Spectral geometry and scattering theory for certain complete surfaces of finite volume

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MPI/91-21

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Introduction

In this paper we study complete surfaces (M, g) which have finite area and hyberbolic ends or, what is the same, complete surfaces of finite area whose Gaussian curvature equals -1 in the complement of some compact subset of M. Such a surface has a decomposition into a compact surface with smooth boundary and a finite number of ends called *cusps*. Examples are compactly supported conformal deformations of hyperbolic surfaces of finite area. Let Δ be the Laplace operator on M associated to the metric g. Since g is complete, Δ, regarded as an operator in $L^2(M)$ with domain $C_0^\infty(M)$, is essentially self-adjoint [Ch] and we shall denote its unique self-adjoint extension by $\overline{\Delta}$.

If M is compact then the spectrum of $\overline{\Delta}$ is a discrete sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots \to \infty$$

where each eigenvalue λ_j has finite multiplicity. A great deal of work has been done to understand the following problem:

(0.1) *To what extent do the eigenvalues determine the geometric structure of (M, g) and vice versa?*

See for example [Hu], [Mc], [OPS1], [OPS2], [Su], [V], [W]. Part of this theory may be regarded as inverse spectral theory for compact surfaces.

Our purpose is to study the analogous problem for the class of complete surfaces defined above. To start with we have to find appropriate spectral data which can replace the eigenvalues in the compact case. The spectrum of the Laplace operator on a complete surface M of finite area and with hyperbolic ends consists of a sequence of eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and an absolut continuous spectrum which is the interval $[1/4, \infty)$ with multiplicity equal to the number of ends of M. Furthermore, Colin de Verdiere [C2] has shown that, for a generic metric on M, the Laplace operator has only finitely many eigenvalues and all of them lie below the continuous spectrum. Thus eigenvalues are certainly not sufficient for the purpose of spectral geometry.

The additional spectral information is provided by scattering theory. The stationary approach to scattering theory gives rise to a scattering matrix $C(s)$ which is a meromorphic matrix valued function of $s \in \mathbf{C}$. Its coefficients may be interpreted as follows: A plane wave is sent in from a given cusp and scattered by the compact part of the surface, transmitting some part into the other cusps and reflecting another part into the given one. The asymptotic behaviour of the scattered plane waves obtained in this way is described by the scattering matrix $C(s)$. In analogy to quantum mechanics we call poles of $C(s)$ *resonances*. Actually we are working with the poles of the meromorphic function

$$\phi(s) = \det C(s)$$

and call also these poles *resonances*. The resonances are the complementary spectral parameters to the eigenvalues which we are going to use to develop spectral geometry for the surfaces described above. To combine resonances and eigenvalues in a common set we write each eigenvalue λ_j as

$$\lambda_j = s_j(1 - s_j), \quad s_j \in \mathbf{C}$$

and associate to λ_j the points s_j given by this equality. Then we introduce the set $\sigma(M)$ which is the union of the following three sets:

(a) The set of all poles and zeros of $\phi(s)$ in the half-plane $\mathrm{Re}(s) < 1/2$.

(b) The set of all $s_j \in \mathbf{C}$ such that $s_j(1 - s_j)$ is an eigenvalue of Δ.

(c) $\{\frac{1}{2}\}$.

Each point $\eta \in \sigma(M)$ occurs with a certain multiplicity $m(\eta)$ (cf. Definition 5.20). In abuse of notation we call $\sigma(M)$ the *resonance set* and we think of the s_j-th as being resonances corresponding to L^2 bound states. By means of Lax-Phillips scattering theory one can identify $\sigma(M)$ with the spectrum of a certain non-self-adjoint operator $B+\frac{1}{2}I$. Here B is the generator of the Lax-Phillips semi-group $Z(t), t \geq 0$, associated to the hyperbolic wave equation on M. This operator has a compact resolvent and therefore, we can employ standard perturbation theory to study the behaviour of $\sigma(M)$ under perturbations of the metric. For hyberbolic surfaces this approach was first used by Phillips and Sarnak [PS1] to study $\sigma(M)$ as a function on Teichmüller space. The problem analogous to (0.1) can now be stated as follows:

(0.2) *To what extent does the resonance set $\sigma(M)$ determine the geometric structure of (M,g) and vice versa?*

This may be compared with the forward and inverse problem of scattering on the real line (cf. [DT]). The role of the potential is played by the metric on M. But now we have also to deal with the topological structure of the surface.

Of course, we can not expect to get a complete answer to (0.2) at the present state of our knowledge. Even in the compact case there are many open problems related to (0.1). Our purpose in this paper is to develop some of the machinery which is available for compact surfaces and to extend some of the results from the compact case.

Now we shall describe the content of this paper. In section 1 we recall the basic facts about the spectral decomposition of $\overline{\Delta}$ and we introduce the scattering matrix $C(s)$. In section 2 we review some results of [Mü] concerning the heat kernel $K(z_1,z_2,t)$, including the trace formula for the truncated heat kernel and the related asymptotic expansion. Then we study in section 3 the analytic properties of the scattering matrix $C(s)$. This is the forward problem of scattering theory. One of the main results is Theorem 3.20 which says that the determinant $\phi(s)$ of $C(s)$ is a meromorphic function of order ≤ 4. For hyperbolic surfaces this result is due to Selberg [Sel]. Our method is different from his and it is based on Colin de Verdiere's method of the analytic continuation of Eisenstein series [C1]. Another result of section 3 is the following product formula:

$$\phi(s) = q^{s-1/2}\prod_{\rho}\frac{s-1+\overline{\rho}}{s-\rho} \tag{0.3}$$

where ρ runs over all poles of $\phi(s)$, counted with the order, and q is a certain constant. This is again due to Selberg [Sel] if M is hyperbolic. An important consequence of (0.3) is the following formula for the logarithmic derivative of ϕ along the line $\mathrm{Re}(s)=1/2$:

$$\frac{\phi'}{\phi}(1/2+i\lambda) = \log q + \sum_{\rho}\frac{2\mathrm{Re}(\rho)-1}{(1/2-\mathrm{Re}(\rho))^2+(\lambda-\mathrm{Im}(\rho))^2} \tag{0.4}$$

This formula is important for the further investigation of $\sigma(M)$.

In section 4 we study the distribution of poles of $\phi(s)$ and the main result - Theorem 4.23 - can be restated as follows:

$$\#\{\eta\in\sigma(M) \mid |\eta|\leq T\} \sim \frac{\mathrm{Area}(M)}{2\pi}\,T^2 \tag{0.5}$$

as $T\to\infty$. This may be regarded as an analogue of Weyl's formula.

Next we consider in section 5 the following integral

$$\frac{1}{4\pi}\int_{-\infty}^{+\infty} h(\lambda)\frac{\phi'}{\phi}(1/2+i\lambda)d\lambda$$

where h is the Fourier transform of some function g in $C_0^\infty(\mathbf{R})$.

We wish to express this integral in terms of the poles of $\phi(s)$. The Cauchy residue theorem can not be applied directly, because the integral obtained by shifting the contour of integration to the line $\mathrm{Re}(s)=\sigma$ does not disappear in the limit $\sigma\to-\infty$. Nevertheless, using (0.4), we are able to prove the following result (Theorem 5.15):

Let g be an even function in the Schwartz space $\mathcal{S}(\mathbf{R})$. Set $h=\hat{g}$ and $h_+(z)=\int_0^\infty g(y)e^{zy}dy$, $\mathrm{Re}(z)\le 0$. Then

(0.6)

$$-\frac{1}{4\pi}\int_{-\infty}^{+\infty} h(\lambda)\frac{\phi'}{\phi}(1/2+i\lambda)d\lambda=-\frac{\log q}{4\pi}\,g(0)+\frac{1}{2}\sum_{\rho} n(\rho)\{h_+(\rho-1/2)+h_+(\bar{\rho}-1/2)\},$$

where q is the constant occurring in (0.3) , ρ runs over all poles and zeros of $\phi(s)$ in $\mathrm{Re}(s)<1/2$ and $n(\rho)$ is the order of the pole or the negative of the order of the zero of $\phi(s)$ at ρ. If M is hyperbolic and g has compact support with support contained in $(0,\infty)$, this formula was proved by Lax and Phillips [LP,Chapter IX]. For M hyperbolic, (0.6) allows us to rewrite the Selberg trace formula in a way which, up to some inessential terms, resembles the trace formula for a compact surface (cf. Theorem 5.31). In the same way we can rewrite the trace formula for the truncated heat kernel and combined with the asymptotic expansion for the trace of the truncated heat operator we obtain that $\sigma(M)$ determines Area(M), the Euler characteristic $\chi(M)$ of M and the number m of ends of M. In particular, $\sigma(M)$ determines the conformal type (h,m) of M where h is the genus of the surface $\overline{M}$ obtained by compactifying M.

In section 6 we introduce two different zeta functions associated with the Laplace operator Δ. The first one is the *spectral zeta function* $\zeta_\Delta(s)$ which is essentially the Mellin transform of the trace of the truncated heat operator and the second one is the *resonance zeta function*

$$\zeta_B(s)=\sum_{\eta\in\sigma(M)}{}'(1-\eta)^{-s},\quad \mathrm{Re}(s)>2$$

where $\sum'$ means the sum over all $\eta\neq 1$. Here the index B denotes the generator of the Lax-Phillips semi-group associated to the hyperbolic wave equation. According to Theorem 5.27 we have $\sigma(M)=\mathrm{Spec}(B+\frac{1}{2}I)$. Therefore we may regard $\zeta_B(s)$ as the zeta function of the non-self-adjoint operator $B_1=-B+\frac{1}{2}I$.

The meromorphic continuation of $\zeta_\Delta(s)$ is obtained in the well-known way from the asymptotic expansion of the trace of the truncated heat operator . Then we use an extended version of (0.6) to express $\zeta_\Delta(s)$ in terms of $\sigma(M)$. This formula establishes a relation between $\zeta_\Delta(s)$ and an infinite linear combination of resonance zeta functions with shifted argument $\zeta_B(2s+k), k\in\mathbf{N}$ and it leads finally to the meromorphic continuation of $\zeta_B(s)$.

Both zeta functions are holomorphic at $s = 0$ and we can introduce the corresponding determinants

$$\det{}'\Delta = e^{-\zeta'_\Delta(0)} \quad \text{and} \quad \det{}' B_1 = e^{-\zeta'_B(0)}.$$

The complecated relation between $\zeta_\Delta(s)$ and $\zeta_B(s)$ is reduced to the following simple equality for the determinants

$$\det{}'\Delta = \exp\left(\frac{\operatorname{Area}(M)}{8\pi} - \frac{3\pi\gamma}{2}m\right)\det{}' B_1 \tag{0.7}$$

where γ denotes Euler's constant and m is the number of ends of M. For $z \in \mathbf{C}, \operatorname{Re}(z) > 1$, we also introduce regularized determinants $\det(\Delta + z(z-1))$ and $\det(B_1 + (z-1))$ and (0.7) extends to a corresponding identity for these determinants. If the surface is hyperbolic, we use results of I.Efrat [E1], [E2] to express $\det(B_1 + (z-1))$ in terms of $\phi(s)$ and the Selberg zeta function.

Throughout sections 7 and 8 we assume that M is hyperbolic and we study the inverse problem of scattering theory for hyperbolic surfaces. In section 7 we use the version of the Selberg trace formula established in section 5 to show that $\sigma(M)$ determines the length spectrum of the closed geodesics of M and vice versa.

Finally, in section 8 we prove that the resonance set determines a hyperbolic surface of finite area up to finitely many possibilities. For compact hyperbolic surfaces this result is due to H.McKean [Mc]. According to S.Wolpert [W] we also know that a generic compact hyperbolic surface is uniquely determined by the eigenvalues of its Laplacian. In other words, the eigenvalues are moduli for generic compact hyperbolic surfaces. We can ask the same question: *Are the points $\eta \in \sigma(M)$ moduli for a generic hyperbolic surface of finite area?* The answer is very likely to be yes. We may also try to extend T.Sunada's results concerning isospectral manifolds [Su]. In our context two surfaces M_1 and M_2 are called *isospectral* if the resonance sets $\sigma(M_1)$ and $\sigma(M_2)$ coincide. We shall not pursue any of these problems in the present paper, but we shall return to these questions in a forthcoming publication.

We expect that most of the theory developed in this paper can be done for larger classes of surfaces. For example, the condition on the ends can certainly be relaxed. In place of hyperbolic ends we may assume that the metric is asymptotic (in a sense to be made precise) to the metric of constant curvature -1. Since this is technically more complicated we have chosen to work with the surfaces introduced above.

Acknoledgment: This work was done during the author's visit at the Institute for Advanced Study at Princeton and the Max-Planck-Institut für Mathematik at Bonn. I am very grateful to both institutions for financial support and hospitality.

1. The spectral resolution of the Laplacian on admissible surfaces

As in the introduction we let (M, g) be a complete surface of finite area whose Gaussian curvature equals -1 in the complement of some compact subset of M. In other words, (M, g) is a two-dimensional Riemannian manifold which admits a decomposition of the form

$$M = M_0 \cup Z_1 \cup \cdots \cup Z_m,$$

where M_0 is a compact surface with smooth boundary and

$$Z_i \cong [a_i, \infty) \times S^1, \quad i = 1, ..., m,$$

with $a_i > 0$ and the metric on Z_i equals

$$ds^2 = \frac{dy^2 + dx^2}{y^2}$$

where $(y, x) \in [a_i, \infty) \times S^1$. Each end Z_i will be called *cusp* and the surface M will be called *admissible.*

Any admissible surface is diffeomorphic to the complement of a finite number of points $z_1, ..., z_m$ in a compact surface $\overline{M}$. Let h be the genus of $\overline{M}$. The pair (h, m) is called the *conformal type* of the surface M. If the metric g on M has constant curvature -1, then we call M a *hyperbolic* surface. Any hyperbolic surface is of the form $\Gamma\backslash H$ where H is the upper half-plane and Γ is a torsion free discrete subgroup of $SL(2, \mathbf{R})$. Finally, we note that for an admissible surface M the Gauss-Bonnet theorem holds:

$$\chi(M) = \frac{1}{2\pi} \int_M K(z) dz.$$

Here $\chi(M)$ is the Euler characteristic and $K(z)$ the Gaussian curvature of M at $z \in M$ (cf. [CG]).

Now let $\Delta : C^\infty(M) \to C^\infty(M)$ be the Laplace operator on M. We denote by $L^2(M)$ the Hilbert space of measurable functions on M which are square integrable with respect to the measure $d\mu$ defined by the Riemannian metric g. Since (M, g) is complete, it follows from [Ch] that Δ, regarded as an operator in $L^2(M)$ with domain $C_0^\infty(M)$, is essentially self-adjoint. We denote its unique self-adjoint extension by $\overline{\Delta}$. In this section we recall some facts about the spectral resolution of $\overline{\Delta}$. Details are contained in [Mü] and [C2].

The spectrum of $\overline{\Delta}$ is the union of a point spectrum σ_p and a continuous spectrum σ_c. The point spectrum σ_p is a sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...,$$

where each eigenvalue λ_j has finite multiplicity and it is repeated in this sequence according to its multiplicity. For a generic metric on M this sequence is finite (cf. [C2]). Let $N(T)$ be the counting function, i.e.,

$$N(T) = \#\{\lambda_i | \lambda_i \leq T^2\} \tag{1.1}$$

where T is a given positive real number. Then

$$\limsup_{T\to\infty} \frac{N(T)}{T^2} \le \frac{\text{Area(M)}}{4\pi}. \tag{1.2}$$

Remark. This is not the standard definition of the counting function. But in the surface case one usually writes the eigenvalues as $\lambda_j = 1/4 + r_j^2, r_j \in \mathbf{R} \cup i[-1/2, 1/2]$, and counts the number of r_j with $|r_j| \le T$.

The continuous spectrum σ_c is the interval $[1/4, \infty)$ with multiplicity equal to the number of cusps of M. The spectral decomposition of the absolutely continuous part of $\overline{\Delta}$ is described by generalized eigenfunctions $E_i(z,s), i = 1, ..., m$, which have the following properties:

Each $E_i(z,s)$ is a meromorphic function of $s \in \mathbf{C}$ with poles contained in the union of the half-plane $\text{Re}(s) < 1/2$ and the interval $(1/2, 1]$. Furthermore, each $E_i(z,s)$ is a smooth function of $z \in M$ and satisfies

$$\Delta E_i(z,s) = s(1-s)E_i(z,s). \tag{1.3}$$

If we expand $E_i(z,s)$ in a Fourier series on the cusp Z_j then the zeroth Fourier coefficient takes the form

$$\delta_{ij} y_j^s + C_{ij}(s) y_j^{1-s}, \tag{1.4}$$

where $y_j \in [a_j, \infty)$ is the radial variable for the cusp $Z_j \cong [a_j, \infty) \times S^1$. Put

$$C(s) = \big(C_{ij}(s)\big). \tag{1.5}$$

Then $C(s)$ is a $m \times m$ matrix which is a meromorphic function of $s \in \mathbf{C}$ and satisfies

$$C(s)C(1-s) = Id, \quad \overline{C(s)} = C(\bar{s}) \quad \text{and} \quad C(s)^* = C(\bar{s}). \tag{1.6}$$

All poles of $C(s)$ are contained in the union of the half-plane $\text{Re}(s) < 1/2$ and the interval $(1/2, 1]$ and those contained in $(1/2, 1]$ are simple. In analogy to quantum mechanics we shall call $C(s)$ *scattering matrix* and its poles *resonances*. Set

$$\mathbf{E}(z,s) = \begin{pmatrix} E_1(z,s) \\ \vdots \\ E_m(z,s) \end{pmatrix}. \tag{1.7}$$

Regarded as a vector valued function it satisfies the following functional equation

$$\mathbf{E}(z,s) = C(s)\mathbf{E}(z, 1-s). \tag{1.8}$$

Let $L_d^2(M)$ be the subspace of $L^2(M)$ which is spanned by the eigenfunctions of Δ and let $\varphi_0, \varphi_1, ...$ be an orthonormal basis for $L_d^2(M)$ consisting of eigenfunctions with eigenvalues $0 = \lambda_0 < \lambda_1 \le ...$. Then the Fourier expansion of a given $f \in C_0^\infty(M)$ takes the form

$$\begin{aligned} f(z) &= \sum_j (\varphi_j, f)\varphi_j \\ &+ \frac{1}{4\pi} \sum_{k=1}^m \int_{-\infty}^{+\infty} E_k(z, 1/2 + i\lambda) \int_M E_k(w, 1/2 - i\lambda) f(w) d\mu(w) d\lambda. \end{aligned} \tag{1.9}$$

2. The heat kernel and Weyl's formula.

In this section we review some results concerning the heat kernel of the Laplace operator Δ on an admissible surface M. In section 4 of [Mü] we constructed a unique kernel $K(z_1, z_2, t)$ for the heat operator $\exp(-t\bar{\Delta})$ which satisfies the following properties:

(a) $K(z_1, z_2, t)$ is a smooth function on $M \times M \times \mathbf{R}^+$ which is symmetric with respect to (z_1, z_2) and satisfies the semi-group property.

