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Higher Spherical Polynomials

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Contents

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Abstract. In this paper, we study a new class of special functions. Specifically, we study a vector space $\mathcal{P}^{(n)}(d)$ $(n \in \mathbb{N},\ d \in \mathbb{C})$ of polynomials $P(T)$ in $n \times n$ symmetric matrices $T = (t_{ij})$. For integral $d \geq n$ these are the polynomials on $(\mathbb{R}^d)^n$ that are invariant with respect to the diagonal action of $O(d)$ and harmonic on each $\mathbb{R}^d$, and for general d they are the polynomial solutions of a certain system of differential equations. For $n = 2$ these are the classical Legendre and Gegenbauer polynomials, but for $n > 2$ they are new and are interesting for several reasons, including an application to the theory of Siegel modular forms and the fact that for $n = 3$ the associated system of differential equations is holonomic. When $n = 3$, each homogeneous component of $\mathcal{P}^{(n)}(d)$ is one-dimensional, so there is a canonical basis. Even here the structure of the polynomials turns out to be very subtle. For $n > 3$ there is no obvious basis. We construct two canonical bases, dual to one another, and a generating function (generalizing the classical one for $n = 2$, and algebraic if $n = 3$ but not in general) for one of them. We also provide tables that illustrate some of the idiosyncracies of the theory.

Introduction

For two natural numbers n and d and a scalar product $(\ ,\ )$ on $\mathbb{R}^d$, we have a map β_n from $(\mathbb{R}^d)^n$ to the space $\mathcal{S}_n$ of real symmetric $n \times n$ matrices, given by

$$\beta_n : \quad (x_1, \dots, x_n) \mapsto T = (t_{ij})_{1 \leq i,j \leq n}\,, \quad t_{ij} = (x_i, x_j)\,. \tag{1}$$

Thus any polynomial P on $\mathcal{S}_n$ defines a polynomial $\widetilde{P} = P \circ \beta_n$ on $(\mathbb{R}^d)^n$. If $d \geq n$, then we denote by $\mathcal{P}^{(n)}(d)$ the space of polynomials P on $\mathcal{S}_n$ for which $\widetilde{P}$ is harmonic with respect to x_i for each $i = 1, \dots, n$. For each multidegree $\mathbf{a} = (a_1, \dots, a_n) \in (\mathbb{Z}_{\geq 0})^n$ we denote by $\mathcal{P}_{\mathbf{a}}(d)$ the subspace of $P \in \mathcal{P}^{(n)}(d)$ for which $\widetilde{P}$ is a homogeneous of degree a_i with respect to x_i for each $i = 1, \dots, n$. (Here the superscript "(n)" can be omitted since n is just the length of $\mathbf{a}$.) Later in the paper we will generalize the definitions of both $\mathcal{P}^{(n)}(d)$ and $\mathcal{P}_{\mathbf{a}}(d)$ to arbitrary complex values of d. The elements of $\mathcal{P}^{(n)}(d)$ will be called *higher spherical polynomials.* There are several motivations for studying them:

(i) If $n = 2$, then $\mathcal{P}_{(a_1,a_2)}(d)$ is one-dimensional if $a_1 = a_2$ (it is 0 otherwise) and is then spanned by a polynomial in $t_{11}t_{22}$ and t_{12} which is the homogeneous version of one of the classical families of orthogonal polynomials (Legendre polynomials if $d = 3$, Chebyshev polynomials if $d = 2$ or $d = 4$, and Gegenbauer polynomial for d arbitrary). These classical polynomials occur in many places in mathematics and mathematical physics and have many nice properties: differential equations, orthogonality properties, recursions, generating functions, closed formulas, etc. It is natural to try to generalize these.

(ii) The polynomials in $\mathcal{P}^{(n)}(d)$ have an application in the theory of Siegel modular forms. This is described in detail in [7] (similar differential operators are given in [2] and [4]) and will not be pursued in this paper, but was our original reason for studying this particular space of polynomials. Briefly, the connection is as follows. If $P \in \mathcal{P}_{\mathbf{a}}(d)$ ($d \geq n$) and if

$$F(Z) = \sum_T c(T)\, e^{2\pi i\, \mathrm{Tr}(TZ)} \qquad (Z \in \mathfrak{H}_n = \text{Siegel upper half-space})$$

is a holomorphic Siegel modular form of weight $d/2$, then the function

$$P\Big(\frac{1+\delta_{ij}}{2}\,\frac{\partial}{\partial z_{ij}}\Big)F(Z)\Big|_{\mathfrak{H}_1^n} = \sum_T c(T)\, P(T)\, e^{2\pi i(t_{11}z_1+\ldots+t_{nn}z_n)} \qquad (z_1,\ldots,z_n \in \mathfrak{H}_1) \tag{2}$$

is an elliptic modular form of weight $d/2+a_i$ in z_i for each $i=1,\ldots,n$. (By considering images $g(\mathfrak{H}^n) \subset \mathfrak{H}_n$, where g is a suitable element of $\mathrm{Sp}_{2n}(\mathbb{R})$, one also gets maps from Siegel modular forms of degree n to Hilbert modular forms on totally real number fields of degree n, or to products of Hilbert modular forms on several totally real fields with degrees adding up to n.) We can understand why this should be true by taking F to be a Siegel theta series, say

$$F(Z) = \sum_{(x_1,\ldots,x_n)\in L^n} e^{2\pi i\, \mathrm{Tr}(\,\beta_n(x_1,\ldots,x_n)\,Z)}$$

where $L \subset \mathbb{R}^d$ is a lattice on which the quadratic form (x,x) takes rational values. Then the function defined by (2) is the series

$$\sum_{x_1\in L}\cdots\sum_{x_n\in L} \widetilde{P}(x_1,\ldots,x_n)\; q_1^{(x_1,x_1)}\ldots q_n^{(x_n,x_n)} \qquad (q_i = e^{2\pi\sqrt{-1}\,z_i})$$

and for $P \in \mathcal{P}_{\mathbf{a}}(d)$ this is a modular form of weight $d/2+a_i$ in z_i by the classical Hecke-Schoeneberg theory of theta series with harmonic polynomial coefficients. A deeper direct application of our theory to critical values of triple L function is found in [10] (see also [5]) and another direct application is given in [3]. There are various other theories of differential operators and applications of a similar kind (for example, see [8], [9], [11]).

(iii) The spaces $\mathcal{P}_{\mathbf{a}}(d)$ can be looked at from the point of view of representation theory. Specifically, the map $P \mapsto \widetilde{P}$ gives an isomorphism between $\mathcal{P}_{\mathbf{a}}(d)$ and the space of "invariant harmonic polynomials," i.e., polynomials in n variables $x_i \in \mathbb{R}^d$ which are harmonic and homogeneous (of degree a_i) in each x_i and invariant under the simultaneous action of $O(d)$, the orthogonal group of the scalar product $(\ ,\ )$, on all x_i; this space can be analyzed in terms of the classical representation theory of $O(d)$. However, apart from a brief mention of this in connection with the dimension formula proved in §2, we will not use this interpretation, but will work directly with the functions P on $\mathcal{S}_n$, since this is more elementary, gives more explicit results, and is applicable in greater generality.

(iv) If $n=3$, then the dimension of $\mathcal{P}_{\mathbf{a}}(d)$ is always 0 or 1, so that the higher spherical polynomials are particularly canonical in this case. The higher spherical polynomials of varying multidegree give an orthogonal basis of L^2 functions in three variables with respect to a certain scalar product, and the coefficients of these polynomials also turn out to be combinatorially very interesting expressions, including as special cases the classical $3j$-symbol (or Clebsch-Gordan coefficients) of quantum mechanics.

(v) In a sequel to this paper [12], we will show that the system of differential equations defining the higher spherical polynomials (generalized Legendre differential equations) in the case $n=3$ is equivalent to an integrable Pfaffian system of rank 8, so that there are precisely eight linearly independent solutions, generalizing the Legendre functions of the first and second kinds in the classical case $n=2$. In the general theory of differential equations, such systems are very rare: systems of linear partial differential equations with polynomial coefficients generally have a solution space which is either zero- or infinite-dimensional, and it is non-trivial to construct Pfaffian systems satisfying the necessary integrability conditions. The system arising from the study of higher spherical polynomials is therefore also interesting from this point of view.

The contents of the paper are as follows. Chapter I describes the general theory, for arbitrary n and d. In the first two sections, which are basic for everything that follows, we write down the differential equations defining the space $\mathcal{P}_n(d)$ (now for any $d \in \mathbb{C}$), prove a formula for the dimensions of its homogeneous pieces $\mathcal{P}_{\mathbf{a}}(d)$, and construct a specific basis $P_{\boldsymbol{\nu}}^M$ ("monomial basis") characterized by the property that its restriction to the subspace $t_{11} = \cdots = t_{nn} = 0$ consists of monomials. We also give a brief description of the behavior of $\mathcal{P}_n(d)$ at certain special values of d (in $2\mathbb{Z}_{\le 0}$) where its dimension changes. Sections 3 and 4 contain the definition of a scalar product with respect to which the various spaces $\mathcal{P}_{\mathbf{a}}(d)$ are mutually orthogonal, generalizing one of the basic properties of the classical polynomials for $n = 2$, and of a Lie algebra $\mathfrak{g}$, isomorphic to $\mathfrak{sp}(n, \mathbb{R})$, of differential operators on polynomials on $\mathcal{S}_n$. This is then applied in §5 to construct two further canonical bases of $\mathcal{P}_n(d)$, the "ascending" and the "descending" basis, whose elements are inductively defined by the property that two adjacent basis elements are related by the action of a specific element of the universal enveloping algebra of $\mathfrak{g}$. The first of these two bases is proportional in each fixed multidegree to the monomial basis, and the two bases are mutually dual with respect to the canonical scalar product. Finally, in §6 we study the situation when d is a positive integer less than n and describe the relationship between the spaces $\mathcal{P}_n(d)$ and $\{P \mid \widetilde{P}$ is harmonic in each $x_i\}$, which now no longer coincide, in this case.

Chapter II is devoted to explicit constructions of higher spherical polynomials. In §7 we use the relationship between $O(4)$ and $GL(2)$ to construct specific elements of $\mathcal{P}_n(d)$ when $d = 4$ and show that they can be obtained as the coefficients of an explicit algebraic generating function. The next section, which is quite long, concerns the case $n = 3$. This case is especially interesting both because it is the first one beyond the classical case and because the dimension of $\mathcal{P}_{\mathbf{a}}(d)$ here is always 0 or 1, so that the polynomials are especially canonical. Specifically, the dimension is 1 if and only if $a_1 + a_2 + a_3 - 2\max\{a_1, a_2, a_3\}$ is even and non-negative, i.e., if we can write

$$a_1 = \nu_2 + \nu_3, \qquad a_2 = \nu_1 + \nu_3, \qquad a_3 = \nu_1 + \nu_2 \tag{3}$$

for some triple $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ of integers $\nu_i \ge 0$, which we then call the *index* of the polynomial. In Subsection **A** we generalize the generating function given for $d = 4$ in §7 (now specialized to $n = 3$) to construct an explicit generating function for $n = 3$ and generic values of $d \in \mathbb{C}$ whose coefficients give polynomials $P_{\boldsymbol{\nu}}$ generating the 1-dimensional space $\mathcal{P}_{\mathbf{a}}(d)$. The next three subsections give explicit formulas for the scalar products of these special polynomials, their relations to the three general bases (monomial, ascending and descending), the recursion relations satisfied by their coefficients, and for the coefficients themselves, while Subsection **E** discusses the geometry and symmetry of the 3-dimensional space on which these polynomials naturally live. Section 9 contains the construction of generating functions for arbitrary n and d whose coefficients span the spaces of higher spherical polynomials of any given multidegree (and in fact turn out to be proportional to the elements of the descending basis). We also describe how these coefficients can be obtained inductively by the repeated application of certain differential operators and give a number of special cases and explicit formulas. Finally, in §10 we study the bases of higher spherical polynomials coming from these generating functions in the case $n = 4$ and show that the generating functions themselves, which were given by algebraic expressions for $n \le 3$, now usually are *not* algebraic, but that for even integral values of $d \ge 4$ they become algebraic after being differentiated a finite number of times. The paper ends with some tables of higher spherical polynomials and their scalar products for $3 \le n \le 5$, revealing many interesting regularities and irregularities of these objects.

A list of the principal notations used (omitting standard ones like $\mathbb{N} = \mathbb{Z}_{>0}$) is given at the end of the paper. Theorems are numbered consecutively throughout the paper, but propositions, proposition-definitions, lemmas and corollaries are numbered separately in each section, and not numbered at all if there is only one of them in a given section. Readers who are primarily interested in the $n = 3$ case can read just §8, in which we have repeated a few definitions and notations so that it can be read independently of the preceding sections.

Finally, we should perhaps also say a few words about the history of this paper, since it has been in preparation for a very long time and earlier versions circulated for several years and were quoted by several authors. The project originally started well over twenty years ago with a letter sent by the second author to B. Gross and S. Kudla in connection with a plan (later abandoned) to compute the arithmetic height pairings of modularly embedded curves in the modular 3-fold $(\mathfrak{H}/SL(2,\mathbb{Z}))^3$. This letter concerned spherical polynomials on $GL(2)\backslash GL(2)^3/GL(2)$, corresponding to the case $n = 3$, $d = 4$ of the present paper, and gave for that case the generating function for the polynomials and some explicit calculations of their coefficients (now given in Sections 7 and 8). During the second author's visits to Kyushu in the early 1990's the first author suggested the formulation in terms of orthogonal groups and the generalization to arbitrary n and d, developed a more general theory (later published separately in [7]), and obtained a number of the results given in this paper, including the generating function for $n = 3$ and arbitrary d. The preliminary version mentioned above was primarily concerned with the case $n = 3$ and corresponded to parts of Sections 1–3 and all of Sections 7 and 8, while the remaining results in Sections 2 and 3 and all the material in Sections 4, 5, 6, 9 and 10 was found and written up much later. We delayed publication for many years as new results for higher n emerged, in order to be able to present a more complete and coherent picture.

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§1. The generalized Legendre differential equation

The definition of $\mathcal{P}_{\mathbf{a}}(d)$ which was given in the introduction makes sense only if d is a positive integer and gives a finite-dimensional space only if $d \ge n$, since for $d < n$ the map $P \mapsto \widetilde{P}$ is not injective (because the (x_i, x_j) are then algebraically dependent, the relations being the vanishing of all $(d+1) \times (d+1)$ minors of T). In this section we generalize the definition of $\mathcal{P}_{\mathbf{a}}(d)$, first to $d < n$ and then even to complex values of d, by thinking of the elements of $\mathcal{P}_{\mathbf{a}}(d)$ as polynomial solutions of a certain system of differential equations generalizing the classical Legendre equation. This system of differential equations will be used in §2 to calculate the dimension of $\mathcal{P}_{\mathbf{a}}(d)$. As illustrations of the general situation we discuss the two examples $n = 2$, $\mathbf{a} = (a, a)$, and $n = 4$, $\mathbf{a} = (2, 2, 2, 2)$ in some detail. At the end of the section we also give a brief discussion of the inhomogeneous version of the higher spherical polynomials and their differential equations.

First, we must say something about the coordinates on the space $\mathcal{S}_n$. Of course we could use the $n(n+1)/2$ independent numbers t_{ij} with $1 \le i \le j \le n$, but this does not respect the symmetry. Instead, we will use all n^2 components t_{ij} of $T \in \mathcal{S}_n$ as variables, with $t_{ij} = t_{ji}$, and will write elements[1] of $\mathbb{C}[\mathcal{S}_n]$ symmetrically as polynomials in the variables $\sqrt{t_{ij}t_{ji}}$ $(= t_{ij})$, i.e. we take as a basis for $\mathbb{C}[\mathcal{S}_n]$ the set of monomials $T^{\boldsymbol{\nu}} := \prod_{i,j=1}^{n} t_{ij}^{\nu_{ij}/2}$ with $\boldsymbol{\nu}$ ranging over the set

$$\mathcal{N} \;=\; \mathcal{N}_n \;=\; \bigl\{\boldsymbol{\nu} = (\nu_{ij})_{1 \le i,j \le n} \mid \nu_{ij} = \nu_{ji} \in \mathbb{Z}_{\ge 0}\,, \quad \nu_{ii} \equiv 0 \pmod 2\bigr\}$$

of even symmetric $n \times n$ matrices with non-negative entries. There is a canonical isomorphism $\mathbb{C}[\mathcal{S}_n] \cong \mathbb{C}^{\mathcal{N}}$ given by mapping a polynomial $\sum C_{\boldsymbol{\nu}}\, T^{\boldsymbol{\nu}}$ to its set of coefficients $\{C_{\boldsymbol{\nu}}\}$. For the differentiation operator with respect to t_{ij} we take $\partial_{ij} = (1 + \delta_{ij})\partial/\partial t_{ij}$. (Note: We had to do something similar in eq. (2) for the same reason, namely, that the coordinates z_{ij} are not independent variables. There we also included a normalizing factor 1/2, because for Siegel modular forms the diagonal elements are the most important ones and one wants a natural generalization of $n = 1$, but in our situation the non-diagonal entries of the matrix are equally important and we have chosen a normalization which preserves the lattice of polynomials with integral coefficients.) The action of ∂_{ij} on our chosen basis of $\mathbb{C}[\mathcal{S}_n]$ is given by $T^{\boldsymbol{\nu}} \mapsto \nu_{ij}\, T^{\boldsymbol{\nu} - \mathbf{e}_{ij}}$. Here $\mathbf{e}_{ij}$ denotes the $n \times n$ matrix with (a, b)-component $\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}$(i.e., the matrix with (i, j) and (j, i) entries equal to 1 and all other entries equal to 0 if $i \ne j$, and with (i, i)-entry 2 and all other entries 0 if $i = j$), so that $\mathcal{N}$ is the free abelian semigroup generated by the $\mathbf{e}_{ij}$.

We define the *multidegree* of the monomial $T^{\boldsymbol{\nu}}$ to be the vector $\boldsymbol{\nu}\cdot\mathbf{1} \in \mathbb{Z}_{\ge 0}^{n}$, where $\mathbf{1}$ is the vector of length n with all components equal to 1, and denote by $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ the subspace of $\mathbb{C}[\mathcal{S}_n]$ spanned by all monomials of given multidegree $\mathbf{a}$. Thus

$$\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \;\cong\; \mathbb{C}^{\mathcal{N}(\mathbf{a})} \qquad \bigl(\mathcal{N}(\mathbf{a}) := \{\boldsymbol{\nu} \in \mathcal{N} \mid \boldsymbol{\nu}\cdot\mathbf{1} = \mathbf{a}\}\bigr)$$

with respect to the above isomorphism $\mathbb{C}[\mathcal{S}_n] \cong \mathbb{C}^{\mathcal{N}}$, while more intrinsically

$$\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} = \bigl\{P \in \mathbb{C}[\mathcal{S}_n] \mid P(\lambda T \lambda) = \lambda^{\mathbf{a}} P(T) \quad \text{for all } \lambda \in \operatorname{diag}(\mathbb{C}^n)\bigr\}\,, \tag{4}$$

where $\lambda^{\mathbf{a}} = \prod \lambda_i^{a_i}$ for $\lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. For $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ and $(\mathbb{R}^d, (\ ,\ ))$ as in the introduction, the function $\widetilde{P} :\ (\mathbb{R}^d)^n \to \mathbb{C}$ has multidegree $\mathbf{a}$. Hence, if $d \ge n$ (so that the map $P \mapsto \widetilde{P}$ is

[1] Abusing notation slightly, we will use $K[\mathbb{R}^d]$ or $K[\mathbb{C}^d]$ (or more generally $K[V]$) to denote the algebra of polynomials in d variables (or more generally of polynomials on an arbitrary vector space V) with coefficients in a field K.

injective), then the space $\mathcal{P}_{\mathbf{a}}(d)$ as defined in the introduction is a subspace of the space $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$. Specifically, it is the space of all polynomials $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ for which the polynomial $\widetilde{P}$ (which is now automatically multihomogeneous of multidegree $\mathbf{a}$ in the x's) is harmonic with respect to each x_i. We would like to define $\mathcal{P}_{\mathbf{a}}(d)$ for all values of d as a subspace of $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$. To do this, we must express the condition that $\widetilde{P}$ is multiharmonic directly in terms of the polynomial P. The answer will be a system of differential equations with coefficients depending polynomially on d; these will then make sense for any complex number d.

The calculation is straightforward. The harmonicity conditions are $\Delta_i \widetilde{P} = 0$ for $1 \le i \le n$, where Δ_i denotes the Laplacian[2] with respect to the variable $x_i \in \mathbb{R}^d$ (and with respect to the chosen scalar product in $\mathbb{R}^d$). Let $x_{i\alpha}$ $(\alpha = 1, \dots, d)$ be the coordinates of x_i in a coordinate system for which the scalar product (and hence also the Laplacian) is the standard one. Then $t_{ij} = \sum_\alpha x_{i\alpha} x_{j\alpha}$, so

$$\frac{\partial \widetilde{P}}{\partial x_{i\alpha}} = \sum_{j=1}^{n} x_{j\alpha} \left(\partial_{ij} P\right)^{\sim},$$

with ∂_{ij} defined as above. Applying this formula a second time, we get

$$\frac{\partial^2 \widetilde{P}}{\partial x_{i\alpha}^2} = \left(\partial_{ii} P\right)^{\sim} + \sum_{j,k=1}^{n} x_{j\alpha}\, x_{k\alpha} \left(\partial_{ij}\partial_{ik} P\right)^{\sim},$$

and summing this over α gives $\Delta_i(\widetilde{P}) = (D_i P)^{\sim}$, where

$$D_i = D_i^{(d)} = d\, \partial_{ii} + \sum_{j,k=1}^{n} t_{jk}\, \partial_{ij} \partial_{ik} \qquad (1 \le i \le n)\,. \tag{5}$$

(Recall that $\partial_{ii} = 2\,\partial/\partial t_{ii}$.) We therefore have the following:

Proposition-Definition. *For $n \in \mathbb{N}$, $\mathbf{a} \in \mathbb{Z}^n_{\ge 0}$, and arbitrary $d \in \mathbb{C}$, set*

$$\mathcal{P}_{\mathbf{a}}(d) = \left\{ P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \mid D_1 P = \ldots = D_n P = 0 \right\}$$

with $D_i = D_i^{(d)}$ as in (5). *For integral $d \ge n$, this agrees with the definitions given in the introduction.*

Remark. The space $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ is itself the solution set of the system of linear differential equations $E_i P = a_i P$ $(1 \le i \le n)$, where E_i is the Euler operator $\sum_{j=1}^n t_{ij} \partial_{ij}$. Thus one could also define $\mathcal{P}_{\mathbf{a}}(d)$ as the subspace of $\mathbb{C}[\mathcal{S}_n]$ annihilated by the operators $D_i^{(d)}$ and $E_i - a_i$ for $1 \le i \le n$. We also observe that the spaces $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ and $\mathcal{P}_{\mathbf{a}}(d)$ are 0 unless the integer $\mathbf{a} {\cdot} \mathbf{1} = a_1 + \cdots + a_n$ (*total degree*) is even. From now on we will usually assume this.

Since the space $\mathcal{P}_{\mathbf{a}}(d)$ is always a subset of $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$, its dimension for any $d \in \mathbb{C}$ is at most $N(\mathbf{a}) := \#\mathcal{N}(\mathbf{a})$, the number of nonnegative even symmetric matrices with row sums $a_1, \dots, a_n$. We will show in §2 that (except possibly for d belonging to a finite set of nonpositive even integers) the dimension of $\mathcal{P}_{\mathbf{a}}(d)$ is independent of d and equals $N_0(\mathbf{a})$, the cardinality of the set $\mathcal{N}_0(\mathbf{a}) = \left\{ \boldsymbol{\nu} \in \mathcal{N}(\mathbf{a}) \mid \nu_{ii} = 0\ (\forall i) \right\}$. We illustrate this with two examples.

Example 1: $n = 2$. In this case the operator $t_{11} D_1^{(d)} - t_{22} D_2^{(d)}$ acts on polynomials of bidegree (a_1, a_2) as multiplication by $(a_1 - a_2)(a_1 + a_2 + d - 2)$, so if d is a positive integer, or

[2] Note that, since everything is purely algebraic, we could also work over $\mathbb{C}$ rather than $\mathbb{R}$, but the condition for an element of $\mathbb{C}[(\mathbb{C}^d)^n]$ to be harmonic in the ith variable $z_i = (z_{i1}, \dots, z_{id}) \in \mathbb{C}^d$ would still be with respect to the complexified real Laplacian $\sum_\alpha \partial^2/\partial z_{i\alpha}^2$, not the complex Laplacian $\sum_\alpha \partial^2/\partial z_{i,\alpha} \partial \bar{z}_{i,\alpha}$.

indeed any complex number except $2-a_1-a_2$, then $\mathcal{P}_{\mathbf{a}}(d)$ can be non-zero only if $\mathbf{a}=(a,a)$ for some $a\in\mathbb{Z}_{\ge 0}$, as was already mentioned in §1. If $\mathbf{a}=(a,a)$, then $\mathcal{N}(\mathbf{a})$ is the set of matrices $\left(\begin{smallmatrix} 2l & a-2l \\ a-2l & 2l\end{smallmatrix}\right)$ with $0\le l\le a/2$, so $N(\mathbf{a})=[a/2]+1$, while $\mathcal{N}_0(\mathbf{a})$ contains only the matrix $\left(\begin{smallmatrix} 0 & a \\ a & 0\end{smallmatrix}\right)$, so $N_0(\mathbf{a})=1$. If $P(T)=\sum_{l=0}^{[a/2]} c_l(t_{11}t_{22})^l(t_{12}t_{21})^{(a-2l)/2}$ is a polynomial in $\mathbb{C}[\mathcal{S}_2]$, then we find after a short calculation that each of the two equations $D_1^{(d)}P=0$ and $D_2^{(d)}P=0$ is equivalent to the recursion

$$4l(a-l+\frac{d}{2}-1)\,c_l+(a-2l+2)(a-2l+1)\,c_{l-1}=0 \qquad (1\le l\le [a/2]\,)$$

and that this recursion has a one-dimensional space of solutions for every complex number d, even though the recursion itself behaves slightly differently for certain exceptional values of d (namely, the even integers in the interval $[4-2a,2-a]$). Thus the dimension of $\mathcal{P}_{\mathbf{a}}(d)$ is always equal to $N_0(\mathbf{a})=1$, as claimed, with the generator of the space being the homogeneous form of a Gegenbauer polynomial (respectively a Legendre polynomial if $d=3$ or a Chebyshev polynomial of the first or second kind if $d=2$ or 4). We also see in this example that the dimension formula $\dim\mathcal{P}_{\mathbf{a}}(d)=N_0(\mathbf{a})$ fails in the exceptional case $d=2-a_1-a_2$, the simplest example being $\mathbf{a}=(2,0)$ and $d=0$, when $\mathcal{P}_{\mathbf{a}}(d)$ contains the polynomial t_{11} and hence has dimension $N(\mathbf{a})=1$, whereas $N_0(\mathbf{a})=0$. In fact, $\mathcal{P}_{(a_1,a_2)}(2-a_1-a_2)$ has dimension 1 for any $a_1,\ a_2\ge 0$ with $a_1\equiv a_2$ (mod 2), while $N_0(a_1,a_2)=0$ unless $a_1=a_2$.

Example 2: $n=4$**,** $\mathbf{a}=(2,2,2,2)$**.** In this case $N(\mathbf{a})=17$, a basis for $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ being given by the monomials $t_{11}t_{22}t_{33}t_{44}$, $t_{11}t_{22}t_{34}^2$, $t_{11}t_{23}t_{34}t_{42}$, $t_{13}t_{14}t_{23}t_{24}$, $t_{12}^2t_{34}^2$ and their permutations $(1+6+4+3+3=17)$. The system of differential equations $D_1P=D_2P=D_3P=D_4P=0$ imposes $4\times 5=20$ conditions. (For instance, D_1 maps P to the 5-dimensional space spanned by $t_{22}t_{33}t_{44}$, $t_{22}t_{34}^2$, $t_{33}t_{24}^2$, $t_{44}t_{23}^2$, and $t_{23}t_{34}t_{42}$.) *A priori* one would not expect a system of 20 linear equations in 17 unknowns to have any solutions at all, but our claim $\dim\mathcal{P}_{\mathbf{a}}(d)=N_0(\mathbf{a})$ implies that there should in fact be $N_0(\mathbf{a})=6$ linearly independent ones, i.e. the rank of the 20×17 matrix of coefficients should be only 11. Writing down this matrix, we find that this is indeed the case and that the solution space is spanned by the following:

i) $(t_{11}t_{22}-d\,t_{12}^2)(t_{33}t_{44}-d\,t_{34}^2),\ (t_{11}t_{33}-d\,t_{13}^2)(t_{22}t_{44}-d\,t_{24}^2),\ (t_{11}t_{44}-d\,t_{14}^2)(t_{22}t_{33}-d\,t_{23}^2)\,,$

ii) $t_{12}t_{34}(t_{13}t_{24}-t_{14}t_{23}),\quad t_{14}t_{23}(t_{12}t_{34}-t_{13}t_{24}),\quad t_{13}t_{24}(t_{14}t_{23}-t_{12}t_{34})\,,$

iii) $t_{11}t_{23}t_{34}t_{42}+t_{22}t_{13}t_{34}t_{41}+t_{33}t_{12}t_{24}t_{41}+t_{44}t_{12}t_{23}t_{31}-(t_{12}^2t_{34}^2+t_{13}^2t_{24}^2+t_{14}^2t_{23}^2)-d\,t_{12}t_{13}t_{24}t_{34}\,.$

(This list actually contains seven solutions, but the three in group (ii) have sum zero and we have given all three only to preserve the symmetry, while the solution (iii), although written asymmetrically, is symmetric in $i=1,\dots,4$ modulo linear combinations of the solutions (ii).)

