only computational.

9. Discriminants of almost regular quaternary lattices

In this section we shall study the discriminants of the lattices in $\mathcal{E}_p$.

Theorem 9.1. *For each prime p there is a finite set A_p of integers relatively prime to p, such that an integer d is the discriminant of a primitive almost regular p-anisotropic quaternary $\mathbb{Z}$-lattice if and only if $d = rp^{2n}$ for some $r \in A_p$ and some integer $n \geq 1$ (if $p > 2$), and $n \geq 0$ (if $p = 2$).*

Proof. Theorem 1.3 implies that

$$A_p = \left\{ \frac{d(L)}{p^2} \;\middle|\; L \in \mathcal{R}_p \right\},$$

which is finite, and bounded by an effective constant (cf. Theorem 5.4). □

We have seen that $A_p = \{1\}$ for $p = 17, 19, 29, 37$, and $A_p = \{1, 4\}$ for $p = 23,\ 31$ (cf. Theorem 6.1). Clearly, $A_p = \emptyset$ for $p > 37$. The list of all elements of A_p for the remaining 6 primes $p \leq 13$ is not known. However, from the next two results we can get some idea about the shape of A_p for these values of p.

Recall that $D_p = \{q \in P \mid q \text{ divides } d(L), L \in \mathcal{E}_p\}$ and that the set

$$U_2 = \{q \in P \mid q \text{ divides } d(L), L \in \mathcal{U}_2\} = \{2, 3, 5, \ldots, 281, 353\}$$

contains exactly 29 primes which are listed at the beginning of Section 3. Let $D = \bigcup D_p$ be the set of all primes dividing the discriminant of some primitive exceptional almost regular quaternary $\mathbb{Z}$-lattice.

Theorem 9.2. *One has*

$$D = D_2 = U_2 = \{2, 3, 5, \ldots, 281, 353\}.$$
$$D_3 = \{2, 3, 5, 7, 37\}, \quad D_5 = \{2, 3, 5, 11\},$$
$$\{2, p\} \subset D_p \subset \{2, 3, 5, p\} \textit{ for } p = 7, 11, \quad \{2, 3, 13\} \subset D_{13} \subset \{2, 3, 5, 13\},$$
$$D_p = \{2, p\} \textit{ for } p = 23, 31, \quad D_p = \{p\} \textit{ for } p = 17, 19, 29, 37.$$

Proof. Since by Corollary 5.3 one has

$$U_p \subset D_p \subset U_p \cup \{2, 3, 5\},$$

the theorem follows from the direct inspection of the sets U_p, which are known explicitly (cf. Corollary 3.7 for $p \geq 3$). □

Proposition 9.3. *Let ξ_p be the maximal number of primes dividing the discriminant of some lattice in $\mathcal{E}_p$. Then $\xi_p = 1$ for $p = 17, 19, 29, 37$, $\xi_p = 2$ for $p = 23, 31$, $3 \leq \xi_p \leq 4$ for $p = 5, 13$, $2 \leq \xi_p \leq 4$ for $p = 7, 11$, $3 \leq \xi_3 \leq 5$, $4 \leq \xi_2 \leq 5$.*

Proof. Since

$$U_p \subset D_p \subset U_p \cup \{2, 3, 5\},$$

the estimates for $p \geq 3$ can be read directly from the list of the discriminants of the lattices in $\mathcal{U}_p$ given in Section 3. The inspection of the discriminants of the lattices of family $\mathcal{U}_2$ (given in [3] Table 4), implies that for any L in $\mathcal{E}_2$ the discriminant $d(L)$ is of the form

$$d(L) = 4^k \cdot 3^\ell \cdot 5^m \cdot b^s \quad \text{or} \quad 4^k \cdot 3^\ell \cdot 5^m \cdot 7 \cdot 13 \quad \text{or} \quad 4^k \cdot 3^\ell \cdot 5^m \cdot 11 \cdot 19,$$

where $b \in U_2$, $k \geq 0$ is an arbitrary integer, $0 \leq s \leq 2$, and $0 \leq m \leq N$, $0 \leq \ell \leq N$, for some constant N independent of L. The proposition follows. □

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