(b) $(\frac{\partial}{\partial t} + \Delta_{z_1})K(z_1, z_2, t) = 0$.

(c) $\lim_{z_1 \to z_2} K(z_1, z_2, t) = \delta(z_1 - z_2)$, the Dirac delta measure.

(d) Let i be the function on M which is defined as

$$i(z) = \begin{cases} 1, & \text{if } z \in \mathrm{int} M_0; \\ y_j, & \text{if } z \in Z_j \text{ and } z = (y_j, x). \end{cases}$$

For each $T > 0$ there exist constants $C_1, C_2 > 0$ such that

$$|K(z_1, z_2, t)| \leq C_1 (i(z_1) i(z_2))^{1/2} t^{-1} \exp\left(-C_2 \frac{d^2(z_1, z_2)}{t}\right)$$

uniformly for $0 < t < T$ and $z_1, z_2 \in M$. Here $d(z_1, z_2)$ denotes the geodesic distance of z_1 and z_2.

See [DM] for another proof of (d).

For each $j, 1 \leq j \leq m$, let $k_j(z_1, z_2, t) \in C_0^\infty(Z_j \times Z_j \times \mathbf{R}^+)$ be defined as

$$k_j((y, x), (y', x'), t) = \frac{\sqrt{(yy')}}{\sqrt{4\pi t}} \exp\left(-\frac{t}{4} - \frac{\log^2(y/y')}{4t}\right).$$

Then $k_j(z_1, z_2, t)$ is the constant term of the heat kernel $K(z_1, z_2, t)$ on the cusp Z_j. Set

$$k(z_1, z_2, t) = \begin{cases} k_j(z_1, z_2, t), & \text{if } z_1, z_2 \in Z_j \text{ for some } j, 1 \leq j \leq m; \\ 0, & \text{otherwise.} \end{cases}$$

In other words, k is the sum of the constant terms k_j. Then one can show that $K(z, z, t) - k(z, z, t)$ is an absolutely integrable function on M (cf. section 8 in [Mü]). Furthermore, set

$$\phi(s) = \det C(s), \tag{2.1}$$

where $C(s)$ is the scattering matrix and let $d\mu$ be the measure on M associated to the Riemannian metric on M. Then, by Theorem 8.13 in [Mü], we have the following trace formula:

$$\begin{aligned} \int_M (K(z, z, t) - k(z, z, t)) d\mu(z) &= \sum_j e^{-\lambda_j t} - \frac{1}{4\pi} \int_{-\infty}^{+\infty} e^{-(1/4 + i\lambda^2)} \frac{\phi'}{\phi}(1/2 + i\lambda) d\lambda \\ &\quad + \frac{1}{4} e^{-t/4} \mathrm{Tr}(C(1/2)) + \frac{e^{-t/4}}{\sqrt{4\pi t}} \sum_{j=1}^m \log a_j. \end{aligned} \tag{2.2}$$

Using Theorem 8.20 of [Mü] we obtain the following asymptotic expansion as $t \to 0$:

(2.3)
$$\int_M (K(z,z,t) - k(z,z,t))d\mu(z) = \frac{\text{Area}(M)}{4\pi t} + \frac{m}{2}\frac{\log t}{\sqrt{4\pi t}} + \left(\frac{3\gamma m}{2} + \sum_{j=1}^{m} \log a_j\right)\frac{1}{\sqrt{4\pi t}} + \frac{\chi(M)}{6} + O(\sqrt{t}),$$

where $\chi(M)$ is the Euler characteristic of M, m is the number of cusps of M and γ denotes Euler's constant. To obtain (2.3) we simply have to determine the constants occurring in the asymptotic expansion of Theorem 8.20 of [Mü]. In our case we have $\zeta_j(s) = 2\zeta(2s)$ where $\zeta(s)$ denotes the Riemann zeta function. There are also two misprints in the statement of the theorem. Namely $\log a_j$ has to be $2\log a_j$ and $b_{j,N} = \zeta_j(0)+1$. We note that the asymptotic expansion (2.3) exists to all orders. This can be easily extracted from the proof of Theorem 8.20 of [Mü]. It is of the form

$$\int_M (Kz,z,t) - k(z,z,t))d\mu(z) \sim \sum_{k=0}^{\infty} a_k t^{-1+k/2} + \sum_{k=1}^{\infty} b_k t^{-1+k/2}\log t \quad (2.4)$$

as $t \to 0$.

As usually, we write the eigenvalues as

$$\lambda_j = 1/4 + r_j^2 \quad \text{with} \quad r_j \in \mathbf{R} \cup i[-1/2, 1/2]. \quad (2.5)$$

Note that each eigenvalue $\lambda_j \neq 1/4$ determines two points r_j and $-r_j$. Then (2.2) combined with (2.3) gives

$$\sum_j e^{-r_j^2 t} - \frac{1}{4\pi}\int_{\infty}^{\infty} e^{-\lambda^2 t}\frac{\phi'}{\phi}(1/2 + i\lambda)d\lambda = \frac{\text{Area}(M)}{4\pi t} + O\left(\frac{\log t}{\sqrt{t}}\right). \quad (2.6)$$

Let $N(T)$ be the counting function (1.1). If we apply a standard Tauberian theorem to (2.6) we get the following analogue of the Weyl theorem for admissible surfaces

Theorem 2.7. *As $T \to \infty$, we have*

$$N(T) - \frac{1}{4\pi}\int_{-T}^{T} \frac{\phi'}{\phi}(1/2 + i\lambda)d\lambda \sim \frac{\text{Area}(M)}{4\pi}T^2$$

For hyperbolic surfaces this result is due to Selberg [Sel]. In this case the asymptotic expansion can be improved and includes two remainder terms

$$N(T) - \frac{1}{4\pi}\int_{-T}^{T} \frac{\phi'}{\phi}(1/2 + i\lambda)d\lambda = \frac{\text{Area}(M)}{4\pi}T^2 - \frac{m}{\pi}T\log T + \frac{m(1-\log 2)}{\pi}T + O\left(\frac{T}{\log T}\right). \quad (2.8)$$

(cf.[Se2]).

For a generic metric on M the number of eigenvalues is finite and all eigenvalues are contained in $[0, 1/4)$ (cf. [C2]), i.e., there are no eigenvalues embedded in the continuous spectrum. By Theorem 2.7, we get in this case

$$-\frac{1}{4\pi}\int_{-T}^{T}\frac{\phi'}{\phi}(1/2+i\lambda)d\lambda \sim \frac{\operatorname{Area}(M)}{4\pi}T^2 \tag{2.9}$$

as $T \to \infty$. In other words, the spectral information is essentially contained in ϕ. Note that this is analogous to the behaviour of the spectrum of the self-adjoint extension H associated to the Schrödinger operator $-d^2/dx^2 + q$ where $q \in C_0^\infty(\mathbf{R})$.

On the other hand, for congruence subgroups of $SL(2, \mathbf{Z})$ it is known that

$$\int_{-T}^{T}\frac{\phi'}{\phi}(1/2+i\lambda)d\lambda = O(T\log T).$$

This follows from the explicit description of ϕ (cf. [Hx]) in terms of Dirichlet L-functions and standard results from analytic number theory.

3. Analytic properties of the scattering matrix

As before let M be an admissible surface and $\phi(s)$ the determinant of the corresponding scattering matrix. In this section we shall further investigate this meromorphic function. It follows from (1.6) that $\phi(s)$ satisfies

$$(3.1) \qquad \phi(s)\phi(1-s) = 1, \quad \overline{\phi(s)} = \phi(\bar{s}), \quad s \in \mathbf{C}.$$

Note that (3.1) implies

$$(3.2) \qquad |\phi(1/2 + i\lambda)| = 1, \quad \lambda \in \mathbf{R}.$$

Furthermore, we know that the poles of $\phi(s)$ are contained in the union of the half-plane $\mathrm{Re}(s) < 1/2$ and the interval $(1/2, 1]$.

If M is a surface of constant negative curvature, Selberg [Sel] proved that $\phi(s)$ has the following two important properties:

1) *$\phi(s)$ is a meromorphic function of order ≤ 4.*
2) *The poles of $\phi(s)$ are contained in a strip $C_1 < \mathrm{Re}(s) < C_2$.*

We shall extend 1) to all admissible surfaces. Our method to establish 1) is different from the one used by Selberg. It is based on Colin de Verdiere's approach to obtain the analytic continuation of Eisenstein series (cf. [C1]). We briefly recall this method. Colin de Verdiere works with the assumption that the surface has a single cusp, but there is no difficulty to extend everything to the case of several cusps.

Let $M = M_0 \cup Z_1 \cup ... \cup Z_m$ be the decomposition of M into a compact surface M_0 and the hyperbolic ends $Z_j \cong [a_j, \infty) \times S^1, a_j > 0, j = 1, ...m$. Let $b = max\{1, a_1, ..., a_m\}$ and choose $\varphi \in C_0^\infty(\mathbf{R})$ satisfying $\varphi(y) = 0$ for $y < b$ and $\varphi(y) = 1$ for $y > b+1$. For $j, 1 \leq j \leq m$, and $s \in \mathbf{C}$ we set

$$(3.3) \qquad \theta_j(z,s) = \begin{cases} 0, & \text{if } z \in M - Z_j; \\ \varphi(y)y^s, & \text{if } z = (y,x) \in Z_j. \end{cases}$$

Note that, for each $s \in \mathbf{C}, \theta_j(.,s) \in C^\infty(M)$. Put

$$(3.4) \qquad \psi_j = (\Delta - s(1-s))(\theta_j(s)), \quad j = 1, ..., m.$$

Then ψ_j is a smooth function on M with compact support. In particular, it belongs to $L^2(M)$ and, for $\mathrm{Re}(s) > 1$, the generalized eigenfunction $E_i(z,s)$ is given by

$$(3.5) \qquad E_i(z,s) = \theta_i(z,s) - (\overline{\Delta} - s(1-s))^{-1}(\psi_i(.,s))$$

(cf. [C2]). To obtain the analytic continuation we have to introduce a cut-off Laplacian Δ_a. We denote by $H^1(M)$ the first Sobolev space. For $f \in H^1(M)$ we define its constant term $f_j^{(0)}$ in the j-th cusp as the zeroth Fourier coefficient of $f_j = f|_{Z_j}$, i.e,

$$(3.6) \qquad f_j^{(0)}(y) = \int_0^{2\pi} f_j(y,x)dx.$$

Note that $f_j^{(0)}(y)$ exists for allmost all $y \in (a_j, \infty)$ and $f_j^{(0)}$ belongs to $H^1((a_j, \infty))$. Given $a > b$, we introduce the following subspace of the Sobolev space

$$H_a^1(M) = \{f \in H^1(M) | \ f_j^{(0)}|_{(a,\infty)} = 0, \ j = 1, ..., m\}. \tag{3.7}$$

This is a closed subspace of $H^1(M)$. Its closure in $L^2(M)$ will be denoted by $\mathcal{H}_a$. Now consider the quadratic form q_a on $H_a^1(M)$ which is given by

$$q_a(f) = \| df \|^2, \quad f \in H_a^1(M). \tag{3.8}$$

This quadratic form is closed and therefore it is represented by a self-adjoint operator Δ_a acting in the Hilbert space $\mathcal{H}_a$. The operator Δ_a has a pure point spectrum consisting of eigenvalues of finite multiplicity. In particular, the resolvent of Δ_a is a compact operator in $\mathcal{H}_a$. Now assume that $a > b + 2$. Then it follows from (3.3) and (3.4) that each $\psi_i, i = 1, ..., m$, belongs to $\mathcal{H}_a$. Hence we can define the following functions

$$F_i(z, s) = \theta_i(z, s) - (\Delta_a - s(1-s))^{-1}(\psi_i(., s)), \quad i = 1, ..., m. \tag{3.9}$$

Since the resolvent of Δ_a is compact, each $F_i(z, s)$ is a meromorphic function of $s \in \mathbf{C}$. As a function of $z \in M$ it is smooth in the complement of the curves $\{a_j\} \times S^1 \subset Z_j \subset M, j = 1, ..., m$. Moreover the nonzero Fourier coefficients of $F_i(z, s)|_{Z_j}$ are smooth on (a_j, ∞). The zeroth Fourier coefficient $F_{i,j}^{(0)}(y, s)$ of $F_i(., s)|_{Z_j}$ has the form

$$F_{i,j}^{(0)}(y, s) = \begin{cases} \delta_{ij} y^s, & \text{if } y > a; \\ A_{ij}(s) y^s + B_{ij}(s) y^{1-s}, & \text{if } a_j \leq y \leq a, \end{cases} \tag{3.10}$$

where $A_{ij}(s)$ and $B_{ij}(s)$ are meromorphic functions of $s \in \mathbf{C}$.

Let $\chi_{a,j}$ be the characteristic function of $[a, \infty) \times S^1$ regarded as a submanifold of $Z_j \cong [a_j, \infty) \times S^1$ and set

$$G_i(z, s) = F_i(z, s) + \sum_{j=1}^{m} \chi_{a,j}(z) \{ A_{ij}(s) y_j^s + B_{ij}(s) y_j^{1-s} - \delta_{ij} y_j^s \}$$

where y_j denotes the radial variable with respect to the cusp Z_j. This is a meromorphic function of $s \in \mathbf{C}$. Now set

$$A(s) = (A_{ij}(s)), \ B(s) = (B_{ij}(s))$$

and

$$\mathbf{G}(z, s) = \begin{pmatrix} G_1(z, s) \\ \vdots \\ G_m(z, s) \end{pmatrix}.$$

One can show that $\det A(s) \not\equiv 0$. Therefore $A(s)^{-1}$ is a meromorphic function of $s \in \mathbf{C}$. Furthermore, for $\mathrm{Re}(s) > 1$, one has

$$\mathbf{E}(z, s) = A(s)^{-1} \mathbf{G}(z, s) \tag{3.11}$$

where $\mathbf{E}(z,s)$ is defined by (1.7). The right hand side provides the analytic continuation of $\mathbf{E}(z,s)$. Moreover, the scattering matrix is given by

$$C(s) = A(s)^{-1} \circ B(s), \quad s \in \mathbf{C}. \tag{3.12}$$

We shall employ this description of the scattering matrix to show that $\phi(s)$ is of order ≤ 4. Let

$$0 < \mu_0(a) \leq \mu_1(a) \leq \ldots$$

be the eigenvalues of Δ_a. It follows from Theorem 5 in [C2] that zero is not an eigenvalue of Δ_a. Moreover, by the estimations on pp. 96, 97 in [C2], there exists $C > 0$ such that

$$\#\{\mu_j(a) \mid \mu_j(a) \leq \lambda\} \leq C(1+\lambda), \quad \lambda \geq 0. \tag{3.13}$$

This implies that

$$\sum_{j=0}^{\infty} \mu_j(a)^{-\sigma} < \infty \tag{3.14}$$

for $\sigma > 1$. Given $p \in \mathbf{N}$, let

$$e(u,p) = (1-u)\exp\left(u + \frac{u^2}{2} + \ldots + \frac{u^p}{p}\right), \quad u \in \mathbf{C}.$$

By (3.14), the infinite product

$$\tilde{P}(z) = \prod_{j=0}^{\infty} e(\frac{z}{\mu_j}, 1)$$

converges uniformly on compact subsets of $\mathbf{C}$ and $\tilde{P}(z)$ is an entire function of order 1 whose zeros are $\mu_0, \mu_1, \ldots$ (cf. pp. 18-19 in [Bo]). For $s \in \mathbf{C}$ put

$$P(s) = (s - 1/2)\tilde{P}(s(1-s)).$$

Lemma 3.15. *Let $A_{ij}(s)$ and $B_{ij}(s), i,j = 1,\ldots,m$, be the meromorphic functions defined by (3.10). Then $P(s)A_{ij}(s)$ and $P(s)B_{ij}(s)$ are entire.*

Proof. First recall that $(\Delta_a - zId)^{-1}$ is a meromorphic function of $z \in \mathbf{C}$ with simple poles at $z = \mu_0, \mu_1, \ldots$. By (3.9) it follows that $\tilde{P}(s(1-s))F_i(z,s), i = 1,\ldots,m$, is an entire function of s. Its constant term along Z_j is also entire. In view of (3.10) this implies that $\tilde{P}(s(1-s))(A_{ij}(s)y^s + B_{ij}(s)y^{1-s})$ is entire for $a_j \leq y \leq a$. Hence $\tilde{P}(s(1-s))A_{ij}(s)$ and $\tilde{P}(s(1-s))B_{ij}(s)$ are holomorphic on $\mathbf{C} - \{1/2\}$ and they can have at most simple poles at $s = 1/2$. Q.E.D.