We end this section by saying something about the inhomogeneous versions of our higher spherical polynomials. In the classical $n=2$, the polynomials in $\mathcal{P}_{\mathbf{a}}(d)$ (here $\mathbf{a}=(a,a)$ for some integer $a\ge 0$) are interpreted as polynomials of one variable via the correspondence

$$P\in\mathcal{P}_{(a,a)}(d)\quad\Rightarrow\quad P\begin{pmatrix} t_{11} & t_{12}\\ t_{12} & t_{22}\end{pmatrix}\;=\;\bigl(t_{11}t_{22}\bigr)^{a/2}\,p\biggl(\frac{t_{12}}{\sqrt{t_{11}t_{22}}}\biggr),\qquad p(t)\;=\;P\begin{pmatrix} 1 & t\\ t & 1\end{pmatrix},$$

in which case the two differential equations defining $\mathcal{P}_{\mathbf{a}}(d)$ are equivalent to a single differential equation (Legendre or Gegenbauer differential equation) for the polynomial $p(t)$. In the same way, for $n>2$ we can pass back and forth via homogenization and dehomogenization between the spaces of multihomogeneous polynomials of multidegree $\mathbf{a}$ on $\mathcal{S}_n$ and polynomials of multidegree $\le \mathbf{a}$ and multi-parity $\mathbf{a}$ on $\mathcal{S}_n^1 = \{T \in \mathcal{S}_n \mid t_{ii} = 1\ (\forall\ \mathrm{i})\}$, the correspondence being given by

$$\begin{gathered} P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \quad \leftrightarrow \quad p = P\big|_{\mathcal{S}_n^1} \in \mathbb{C}[\mathcal{S}_n^1]\,, \quad P(T) = \delta(T)^{\mathbf{a}/2}\, p\big(T^{(1)}\big)\,, \\ \text{where} \quad \delta(T)^{\mathbf{a}/2} = \prod_{i=1}^{n} t_{ii}^{a_i/2}\,, \quad T^{(1)} = \big(\tau_{ij}\big)\,, \quad \tau_{ij} = \frac{t_{ij}}{\sqrt{t_{ii}t_{jj}}}\,. \end{gathered} \tag{6}$$

(The conditions on the degree and parity of p ensure that the function of T defined by the expression on the right is a polynomial.) Just as in the classical case, we have (eq. (10) below) that every polynomial on $\mathcal{S}_n^1$ is the restriction of a unique higher spherical polynomial (element of $\mathcal{P}_n(d)$). More interestingly, the system of differential equations in $n(n+1)/2$ variables defining the space $\mathcal{P}_{\mathbf{a}}(d)$ will correspond to a system of n diffential equations in the $n(n-1)/2$ coordinates of $\mathcal{S}_n^1$. For instance, for $n = 3$, if we define coordinates t_i on $\mathcal{S}_3^1$ by

$$\mathcal{S}_3^1 \quad \ni \quad T \;=\; \begin{pmatrix} 1 & t_3 & t_2 \\ t_3 & 1 & t_1 \\ t_2 & t_1 & 1 \end{pmatrix} \tag{7}$$

(so that $t_1 = \tau_{23}$ etc.), then the condition on a polynomial $Q(t_1, t_2, t_3)$ to be the restriction to $\mathcal{S}_3^1$ of an element of $\mathcal{P}_{\mathbf{a}}(d)$ is that Q satisfy the system of differential equations given by

$$\begin{gathered} (1 - t_2^2)\,\frac{\partial^2 Q}{\partial t_2^2} \;+\; 2\,(t_1 - t_2 t_3)\,\frac{\partial^2 Q}{\partial t_2 \partial t_3} \;+\; (1 - t_3^2)\,\frac{\partial^2 Q}{\partial t_3^2} \\ -\;(d-1)\left(t_2\frac{\partial Q}{\partial t_2} \;+\; t_3\frac{\partial Q}{\partial t_3}\right) \;+\; a_1(a_1 + d - 2)\,Q \;=\; 0 \end{gathered} \tag{8}$$

and its two cyclic permutations. This system of differential equations will be studied in detail in [12], while some special properties of the coordinates (t_1, t_2, t_3) defined by (7) for the case $n = 3$ will be discussed in §8.**E** of the present paper.

§2. Dimension formula and decomposition theorem

Let $\mathcal{N}(\mathbf{a})$ and $\mathcal{N}_0(\mathbf{a}) \subset \mathcal{N}(\mathbf{a})$ be as in the last section and $\Phi : \mathcal{P}_{\mathbf{a}}(d) \to \mathbb{C}^{\mathcal{N}_0(\mathbf{a})}$ the natural projection map (the composite of the inclusion $\mathcal{P}_{\mathbf{a}}(d) \subseteq \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \cong \mathbb{C}^{\mathcal{N}(\mathbf{a})}$ with the projection map $\mathbb{C}^{\mathcal{N}(\mathbf{a})} \to \mathbb{C}^{\mathcal{N}_0(\mathbf{a})}$), which corresponds to the operation of restricting a polynomial $P(T)$ to the subspace $\mathcal{S}_n^0 \subset \mathcal{S}_n$ of matrices T whose diagonal coefficients vanish. The main result of this section is:

Theorem 1. *Let $\mathbf{a} \in \mathbb{Z}_{\ge 0}^n$ be a multidegree. Then for any complex number d not belonging to the finite subset*

$$\Xi(\mathbf{a}) \;:=\; 2\mathbb{Z} \cap \bigcup_{a_i \ge 2} [4 - 2a_i, 2 - a_i]$$

of $2\,\mathbb{Z}_{\le 0}$, the map $\Phi : \mathcal{P}_{\mathbf{a}}(d) \to \mathbb{C}^{\mathcal{N}_0(\mathbf{a})}$ is an isomorphism. In particular, the dimension of $\mathcal{P}_{\mathbf{a}}(d)$ for such d equals $N_0(\mathbf{a})$, the cardinality of $\mathcal{N}_0(\mathbf{a})$.

Proof. Let $P(T) = \sum_{\boldsymbol{\nu}} C(\boldsymbol{\nu}) T^{\boldsymbol{\nu}}$ be an element of $\mathcal{P}_{\mathbf{a}}(d)$, i.e. P is annihilated by each of the operators $D_i = D_i^{(d)}$ defined in (5). The action of D_i on monomials is given by

$$D_i(T^{\boldsymbol{\nu}}) = \nu_{ii}\,(2a_i - \nu_{ii} + d - 2)\,T^{\boldsymbol{\nu} - \mathbf{e}_{ii}} + \sum_{j,k \ne i} \nu_{ij}(\nu_{ik} - \delta_{jk})\,T^{\boldsymbol{\nu} - \mathbf{e}_{ij} - \mathbf{e}_{ik} + \mathbf{e}_{jk}}\,,$$

where the sum runs over $j,\, k \in \{1,\dots,\widehat{i},\dots,n\}$ and $\mathbf{e}_{ab}$ has the same meaning as in §2, and using this we can rewrite the differential equation $D_iP = 0$ as a recursion formula for the coefficients, as we did for the example $n = 2$ in the last section. This recursion for $i = 1$, together with the assumption that $2a_1 - \nu_{11} + d - 2$ does not vanish for any even integer $\nu_{11} \in (0, a_1]$, shows that the coefficients $C(\boldsymbol{\nu})$ with $\nu_{11} > 0$ can be expressed as linear combinations of those with a smaller value of ν_{11} and hence that P is determined by the $C(\boldsymbol{\nu})$ with $\nu_{11} = 0$. Similarly, the equation $D_2P = 0$ shows that the coefficients $C(\boldsymbol{\nu})$ with $\nu_{11} = 0$ and $\nu_{22} > 0$ are combinations of $C(\boldsymbol{\nu}')$ with $\nu'_{11} = 0$ and $\nu'_{22} < \nu_{22}$, so that they are determined by the coefficients with $\nu_{22} = 0$. Continuing in this way, we see by induction that the coefficients $C(\boldsymbol{\nu})$ with $\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})$ determine P completely. Hence Φ on $\mathcal{P}_{\mathbf{a}}(d)$ is injective and $\dim \mathcal{P}_{\mathbf{a}}(d) \le N_0(\mathbf{a})$.

To show directly that Φ is also surjective, we would have to show that the system of equations $D_iP = 0$ can be solved for any choice of the coefficients $C(\boldsymbol{\nu})$ with $\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})$. But this is not at all obvious, as the examples in §2 make clear. Instead, we will prove by induction on i the stronger statement that the dimension of $K_i = K_i(\mathbf{a}) := \operatorname{Ker}(D_1) \cap \cdots \cap \operatorname{Ker}(D_i)$ is exactly $N^{(i)}(\mathbf{a})$ for each i between 0 and n, where $N^{(i)}(\mathbf{a})$ is the number of matrices in $\mathcal{N}(\mathbf{a})$ with $\nu_{11} = \ldots = \nu_{ii} = 0$. The case $i = n$ gives the assertion we want. The main observation for the proof is that the various operators D_i commute. This is obvious when d is a positive integer because of the interpretation of the D_i's as the Laplacians with respect to disjoint sets of variables $x_i \in \mathbb{R}^d$, and it then remains true for all d because the coefficients of the operators are polynomials in d and no polynomial has infinitely many zeros. It follows that D_i maps K_{i-1} to K_{i-1} or more precisely, since D_i clearly decreases $\mathbf{a} = \boldsymbol{\nu} \cdot \mathbf{1}$ by $2\mathbf{e}_i$ (where $\mathbf{e}_i$ as before denotes the vector with 1 in the ith place and 0's elsewhere), $K_{i-1}(\mathbf{a})$ to $K_{i-1}(\mathbf{a} - 2\mathbf{e}_i)$. If we therefore assume inductively that $\dim K_{i-1}(\mathbf{a}) = N^{(i-1)}(\mathbf{a})$ for all $\mathbf{a}$, then it follows that

$$\begin{aligned}\dim K_i(\mathbf{a}) &= \dim \operatorname{Ker}\big(\, D_i : \, K_{i-1}(\mathbf{a}) \to K_{i-1}(\mathbf{a} - 2\mathbf{e}_i)\big) \\ &\ge \dim K_{i-1}(\mathbf{a}) - \dim K_{i-1}(\mathbf{a} - 2\mathbf{e}_i) \\ &= N^{(i-1)}(\mathbf{a}) - N^{(i-1)}(\mathbf{a} - 2\mathbf{e}_i) \;=\; N^{(i)}(\mathbf{a}),\end{aligned}$$

and since the previous argument gives the reverse inequality, this completes the proof. □

Remark 1. The definitions of $N(\mathbf{a})$ and $N_0(\mathbf{a})$ are equivalent to the generating functions

$$\sum_{\mathbf{a}\ge 0} N(\mathbf{a})\, z_1^{a_1} \cdots z_n^{a_n} = \prod_{1\le i\le j\le n} \frac{1}{1 - z_iz_j}\,, \qquad \sum_{\mathbf{a}\ge 0} N_0(\mathbf{a})\, z_1^{a_1} \cdots z_n^{a_n} = \prod_{1\le i< j\le n} \frac{1}{1 - z_iz_j}\,.$$

Remark 2. There are simple expressions for $N_0(\mathbf{a})$ if $n \le 4$. First note that, for any n, a necessary condition for $N_0(\mathbf{a}) \ne 0$ is that the sum of the a_i's is even and that no a_i is strictly bigger than the sum of the others, i.e., that the number $\delta := \frac{1}{2}\sum\limits_i a_i - \max\limits_i a_i$ is a nonnegative integer. Assuming that $\mathbf{a}$ satisfies this condition, we have the following formulas for $n \le 4$:

$$N_0(a_1, a_2) = 1, \qquad N_0(a_1, a_2, a_3) = 1, \qquad N_0(a_1, a_2, a_3, a_4) = \frac{(s + 1)(s + 2)}{2}\,,$$

where s in the last case is $\min\{a_1, a_2, a_3, a_4, \delta\}$. Note that for $n = 2$ and $n = 3$ the condition $\delta \ge 0$ is equivalent to $a_1 = a_2$ and to the triangle inequality, respectively. For general values of n, there is no simple formula for $N_0(\mathbf{a})$, but assuming that $n \ge 4$ and that no a_i vanishes, we have $N_0(\mathbf{a}) = 1$ if and only if $\delta = 0$. Under the same assumption, one has the following partial results:

δ	$N_0(\mathbf{a})$
< 0	0
0	1
1	$\binom{n-1}{2}$
2	$\binom{M+1}{2}$ where $M = \binom{n-1}{2} - \#\{i \mid a_i = 1\}$
≥ 3	$n^{2\delta}/2^{\delta}\delta! + \mathrm{O}\big(n^{2\delta-1}\big)$

Notice that the dimension is a triangular number whenever $n \leq 4$ or $\delta \leq 2$, but this is not true in general, the smallest counterexample being $N_0(2,2,2,2,2) = 22$.

Remark 3. As mentioned in the introduction, in the case where d is an integer $\geq n$, one can also see the dimension formula via the interpretation of $\mathcal{P}_{\mathbf{a}}(d)$ as the $O(d)$-invariant subspace of $\mathcal{H}_{a_1}(\mathbb{R}^d) \otimes \ldots \otimes \mathcal{H}_{a_n}(\mathbb{R}^d)$, where $\mathcal{H}_a(\mathbb{R}^d)$ denotes the space of homogeneous harmonic polynomials of degree a in $\mathbb{R}^d$ and $O(d)$ acts diagonally on $(\mathbb{R}^d)^n$. Let ρ_{a_i} be the symmetric tensor representation of $GL(d)$ of degree a_i, that is, the representation on degree a_i homogeneous polynomials. We denote by χ_{a_i} the character of ρ_{a_i}. Now consider the tensor product representation $\otimes_{i=1}^n \rho_{a_i}$ of $GL(d)$. We take the restriction of this representation to $O(d)$. For $d \geq n$, the isotypic component of the trivial representation is spanned by the linearly independent polynomials $\prod_{i,j=1}^n (x_i, x_j)^{\nu_{ij}/2}$ with $\nu_{ij} \in \mathbb{Z}_{\geq 0}$, $\nu_{ii} \in 2\mathbb{Z}$ and $a_i = \sum_{j=1}^n \nu_{ij}$, by classical invariant theory (Weyl [16], pages 53 and 75). Their number is exactly $N(\mathbf{a})$. On the other hand, the character of the representation of $O(d)$ on harmonic polynomials of degree a_i is $\chi_{a_i} - \chi_{a_i-2}$. What we want to know is the multiplicity of the trivial representation of $O(d)$ in $\prod_{i=1}^n (\chi_{a_i} - \chi_{a_i-2})$. Hence, expanding the parenthesis and counting the multiplicities of the trivial character for each product, we can see that the multiplicity in $\prod_{i=1}^n (\chi_{a_i} - \chi_{a_i-2})$ is the coefficient of $\prod z_i^{a_i}$ in

$$\prod_{i=1}^n (1 - z_i^2) \prod_{1 \leq i \leq j \leq n} (1 - z_i z_j)^{-1} = \prod_{1 \leq i < j \leq n} (1 - z_i z_j)^{-1},$$

as asserted.

Remark 4. From the proof of Theorem 1 we deduce the analogue for the operators D_i of the classical fact that any collection of functions $f_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$ satisfying $\partial f_i/\partial x_j = \partial f_j/\partial x_i$ for all i and j is the collection of partial derivatives $\partial g/\partial x_i$ of a single function g. Here the assertion is that, if $d \notin \Xi(\mathbf{a})$, then any collection of functions $P_i \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}-2\mathbf{e}_i}$ $(i = 1, \ldots, n)$ satisfying $D_i(P_j) = D_j(P_i)$ for all i and j has the form $P_i = D_i(Q)$ for some $Q \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$. Indeed, the proof of Theorem 1 implies (in the notations there) that the map $D_i : K_{i-1}(\mathbf{a}) \to K_{i-1}(\mathbf{a} - 2\mathbf{e}_i)$ is surjective for all i. Applying this with $i = 1$ (where $K_0(\mathbf{a}) = \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ and $K_0(\mathbf{a} - 2\mathbf{e}_1) = \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}-2\mathbf{e}_1}$), we can find $Q_1 \in \mathbb{C}[S]_{\mathbf{a}}$ such that $D_1 Q_1 = P_1$. Then $D_1(P_2 - D_2 Q_1) = D_2 P_1 - D_2 P_1 = 0$, so $P_2 - D_2 Q_1 \in K_1(\mathbf{a} - 2\mathbf{e}_2)$. Now the surjectivity of $D_2 : K_1(\mathbf{a}) \to K_1(\mathbf{a} - 2\mathbf{e}_2)$ implies that there is a $Q_2 \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ with $D_1 Q_2 = 0$ and $D_2 Q_2 = P_2 - D_2 Q_1$, or equivalently $D_1(Q_1 + Q_2) = P_1$ and $D_2(Q_1 + Q_2) = P_2$. Continuing in this way, we obtain the statement claimed, which will be used again in the Remark following Proposition 2 below and at several later points in the paper.

Theorem 1 has two interesting consequences. The first one, which is a refinement of the dimension statement, is an analogue of the classical fact that every polynomial on $\mathbb{R}^d$ has a unique decomposition into summands of the form $(x, x)^j P_j(x)$ with each $P_j(x)$ harmonic.

Corollary. *For any* $d \in \mathbb{C} \smallsetminus 2\mathbb{Z}_{\leq 0}$ *we have*

$$\mathbb{C}[\mathcal{S}_n] = \mathcal{P}^{(n)}(d)\,[t_{11}, t_{22}, \ldots, t_{nn}]\,.$$

More precisely, for $\mathbf{a}$ *and* d *be as in Theorem 1, the space* $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ *of all polynomials on* $\mathcal{S}_n$ *of multidegree* $\mathbf{a}$ *has a direct sum decomposition as*

$$\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \;=\; \bigoplus_{\mathbf{0}\le\mathbf{m}\le\mathbf{a}/2} \delta(T)^{\mathbf{m}}\,\mathcal{P}_{\mathbf{a}-2\mathbf{m}}(d)\,, \tag{9}$$

where the sum ranges over all $\mathbf{m}\in\mathbb{Z}^n$ *satisfying* $0\le m_i\le a_i/2$ *and* $\delta(T)^{\mathbf{m}}=\prod_{i=1}^n t_{ii}^{m_i}$ *.*

Proof. The proof of Theorem 1 shows that the map $\Phi_i:\,K_i(\mathbf{a})\to\mathbb{C}^{\mathcal{N}^{(i)}(\mathbf{a})}$, where $\mathcal{N}^{(i)}(\mathbf{a})$ is the set of $\boldsymbol{\nu}\in\mathcal{N}(\mathbf{a})$ with $\nu_{11}=\cdots=\nu_{ii}=0$, is injective (i.e., any $P\in K_i(\mathbf{a})$ is determined by its coefficients $C(\boldsymbol{\nu})$ with $\boldsymbol{\nu}\in\mathcal{N}^{(i)}(\mathbf{a})$) and surjective (for dimension reasons), and that the sequence $0\to K_i(\mathbf{a})\to K_{i-1}(\mathbf{a})\stackrel{D_i}{\to}K_{i-1}(\mathbf{a}-2\mathbf{e}_i)\to 0$ is exact. Now let $P(T)$ be an arbitrary element of $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}=\mathbb{C}^{\mathcal{N}(\mathbf{a})}$. Its canonical projection to $\mathbb{C}^{\mathcal{N}^{(1)}(\mathbf{a})}$ equals $\Phi_1(Q_0)$ for a unique element Q_0 of $K_1(\mathbf{a})$. But this means that the restrictions of P and Q_0 to $t_{11}=0$ agree, so we have $P(T)=Q_0(T)+t_{11}P_1(T)$ for some $P_1\in\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}-2\mathbf{e}_1}$, and this decomposition is unique. Applying the same argument to P_1, we get $P_1=Q_1+t_{11}P_2$ with $Q_1\in K_1(\mathbf{a}-2\mathbf{e}_1)$ and $P_2\in\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}-4\mathbf{e}_1}$, and continuing in this way we find that P has a unique decomposition of the form $P(T)=\sum_{0\le m_1\le a_1/2}t_{11}^{m_1}Q_{m_1}(T)$ with $Q_{m_1}\in K_1(\mathbf{a}-2m_1\mathbf{e}_1)$ for each m_1. Now we use the same argument to show that each Q_{m_1} can be decomposed uniquely as $\sum_{0\le m_2\le a_2/2}t_{22}^{m_2}Q_{m_1,m_2}(T)$ with $Q_{m_1,m_2}\in K_2(\mathbf{a}-2m_1\mathbf{e}_1-2m_2\mathbf{e}_2)$, etc., obtaining in the end precisely the direct sum decomposition asserted by the corollary. $\square$

The decomposition (9), which will be shown in the next section to be orthogonal with respect to a natural scalar product on $\mathbb{C}[\mathcal{S}_n]$, is one of the key properties of higher spherical polynomials. Note that it can also be formulated in terms of the inhomogeneous coordinates discussed in the final paragraph of §1 as the assertion that the restriction of the projection map $\mathbb{C}[\mathcal{S}_n]\to\mathbb{C}[\mathcal{S}_n^1]$ to the space of higher spherical polynomials is an isomorphism:

$$\mathbb{C}[\mathcal{S}_n^1] \;=\; \mathcal{P}_n(d)\big|_{\mathcal{S}_n^1} \;=\; \bigoplus_{\mathbf{a}} \mathcal{P}_{\mathbf{a}}(d)\big|_{\mathcal{S}_n^1}\,. \tag{10}$$

For $n=2$ this is just the classical decomposition of polynomials in one variable into linear combinations of Legendre or Gegenbauer polynomials

The second consequence of Theorem 1 which we want to mention is that it gives us canonical bases of the spaces $\mathcal{P}_{\mathbf{a}}(d)$ and $\mathcal{P}^{(n)}(d)$. Let $\mathcal{N}_0=\bigcup\mathcal{N}_0(\mathbf{a})$ be the set of all matrices $\boldsymbol{\nu}\in\mathcal{N}$ with all $\nu_{ii}=0$. Then by pulling back the basis of monomials $T^{\boldsymbol{\nu}}$ under the isomorphism Φ of the theorem, we obtain:

Proposition 1. *Let* $\mathbf{a}\in\mathbb{Z}_{\ge0}^n$ *and* $d\in\mathbb{C}\smallsetminus\Xi(\mathbf{a})$*. Then for every* $\nu\in\mathcal{N}_0(\mathbf{a})$ *there is a unique element* $P_{\boldsymbol{\nu},d}^M\in\mathcal{P}^{(n)}(d)$ *whose restriction to* $\mathcal{S}_n^0$ *is the monomial* $T^{\boldsymbol{\nu}}$*, and these polynomials form a basis of* $\mathcal{P}_{\mathbf{a}}(d)$*.*

The polynomials $P_{\boldsymbol{\nu}}^M(T)$ (the index d will usually be dropped when no confusion can result) will be called the *monomial basis* of $\mathcal{P}_n(d)$. In Section 5 we will study this basis (and its dual basis with respect to a canonical scalar product) in more detail. We will also give explicit constructions of these polynomials there. This also gives another proof of Theorem 1 and its corollary.

Finally, let us make a few remarks concerning the degenerate case $d\in\Xi(\mathbf{a})$. In this case the dimension of $\mathcal{P}_{\mathbf{a}}(d)$ can be larger than $N_0(\mathbf{a})$, as we saw in Example 1. However, there is a natural way to define a subspace $\mathcal{P}_{\mathbf{a}}^*(d)$ of $\mathcal{P}_{\mathbf{a}}(d)$ of the correct dimension, because it is not hard to see that the subspaces $\mathcal{P}_{\mathbf{a}}(d')\subset\mathbb{C}[\mathcal{S}_n]$ have a limiting value as $d'\to d$. For example, the six polynomials in $\mathcal{P}_{(2,2,2,2)}(d)$ given in Example 2 become linearly dependent if $d = 0$ (the three polynomials in i) become equal), but we can make a change of basis over $\mathbb{Q}(d)$ to give six other polynomials, still with coefficients depending polynomially on d, which are linearly independent for all d. (Replace the second and third solutions in i) by their differences with the first solution divided by d.) This means that if we replace the space $\mathcal{P}_{\mathbf{a}}(d)$ by the limiting space

$$\mathcal{P}^*_{\mathbf{a}}(d) \ := \ \lim_{\substack{d'\to d \\ d'\neq d}} \mathcal{P}_{\mathbf{a}}(d') \tag{11}$$

whenever d belongs to $\Xi(\mathbf{a})$, then the dimension formula $\dim \mathcal{P}_{\mathbf{a}}(d) = N_0(\mathbf{a})$ holds for all $d \in \mathbb{C}$ and the map $\mathcal{P}^*_{\mathbf{a}}$ from $\mathbb{C}$ to the Grassmannian of $N_0(\mathbf{a})$-dimensional subspaces of $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ is continuous, and in fact analytic, everywhere. For this reason, we could equally reasonably have taken $\mathcal{P}^*_{\mathbf{a}}(d)$ rather than $\cap \mathrm{Ker} D_i$ as the definition of $\mathcal{P}_{\mathbf{a}}(d)$ for values of d belonging to $\Xi(\mathbf{a})$. On the other hand, $\mathcal{P}^*_{\mathbf{a}}(d)$ is not as nicely behaved as $\mathcal{P}_{\mathbf{a}}(d)$ for generic d: the "constant term" map from $\mathcal{P}^*_{\mathbf{a}}(d)$ to $\mathbb{C}^{\mathcal{N}_0(\mathbf{a})}$ is no longer an isomorphism, even though the dimensions are equal, and the decomposition (9) also breaks down, as the following examples show.

Example 1: If $n = 2$ and $\mathbf{a} = (2, 0)$, then we have $\mathcal{P}_{\mathbf{a}}(d) = 0$ for $d \neq 0$ and $\mathcal{P}_{\mathbf{a}}(0) = \mathbb{C}\, t_{11}$, but we have $\mathcal{P}^*_{\mathbf{a}}(0) = 0$.

Example 2: If $n = 2$ and $\mathbf{a} = (4, 4)$, then for $d \notin \Xi(\mathbf{a}) = \{-2, -4\}$, the space $\mathcal{P}_{\mathbf{a}}(d)$ is spanned by $(d+4)(d+2)t_{12}^4 - 6(d+2)t_{11}t_{22}t_{12}^2 + 3t_{11}^2 t_{22}^2$, while $\mathcal{P}_{\mathbf{a}}(-2)$ and $\mathcal{P}_{\mathbf{a}}(-4)$ are spanned by $t_{11}^2 t_{22}^2$ and $4t_{11}t_{22}t_{12}^2 + t_{11}^2 t_{22}^2$, respectively. So in this case $\mathcal{P}_{\mathbf{a}}(-2) = \mathcal{P}^*_{\mathbf{a}}(-2)$ and $\mathcal{P}_{\mathbf{a}}(-4) = \mathcal{P}^*_{\mathbf{a}}(-4)$. In particular, for $d = -2$ and -4, there exists no monomial basis.

We end this section by giving an alternative, purely algebraic, definition of $\mathcal{P}^*_{\mathbf{a}}(d)$.

Proposition 2. *For $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ and arbitrary $d \in \mathbb{C}$ the space $\mathcal{P}^*_{\mathbf{a}}(d)$ defined by (11) coincides with the space of polynomials P belonging to a sequence $(\ldots, P_2, P_1, P_0 = P, P_{-1} = 0)$ of polynomials $P_r \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ satisfying the system of differential equations*

$$D_i(P_r) = \partial_{ii}(P_{r-1}) \qquad (r = 0,\, 1,\, 2, \ldots, \quad 1 \leq i \leq n)\,. \tag{12}$$

Moreover, the polynomials P_r can be chosen such that only finitely many of them are non-zero.

Proof. By the discussion preceding the statement of the proposition, any $P \in \mathcal{P}^*_{\mathbf{a}}(d)$ can be deformed to a (convergent or formal) power series $P(T;\varepsilon) \in \mathbb{C}[\mathcal{S}_n][[\varepsilon]]$ belonging to $\mathcal{P}^*_{\mathbf{a}}(d-\varepsilon)$ for any (complex or infinitesimal) value of ε. (Choose a basis $\{P^{(i)}(T, d')\}_{1\leq i\leq N_0(\mathbf{a})}$ of $\mathcal{P}^*_{\mathbf{a}}(d')$ for all d' near to d, or even for all $d' \in \mathbb{C}$, where each $P^{(i)}(T, d')$ is a polynomial in T and d', write $P(T)$ as $\sum_i \alpha_i P^{(i)}(T, d)$, choose any convergent or formal power series $a_i(\varepsilon)$ with $a_i(0) = \alpha_i$, and set $P(T;\varepsilon) = \sum_i a_i(\varepsilon) P^{(i)}(T, d-\varepsilon)$.) Write $P(T;\varepsilon)$ as $\sum_{r\geq 0} P_r(T)\,\varepsilon^r$. From the definition (5) we get

$$D_i^{(d-\varepsilon)}\big(P(T;\varepsilon)\big) \ = \ \big(D_i^{(d)} \ - \ \varepsilon \partial_{ii}\big)\Big(\sum_{r=0}^{\infty} P_r(T)\,\varepsilon^r\Big) \ = \ \sum_{r=0}^{\infty}\big(D_i^{(d)}(P_r) \ - \ \partial_{ii}(P_{r-1})\big)\,\varepsilon^r$$

for each $1 \leq i \leq n$, so the condition $P(T;\varepsilon) \in \mathcal{P}^*_{\mathbf{a}}(d-\varepsilon)$ is equivalent to the system of differential equations (12). If we choose the power series $a_i(\varepsilon)$ in the above argument to be polynomials (e.g., constant polynomials) in ε, then $P(T;\varepsilon) \in \mathbb{C}[\mathcal{S}_n][\varepsilon]$ and the polynomials $P_r(T)$ vanish for all but finitely many r. $\square$

Remark. In the generic case $d \notin \Xi(\mathbf{a})$, we can also deduce the existence of polynomials $P_r(T)$ as above for any $P \in \mathcal{P}_{\mathbf{a}}(d)$ by using the fact given in Remark **4** above. Indeed, if we set $P_{-1} = 0$ and $P_0 = P$, then (12) is satisfied for $r = 0$ by the definition of $\mathcal{P}_{\mathbf{a}}(d)$. Now suppose by induction that for some $R \geq 0$ we have found polynomials ($P_R, \ldots, P_0 = P, P_{-1} = 0$) satisfying equation (12) for $0 \leq r \leq R$. Then from the commutation relation

$$[D_i, \partial_{jj}] = \sum_{k,\ell} [t_{k\ell}, \partial_{jj}]\, \partial_{ik}\partial_{i\ell} = -2\, \partial_{ij}^2$$

we find that $D_i(\partial_{jj}(P_R)) = \partial_{jj}\partial_{ii}(P_{R-1}) - 2\partial_{ij}^2(P_R)$, which is symmetric in i and j, so the result of Remark **4** implies there exists some $P_{R+1} \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ with $D_j(P_{R+1}) = \partial_{jj}(P_R)$ for all j, completing the induction. We also observe that the freedom in choosing each new polynomial P_r $(r \geq 1)$ in the inductive system (12) is precisely $\mathcal{P}_{\mathbf{a}}(d)$, corresponding to the freedom of the choice of $a_i(\varepsilon) \in \alpha_i + \varepsilon\, \mathbb{C}[[\varepsilon]]$ in the proof above.

§3. Scalar product and orthogonality

One of the most important properties of the classical Legendre and Gegenbauer polynomials is that they are orthogonal with respect to a suitable scalar product. In this section we prove an analogous property for the higher spherical polynomials by constructing natural scalar products on the spaces $\mathbb{C}[\mathcal{S}_n]$ and $\mathbb{C}[\mathcal{S}_n^1]$, proportional to each other in each fixed multidegree, such that the direct sum decompositions (9) and (10) given in the last section are orthogonal. For $n = 2$ this is equivalent to the classical orthogonality property just mentioned, and for $n = 3$ it is exactly analogous, because each summand in (9) and (10) has dimension ≤ 1, so that we again obtain a canonical orthogonal basis of the space of all polynomials on $\mathcal{S}_n^1$. For $n \geq 4$ the spaces $\mathcal{P}_{\mathbf{a}}(d)$ in general have dimension greater than 1, and we could not find any natural orthogonal basis. However, it will turn out that there are two natural bases of $\mathcal{P}_{\mathbf{a}}(d)$, one of them being the "monomial basis" constructed in the last section, that are dual to each other (at least up to scalar factors; the normalizations of both bases are not entirely canonical) with respect to the scalar product. This will be discussed in §5.

The definitions of both scalar products will be motivated by the case when d is an integer $\geq n$, so that $\mathcal{P}_{\mathbf{a}}(d)$ can be identified with $(\otimes \mathcal{H}_{a_i}(\mathbb{R}^d))^{O(d)}$. In the classical theory of harmonic polynomials, one defines a scalar product $(\ ,\ )_{S^{d-1}}$ on $\mathbb{C}[\mathbb{R}^d]$ by

$$(f,\, g)_{S^{d-1}} = \frac{1}{\mathrm{Vol}(S^{d-1})} \int_{S^{d-1}} f(x)\, \overline{g(x)}\, dx \qquad (f,\, g \in \mathbb{C}[\mathbb{R}^d])\,,$$

where dx denotes the standard volume form on the sphere, and shows that the spaces $\mathcal{H}_a(\mathbb{R}^d)$ and $\mathcal{H}_a(\mathbb{R}^d)$ are orthogonal with respect to this inner product if $a \neq b$. Furthermore, the restriction map to S^{d-1} is injective on $\mathcal{H}_a(\mathbb{R}^d)$ and the space of all polynomial functions on S^{d-1} is the direct sum of these spaces. (In fact the space $\mathcal{H}_a(\mathbb{R}^d)_{S^{d-1}}$ is just the eigenspace of the Laplacian operator $\Delta_{S^{d-1}}$ of the Riemannian manifold S^{d-1} with eigenvalue $a(a+d-2)$, and this orthogonal decomposition simply corresponds to the spectral decomposition of $L^2(S^{d-1})$ with respect to the Laplacian.)

For purposes of calculation it is convenient to relate the scalar product $(\ ,\ )_{S^{d-1}}$ to a second scalar product $(\ ,\ )_{\mathbb{R}^d}$ on $\mathbb{C}[\mathbb{R}^d]$, defined by

$$(f, g)_{\mathbb{R}^d} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)\, \overline{g(x)}\, e^{-\|x\|^2/2}\, dx\,,$$

where the normalizing factor $(2\pi)^{-d/2}$ has been chosen so that $(1,1)_{\mathbb{R}^d} = 1$. We can write the definitions of $(f,g)_{S^{d-1}}$ and $(f,g)_{\mathbb{R}^d}$ as $\mathsf{E}_{S^{d-1}}\big[f\overline{g}\big]$ and $\mathsf{E}_{\mathbb{R}^d}\big[f\overline{g}\big]$, where $\mathsf{E}_{S^{d-1}}\big[f\big] = (f,1)_{S^{d-1}}$ and $\mathsf{E}_{\mathbb{R}^d}\big[f\big] = (f,1)_{\mathbb{R}^d}$ denote the expectation values of f with respect to the probability measure

$\mathrm{Vol}(S^{d-1})^{-1}dx$ and $(2\pi)^{-d/2}e^{-\|x\|^2/2}dx$ on S^{d-1} and $\mathbb{R}^d$, respectively. For (the restrictions to S^{d-1} of) homogeneous polynomials these two measures are proportional: if $f \in \mathbb{C}[\mathbb{R}^d]_a$, then

$$\begin{aligned} \mathsf{E}_{\mathbb{R}^d}\big[f\big] &= (2\pi)^{-d/2} \int_0^\infty e^{-r^2/2} \left(\int_{S^{d-1}} f(rx)\, d(rx) \right) dr \\ &= \frac{\mathrm{Vol}(S^{d-1})}{(2\pi)^{d/2}} \int_0^\infty r^{a+d-1}\, e^{-r^2/2}\, dr \;\cdot\; \mathsf{E}_{S^{d-1}}\big[f\big] \\ &= \varepsilon_a(d)\, \mathsf{E}_{S^{d-1}}\big[f\big] \end{aligned} \tag{13}$$

with

$$\varepsilon_a(d) \;=\; 2^{a/2}\,\Gamma\big(\frac{a+d}{2}\big)/\Gamma\big(\frac{d}{2}\big) \;=\; d(d+2)\cdots(d+a-2) \qquad (a \text{ even}). \tag{14}$$

(The definition of $\varepsilon_a(d)$ for a odd does not matter because in that case $f(x)$ is odd and both measures under consideration vanish trivially.)

The scalar products $(\ ,\ )_{\mathbb{R}^d}$ and $(\ ,\ )_{S^{d-1}}$ on $\mathbb{C}[\mathbb{R}^d]$ extend in the obvious way to scalar products $(\ ,\ )_{(\mathbb{R}^d)^n}$ and $(\ ,\ )_{(S^{d-1})^n}$ on $\mathbb{C}[(\mathbb{R}^d)^n]$ and induce scalar products $(\ ,\ )_d$ and $(\ ,\ )_d^1$ on $\mathbb{C}[\mathcal{S}_n]$ by the formulas (in which elements of $(\mathbb{R}^d)^n$ are thought of as $n \times d$ matrices)

$$(P,Q)_d \;=\; (\widetilde{P},\widetilde{Q})_{(\mathbb{R}^d)^n} \;=\; (2\pi)^{-nd/2} \int_{\mathbb{R}^{dn}} P(XX^t)\,\overline{Q(XX^t)}\, e^{-\mathrm{tr}(XX^t)/2}\, dX\,, \tag{15}$$

$$(P,\,Q)_d^{(1)} \;=\; (\widetilde{P},\,\widetilde{Q})_{(S^{d-1})^n} \;=\; \frac{1}{\mathrm{Vol}(S^{d-1})^n} \int_{(S^{d-1})^n} P(XX^t)\,\overline{Q(XX^t)}\, dX \tag{16}$$

for $P,\,Q \in \mathbb{C}[\mathcal{S}_n]$, where $\widetilde{P} = P \circ \beta_n$, $\widetilde{Q} = Q \circ \beta_n$ as usual. The properties reviewed above then immediately imply the corresponding properties for these new scalar products, namely, that the spaces $\mathcal{P}_{\mathbf{a}}(d) = \big(\otimes \mathcal{H}_{a_i}(\mathbb{R}^d)\big)^{O(d)}$ and $\mathcal{P}_{\mathbf{b}}(d) = \big(\otimes \mathcal{H}_{b_i}(\mathbb{R}^d)\big)^{O(d)}$ are orthogonal to each other if the multi-indices $\mathbf{a}$ and $\mathbf{b}$ are distinct, and that the two scalar products are proportional in each multidegree:

$$(P,Q)_d \;=\; \varepsilon_{\mathbf{a}+\mathbf{b}}(d)\,(P,Q)_d^{(1)} \qquad \text{for } P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}},\, Q \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{b}}\,. \tag{17}$$

with $\varepsilon_{\mathbf{a}}(d) \in \mathbb{Z}[d]$ defined by

$$\varepsilon_{\mathbf{a}}(d) \;=\; \prod_{i=1}^n \varepsilon_{a_i}(d) \qquad (\mathbf{a} \in \mathbb{Z}^n_{\geq 0})\,. \tag{18}$$

(Again this definition is relevant only if $\mathbf{a}$ is divisible by 2, since, as one can see easily, polynomials whose multidegrees are not congruent modulo 2 are orthogonal with respect to both scalar products.) We now show that these definitions can be extended to arbitrary complex values of d, with the same properties.

Denote by $\mathcal{S}_n^+ \subset \mathcal{S}_n$ the space of positive definite symmetric $n \times n$ matrices and by $\mathcal{S}_n^{1,+}$ its intersection with the space $\mathcal{S}_n^1$ of $n \times n$ matrices with all diagonal coefficients equal to 1. The spaces $\mathcal{S}_n$ and $\mathcal{S}_n^1$ are isomorphic to the Euclidean spaces $\mathbb{R}^{n(n+1)/2}$ and $\mathbb{R}^{n(n-1)/2}$, respectively, and have natural Lebesgue measures $dT = \prod_{1\leq i\leq j\leq n} dt_{ij}$ and $d^1T = \prod_{1\leq i<j\leq n} dt_{ij}$ which we can restrict to their open subsets $\mathcal{S}_n^+$ and $\mathcal{S}_n^{1,+}$. For $\Re(d)$ sufficiently large we define

$$(P,Q)_d \;=\; c_n(d) \int_{\mathcal{S}_n^+} e^{-\mathrm{tr}(T)/2}\, P(T)\,\overline{Q(T)}\, \det(T)^{(d-n-1)/2}\, dT \tag{19}$$

and

$$(P,Q)_d^1 \;=\; c_n^1(d) \int_{\mathcal{S}_n^{1,+}} P(T)\,\overline{Q(T)}\, \det(T)^{(d-n-1)/2}\, d^1T\,, \tag{20}$$

where $c_n(d)$ and $c_n^1(d)$ are normalizing constants defined by

$$2^{nd/2}\, c_n(d) \;=\; \Gamma\Big(\frac{d}{2}\Big)^{-n} c_n^1(d) \;=\; \pi^{-n(n-1)/4} \prod_{i=0}^{n-1} \Gamma\Big(\frac{d-i}{2}\Big)^{-1}\,. \tag{21}$$

Theorem 2. *(a) The integrals in* (19) *and* (20) *converge absolutely for* $d \in \mathbb{C}$ *with* $\Re(d) > n-1$ *and their right-hand sides agree with* (15) *and* (16) *if* d *is an integer* $\geq n$.
(b) The values of $(P,Q)_d$ *or* $(P,Q)_d^1$ *for* P *and* Q *multihomogeneous are related as in* (17) *for all complex values of* d*, with* $\varepsilon_{\mathbf{a}}(d)$ *defined by equations* (14) *and* (18).
(c) For fixed $P, Q \in \mathbb{C}[\mathcal{S}_n]$ *and variable* d*, the scalar product* $(P,Q)_d$ *is a polynomial in* d*, belonging to* $\mathbb{Z}[d]$ *if* P *and* Q *have integral coefficients, and* $(1,1)_d = 1$.
(d) The spaces $\mathcal{P}_{\mathbf{a}}(d)$ *and* $\mathcal{P}_{\mathbf{b}}(d)$ *with distinct multi-indices* $\mathbf{a}$ *and* $\mathbf{b}$ *are orthogonal.*

Remark 1. Parts (b) and (c) give the holomorphic continuation of $(\ ,\ )_d$ and the meromorphic continuation of $(\ ,\ )_d^1$ to all complex values of d, with the poles of the latter occurring only for $d \in \{0, -2, -4, \dots\}$. These are the same exceptional values as in Theorem 1.

Remark 2. Part (d) (in which we do not have to specify which scalar product is meant, by part (b)) implies that both the direct sum decomposition $\mathbb{C}[\mathcal{S}_n] = \bigoplus_{\mathbf{a}} \mathcal{P}_{\mathbf{a}}(d) \otimes \mathbb{C}[t_{11}, \dots, t_{nn}]$ given by the corollary to Theorem 1 and the direct sum decomposition of $\mathbb{C}[S_n]_{\mathbf{a}}$ given in (9) are orthogonal with respect to either scalar product, and that the direct sum decomposition of $\mathbb{C}[S_n^1]$ given in (10) is orthogonal with respect to the scalar product $(\ ,\ )_d^1$ on $\mathbb{C}[S_n^1]$.

Remark 3. If d is real and $> n-1$, then the convergent integral representation (19) shows that the scalar product $(\ ,\ )_d$ on $\mathbb{C}[\mathcal{S}_n]$ is positive definite and hence non-degenerate. For other values of d, $(\ ,\ )_d$ may be degenerate. We will show in Theorem 11 that this can happen only if d is an integer. Numerical examples can be found at the end of §5 and in Tables 2 and 3 at the end of the paper.

Remark 4. Under the correspondence $P \leftrightarrow p$ between functions in $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ and $p \in \mathbb{C}[\mathcal{S}_n^{1,+}]$ given in (6), we have $D_i(P) \leftrightarrow \big(\mathbb{D}_i + a_i(a_i+d-2)\big)p$ for some second order differential operators $\mathbb{D}_i$ (depending on d but not on $\mathbf{a}$) on $\mathcal{S}_n^{1,+}$. (Compare (8) for the case $n=3$, $i=1$.) The spaces $\mathcal{P}_{\mathbf{a}}(d)$ therefore correspond to spaces of simultaneous eigenfunctions of the operators $\mathbb{D}_i$, with eigenvalues $a_i(a_i+d-2)$. If d is real and $> n-1$, then we can define a Hilbert space $L^2(\mathcal{S}_n^{1,+})$ using the positive definite scalar product (20). Since $\mathcal{S}_n^{1,+}$ is compact and the restrictions to $\mathcal{S}_n^1$ of polynomials on $\mathcal{S}_n$ separate points, these restrictions are dense in $L^2(\mathcal{S}_n^{1,+})$ by the Stone-Weierstrass theorem. It follows that $L^2(\mathcal{S}_n^{1,+})$ has an orthogonal Hilbert space direct sum decomposition as $\bigoplus_{\mathbf{a}} \mathcal{P}_{\mathbf{a}}(d)\big|_{\mathcal{S}_n^{1,+}}$ and that this is simply the eigenspace decomposition of the collection of commuting self-adjoint operators $\mathbb{D}_i$.

Proof of Theorem 2. We begin by proving (b), the relationship between the two scalar products $(P,Q)_d$ and $(P,Q)_d^1$ when P and Q are multihomogeneous, after which we can work only with $(P,Q)_d$, which is easier. We can also simplify by observing that $(P,Q)_d$ and $(P,Q)_d^1$ can be written as $\mathsf{E}_d[P\bar{Q}]$ and $\mathsf{E}_d^1[P\bar{Q}]$, respectively, where E_d and E_d^1 are the maps from $\mathbb{C}[\mathcal{S}_n]$ to $\mathbb{C}$ defined by $\mathsf{E}_d[P] = (P,1)_d$ and $\mathsf{E}_d^1[P] = (P,1)_d^1$. This is more convenient since these functions depend on only one rather than on two arguments. The statement of (b) then becomes that $\mathsf{E}_d[P] = \varepsilon_{\mathbf{a}}(d)\, \mathsf{E}_d^1[P]$ for $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$. To prove it, we observe that every $T \in \mathcal{S}_n^+$ can be written uniquely as $\lambda T_1 \lambda$ where λ is a diagonal matrix with positive entries and $T_1 \in \mathcal{S}_n^1$, the values of λ and T being given by $t_{ii} = \lambda_i^2$ and $t_{ij} = \lambda_i\lambda_j\tau_{ij}$ for $i \neq j$, where τ_{ij} are the entries of T_1. We then have

$$dT \;=\; \prod_{i=1}^{n}(2\lambda_i\, d\lambda_i) \cdot \prod_{i<j}(\lambda_i\lambda_j d\tau_{ij}) \;=\; (2\lambda_1 \cdots \lambda_n)^n\, dT_1\, d\lambda\,,$$

and of course $\det(T) = (\lambda_1 \cdots \lambda_n)^2 \det(T_1)$, so (4) and the definitions of E_d and E_d^1 give

$$\begin{aligned}\mathsf{E}_d[P] &= 2^n\, c_n(d) \int_{\mathbb{R}_+^n} \int_{\mathcal{S}_n^{1,+}} e^{-\|\lambda\|^2/2}\, \lambda^{\mathbf{a}}\, P(T_1)\, (\lambda_1 \cdots \lambda_n)^{d-1}\, \det(T_1)^{(d-n-1)/2}\, dT_1\, d\lambda \\ &= \mathsf{E}_d^1[P] \cdot \frac{c_n(d)}{c_n^1(d)} \prod_{i=1}^n \biggl(2 \int_0^\infty e^{-\lambda^2/2}\, \lambda^{d+a_i-1}\, d\lambda \biggr) \;=\; \varepsilon_{\mathbf{a}}(d)\, \mathsf{E}_d^1[P]\end{aligned}$$

for $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$, where the last equality follows from by (18), (14) and the first of equations (21) because the expression in parentheses equals $2^{d/2}\Gamma(d/2)\, \varepsilon_{a_i}(d)$ for a_i even and because both $\mathsf{E}_d[P]$ and $\mathsf{E}_d^1[P]$ vanish if a_i is odd (as one sees by replacing T by $\lambda T \lambda$, where λ is the diagonal matrix with $\lambda_i = -1$ and $\lambda_j = 1$ for $j \neq i$).

We next show the equality of the right-hand sides of (19) and (15) when d is an integer $\geq n$. (The equality of the right-hand sides of (20) and (16) then also follows since we have proved the proportionality result (17) for both definitions of the scalar products.) We must show that

$$c_n(d) \int_{\mathcal{S}_n^+} F(T)\, \det(T)^{(d-n-1)/2}\, dT \;=\; (2\pi)^{-nd/2} \int_{(\mathbb{R}^d)^n} F(XX^t)\, dX\,. \tag{22}$$

for rapidly decreasing functions $F : \mathcal{S}_n^+ \to \mathbb{C}$. (Apply this to $F(T) = P(T)\, \overline{Q(T)}\, e^{-\mathrm{tr}(T)/2}$.) To do this, we will decompose the Lebesgue measure on $\mathbb{R}^{dn}$ into a T-part and another part. We first decompose X as SV where S is a lower triangular $n \times n$ matrix with positive diagonal entries and V an $n \times d$ matrix with orthonormal rows, i.e., we write each x_i $(1 \leq i \leq n)$ as $\sum_{1 \leq j \leq i} s_{ij}\, v_j$ with $s_{ii} > 0$ and $v_1, \ldots, v_n$ orthogonal unit vectors in $\mathbb{R}^d$ (Gram-Schmidt orthonormalization process). The numbers s_{ij} and the components of the vectors v_j can be taken as local coordinates, so we can decompose dX into an S-part and a V-part. The vector v_1 belongs to the unit sphere in $\mathbb{R}^d$, so for the vector $x_1 = s_{11} v_1$ we get $dx_1 = s_{11}^{d-1}\, ds_{11}\, d\mu_{d-1}$, where $d\mu_{d-1}$ denotes the appropriately normalized standard measure on the sphere S^{d-1}. Once we have fixed v_1, then the vector v_2 belongs to a $(d-2)$-dimensional sphere and for the vector $x_2 = s_{21} v_1 + s_{22} v_2$ we get $dx_2 = s_{22}^{d-2}\, ds_{21} ds_{22}\, d\mu_{d-2}$. Continuing the process, we get

$$dX \;=\; \prod_{1 \leq i \leq n} s_{ii}^{d-i} \prod_{1 \leq j \leq i \leq n} ds_{ij}\; dV\,,$$

where $dV := d\mu_{d-1}\, d\mu_{d-2} \cdots d\mu_{d-n}$ is the measure on the V-coordinates. By definition, we have $T = XX^t = SVV^tS^t = SS^t$. This gives

$$dT \;=\; \prod_{i \leq j} dt_{ij} \;=\; 2^n \prod_{i=1}^n s_{ii}^{n-i+1} \prod_{i \geq j} ds_{ij}\,. \tag{23}$$

(We have $t_{ii} = s_{ii}^2 + \cdots$ and $t_{ij} = s_{ii} s_{ji} + \cdots$ for $j > i$, where $\cdots$ denotes terms of higher index if we order the pairs (i,j) lexicographically.) Comparing the last two formulas and noting that $\prod_{i=1}^n s_{ii} = \det(T)^{1/2}$, we get

$$dX = 2^{-n}\, \det(T)^{\frac{d-n-1}{2}} \prod_{i \leq j} dT\, dV\,,$$

and since V runs over $\prod_{i=0}^{n-1} S^{d-i-1}$ this implies (22), with $c_n(d)$ given by

$$c_n(d) \;=\; 2^{-n}\, (2\pi)^{-nd/2} \prod_{i=0}^{n-1} \mathrm{Vol}(S^{d-i-1}) \;=\; 2^{-nd/2}\, \pi^{-n(n-1)/4} \prod_{i=0}^{n-1} \Gamma\bigl(\frac{d-i}{2}\bigr)^{-1}\,,$$

in agreement with the formula in (21).

We now see that essentially the same calculation works also for non-integral values of d, because the decomposition $T = SS^t$ did not involve the V-part of the variable $X \in \mathbb{R}^{nd}$ but only the $n \times n$ matrix S. The maps $S \mapsto SS^t$ defines a bijection between the space $\mathfrak{L}_n^+$ of lower triangular $n \times n$ matrices S with positive diagonal entries and the space $\mathcal{S}_n$ of positive definite symmetric $n \times n$ matrices. The space $\mathfrak{L}_n^+$ is an open subset of a Euclidean space with coordinates s_{ij} $(1 \le j \le i \le n)$ and natural Lebesgue measure $dS = \prod_{j \ge i} ds_{ij}$, so by (23) we find

$$\mathsf{E}_d[P] \;=\; 2^n\, c_n(d) \int_{\mathfrak{L}_n^+} e^{-\mathrm{tr}(SS^t)/2}\, \widehat{P}(S) \prod_{i=1}^{n} s_{ii}^{d-i}\, dS \;=:\; \mathsf{E}_d^{\mathfrak{L}}[\widehat{P}] \tag{24}$$

for any $P \in \mathbb{C}[\mathcal{S}_n]$, where $\widehat{P} \in \mathbb{C}[\mathfrak{L}_n]^{\mathrm{ev}}$ (the space of polynomials on lower triangular $n \times n$ matrices that are invariant under $S \mapsto S\lambda$ for any diagonal matrix λ with all entries ± 1) is defined by $\widehat{P}(S) = P(SS^t)$. (The formula for $\widehat{P}(S)$ is the same as the formula for $\widetilde{P}(S)$ in the special case $n = d$ if we consider $\mathfrak{L}_n$ as a subspace of the space $(\mathbb{R}^n)^n$ of all $n \times n$ matrices, but the roles of X and S are quite different and we prefer to use a different notation.) This formula makes it clear that the integral converges for all polynomials if (and only if) d is a complex number with real part $> n-1$, since for convergence in (24) we need $d - i > -1$ for all $1 \le i \le n$. This completes the proof of part (a) of the theorem, but part (c) now also follows immediately, since we have the explicit formula

$$\begin{aligned}\mathsf{E}_d^{\mathfrak{L}}\Bigl[\prod_{j \le i} s_{ij}^{m_{ij}}\Bigr] \;&=\; 2^n c_n(d) \prod_{1 \le j < i \le n} \Bigl(\int_{-\infty}^{\infty} s^{m_{ij}} e^{-s^2/2}\, ds\Bigr) \cdot \prod_{1 \le i \le n} \Bigl(\int_0^{\infty} s^{m_{ii}+d-i} e^{-s^2/2}\, ds\Bigr) \\ &=\; \begin{cases} \displaystyle\prod_{1 \le j < i \le n} (m_{ij}-1)!! \cdot \prod_{1 \le i \le n} \varepsilon_{m_{ii}}(d-i+1) & \text{if all } m_{ij} \text{ are even} \\ 0 & \text{otherwise,} \end{cases}\end{aligned} \tag{25}$$

where $(m-1)!!$ is defined as usual as 1 for $m = 0$ and as $1 \times 3 \times \cdots \times (m-1)$ for $m > 0$ odd. (Here one has to note that the integral over ds_{ij} for $j < i$ vanishes if m_{ij} is odd because we are integrating an odd function over all of $\mathbb{R}$, and that if all of the m_{ij} with $j \ne i$ are even then the diagonal entries are always even because for a monomial $\prod s_{ij}^{m_{ij}}$ in $\mathbb{C}[\mathfrak{L}_n]^{\mathrm{ev}}$ the sum of the exponents m_{ij} for any fixed value of j is even. We are therefore in the case where the second formula in equation (14) can be applied.) It thus follows that $\mathsf{E}_d[P] = \mathsf{E}_d^{\mathfrak{L}}[\widehat{P}] \in \mathbb{Z}[d]$ for any polynomial $P \in \mathbb{Z}[\mathcal{S}_n]$, and that $\mathsf{E}_d[1] = 1$, proving (c).

Finally, we observe that the orthogonality statement (d), the most important part of the theorem, now follows formally from the other parts, since we already know that it is true when d is an integer $\ge n$ and a polynomial which vanishes for infinitely many values of its argument must be identically zero. A more direct proof for arbitrary d will follow from results given in the next section, where we will show that

$$\bigl(a_i - b_i\bigr)\, (P, Q)_d \;=\; \bigl(D_i P,\, Q\bigr)_d - \bigl(P,\, D_i Q\bigr)_d \tag{26}$$

for every $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$, $Q \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{b}}$ and every $1 \le i \le n$, and hence $(a_i - b_i)(P, Q)_d = 0$ if $P, Q \in \mathrm{Ker}(D_i)$. $\square$

We now discuss the actual calculation of the scalar products. In the case when $d \ge n$ is an integer, this is easy because the expression $\mathsf{E}_{\mathbb{R}^d}\bigl[f\bigr]$ can be calculated trivially for monomials. (For $d = 1$ one has $\mathsf{E}_{\mathbb{R}}\bigl[x^m\bigr] = (m-1)!!$ for m even and $\mathsf{E}_{\mathbb{R}}\bigl[x^m\bigr] = 0$ for m odd, and the values for $\mathbb{R}^d$ or $(\mathbb{R}^d)^n$ are obtained from this by multiplicativity.) For d complex the integral (19), although it is convergent if d is large enough, is useless for computations since it involves a complex power of a polynomial with many terms. We mention three[3] quite different effective methods—each of which could be used as an alternative definition instead of the integral (19)—for calculating the scalar products $(P,Q)_d$ explicitly as polynomials in d.