We shall now estimate the order of growth of $P(s)A_{ij}(s)$ and $P(s)B_{ij}(s), i,j = 1,\ldots,m$. First we need an auxiliary lemma. For each $j \in \mathbf{N}$, put

$$\tilde{P}_j(z) = \prod_{k \neq j} e(\frac{z}{\mu_k}, 1).$$

Lemma 3.16. *There exists a constant $C > 0$ such that*

$$\left|\tilde{P}_j(z)\right| \leq e^{C|z|^2}, \quad z \in \mathbf{C}, j \in \mathbf{N}.$$

Proof. We have

$$\log|\tilde{P}_j(z)| = \left(\sum_{\substack{\mu_k \leq 2|z| \\ k \neq j}} + \sum_{\substack{\mu_k > 2|z| \\ k \neq j}}\right) \log\left|e(\frac{z}{\mu_k}, 1)\right| = S_1 + S_2.$$

To estimate S_1 observe that $|z|/\mu_k \geq 1/2$ and

$$\log\left|1 - \frac{z}{\mu_k}\right| \leq \frac{|z|}{\mu_k}.$$

Hence

$$\log\left|e(\frac{z}{\mu_k}, 1)\right| \leq 4\frac{|z|^2}{\mu_k^2}.$$

Together with (3.14) we obtain

$$S_1 \leq 4|z|^2 \sum_{\mu_k \leq 2|z|} \mu_k^{-2} = C_1|z|^2.$$

Now consider S_2. In this case $|z|/\mu_k < 1/2$. Using 2.6.3 in [Bo], we get

$$\log\left|e(\frac{z}{\mu_k}, 1)\right| \leq 2\frac{|z|^2}{\mu_k^2}$$

and, by (3.14),

$$S_2 \leq 2|z|^2 \sum_{2|z| < \mu_k} \mu_k^{-2} = C_2|z|^2.$$

Q.E.D.

Lemma 3.17. *Let $R_a(s) = (\Delta_a - s(1-s))^{-1}$ and let $\psi_i(s)$ be defined by (3.4), where $s \in \mathbf{C}$. Then $P(s)R_a(s)(\psi_i(s))$ is entire and there exist constants $C_1, C_2 > 0$ such that , for $f \in L^2(M), s \in \mathbf{C}$ and i=1,...,m,*

$$|P(s)||(R_a(s)(\psi_i(s)), f)| \leq C_1 \exp(C_2|s|^4) \parallel f \parallel.$$

Proof. Let $T_{a,j} \in \mathcal{D}'(M), j = 1, ..., m$, be defined by $\langle T_{a,j}|g\rangle = g_j^{(0)}(a)$ where $g_j^{(0)}$ is given by (3.6). If we extend Theorem 1 in [C2] to the case of several cusps, it follows that the domain of Δ_a consists of all f in $H_a^1(M)$ for which there exist $C_j \in \mathbf{C}, j = 1, ..., m$, such that $\Delta f - \sum_{j=1}^m C_j T_{a,j}$ is contained in $L^2(M)$. Moreover, if they exist the constants C_j are uniquely determined and $\Delta_a f = \Delta f - \sum_{j=1}^m C_j T_{a,j}$ where Δf is defined in the distributional sense. Using the definition of $\psi_i(s)$ it follows immediately that $\psi_i(s)$ belongs to the domain of Δ_a and $\Delta_a \psi_i(s) = \Delta\psi_i(s)$. Now let $\{\varphi_j\}_{j\in\mathbf{N}}$ be an orthonormal basis of eigenfunctions of Δ_a corresponding to the eigenvalues $\mu_0 \leq \mu_1 \leq \dots$. Using the observation above, we get

$$\mu_j(\psi_i(s), \varphi_j) = (\Delta\psi_i(s), \varphi_j), \quad j \in \mathbf{N}, i = 1, ..., m. \tag{3.18}$$

Let $f \in L^2(M)$. Then by (3.18)

$$(R_a(s)(\psi_i(s)), f) = \sum_{j=0}^{\infty} \frac{(\Delta\psi_i(s), \varphi_j)(\varphi_j, f)}{\mu_j(\mu_j - s(1-s))}.$$

Note that by (3.14) the series is absolutely convergent. By (3.14) and Lemma 3.16, it follows that the right hand side, multiplied by $P(s)$, is entire and can be estimated by

$$C_1 \ \| \ \Delta\psi_i(s) \ \| \ \| \ f \ \| \ \exp(C_2|s|^4)$$

for certain constants $C_1, C_2 > 0$. Using the definition of $\psi_i(s)$, one can estimate $\| \ \Delta\psi_i(s) \ \|$ by $\exp(C|s|)$ for some $C > 0$.Q.E.D.

Let $f \in C_0^\infty(M)$ with supp f contained in $(a_j, a) \times S^1 \subset Z_j$. Using (3.9) and Lemma 3.17, it follows that there exist constants $C_3, C_4 > 0$ (which depend on a) such that

$$|P(s)| \ |(F_i(s), f)| \leq C_3 \exp(C_4|s|^4) \ \| \ f \ \| \ . \tag{3.19}$$

If we assume that f depends only on the radial variable y, i.e., $f \in C^\infty(\mathbf{R})$ with supp f contained in (a_j, a) then, by (3.10), we obtain

$$(F_i(s), f) = A_{ij}(s) \int_{a_j}^{a} y^s \overline{f(y)} \ \frac{dy}{y^2} + B_{ij}(s) \int_{a_j}^{a} y^{1-s} \overline{f(y)} \ \frac{dy}{y^2}.$$

Now we make a special choice for f. Let $g \in C^\infty(\mathbf{R})$ with supp $g \subset (a_j, a)$ and set

$$f(y) = -y^{\bar{s}+1} \ \frac{d}{dy}(y^{-2\bar{s}+2} g(y)).$$

Then the second integral involving f vanishes and the first one equals $(2s-1)\int_{\mathbf{R}} g(y)dy$. Assume that $g \geq 0, g \neq 0$. Together with (3.19) we obtain

$$|P(s)A_{ij}(s)| \leq C \ \exp(c|s|^4), \quad s \in \mathbf{C},$$

for some constants $C, c > 0$ and $i, j = 1, ..., m$. In the same way we get

$$|P(s)B_{ij}(s)| \leq C \ \exp(c|s|^4), \quad s \in \mathbf{C}, i, j = 1, ..., m.$$

Combining our results, we have proved that $P(s)^m \det A(s)$ and $P(s)^m \det B(s)$ are entire functions of order ≤ 4 and ,by (3.12), we get

Theorem3.20. *Let $\phi(s)$ be the determinant of the scattering matrix associated to the Laplacian on an admissible surface M. There exist entire functions $F_1(s)$ and $F_2(s)$ of order ≤ 4 such that*

$$\phi(s) = \frac{F_1(s)}{F_2(s)}.$$

To continue the investigation of the scattering matrix we need the following

Lemma 3.21. *There exists $q_1 > 1$ and $\sigma_0 > 1$ such that*

$$q_1^{-\mathrm{Re}(s)} |\phi(s)| \leq \frac{1}{2}$$

for $\mathrm{Re}(s) \geq \sigma_0$.

Proof. Let $\mathrm{Re}(s) > 1$. By (3.3) we have $\theta_i(z,s) = y_i^s$ for $z = (y_i, x) \in Z_i,\ y_i > b+1$. Using (1.4) and (3.5), it follows that the zeroth Fourier coefficient of $(\overline{\Delta} - s(1-s))^{-1}(\psi_i(s))$ on Z_j equals

$$-y_j^{1-s} C_{ij}(s)$$

for $y_j > b+1$. Now observe that

$$\| (\overline{\Delta} - s(1-s))^{-1} \| = \frac{1}{\mathrm{dist}(s(1-s), \mathrm{Spec}(\overline{\Delta}))} \tag{3.22}$$

(cf. Ch.V,3.8 in [K]). But $\mathrm{Spec}(\overline{\Delta}) \subset [0,\infty)$. This implies

$$\mathrm{dist}(s(1-s), \mathrm{Spec}(\overline{\Delta})) \geq |s| \tag{3.23}$$

for $\mathrm{Re}(s) \geq 2$. Put $p = b+1$. A simple computation shows that

$$\| \psi_i(s) \| \leq C\ |s|\ p^{\mathrm{Re}(s)}, \quad i = 1, ..., m, \tag{3.24}$$

for $\mathrm{Re}(s) \geq 2$ and some constant $C > 0$. Using the description of the constant term of $(\overline{\Delta} - s(1-s))^{-1}(\psi_i(s))$ given above and (3.22)-(3.24) we obtain

$$\frac{p^{-\mathrm{Re}(s)+1/2}}{\sqrt{2\mathrm{Re}(s) - 1}}\ |C_{ij}(s)| < \| (\overline{\Delta} - s(1-s))^{-1}(\psi_i(s)) \| \leq C\ p^{\mathrm{Re}(s)} \tag{3.25}$$

for $\mathrm{Re}(s) \geq 2$. Since $p > 1$, (3.25) implies

$$|C_{ij}(s)| \leq C_1\ p^{2\mathrm{Re}(s)}, \quad i,j = 1, ..., m,$$

for $\mathrm{Re}(s) \geq 2$. This implies

$$|\det\ C(s)| \leq C_2\ p^{2m\mathrm{Re}(s)}, \quad \mathrm{Re}(s) \leq 2.$$

Put $q_1 = p^{4m}$ and $\sigma_0 = \log 2\, C_2(2m \log p)^{-1}$. Then our lemma follows with these constants. Q.E.D.

Now we can proceed in essentially the same way as on pp. 655 - 656 in [Sel] and factorize $\phi(s)$. At some place the method of Selberg has to be modified, because (8.3) in [Sel] is not available in our case. For the convenience of the reader we include details.

Let $\sigma_1, ..., \sigma_h \in (1/2, 1]$ be the poles of $\phi(s)$ in $\mathrm{Re}(s) \geq 1/2$. Put

$$\xi(s) = q_1^{-s} \prod_{i=1}^{h} \frac{s + 1/2 - \sigma_i}{s - 1/2 + \sigma_i}\, \phi(s + 1/2). \tag{3.26}$$

Then $\xi(s)$ has the following properties:

1) $\xi(s)\xi(-s) = 1, s \in \mathbf{C}$.

2) $|\xi(s)| = 1 \quad$ for $\mathrm{Re}(s) = 0$.

3) $\xi(s)$ is holomorphic in the half-plane $\mathrm{Re}(s) > 0$ and satisfies $|\xi(s)| \leq 1$ for $\mathrm{Re}(s) \geq 0$.

1) and 2) follow from (3.1). The first part of 3) is clear from the definition of $\xi(s)$. To prove the second part consider any strip $S_\sigma = \{s \in \mathbf{C} |\ 0 \leq \mathrm{Re}(s) \leq \sigma\}, \sigma > 0$. By Lemma 8.8 in [Mü], $\phi(s)$ is bounded in the domain $1/2 \leq \mathrm{Re}(s) \leq \sigma + 1/2, |\mathrm{Im}(s)| \geq 1$ and therefore, $\xi(s)$ is bounded in S_σ. If σ is sufficiently large it follows from Lemma 3.21 that $|\xi(s)| < 1$ on the vertical line $\mathrm{Re}(s) = \sigma$. Finally, by (3.1), $\xi(s)$ satisfies $\overline{\xi(s)} = \xi(\bar{s}), s \in \mathbf{C}$. Combining these observations with 2) and a Phragmen - Lindelöf type theorem, we obtain the desired result.

Next consider the series

$$\sum_{\eta} \frac{\mathrm{Re}(\eta)}{|\eta|^2} \tag{3.27}$$

where η runs over all zeros, counted with the order, of $\xi(s)$ in the half-plane $\mathrm{Re}(s) > 0$. Then we have

Lemma 3.28. *The series (3.27) converges.*

Proof. By 3), $\xi(s)$ is analytic in the half-plane $\mathrm{Re}(s) > 0$ and continuous and bounded in $\mathrm{Re}(s) \geq 0$. The convergence follows from Carleman's theorem [T, section 3.71]. Q.E.D.

Corollary 3.29. *Let ρ run over all poles, counted with the order , of $\phi(s)$ in* $\mathrm{Re}(s) < 1/2$. *Then*

$$\sum_{\rho} \frac{1 - 2\mathrm{Re}(\rho)}{|\rho - 1/2|^2} < \infty.$$

Proof. This follows from Lemma 3.28, (3.26) and (3.1). Q.E.D.

Now observe that, by Theorem 3.20,

$$\xi(s) = \frac{H_1(s)}{H_2(s)}, \quad s \in \mathbf{C}$$

where $H_1(s)$ and $H_2(s)$ are entire functions of order ≤ 4. Let η be a zero of $\xi(s)$. Then it follows from (3.1) that $\bar{\eta}$ is a zero and $-\eta, -\bar{\eta}$ are poles of $\xi(s)$. By Hadamard's factorization theorem we get

$$\xi(s) = e^{P(s)} \frac{\prod_\eta e\left(\frac{s}{\eta}, 4\right) e\left(\frac{s}{\bar{\eta}}, 4\right)}{\prod_\eta e\left(\frac{s}{-\eta}, 4\right) e\left(\frac{s}{-\bar{\eta}}, 4\right)} \tag{3.30}$$

where η runs over half the zeros of $\xi(s)$ in $\mathrm{Re}(s) > 0$ and we have chosen one representative for each pair $\{\eta, \bar{\eta}\}$ of zeros. Moreover $P(s)$ is a polynomial in s of degree ≤ 4 and $e(z, 4)$ is the Weierstrass elementary factor defined above.

Now consider the expression

$$I_k(\eta) = \frac{1}{\eta^k} + \frac{1}{\bar{\eta}^k} - \frac{1}{(-\eta)^k} - \frac{1}{(-\bar{\eta})^k}$$

for $1 \leq k \leq 4, \eta \in \mathbf{C}$. If k is even then $I_k = 0$. For $k = 1$ we have $I_1(\eta) = 4\mathrm{Re}(\eta)/|\eta|^2$ and it follows from Lemma 3.28 that

$$\sum_\eta |I_1(\eta)| < \infty. \tag{3.31}$$

It remains to investigate I_3. Put $\eta = |\eta|\, e^{i\vartheta}$. Then

$$I_3(\eta) = \frac{4}{|\eta|^3} \cos(3\vartheta).$$

Now $|\cos(3\vartheta)| \leq 4|\cos\vartheta|$. Hence $|I_3(\eta)| \leq 4|\eta|^{-2}|I_1(\eta)|$. Together with (3.31) this implies

$$\sum_\eta |I_3(\eta)| < \infty. \tag{3.32}$$

In view of (3.31) and (3.32) the exponential factors in (3.30) can be combined to give

$$\xi(s) = e^{Q(s)} \prod_\eta \frac{(s-\eta)\,(s-\bar{\eta})}{(s+\eta)\,(s+\bar{\eta})}$$

for some polynomial $Q(s)$ of degree ≤ 4. The infinite product can be rewritten as

$$\prod_\eta \left(1 - 4s\frac{\mathrm{Re}(\eta)}{(s+\eta)\,(s+\bar{\eta})}\right)$$

and by Lemma 3.28, this product is absolutely convergent.

Now consider $Q(s)$. The equation $\xi(i\lambda)\xi(-i\lambda) = 1, \lambda \in \mathbf{R}$, implies $Q(i\lambda) + Q(-i\lambda) = 2\pi i l$ for some $l \in \mathbf{Z}$. Thus

$$Q(s) = a_3 s^3 + a_1 s + \pi i l.$$

By (3.1),$\overline{\xi(s)} = \xi(\bar{s})$. Hence $a_1, a_2 \in \mathbf{R}$. Assume that $a_3 \neq 0$. If $a_3 > 0$ then $\xi(\sigma) \sim \exp(a_3\sigma^3)$ for $\sigma \in \mathbf{R}, \sigma \to \infty$. This contradicts $|\xi(s)| \leq 1$ in $\operatorname{Re}(s) \geq 0$. Next assume that $a_3 < 0$. Then we can choose s in the half-plane $\operatorname{Re}(s) > 0$ so that $\operatorname{Re}(s^3) < 0$ and $\operatorname{Re}(s^3)$ tends to $-\infty$ as $s \to \infty$. Again we get $|\xi(s)| \to \infty$. Thus $Q(s) = a_1 s + \pi i l$. Repeating this argument it follows that $a_1 \leq 0$. Put

$$q = q_1 \ \exp a_1 \tag{3.33}$$

where q_1 is the constant from Lemma 3.21. Combining our results and using the definition (3.26) of $\xi(s)$ we obtain

Theorem 3.34. *Let ρ run over all poles of $\phi(s)$, counted with the order, and let q be given by (3.33). Then*

$$\phi(s) = \phi(1/2)\ q^{s-1/2} \prod_\rho \frac{s - 1 + \bar{\rho}}{s - \rho}.$$

This allows us to compute the logarithmic derivative of $\phi(s)$.