(i) The first way is to use the formula (24) together with the explicit evaluation (25). This can be programmed very easily.

(ii) The second method is to use rules that will be given in the next section (Proposition 1) to write the value of E_d on any polynomial of the form $t_{ij}P$ in terms of its values on P and derivatives of P. This gives E_d for all monomials inductively starting from the value $\mathsf{E}_d[1]=1$.

(iii) The third, and in some sense most explicit, method comes from the following theorem, which gives a generating function for all monomials $P(T)=T^{\boldsymbol{\nu}}$. To state it, we introduce the notation

$$\boldsymbol{\nu}! \;=\; \prod_{1\le i\le n} \nu_{ii}!! \;\cdot\; \prod_{1\le i<j\le n} \nu_{ij}! \tag{27}$$

for $\boldsymbol{\nu}=\bigl(\nu_{ij}\bigr)_{1\le i,\,j\le n}\in\mathcal{N}_n$, where $\nu!!$ is defined as $2^{\nu/2}(\nu/2)!$ for ν even. Then we have:

Theorem 3. *Let Y be a symmetric matrix with complex entries with $1-Y\overline{Y^t}>0$. Then*

$$\sum_{\boldsymbol{\nu}\in\mathcal{N}_n} \mathsf{E}_d\bigl[T^{\boldsymbol{\nu}}\bigr]\,\frac{Y^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \;=\; \det\bigl(1\,-\,Y\bigr)^{-d/2} \;=\; \exp\biggl(\frac{d}{2}\sum_{r=1}^{\infty}\frac{\operatorname{tr}(Y^r)}{r}\biggr)\,. \tag{28}$$

Proof. Both series in (28) converge exponentially under the condition stated, since then all eigenvalues of Y are less than 1 in absolute value. (This is actually not very important for us, since in applications of the theorem Y will always be a formal variable.) By analytic continuation, we can assume that Y is real. The second equality in (28) is standard, so we need only prove the first. The basic calculation is

$$\sum_{\boldsymbol{\nu}\in\mathcal{N}} \frac{A^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \;=\; \prod_{1\le i\le n}\biggl(\sum_{\nu=0}^{\infty}\frac{a_{ii}^{\nu}}{2^{\nu}\nu!}\biggr)\;\cdot\;\prod_{1\le i<j\le n}\biggl(\sum_{\nu=0}^{\infty}\frac{a_{ij}^{\nu}}{\nu!}\biggr) \;=\; \exp\biggl(\frac{1}{2}\sum_{1\le i,j\le n} a_{ij}\biggr)\,,$$

valid for any symmetric $n\times n$ matrix A. Applying it to the matrix with entries $a_{ij}=t_{ij}y_{ij}$ gives

$$\sum_{\boldsymbol{\nu}\in\mathcal{N}} \frac{T^{\boldsymbol{\nu}}Y^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \;=\; e^{\operatorname{tr}(TY)/2} \tag{29}$$

for any T, $Y\in\mathcal{S}_n$ and hence

$$\sum_{\boldsymbol{\nu}\in\mathcal{N}_n} \mathsf{E}_d\bigl[T^{\boldsymbol{\nu}}\bigr]\,\frac{Y^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \;=\; c_n(d)\int_{\mathcal{S}_n^+} e^{-\operatorname{tr}(T(1-Y))/2}\,\det(T)^{(d-n-1)/2}\,dT$$

for sufficiently small $Y\in\mathcal{S}_n$. We can write $1-Y=U^2$ for some invertible $n\times n$ matrix $U\in\mathcal{S}_n$. Then $\operatorname{tr}(T(1-Y))=\operatorname{tr}(UTU)$. The change of variables from T to UTU is a bijection from $\mathcal{S}_n^+$ to itself and sends $\det(T)$ to $\det(U)^2\det(T)$ and dT to $\det(U)^{n+1}dT$, so the value of the expression on the right is $\det(U)^{-d}$ times its value for $U=1$, which equals 1 by the definition of the normalizing constant $c_n(d)$. $\square$

[3] Another effective method uses the Corollary to Proposition 2 in §4 below. All values given in the tables at the end of the paper were computed independently from at least two of these four formulas.

We remark that a related, but somewhat simpler proof of Theorem 3 can be given in the case when $d \ge n$ is an integer (which suffices to prove the general case since (28) is equivalent to a collection of polynomial identities in d). Again using the identity (29), we find in this case

$$\begin{aligned}\sum_{\boldsymbol{\nu}\in\mathcal{N}_n} \mathsf{E}_d\big[T^{\boldsymbol{\nu}}\big]\,\frac{Y^{\boldsymbol{\nu}}}{\boldsymbol{\nu}!} \;&=\; (2\pi)^{-nd/2}\int_{(\mathbb{R}^d)^n} e^{-\mathrm{tr}(XX^t(1-Y))/2}\,dX\\ &=\; (2\pi)^{-nd/2}\int_{(\mathbb{R}^n)^d} e^{-\mathrm{tr}(X(1-Y)X^t)/2}\,dX \qquad\qquad (X\to X^t)\\ &=\; \Big((2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-x(1-Y)x^t/2}\,dx\Big)^d \;=\; \det(1-Y)^{-d/2}\,,\end{aligned}$$

where we have used the standard identity $\pi^{-n/2}\int_{\mathbb{R}^n} e^{-xAx^t}dx = (\det A)^{-1/2}$ for $A\in\mathcal{S}_n^+$.

As a simple example of the application of the theorem, we have

$$\sum_{m=0}^{\infty} \mathsf{E}_d\big[t_{ii}^m\big]\,\frac{y^m}{2^m m!} \;=\; (1-y)^{-d/2} \;=\; \sum_{m=0}^{\infty}\frac{\varepsilon_{2m}(d)}{2^m m!}\,y^m\,.$$

Of course here we knew the answer already since $\mathsf{E}_d\big[t_{ii}^m\big] = \varepsilon_{2m}(d)\,\mathsf{E}_d^1\big[t_{ii}^m\big] = \varepsilon_{2m}(d)\,\mathsf{E}_d^1[1] = \varepsilon_{2m}(d)$. As a less trivial example, we take $Y = y\,\mathbf{e}_{ij}$ ($\mathbf{e}_{ij}$ as in §1) for $i\ne j$ to get

$$\sum_{m=0}^{\infty} \mathsf{E}_d\big[t_{ij}^m\big]\,\frac{y^m}{m!} \;=\; \det\big(1 - y\,\mathbf{e}_{ij}\big)^{-d/2} \;=\; \big(1-y^2\big)^{-d/2}$$

and hence

$$\mathsf{E}_d\big[t_{ij}^m\big] \;=\; \begin{cases} (m-1)!!\,\cdot\,2^{m/2}\,(d/2)_{m/2} & \text{if } m \text{ is even,}\\ 0 & \text{if } m \text{ is odd.}\end{cases} \tag{30}$$

§4. The Lie algebra of mixed Laplacians

Let us return for the moment to the case that d is an integer $\ge n$ and that our polynomials on $\mathcal{S}_n$ correspond via $P\mapsto\widetilde{P}$ to the $O(d)$-invariant polynomials on $(\mathbb{R}^d)^n$. Then as well as the "pure" Laplace operators

$$\Delta_i = \Delta_{ii} = \Big(\frac{\partial}{\partial x_i}\Big)\Big(\frac{\partial}{\partial x_i}\Big)^t = \sum_{\alpha=1}^{d}\frac{\partial^2}{\partial x_{i\alpha}^2} \qquad (1\le i\le n)$$

on $\mathbb{C}[(\mathbb{R}^d)^n]$ that were used to define $\mathcal{P}_{\mathbf{a}}(d)$, we also have the "mixed" Laplace operators

$$\Delta_{ij} = \Big(\frac{\partial}{\partial x_i}\Big)\Big(\frac{\partial}{\partial x_j}\Big)^t = \sum_{\alpha=1}^{d}\frac{\partial^2}{\partial x_{i\alpha}\,\partial x_{j\alpha}} \qquad (1\le i,\,j\le n)\,.$$

The same calculation as for Δ_i shows that these correspond under the map (1) to the differential operators

$$D_{ij} = D_{ij}^{(d)} = d\,\partial_{ij} + \sum_{k,l=1}^{n} t_{kl}\,\partial_{ik}\partial_{jl}\,. \tag{31}$$

The D_{ij}, which can now be defined for this equation also for $d<n$ or $d\notin\mathbb{Z}$, commute with one other and in particular with the pure Laplacians $D_i = D_{ii}$, so they preserve the condition

of being harmonic, i.e. D_{ij} $(i \neq j)$ maps $\mathcal{P}_{\mathbf{a}}(d)$ to $\mathcal{P}_{\mathbf{a}-\mathbf{e}_i-\mathbf{e}_j}(d)$, where $\mathbf{e}_i$ is the vector with 1 in the ith place and 0's elsewhere. These "intertwining" operators will turn out to be very useful.

In exactly the same way, as well as the operators $E_i : \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \to \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ corresponding to the Euler operators $\sum_\alpha x_{i,\alpha}\partial/\partial x_{i,\alpha}$ (Remark following the Proposition-Definition in §1) we also have maps $E_{ij} : \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \to \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}+\mathbf{e}_i-\mathbf{e}_j}$ for $i \neq j$, given explicitly by $E_{ij} = \sum_{k=1}^n t_{ik}\,\partial_{jk}$, which correspond when d is integral and $T = XX^t$ to the "mixed Euler operators" $\sum_\alpha x_{i,\alpha}\partial/\partial x_{j,\alpha}$ (because these are again $O(d)$-invariant). Finally, we have operators $F_{ij} : \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \to \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}+\mathbf{e}_i+\mathbf{e}_j}$ for all $i,\, j = 1, \ldots, n$ given by multiplication with t_{ij}. In this section we will show that the vector space spanned by all of these operators and by the identity is closed under commutators and also under adjoints with respect to the scalar product defined in the previous section.

To check the Lie property involves computing a large number of different commutators. We have six kinds of commutators ($[D, D]$, $[D, E]$, $[D, F]$, $[E, E]$, $[E, F]$ and $[F, F]$) and for each one a number of different special cases depending on which indices are distinct and which are equal. For instance, to give the commutators of the operators D_{ij} with F_{kl} we have to consider seven different cases, obtaining after somewhat tedious computations the seven formulas

$$[D_{ii}, F_{ii}] = 4E_i + 2d, \quad [D_{ii}, F_{ik}] = 2E_{ki}, \quad [D_{ii}, F_{kl}] = 0,$$
$$[D_{ij}, F_{ij}] = E_i + E_j + d, \quad [D_{ij}, F_{ii}] = 2E_{ij}, \quad [D_{ij}, F_{ik}] = E_{kj}, \quad [D_{ij}, F_{kl}] = 0$$

for $i \neq j$ and $k,\, l \neq i,\, j$. If we modify the definition of E_{ij} for $i = j$ by setting $E_{ii} = E_i + d/2$ rather than by the more natural-seeming formula $E_{ii} = E_i$, then the scalar terms "$2d$" and "d" in these equations drop out and the seven cases can be written in a uniform way. Moreover, somewhat surprisingly, the corresponding simplification also occurs in all the other types of commutators if we make the same substitution; in other words, the vector space spanned by the operators D_{ij}, E_{ij} and F_{ij} is already closed under commutators, without having to include also the identity operator. More precisely, we have:

Theorem 4. *For fixed $n \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{C}$, the vector space $\mathfrak{g} \subset \mathrm{End}(\mathbb{C}[\mathcal{S}_n])$ spanned by*

$$D_{ij} = d\partial_{ij} + \sum_{k,l=1}^n t_{kl}\partial_{ik}\partial_{jl}\,, \quad E_{ij} = \frac{d}{2}\,\delta_{ij} + \sum_{k=1}^n t_{ik}\partial_{jk}\,, \quad F_{ij} \;=\; t_{ij} \quad (1 \leq i,\, j \leq n) \tag{32}$$

is a Lie subalgebra, with commutators given by

$$[D_{ij}, D_{kl}] = 0, \quad [D_{ij}, E_{kl}] = \delta_{ik}D_{jl} + \delta_{jk}D_{il}, \quad [D_{ij}, F_{kl}] = \delta_{ik}E_{lj} + \delta_{il}E_{kj} + \delta_{jk}E_{li} + \delta_{jl}E_{ki},$$
$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{il}E_{kj}, \quad [E_{ij}, F_{kl}] = \delta_{jk}F_{il} + \delta_{jl}F_{ik}, \quad [F_{ij}, F_{kl}] = 0 \quad (1 \leq i, j, k, l \leq n)\,.$$

Proof. As already indicated, this is just a straightforward but lengthy computation, with many cases to be checked. We give the proof of the second commutation relation as an example. From the definitions we have

$$[D_{ij}, E_{kl}] \;=\; d\sum_m [\partial_{ij},\, t_{km}\partial_{lm}] \;+\; \sum_{r,s,m} [t_{rs}\partial_{ir}\partial_{js},\, t_{km}\partial_{lm}]\,,$$

where all indices run from 1 to n. The first term is equal to $d\sum_m(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\partial_{lm} = d\,(\delta_{ik}\partial_{jl} + \delta_{jk}\partial_{il})$, and the second to

$$\sum_{r,\,s,\,m} \big[(\delta_{jk}\delta_{sm} + \delta_{jm}\delta_{ks})t_{rs}\partial_{ir}\partial_{lm} + (\delta_{ik}\delta_{rm} + \delta_{im}\delta_{rk})t_{rs}\partial_{js}\partial_{lm} - (\delta_{lr}\delta_{ms} + \delta_{ls}\delta_{mr})t_{km}\partial_{ir}\partial_{js}\big]$$
$$= \;\delta_{jk}\sum_{r,\,s} t_{rs}\partial_{ir}\partial_{ls} + (1-1)\sum_r t_{kr}\partial_{ir}\partial_{jl} + \delta_{ik}\sum_{r,\,s} t_{rs}\partial_{jr}\partial_{ls} + (1-1)\sum_s t_{ks}\partial_{js}\partial_{il}\,.$$

Adding these gives the desired result $[D_{ij}, E_{kl}] = \delta_{ik} D_{jl} + \delta_{jk} D_{il}$. The other statements in the theorem can be checked in the same way. Alternatively, since each of the commutation relations among the generators of $\mathfrak{g}$ is polynomial in d and therefore need only be checked for values of $d \in \mathbb{Z}_{\geq n}$, we can work instead with the mixed Laplace and Euler operators and the operations of multiplying by (x_i, x_j) on the space of polynomials in $(\mathbb{R}^d)^n$ and check the commutation relations there; then the calculations are different but the final result is the same. $\square$

Observe that the operators D_{ij} and F_{ij} are symmetric in the two indices i and j, whereas E_{ij} is not, so the dimension of the Lie algebra $\mathfrak{g}$ is equal to $\dfrac{n(n+1)}{2} + n^2 + \dfrac{n(n+1)}{2} = 2n^2 + n$. This is the same as the dimension of the simple Lie algebra

$$\mathfrak{sp}(2n, \mathbb{R}) = \{X \in M_{2n}(\mathbb{R}) \mid XJ_n = -J_n X^t\}\,, \qquad J_n = \begin{pmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{pmatrix},$$

and indeed (working over $\mathbb{R}$) we have an isomorphism $\mathfrak{g} \cong \mathfrak{sp}(2n, \mathbb{R})$, as we can see explicitly by making the assignments

$$D_{ij} \;\mapsto \begin{pmatrix} 0 & 0 \\ -e_{ij} - e_{ji} & 0 \end{pmatrix}, \qquad E_{ij} \;\mapsto \begin{pmatrix} e_{ij} & 0 \\ 0 & -e_{ji} \end{pmatrix}, \qquad F_{ij} \;\mapsto \begin{pmatrix} 0 & e_{ij} + e_{ji} \\ 0 & 0 \end{pmatrix}$$

(where e_{ij} $(1 \leq i,\, j \leq n)$ is the elementary $n \times n$ matrix with 1 in the (i,j)th place and zeros elsewhere) and comparing the commutators in $\mathfrak{g}$ as given in the proposition with those in the Lie algebra $\mathfrak{sp}(2n)$. A better understanding of why $\mathfrak{sp}(2n)$ occurs comes from the discussion given in §1 about the connection between higher spherical polynomials and Siegel modular forms. The Lie group $Sp(2n, \mathbb{R})$ acts on the space of holomorphic functions $F : \mathfrak{H}_n \to \mathbb{C}$ by $(F|g)(Z) = \det(CZ + D)^{-d/2} F((AZ + B)(CZ + D)^{-1})$ for $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in G$. (This is the action used to define Siegel modular forms of weight $d/2$.) The action of G induces an action $F \mapsto F|X$ of its Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ via $F|e^{\varepsilon X} = F + (F|X)\varepsilon + \mathrm{O}(\varepsilon^2)$, and from the calculation (setting $\varepsilon^2 = 0$ and $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$)

$$\begin{aligned} F\big|\left(\begin{smallmatrix} 1+\varepsilon A & \varepsilon B \\ \varepsilon C & 1+\varepsilon D \end{smallmatrix}\right)(Z) \;&\equiv\; \det\bigl(1 + \varepsilon(CZ + D)\bigr)^{-d/2} F\bigl((Z + \varepsilon(AZ + B))(1 + \varepsilon(CZ + D))^{-1}\bigr) \\ &\equiv\; \Bigl(1 - \varepsilon \frac{d}{2} \operatorname{tr}(CZ + D)\Bigr) F\bigl(Z + \varepsilon(AZ + B - Z(CZ + D))\bigr) \end{aligned}$$

we see that this action is given explicitly by

$$F|\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)(Z) \;=\; -\frac{d}{2} \operatorname{tr}(CZ + D)\, F(Z) \;+\; \sum_{i,\,j=1}^{n} \bigl(AZ + B - Z(CZ + D)\bigr)_{ij}\, \partial^*_{ij} F(Z)\,,$$

where $\partial^*_{ij} = \frac{1}{2}(1 + \delta_{ij})\partial/\partial z_{ij}$. Using the above isomorphism $\mathfrak{g} \cong \mathfrak{sp}(2n, \mathbb{R})$, we then find

$$D_{ij} \;\mapsto\; d z_{ij} + \sum_{k,l=1}^{n} z_{ik} z_{jl}\, \partial^*_{kl}\,, \qquad E_{ij} \;\mapsto\; \frac{d}{2}\, \delta_{ij} + \sum_{k=1}^{n} z_{ik}\, \partial^*_{jk}\,, \qquad F_{ij} \;\mapsto\; \partial^*_{ij}$$

and this agrees with the definition of the generators in $\mathfrak{g}$ if we remember from the discussion in §1 that the variables T (argument of the higher spherical polynomials) and Z (argument of the Siegel modular forms) are dual variables with respect to the Fourier expansion, so that t_{ij} and ∂_{ij} correspond to ∂^*_{ij} and z_{ij}. In fact, we can use this correspondence to see directly that the generators of $\mathfrak{g}$ correspond to the standard generators of $\mathfrak{sp}(2n)$, thus obtaining yet a third proof of the commutation relations in Theorem 2. This is certainly related to the Weil representation ([15], [13]), but we will not discuss this any further here because the identification of $\mathfrak{g}$ with $\mathfrak{sp}(2n,\mathbb{R})$ will not play any role in this paper.

We now relate the operators in $\mathfrak{g}$ to the scalar products that were defined in §3. Since each of the generators in (32) is homogeneous, in the sense that it sends $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ to $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}+\boldsymbol{\delta}}$ for some fixed $\boldsymbol{\delta}\in\mathbb{Z}^n$, and since the two scalar products $(\ ,\ )_d$ and $(\ ,\ )_d^1$ are proportional in each multi-degree, we only have to give the formulas for $(\ ,\ )_d$, where they are simpler. Recall from §3 that $(P,Q)_d=\mathsf{E}_d[PQ]$ for the linear map $\mathsf{E}_d:\mathbb{C}[\mathcal{S}_n]\to\mathbb{C}$ defined by $\mathsf{E}_d[P]=\mathsf{E}_{(\mathbb{R}^d)^n}\big[\widetilde{P}\big]$.

Proposition 1. *For arbirary $d\in\mathbb{C},\ i,\,j\in\{1,\dots,n\}$ and $P\in\mathbb{C}[\mathcal{S}_n]$ we have*

$$\mathsf{E}_d[F_{ij}P]\;=\;\mathsf{E}_d\big[\big(E_{ij}\,+\,\frac{d}{2}\,\delta_{ij}\big)P\big]\;=\;\mathsf{E}_d[\big(D_{ij}\,+\,d\,\delta_{ij}\big)P]\,. \tag{33}$$

Proof. Since each expression in (33) is a polynomial in d for any P, we may assume that $d\in\mathbb{Z}_{\ge n}$ and that $(\ ,\ )_d$ is defined by (15). Note first that the operator $\mathsf{E}_{\mathbb{R}}\big[f\big]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,e^{-x^2/2}\,dx$ (the special case $n=d=1$ of E_d) satisfies the relation $\mathsf{E}_{\mathbb{R}}\big[xf\big]=\mathsf{E}_{\mathbb{R}}\big[f'\big]$ for any $f\in\mathbb{C}[x]$, as one can see either from the explicit formulas $\mathsf{E}_{\mathbb{R}}\big[x^{2m}\big]=(2m-1)!!$ and $\mathsf{E}_{\mathbb{R}}\big[x^{2m-1}\big]=0$ or, more naturally, by using the identity

$$\mathsf{E}_{\mathbb{R}}\big[f'\big]\;-\;\mathsf{E}_{\mathbb{R}}\big[xf\big]\;=\;\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\Big(e^{-x^2/2}\,f(x)\Big)\;=\;0\,,$$

i.e., by integration by parts. This immediately gives $\mathsf{E}_{\mathbb{R}^d}\big[x_\alpha f\big]=\mathsf{E}_{\mathbb{R}^d}\big[\partial f/\partial x_\alpha\big]$ $(\alpha=1,\dots,d)$. for any $f\in\mathbb{C}[\mathbb{R}^d]$ and hence

$$\mathsf{E}_{(\mathbb{R}^d)^n}\big[x_{i\alpha}f\big]\;=\;\mathsf{E}_{(\mathbb{R}^d)^n}\big[\frac{\partial f}{\partial x_{i\alpha}}\big]\qquad\quad(i=1,\dots,n,\quad\alpha=1,\dots,d)\,. \tag{34}$$

for any $f\in\mathbb{C}[(\mathbb{R}^d)^n]$. These formulas allow us to compute $\mathsf{E}_{\mathbb{R}}$, $\mathsf{E}_{\mathbb{R}^d}$ and $\mathsf{E}_{(\mathbb{R}^d)^n}$ on all monomials (and hence on all polynomials) by induction on the degree.

Using (34) twice, we find, for all $F\in\mathbb{C}[(\mathbb{R}^d)^n]$ and all $1\le i,j\le n$, $1\le\alpha\le d$,

$$\begin{aligned}\mathsf{E}_{(\mathbb{R}^d)^n}\big[x_{i\alpha}x_{j\alpha}F(X)\big]\;&=\;\mathsf{E}_{(\mathbb{R}^d)^n}\big[\frac{\partial}{\partial x_{j\alpha}}\big(x_{i\alpha}F\big)\big]\\ &=\;\mathsf{E}_{(\mathbb{R}^d)^n}\big[\delta_{ij}F\,+\,x_{i\alpha}\frac{\partial F}{\partial x_{j\alpha}}\big]\;=\;\mathsf{E}_{(\mathbb{R}^d)^n}\big[\delta_{ij}F\,+\,\frac{\partial^2F}{\partial x_{i\alpha}\partial x_{j\alpha}}\big]\end{aligned}$$

and hence, summing over α,

$$\mathsf{E}_{(\mathbb{R}^d)^n}\big[\,(x_i,x_j)\,F\,\big]\;=\;\mathsf{E}_{(\mathbb{R}^d)^n}\big[\,d\,\delta_{ij}F\,+\,\sum_{\alpha=1}^{d}x_{i\alpha}\frac{\partial F}{\partial x_{j\alpha}}\,\big]\;=\;\mathsf{E}_{(\mathbb{R}^d)^n}\big[\,d\,\delta_{ij}\,F\,+\,\Delta_{ij}F\,\big]\,. \tag{35}$$

Applying this to $F=\widetilde{P}$ with $P\in\mathbb{C}[\mathcal{S}_n]$, we obtain (33) for all $d\in\mathbb{Z}_{\ge n}$ and hence for all d. $\square$

Notice that either of the equalities $\mathsf{E}_d[t_{ij}P]=\mathsf{E}_d[(E_{ij}+\frac{d}{2}\delta_{ij})P]$ or $\mathsf{E}_d[t_{ij}P]=\mathsf{E}_d[(D_{ij}+d\delta_{ij})P]$ in (33), together with the normalization $\mathsf{E}_d[1]=1$, defines E_d uniquely, because the argument of E_d on the right has lower degree than the one on the left. Proposition 1 therefore gives us a new definition (to be added to the three previous definitions (19), (24)/(25) and (28)) of the functional $\mathsf{E}_d:\mathbb{C}[\mathcal{S}_n]\to\mathbb{C}$ and hence of the scalar product $(\ ,\ )_d$ for arbitrary $d\in\mathbb{C}$. Notice also that the first equality in (33) for that $i=j$ gives the useful identity

$$\mathsf{E}_d[t_{ii}P]\;=\;(d+a_i)\,\mathsf{E}_d[P]\qquad\quad\text{for }P\in\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}\,. \tag{36}$$

This also follows from (17), since $\mathsf{E}_d[t_{ii}P]=\varepsilon_{\mathbf{a}+2\mathbf{e}_i}(d)\mathsf{E}_d^1[t_{ii}P]=\varepsilon_{\mathbf{a}+2\mathbf{e}_i}(d)\mathsf{E}_d^1[P]=(d+a_i)\mathsf{E}_d[P]$.

Proposition 1 describes the relation between the Lie algebra $\mathfrak{g}$ and the functional $P\mapsto(P,1)_d$. The following more precise result describes its relation to the whole scalar product.

Proposition 2. *The Lie algebra $\mathfrak{g}$ is equal to its own adjoint with respect to the action on the space $\mathbb{C}[\mathcal{S}_n]$ with the inner product $(P,Q)_d$. Explicitly, the adjoint operators of the standard basis elements of $\mathfrak{g}$ are given by*

$$D_{ij}^* \;=\; D_{ij} - E_{ij} - E_{ji} + F_{ij}\,, \qquad E_{ij}^* \;=\; -E_{ij} + F_{ij}\,, \qquad F_{ij}^* \;=\; F_{ij}\,. \tag{37}$$

Proof. The third equality in (37) is trivial since $(t_{ij}P)\,Q = P\,(t_{ij}Q)$ for any P and Q; notice that it does not matter whether we write "Q" or "$\overline{Q}$" since the integrations defining the scalar products are taken over real-valued variables. The second equality (which includes the third since $E_{ij} + E_{ij}^*$ is self-adjoint) follows from the first equality in (33) because $E_{ij} - \frac{d}{2}\delta_{ij}$ is a derivation. To prove the first equality, we assume once again that $d \in \mathbb{Z}_{\geq n}$ and that the scalar product is defined by (15). For F and G in $\mathbb{C}[(\mathbb{R}^d)^n]$ we have

$$\Delta_{ij}(FG) \;=\; \Delta_{ij}(F)\,G \,+\, F\,\Delta_{ij}(G) \,+\, \sum_{\alpha=1}^{d}\Bigl(\frac{\partial F}{\partial x_{i\alpha}}\,\frac{\partial G}{\partial x_{j\alpha}} \,+\, \frac{\partial F}{\partial x_{j\alpha}}\,\frac{\partial G}{\partial x_{i\alpha}}\Bigr)$$

and hence, denoting by $\equiv$ congruence modulo the kernel of E_d and using (34) and (35),

$$\begin{aligned} F\,\Delta_{ij}(G) \,-\, \Delta_{ij}(F)\,G \;&=\; -\Delta_{ij}(FG) \,+\, \sum_{\alpha=1}^{d}\biggl[\frac{\partial}{\partial x_{i\alpha}}\Bigl(F\,\frac{\partial G}{\partial x_{j\alpha}}\Bigr) \,+\, \frac{\partial}{\partial x_{j\alpha}}\Bigl(F\,\frac{\partial G}{\partial x_{i\alpha}}\Bigr)\biggr] \\ &\equiv\; -\bigl((x_i,x_j) - d\,\delta_{ij}\bigr)\,FG \,+\, F\,\sum_{\alpha=1}^{d}\Bigl(x_{i\alpha}\,\frac{\partial G}{\partial x_{j\alpha}} \,+\, x_{j\alpha}\,\frac{\partial G}{\partial x_{i\alpha}}\Bigr) \\ &=\; F\,\bigl(E_{ij} \,+\, E_{ji} \,-\, F_{ij}\bigr)\,G\,. \end{aligned}$$

The identity now follows by applying $\mathsf{E}_{(\mathbb{R}^d)^n}$ to both sides and replacing F and G by $\widetilde{P}$ and $\widetilde{Q}$ with P, $Q \in \mathbb{C}[\mathcal{S}_n]$. $\square$

Observe that equation (26), which was used in the last section to give a second and more direct proof of the orthogonality of $\mathcal{P}_{\mathbf{a}}(d)$ and $\mathcal{P}_{\mathbf{b}}(d)$ for $\mathbf{a} \neq \mathbf{b}$, follows from (36) and the special case $i = j$ of the first identity in (37).