Corollary 3.35. *Let the notation be the same as in Theorem 3.34. Then*

$$\frac{\phi'}{\phi}(1/2 + i\lambda) = \log q + \sum_\rho \frac{2\operatorname{Re}(\rho) - 1}{(1/2 - \operatorname{Re}(\rho))^2 + (\lambda - \operatorname{Im}(\rho))^2}, \quad \lambda \in \mathbf{R}. \tag{3.36}$$

The convergence of the series on the right hand side follows from Corollary 3.29. Indeed, let $\zeta = 1/2 + i\lambda$. Then

$$\sum_{|\rho| > 2|\zeta|} \frac{1 - 2\operatorname{Re}(\rho)}{|\rho - \zeta|^2} \leq 4 \sum_\rho \frac{1 - 2\operatorname{Re}(\rho)}{|\rho|^2}$$

which is convergent by Corollary 3.29, because only finitely many poles occur in $\operatorname{Re}(s) > 1/2$. The same argument gives

Lemma 3.37. *The series on the right hand side of (3.36) is uniformly convergent for λ in any finite interval $[-T, T]$.*

4.On the distribution of poles of the scattering matrix

We continue in this section with the investigation of the poles of $\phi(s)$. To begin with we shall estimate the winding number of $\phi(1/2+i\lambda)$:

Proposition 4.1. *There exists a constant* $C>0$ *such that*

$$\left|\int_{-T}^{T}\frac{\phi'}{\phi}(1/2+i\lambda)d\lambda\right|\leq C\,T^2$$

for all $T\geq 0$.

Proof. Let $a>\max\{a_1,...,a_m\}$ and set

$$C_a(s)=a^{1-2s}C(s).$$

By (1.6), $C_a(s)$ is unitary for $\mathrm{Re}(s)=1/2$ and hence, can be diagonalized. Moreover, $C_a(s)$ is holomorphic in a neighborhood of $\mathrm{Re}(s)=1/2$. Therefore we can apply Rellich's theorem [Bau,p.142] which implies that there exist real-valued real analytic functions $\beta_1(\lambda),...,\beta_m(\lambda)$ of $\lambda\in\mathbf{R}$ such that $e^{i\beta_1(\lambda)},...,e^{i\beta_m(\lambda)}$ are the eigenvalues of $C_a(1/2+i\lambda)$. Each $\beta_j(\lambda)$ is only determined up to $2\pi\mathbf{Z}$. Furthermore, the functional equation (1.6) implies $C_a(1/2)^2=Id$. Hence $\beta_j(0)=\pi n_j, n_j\in\mathbf{Z}, j=1,...,m$. Put

$$\tilde\beta_j(\lambda)=\int_0^\lambda\frac{d}{du}\beta_j(u)\,du,\quad j=1,...,m.$$

Then we can choose either $\beta_j=\tilde\beta_j$ or $\beta_j=\tilde\beta_j+\pi$ and we get

$$\left|\int_{-T}^{T}\frac{d}{ds}\log\ \det C_a(1/2+i\lambda)d\lambda\right|\leq 2\sum_{j=1}^{m}|\tilde\beta_j(T)|\leq 2m\ \max_j|\tilde\beta_j(T)|. \tag{4.2}$$

Let $n_j(T)$ be the number of points $w\in[0,T]$ such that $e^{i\beta_j(w)}=-1$, i.e., $\beta_j(w)=(2k+1)\pi$ for some $k\in\mathbf{Z}$. Obviously we have

$$|\tilde\beta_j(T)|\leq 4\pi n_j(T),\quad j=1,...,m. \tag{4.3}$$

Let $n(T)$ be the number of $w\in[0,T]$ such that $C_a(1/2+iw)$ has at least one eigenvalue equal to -1. Then $n_j(T)\leq n(T), j=1,...,m$, and by (4.2) and (4.3), we get

$$\left|\int_{-T}^{T}\frac{d}{ds}\log\ \det C_a(1/2+i\lambda)\ d\lambda\right|\leq 8\pi m\,n(T). \tag{4.4}$$

Thus it is sufficient to estimate $n(T)$. For hyperbolic surfaces Lax and Phillips proved in [LP,pp. 205-216] that $n(T)$ is bounded by the number of eigenvalues of Δ_a which are less than $1/4+T^2$. Their method extends without any difficulty to our case.

Assume that $(b_1, ..., b_m)$ is an eigenvector of $C_a(1/2+iw)$ with eigenvalue -1 for some $w \in [0,T]$. Then

$$(4.5) \qquad b_j a^{1/2+iw} + \sum_{k=1}^{m} b_k C_{kj}(1/2+iw) a^{1/2-iw} = 0.$$

Set

$$(4.6) \qquad \varphi(z) = \sum_{j=1}^{m} b_j E_j(z, 1/2+iw).$$

Then φ is a C^∞ function on M and , by (1.3), it satisfies

$$(4.7) \qquad \Delta\varphi = (1/4+w^2)\varphi.$$

Moreover, by (1.4), the constant term $\varphi_j^{(0)}$ of φ on the j-th cusp is given by

$$(4.8) \qquad \varphi_j^{(0)}(y_j) = b_j y_j^{1/2+iw} + \sum_{k=1}^{m} b_k C_{kj}(1/2+iw)\, y_j^{1/2-iw}.$$

This shows that $\varphi \not\equiv 0$. Now observe that in view of (4.5), the constant terms satisfy

$$(4.9) \qquad \varphi_j^{(0)}(a) = 0, \quad j = 1, ..., m.$$

Let $\chi_{a,j}$ be the characteristic function of $[a,\infty) \times S^1$ regarded as a submanifold of $Z_j \cong [a_j, \infty) \times S^1$. Set

$$(4.10) \qquad \varphi_a = \varphi - \sum_{j=1}^{m} \chi_{a,j} \varphi_j^{(0)}.$$

Then φ_a is smooth except for the 0-th Fourier coefficients which, by (4.9), are continous and smooth for $y_j \neq a, j = 1, ..., m$. Hence φ_a belongs to the Sobolev space $H^1(M)$. Using (4.7), (4.10) and the description of the domain of Δ_a (cf. proof of Lemma 3.17), a direct computation shows that φ_a belongs to the domain of Δ_a and

$$\Delta_a \varphi_a = (1/4+w^2)\varphi_a.$$

Let

$$N_a(\lambda) = \#\{\mu_j(a) \mid \mu_j(a) \leq \lambda\}$$

where $\mu_0(a) \leq \mu_1(a) \leq ...$ are the eigenvalues of Δ_a. Then our discussion above implies that

$$(4.11) \qquad n(T) \leq N_a(1/4+T^2).$$

Combining (4.4), (4.11) and (3.13) gives the desired estimate. Q.E.D.

Now we can proceed and estimate the number of poles of $\phi(s)$ in the half-plane $\mathrm{Re}(s) < 1/2$. If we use Corollary 3.35 combined with Proposition 4.1, it follows that

$$\int_{-2T}^{2T} \sum_{\rho} \frac{1 - 2\mathrm{Re}(\rho)}{(1/2 - \mathrm{Re}(\rho))^2 + (\lambda - \mathrm{Im}(\rho))^2}\, d\lambda \leq C_1\, T^2 \tag{4.12}$$

for $T \geq 1$. Here ρ runs over all poles of $\phi(s)$ in $\mathrm{Re}(s) < 1/2$, counted with the order of the poles. Since all terms on the left hand side are positive, we obtain

$$\sum_{|\rho|<T} \int_{-2T}^{2T} \frac{1 - 2\mathrm{Re}(\rho)}{(1/2 - \mathrm{Re}(\rho))^2 + (\lambda - \mathrm{Im}(\rho))^2}\, d\lambda \leq C_1\, T^2 \tag{4.13}$$

for $T \geq 1$. Since $|\mathrm{Re}(\rho)| < T$ and $|\mathrm{Im}(\rho)| < T$, each of the integrals occurring in (4.13) is bounded from below by $\int_{-1}^{1}(1+\lambda^2)^{-1} d\lambda$. Thus

$$\sum_{|\rho|<T} 1 \leq C_2\, T^2, \quad T \geq 1.$$

This proves

Theorem 4.14. *Let $N_p(T)$ be the number of poles ρ of $\phi(s)$ satisfying $|\rho| < T$. There exists a constant $C > 0$ such that*

$$N_p(T) \leq C\, T^2, \quad T \geq 1.$$

An immediate consequence is the following

Corollary 4.15. *For $\mathrm{Re}(s) > 2$,*

$$\sum_{\rho} |\rho|^{-s} < \infty$$

where ρ runs over the poles of $\phi(s)$, counted with the order of the poles.

Next we shall relate $N_p(T)$ to the winding number of ϕ. For a hyperbolic surface of finite area Selberg has shown that

$$-\frac{1}{2\pi} \int_0^T \frac{\phi'}{\phi}(1/2 + i\lambda) d\lambda = \frac{1}{2} N_p(T) + O(T) \tag{4.16}$$

Actually, Selberg uses a slightly different counting function $\tilde{N}_p(T)$ which is the number of poles of $\phi(s)$ in $0 \leq \mathrm{Im}(s) \leq T$. However it is not difficult to show that $N_p(T) = 2\tilde{N}_p(T) + O(T)$ (cf. (0.15) in [Se2]). Combined with (2.8) this implies

$$N(T) + \frac{1}{2} N_p(T) = \frac{\mathrm{Area}(M)}{4\pi} T^2 - \frac{m}{\pi} T \log T + O(T) \quad \text{as } T \to \infty. \tag{4.17}$$

We shall establish a similar but weaker result for an arbitrary admissible surface M. By Corollary 3.35 and Lemma 3.37 we have

$$(4.18)\qquad -\frac{1}{4\pi}\int_{-T}^{T}\frac{\phi'}{\phi}(1/2+i\lambda)d\lambda=\sum_{\rho=\beta+i\gamma}\frac{1}{4\pi}\int_{-T}^{T}\frac{1-2\beta}{(\beta-1/2)^2+(\gamma-\lambda)^2}\,d\lambda\;+O(T)$$

where ρ runs over all poles of $\phi(s)$. Assume that $\beta<1/2$. Then

$$\int_{-T}^{T}\frac{1-2\beta}{(\beta-1/2)^2+(\gamma-\lambda)^2}\,d\lambda=2\left\{\arctan\Big(\frac{T-\gamma}{1/2-\beta}\Big)+\arctan\Big(\frac{T+\gamma}{1/2-\beta}\Big)\right\}.$$

Now we use that

$$\arctan x+\arctan y=\arctan\frac{x+y}{1-xy}+\begin{cases}0, & \text{if } xy<1;\\ \pi, & \text{if } xy>1, x>0.\end{cases}$$

Then we get

$$\int_{-T}^{T}\frac{1-2\beta}{(\beta-1/2)^2+(\gamma-\lambda)^2}\,d\lambda=2\ \arctan\left\{\frac{(1-2\beta)T}{|\rho-1/2|^2}\Big(1-\frac{T^2}{|\rho-1/2|^2}\Big)^{-1}\right\}$$
$$+\begin{cases}0, & \text{if } |\rho-1/2|>T;\\ 2\pi, & \text{if } |\rho-1/2|<T.\end{cases}$$

Combined with (4.18) this implies

$$(4.19)\qquad -\frac{1}{4\pi}\int_{-T}^{T}\frac{\phi'}{\phi}(1/2+i\lambda)\,d\lambda=$$
$$=\frac{1}{2}N_p(T)+\frac{1}{2\pi}\sum_{\substack{\rho\\ \mathrm{Re}(\rho)<1/2}}\arctan\left\{\frac{1-2\mathrm{Re}(\rho)}{|\rho-1/2|^2}\,T\left(1-\frac{T^2}{|\rho-1/2|^2}\right)^{-1}\right\}+O(T).$$

We split the sum over ρ as follows

$$(4.20)\qquad \sum_{|T-|\rho-1/2||>\sqrt{T}}+\sum_{|T-|\rho-1/2||<\sqrt{T}}.$$

To estimate the first sum we remark that $|\arctan x|\le|x|$. Furthermore, if $|T-|\rho-1/2||>\sqrt{T}$ then

$$\left|1-\frac{T^2}{|\rho-1/2|^2}\right|^{-1}\le 2\sqrt{T}.$$

This implies that the first sum can be estimated by

$$2\sum_{\mathrm{Re}(\rho)<1/2}\frac{1-2\mathrm{Re}(\rho)}{|\rho-1/2|^2}\,T^{3/2}$$

By Corollary 3.29, the series over ρ converges and therefore, the first sum in (4.20) is $O(T^{3/2})$. Since $|\arctan x| \leq \pi/2$ the second sum can be estimated by

$$\frac{1}{4}(N_p(T+\sqrt{T}) - N_p(T-\sqrt{T})).$$

Lemma 4.21. *We have*

$$N_p(T+\sqrt{T}) - N_p(T-\sqrt{T}) = o(T^2)$$

.

Proof. By Theorem 4.14 we know that

$$0 \leq \frac{N_p(T \pm \sqrt{T})}{T^2} \leq C, \quad T \geq 0,$$

for some constant $C > 0$. Let $a \geq 0$ be any point of accumulation of $N_p(T+\sqrt{T})\, T^{-2}$. Then a is also a point of accumulation of $N_p(T-\sqrt{T})\, T^{-2}$ and vice versa. Hence

$$\limsup_{T\to\infty} \frac{(N_p(T+\sqrt{T}) - N_p(T-\sqrt{T}))}{T^2} = 0$$

Q.E.D.

This lemma implies that the second sum is $o(T^2)$. Together with (4.19) we obtain

$$-\frac{1}{4\pi}\int_{-T}^{T} \frac{\phi'}{\phi}(1/2+i\lambda)\, d\lambda = \frac{1}{2}N_p(T) + o(T^2). \tag{4.22}$$

Remark. For a hyperbolic surface we know from [Se2,(0.15)] that the remainder term is actually $O(T)$. We conjecture that this is true for an arbitrary admissible surface.

If we combine Theorem 2.7 and (4.22), we get

Theorem 4.23. *As $T \to \infty$,*

$$N(T) + \frac{1}{2}N_p(T) \sim \frac{\mathrm{Area}(M)}{4\pi}\, T^2.$$

This is another analogue of Weyl's formula for an admissible surface. In contrast to Theorem 2.7 we are now dealing with a discrete set of spectral parameters. Theorem 4.23 also suggests that the right set of spectral parameters for an admissible surface is the union of the eigenvalues and the set of poles of $\phi(s)$. On the spectral side this set should replace the eigenvalues in the case of a compact surface. We shall return to this point in the next section.

In view of (4.17) we expect that the remainder term in the asymptotic formula of Theorem 4.23 is $O(T\ \log T)$. We also remark that in general it is very difficult to study $N(T)$ and $N_p(T)$ separately. However Colin de Verdiere [C2] has shown that a generic compactly supported conformal deformation of the metric will destroy all embedded eigenvalues and convert them into poles of $\phi(s)$. In this case there are only finitely many eigenvalues which are all contained in $[0, 1/4)$. Hence we get

Corollary 4.24. *For a generic metric on M,*

$$N_p(T) = \frac{\text{Area}(M)}{2\pi}\, T^2 + o(T^2)$$

as $T \to \infty$.

On the other hand, if $M = \Gamma(N)\backslash H$ where $\Gamma(N)$ denotes the principal congruence subgroup of level N of $SL(2, \mathbf{Z})$, then the determinant $\phi(s)$ of the scattering matrix can be computed explicitely [Hx] and it turns out that, up to a Gamma factor, $\phi(s)$ is the product of certain Dirichlet L-series. Standard results from analytic number theory [P] imply then

$$-\frac{1}{4\pi}\int_{-T}^{T} \frac{\phi'}{\phi}(1/2 + i\lambda)\, d\lambda = O(T\ \log T)$$

and, by (4.16) and (4.17), we get

$$N_p(T) = O(T\ \log T) \tag{4.25}$$

$$N(T) = \frac{\text{Area}(M)}{4\pi}\, T^2 + O(T\ \log T). \tag{4.26}$$

In view of the results of Colin de Verdiere [C2] and Phillips-Sarnak [PS2] one may conjecture that (4.25) is the minimal growth for an admissible surface with $\chi(M) < 0$ and that in this case $N(T)$ has the maximal possible growth.

For further results about the distribution of poles of $\phi(s)$ in the hyperbolic case we refer the reader to [Se2].

Another important feature of the scattering matrix for a hyperbolic surface is property 2) mentioned at the begining of section 3.We do not know if this continues to hold for an arbitrary admissible surface.

5. The resonance set and a trace formula

Let $g \in C_0^\infty(\mathbf{R}^+)$ and let $h = \hat{g}$. For a hyperbolic surface $M = \Gamma\backslash H$, Lax and Phillips proved in section 9 of [LP] the following mini-trace formula

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty} h(y)\frac{\phi'}{\phi}(1/2+i\lambda)\,d\lambda = \sum_{\rho} h(i(1/2-\rho)) - \sum_{j=1}^{n} h(i(1/2-\sigma_j)). \tag{5.1}$$

Here ρ runs over the poles of $\phi(s)$ in $\mathrm{Re}(s) < 1/2, \sigma_1, ..., \sigma_n \in (1/2, 1]$ are the poles of $\phi(s)$ in $\mathrm{Re}(s) > 1/2$ and each pole is counted according to its order. For (5.1) to hold it is important that the support of g is contained in $(0,\infty)$. We wish to extend (5.1) to all admissible surfaces and we want a formula which works for functions like $h(\lambda) = (a^2+\lambda^2)^{-s}$ or $h(\lambda) = \exp(-(a^2+\lambda^2)t), a > 0$. Our approach to this problem is based on Corollary 3.35.

Let $g \in C_0^\infty(\mathbf{R})$ and assume that g is even. Let $h = \hat{g}$ and $T > 0$. From (3.36) and Lemma 3.37 it follows that

$$\begin{aligned}\int_{-T}^{T} h(\lambda)\frac{\phi'}{\phi}(1/2+i\lambda)\,d\lambda &= \log q\int_{-T}^{T} h(\lambda)\,d\lambda \\ &+ \sum_{\rho=\beta+i\gamma}\int_{-T}^{T} h(\lambda)\frac{2\beta-1}{(\beta-1/2)^2+(\lambda-\gamma)^2}\,d\lambda.\end{aligned} \tag{5.2}$$

If we integrate by parts, the left hand side equals

$$h(T)\int_{-T}^{T}\frac{\phi'}{\phi}(1/2+i\lambda)\,d\lambda - \int_{-T}^{T} h'(\lambda)\int_0^{\lambda}\frac{\phi'}{\phi}(1/2+iu)du\,d\lambda. \tag{5.3}$$

Since h is rapidly decreasing it follows from Proposition 4.1 that the limit as $T \to \infty$ of (5.3) exists and therefore

$$\int_{-\infty}^{\infty} h(\lambda)\frac{\phi'}{\phi}(1/2+i\lambda)\,d\lambda = \lim_{T\to\infty}\int_{-T}^{T} h(\lambda)\frac{\phi'}{\phi}(1/2+i\lambda)\,d\lambda$$

is well-defined.

To compute the right hand side , we consider the individual integrals

$$\int_{-\infty}^{\infty} h(\lambda)\frac{2\beta-1}{(\beta-1/2)^2+(\lambda-\gamma)^2}\,d\lambda. \tag{5.4}$$

We would like to shift the contour of integration and apply the residue theorem to compute (5.4). However this works only if the support of g is contained in $(0,\infty)$. We proceed as follows: Given $0 < \varepsilon < 1/2$ let $\chi_\varepsilon \in C^\infty(\mathbf{R}^+)$ be such that $0 \le \chi_\varepsilon \le 1$, $\chi_\varepsilon(y) = 1$ for $2\varepsilon < y < (2\varepsilon)^{-1}$ and $\chi_\varepsilon(y) = 0$ for $y < \varepsilon$ and $y > 1/\varepsilon$.

Set

$$h_\varepsilon(z) = \int_0^\infty \chi_\varepsilon(y) g(y) e^{iyz}\, dy.$$

Then, for $\mathrm{Im}(z) \geq 0$,

$$\lim_{\varepsilon \to 0} h_\varepsilon(z) = \int_0^\infty g(y) e^{iyz}\, dy \tag{5.5}$$

uniformly in the half-plane $\mathrm{Im}(z) \geq 0$. Furthermore,

$$\int_{-\infty}^\infty (h_\varepsilon(\lambda) + h_\varepsilon(-\lambda)) \frac{2\beta - 1}{(\beta - 1/2)^2 + (\lambda - \gamma)^2}\, d\lambda \tag{5.6}$$

converges to (5.4) as $\varepsilon \to 0$. Now observe that (5.6) can be written as

$$\frac{1}{i} \int_{\mathrm{Re}(s)=1/2} \big(h_\varepsilon(i(1/2 - s)) + h_\varepsilon(-i(1/2 - s))\big) \frac{1 - 2\mathrm{Re}(\rho)}{(s - \rho)\,(s - 1 + \bar{\rho})}\, ds. \tag{5.7}$$

Assume that $\mathrm{Re}(\rho) < 1/2$ and let $\sigma > \mathrm{Re}(1 - \rho)$. By the residue theorem, (5.7) equals

$$\begin{aligned} &-2\pi\, h_\varepsilon(i(1/2 - \rho)) - h_\varepsilon(i(1/2 - \bar{\rho})) \\ &+\frac{1}{i} \int_{\mathrm{Re}(s)=-\sigma} h_\varepsilon(i(1/2 - s)) \frac{1 - 2\mathrm{Re}(\rho)}{(s - \rho)\,(s - 1 + \bar{\rho})}\, ds \\ &+\frac{1}{i} \int_{\mathrm{Re}(s)=\sigma} h_\varepsilon(-i(1/2 - s)) \frac{1 - 2\mathrm{Re}(\rho)}{(s - \rho)\,(s - 1 + \bar{\rho})}\, ds. \end{aligned} \tag{5.8}$$

From the definiton of h_ε it follows that, for $\mathrm{Im}(z) > 0$,

$$|h_\varepsilon(z)| \leq e^{-\mathrm{Im}(z)\varepsilon}.$$

This shows that the itegrals in (5.8) can be made arbitrarily small as $\sigma \to \infty$ and therefore, they are identically zero. Thus (5.6) equals

$$-2\pi h_\varepsilon(i(1/2 - \rho)) - 2\pi h_\varepsilon(i(1/2 - \bar{\rho})).$$

Since $\mathrm{Im}(i(1/2 - \rho)) = 1/2 - \mathrm{Re}(\rho) > 0$, it follows from (5.5) that the limit as $\varepsilon \to 0$ exists and (5.4) equals

$$-2\pi \int_0^\infty g(y) e^{(\rho - 1/2)y} dy - 2\pi \int_0^\infty g(y) e^{(\bar{\rho} - 1/2)y} dy. \tag{5.9}$$

The case $\mathrm{Re}(\rho) > 1/2$ can be treated in the same way. Any such pole is real and (5.4) equals

$$4\pi \int_0^\infty g(y) e^{(1/2 - \rho)y} dy. \tag{5.9'}$$

Note that there are only finitely many poles in $\mathrm{Re}(s) > 1/2$.

Next we shall investigate the convergence of the series which we obtain by summing (5.9) over all poles ρ. If we integrate by parts and employ the fact that $g'(0) = 0$, it follows that (5.9) can be estimated by

$$2\pi \frac{1-2\mathrm{Re}(\rho)}{|\rho - 1/2|^2} + \frac{C}{|\rho - 1/2|^3}$$

where the constant C is independent of ρ. Hence, by Corollary 3.29 and Corollary 4.15, the sum of (5.9) over all $\rho, \mathrm{Re}(\rho) < 1/2$, is absolutely convergent and therefore, the series

$$\sum_{\rho=\beta+i\gamma} \int_{-\infty}^{\infty} h(\lambda) \frac{2\beta - 1}{(\beta - 1/2)^2 + (\lambda - \gamma)^2}\, d\lambda \tag{5.10}$$

is absolutely convergent.

Our next goal is to show that the limit as $T \to \infty$ of the series on the right hand side of (5.2) equals (5.10). For this purpose we observe that the series (5.10) remains absolutely convergent if we replace h by $(1+\lambda^2)^{-3}$. This function is not of the form $\hat{g}, g \in C_0^\infty(\mathbf{R})$, but the method used above can be easily extended to cover this case too. Let $R > 0$. Then

$$\begin{aligned}\Bigg| \sum_{\substack{|\rho|>R \\ \rho=\beta+i\gamma}} \int_{-T}^{T} h(\lambda) \frac{2\beta-1}{(\beta-1/2)^2 + (\lambda-\gamma)^2}\, d\lambda \Bigg| \\ \leq C \sum_{\substack{|\rho|>R \\ \rho=\beta+i\gamma}} \int_{-\infty}^{\infty} (1+\lambda^2)^{-3} \frac{|2\beta-1|}{(\beta-1/2)^2+(\lambda-\gamma)^2}\, d\lambda\end{aligned}$$

for all $T \geq 0$. By the remark above, for every $\varepsilon > 0$, we can choose R_0 such that the right hand side is $< \varepsilon$ for $R \geq R_0$. This gives the desired result. Set

$$h_+(z) = \int_0^\infty g(y) e^{zy} dy, \quad z \in \mathbf{C}.$$

Combining our results we have established the following equality

$$\begin{aligned}-\frac{1}{4\pi} \int_{-\infty}^{\infty} h(\lambda) \frac{\phi'}{\phi}(1/2 + i\lambda)\, d\lambda = -\frac{\log q}{4\pi}\, g(0) \\ + \frac{1}{2} \sum_{\rho} \Big\{ h_+(\rho - 1/2) + h_+(\bar{\rho} - 1/2) \Big\} - \sum_{j=1}^{n} h_+(1/2 - \sigma_j)\end{aligned} \tag{5.11}$$

where ρ runs over all poles of $\phi(s)$ in $\mathrm{Re}(s) < 1/2$, counted with the order of the pole, and $\sigma_1, ..., \sigma_n$ are the poles of $\phi(s)$ in $\mathrm{Re}(s) > 1/2$, also counted with the order of the corresponding pole. This agrees with the formula of Lax and Phillips if the support of g is contained in $(0, \infty)$. In this case the series $\sum_\rho h_+(\rho - 1/2)$ is absolutely convergent.

It is not difficult to extend (5.11) to a larger class of functions, say to all even g in the Schwartz space $\mathcal{S}(\mathbf{R})$. Let $g \in \mathcal{S}(\mathbf{R})$ be even. We choose $\chi \in C^\infty(\mathbf{R})$ satisfying $0 \le \chi \le 1, \chi(y) = 1$ for $|y| < 1/2$ and $\chi(y) = 0$ for $|y| > 1$. For $\varepsilon > 0$ we define χ_ε by $\chi_\varepsilon(y) = \chi(\varepsilon y)$. Then (5.11) can be applied to $g_\varepsilon = \chi_\varepsilon g$ and we have to investigate the behaviour of (5.11) as $\varepsilon \to 0$. Let $h = \hat{g}$ and, for $\varepsilon > 0$, let $h_\varepsilon = \hat{g}_\varepsilon$. If we integrate by parts and use Proposition 4.1, it follows that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} h_\varepsilon(\lambda) \frac{\phi'}{\phi}(1/2 + i\lambda)\, d\lambda = \int_{-\infty}^{\infty} h(\lambda) \frac{\phi'}{\phi}(1/2 + i\lambda)\, d\lambda. \tag{5.12}$$

Now consider the right hand side of (5.11). Set

$$h_+^\varepsilon(z) = \int_0^\infty \chi_\varepsilon(y) g(y) e^{zy} dy, \quad z \in \mathbf{C},$$

and

$$h_+(z) = \int_0^\infty g(y) e^{zy} dy, \quad \mathrm{Re}(z) \le 0.$$

Assume that $\mathrm{Re}(z) \le 0$. Integrating by parts it follows that there exists a constant $C > 0$ such that

$$\left| h_+^\varepsilon(z) + \frac{g(0)}{z} \right| \le \frac{C}{|z|^3}, \tag{5.13}$$

for all $\varepsilon > 0$ and

$$\left| h_+(z) + \frac{g(0)}{z} \right| \le \frac{C}{|z|^3}. \tag{5.14}$$

This implies

$$\sum_\rho \left| h_+^\varepsilon(\rho - 1/2) + h_+^\varepsilon(\bar\rho - 1/2) \right| \le C_1 \sum_\rho \frac{1 - 2\mathrm{Re}(\rho)}{|\rho - 1/2|^2} + C_2 \sum_\rho \frac{1}{|\rho - 1/2|^3}$$

which is finite by Corollary 3.29 and Corollary 4.15. The constants C_1 and C_2 are independent of ε. From (5.14) follows in the same way that

$$\sum_\rho \big(h_+(\rho - 1/2) + h_+(\bar\rho - 1/2) \big)$$

is absolutely convergent and it is the limit as $\varepsilon \to 0$ of

$$\sum_\rho \big(h_+^\varepsilon(\rho - 1/2) + h_+^\varepsilon(\bar\rho - 1/2) \big).$$

Finally we observe that by (3.1), $s \mapsto 1 - s$ sets up a one-to-one correspondence between the poles of $\phi(s)$ in $\mathrm{Re}(s) > 1/2$ and the zeros of $\phi(s)$ in $\mathrm{Re}(s) < 1/2$. Let ρ be a pole or a zero of $\phi(s)$. Then we denote by $n(\rho)$ the order of the pole ρ or the negative of the order of the zero ρ. With this notation we can summarize our results as follows:

Theorem 5.15. *Let $g \in \mathcal{S}(\mathbf{R})$ be even. Set $h = \hat{g}$ and $h_+(z) = \int_0^\infty g(y)e^{zy}dy$, $\mathrm{Re}(z) \leq 0$. Then*

$$(5.16)\qquad \begin{aligned} -\frac{1}{4\pi}\int_{-\infty}^{\infty} h(\lambda)\frac{\phi'}{\phi}(1/2+i\lambda)\ \lambda &= -\frac{\log q}{4\pi}\ g(0) \\ &\quad + \frac{1}{2}\sum_\rho n(\rho)\{h_+(\rho-1/2)+h_+(\bar\rho-1/2)\} \end{aligned}$$

where ρ runs over all poles and zeros of $\phi(s)$ in $\mathrm{Re}(s) < 1/2$.

We apply Theorem 5.15 to the function $g(y) = (4\pi t)^{-\frac{1}{2}}\exp(-y^2/4t)$. According to [B1], p.146, we have

$$\frac{1}{\sqrt{4\pi t}}\int_0^\infty e^{-y^2/4t}e^{zy}dy = \frac{1}{2}\,e^{tz^2}\mathrm{Erfc}(-\sqrt{t}\ z)$$

where Erfc denotes the complementary error function. Thus, by (5.16)

$$(5.17)\qquad \begin{aligned} -\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-(\frac{1}{4}+\lambda^2)t}\ \frac{\phi'}{\phi}(1/2+i\lambda)\ d\lambda &= -\frac{\log q}{(4\pi)^{3/2}}\frac{e^{-t/4}}{\sqrt{t}} \\ + \frac{1}{4}\sum_\rho n(\rho)\{e^{-t\rho(1-\rho)}\mathrm{Erfc}(\sqrt{t}\ (1/2-\rho)) &+ e^{-t\bar\rho(1-\bar\rho)}\mathrm{Erfc}(\sqrt{t}\ (1-\bar\rho))\}. \end{aligned}$$

Now let λ_j be an eigenvalue of Δ. As usually, write $\lambda_j = s_j(1-s_j)$ with $s_j \in \mathbf{C}, \mathrm{Re}(s_j) \geq 1/2$ and $\mathrm{Im}(s_j) \geq 0$. First assume that $\lambda_j \geq 1/4$. Then $\mathrm{Re}(s_j) = 1/2$ and it follows that

$$(5.18)\qquad e^{-\lambda_j t} = \frac{1}{2}\ e^{-s_j(1-s_j)t}\ \mathrm{Erfc}(\sqrt{t}\ (1/2-s_j)) + \frac{1}{2}e^{-\bar s_j(1-\bar s_j)t}\ \mathrm{Erfc}(\sqrt{t}\ (1/2-\bar s_j)).$$

On the other hand, if $\lambda_j < 1/4$ then $\mathrm{Im}(s_j) = 0$ and $s_j \in (1/2, 1]$. In this case we get

$$(5.19)\qquad e^{-\lambda_j t} = \frac{1}{2}\ e^{-s_j(1-s_j)t}(\mathrm{Erfc}(\sqrt{t}\ (s_j-1/2)) + \mathrm{Erfc}(\sqrt{t}\ ((1-s_j)-1/2)).$$

This allows us to combine the contribution of the eigenvalues and resonances to the trace formula (2.2) in a single formula. For this purpose we assign to each point η in $\mathbf{C}$ a certain multiplicity $m(\eta)$ as follows:

1) If $\mathrm{Re}(\eta) \geq 1/2, \eta \neq 1/2$, we define $m(\eta)$ to be the dimension of the eigenspace of Δ for the eigenvalue $\eta(1-\eta)$.

2) If $\mathrm{Re}(\eta) < 1/2$ then $m(\eta)$ is the dimension of the eigenspace for the eigenvalue $\eta(1-\eta)$ plus the order of the pole or the negative of the order of the zero of $\phi(s)$ at η.

3) $m(1/2)$ equals $(\mathrm{Tr}(C(1/2))+m)/2$, where m is the number of cusps of M, plus twice the dimension of the eigenspace with eigenvalue $1/4$.

4) For all other points η in $\mathbf{C}$ which are not among 1) - 3) we set $m(\eta) = 0$.

Definition 5.20. The resonance set $\sigma(M)$ is the set of all η in **C** such that $m(\eta) > 0$.

Remarks.

1) Note that , by (1.6) , $C(1/2)^2 = Id$. Therefore the eigenvalues of $C(1/2)$ are ± 1. Hence $(\mathrm{Tr}(C(1/2)) + m)/2$ equals the number of eigenvalues $+1$ of $C(1/2)$.

2) For a hyperbolic surface $M = \Gamma\backslash H$ the resonance set was first introduced by R.Phillips and P.Sarnak in [PS1]. They actually use a slightly different definition called the *singular set* $\sigma(\Gamma)$. It is related to our definition by $\eta \in \sigma(M) \mapsto i(1/2 - \eta) \in \sigma(\Gamma)$.

Assumption. To simplify notation we shall use the following convention from now on. Whenever we sum over $\sigma(M)$ or some subset of $\sigma(M)$ we count each point η in this subset according to its multiplicity $m(\eta)$.