As a further consequence of Proposition 2, we get the following nice formula for the scalar product of §3 with respect to the monomial basis of $\mathcal{P}(d)$.

Corollary. *For any two multi-indices $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ of the same total degree, the scalar product of $P_{\boldsymbol{\mu}}^M$ and $P_{\boldsymbol{\nu}}^M$ is given by*

$$\bigl(P_{\boldsymbol{\mu}}^M,\, P_{\boldsymbol{\nu}}^M\bigr)_d \;=\; D^{\boldsymbol{\nu}}\bigl(P_{\boldsymbol{\mu}}^M\bigr)\,, \tag{38}$$

where $D^{\boldsymbol{\nu}}$ denotes $\prod_{i<j} D_{ij}^{\nu_{ij}}$.

Proof. Let $\Pi = \Pi_{\mathbf{a}}^{(d)}$ be the projection from $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ onto the summand $\mathcal{P}_{\mathbf{a}}(d)$ in the decomposition (9). (We will give an explicit formula for this projection operator in eq. (44) of §5 using the results of this section, but we do not need it here.) Then we have

$$(D_{ij}P,\, Q)_d \;=\; (P,\, t_{ij}Q)_d \;=\; (P,\, \Pi(t_{ij}Q))_d \qquad \text{for } P \in \mathcal{P}_{\mathbf{a}}(d)\,,\;\; Q \in \mathcal{P}_{\mathbf{a}-\mathbf{e}_i-\mathbf{e}_j}(d)\,, \tag{39}$$

where the second equality follows from the definition of Π together with the second remark after Theorem 2, and the first from the first formula in (37) together with the fact that spherical polynomials are orthogonal to all polynomials of smaller degree (which is needed because the operator D^*_{ij}, like E^*_{ij}, is not homogeneous). On the other hand, for $i \neq j$, the definition of the monomial basis implies that

$$P^M_{\boldsymbol{\nu}} \;=\; \Pi^{(d)}_{\mathbf{a}}\big(t_{ij} P^M_{\boldsymbol{\nu}-\mathbf{e}_{ij}}\big) \qquad \text{for all } \;\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})\,, \tag{40}$$

since $t_{ij} P^M_{\boldsymbol{\nu}-\mathbf{e}_{ij}}$ differs from $T^{\boldsymbol{\nu}}$ by something belonging to the ideal of $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ generated by the t_{kk} and since the decomposition (9) implies (by induction on the total degree) that $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ is the direct sum of $\mathcal{P}_{\mathbf{a}}(d)$ and this ideal. Combining equations (40) and (39), we find

$$\big(P^M_{\boldsymbol{\mu}},\, P^M_{\boldsymbol{\nu}}\big)_d \;=\; \big(P^M_{\boldsymbol{\mu}},\, \Pi(t_{ij} P^M_{\boldsymbol{\nu}-\mathbf{e}_{ij}})\big)_d \;=\; \big(D_{ij}(P^M_{\boldsymbol{\mu}}),\, P^M_{\boldsymbol{\nu}-\mathbf{e}_{ij}})\big)_d\,.$$

Applying this identity repeatedly we find finally that this scalar product equals $(D^{\boldsymbol{\nu}}(P_{\boldsymbol{\mu}}), P^M_{\mathbf{0}})$. But $D^{\boldsymbol{\nu}}(P_{\boldsymbol{\mu}})$ is a constant (polynomial of degree 0), $P^M_{\mathbf{0}} = 1$, and $(1,1)_d$ has been normalized to be equal to 1. $\square$

Proposition 2 shows that each of the 3- or 4-dimensional subspaces $\langle D_{ij},\, E_{ij},\, E_{ji},\, F_{ij}\rangle$ of which $\mathfrak{g}$ is composed is already closed under taking adjoints. More precisely, the elements

$$\begin{aligned}
D^+_{ij} \;&=\; D^+_{ji} \;=\; D_{ij} \,-\, \frac{1}{2}\big(E_{ij} \,+\, E_{ji}\big) \qquad &&(1 \le i \le j \le n)\,,\\
F^+_{ij} \;&=\; F^+_{ji} \;=\; F_{ij} &&(1 \le i \le j \le n)\,,\\
E^-_{ij} \;&=\; -E^-_{ji} \;=\; E_{ij} \,-\, E_{ji} &&(1 \le i < j \le n)\,,\\
F^-_{ij} \;&=\; F^-_{ji} \;=\; F_{ij} \,-\, E_{ij} \,-\, E_{ji} &&(1 \le i \le j \le n)
\end{aligned} \tag{41}$$

give a basis of $\mathfrak{g}$ consisting of selfdual or anti-selfdual elements. In particular, the subspaces $\mathfrak{g}^{\pm} = \{X \in \mathfrak{g} \mid X^* = \pm X\}$ of $\mathfrak{g}$ have dimensions $n^2 + n$ and n^2, respectively, and $\mathfrak{g}^-$ is a sub-Lie algebra of $\mathfrak{g}$, since $[X,Y]^* = -[X^*, Y^*]$. More concretely we have

$$\mathfrak{g}^- \;=\; \left\{\begin{pmatrix} A & \tfrac{1}{2}(A + A^t) \\ 0 & -A^t \end{pmatrix}\right\} \;\cong\; \mathfrak{gl}(n)\,.$$

Finally, we remark that instead of proving Propositions 1 and 2 for integral values of $d \geq n$ and deducing the general case by polynomiality, one could prove them directly for all $d \in \mathbb{C}$ by using the description of $(\;,\;)_d$ in (24) together with the identity (again easily proved by integration by parts)

$$\mathsf{E}^{\mathfrak{L}}_d\big[s_{ij}\, F\big] \;=\; \mathsf{E}^{\mathfrak{L}}_d\Big[\frac{\partial F}{\partial s_{ij}} \,+\, \delta_{ij}\,\frac{d-i}{s_{ii}}\, F\Big] \qquad\quad \text{for all } \; F \in \mathbb{C}[\mathfrak{L}_n]^{\mathrm{ev}}\,.$$

However, this calculation is much more complicated than the one with the X-variables because the expressions for the operators E_{ij} and D_{ij} in terms of the S-variables are no longer polynomial, but involve powers of the diagonal elements s_{kk} of S in their denominators.

§5. The two canonical bases of $\mathcal{P}_{\mathbf{a}}(d)$

In §2 (Proposition 1 and the following remark) we observed that Theorem 1 implies the existence of a canonical basis $\{P^M_{\boldsymbol{\nu}}\}_{\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})}$ of $\mathcal{P}_{\mathbf{a}}(d)$, characterized by the property that $P^M_{\boldsymbol{\nu}}(T)$ becomes equal to $T^{\boldsymbol{\nu}}$ if one sets all t_{ii} equal to 0. In this section we will study this monomial basis in detail. First, we will give an explicit construction of the polynomials $P^M_{\boldsymbol{\nu}}$ in terms of a certain "harmonic projection operator" (Proposition 1). This at the same time gives a second, more constructive, proof of Theorem 1 and its corollary. Next, as an application of the Lie algebra $\mathfrak{g}$ of differential operators introduced above, we will give an inductive construction of the polynomials $P_{\boldsymbol{\nu}}^M$ in terms of certain "raising operators" belonging to the universal enveloping algebra of $\mathfrak{g}$. We then discuss the dual basis $\{P_{\boldsymbol{\nu}}^D\}$ of the basis $\{P_{\boldsymbol{\nu}}^M\}$ with respect to the scalar product from §3 (for generic values of $d \in \mathbb{C}$) and show that its members satisfy (and in fact are determined by) the recursive property $D_{ij}(P_{\boldsymbol{\nu}}^D) = P_{\boldsymbol{\nu}-\mathbf{e}_{ij}}^D$, where D_{ij} are the "mixed Laplace operators" from §4. Finally, again using the scalar product, we will show that each product $t_{ij}P_{\boldsymbol{\nu}}^M(T)$ can be written as a linear combination with constant coefficients of a bounded number of basis elements $P_{\boldsymbol{\nu}'}^M(T)$. This gives another recursive way to obtain the basis elements, generalizing the classical recursion relations for Legendre and Gegenbauer polynomials.

We start with the construction of the $P_{\boldsymbol{\nu}}^M$ by a projection operator. Denote by $\mathcal{H}(\mathbb{R}^d) = \operatorname{Ker}(\Delta) \subset \mathbb{C}[\mathbb{R}^d]$ the subspace of harmonic polynomials on $\mathbb{R}^d$. As we mentioned before the corollary to Theorem 1 in §2, it is well-known that every polynomial P on $\mathbb{R}^d$ can be written uniquely as

$$P(x) \;=\; \sum_{j\ge 0} P_m(x)\,(x,x)^m \tag{42}$$

where each P_m is harmonic (and homogeneous of degree $a-2m$ if P is homogeneous of degree a). Denote by $\pi^{(d)} : \mathbb{C}[\mathbb{R}^d] \to \mathcal{H}(\mathbb{R}^d)$ the projection map sending P to P_0, i.e., to the unique harmonic polynomial which is congruent to P modulo the ideal in $\mathbb{C}[\mathbb{R}^d]$ generated by (x,x). The following formula for $\pi^{(d)}$ is surely well-known, but we include its proof for lack of a convenient reference.

Lemma. *Suppose that $P \in \mathbb{C}[\mathbb{R}^d]$ is homogeneous of degree a. Then*

$$\pi^{(d)}(P(x)) \;=\; \sum_{0\le j\le a/2} \frac{1}{4^j\, j!\,(2-a-d/2)_j}\,\Delta^j(P(x))\,(x,x)^j\,, \tag{43}$$

where $(x)_j = x(x+1)\cdots(x+j-1)$ is the ascending Pochhammer symbol.

Proof. We have $\Delta(PQ) = \Delta(P)Q + 2\sum_\alpha \frac{\partial P}{\partial x_\alpha}\frac{\partial Q}{\partial x_\alpha} + P\Delta(Q)$ for any P, $Q \in \mathbb{C}[\mathbb{R}^d]$. Apply this to $Q(x) = (x,x)^j$, for which we have

$$\frac{\partial Q}{\partial x_\alpha} \;=\; 2j\,x_\alpha\,(x,x)^{j-1}\,, \qquad \frac{\partial^2 Q}{\partial x_\alpha^2} \;=\; 2j\,(x,x)^{j-1} \;+\; 4j(j-1)\,x_\alpha^2\,(x,x)^{j-2}\,,$$

to get

$$\Delta\bigl(P(x)(x,x)^j\bigr) \;=\; \Delta(P(x))\,(x,x)^j \;+\; 4j\,\bigl(E(P(x)) \;+\; (d/2+j-1)\,P(x)\bigr)\,(x,x)^{j-1}\,,$$

where $E = \sum_\alpha x_\alpha \frac{\partial}{\partial x_\alpha}$ is the Euler operator on $\mathbb{C}[\mathbb{R}^d]$. Replacing P by $\Delta^j(P)$ in this formula, where P is homogeneous of degree a, we find

$$\Delta\bigl(\Delta^j(P(x))\,(x,x)^j\bigr) \;=\; \Delta^{j+1}(P(x))\,(x,x)^j \;+\; 4j\,(a+d/2-j-1)\,\Delta^j(P(x))\,(x,x)^{j-1}\,,$$

since $\Delta^j(P)$ has degree $a-2j$. It follows that the expression $\sum \gamma_j\,\Delta^j(P(x))\,(x,x)^j$ is in the kernel of Δ if $\gamma_{j-1} + 4j(a+d/2-j-1)\gamma_j = 0$ for all $j \ge 1$, and combining this recursion with the condition $\gamma_0 = 1$, which ensures that this expression is congruent to $P(x)$ modulo (x,x), gives the formula stated. □

Now using the correspondence between the operators D_i on $\mathbb{C}[\mathcal{S}_n]$ and Δ_i on $\mathbb{C}[(\mathbb{R}^d)^n]$, we find from the lemma that for each $\mathbf{a} \in \mathbb{Z}_{\ge 0}^n$ the product operator

$$\Pi \;=\; \Pi_{\mathbf{a}}^{(d)} \;=\; \prod_{i=1}^{n}\biggl(\sum_{0\le j\le a_i/2} \frac{t_{ii}^j\,D_i^j}{4^j\,j!\,(2-a_i-d/2)_j}\biggr) \tag{44}$$

(in which the order of the product does not matter since the operators for different i commute) sends any $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ to the unique element of $\mathcal{P}_{\mathbf{a}}(d)$ which is congruent to P modulo the ideal $(t_{11}, \ldots, t_{nn})$ of $\mathbb{C}[\mathcal{S}_n]$, i.e., it gives the projection onto the $\boldsymbol{\mu} = \mathbf{0}$-component of the direct sum decomposition (9). (More precisely, the lemma gives this when d is an integer $> n$, and the general case follows by "analytic continuation" since it is equivalent to a collection of identities between rational functions of d, each true for infinitely many values of d.) In particular, we have the following extension of eq. (40):

Proposition 1. *The monomial basis of $\mathcal{P}_{\mathbf{a}}(d)$ for $d \in \mathbb{C} \smallsetminus \Xi(\mathbf{a})$ is given by*

$$P^M_{\boldsymbol{\nu},d} \;=\; \Pi^{(d)}_{\mathbf{a}}\big(T^{\boldsymbol{\nu}}\big) \qquad (\nu \in \mathcal{N}_0(\mathbf{a}))\,, \tag{45}$$

where $\Pi^{(d)}_{\mathbf{a}} : \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \to \mathcal{P}_{\mathbf{a}}(d)$ is the operator defined by equation (44).

We observe that the same argument as was used in the above calculation leads to a second proof of Theorem 1 of §2 and to an explicit form of the decomposition (44) of polynomials on $\mathcal{S}_n$. Indeed, almost the same argument as the one used to prove (43) shows that the mth component $P_m(x)$ of the decomposition (42) of a polynomial $P \in \mathbb{C}[\mathbb{R}^d]_a$ is given by the formula

$$P_m(x) \;=\; \frac{1}{4^m m!\,(a+d/2-2m)_m} \sum_{0\le j\le a/2-m} \frac{(x,x)^j\,\Delta^{m+j}(P(x))}{4^j j!\,(2m+2-a-d/2)_j}\,. \tag{46}$$

(One first checks that the expression on the right is annihilated by Δ either by repeating the calculation in the lemma or by observing that the sum is simply $\pi^{(d)}(\Delta^m P(x))$, and then verifies the equality (42) for the polynomials defined by (46) using a simple binomial coefficient identity.[4]) We immediately deduce:

Proposition 2. *The projection from $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ to $\mathcal{P}_{\mathbf{a}-2\mathbf{m}}$ ($\mathbf{0} \le \mathbf{m} \le \mathbf{a}/2$) defined by the direct sum decomposition* (9) *is given explicitly by the operator*

$$\Pi^{(d)}_{\mathbf{a},\mathbf{m}} \;=\; \prod_{i=1}^{n}\Bigg(\frac{1}{4^{m_i}\,m_i!\,(a_i+d/2-2m_i)_{m_i}} \sum_{0\le j\le a_i/2-m_i} \frac{t_{ii}^j\,D_i^{m_i+j}}{4^j\,j!\,(2m_i+2-a_i-d/2)_j}\Bigg)$$

and is, up to a scalar factor, equal to the map $P \mapsto \Pi^{(d)}_{\mathbf{a}-2\mathbf{m}}\big(D_1^{m_1}\cdots D_n^{m_n}P\big)$ with $\Pi^{(d)}_{\mathbf{a}}$ as in (40).

We now turn to the construction of the monomial basis, starting with the constant function $P^M_{\mathbf{0}} = 1$, using "raising operators." For all $i \ne j$ we define elements $R_{ij} = R_{ji}$ in the universal enveloping algebra of $\mathfrak{g}$ by

$$R_{ij} \;=\; F_{ii}F_{jj}D_{ij} \;-\; 2(E_{jj}-2)F_{ii}E_{ji} \;-\; 2(E_{ii}-2)F_{jj}E_{ij} \;+\; 4(E_{ii}-2)(E_{jj}-2)F_{ij}\,. \tag{47}$$

An alternative way to write this definition, since E_{ij}, E_{ji} and F_{ij} change the multidegree, is

$$R_{ij} \;=\; F_{ii}F_{jj}D_{ij} \;-\; 2F_{ii}E_{ji}(E_{jj}-1) \;-\; 2F_{jj}E_{ij}(E_{ii}-1) \;+\; 4F_{ij}(E_{ii}-1)(E_{jj}-1)\,. \tag{48}$$

As operators on $\mathbb{C}[\mathcal{S}_n]$, these operators map $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ to $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}+\mathbf{e}_i+\mathbf{e}_j}$ and in particular increase the total degree by 2. We will explain the origin of the formulas (47) and (48) below.

[4] namely (with $n = m+j$ and $A = a+d/2-1$), that $\sum_{m=0}^{n}(A-2m)\binom{A}{m}\binom{2n-A}{n-m} = 0$ if $n > 0$.

Theorem 5. *The raising operators R_{ij} commute with one another and map the space $\mathcal{P}^{(n)}(d)$ of higher spherical polynomials to itself. For $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^n$ and $\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})$ the monomial basis element $P^M_{\boldsymbol{\nu}} \in \mathcal{P}_{\mathbf{a}}(d)$ is given by*

$$P^M_{\boldsymbol{\nu}} \;=\; \frac{1}{\varepsilon_{2\mathbf{a}}(d-2)}\,\mathbf{R}^{\boldsymbol{\nu}}(1) \qquad (d \in \mathbb{C},\;\; d \notin 2\,Z_{\leq 1}), \tag{49}$$

where $\varepsilon_{2\mathbf{a}}(d-2)$ is defined by equations (14) *and* (18) *and $\mathbf{R}^{\boldsymbol{\nu}} = \prod_{i<j} R_{ij}^{\nu_{ij}}$.*

Proof. The identities

$$\big[R_{ij},\,R_{kl}\big] \;=\; 0\,, \qquad D_k\,R_{ij} \;=\; R^{[k]}_{ij}\,D_k \qquad (i \neq j,\;\; k \neq l) \tag{50}$$

in the universal enveloping algebra of $\mathfrak{g}$, where $R^{[k]}_{ij}$ is defined like R_{ij} but with $E_{ii}-2$ and $E_{jj}-2$ in (47) replaced by $E_{ii}-2+2\delta_{ik}$ and $E_{jj}-2+2\delta_{jk}$, can be checked directly from the commutation relations given in §4. (However, this is very tedious. Two better ways will be given at the end of the proof.) These identities prove the first two statements of the theorem. For the last one, it suffices by induction to show that

$$R_{ij}\big(P^M_{\boldsymbol{\nu}}\big) \;=\; (d+2a_i-2)\,(d+2a_j-2)\,P^M_{\boldsymbol{\nu}+\mathbf{e}_{ij}} \tag{51}$$

But this is almost obvious: The first three terms in (48) are divisible by t_{ii} or t_{jj} and the operator $4(E_{ii}-1)(E_{jj}-1)$ acts on $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ as multiplication by $(d+2a_i-2)(d+2a_j-2)$, so

$$\begin{aligned} R_{ij}\big(P^M_{\boldsymbol{\nu}}\big)(T)\Big|_{t_{11}=\cdots=t_{nn}=0} \;&=\; (d+2a_i-2)\,(d+2a_j-2)\,t_{ij}P^M_{\boldsymbol{\nu}}(T)\Big|_{t_{11}=\cdots=t_{nn}=0} \\ &=\; (d+2a_i-2)\,(d+2a_j-2)\,T^{\boldsymbol{\nu}+\mathbf{e}_{ij}}\,, \end{aligned}$$

and now (51) follows from the definition of the monomial basis and the fact that R_{ij} sends $\mathcal{P}^{(n)}(d)$ to itself. $\square$

The proof just given is short, but has two disadvantages: the definition (47) (or (48)) is completely unmotivated, and the commutation relations (50) are proved by a lengthy brute force calculation. We describe two other approaches that are more illuminating and also illustrate ideas that will be used again later.

(I) For $i,\,j \in \{1,\ldots,n\}$ and $\mathbf{a} \in \mathbb{C}^n$ we define an operator $R_{ij}(\mathbf{a})$ by

$$R_{ij}(\mathbf{a}) \;=\; \delta(T)^{\mathbf{a}+\mathbf{e}_i+\mathbf{e}_j-(2-d)\mathbf{1}/2}\,D_{ij}\;\delta(T)^{(2-d)\mathbf{1}/2-\mathbf{a}}\,, \tag{52}$$

where $\delta(T)^{\mathbf{m}}$ is defined as in the Corollary to Theorem 1. (Note that $\delta(T)^{\mathbf{m}}$ is not well-defined for non-integral $\mathbf{m}$, but the right-hand side of (52) is well-defined because the ambiguity of phase of the two $\delta(T)^*$ terms cancel out.) This definition is motivated by the symmetry of the inhomogeneous form of the differential system defining $\mathcal{P}_{\mathbf{a}}(d)$ under $\mathbf{a} \mapsto (2-d)\mathbf{1}-\mathbf{a}$, a symmetry which is implicit in Remark 4 of §3 and explicit in equation (8) in the case $n=3$. A direct calculation shows that

$$P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \quad\Rightarrow\quad R_{ij}(\mathbf{a})\,P \;=\; \begin{cases} t_{ii}^2 D_i(P) & \text{if } i=j, \\ R_{ij}(P) & \text{if } i \neq j. \end{cases} \tag{53}$$

This makes the fact that the operators R_{ij} commute almost obvious, because applying R_{ij} to $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ is the same as applying D_{ij} to $\delta(T)^{(2-d)\mathbf{1}/2-\mathbf{a}}\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$, and the operators D_{ij} all commute with one another, and an exactly similar argument shows that the operators R_{ij} map spherical polynomials to spherical polynomials, because (53) shows that the spherical polynomials in $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ are precisely the polynomials killed by all $R_{kk}(\mathbf{a})$.

(II) The second approach is to prove equation (51) directly using the harmonic projection operators $\Pi_{\mathbf{a}}^{(d)}$ discussed in the first part of this section. This equation then implies equation (49) and the facts that the operators R_{ij} preserve $\mathcal{P}^{(n)}(d)$ and (at least when restricted to this space) that they commute with one another (because $\boldsymbol{\nu}+\mathbf{e}_{ij}+\mathbf{e}_{kl}=\boldsymbol{\nu}+\mathbf{e}_{kl}+\mathbf{e}_{ij}$). Because of equation (40), all we have to do to establish (51) (in which we take $i=1$, $j=2$ for convenience) is to show that

$$\begin{aligned}(d+2a_1-2)(d+2a_2-2)\,\Pi_{\mathbf{a}}^{(d)}\bigl(t_{12}P\bigr) \;=\; 4(d+2a_1-2)(d+2a_2-2)t_{12}P\\ -\;2(d+2a_2-2)t_{11}E_{21}(P)\;-\;2(d+2a_1-2)t_{22}E_{12}(P)\;+\;t_{11}t_{22}\,D_{12}(P)\end{aligned}\tag{54}$$

for all $P\in\mathcal{P}_{\mathbf{a}}(d)$. We do this using the commutation relations given in Theorem 4.

Denote by $\pi_k=\pi_k^{(d)}$ the kth factor in (44), so that $\Pi_{\mathbf{a}}^{(d)}=\pi_1\cdots\pi_n$. Since π_k commutes with F_{12} for $k\ge 3$ and projects P to itself, we have $\Pi_{\mathbf{a}}^{(d)}\bigl(t_{12}P\bigr)=\pi_1\pi_2\bigl(t_{12}P\bigr)$. From the commutation relations given in Theorem 4 and the fact that $D_i=D_{ii}$ annihilates P for all i, we find

$$D_2\bigl(t_{12}P\bigr)\;=\;\bigl[D_2,F_{12}\bigr](P)\;=\;2\,E_{12}(P)\,,\qquad D_2^2\bigl(t_{12}P\bigr)\;=\;2\,\bigl[D_2,E_{12}\bigr](P)\;=\;0\,,$$

so that only the first two terms in the series defining π_2 contribute to $\pi_2(t_{12}P)$ and we have

$$\pi_2\bigl(t_{12}P\bigr)\;=\;t_{12}P\;-\;\frac{1}{2(2a_2+d-2)}\,t_{22}E_{12}(P)\;.$$

By the same argument, and since π_1 commutes with F_{22}, we find

$$\pi_1\pi_2\bigl(t_{12}P\bigr)\;=\;F_{12}(P)\;-\;\frac{1}{2(2a_1+d-2)}\,t_{11}E_{21}(P)\;-\;\frac{1}{2(2a_2+d-2)}\,t_{22}\,\pi_1\bigl(E_{12}P\bigr)\,,$$

and since an exactly similar argument using the commutation relation $[D_1,E_{12}]=2D_{12}$ gives

$$\pi_1\bigl(E_{12}P\bigr)\;=\;E_{12}(P)\;-\;\frac{1}{2(a_1+d-2)}\,t_{11}D_{12}(P)\,,$$

this completes the proof of (54) and hence also of (49). It also explains the motivation for the artificial-looking definition of R_{ij}: this operator is the simplest element of the universal enveloping algebra of the Lie algebra of §4 that is (up to a scalar factor in each homogeneous component) adjoint to the mixed Laplace operator D_{ij} with respect to the restriction of the scalar product $(\,\cdot\,,\,\cdot\,)_d$ to $\mathcal{P}(d)$.

We now come to the definition of the second canonical basis of $\mathcal{P}^{(n)}(d)$. Theorem 5 tells us that, for generic d, the monomial basis $\bigl\{P_{\boldsymbol{\nu}}^M(T)\mid\boldsymbol{\nu}\in\mathcal{N}_0(\mathbf{a})\bigr\}$ of $\mathcal{P}^{(n)}(d)$ is the same, up to scalar factors, as the "ascending basis" defined by taking 1 as the basis element of $\mathcal{P}_{\mathbf{0}}(d)$ and then applying the raising operators R_{ij} to obtain the higher basis elements by induction. The descending basis is defined, again for generic d, by a similar process in reverse.

Proposition-Definition. *For generic d, there exists a unique basis $\bigl\{P_{\boldsymbol{\nu}}^D(T)\mid\boldsymbol{\nu}\in\mathcal{N}_0(\mathbf{a})\bigr\}$ of $\mathcal{P}^{(n)}(d)$, called the* **descending basis***, characterized by the property*

$$D_{ij}\bigl(P_{\boldsymbol{\nu}}^D\bigr)\;=\;P_{\boldsymbol{\nu}-\mathbf{e}_{ij}}^D\qquad(i\neq j)\tag{55}$$

(*where the right-hand side is to be taken as 0 if $\nu_{ij}=0$*) *and the initial condition $P_{\mathbf{0}}^D(T)=1$.*

This follows immediately from the following lemma, in which $\mathcal{P}_{\mathbf{a}}(d)=\{0\}$ if $a_i<0$ for some i.

Lemma. *Let* $\mathbf{a} \neq \mathbf{0}$ *and* d *a complex number such that the inner product in* $\mathcal{P}_{\mathbf{a}}(d)$ *is non-degenerate. Suppose that for all* $i \neq j$ *we are given polynomials* $G_{ij} = G_{ji} \in \mathcal{P}_{\mathbf{a}-\mathbf{e}_i-\mathbf{e}_j}(d)$ *such that* $D_{kl}(G_{ij}) = D_{ij}(G_{kl})$ *for all* $i \neq j$ *and* $k \neq l$. *Then there exists a unique polynomial* $G \in \mathcal{P}_{\mathbf{a}}(d)$ *such that* $D_{ij}(G) = G_{ij}$ *for all* $i \neq j$.

Proof. The proof is similar to that of Theorem 1. We will show by induction on $|S|$ that, for any set S of (unordered) pairs (i, j) with $1 \leq i, j \leq n$, $i \neq j$, the following statements hold:

(i) The dimension of the space $K_{\mathbf{a}}(S) := \bigcap_{(i,j)\in S} \mathrm{Ker}(D_{ij})$, where $\mathrm{Ker}(D_{ij}) = \{P \in \mathcal{P}_{\mathbf{a}}(d) \mid D_{ij}P = 0\}$, is equal to the number of $\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})$ with $\nu_{ij} = 0$ for all $(i, j) \in S$.

(ii) The map $D_{ij} : K_{\mathbf{a}}(S) \to K_{\mathbf{a}-\mathbf{e}_i-\mathbf{e}_j}(S)$ is surjective whenever $(i, j) \notin S$.

(iii) Assume that for all $(i, j) \in S$, polynomials $G_{ij} \in \mathcal{P}_{\mathbf{a}-\mathbf{e}_i-\mathbf{e}_j}(d)$ are given which satisfy $D_{kl}(G_{ij}) = D_{ij}(G_{kl})$ for all $(k, l) \in S$. Then there exists $G \in \mathcal{P}_{\mathbf{a}}(d)$ such that $D_{ij}(G) = G_{ij}$ for all $(i, j) \in S$.