If we employ (2.3) we obtain the following asymptoytic expansion as $t \to 0$:

$$\begin{aligned}(5.22)\quad &\frac{1}{4}\sum_{\eta\in\sigma(M)}\left\{e^{-t\eta(1-\eta)}\,\mathrm{Erfc}(\sqrt{t}\,(1/2-\eta)) + e^{-t\bar\eta(1-\bar\eta)}\,\mathrm{Erfc}(\sqrt{t}\,(1/2-\bar\eta))\right\}\\ &= \frac{\mathrm{Area}(M)}{4\pi t} + \frac{m}{2}\frac{\log t}{\sqrt{4\pi t}} + \left(\frac{3\gamma m}{2} + \frac{\log q}{4\pi}\right)\frac{1}{\sqrt{4\pi t}} + \frac{\chi(M)}{6} + \frac{m}{4} + O(\sqrt{t}).\end{aligned}$$

This implies

Theorem 5.23. *Let M be an admissible surface. The resonance set $\sigma(M)$ determines* $\mathrm{Area}(M)$ *, the number m of cusps of M and $\chi(M)$. In particular, the conformal type (h, m) of M is determined by the resonance set.*

This is the analogue of a known result for compact surfaces. Another property satisfied by the resonance set is the following asymptotic formula

$$(5.24)\qquad \#\{\eta \in \sigma(M) \mid |\eta| \le T\} = \frac{\mathrm{Area}(M)}{2\pi}\,T^2 + o(T^2), \quad T \to \infty,$$

which is an immediate consequence of Theorem 4.23. This is our final version of Weyl's law for admissible surfaces. For later use we note that (5.24) implies

Proposition 5.25. *For* $\mathrm{Re}(s) > 2$, *the series*

$$\sum_{\eta\neq 0}|\eta|^{-s}$$

is absolutely convergent.

For hyperbolic surfaces Phillips and Sarnak showed in [PS1] that the resonance set can be identified with the spectrum of the generator of a cut-off wave equation. This is very useful, because one may employ standard techniques of perturbation theory to study the resonance set. The approach used by Phillips and Sarnak to prove the result above is based on Lax-Phillips scattering theory applied to the hyperbolic wave equation [LP]. The assumption that the surface is hyperbolic is not essential for this theory to work, because at most places one uses only the structure of the cusps. An important tool to establish the main result of [PS1] is the mini-trace formula (5.1). By Theorem 5.15, this formula is available for all admissible surfaces. Hence the main result of [PS1] extends to admissible surfaces.

To make the statement precise we have to introduce some notation. Set

$$L = -\Delta + \tfrac{1}{4}I.$$

The hyperbolic wave equation is then

$$\frac{\partial^2}{\partial t^2}u = Lu$$

with initial values $f = \{f_1, f_2\}$

$$u(z,0) = f_1(z) \quad \text{and} \quad \frac{\partial}{\partial t}u(z,0) = f_2(z).$$

Let $\mathcal{H}_G$ be the completion of $C_0^\infty(M) \times C_0^\infty(M)$ with respect to the modified energy norm G (cf. section 5 of [LP]). One may rewrite the wave equation in the form

$$\frac{\partial}{\partial t}f = Af$$

where the infinitesimal generator A is given by

$$A = \begin{pmatrix} 0 & I \\ L & 0 \end{pmatrix}$$

defined as the closure of A , restricted to $C_0^\infty(M) \times C_0^\infty(M)$. The operator A generates a group of bounded operators $U(t)$ in $\mathcal{H}_G$.

The incoming subspace $\mathcal{D}_-$ and the outgoing subspace $\mathcal{D}_+$ of $\mathcal{H}_G$ are defined in the same way as in [LP], [PS1]. Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{D}_- \bigoplus \mathcal{D}_+$ in $\mathcal{H}_G$ and let P denote the G-orthogonal projection of $\mathcal{H}_G$ onto $\mathcal{H}$. Let $a > \max\{a_1, ..., a_m\}$. Then P is given by

$$Pf = f \quad \text{except for the zero Fourier coefficients in each cusp}$$
$$(Pf)_j^{(0)}(y) = \{(f_1)_j^{(0)}(a)\,(y/a)^{1/2}, 0\} \text{ for } y > a \text{ in the cusp } Z_j.$$

Set

$$Z(t) = P\,U(t)\,P, \quad t \geq 0. \tag{5.26}$$

As in [LP], Theorem 2.7, it can be shown that $Z(t)$ is a strongly continuous semigroup of operators in $\mathcal{H}$. Let B denote the infinitesimal generator of $Z(t)$. Then B has pure point spectrum of finite multiplicity and one has

Theorem 5.27. *Let M be an admissible surface and B the generator of the corresponding semigroup (5.26). Then*

$$\sigma(M) = \mathrm{Spec}(B + \tfrac{1}{2}I)$$

and, for each $\eta \in \sigma(M), m(\eta)$ equals the dimension of the generalized eigenspace of B with eigenvalue $\eta - 1/2$.

Thus in place of $\sigma(M)$ we may as well study the operator B. For example, let $g_u, u \in (-\varepsilon, \varepsilon)$, be a real analytic family of metrics on an admissible surface M and assume that $g_u = g_0, u \in (-\varepsilon, \varepsilon)$, outside a fixed compact subset of M. Let B_u be the infinitesimal generator of the cut-off wave equation on (M, g_u). In the same way as in [PS1] one can prove

Theorem 5.28. *The resolvent $(B_u - \lambda I)^{-1}$ is real analytic in u on the resolvent set of B_0 for $|u|$ sufficiently small.*

This theorem tells us how the resonance set $\sigma(M, g_u)$ varies with respect to u.

At the end of this section we consider the case of a hyperbolic surface $M = \Gamma\backslash H$. Then (5.16) allows us to rewrite the Selberg trace formula in a way which resembles the trace formula for a compact hyperbolic surface.

As above let λ_j be an eigenvalue of Δ and assume that $\lambda_j = s_j(1 - s_j)$. If $s_j = \frac{1}{2} + ir_j, r_j \in \mathbf{R}$, then it follows from the definition of h and h_+ that

$$h(r_j) = h_+(s_j - 1/2) + h_+(\bar{s}_j - 1/2) \tag{5.29}$$

and, if $s_j \in (1/2, 1]$ and $s_j = \frac{1}{2} + ir_j$, we have

$$h(r_j) = h_+(s_j - 1/2) + h_+((1 - s_j) - 1/2). \tag{5.30}$$

Let $\sigma(\Gamma)$ denote the resonance set in the present case. Then the Selberg trace formula [H1], [H3] can be rewritten as

Theorem 5.31. *Let $g \in C_0^\infty(\mathbf{R})$ be even and set $h = \hat{g}$, $h_+(z) = \int_0^\infty g(y)e^{zy}dy, z \in \mathbf{C}$. Then*

$$\frac{1}{2} \sum_{\eta \in \sigma(\Gamma)} \big(h_+(\eta - 1/2) + h_+(\bar{\eta} - 1/2)\big) = \frac{\mathrm{Area}(M)}{4\pi} \int_{-\infty}^{\infty} h(\lambda)\lambda \tanh(\pi\lambda)\, d\lambda$$

$$+ \sum_{k=1}^{\infty} \sum_{\{\gamma\}} \frac{\log N(\gamma)}{N(\gamma)^{k/2} - N(\gamma)^{-k/2}}\, g(k \log N(\gamma)) - \frac{m}{2\pi} \int_{-\infty}^{\infty} h(\lambda)\, \frac{\Gamma'}{\Gamma}(1 + i\lambda)\, d\lambda \tag{5.32}$$

$$-g(0)\Big(m \log 2 + \frac{\log q}{4\pi}\Big)$$

where $\{\gamma\}$ runs over all primitive hyperbolic conjugacy classes in Γ and $N(\gamma)$ is the norm of the hyperbolic element γ (cf. [H1]).

Remark. In our case Γ has no fixed points. If we consider discrete subgroups of $SL(2, \mathbf{R})$ which are not torsion free we have to add the usual fixed point contribution to the right hand side of (5.32).

Apart from the inessential last two terms the right hand side of (5.32) agrees with the right hand side of the Selberg trace formula for a compact quotient.

6. The zeta function and the determinant of the Laplacian

The determinant of the Laplacian on a compact surface is a very useful functional in spectral geometry (cf. [OPS1], [OPS2]). We wish to have a similar functional available for admissible surfaces. Since for a compact surface the determinant is defined via the zeta function of the Laplacian we begin with the study of the zeta function of Laplacians on admissible surfaces . To define the zeta function we take the Mellin transform of (2.2) and collect all terms which involve spectral invariants. The resulting analytic function is the *spectral zeta function*

$$(6.1)\ \zeta_\Delta(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s} - \frac{1}{4\pi} \int_{-\infty}^{\infty} (1/4+\lambda^2)^{-s}\, \frac{\phi'}{\phi}(1/2+i\lambda)\, d\lambda + \tfrac{1}{4}\{\mathrm{Tr}(C(\tfrac{1}{2})) + m\}\left(\tfrac{1}{4}\right)^{-s}.$$

Actually we have modified the last term by adding $\frac{m}{4}(\frac{1}{4})^{-s}$. The reason for it will become clear below when we introduce the second kind of zeta functions –the *resonance zeta functions*–and relate $\zeta_\Delta(s)$ to them.

By (1.2) and Proposition 4.1, the series and the integral are absolutely convergent for $\mathrm{Re}(s) > 1$. Furthermore, by (2.2) we have

$$(6.2)\ \ \zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \int_M \left(K(z,z,t) - k(z,z,t)\right) d\mu(z)\, dt - \frac{4^{s-1}}{\sqrt{\pi}} \sum_{j=1}^{m} \log a_j + \tfrac{m}{4} 4^s.$$

Using (2.3) and (2.4), we obtain

Theorem 6.3. *$\zeta_\Delta(s)$ admits a meromorphic extension to* **C**. *The only poles in the half-plane* $\mathrm{Re}(s) > -1/2$ *occur at* $s = 1$ *and* $s = 1/2$. *The pole at* $s = 1$ *is simple with residue* $\mathrm{Area}(M)/4\pi$ *and the pole at* $s = 1/2$ *has order* 2. *Let* γ *be Euler's constant. The coefficients of* $(s-1/2)^{-2}$ *and* $(s-1/2)^{-1}$ *in the Laurent expansion at* $s = 1/2$ *are* $3m\gamma/2$ *and* $m/2$, *respectively. In particular,* $\zeta_\Delta(s)$ *is holomorphic at* $s = 0$.

Remarks.

1)Since the asymptotic expansion (2.4) involves fractional powers of t and $\log t, \zeta_\Delta(s)$ has in general infinite many poles which may have order > 1. This is the main difference to the zeta function of the Laplacian on a compact surface.

2) The right hand side of (6.2) may be regarded as a relative zeta function in the following sense: The kernel $k(z_1, z_2, t)$ is the heat kernel of the self-adjoint operator obtained by restricting the Laplacian to the zero Fourier coefficients on each cusp and imposing Dirichlet boundary conditions. In other words, this operator is the direct sum of operators of the form $-y^2\, d^2/dy^2$ acting in the Hilbert space $L^2([a,\infty), dy/y^2)$ with an appropriate choice of the domain. In this sense we compare the Laplacian with the direct sum of 1-dimensional Laplacians on a half-line and the difference of the corresponding heat operators turns out to be trace class. A similar definition of a relative zeta function was used by R.Lundelius [L] in his thesis. The advantage of our definition is that (6.1) is an intrinsic definition.

3)For hyperbolic surfaces I.Efrat introduced a zeta function similar to (6.1) [E2].

Using the zeta function $\zeta_\Delta(s)$ we define the regularized determinant of the Laplacian Δ in the same way as in the compact case as

$$\det{}'\Delta = e^{-\zeta'_\Delta(0)}. \tag{6.4}$$

In view of (3.36), the zeta function is completely determined by $\sigma(M)$ and q and our next goal is to find an explicit formula expressing the zeta function in terms of $\sigma(M)$. Note that (5.16) is not applicable, because the functions h and g are not in $\mathcal{S}(\mathbf{R})$, but we may proceed in the same way as in the previous section and extend (5.16) so that the zeta function is included.

The function h equals now $(1/4+\lambda^2)^{-s}$ and therefore, for $\mathrm{Re}(s)>1$, we have

$$g(y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(1/4+\lambda^2)^{-s}\,e^{-i\lambda y}\,d\lambda=\frac{1}{\sqrt{\pi}\Gamma(s)}\,|y|^{s-1/2}\,K_{s-1/2}\Big(\frac{|y|}{2}\Big)$$

(cf. [B1],p.11,(7)), where K_ν denotes the modified Bessel function. For $\mathrm{Re}(\nu)>-1/2$ we have the following integral representation of K_ν:

$$K_\nu(y)=\frac{\pi}{2}\frac{y^\nu e^{-y}}{\Gamma(\nu+1/2)}\int_0^\infty e^{-yt}\,\big(t\,(1+t/2)\big)^{\nu-1/2}\,dt$$

(cf. [MO]), which shows that g is rapidly decreasing at infinity, but it is not smooth at the origin. However, smoothness at the origin is not necessary to derive (5.16). We need only formulas (5.13) and (5.14) and for these to hold it is sufficient to know that g is three times continuously differntiable at the origin. If $\mathrm{Re}(s)>2$ then g is contained in $C^3(\mathbf{R})$. Furthermore, by formula (23) on p.131 in [B2], we have

$$h_+(z)=\frac{\Gamma(2s)}{\Gamma(s)\,\Gamma(s+1)}\,(1/2-z)^{-2s}\,F\Big(2s,s;s+1;\frac{z+1/2}{z-1/2}\Big),\quad \mathrm{Re}(z)<0,$$

where $F(\alpha,\beta;\gamma;w)$ denotes the hypergeometric series. Now recall that the hypergeometric series admits an analytic continuation to $\mathbf{C}-[1,\infty)$. If we employ arguments similar to those used to prove Theorem 5.15, we obtain

$$\begin{aligned}\zeta_\Delta(s)=\frac{1}{2}\,\frac{\Gamma(2s)}{\Gamma(s)\,\Gamma(s+1)}\sum_{\eta\neq1}\Big\{&(1-\eta)^{-2s}\,F\Big(2s,s;s+1;\frac{\eta}{\eta-1}\Big)\\ &+(1-\bar\eta)^{-2s}\,F\Big(2s,s;s+1;\frac{\bar\eta}{\bar\eta-1}\Big)\Big\}-\frac{\log q}{4\sqrt{2\pi}}\frac{\Gamma(2s-1)}{\Gamma(s)^2}\end{aligned}\tag{6.5}$$

for $\mathrm{Re}(s)>2$. Note that $\mathrm{Re}(\eta)<1$ for all $\eta\in\sigma(M)-\{1\}$. Therefore, $(1-\eta)^{-2s}$ is well-defined as $\exp(-2s\,\log(1-\eta))$ where $\log z$ is the branch of the logarithm which satisfies $\log 1=0$. Furthermore note that $\eta/(\eta-1)\in\mathbf{C}-[1,\infty)$ for $\eta\in\sigma(M)-\{1\}$ and on this domain the hypergeometric function is defined by analytic continuation. The series (6.5) is absolutely convergent for $\mathrm{Re}(s) > 2$.

We consider a single term of this series and compute its derivative at $s = 0$. If η is a pole or a zero of ϕ in $\mathrm{Re}(s) < 1/2$ then $|\eta/(\eta-1)| < 1$ and $F(2s, s; s+1; \eta/(\eta-1))$ is given by the usual power series and from its definition follows immediately that

$$\frac{d}{ds} F\Big(2s, s; s+1; \frac{\eta}{\eta-1}\Big) \Big|_{s=0} = 0.$$

Furthermore, $\Gamma(2s)\,(\Gamma(s)\,\Gamma(s+1))^{-1} = 1/2\;\Gamma(2s+1)\,\Gamma(s+1)^{-2}$ and a direct computation shows that the derivative at $s = 0$ vanishes. Hence the derivative of a single term of the series (6.5) equals $-\log|1-\eta|$, so that formally

$$\zeta'_\Delta(0) = -\sum_{\eta\neq 1} \log|1-\eta| + \frac{\log q}{8\sqrt{2\pi}}.$$

This suggests to introduce the following zeta function

$$\zeta_B(s) = \sum_{\eta\neq 1} (1-\eta)^{-s} \tag{6.6}$$

which by Proposition 5.25 converges for $\mathrm{Re(s)} > 2$. Here B denotes the generator of the Lax-Phillips semi-group $Z(t), t \geq 0$, associated to the hyperbolic wave equation on M. We call $\zeta_B(s)$ the ***resonance zeta function***. As explained in the introduction, we may regard $\zeta_B(s)$ as the zeta function of the non-self-adjoint operator $B_1 = -B + \frac{1}{2}I$. Our next goal is to show that $\zeta_B(s)$ admits a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s = 0$ and to compare its derivative at $s = 0$ with $\zeta'_\Delta(0)$. This is then the rigorous version of the formal equality above.

We employ formula (1) of section 2.10 in [B3] for the analytic continuation of the hypergeometric series . It gives

$$\frac{\Gamma(2s)}{\Gamma(s)\,\Gamma(s+1)}\,(1-\eta)^{-2s}\,F\Big(2s, s; s+1; \frac{\eta}{\eta-1}\Big)$$

$$= \frac{\Gamma(2s)\,\Gamma(1-2s)}{\Gamma(s)\,\Gamma(1-s)}\,(1-\eta)^{-2s}\,F\Big(2s, s; 2s; \frac{1}{1-\eta}\Big) \tag{6.7}$$

$$+\frac{\Gamma(2s-1)}{\Gamma(s)^2}\,\frac{1}{1-\eta}\,F\Big(1-s, 1; 2-2s; \frac{1}{1-\eta}\Big)$$

which holds if both sides are defined. For example, this is the case if $|1-\eta| > 1$ and $\mathrm{Re(s)} \leq 1/2$. Let

$$\sigma_1 = \sigma(M) - \{\eta \in \sigma(M) \mid |1-\eta| \leq 1\}.$$

Then we consider the following series

$$\sum_{\eta\in\sigma_1}(1-\eta)^{-2s}\, F\Big(2s,s;2s;\frac{1}{1-\eta}\Big).$$

For each $s \in \mathbf{C}$, there exists $C > 0$ such that $|F(2s,s;2s;1/(1-\eta))| \le C$ for $\eta \in \sigma_1$. Hence, by Proposition 5.25, the series is absolutely convergent for Re(s) > 1. Since(6.5) is absolutely convergent for Re(s) > 2, it follows that

$$\sum_{\eta\in\sigma_1}\Big\{\frac{1}{1-\eta}\, F\Big(1-s,1;2-2s;\frac{1}{1-\eta}\Big)+\frac{1}{1-\bar\eta}\, F\Big(1-s,1;2-2s;\frac{1}{1-\bar\eta}\Big)\Big\} \tag{6.8}$$

is also absolutely convergent for Re(s) > 2. Using the definition of the hypergeometric series and Proposition 5.25, it follows that

$$\sum_{\eta\neq1}\Big\{\frac{1}{1-\eta}+\frac{1}{1-\bar\eta}+\frac{1}{2}\,\frac{1}{(1-\eta)^2}+\frac{1}{2}\,\frac{1}{(1-\bar\eta)^2}\Big\}$$

is absolutely convergent too. This series equals

$$\sum_{\eta\neq1}\Big(\frac{1-2\mathrm{Re}(\eta)}{|1-\eta|^2}+2\frac{(1-\mathrm{Re}(\eta))^2}{|1-\eta|^4}\Big)$$

If $\eta(1-\eta)$ is an eigenvalue $\ge 1/4$, then $\mathrm{Re}(\eta)=1/2$. Hence, by Corollary 3.29

$$\sum_{\eta\neq1}\frac{1-2\mathrm{Re}(\eta)}{|1-\eta|^2}<\infty. \tag{6.9}$$

This implies

Proposition 6.10. *The series*

$$\sum_{\eta\neq1}\frac{(1-\mathrm{Re}(\eta))^2}{|1-\eta|^4}$$

is convergent.

Remark. For a hyperbolic surface we know that there exists $C > -\infty$ such that $C \le \mathrm{Re}(\eta) \le 1$ for all $\eta \in \sigma(M)$. Proposition 6.10 is then a consequence of Proposition 5.25. However, for an arbitrary admissible surface we don't have this estimate and Proposition 6.10 sheds some new light on the distribution of poles of $\phi(s)$ in the general case. For example, it implies that, for any $\varepsilon > 0$,

$$\sum_{\substack{\eta\neq1\\ |\mathrm{Re}\eta|>\varepsilon|\mathrm{Im}\eta|}}\frac{1}{|\eta|^2}<\infty.$$

This can be restated as follows. Given $T > 0$ and $\varepsilon > 0$, let

$$N(T,\varepsilon) = \#\{\eta \in \sigma(M) \mid |\eta| \le T, |\mathrm{Re}\eta| > \varepsilon|\mathrm{Im}\eta|\}.$$

Then

$$\limsup_{T\to\infty} \frac{N(T,\varepsilon)}{T^2} = 0.$$

In other words, the main contribution to the asymptotic formula (5.24) comes from resonances which are "close" to the line Re(s) = 1/2.

Now (6.9) and Proposition 6.10 combined with Proposition 5.25 imply that the series (6.8) is absolutely convergent for all $s \neq k + \frac{1}{2}$, $k \in \mathbf{N}$, and defines a meromorphic function on $\mathbf{C}$ with simple poles at $s = k + \frac{1}{2}$, $k \in \mathbf{N}$.

Lemma 6.11. *The series*

$$\sum_{\eta\in\sigma_1} (1-\eta)^{-2s}\, F\Big(2s, s; 2s; \frac{1}{1-\eta}\Big)$$

is absolutely convergent for Re(s) > 1 *and admits a meromorphic continuation to* $\mathbf{C}$. *The only poles in the half-plane* Re(s) > −1/2 *occur at* $s = 1$ *and* $s = 1/2$ *and they are simple. The residue at* $s = 1$ *equals* −Area(M)/2π *and the residue at* $s = 1/2$ *equals* $-3\pi m\gamma$ *where* γ *is Euler's constant.*

Proof. The fact that the series is absolutely convergent for Re(s) > 1, was proved above. Furthermore, by (6.5) and (6.7), we have for Re(s) > 1

$$(6.12)\quad \sum_{\eta\in\sigma_1} (1-\eta)^{-2s}\, F\Big(2s, s; 2s; \frac{1}{1-\eta}\Big) = \frac{\Gamma(s)\,\Gamma(1-s)}{\Gamma(2s)\,\Gamma(1-2s)}\,\tilde{\zeta}_\Delta(s) - \frac{\Gamma(1-s)}{\Gamma(s)\,\Gamma(2-2s)}\,\frac{\log q}{4\sqrt{2\pi}}$$

$$+\frac{\Gamma(1-s)}{\Gamma(s)\,\Gamma(2-2s)}\,\frac{1}{2}\sum_{\eta\in\sigma_1}\Big\{\frac{1}{1-\eta}\, F\Big(1-s, 1; 2-2s; \frac{1}{1-\eta}\Big) + \frac{1}{1-\bar\eta}\, F\Big(1-s, 1; 2-2s; \frac{1}{1-\bar\eta}\Big)\Big\}$$

where $\tilde{\zeta}_\Delta(s)$ is the right hand side of (6.5) with the sum running over $\eta \in \sigma_1$ in place of $\eta \neq 1$. Note that $\tilde{\zeta}_\Delta(s)$ differs from $\zeta_\Delta(s)$ by an entire function. By Theorem 6.3 and the observation above, the right hand side is a meromorphic function on $\mathbf{C}$. Now recall that $1/\Gamma(s)$ is entire with zeros at the negative integers. This shows that the last two terms on the right hand side of (6.12) are entire functions. The rest follows from Theorem 6.3. Q.E.D.

Corollary 6.13. *The series*

$$\zeta_B(s) = \sum_{\eta \neq 1} (1-\eta)^{-s}$$

is absolutely convergent for Re(s) > 2 *and admits a meromorphic continuation to* **C**. *The only poles in the half-plane* Re(s) > −1 *occur at* $s = 2$ *and* $s = 1$ *and they are simple. The residue at* $s = 2$ *equals* −Area(M)/π *and the residue at* $s = 1$ *equals* Area(M)/$2\pi - 6\pi m\gamma$.

Proof. It suffices to prove the Corollary for the following partial zeta function

$$\zeta_1(s) = \sum_{\eta \in \sigma_1} (1-\eta)^{-s}. \tag{6.14}$$

The first statement follows from Proposition 5.25. Suppose that $\zeta_1(s)$ has been analytically continued to the half-plane Re(s) > 2−k, k ∈ **N**. Using the definition of the hypergeometric series and Lemma 6.11, it follows that, for Re(s) > 1,

$$\begin{aligned} \zeta_1(s) &= -\frac{s}{2}\,\zeta_1(s+1) - \cdots - \frac{(s/2)_k}{k!}\,\zeta_1(s+k) \\ &\quad - \sum_{\eta \in \sigma_1} (1-\eta)^{-s-k-1} \sum_{j=k+1}^{\infty} \frac{(s/2)_j}{j!}(1-\eta)^{k+1-j} + F(s) \end{aligned} \tag{6.15}$$

where $F(s)$ is a meromorphic function on **C**. The double series is absolutely convergent for Re(s) > 2 − k − 1 and defines a holomorphic function on this half-plane. This follows from Proposition 5.25. All remaining terms on the right hand side are meromorphic on Re(s) > 2 − k − 1. Hence $\zeta_1(s)$ admits a meromorphic continuation to Re(s) > 2 − k − 1.

To determine the poles of $\zeta_1(s)$ in Re(s) > −1 we first note that, by Proposition 5.25 and Lemma 6.11, the right hand side is holomorphic in Re(s) > 2 and the only pole in Re(s) > 3/2 occurs at $s = 2$ which is the pole of $F(s)$ at $s = 2$. If we repeat this argument, we can determine all poles and their residues in Re(s) > −1. Q.E.D.

By Corollary 6.13, $\zeta_B(s)$ is holomorphic at $s = 0$ and therefore we may compute its first derivative at $s = 0$. We use again the partial zeta function $\zeta_1(s)$. By (6.14)

$$\frac{d}{ds}\zeta_B(s)\Big|_{s=0} = \frac{d}{ds}\zeta_1(s)\Big|_{s=0} - \sum_{0<|1-\eta|\leq 1} \log|1-\eta|.$$

Furthermore, by (6.15)

$$\begin{aligned} \frac{d}{ds}\zeta_1(s)\Big|_{s=0} &= -\frac{1}{2}\frac{d}{ds}(s\zeta_1(s+1))\Big|_{s=0} - \frac{1}{8}\frac{d}{ds}(s(s+2)\zeta_1(s+2))\Big|_{s=0} \\ &\quad - \frac{1}{2}\sum_{k=3}^{\infty}\frac{1}{k}\,\zeta_1(k) + \frac{d}{ds}F(s)\Big|_{s=0}. \end{aligned}$$

To compute $d/dsF(s)|_{s=0}$ we shall employ (6.12). This gives

$$\frac{d}{ds}F(s)\big|_{s=0}=\frac{d}{ds}\zeta_\Delta(s)\big|_{s=0}-\frac{\log q}{8\sqrt{2\pi}}+\frac{1}{2}\sum_{k=3}^{\infty}\frac{1}{k}\,\zeta_1(k)+\sum_{0<|1-\eta|\le 1}\log|1-\eta|$$
$$+\frac{1}{2}\sum_{\eta\in\sigma_1}\left\{\frac{1}{1-\eta}+\frac{1}{1-\bar\eta}+\frac{1}{2}\,\frac{1}{(1-\eta)^2}+\frac{1}{2}\,\frac{1}{(1-\bar\eta)^2}\right\}.$$

To compute the remaining terms we use again (6.15) with s replaced by $s+1$ and $s+2$, respectively. The final result is

$$\frac{d}{ds}\zeta_B(s)\big|_{s=0}=\frac{d}{ds}\zeta_\Delta(s)\big|_{s=0}+\frac{1}{8}\,\mathrm{Res}_{s=2}\zeta_B(s)+\pi\,\frac{d}{ds}(s^2\zeta_\Delta(s+1/2))\big|_{s=0}.$$

Finally, we use Theorem 6.3 and Corollary 6.13 to compute the last two terms on the right hand side. This proves

Proposition 6.16. *We have*

$$\frac{d}{ds}\zeta_B(s)\big|_{s=0}=\frac{d}{ds}\zeta_\Delta(s)\big|_{s=0}-\frac{\mathrm{Area(M)}}{8\pi}+\frac{3\pi\gamma}{2}\,m$$

where γ denotes Euler's constant.

Since $\zeta_B(s)$ is the zeta function of the operator $B_1=-B+\frac{1}{2}I$, we can introduce its regularized determinant by

$$\det{}'B_1=e^{-\zeta_B'(0)}. \tag{6.17}$$

Note that formally

$$\det{}'B_1=\prod_{\lambda_j\neq 0}\lambda_j\,\prod_{\rho\neq 1}|1-\rho|^2,$$

where ρ runs over the poles of $\phi(s)$. By Proposition 6.16, we get

Corollary 6.18. *We have the identity*

$$\det{}'\Delta=\exp\!\left(\frac{\mathrm{Area}(M)}{8\pi}-\frac{3\pi\gamma}{2}m\right)\det{}'B_1.$$

This equality is important for the further investigation of the determinant. For example, $\det'\Delta$ is well-suited for deriving variational formulas similar to those in [OPS1]. On the other hand, $\det' B_1$ is defined via the resonance zeta function and this makes it transparent how the determinant depends on the resonance set.

Let $z \in \mathbf{C}, \mathrm{Re(z)} > 1$. In the same way as above one can define the determinant of the operator $\Delta + z(z-1)I$. For Re(s) > 1, the corresponding spectral zeta function is defined as

$$\zeta_\Delta(z,s) = \sum_{\lambda_j}(\lambda_j + z(z-1))^{-s} - \frac{1}{4\pi}\int_{-\infty}^{\infty}\left((z-1/2)^2+\lambda^2\right)^{-s}\frac{\phi'}{\phi}(1/2+i\lambda)\,d\lambda$$
$$+\frac{\mathrm{Tr}(C(1/2))+m}{4}(z-1/2)^{-2s}.$$

As a function of s, $\zeta_\Delta(z,s)$ admits a meromorphic continuation to $\mathbf{C}$ which is holomorphic at $s=0$. Then we define the determinant of $\Delta + z(z-1)$ to be

$$\det(\Delta + z(z-1)) = \exp\Big(-\frac{\partial}{\partial s}\zeta_\Delta(z,s)\big|_{s=0}\Big). \tag{6.19}$$

Similarly we can also introduce the following resonance zeta function

$$\zeta_B(z,s) = \sum_{\eta\in\sigma(M)}(z-\eta)^{-s}.$$

Since $\mathrm{Re}(\eta) \le 1$ for all $\eta \in \sigma(M)$, the complex powers are well-defined and the series is absolutely and uniformly convergent on any compact subset of Re(s) > 1. As above, $\zeta_B(z,s)$ may be regarded as the zeta function of the operator $B_1 + (z-1)$. Furthermore, $\zeta_B(z,s)$ can be analytically continued to a meromorphic function of $s \in \mathbf{C}$ which is holomorphic at $s=0$ and the same method that we used to prove Proposition 6.16 gives

$$\frac{d}{ds}\zeta_B(z,s)\big|_{s=0} = \frac{d}{ds}\zeta_\Delta(z,s)\big|_{s=0} - \frac{\mathrm{Area(M)}}{8\pi}(2z-1)^2 + \frac{3\pi\gamma m}{2}(2z-1). \tag{6.20}$$

Set

$$\det(B_1 + (z-1)) = \exp\Big(-\frac{\partial}{\partial s}\zeta_B(z,s)\big|_{s=0}\Big).$$

Then (6.20) leads to the following relation between the two determinants

Proposition 6.21. *We have the identity*

$$\det(\Delta + z(z-1)) = e(z)\ \det(B_1 + (z-1))$$

where

$$e(z) = \exp\Big(\frac{\mathrm{Area(M)}}{8\pi}(2z-1)^2 - \frac{3\pi\gamma m}{2}(2z-1)\Big).$$

Now let $\Gamma\backslash H$ be a hyperbolic surface of finite area. Recall that the Selberg zeta function of $\Gamma\backslash H$ is defined as

$$Z(s) = \prod_{\{\gamma\}} \prod_{k=0}^{\infty} \left(1 - e^{-l(\gamma)(s+k)}\right), \quad \mathrm{Re(s)} > 1,$$

where the $\{\gamma\}$ run through the primitive hyperbolic conjugacy classes in Γ and $l(\gamma)$ denotes the length of the closed geodesic on $\Gamma\backslash H$ which is determined by $\{\gamma\}$(cf. [H3], [Sel],[Mc]).

The Selberg zeta function $Z(s)$ has a meromorphic continuation to $\mathbf{C}$. I.Efrat has shown how the determinant $\det(\Delta + z(z-1))$ is related to $Z(s)$ [E1], [E2], and therefore, we can also express $\det(B_1 + (z-1))$ by the Selberg zeta function. We simply have to use (1.7) in [E1] combined with Proposition 6.21. However note that in the corrected version of (1.7) in [E2] one has to use the determinant defined by (6.19) multiplied by $(z-1/2)^{(m-\mathrm{Tr}C(1/2))}$. Using these remarks we get

Theorem 6.22. *We have*

$$\det{}^2(B_1 + (z-1)) = \phi(z)\; Z(z)^2\; Z_\infty(z)^2\; \Gamma(z+1/2)^{-2m}\; e^{c(2z-1)+d}$$

where

$$c = m(3\pi\gamma - \log 2)$$
$$d = \frac{\mathrm{Area(M)}}{\pi}(2\zeta'(-1) - \log(\sqrt{2\pi})) + 2m\;\log\sqrt{2\pi}$$

Here ζ denotes the Riemann zeta function and

$$Z_\infty(s) = \left((2\pi)^s\; \Gamma_2(s)^2/\Gamma(s)\right)^{\mathrm{Area(M)}/2\pi}$$

with $\Gamma_2(s)$ being the double Gamma function (cf.[Bar]).

Note that Theorem 6.22 implies that $\det^2(B_1 + (z-1))$ admits a meromorphic continuation to $\mathbf{C}$.