We denote by $\Pi = \Pi^{(d)}_{\mathbf{a}}$ the harmonic projection. We saw earlier (eq. (39)) that the scalar product $(Q, D_{ij}P)_d$ equals $(\Pi(t_{ij}Q), P)_d$ for any P in $\mathcal{P}_{\mathbf{a}}(d)$ and Q in $\mathcal{P}_{\mathbf{a}-\mathbf{e}_i-\mathbf{e}_j}(d)$. It follows that $D_{ij}P = 0$ if and only if P is orthogonal to all monomial basis polynomials $P^M_{\boldsymbol{\nu}}$ with $\nu_{ij} > 0$, and hence that the dimension of $K_{\mathbf{a}}(S)$ equals $N_0(\mathbf{a})$ minus the number of $\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})$ with $\nu_{ij} > 0$ for some pair $(i, j) \in S$, which is precisely the statement of (i). Statement (ii) follows immediately from (i) by comparing the dimensions. We now prove (iii) by induction on the cardinality of S. The claim is true when $|S| = 1$ since $D_{ij} : \mathcal{P}_{\mathbf{a}}(d) \to \mathcal{P}_{\mathbf{a}-\mathbf{e}_i-\mathbf{e}_j}(d)$ is surjective by (ii). Now assume that (iii) is true for some S and that G_{ij} are given for $(i, j) \in S \cup \{(k, l)\}$ and satisfy the compatibility condition. By the inductive assumption, there exists $G_0 \in \mathcal{P}_{\mathbf{a}}(d)$ such that $D_{ij}(G_0) = G_{ij}$ for any $(i, j) \in S$. We have $D_{ij}(G_{kl} - D_{kl}(G_0)) = D_{kl}(G_{ij} - D_{ij}(G_0)) = 0$, so $G_{kl} - D_{kl}(G_0) \in K_{\mathbf{a}-e_k-e_l}(S)$. Since D_{kl} maps $K_{\mathbf{a}}(S)$ to $K_{\mathbf{a}-e_k-e_l}(S)$ surjectively by (ii), there exists $G_1 \in K_{\mathbf{a}}(S)$ such that $D_{kl}(G_1) = G_{kl} - D_{kl}(G_0)$. So if we set $G = G_0 + G_1$ then $D_{kl}(G) = G_{kl}$ and, since $D_{ij}G_1 = 0$ for any $(i, j) \in S$, also $D_{ij}(G) = G_{ij}$. The uniqueness of G follows from (i), which implies that the dimension of $K_{\mathbf{a}}(S)$ is 0 if S is the set of all (i, j) with $i \neq j$ and $\mathbf{a} \neq 0$. $\square$

Remark. The Lemma, and therefore also the Proposition-Definition, apply only to generic values of d for which the scalar product on $\mathcal{P}_{\mathbf{a}}(d)$ is non-degenerate. In fact this assumption holds for all $d \notin \mathbb{Z}_{<n}$. This will follow from the results in §9, where we will give an independent proof of the existence of $\{P^D_{\boldsymbol{\nu}}\}$ satisfying (55) using a generating function. To prove it directly, we would need an *a priori* proof that $\bigcap_{i,j} \mathrm{Ker}(D_{ij}) = \{0\}$ for $d \notin \mathbb{Z}$. We were not able to give this, since the argument used for Theorem 1 does not generalize in any obvious way. For example, for $\mathbf{a} = (1, 1, 1, 1)$, then the space of harmonic polynomials is spanned by the three monomials $t_{12}t_{34}$, $t_{13}t_{24}$ and $t_{14}t_{23}$; their images under D_{12} are t_{34} times d, 1 and 1, respectively, so that there would seem to be a problem only if $d = 0$, but in fact the trouble occurs for the two values $d = -2$ (where there is a 1-dimensional radical, with basis the sum of the three polynomials above) and $d = 1$ (where the radical is 2-dimensional, spanned by their differences).

Theorem 6. *The monomial and descending bases of* $\mathcal{P}_{\mathbf{a}}(d)$ *are dual to one another with respect to the scalar product.*

Proof. Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathcal{N}_0$. We have to show that $\big(P^M_{\boldsymbol{\nu}}, P^D_{\boldsymbol{\mu}}\big)_d = \delta_{\boldsymbol{\nu},\boldsymbol{\mu}}$. We can assume that $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ have the same multidegree $\mathbf{a}$, since otherwise the inner product vanishes automatically. We proceed by induction on the total degree. If $\mathbf{a} = \mathbf{0}$, then $P^M_{\boldsymbol{\nu}} = P^D_{\boldsymbol{\mu}} = 1$ and $(1, 1)_d = 1 = \delta_{\boldsymbol{\nu},\boldsymbol{\mu}}$. Otherwise, we have $\nu_{i,j} > 0$ for some $i \neq j$ and we find, using the properties of the projection operator Π, equation (39) and the induction assumption,

$$\begin{aligned}\big(P^M_{\boldsymbol{\nu}}, P^D_{\boldsymbol{\mu}}\big)_d &= \big(\Pi(t_{ij}P^M_{\boldsymbol{\nu}-\mathbf{e}_{ij}}), P^D_{\boldsymbol{\mu}}\big)_d = \big(t_{ij}P^M_{\boldsymbol{\nu}-\mathbf{e}_{ij}}, P^D_{\boldsymbol{\mu}}\big)_d \\ &= \big(P^M_{\boldsymbol{\nu}-\mathbf{e}_{ij}}, D_{ij}P^D_{\boldsymbol{\mu}}\big)_d = \big(P^M_{\boldsymbol{\nu}-\mathbf{e}_{ij}}, P^D_{\boldsymbol{\mu}-\mathbf{e}_{ij}}\big)_d = \delta_{\boldsymbol{\nu},\boldsymbol{\mu}}\,. \qquad \square\end{aligned}$$

In the preceding theorem we did not specify what d was. In fact there are two possible interpretations. One is to assume, as we did in the preceding Proposition-Definition and Lemma, that d is a specific complex number for which the scalar product is non-degenerate (i.e., in view of the Remark above, that $d \notin \mathbb{Z}_{<n}$). The other is to consider d as a variable. In that case the coefficients of the canonical basis elements $P_{\boldsymbol{\nu}}^M$ and $P_{\boldsymbol{\nu}}^D$, and the coefficients of the scalar products $(X,Y)_d$ for any $X,\, Y \in \mathbb{Q}[\mathcal{S}_n]$, are elements of the field $\mathcal{K} := \mathbb{Q}(d)$, and Theorem 6 becomes an identity over the field $\mathcal{K}$. To make this clearer, take the example $n = 4$, $\mathbf{a} = (1,1,1,1)$, considered above. Here $N_0(\mathbf{a}) = 3$, $\mathcal{N}_0(\mathbf{a})$ consists of the three 4×4 matrices $\boldsymbol{\nu}_1 = \mathbf{e}_{12} + \mathbf{e}_{34}$, $\boldsymbol{\nu}_2 = \mathbf{e}_{13} + \mathbf{e}_{24}$, $\boldsymbol{\nu}_3 = \mathbf{e}_{14} + \mathbf{e}_{23}$, and the space $\mathcal{P}_{\mathbf{a}}(d)$ is spanned by the three monomials $P_1(T) = t_{12}t_{34}$, $P_2(T) = t_{13}t_{24}$ and $P_3(T) = t_{14}t_{23}$. (This is obviously the monomial basis, which here consists of actual monomials.) The corresponding Gram matrix is given by

$$\big(P_i,\, P_j\big)_{1\le i,\, j\le 3} \;=\; \begin{pmatrix} d^2 & d & d \\ d & d^2 & d \\ d & d & d^2 \end{pmatrix} \tag{56}$$

and the descending basis by

$$\begin{pmatrix} P_1^D(T) \\ P_2^D(T) \\ P_3^D(T) \end{pmatrix} \;=\;=\; \frac{1}{d(d-1)(d+2)} \begin{pmatrix} d+1 & -1 & -1 \\ -1 & d+1 & -1 \\ -1 & -1 & d+1 \end{pmatrix} \begin{pmatrix} P_1(T) \\ P_2(T) \\ P_3(T) \end{pmatrix} \;,$$

in which, by virtue of Theorem 6, the 3×3 matrix is just the inverse of the matrix in (56). Many more examples can be found in Table 2 in the appendix at the end of the paper.

We make one final observation. Define degree-preserving operators C_{ij} $(i,j = 1,\dots,n)$ by

$$C_{ij} \;=\; F_{ij}D_{ij} \;-\; E_{ij}E_{ji} \;+\; E_{ii}\,.$$

From the commutation relations given in Theorem 2, we find that $C_{ij} = C_{ji}$ (because $[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$) and that the C_{ij} commute with all D_k. The latter property implies that the operators C_{ij} act on each space $\mathcal{P}_{\mathbf{a}}(d)$. For $i = j$ the operator $C_{ii} = F_{ii}D_i - E_{ii}(E_{ii} - 1)$ corresponds to the usual Casimir operator in the Lie algebra $\mathfrak{sl}(2)$ generated by the three operators D_i, E_{ii} and F_{ii} on $(\mathbb{R}^d)_i$, so we can think of the C_{ij} as a kind of "mixed Casimir operators." On $\mathcal{P}_{\mathbf{a}}(d)$, the operator C_{ii} is equal to the scalar $-(a_i + d/2)(a_i + d/2 - 1)$ and the mixed operator C_{ij} is given by the commutator relation

$$\big[D_{ij},\, R_{ij}\big] \;=\; 4(a_i + a_j - 2)\,(C_{ij} - a_i - a_j + a_ia_j)\,,$$

but the operators C_{ij} for $i \neq j$ do not act as scalars and do not even commute with each other. We do not know whether these operators have interesting applications.

§6. Higher spherical polynomials and invariant harmonic polynomials

In this section we look in more detail at what happens with the spaces $\mathcal{P}^{(n)}(d)$ when d is a positive integer smaller than n.

Our original motivation for the definition of $\mathcal{P}^{(n)}(d)$ was to consider functions $P(T)$ whose pull-back $\widetilde{P} = P \circ \beta_n$ to $(\mathbb{R}^d)^n$ is harmonic in each component $x_i \in \mathbb{R}^d$. Because the map $\beta_n^* : P \mapsto \widetilde{P}$ is not injective for $d < n$, we did not use this as the definition of $\mathcal{P}^{(n)}(d)$ in that case, but instead defined $\mathcal{P}^{(n)}(d)$ as $\bigcap_{i=1}^n \mathrm{Ker}(D_i)$. In this section we study the relationship of the three vector spaces

$$\begin{aligned} &\text{(i)} && \mathcal{P}_{\mathbf{a}}(d) \;=\; \big\{P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \mid D_1P = \cdots = D_nP = 0\big\}\,, \\ &\text{(ii)} && \mathcal{H}_{\mathbf{a}}(d) \;=\; \big(\mathcal{H}_{a_1}(\mathbb{R}^d) \otimes \ldots \otimes \mathcal{H}_{a_n}(\mathbb{R}^d)\big)^{O(d)}\,, \\ &\text{(iii)} && \mathcal{V}_{\mathbf{a}}(d) \;=\; \big\{P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \mid \widetilde{P} \in \mathcal{H}_{\mathbf{a}}(d)\big\}\,, \end{aligned}$$

where $\mathcal{H}_{a_i}(\mathbb{R}^d)$ in (ii) is defined as in Remark 3 of §2. For $d \ge n$, all three spaces are isomorphic. In general, we have $\mathcal{P}_{\mathbf{a}}(d) \subseteq \mathcal{V}_{\mathbf{a}}(d)$ (because $\Delta_i(\widetilde{P}) = D_i(P)^\sim$), and we will show that the map $\beta_n^* : \mathcal{P}_{\mathbf{a}}(d) \to \mathcal{H}_{\mathbf{a}}(d)$ is surjective (Theorem 7). In particular, we have

$$\dim \mathcal{H}_{\mathbf{a}}(d) \;\le\; \dim \mathcal{P}_{\mathbf{a}}(d) \;\le\; \dim \mathcal{V}_{\mathbf{a}}(d)\,. \tag{57}$$

At the end of this section we will show that in general all three dimensions are distinct, and will determine all three of them exactly in the special case when $d = n-1$.

Let us define two further spaces

$$\begin{aligned} \mathcal{K}_{\mathbf{a}}(d) \;&=\; \operatorname{Ker}\bigl(\beta_n^* \,:\, \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \to \bigl(\mathbb{C}[\mathbb{R}^d]_{a_1} \otimes \cdots \otimes \mathbb{C}[\mathbb{R}^d]_{a_n}\bigr)^{O(d)}\bigr)\,, \\ \mathcal{K}'_{\mathbf{a}}(d) \;&=\; \mathcal{K}_{\mathbf{a}}(d) \cap \mathcal{P}_{\mathbf{a}}(d)\,. \end{aligned}$$

Clearly $\mathcal{K}_{\mathbf{a}}(d) \subseteq \mathcal{V}_{\mathbf{a}}(d)$.

Theorem 7. *The diagram*

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K}_{\mathbf{a}}(d) & \longrightarrow & \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} & \xrightarrow{\beta_n^*} & \bigl(\bigotimes_{i=1}^n \mathbb{C}[\mathbb{R}^d]_{a_i}\bigr)^{O(d)} & \longrightarrow & 0 \\ & & \| & & \bigcup & & \bigcup & & \\ 0 & \to & \mathcal{K}_{\mathbf{a}}(d) & \longrightarrow & \mathcal{V}_{\mathbf{a}}(d) & \xrightarrow{\beta_n^*} & \mathcal{H}_{\mathbf{a}}(d) & \longrightarrow & 0 \\ & & \bigcup & & \bigcup & & \| & & \\ 0 & \to & \mathcal{K}'_{\mathbf{a}}(d) & \longrightarrow & \mathcal{P}_{\mathbf{a}}(d) & \xrightarrow{\beta_n^*} & \mathcal{H}_{\mathbf{a}}(d) & \longrightarrow & 0 \end{array}$$

is commutative with exact rows.

Proof. Only the surjectivity of β_n^* in each of the three rows has to be proved, since the spaces on the left are by definition the kernel of β_n^* in each case. The surjectivity of β_n^* in the first row is a classical result ("first fundamental theorem of invariant theory," [16]), and the surjectivity in the second row an immediate consequence of this, since $\mathcal{V}_{\mathbf{a}}(d)$ is defined as the inverse image of $\mathcal{H}_{\mathbf{a}}(d)$ in $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$. Thus the real assertion of the theorem is the statement that the map $\beta_n^* : \mathcal{P}_{\mathbf{a}}(d) \to \mathcal{H}_{\mathbf{a}}(d)$ is surjective for all n and d (or equivalently, that the space $\mathcal{V}_{\mathbf{a}}(d)$ is the sum of its two subspaces $\mathcal{K}_{\mathbf{a}}(d)$ and $\mathcal{P}_{\mathbf{a}}(d)$). We prove this using the two following lemmas.

Lemma 1. *For any $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ and $\mathbf{m} \in \mathbb{Z}^n_{\ge 0}$ we have*

$$D_i\bigl(\delta(T)^{\mathbf{m}}\, P(T)\bigr) \;=\; \delta(T)^{\mathbf{m}}\, D_i(P(T)) + 2m_i(d{+}2a_i{+}2m_i{-}2)\,\delta(T)^{\mathbf{m}-\mathbf{e}_i}\, P(T) \qquad (i = 1, \ldots, n)\,.$$

Proof. This follows by induction on the non-negative integer m_i from the commutation relations $[D_i, F_{ii}] = 4E_{ii} = 4E_i + 2d$ and $[D_i, F_{jj}] = 0$ $(i \ne j)$. □

Lemma 2. *Let $P = \sum_{\mathbf{m}} \delta(T)^{\mathbf{m}}\, P_{\mathbf{m}}(T)$ be the decomposition of a polynomial $P \in \mathbb{C}[\mathcal{S}_n]$ given by the corollary to Theorem* 1, *where d is a positive integer. If $\widetilde{P} = 0$, then $\widetilde{P}_{\mathbf{m}} = 0$ for every* $\mathbf{m}$.

Proof. This follows immediately from the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} & \xleftarrow{\cong} & \bigoplus_{\mathbf{0} \le \mathbf{m} \le \frac{1}{2}\mathbf{a}} \mathcal{P}_{\mathbf{a}-2\mathbf{m}}(d) \\ \beta_n^* \downarrow & & \downarrow \beta_n^* \\ \mathbb{C}[\mathbb{R}^d]_{a_1} \otimes \cdots \otimes \mathbb{C}[\mathbb{R}^d]_{a_n} & \xleftarrow{\cong} & \bigoplus_{\mathbf{0} \le \mathbf{m} \le \frac{1}{2}\mathbf{a}} \mathcal{H}_{a_1 - 2m_1}(\mathbb{R}^d) \otimes \cdots \otimes \mathcal{H}_{a_n - 2m_n}(\mathbb{R}^d) \end{array}$$

in which the horizontal isomorphisms are those given by the decompositions (9) and (42). $\square$

Returning to the proof of Theorem 7, let $F \in \mathcal{H}_{\mathbf{a}}(d)$ and choose $P \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$ with $\widetilde{P} = F$. By definition we have $P \in \mathcal{V}_{\mathbf{a}}(d)$ and by the relationship between D_i and the Laplacian Δ_i on $(\mathbb{R}^d)_i$ discussed in §1 we have $(D_iP)^\sim = \Delta_i(\widetilde{P}) = \Delta_i(F) = 0$ for all i. Let $P = \sum_{\mathbf{m}} \delta(T)^{\mathbf{m}} P_{\mathbf{m}}(T)$ be the decomposition of P given by (9). Applying D_i to both sides of this equality and using Lemma 1 (with P replaced by $P_{\mathbf{m}}$ and $\mathbf{a}$ by $\mathbf{a} - 2\mathbf{m}$), we find

$$D_i(P) \;=\; \sum_{\substack{\mathbf{m}\geq 0\\ m_i\geq 1}} 2m_i\,(d + 2a_i - 2m_i - 2)\,\delta(T)^{\mathbf{m}-\mathbf{e}_i}\,P_{\mathbf{m}}(T)\,.$$

The fact that $D_i(P)^\sim = 0$ and Lemma 2 then imply that $\widetilde{P}_{\mathbf{m}} = 0$ for all $\mathbf{m}$ with $m_i > 0$, because the factor $2m_i(d + 2a_i - 2m_i - 2)$ is non-zero for $d > 0$ and $0 < m_i \leq a_i/2$, and since this is true for every i, it follows that $\widetilde{P}_{\mathbf{m}} = 0$ for all $\mathbf{m} \neq \mathbf{0}$, so $F = \widetilde{P} = \widetilde{P}_{\mathbf{0}} \in \beta_n^*\big(\mathcal{P}_{\mathbf{a}}(d)\big)$. $\square$

From Theorem 7 we see that $\mathcal{V}_{\mathbf{a}}(d) = \mathcal{K}_{\mathbf{a}}(d) + \mathcal{P}_{\mathbf{a}}(d)$ and hence that

$$\dim \mathcal{H}_{\mathbf{a}}(d) \;=\; \dim \mathcal{V}_{\mathbf{a}}(d) \,-\, \dim \mathcal{K}_{\mathbf{a}}(d) \;=\; \dim \mathcal{P}_{\mathbf{a}}(d) \,-\, \dim \mathcal{K}'_{\mathbf{a}}(d) \tag{58}$$

so that the claim that both inequalities in (57) are in general strict is equivalent to the claim that both inclusions $0 \subseteq \mathcal{K}'_{\mathbf{a}}(d) \subseteq \mathcal{K}_{\mathbf{a}}(d)$ are in general proper. We will prove this by computing all three spaces in question in the special case $d = n - 1$.

Proposition 1. *For $d = n-1$ we have the dimension formulas*

$$\begin{aligned} \dim \mathcal{H}_{\mathbf{a}}(n-1) \;&=\; N_0(\mathbf{a}) \,-\, N_0(\mathbf{a}-\mathbf{2})\,,\\ \dim \mathcal{P}_{\mathbf{a}}(n-1) \;&=\; N_0(\mathbf{a})\,,\\ \dim \mathcal{V}_{\mathbf{a}}(n-1) \;&=\; N_0(\mathbf{a}) \,+\, N(\mathbf{a}-\mathbf{2}) \,-\, N_0(\mathbf{a}-\mathbf{2})\,, \end{aligned}$$

where $\mathbf{2} = 2\cdot\mathbf{1} = (2,\dots,2)$.

Proof. In general, it is known ("second fundamental theorem of invariant theory") that for $d < n$ the kernel of the map β_n^* is the ideal in $\mathbb{C}[\mathcal{S}_n]$ generated by all $(d+1)\times(d+1)$ minors of the coordinate $T \in \mathcal{S}_n$. In particular, if $d = n-1$ then this kernel is the ideal generated by the single polynomial $\mathbf{D} \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{2}}$ defined by $\mathbf{D}(T) = \det T$. In other words, we have

$$\mathcal{K}_{\mathbf{a}}(n-1) \;=\; \mathbf{D}\cdot\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}-\mathbf{2}}\,. \tag{59}$$

To complete the proof, we will show that

$$\mathcal{K}'_{\mathbf{a}}(n-1) \;=\; \mathbf{D}\cdot\mathcal{P}_{\mathbf{a}-\mathbf{2}}(n+3)\,. \tag{60}$$

Equations (59) and (60) and Theorem 1 give $\dim\mathcal{K}_{\mathbf{a}}(n-1) = N(\mathbf{a}-\mathbf{2})$ and $\dim\mathcal{K}'_{\mathbf{a}}(n-1) = N_0(\mathbf{a}-\mathbf{2})$, and together with the exact sequences of Theorem 7 these imply the dimension formulas given in the proposition.

To prove (60), we have to compute the effect of the ith Laplace operator D_i on products of the form $P = \mathbf{D}\cdot Q$. Since the formulas depend on the value of d, we will temporarily write $D_i^{(d)}$ rather than simply D_i to denote the differential operator on $\mathbb{C}[\mathcal{S}_n]$ defined by (5).

Lemma 3. *We have*

$$D_i^{(n-1)}(\mathbf{D}) \;=\; 0 \qquad (1 \le i \le n) \tag{61}$$

and more generally

$$D_i^{(n-1)}(\mathbf{D}\,Q) \;=\; \mathbf{D}\,D_i^{(n+3)}(Q) \qquad (1 \le i \le n) \tag{62}$$

for any polynomial $Q \in \mathbb{C}[\mathcal{S}_n]$.

Proof. The partial derivatives of $\mathbf{D}$ are easily seen to be given by

$$\partial_{ij}\mathbf{D} \;=\; 2\,\mathbf{D}_{ij} \qquad (1 \le i,\, j \le n)$$

where $\mathbf{D}_{ij}(T)$ is the (i,j)-cofactor of T. (In checking this for $i=j$, one has to remember that $\partial_{ii} = 2\partial/\partial t_{ii}$.) Using this and the formula

$$\sum_{k=1}^{n} t_{jk}\,\mathbf{D}_{ik} \;=\; \delta_{ij}\,\mathbf{D}$$

(the product of T and its adjoint matrix equals $\det(T)$ times the identity matrix), we find

$$\begin{aligned} D_i^{(d)}(\mathbf{D}) \;&=\; 2d\,\mathbf{D}_{ii} \,+\, \sum_{j,\,k=1}^{n} t_{jk}\,\partial_{ij}(\,2\,\mathbf{D}_{ik}) \\ &=\; 2d\,\mathbf{D}_{ii} \,+\, 2\sum_{j,\,k=1}^{n} \bigl(\partial_{ij}(t_{jk}\,\mathbf{D}_{ik}) \,-\, \partial_{ij}(t_{jk})\,\mathbf{D}_{ik}\bigr) \\ &=\; 2d\,\mathbf{D}_{ii} \,+\, 2\sum_{j=1}^{n}\partial_{ij}\Biggl(\sum_{k=1}^{n} t_{jk}\,\mathbf{D}_{ik}\Biggr) \,-\, 2\sum_{j,\,k=1}^{n} \delta_{ik}(1+\delta_{ij})\,\mathbf{D}_{ik} \\ &=\; 2d\,\mathbf{D}_{ii} \,+\, 4\,\mathbf{D}_{ii} \,-\, 2\sum_{j=1}^{n}\,(1+\delta_{ij})\,\mathbf{D}_{ii} \\ &=\; 2(d+1-n)\,\mathbf{D}_{ii}\,, \end{aligned}$$

and assertion (61) follows. (Another proof, more in keeping with the contents of this section, is as follows. Let $\mathbf{D} = \sum_{\mathbf{0}\le\mathbf{m}\le\mathbf{1}} \delta(T)^{\mathbf{m}}\,\mathbf{D}_{\mathbf{m}}$ with $\mathbf{D}_{\mathbf{m}} \in \mathcal{P}_{\mathbf{2}-2\mathbf{m}}(d-1)$ be the canonical decomposition of $\mathbf{D}$ as given by Theorem 1. From $\widetilde{\mathbf{D}} = 0$ and Lemma 2 we deduce that $\widetilde{\mathbf{D}_{\mathbf{m}}} = 0$ for all $\mathbf{m}$. But then $\mathbf{D}_{\mathbf{m}} \in \operatorname{Ker}(\beta_n^*) = \mathbf{D}\cdot\mathbb{C}[\mathcal{S}_n]$, and since the degree of $\mathbf{D}_{\mathbf{m}}$ is smaller than that of $\mathbf{D}$ for $\mathbf{m} \ne \mathbf{0}$, this implies that $\mathbf{D}_{\mathbf{m}} = 0$ for all $\mathbf{m} \ne \mathbf{0}$ and hence that $\mathbf{D} = \mathbf{D}_{\mathbf{0}} \in \mathcal{P}_{\mathbf{2}}(n-1)$.) To prove the second assertion, we apply the easily proved general formula

$$D_i^{(d)}(PQ) \;=\; D_i^{(d)}(P)\,Q \,+\, 2\sum_{j,k=1}^{n} t_{jk}\,\partial_{ij}(P)\,\partial_{ik}(Q) \,+\, P\,D_i^{(d)}(Q)$$

with $d = n-1$, $P = \mathbf{D}$ and use (61) to find

$$\begin{aligned} D_i^{(n-1)}(\mathbf{D}\,Q) \;&=\; 2\sum_{k=1}^{n}\Biggl(2\sum_{j=1}^{n} t_{jk}\,\mathbf{D}_{ij}\Biggr)\,\partial_{ik}(Q) \,+\, \mathbf{D}\,D_i^{(n-1)}(Q) \\ &=\; \mathbf{D}\,\bigl(D_i^{(n-1)}(Q) \,+\, 4\partial_{ii}(Q)\bigr) \;=\; \mathbf{D}\,D_i^{(n+3)}(Q)\,. \qquad \square \end{aligned}$$

Formula (60) follows immediately: If $P \in \mathcal{K}_{\mathbf{a}}(n-1)$, we can write $P = \mathbf{D}\,Q$ with $Q \in \mathbb{C}[\mathcal{S}_n]_{\mathbf{a}-\mathbf{2}}$, and equation (62) shows that $P \in \mathcal{P}_{\mathbf{a}}(n-1)$ if and only if $Q \in \mathcal{P}_{\mathbf{a}-\mathbf{2}}(n+3)$. $\square$

We end this section with a final remark. As we mentioned before the Corollary of Theorem 1 and have used several times, given a non-degenerate quadratic form $Q(x) = (x,x)$ on $\mathbb{R}^d$ we have a direct sum decomposition $\mathbb{C}[\mathbb{R}^d] = \bigoplus_{m\ge 0} Q(x)^m\,\mathcal{H}(\mathbb{R}^d)$ of the space $\mathbb{C}[\mathbb{R}^d]$ and hence a direct sum decomposition

$$\mathbb{C}[\mathbb{R}^d]^{\otimes n} \;=\; \bigoplus_{\mathbf{m}\ge 0}\left(\prod_{i=1}^{n} Q(x_i)^{m_i}\right)\mathcal{H}(\mathbb{R}^d)^{\otimes n} \;=\; \mathcal{H}(\mathbb{R}^d)^{\otimes n}\,\oplus\, I_n\,,$$

of its nth tensor power, where I_n is the ideal generated by the polynomials $Q(x_1),\dots,Q(x_n)$ in $\mathbb{C}[\mathbb{R}^{nd}] = \mathbb{C}[x_1,\dots,x_n]$, the splitting being given explicitly by the product of the projection operators $\pi_i^{(d)}$ $(1\le i\le n)$ defined in (43). In particular, a polynomial $P(x_1,\dots,x_n)$ that is harmonic with respect to each variable $x_i\in\mathbb{R}^d$ and that belongs to the ideal I_n vanishes. But, at least for $d\ge 3$, the elements of I_n are precisely the polynomials vanishing on the discriminant variety $\mathcal{D}_n = \{(x_1,\dots,x_n)\in\mathbb{C}^{dn}\mid Q(x_1)=\cdots=Q(x_n)=0\}$. (*Proof*: The quadratic form Q is irreducible because a reducible form $Q(x) = (a^tx)(b^tx)$ corresponds to the symmetric matrix a^tb+ab^t of rank ≤ 2, so cannot be non-degenerate if $d\ge 3$. But then $\mathcal{D}_n$ is the product of n irreducible varieties and hence irreducible, so its associated ideal I_n is prime and hence equal to its own radical. The assertion then follows from Hilbert's Nullstellensatz.) This proves the following statement, which will be used at the end of the next section in the case when $n\ge d=4$.

Proposition 2. *A polynomial belonging to $\mathcal{H}(\mathbb{R}^d)^{\otimes n}$ is completely determined by its restriction to the set $\{Q(x_1)=\cdots=Q(x_n)=0\}\subset\mathbb{C}^{dn}$. In particular, if this restriction is $O(d)$-invariant, then so is the original polynomial.*

§7. Construction of invariant harmonic polynomials for $d = 4$

In this section we will give a construction of higher spherical polynomials for the special case $d = 4$ by identifying $\mathbb{C}^d$ in this case with the space $M(2,\mathbb{C})$ of 2×2 complex matrices,[5] with the quadratic form $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \det(g) = ad - bc$. The group $G = SL(2,\mathbb{C})$ acts on $M(2,\mathbb{C})$ by both right and left multiplication, and the combined action of $G\times G$ on $M(2,\mathbb{C}) \cong \mathbb{C}^4$ identifies $(G\times G)/\{\pm 1\}$ with the special orthogonal group $SO(4)$.

Let V_1 denote the standard 2-dimensional representation of G and V_a ($a \in \mathbb{N}$) its ath symmetric power, the $(a+1)$-dimensional space of homogeneous polynomials of degree a in two variables x and y. We have a G-invariant scalar product in V_a given by

$$\langle x^p y^q,\, x^{p'} y^{q'} \rangle = (-1)^p\, p!\, q!\, \delta_{p,q'} \qquad (p + q = p' + q' = a)\,, \tag{63}$$

so we obtain by tensor product a G^n-invariant scalar product $\langle\ ,\ \rangle$ on $V_{\mathbf{a}} = V_{a_1}\otimes\cdots\otimes V_{a_n}$. Now for each multi-index $\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})$ we have the G-invariant vector

$$w_{\boldsymbol{\nu}} = \prod_{1\le i<j\le n} \frac{(x_i y_j - x_j y_i)^{\nu_{ij}}}{\nu_{ij}!} \ \in\ V_{\mathbf{a}}\,,$$

where (x_i, y_i) are the coordinates on V_{a_i}. We define a function $F_{\boldsymbol{\nu}}$ on $M_2(\mathbb{C})^n$ by

$$F_{\boldsymbol{\nu}}(\mathbf{g}) = \langle \mathbf{g} w_{\boldsymbol{\nu}}, w_{\boldsymbol{\nu}} \rangle \qquad \bigl(\,\mathbf{g} = (g_1,\dots,g_n) \in M_2(\mathbb{C})^n\,\bigr)\,,$$

and more generally set $F_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g}) = \langle \mathbf{g} w_{\boldsymbol{\mu}}, w_{\boldsymbol{\nu}} \rangle$ for any $\boldsymbol{\mu},\ \boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})$.

Proposition 1. *The polynomial $F_{\boldsymbol{\mu},\boldsymbol{\nu}}$ is a homogeneous harmonic polynomial of degree a_i with respect to $g_i \in M_2(\mathbb{C}) \cong \mathbb{C}^4$ for each index $i = 1,\dots,n$. It is $SO(4)$-invariant for any $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, and is $O(4)$-invariant if $\boldsymbol{\mu} = \boldsymbol{\nu}$.*

Proof. The homogeneity property is obvious. For the harmonicity we must show that $\Delta_i F_{\boldsymbol{\mu},\boldsymbol{\nu}} = 0$ for each i, where $g_i = \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)$, $\Delta_i = 4\bigl(\frac{\partial^2}{\partial a_i \partial d_i} - \frac{\partial^2}{\partial b_i \partial c_i}\bigr)$. This follows from the observation that Δ_i annihilates $F(a_i x + b_i y, c_i x + d_i y)$ for any twice differentiable function $F(x,y)$, whence $\Delta_i(\mathbf{g} w_{\boldsymbol{\mu}}) = 0$ and *a fortiori* $\Delta_i(\langle \mathbf{g} w_{\boldsymbol{\mu}}, w_{\boldsymbol{\nu}}\rangle) = 0$ for any $\boldsymbol{\nu}$. Finally, since $w_{\boldsymbol{\nu}}$ is (left) invariant under the diagonal action of G on $V_{\mathbf{a}}$ and the scalar product $\langle\ ,\ \rangle$ is G-invariant, it is clear that $F_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g}) = \langle \mathbf{g} w_{\boldsymbol{\mu}}, w_{\boldsymbol{\nu}} \rangle$ is invariant under both right and left multiplication of $\mathbf{g}$ by elements of G, i.e., it is invariant under the action of $SO(4)$. The G-invariance and $(-1)^a$-symmetry of the scalar product on V_a and the assumption that $a_1 + \cdots + a_n$ is even imply that $F_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g}^*) = F_{\boldsymbol{\nu},\boldsymbol{\mu}}(\mathbf{g})$, where $\mathbf{g} \mapsto \mathbf{g}^*$ is the involution on $M_2(\mathbb{C})^n$ induced by the involution $* : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ of $M_2(\mathbb{C})$. Since this involution represents the non-trivial coset of $SO(4)$ in $O(4)$, it follows that the polynomials $F_{\boldsymbol{\mu},\boldsymbol{\nu}} + F_{\boldsymbol{\nu},\boldsymbol{\mu}}$, and hence in particular also the polynomials $F_{\boldsymbol{\nu}} = F_{\boldsymbol{\nu},\boldsymbol{\nu}}$, are $O(4)$-invariant. $\square$

We next define a generating function which has all of the polynomials $F_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g})$ as its coefficients. Let $\mathbf{U} = (U_{ij})$ and $\mathbf{V} = (V_{ij})$ be antisymmetric $n\times n$ matrices with variable coefficients (so that each is coordinatized by $n(n-1)/2$ independent variables U_{ij}, V_{ij} with $i < j$) and set

$$F(\mathbf{g}; \mathbf{U}, \mathbf{V}) = \sum_{\boldsymbol{\mu},\,\boldsymbol{\nu}} F_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g})\, \mathbf{U}^{\boldsymbol{\mu}} \mathbf{V}^{\boldsymbol{\nu}}\,,$$

[5]Here we work over $\mathbb{C}$ rather than $\mathbb{R}$ (cf. footnote 2 in §1) because the determinant form on 2×2 matrices is not isomorphic to the standard quadratic form $\sum_1^4 x_i^2$ over $\mathbb{R}$, but is over $\mathbb{C}$.

where $\mathbf{U}^{\boldsymbol{\mu}}$ means $\prod_{i<j} U_{ij}^{\mu_{ij}}$ and we set $F_{\boldsymbol{\mu},\boldsymbol{\nu}}$ equal to 0 if $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ have different multidegrees. The homogeneity of the $F_{\boldsymbol{\mu},\boldsymbol{\nu}}$ implies that this generating function is unchanged if we replace g_i by g_i/λ_i and U_{ij} by $\lambda_i\lambda_j U_{ij}$ for any non-zero constants λ_i, so we may restrict ourselves to the case when all the g_i are unimodular.

Theorem 8. *The generating function $F(\mathbf{g};\mathbf{U},\mathbf{V})$ for $\mathbf{g}\in SL(2,\mathbb{R})^n$ is given by*

$$F(\mathbf{g};\mathbf{U},\mathbf{V}) \;=\; D(\mathbf{g};\mathbf{U},\mathbf{V})^{-1/2}\,,$$

where $D(\mathbf{g};\mathbf{U},\mathbf{V})$, a polynomial in the coefficients a_i, b_i, c_i, d_i of $\mathbf{g}$ and U_{ij}, V_{ij} of $\mathbf{U}$ and $\mathbf{V}$, is defined as the determinant of the symmetric $4n\times 4n$ matrix

$$\begin{pmatrix} \mathbf{0_n} & \mathbf{U} & \mathbf{a} & \mathbf{b} \\ -\mathbf{U} & \mathbf{0_n} & \mathbf{c} & \mathbf{d} \\ \mathbf{a} & \mathbf{c} & \mathbf{0_n} & \mathbf{V} \\ \mathbf{b} & \mathbf{d} & -\mathbf{V} & \mathbf{0_n} \end{pmatrix},$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are the $n\times n$ diagonal matrices with diagonal entries a_i, b_i, c_i, d_i, respectively.

Proof. Extend the scalar product (63) to be 0 for polynomials of different degree, i.e., replace the right-hand side of (63) by $\langle x^py^q,\, x^{p'}y^{q'}\rangle = (-1)^p p!q!\delta_{pq'}\delta_{p'q}$. Then $\langle u,v\rangle = 0$ for $u\in W_{\boldsymbol{\nu}}$, $v\in W_{\boldsymbol{\mu}}$ with $\boldsymbol{\nu}\cdot\mathbf{1}\neq\boldsymbol{\mu}\cdot\mathbf{1}$, so that the definition of $F_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g})$ as $\langle \mathbf{g}w_{\boldsymbol{\mu}}, w_{\boldsymbol{\nu}}\rangle$ holds for all $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. From the definition of $w_{\boldsymbol{\mu}}$ we have

$$\sum_{\boldsymbol{\mu}} w_{\boldsymbol{\mu}}\,\mathbf{U}^{\boldsymbol{\mu}} = \prod_{1\le i<j\le n} e^{U_{ij}(x_iy_j-x_jy_i)} = \exp\bigl(\sum_{i,j} U_{ij}x_iy_j\bigr)\,,$$

and hence

$$F(\mathbf{g};\mathbf{U},\mathbf{V}) = \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} \langle w^*_{\boldsymbol{\mu}}, w_{\boldsymbol{\nu}}\rangle\, \mathbf{U}^{\boldsymbol{\mu}}\mathbf{V}^{\boldsymbol{\nu}} = \bigl\langle \exp\bigl(\sum_{i,j} U_{ij}x_i^*y_j^*\bigr),\, \exp\bigl(\sum_{i,j} V_{ij}x_iy_j\bigr)\bigr\rangle\,,$$

where $\begin{pmatrix} x_i^* \\ y_i^* \end{pmatrix} := g_i \begin{pmatrix} x_i \\ y_i \end{pmatrix}$. Using the identity (easily proved using polar coordinates)

$$\delta_{pq}\,p! \;=\; \frac{1}{\pi}\int_{\mathbb{C}} x^p\bar{x}^q e^{-|x|^2}\,d\mu_x \qquad (p,\,q\in\mathbb{Z}_{\ge 0}),$$

where $d\mu_x$ denotes the standard Lebesgue measure in $\mathbb{C}$, we obtain the integral representation

$$\langle x^py^q, x^{p'}y^{q'}\rangle = \frac{1}{\pi^2}\int_{\mathbb{C}^2} (-x)^p y^q \bar{y}^{p'}\,\bar{x}^{q'}\, e^{-|x|^2-|y|^2}\,d\mu_x d\mu_y$$

or (changing x to $-x$) $\langle f_1,\, f_2\rangle = \pi^{-2}\int_{\mathbb{C}^2} f_1(x,y)f_2(\bar{y},-\bar{x})e^{-|x|^2-|y|^2}\,d\mu_x d\mu_y$ for the scalar product $\langle\cdot,\cdot\rangle$ on $\bigoplus_a V_a$. This gives the integral representation

$$F(\mathbf{g};\mathbf{U},\mathbf{V}) = \frac{1}{\pi^{2n}}\int_{\mathbb{C}^{2n}} \exp\Bigl(-\Bigl[\sum_{i=1}^{n}(x_i\bar{x}_i+y_i\bar{y}_i) \;-\; \sum_{i,j}\bigl(U_{ij}x_i^*y_j^* + V_{ij}\bar{x}_i\bar{y}_j\bigr)\Bigr]\Bigr)\,d\mu$$

for $F(\mathbf{g};\mathbf{U},\mathbf{V})$, where $d\mu = d\mu_{x_1}\cdots d\mu_{y_n}$ is the Lebesgue measure on $\mathbb{C}^{2n}$. Now this integral equals $D^{-1/2}$ where D is the determinant of the quadratic form in square brackets, considered as a form of rank $4n$ over $\mathbb{R}$. The matrix representing this quadratic form with respect to the variables $(y_1^*, \ldots, y_n^*, -x_1^*, \ldots, -x_n^*, \bar{y}_1, \ldots, \bar{y}_n, -\bar{x}_1, \ldots, -\bar{x}_n)$ is half of the one given in the proposition, where we have to remember that $g^t\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)g = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. But we are taking the usual Lebesgue measure with respect to real parts and imaginary parts of x_i and y_i for the integral and we have $\Re(x) = (x+\bar{x})/2$, $\Im(x) = (x-\bar{x})/2i$, so the factors of 2 cancel. $\square$

Theorem 8 in principle gives us a way to calculate the polynomials $F_{\boldsymbol{\mu},\boldsymbol{\nu}}$ for arbitrary n and for arbitrary $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, but these polynomials can be written uniquely in terms of the numbers $t_{ij} = \mathrm{Tr}(g_i\overline{g_j})$ only if $\boldsymbol{\mu} = \boldsymbol{\nu}$ and $n \le 4$, because if $\boldsymbol{\mu} \ne \boldsymbol{\nu}$ then they are only $SO(4)$- rather than $O(4)$-invariant, while if $n > 4$ then there are non-trivial relations among the t_{ij} (vanishing of all 5×5 minors) which mean that the representation of $F_{\boldsymbol{\nu}}(\mathbf{g})$ as a polynomial in the t_{ij} is not unique. Therefore the most important cases of the theorem are $n = 3$ (which will be treated in detail in the next section) and $n = 4$ (where we have not been able to do any interesting calculations). However, using the results of §5 we can prove the following result for arbitrary values of n.

Proposition 2. *The polynomials $F_{\boldsymbol{\nu}}$ for $d = 4$ for any value of n coincide up to a scalar factor with $\widetilde{P_{\boldsymbol{\nu}}^M}$, the pull-backs under β_n of the monomial basis polynomials $P_{\boldsymbol{\nu}}^M \in \mathcal{P}^{(n)}(4)$.*

Proof. Since both $F_{\boldsymbol{\nu}}$ and $\widetilde{P_{\boldsymbol{\nu}}^M}$ are harmonic with respect to each variable $g_i \in M_2(\mathbb{R}) = \mathbb{R}^4$, it suffices by Proposition 2 of §6 to show that they agree up to a constant factor when all the g_i have determinant 0. In view of the definition of $P_{\boldsymbol{\nu}}^M$, this means that we must prove that

$$\big\langle \mathbf{g}w_{\boldsymbol{\nu}},\, w_{\boldsymbol{\nu}} \big\rangle \;=\; \frac{\mathbf{a!}}{\boldsymbol{\nu}!^2} \prod_{1\le i<j\le n} \big(g_i,\, g_j\big)^{\nu_{ij}} \qquad \text{if } \det(g_1) = \cdots = \det(g_n) = 0\,, \tag{64}$$

where we have now inserted the constant, using the abbreviations $\boldsymbol{\nu}! = \prod_{i<j} \nu_{ij}!$, $\mathbf{a}! = \prod_i a_i!$.

To prove (64), we first observe that a 2×2 matrix of determinant 0 has rank ≤ 1 and hence can be written as $\boldsymbol{\xi}\,\mathbf{r}^t$ for some column vectors $\boldsymbol{\xi}$ and $\mathbf{r}$. So we can write each g_i as $\boldsymbol{\xi}_i\mathbf{r}_i^t$ for some column vectors $\boldsymbol{\xi}_i = \left(\begin{smallmatrix} \xi_i \\ \eta_i \end{smallmatrix}\right)$ and $\mathbf{r}_i = \left(\begin{smallmatrix} r_i \\ s_i \end{smallmatrix}\right)$. We first claim that the scalar product (g_i, g_j) is then given as the product of the two determinants $|\boldsymbol{\xi}_i\,\boldsymbol{\xi}_j|$ and $|\mathbf{r}_i\,\mathbf{r}_j|$. This can be proved either by multiplying everything out and checking or, slightly more elegantly, by observing that

$$\big(g_i,\, g_j\big) = \mathrm{tr}(g_i^* g_j) = \mathrm{tr}(-Jg_i^tJg_j) = \mathrm{tr}(-J\mathbf{r}_i\boldsymbol{\xi}_i^tJ\boldsymbol{\xi}_j\mathbf{r}_j^t) = (-\boldsymbol{\xi}_i^tJ\boldsymbol{\xi}_j)\,\mathrm{tr}(J\mathbf{r}_i\mathbf{r}_j^t) = |\boldsymbol{\xi}_i\,\boldsymbol{\xi}_j|\,|\mathbf{r}_i\,\mathbf{r}_j|\,.$$

(Here $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ as usual.) Next, we observe that $g_i\left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right) = (r_ix_i + s_iy_i)\,\boldsymbol{\xi}_i$, so we have

$$\mathbf{g}w_{\boldsymbol{\nu}} \;=\; \frac{1}{\boldsymbol{\nu}!} \prod_{i=1}^{n} \big(r_ix_i + s_iy_i\big)^{a_i} \cdot \prod_{1\le i<j\le n} \big|\boldsymbol{\xi}_i\,\boldsymbol{\xi}_j\big|^{\nu_{ij}}$$

Finally, the definition of the G-invariant scalar product in V_a as defined in (63) is easily seen to be equivalent to

$$\big\langle (rx+sy)^a,\, P(x,y) \big\rangle \;=\; a!\, P(s,-r) \qquad \text{for all } P \in V_a\,.$$

Combining these three facts, we get the desired equality (64):

$$\begin{aligned} \big\langle \mathbf{g}w_{\boldsymbol{\nu}},\, w_{\boldsymbol{\nu}} \big\rangle \;&=\; \frac{1}{\boldsymbol{\nu}!} \prod_{i<j} \big|\boldsymbol{\xi}_i\,\boldsymbol{\xi}_j\big|^{\nu_{ij}} \cdot \Big\langle \prod_{i=1}^{n} (r_ix_i + s_iy_i)^{a_i},\, w_{\boldsymbol{\nu}} \Big\rangle \\ &=\; \frac{\mathbf{a!}}{\boldsymbol{\nu}!^2} \prod_{i<j} \Big(\big|\boldsymbol{\xi}_i\,\boldsymbol{\xi}_j\big|\, \big|\mathbf{r}_i\,\mathbf{r}_j\big| \Big)^{\nu_{ij}} \;=\; \frac{\mathbf{a!}}{\boldsymbol{\nu}!^2} \prod_{i<j} \big(g_i,\, g_j\big)^{\nu_{ij}}\,. \end{aligned} \qquad \square$$

Remark. The proof of Proposition 2 did not use the $O(4)$-invariance of $F_{\boldsymbol\nu}$, and hence gives another proof of this invariance, because of the second statement in Proposition 2 of §6.

§8. Higher spherical polynomials for $n = 3$

In the case $n = 3$, we have the extra fact that the polynomials $F_{\boldsymbol\mu,\boldsymbol\nu}$ all vanish except those $\boldsymbol\mu = \boldsymbol\nu$, since there is only one triple $\boldsymbol\nu$ with any given multidegree $\mathbf{a}$, as given in (3), and the polynomials $F_{\boldsymbol\mu,\boldsymbol\nu}$ with indices of different multidegrees vanish by definition. This case, which will be treated in detail in this section, is therefore particularly interesting.

It will be convenient to rename the coordinates of $\mathcal{S}_n$ for $n = 3$. We set

$$2T = \begin{pmatrix} 2m_1 & r_3 & r_2 \\ r_3 & 2m_2 & r_1 \\ r_2 & r_1 & 2m_3 \end{pmatrix}. \tag{65}$$

We will use the notations $\mathbf{m}$ and $\mathbf{r}$ to denote the triples (m_1, m_2, m_3) and (r_1, r_2, r_3) and will consistently use the evident vector notation, e.g. $\mathbf{m}^{\boldsymbol\lambda}$ for a triple $\boldsymbol\lambda = (\lambda_1, \lambda_2, \lambda_3)$ denotes $\prod m_i^{\lambda_i}$. The indices i are taken modulo 3. We write a typical element of $\mathcal{N}_0(\mathbf{a})$ as $\begin{pmatrix} 0 & \nu_3 & \nu_2 \\ \nu_3 & 0 & \nu_1 \\ \nu_2 & \nu_1 & 0 \end{pmatrix}$ with $\nu_i \ge 0$, so that ν_i is the same as what was denoted $\nu_{i+1,i+2}$ in the last section. According to the result of §2, we know that $\mathcal{P}_{\mathbf{a}}(d)$ is spanned by a unique (up to a scalar multiple) polynomial $P_{\boldsymbol\nu}(T) = P_{\boldsymbol\nu}(\mathbf{m}, \mathbf{r})$ whenever $\mathbf{a}$ and $\boldsymbol\nu$ are related as in (3). It is easily seen that $P_{\boldsymbol\nu}$ is a homogeneous polynomial of tridegree $\boldsymbol\nu$ if we assign to r_1, r_2, r_3, m_1, m_2, and m_3 the tridegrees (1,0,0), (0,1,0), (0,0,1), (−1,1,1), (1,−1,1), and (1,1,−1), respectively.

In terms of the coordinates (65), the differential operator D_1 computed in §1 is given by

$$\frac{1}{4}D_1 = \left(\frac{d}{2} + m_1\frac{\partial}{\partial m_1} + r_2\frac{\partial}{\partial r_2} + r_3\frac{\partial}{\partial r_3}\right)\frac{\partial}{\partial m_1} + m_3\frac{\partial^2}{\partial r_2^2} + r_1\frac{\partial^2}{\partial r_2 \partial r_3} + m_2\frac{\partial^2}{\partial r_3^2} \tag{66}$$

(and D_2 and D_3 by the same formula with the indices permuted cyclically).

This section, which is fairly long, will be divided into five subsections. In the first, we construct generating functions whose coefficients give us a canonical generator $P_{\boldsymbol\nu}(T)$ of the 1-dimensional space $\mathcal{P}_{\mathbf{a}}(d)$. In the second, we compute the relation between the new basis elements and the ones constructed in §5 and use this to compute the scalar products of the $P_{\boldsymbol\nu}$ with themselves, while the third and fourth subsections contain various recursion relations and explicit formulas for the coefficients of these polynomials. The last subsection contains a brief discussion of the $n = 3$ case of the inhomogeneous coordinates defined at the end of §1, of the related angular coordinates, and of a somewhat surprising extra symmetry of these coordinates.

A. Generating function. We begin by writing out the generating function of Proposition 2 explicitly for $n = 3$. This generating function will then be generalized to arbitrary d.

For the case $d = 4$, we can choose $P_{\boldsymbol\nu}(T)$ canonically as $F_{\boldsymbol\nu}(\mathbf{g}) = \langle \mathbf{g}w_{\boldsymbol\nu}, w_{\boldsymbol\nu}\rangle$ defined as in §6, where $\mathbf{g} = (g_1, g_2, g_3)$ is a triple of matrices in $M_2(\mathbb{R})$ related to the 3×3 matrix T by $m_i = \det(g_i)$, $r_i = \mathrm{Tr}(g_{i+1}g_{i+2}^*)$. We introduce dummy variables X_1, X_2, X_3 and define $\mathbf{X}^{\boldsymbol\nu}$ as usual as $X_1^{\nu_1}X_2^{\nu_2}X_3^{\nu_3}$.

Proposition 1. *The polynomials $P_{\boldsymbol\nu}(T)$ for $n = 3$, $d = 4$ are given by the generating function*

$$\sum_{\boldsymbol\nu} P_{\boldsymbol\nu}(T)\,\mathbf{X}^{\boldsymbol\nu} = \frac{1}{\sqrt{\Delta_0(T,\mathbf{X})^2 - 4d(T)X_1X_2X_3}}, \tag{67}$$

where

$$\Delta_0(T,\mathbf{X}) \;=\; 1-\sum_{i=1}^{3}(r_iX_i - m_ir_iX_{i+1}X_{i+2} - m_{i+1}m_{i+2}X_i^2)$$

and

$$d(T) \;=\; 4\,\det(T) \;=\; \prod_{i=1}^{3} r_i \;-\; \sum_{i=1}^{3} m_ir_i^2 \;+\; 4\prod_{i=1}^{3} m_i\,.$$

Proof. This is just Theorem 8 for the special case $n=3$, since we have already seen that in that case the terms with $\boldsymbol{\mu}\neq\boldsymbol{\nu}$ in the generating function vanish. To compute the generating function explicitly we must calculate the 12×12 determinant $D(\mathbf{g};\mathbf{U},\mathbf{V})$ defined in Theorem 8. By direct calculation we find $D(\mathbf{g};\mathbf{U},\mathbf{V}) = \Delta_0(T,\mathbf{X})^2 - 4d(T)X_1X_2X_3$ when all $m_i=1$, where T is determined by $\mathbf{g}$ as explained above and $X_i = U_{i+1,i+2}V_{i+1,i+2}$, and the general case then follows by homogeneity. $\square$

Proposition 1 makes it easy to compute any polynomial $P_{\boldsymbol{\nu}}$ explicitly. Here is a table for all $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}|\leq 4$, where we give only one representative for each $\mathfrak{S}_3$-orbit of indices $\boldsymbol{\nu}$:

$$\begin{aligned}
&P_{000}=1\,;\qquad P_{100}=r_1\,;\qquad P_{200}=r_1^2-m_2m_3\,,\quad P_{110}=2r_1r_2-m_3r_3\,;\\
&P_{300}=r_1^3-2m_2m_3r_1\,,\qquad P_{210}=3r_1^2r_2-2m_3(r_1r_3+m_2r_2)\,,\\
&\quad P_{111}=8r_1r_2r_3-4(m_1r_1^2+m_2r_2^2+m_3r_3^2)+8m_1m_2m_3\,;\\
&P_{400}=r_1^4-3m_2m_3r_1^2+m_2^2m_3^2\,,\qquad P_{310}=4r_1^3r_2-3m_3r_1^2r_3-6m_2m_3r_1r_2+2m_2m_3^2r_3\,,\\
&\quad P_{220}=6r_1^2r_2^2-6m_3r_1r_2r_3-3m_3(m_1r_1^2+m_2r_2^2)+m_3^2r_3^2+2m_1m_2m_3^2\,,\\
&\quad P_{211}=18r_1^2r_2r_3-9m_1r_1^3-12r_1(m_2r_2^2+m_3r_3^2)-4m_2m_3r_2r_3+26m_1m_2m_3r_1\,.
\end{aligned}$$

We now state a generalization of Proposition 1 which gives explicit higher spherical polynomials for $n=3$ and for arbitrary values of $\boldsymbol{\nu}$ and d as the coefficients of a generating function.

Theorem 9. *For $T\in\mathcal{S}_3$ as in (65) and $\mathbf{X}=(X_1,X_2,X_3)$ let $\Delta_0(T,\mathbf{X})$ and $d(T)$ be as in Proposition 1 and set*

$$R(T,\mathbf{X}) = \frac{\Delta_0(T,\mathbf{X})+\sqrt{\Delta_0(T,\mathbf{X})^2-4d(T)X_1X_2X_3}}{2}\,. \tag{68}$$

For any $d\in\mathbb{C}$, define polynomials $P_{\boldsymbol{\nu},d}(T)$ ($\boldsymbol{\nu}=(\nu_1,\nu_2,\nu_3)\in\mathbb{Z}^3_{\geq 0}$) by the generating function

$$\sum_{\boldsymbol{\nu}} P_{\boldsymbol{\nu},d}(T)\,\mathbf{X}^{\boldsymbol{\nu}} = \frac{R(T,\mathbf{X})^{-s}}{\sqrt{\Delta_0(T,\mathbf{X})^2-4d(T)X_1X_2X_3}}\,. \tag{69}$$

where $s=d/2-2$. Then $P_{\boldsymbol{\nu},d}(T)$ belongs to the space $\mathcal{P}_{\mathbf{a}}(d)$, where $\mathbf{a}$ is related to $\boldsymbol{\nu}$ by (3), and generates this space if $d\notin\{2,\,0,\,-2,\,-4,\,\dots\}$ or if $d=2$ and all ν_i are strictly positive.

Proof. Using the formula for D_1 given in (66), we find by direct calculation that

$$\begin{aligned}
&\frac{1}{4m_2m_3-r_1^2}\,D_1\biggl(\frac{d(T)^n}{\Delta_0(T,\mathbf{X})^{2n+s+1}}\biggr)\\
&\qquad = n(n+s)\,\frac{d(T)^{n-1}}{\Delta_0(T,\mathbf{X})^{2n+s+1}} - (2n+s+1)(2n+s+2)\,\frac{d(T)^n\,X_1X_2X_3}{\Delta_0(T,\mathbf{X})^{2n+s+3}}\,.
\end{aligned}$$

Applying this identity to the expansion

$$\frac{1}{\sqrt{\Delta_0^2-4d(T)X}}\left(\frac{\Delta_0+\sqrt{\Delta_0^2-4d(T)X}}{2}\right)^{-s} = \sum_{n=0}^{\infty}\binom{2n+s}{n}\frac{(d(T)X)^n}{\Delta_0^{2n+s+1}} \tag{70}$$

shows that $D_1(P_{\boldsymbol{\nu},d})=0$. The vanishing of the other D_i follows in the same way or by symmetry. The final statement of the theorem follows because $\dim\mathcal{P}_{\mathbf{a}}(d)=1$ for $d\notin 2\mathbb{Z}_{\le 0}$ and since $P_{\boldsymbol{\nu},d}\not\equiv 0$ for $-s\notin\mathbb{N}$ by the calculation of its leading term as given in Proposition 2 below. $\square$

Using Theorem 9, we can compute the polynomials $P_{\boldsymbol{\nu},d}$ of any given degree, a typical value (generalizing to arbitrary s the one given above for $s=0$) being

$$\begin{aligned} P_{211,d}(T) = {}& \tfrac12(s+2)^2(s+3)^2r_1^2r_2r_3-\tfrac12(s+2)(s+3)^2m_1r_1^3-(s+2)^2(s+3)(m_2r_2^2+m_3r_3^2)r_1\\ &-(s+1)(s+2)^2m_2m_3r_2r_3+(s+2)(5s+13)m_1m_2m_3r_1\,. \end{aligned}$$

The polynomials $P_{\boldsymbol{\nu},d}(T)$ for all $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}|\le 6$ are given in Table 1 at the end of the paper.