7. The length spectrum and the resonance set

In this section we consider only hyperbolic surfaces $\Gamma\backslash H$. Recall that the *length spectrum* of $\Gamma\backslash H$ is defined as follows: Each hyperbolic conjugacy class $\{\gamma\}$ in Γ determines a unique closed geodesic in $\Gamma\backslash H$ whose length $l(\gamma)$ is given by

$$l(\gamma) = 2\ \cosh^{-1} \tfrac{1}{2}\mathrm{tr}(\gamma).$$

Note that $l(\gamma) = \log N(\gamma)$ where $N(\gamma) \in (1,\infty)$ is the norm of γ. The length spectrum is then by definition the set of all $l(\gamma)$ where $\{\gamma\}$ runs over all hyperbolic conjugacy classes in Γ. Each $l(\gamma)$ is counted with multiplicity $m(\gamma)$ which is the number of different hyperbolic conjugacy classes with length $l(\gamma)$. As pointed out by Selberg [Sel] and [Hu], for a compact hyperbolic surface $\Gamma\backslash H$ the Selberg trace formula has the following important consequence:

(7.1) *The eigenvalues of Δ determine the length spectrum of $\Gamma\backslash H$ and vice versa.*

We shall employ the trace formula (5.32) to establish a similar result for a non compact hyperbolic surface $\Gamma\backslash H$ of finite area. The role of the eigenvalues is now played by the resonance set $\sigma(\Gamma)$ and the statement corresponding to (7.1) is the following

Proposition 7.2. *Let $\Gamma\backslash H$ be a hyperbolic surface of finite area. The resonance set $\sigma(\Gamma)$ determines the length spectrum of $\Gamma\backslash H$ and vice versa. Moreover, the length spectrum also determines* $\mathrm{Area}(\Gamma\backslash H)$ *and the number m of cusps of $\Gamma\backslash H$.*

Proof. We apply (5.32) to $g(y) = \exp(-y^2/4t)$. By Proposition 5.24, $\mathrm{Area}(\Gamma\backslash H)$ and m are determined by $\sigma(\Gamma)$. Therefore, $\sigma(\Gamma)$ determines also

$$\frac{\log q}{4\pi} + \sum_{k=1}^{\infty} \sum_{\{\gamma\}} \frac{l(\gamma)}{\sinh \frac{1}{2} l(\gamma^k)}\ e^{-|l(\gamma^k)|^2/4t}$$

and , as explained in section 3.4 of [Mc], from the latter one can determine $\log q$ and the length spectrum. To prove the converse we apply the trace formula to the function

$$g(y) = |y|^{n-1}\ e^{-s|y|}, \quad n > 2, \mathrm{Re}(s) > 1/2.$$

Note that g is not smooth at the origin. However, in the same way as above, we can extend the trace formula to cover this case. Then (5.32), extended to g, gives

$$\sum_{\eta\in\sigma(\Gamma)} (s+1/2-\eta)^{-n} = \frac{\mathrm{Area}(\Gamma\backslash H)}{4\pi} \int_{-\infty}^{\infty} \left\{(s-i\lambda)^{-n} + (s+i\lambda)^{-n}\right\}\ \lambda\ \tanh(\pi\lambda)\ d\lambda$$

$$(7.3) \qquad + \sum_{\{\gamma\}} \frac{\log N(\gamma_0) \log^{n-1} N(\gamma)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}}\ N(\gamma)^{-s}$$

$$-\frac{m}{2\pi} \int_{-\infty}^{\infty} \left\{(s-i\lambda)^{-n} + (s+i\lambda)^{-n}\right\}\ \frac{\Gamma'}{\Gamma}(1+i\lambda)\ d\lambda.$$

Here $\{\gamma\}$ runs over all hyperbolic conjugacy classes in Γ and $\{\gamma_0\}$ denotes the primitive conjugacy class associated to $\{\gamma\}$. The series over $\{\gamma\}$ is convergent for $\mathrm{Re}(s) > 1/2$ (cf. p.429 in [H3]). Since $n > 2$, the left hand side is absolutely convergent by Proposition 5.25 and the convergence is uniform for s in any compact subset of $\mathbf{C}$ which contains no points of $\sigma(\Gamma)$. Therefore, the left hand side is a meromorphic function of $s \in \mathbf{C}$ whose poles are the points $\eta - 1/2, \eta \in \sigma(\Gamma)$. By shifting the contour of integration appropriately it follows that the first and the third term on the right hand side also admit meromorphic continuations to $\mathbf{C}$ with poles at $k + 1/2, k \in \mathbf{Z}$, and $k, k \in \mathbf{Z}$, respectively. Since $\sigma(\Gamma)$ is contained in $\mathrm{Re}(s) < 2$, this implies that the length spectrum determines $\mathrm{Area}(M)$ and m. Hence it also determines the meromorphic function on the left hand side of (7.3) and therefore its pole divisor, i.e., $\sigma(\Gamma)$. Q.E.D.

8.Moduli and the spectrum

Let $\Gamma\backslash H$ be a compact hyperbolic surface. I.M.Gelfand raised in [G] the following question:

To what extent is the surface $\Gamma\backslash H$ determined by the eigenvalues of its Laplace operator?

H.P.McKean [Mc] has shown that a compact hyperbolic surface is determined by its eigenvalue spectrum up to finitely many possibilities. Then S.Wolpert [W] proved that in the Teichmüller space $\mathcal{T}_h$ of compact Riemann surfaces of genus h there exists a proper real analytic subvariety $\mathcal{V}_h$ which is invariant under the extended Teichmüller modular group Γ_h such that a surface $\Gamma\backslash H \in \mathcal{T}_h/\Gamma_h$ is uniquely determined by its eigenvalue spectrum if and only if $\Gamma\backslash H$ is contained in the complement of $\mathcal{V}_h/\Gamma_h$. Thus a generic compact hyperbolic surface is uniquely determined by its eigenvalue spectrum. There exist also examples of non-isometric isospectral surfaces [V], [Su]. Actually, T.Sunada [Su] has shown that $\mathcal{V}_h$ has positive dimension for special values of h.

In this section we shall briefly dicuss the same problem for noncompact hyperbolic surfaces $\Gamma\backslash H$ of finite area. The question is then:

To what extent is the surface $\Gamma\backslash H$ determined by its resonance set $\sigma(\Gamma)$?

We do not intend to answer this question in all generality in the present paper, but we shall return to this point in a forthcoming paper. Here we shall extend McKean's result to the noncompact case.

By Proposition 5.23 and Proposition 7.2, the resonance set $\sigma(\Gamma)$ determines the conformal type (g,m) of the surface $\Gamma\backslash H$ and its length spectrum, or what is the same, the common absolute trace $|tr(\gamma)|$ of the elements of each hyperbolic conjugacy class $\{\gamma\}$ in Γ. Now we recall some facts from the theory of Fricke and Klein [FK]. Note that , by assumption, Γ is a torsion free discrete subgroup of $SL(2,\mathbf{R})$ of cofinite area. A *standard set of generators* for Γ consists of hyperbolic elements $A_i, B_i \in \Gamma, i = 1,...,g$ and parabolic elements $D_j \in \Gamma, j = 1,...m$, which satisfy the single relation

$$D_m \cdots D_1 B_g^{-1} A_g^{-1} B_g A_g \cdots B_1^{-1} A_1^{-1} B_1 A_1 = 1 \tag{8.1}$$

(cf. [Ke]). Then we have the following result which is due to Fricke and Klein if $m = 0$:

Theorem 8.2. *Let Γ be a torsion free discrete subgroup of $SL(2,\mathbf{R})$ of cofinite area. Let (g,m) be the conformal type of $\Gamma\backslash H$ and choose a standard set of generators $\gamma_1,\ldots,\gamma_{2g+m}$ for Γ. Then the single , double and triple traces*

$$\mathrm{tr}(\gamma_i), \mathrm{tr}(\gamma_i\gamma_j), i<j, \mathrm{tr}(\gamma_i\gamma_j\gamma_k), i<j<k, \quad i,j,k = 1,...,2g+m,$$

determine Γ up to a conjugation in $SL(2,\mathbf{R})/\{\pm 1\}$ and a possible reflection

$$\Gamma \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proof. Let Γ and Γ' be two torsion free discrete subgroups of $SL(2,\mathbf{R})$ with cofinite area and of conformal type (g,m). If $g \geq 1$ then we just repeat the proof given by McKean for the compact case (cf. pp.243-244 in [Mc]). It works equally well for $m \neq 0$. What remains is the case $g = 0$. Let$\gamma_1, ..., \gamma_m$ (resp. $\gamma_1', ..., \gamma_m'$) be a set of standard generators for Γ (resp. Γ'). All these elements are parabolic. Suppose that the single, double and triple traces of corresponding elements coincide. After a couple of conjugations, we may assume that

$$\gamma_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \gamma_1'.$$

Let

$$\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \quad \text{and} \quad \gamma_i' = \begin{pmatrix} a_i' & b_i' \\ c_i' & d_i' \end{pmatrix}.$$

Then, for $i > 1$, we have

$$a_i + d_i = 2 = a_i' + d_i'. \tag{8.3}$$

and

$$\mathrm{tr}(\gamma_1\gamma_i) = a_i + d_i + c_i = \mathrm{tr}(\gamma_1'\gamma_i') = a_i' + d_i' + c_i'.$$

Note also that $c_i \neq 0$ for $i > 1$. Hence

$$c_i = c_i' \quad \text{and} \quad c_i \neq 0, i > 1. \tag{8.4}$$

Set

$$k = \begin{pmatrix} 1 & -a_2/c_2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad k' = \begin{pmatrix} 1 & -a_2'/c_2' \\ 0 & 1 \end{pmatrix}.$$

Then, by (8.4),

$$k\gamma_2 k^{-1} = \begin{pmatrix} 0 & -1/c_2 \\ c_2 & 2 \end{pmatrix} = k'\gamma_2' k'^{-1}.$$

Hence, after another couple of conjugations, we have

$$\gamma_1 = \gamma_1' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_2 = \gamma_2' = \begin{pmatrix} 0 & -1/x \\ x & 2 \end{pmatrix}$$

for some $x \neq 0$. Then, for $i > 2$,

$$\mathrm{tr}(\gamma_2\gamma_i) = -\frac{c_i}{x} + b_i x + 2d_i = \mathrm{tr}(\gamma_2'\gamma_i') = -\frac{c_i'}{x} + b_i' + 2d_i'$$

and, by (8.4),

$$(b_i - b_i')x = 2(d_i' - d_i). \tag{8.5}$$

Furthermore, we also have

$$\mathrm{tr}(\gamma_1\gamma_2\gamma_i) = (a_i + b_i)x - \frac{c_i}{x} + 2c_i + 2d_i = \mathrm{tr}(\gamma_1'\gamma_2'\gamma_i') = (a_i' + b_i')x - \frac{c_i'}{x} + 2c_i' + 2d_i'.$$

By (8.3) and (8.4), we get

$$(b_i - b_i')x = (2 - x)(d_i' - d_i).$$

Since $x \neq 0$, this equality combined with (8.5) implies $b_i = b_i', d_i = d_i', i > 2$. Thus $\gamma_i = \gamma_i'$ for all i. Q.E.D.

Actually a smaller number of traces suffices to determine Γ up to conjugation and/or a reflection. This is a consequence of the following remarkable result proved by Fricke and Klein (cf. pp.366 in [FK]):

Proposition 8.6. *Let $g_1, g_2, g_3 \in SL(2, \mathbf{R})$. Then the triple trace $t_{123} = \mathrm{tr}(g_1 g_2 g_3)$ is an algebraic function of degree two of the single and double traces $t_i = \mathrm{tr}(g_i)$ and $t_{ij} = \mathrm{tr}(g_i g_j), i < j$.*

McKean's proof of the fact that a compact hyperbolic surface $\Gamma\backslash H$ is determined by the spectrum of its Laplacian up to finitely many possibilities depends on the following bound due to D.Mumford [Mf]:

$$\mathrm{diam}(\Gamma\backslash H) \cdot \min\{l(\gamma) \mid \gamma \in \Gamma \ \ hyperbolic\} \leq \mathrm{Area}(\Gamma\backslash H)$$

where $\mathrm{diam}(\Gamma\backslash H)$ denotes the diameter of the surface $\Gamma\backslash H$. This bound is not available in the noncompact case and we have to find an appropriate substitute.

Let $p \in \mathbf{R} \cup \{\infty\}$ be a parabolic fixed point of Γ and let Γ_p be the stabilizer of p in Γ. Let U_p be the domain interior to a horocycle through p, chosen so that $\Gamma_p\backslash U_p$ has area equal to 1. If Γ_p is generated by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then

$$U_p = \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 1\}.$$

It is known that two points in U_p are Γ equivalent only if they are Γ_p equivalent. Hence $\Gamma_p\backslash U_p$ is isometric to a subset of $\Gamma\backslash H$. Furthermore, if $p, q \in \mathbf{R} \cup \{\infty\}$ are two different parabolic fixed points of Γ then

$$U_p \cap U_q = \emptyset. \tag{8.7}$$

This can be seen as follows. After conjugation we may assume that $p = \infty$ and Γ_p is generated by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. If $\gamma \in \Gamma - \Gamma_p$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then γ does not fix ∞, hence $c \neq 0$. But then we know that c must satisfy $|c| \geq 1$ (cf. p.58 in [Kr]) which implies (8.7).

Let $\Omega(\Gamma)$ denote the complement in H of the union of all U_p where p runs over the parabolic fixed points of Γ. Then $\Omega(\Gamma)$ is a closed Γ invariant subset of H. Set

$$M_\Gamma = \Gamma\backslash\Omega(\Gamma).$$

This is a compact hyperbolic surface with boundary. The following lemma is a consequence of Lemma 5 in [Be]:

Lemma 8.8. *There exists a constant $c > 0$ which depends only on the conformal type (g, m) of $\Gamma\backslash H$ such that*

$$\operatorname{diam}(M_\Gamma) \cdot \min\{l(\gamma) \mid \gamma \in \Gamma \text{ } hyperbolic\} \leq c.$$

Lemma 8.9. *Let $\gamma_1, ..., \gamma_{2g+m}$ be a standard set of generators for Γ. Then the absolute values of the single, double and triple traces $\operatorname{tr}(\gamma_i)$, $\operatorname{tr}(\gamma_i\gamma_j)$ and $\operatorname{tr}(\gamma_i\gamma_j\gamma_k)$ are bounded by*

$$6 \cosh(c/\min l(\gamma))$$

where c is the same constant as in Lemma 8.8.

Proof. The generators $\gamma_1, ..., \gamma_{2g+m}$ of Γ are connected with a distinguished fundamental domain D for Γ called Fricke polygon (cf. [Ke]). The polygon D is bounded by $4g + 2m$ geodesic arcs in H and, if the sides of D are suitably labeled in order, say

$$a_1, b_1, a_1', b_1', ..., a_g, b_g, a_g', b_g', c_1, c_1', ..., c_m, c_m'$$

then

$$\gamma_1(a_1) = -a_1', \gamma_2(b_1) = -b_1', ..., \gamma_{2g}(b_g) = -b_g', \gamma_{2g+1}(c_1) = -c_1', ..., \gamma_{2g+m}(c_m) = -c_m'.$$

Let $q_1, ..., q_m \in \mathbf{R} \cup \{\infty\}$ denote the fixed points of the parabolic elements $\gamma_{2g+1}, ..., \gamma_{2g+m}$ and set

$$D' = D - \bigcup_{i=1}^{m}(U_p \cap D).$$

Then D' is a fundamental domain for Γ acting on $\Omega(\Gamma)$. Each generator γ_i maps D' to an adjacent fundamental domain $\gamma_i(D')$. This shows that $|\operatorname{tr}(\gamma_i)|, |\operatorname{tr}(\gamma_i\gamma_j)|$ and $|\operatorname{tr}(\gamma_i\gamma_j\gamma_k)|$ are bounded by $6 \cosh(\operatorname{diam}(M_\Gamma))$ and our estimate follows from Lemma 8.8.Q.E.D.

We are now ready to prove the main result of this section

Theorem 8.10. *Let $\Gamma\backslash H$ be a hyperbolic surface of finite area. Then the resonance set $\sigma(\Gamma)$ determines $\Gamma\backslash H$ up to finitely many possibilities.*

Proof. The resonance set determines the conformal type (g, m) of $\Gamma\backslash H$ and the length spectrum of $\Gamma\backslash H$. Hence it determines the numbers

$$|\operatorname{tr}(\gamma)| = 2 \cosh \frac{l(\gamma)}{2}$$

for all hyperbolic elements γ in Γ. Let $\gamma_1, ..., \gamma_{2g+m}$ be a standard set of generators for Γ. Since Γ is torsion free each of the products γ_i, $\gamma_i\gamma_j$, $\gamma_i\gamma_j\gamma_k$ is either parabolic or hyperbolic. Hence we know $|\operatorname{tr}(\gamma_i)|$, $|\operatorname{tr}(\gamma_i\gamma_j)|$, $|\operatorname{tr}(\gamma_i\gamma_j\gamma_k)|$ and those which are > 2 are of the form $2\cosh(l(\gamma)/2)$ for some hyperbolic element $\gamma \in \Gamma$. Now recall that for every $C > 0$ there exist only finitely many hyberbolic conjugacy classes $\{\gamma\}$ in Γ with $l(\gamma) \leq C$ (cf. p.475 in [H3]). Since $\min l(\gamma) = l_1$ is fixed, it follows from Lemma 8.9 that the set of possible values for the single , double and triple traces is finite . The theorem follows from Theorem 8.2. Q.E.D.

If we recall the definition of the resonance set $\sigma(\Gamma)$, then Theorem 8.10 can be restated as follows:

A hyperbolic surface $\Gamma\backslash H$ of finite area is determined , up to finitely many possibilities, by the following numbers:

1) The eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ of the Laplace operator Δ_Γ on $\Gamma\backslash H$.

2) The poles of the determinant $\phi(s)$ of the scattering matrix together with their orders.

3) $\mathrm{Tr}(C(1/2))$ which occurs in the multiplicity of the special point $1/2$.

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