Remark 1. The expressions $\Delta_0(T,\mathbf{X})$ and $d(T)X_1X_2X_3$ appearing in the theorem can be written as $(1-\sigma_1/2)^2-\sigma_2$ and $2\sigma_3$, respectively, where the σ_i are the elementary symmetric polynomials in the eigenvalues (= coefficients of the characteristic polynomial up to sign) of the 3×3 matrix $\begin{pmatrix}0&X_3&X_2\\X_3&0&X_1\\X_2&X_1&0\end{pmatrix}T$. This remark, and the generating function of Theorem 8, will be greatly generalized in §9.

Remark 2. In the above proof, we simply wrote down the right-hand side of (69) as a generalization of (67) and verified that its coefficients were spherical polynomials, which was an easy calculation. The main point here is how to guess the correct formula (69). This was done (roughly) in the following way. We wish to define for all d a generator $P_{\boldsymbol{\nu},d}$ of the 1-dimensional space $\mathcal{P}_{\mathbf{a}}(d)$ in such a way that the sum $\sum_{\boldsymbol{\nu}}P_{\boldsymbol{\nu},d}(T)\mathbf{X}^{\boldsymbol{\nu}}$ is algebraic and agrees with (67) if $d=4$. Because of the one-dimensionality, we know that these functions, if they exist, must be scalar multiples of both the monomial basis elements $P_{\boldsymbol{\nu}}^M$ and the descending basis elements $P_{\boldsymbol{\nu}}^D$ as defined in §5, where the multiples are determined by computing the constant term and the effect of the mixed Laplace operators D_{ij}, respectively. So we begin by computing these things in the already known case $d=4$ (cf. Subsection **B** below, where these computations are done, assuming the definition (69), for all d), and then insert s into the formulas obtained in the simplest way possible such that the coefficients of $P_{\boldsymbol{\nu}}$ do not acquire any denominators (which are prohibited if $\sum P_{\boldsymbol{\nu}}\mathbf{X}^{\boldsymbol{\nu}}$ is to be an algebraic function). This led to formulas (74) and (77) below, after which the generating function (69) was found more or less by doing the calculations of Subsection **B** in the reverse order.

Remark 3. It is interesting to compare the computations for $n=3$ in the last two sections (Proposition 1 and Theorem 9) with the classical case $n=2$. In this case the polynomial $F_{\boldsymbol{\mu},\boldsymbol{\nu}}(\mathbf{g})$ defined in §7 equals $P_a(T)$ if $\boldsymbol{\mu}=\boldsymbol{\nu}=\left(\begin{smallmatrix}0&a\\a&0\end{smallmatrix}\right)$ for some integer $a\ge 0$ and is 0 otherwise, where $T=\left(\begin{smallmatrix}m_1&r/2\\r/2&m_2\end{smallmatrix}\right)$ with $m_i=\det(g_1)$ and $r=\operatorname{tr}(g_1g_2^*)$. The 8×8 determinant $D(\mathbf{g};\mathbf{U},\mathbf{V})$ occurring in Theorem 8, for $\mathbf{U}=\left(\begin{smallmatrix}0&-u\\u&0\end{smallmatrix}\right)$ and $\mathbf{V}=\left(\begin{smallmatrix}0&-v\\v&0\end{smallmatrix}\right)$, is easily calculated to be $(X^2-rX+m_1m_2)^2$, where $X=uv$, so we obtain the generating function $\sum_{a=0}^{\infty}P_a(T)\,X^a=(1-rX+m_1m_2X^2)^{-1}$ for the spherical polynomials $P_a(T)$ when $d=4$. On the other hand, in Example 1 of §1 we calculated a generator of the one-dimensional space $\mathcal{P}_{(a,a)}(d)$ for any $a\ge 0$ and any $d\notin 2\mathbb{Z}\cap[4-2a,2-a]$, obtaining (with a suitable normalization) the formula

$$P_{a,d}\left(\begin{smallmatrix}m_1&r/2\\r/2&m_2\end{smallmatrix}\right) = \sum_{0\le l\le a/2}\binom{a-l}{l}\binom{a-l+d/2-2}{a-l}(-m_1m_2)^l\,r^{a-2l}\,, \tag{71}$$

which in turn is equivalent to the more general generating function

$$\sum_{a=0}^{\infty} P_{a,d}\left(\begin{smallmatrix} m_1 & r/2 \\ r/2 & m_2 \end{smallmatrix}\right) X^a \;=\; (1 - rX + m_1 m_2 X^2)^{1-d/2} \,. \tag{72}$$

The formulas (71) and (72) are of course classical, the polynomials $P_{a,d}$ being called Gegenbauer polynomials in that case (with the special names Legendre polynomials if $d = 3$ and Chebyshev polynomials if $d = 2$ or 4). In §9 we will show how to produce natural generating functions generalizing (72) and (69) that give us canonical bases of $\mathcal{P}_{\mathbf{a}}(d)$ for arbitrary values of n.

Notation. Let us fix the notation for the coefficients of the polynomials $P_{\boldsymbol{\nu}} = P_{\boldsymbol{\nu},d}$. We write T as in equation (65) and denote by $C(\boldsymbol{\lambda};\boldsymbol{\mu})$ the coefficient of $\mathbf{m}^{\boldsymbol{\lambda}}\mathbf{r}^{\boldsymbol{\mu}}$ in $P_{\boldsymbol{\nu}}$. Clearly this coefficient is 0 unless $\boldsymbol{\nu}$, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are related by

$$\nu_i \;=\; \mu_i - \lambda_i + \lambda_{i+1} + \lambda_{i+2}\,, \qquad \lambda_i \;=\; \frac{1}{2}\bigl(\nu_{i+1} + \nu_{i+2} - \mu_{i+1} - \mu_{i+2}\bigr) \tag{73}$$

(here i is considered modulo 3), so that we can omit the subscript $\boldsymbol{\nu}$ in $C(\boldsymbol{\lambda};\boldsymbol{\mu})$, but we can also write $C_{\boldsymbol{\nu}}(\boldsymbol{\lambda};\boldsymbol{\mu})$ if this is needed for emphasis or clarity. We will also almost always omit the "d" from the notation for the coefficients of $P_{\boldsymbol{\nu},d}$, but of course these coefficients depend (polynomially) on this parameter. As usual, we also work with s, where $d = 2s + 4$.

B. Ratios of the various basis elements. In Chapter I, we gave two special bases $\{P_{\boldsymbol{\nu}}^M\}$ and $\{P_{\boldsymbol{\nu}}^D\}$ for $\mathcal{P}_n(d)$. In the special case $n = 3$, we have now given a third basis $\{P_{\boldsymbol{\nu}}\}$ from the generating function (69). Since the (non-zero) spaces $\mathcal{P}_{\mathbf{a}}(d)$ for $n = 3$ are all one-dimensional, all three bases agree up to constants. In this subsection we compute these constants of proportionality and at the same time the constant terms and the norms (with respect to the canonical scalar product) of the polynomials $P_{\boldsymbol{\nu}}$.

Proposition 2. *The higher spherical polynomial $P_{\boldsymbol{\nu}}(T)$ defined by the generating function* (69) *for $n = 3$ is related to the monomial basis by*

$$P_{\boldsymbol{\nu}}(T) \;=\; 2^{\nu_1+\nu_2+\nu_3}\,\frac{(\nu_1+\nu_2+s)!\,(\nu_1+\nu_3+s)!\,(\nu_2+\nu_3+s)!}{\nu_1!\,\nu_2!\,\nu_3!\,(\nu_1+s)!\,(\nu_2+s)!\,(\nu_3+s)!}\,P_{\boldsymbol{\nu}}^M(T)\,. \tag{74}$$

Proof. In view of the definition of the monomial basis, this is equivalent to the statement that the leading coefficient of the polynomial $P_{\boldsymbol{\nu},d}$ is given by

$$C(\mathbf{0};\boldsymbol{\nu}) \;=\; \binom{\nu_1+\nu_2+s}{\nu_1}\binom{\nu_2+\nu_3+s}{\nu_2}\binom{\nu_3+\nu_1+s}{\nu_3}\,. \tag{75}$$

Taking T with $m_1 = m_2 = m_3 = 0$ in (69) and (70), so that $P_{\boldsymbol{\nu}}(T) = C(\mathbf{0};\boldsymbol{\nu})\,\mathbf{r}^{\boldsymbol{\nu}}$, and setting $t_i = r_i X_i$, we find

$$\begin{aligned}
\sum_{\boldsymbol{\nu}} C(\mathbf{0};\boldsymbol{\nu})\,\mathbf{t}^{\boldsymbol{\nu}} &= \sum_{n=0}^{\infty}\binom{2n+s}{n}\frac{(t_1t_2t_3)^n}{(1-t_1-t_2-t_3)^{2n+s+1}} \\
&= \sum_{n,e\ge 0}\binom{2n+s}{n}\binom{2n+s+e}{e}(t_1t_2t_3)^n\,(t_1+t_2+t_3)^e \\
&= \sum_{n,a,b,c\ge 0}\frac{(2n+s+a+b+c)!}{n!\,(n+s)!\,a!\,b!\,c!}\,t_1^{a+n}t_2^{b+n}t_3^{c+n}\,.
\end{aligned}$$

Hence

$$C(\mathbf{0};\boldsymbol{\nu}) \;=\; \sum_{n=0}^{\min(\nu_1,\nu_2,\nu_3)} \frac{(\nu_1+\nu_2+\nu_3+s-n)!}{n!\,(n+s)!\,(\nu_1-n)!\,(\nu_2-n)!\,(\nu_3-n)!}\,. \tag{76}$$

This expression is symmetric in the three arguments ν_i, but more complicated than the formula given in the proposition. To prove that formula, we first break the symmetry. We rewrite (76) as $C(\mathbf{0};\boldsymbol{\nu}) = \binom{\nu_1+\nu_2+s}{\nu_1} C'(\boldsymbol{\nu})$ with $C'(\boldsymbol{\nu})$ defined by

$$C'(\boldsymbol{\nu}) = \sum_{n=0}^{\min(\nu_1,\nu_2,\nu_3)} \binom{\nu_1}{n}\binom{\nu_2+s}{\nu_2-n}\binom{\nu_1+\nu_2+\nu_3+s-n}{\nu_3-n}.$$

Then for fixed ν_1 we find the generating function

$$\begin{aligned}
&\sum_{\nu_2,\,\nu_3\ge 0} C'(\nu_1,\nu_2,\nu_3)\,X^{\nu_2}\,Y^{\nu_3}\\
&\qquad= \sum_{n=0}^{\nu_1}\binom{\nu_1}{n}\sum_{\nu_2=n}^{\infty}\binom{\nu_2+s}{\nu_2-n}X^{\nu_2}\sum_{\nu_3=n}^{\infty}\binom{\nu_1+\nu_2+\nu_3+s-n}{\nu_3-n}Y^{\nu_3}\\
&\qquad= \sum_{n=0}^{\nu_1}\binom{\nu_1}{n}\sum_{\nu_2=n}^{\infty}\binom{\nu_2+s}{\nu_2-n}\frac{X^{\nu_2}\,Y^{n}}{(1-Y)^{\nu_1+\nu_2+s+1}}\\
&\qquad= \sum_{n=0}^{\nu_1}\binom{\nu_1}{n}\frac{X^{n}\,Y^{n}}{(1-Y)^{\nu_1+n+s+1}}\,\frac{1}{(1-X/(1-Y))^{n+s+1}}\\
&\qquad= \frac{1}{(1-Y)^{\nu_1}\,(1-X-Y)^{s+1}}\left(1+\frac{XY}{1-X-Y}\right)^{\nu_1}\\
&\qquad= \frac{(1-X)^{\nu_1}}{(1-X-Y)^{\nu_1+s+1}} \;=\; \sum_{\nu_3=0}^{\infty}\binom{\nu_1+s+\nu_3}{\nu_3}\frac{Y^{\nu_3}}{(1-X)^{\nu_3+s+1}}\\
&\qquad= \sum_{\nu_2=0}^{\infty}\sum_{\nu_3=0}^{\infty}\binom{\nu_1+\nu_3+s}{\nu_3}\binom{\nu_2+\nu_3+s}{\nu_2}X^{\nu_2}\,Y^{\nu_3}
\end{aligned}$$

proving (75). □

We now give the relationship between $P_{\boldsymbol{\nu}}$ and $P^D_{\boldsymbol{\nu}}$. The following proposition is just the special case $n=3$ of equation (135) of §9 below, which says that for any n the higher spherical polynomials defined by a generalization (117) of the generating series (69) are proportional to the descending basis elements, with a proportionality factor depending only on the total weight $k=\frac{1}{2}\,\mathbf{a}\cdot\mathbf{1}$. Recall the Pochhammer symbol $(x)_k = x(x+1)\cdots(x+k-1)$.

Proposition 3. *The polynomial $P_{\boldsymbol{\nu}}(T)$ is related to the descending basis by*

$$P_{\boldsymbol{\nu}}(T) \;=\; 2^k\,(s+2)_k\,(2s+2)_k\,P^D_{\boldsymbol{\nu}}(T)\,, \tag{77}$$

where $k=\nu_1+\nu_2+\nu_3$.

Now combining Proposition 2 and 3 with Theorem 6, we deduce the formula for the scalar product of $P_\nu \in \mathcal{P}_{\mathbf{a}}(d)$ with itself:

Corollary. *For every $\boldsymbol{\nu}=(\nu_1,\nu_2,\nu_3)\in\mathbb{Z}^3_{\ge 0}$ we have*

$$(P_{\boldsymbol{\nu}},P_{\boldsymbol{\nu}})_d \;=\; 2^{2k}\binom{\nu_1+\nu_2+s}{\nu_1}\binom{\nu_2+\nu_3+s}{\nu_2}\binom{\nu_3+\nu_1+s}{\nu_3}\,(s+2)_k\,(2s+2)_k\,, \tag{78}$$

where $d = 2s + 4$ *and* $k = \nu_1 + \nu_2 + \nu_3$. $\square$

Equation (78) can be used to simplify the computation of one of the coefficients in the recursion (82) in Proposition 5 below. Conversely, as explained in the proof of that proposition, one can also obtain the coefficient in question by direct computation and then derive (78) from it, after which one can derive Proposition 3 using Theorem 6 and the fact that all of our bases are *a priori* proportional, without needing to rely on the considerably harder results from §9.

C. Recursion relations. In Subsection **A** we defined canonical polynomials $P_{\boldsymbol{\nu}} \in \mathcal{P}^{(3)}(d)$ by means of a generating function. We now give five different recursion relations for the coefficients of these polynomials, coming from the differential equation, the scalar product, the raising operators, the mixed Laplacians, and the generating function, respectively.

1. Recursion relation coming from the differential equation. The first recursion is for the coefficients of a single polynomial $P_{\boldsymbol{\nu}}$. According to Theorem 1 and its proof, all of the coefficients of $P_{\boldsymbol{\nu}}$ are determined (in fact, overdetermined) recursively by the harmonicity and homogeneity properties from the "constant term" (specialization to $t_{11} = t_{22} = t_{33} = 0$, or to $m_1 = m_2 = m_3 = 0$ in the notation (65)). We make this explicit. In Subsection **A** we gave the formula for the differential operator D_1 in terms of the new coordinates (65), and of course the formulas for the other D_i are obtained by cyclic permutation of the indices. The equation $D_1 P_{\boldsymbol{\nu}} = 0$ gives the recursion (with $s = d/2 - 2$ as before)

$$\begin{aligned} &\lambda_1(\lambda_1 + \mu_2 + \mu_3 + s + 1)\, C(\lambda_1, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3) \\ &\quad + (\mu_2 + 1)(\mu_2 + 2)\, C(\lambda_1 - 1, \lambda_2, \lambda_3 - 1; \mu_1, \mu_2 + 2, \mu_3) \\ &\quad + (\mu_2 + 1)(\mu_3 + 1)\, C(\lambda_1 - 1, \lambda_2, \lambda_3; \mu_1 - 1, \mu_2 + 1, \mu_3 + 1) \\ &\quad + (\mu_3 + 1)(\mu_3 + 2)\, C(\lambda_1 - 1, \lambda_2 - 1, \lambda_3; \mu_1, \mu_2, \mu_3 + 2) \;=\; 0\,. \end{aligned} \tag{79}$$

This gives any $C(\boldsymbol{\lambda}; \boldsymbol{\mu})$ in terms of $C(\boldsymbol{\lambda}'; \boldsymbol{\mu}')$ with $\lambda_1' < \lambda_1$ and hence by induction reduces everything to the case $\lambda_1 = 0$. Now the recurrence given by $D_2 P_{\boldsymbol{\nu}} = 0$ gives

$$\begin{aligned} &\lambda_2(\lambda_2 + \mu_1 + \mu_3 + s + 1)\, C(0, \lambda_2, \lambda_3; \mu_1, \mu_2, \mu_3) \\ &\quad + (\mu_1 + 1)(\mu_1 + 2)\, C(0, \lambda_2 - 1, \lambda_3 - 1; \mu_1 + 2, \mu_2, \mu_3) \\ &\quad + (\mu_1 + 1)(\mu_3 + 1)\, C(0, \lambda_2 - 1, \lambda_3; \mu_1 + 1, \mu_2 - 1, \mu_3 + 1) \;=\; 0\,, \end{aligned} \tag{80}$$

which lets us get down to $\lambda_2 = 0$, and similarly the recurrence from D_3 gives

$$\begin{aligned} &\lambda_3(\lambda_3 + \mu_1 + \mu_2 + s + 1)\, C(0, 0, \lambda_3; \mu_1, \mu_2, \mu_3) \\ &\quad + (\mu_1 + 1)(\mu_2 + 1)\, C(0, 0, \lambda_3 - 1; \mu_1 + 1, \mu_2 + 1, \mu_3 - 1) \;=\; 0\,, \end{aligned} \tag{81}$$

which gets us down to $\boldsymbol{\lambda} = (0, 0, 0)$. But for fixed $\boldsymbol{\nu}$ there is only one coefficient $C(\boldsymbol{\lambda}; \boldsymbol{\mu})$ with $\boldsymbol{\lambda} = (0, 0, 0)$, namely $C(\mathbf{0}; \boldsymbol{\nu})$, which is given by Proposition 2 above. (This is the only point where our specific normalization of the basis element of $\mathcal{P}_{\mathbf{a}}(d)$ for $n = 3$ is used.) Summarizing, we have:

Proposition 4. *The coefficients of* $C(\boldsymbol{\lambda}; \boldsymbol{\mu})$ *are determined by the recursive formulas* (79)–(81) *together with the initial conditions* (75).

We note, however, that determining the coefficients explicitly from the recursions (79)–(81) is quite difficult. We will discuss this problem, and solve it in some cases, in Subsection **D**.

2. Recursion relation coming from the scalar product. In the method just described, we fixed the index $\boldsymbol{\nu}$ and gave a recursive relation for the coefficients $C(\boldsymbol{\lambda}; \boldsymbol{\mu}) = C_{\boldsymbol{\nu}}(\boldsymbol{\lambda}; \boldsymbol{\mu})$ of $P_{\boldsymbol{\nu}}(T)$ by using the differential equation which this polynomial satisfies. A different approach is to give a recursive formula for the polynomials $P_{\boldsymbol{\nu}}$ themselves (and hence implicitly also for their coefficients), as follows:

Proposition 5. *For all* $\nu_1, \nu_2, \nu_3 \geq 0$ *we have*

$$\begin{aligned} \nu_1(\nu_1+s)\,P_{\nu_1,\nu_2,\nu_3} &= (\nu_1+\nu_2+s)(\nu_1+\nu_3+s)\,r_1\,P_{\nu_1-1,\nu_2,\nu_3} \\ &\quad -(\nu_2+1)(\nu_2+s+1)\,m_2\,P_{\nu_1-1,\nu_2+1,\nu_3-1} \\ &\quad -(\nu_3+1)(\nu_3+s+1)\,m_3\,P_{\nu_1-1,\nu_2-1,\nu_3+1} \\ &\quad -(\nu_1+\nu_2+\nu_3+s)(\nu_1+\nu_2+\nu_3+2s)\,m_2m_3\,P_{\nu_1-2,\nu_2,\nu_3} \end{aligned} \tag{82}$$

with the convention that any term having a negative index is to be interpreted as zero.

Proof. The idea is similar to the classical proof of the 3-term recursion for orthogonal polynomials $\{P_\nu(x)\}$ in one variable (corresponding to $n = 2$ in our setup). There one observes that, since the polynomial $P_\nu(x)$ has degree exactly ν, one can write $xP_{\nu-1}(x)$ as a linear combination of $P_{\nu-i}(x)$ $(i = 0, 1, \dots)$, and that the coefficient of $P_{\nu-i}$ in this combination vanishes for $i \geq 3$ because then $(xP_{\nu-1},\, P_{\nu-i}) = (P_{\nu-1},\, xP_{\nu-i}) = 0$ by orthogonality. This gives $xP_{\nu-1} = a_\nu P_\nu + b_\nu P_{\nu-1} + c_\nu P_{\nu-2}$. The coefficient a_ν can be computed by comparing the coefficients of x^ν on both sides. The coefficient b_ν can be computed in a similar way, but by a longer calculation, if one knows the subleading coefficients of the polynomials, and the coefficient c_ν by a similar but even longer calculation if one knows the third coefficients from the top. Alternatively, the coefficients c_ν can be obtained from the calculation

$$c_{\nu+1}\,(P_{\nu-1},\, P_{\nu-1}) \;=\; (xP_\nu,\, P_{\nu-1}) \;=\; (P_\nu,\, xP_{\nu-1}) \;=\; a_\nu(P_\nu,\, P_\nu)$$

if one knows the norms of the polynomials P_ν, or alternatively one can turn this calculation around to obtain a recursive formula for these norms if one knows the c_ν. Here the same arguments apply. Write the product $r_1 P_{\nu-1,\nu_2,\nu_3}(T)$ as a linear combination of other higher spherical polynomials $P_{\boldsymbol{\nu}'}(T)$, where we can use the homogeneity of the polynomials to simplify the formulas by setting $m_1 = m_2 = m_3 = 1$. Then the same scalar product argument as before gives that the coefficients of $P_{\boldsymbol{\nu}'}$ in this linear combination vanish for all multi-indices $\boldsymbol{\nu}'$ except those occurring in (82). The coefficient of P_{ν_1,ν_2,ν_3} in (82) is found easily by comparing the coefficients of $r_1^{\nu_1} r_2^{\nu_2} r_3^{\nu_3}$ on both sides of the equation, using formula (75). The coefficient of $P_{\nu_1-1,\nu_2+1,\nu_3-1}$ is only slightly harder, because its top monomial $r_1^{\nu_1-1} r_2^{\nu_2+1} r_3^{\nu_3-1}$ does not occur in any of the other terms on the right-hand side of (82) and the coefficient of this monomial in P_{ν_1,ν_2,ν_3} (where it corresponds to $\boldsymbol{\lambda} = (0, 1, 0)$) can be obtained easily from (75) and the recursion (81) or as a special case of Proposition 8 below. The calculation for $P_{\nu_1-1,\nu_2-1,\nu_3+1}$ is of course exactly similar. Finally, the coefficient of P_{ν_1-2,ν_2,ν_3} can be obtained in a similar way, though with a little more computation, using the recursion (80) above or Proposition 9 in Subsection **D** below. (Alternatively, they can be obtained with less computation by computing the scalar products of both sides of (82) with P_{ν_1-2,ν_2,ν_3} and using the orthogonality of the higher spherical polynomials together with the explicit formula (78) for their norms. Conversely, as already mentioned at the end of Subsection **B**, we can use the direct computation to obtain alternative proofs of (78) and then of (77) that do not require appealing to the more general results in §9.) The calculations are straightforward, though lengthy, and will be omitted. □

As well as (82) one of course also has the corresponding formulas for any cyclic permutation of the indices i of ν_i, m_i, and r_i, since $P_{\boldsymbol{\nu}}(T)$ is symmetric under these permutations. This result lets one compute all of these polynomials recursively starting with the initial value $P_{0,0,0} = 1$: equation (81) lets one successively reduce the value of ν_1 by (at least) one and hence expresses any P_{ν_1,ν_2,ν_3} in terms of polynomials $P_{0,\nu_2',\nu_3'}$, and then applying equation (81) with the indices cyclically permuted one reduces in turn to polynomials of the form $P_{0,0,\nu_3''}$ and finally to $P_{0,0,0}$.

3. Recursion relation coming from the raising operators. The recursion just given expresses the polynomial $P_{\boldsymbol{\nu}}$ as a linear combination of four polynomials $P_{\boldsymbol{\nu}'}$ with "smaller" values of $\boldsymbol{\nu}'$. The raising operators introduced in §5 permit us to do even better in the sense that we can express $P_{\boldsymbol{\nu}}$ in terms of a single predecessor (which can be chosen at will to be P_{ν_1-1,ν_2,ν_3}, P_{ν_1,ν_2-1,ν_3} or P_{ν_1,ν_2,ν_3-1}), but at the expense of using second order differential operators. More concretely, equation (51) gives us the relation

$$4(\nu_1+\nu_2+s)(\nu_1+\nu_3+s)\,P^M_{\nu_1,\nu_2,\nu_3} \;=\; R_{23}(P^M_{\nu_1-1,\nu_2,\nu_3})\,,$$

where R_{23} is the operator defined in (47) (and of course similar formulas with R_{13} and R_{12}), and then we can use (74) to reexpress this identity in terms of our preferred basis $\{P_{\boldsymbol{\nu}}\}$ as

$$2\nu_1(\nu_1+s)\,P_{\nu_1,\nu_2,\nu_3} \;=\; R_{23}(P_{\nu_1-1,\nu_2,\nu_3})\,. \tag{83}$$

Notice that this recursion, unlike the others discussed here, works equally well for all values of n, not just for $n=3$.

4. Recursion relation coming from the mixed Laplacians. A fourth approach is to use the defining property (55) of the descending basis elements $P^D_{\boldsymbol{\nu}}$, together with the relation (77) between these functions and our preferred basis elements $P_{\boldsymbol{\nu}}$. We do not write this out in detail since it is not particularly illuminating.

5. Recursion relation coming from the generating function. Finally, we can try to compute *all* of the coefficients of the $P_{\boldsymbol{\nu}}(T)$ from the generating function (69), rather than only the constant terms as was done in Proposition 2. A closed formula seems very hard to obtain in this way, but we can obtain yet one more recursive formula for these coefficients, of a somewhat peculiar type, as follows. The quantity $R=R(T,\mathbf{X})$ defined in (68) satisfies the quadratic equation $R^2 - \Delta_0(T,\mathbf{X})R + d(T)X_1X_2X_3 = 0$, where $\Delta_0(T,\mathbf{X})$ and $d(T)$ are given in Proposition 1, so from (69) we deduce the generating function identity

$$\sum_{\boldsymbol{\nu}} P_{\boldsymbol{\nu},d}(T)\,\mathbf{X}^{\boldsymbol{\nu}} \;-\; \Delta_0(T,\mathbf{X})\sum_{\boldsymbol{\nu}} P_{\boldsymbol{\nu},d+2}(T)\,\mathbf{X}^{\boldsymbol{\nu}} \;+\; d(T)X_1X_2X_3\sum_{\boldsymbol{\nu}} P_{\boldsymbol{\nu},d+4}(T)\,\mathbf{X}^{\boldsymbol{\nu}} \;=\; 0\,,$$

and by comparing coefficients we obtain from this a formula for $P_{\boldsymbol{\nu},d}$ as a linear combination of 11 values of $P_{\boldsymbol{\nu}',d+2}$ and $P_{\boldsymbol{\nu}',d+4}$.

D. Explicit formulas for the coefficients of $P_{\boldsymbol{\nu},d}$. In this subsection we compute as far as we are able the coefficients of the polynomials $P_{\boldsymbol{\nu},d}$. The calculations require, and reveal, some quite surprising combinatorial identities. However, the formulas are rather complicated and are not used again, so that the reader who does not like this sort of thing can skip to Susbsection **E** without any loss of continuity. We use the recursions given in the first paragraph of Subsection **B**, i.e., we use (81), (80) and (79) in succession. The first step is easy, since from formulas (81) and (75) we immediately find by induction the formula for $C(\boldsymbol{\lambda};\boldsymbol{\mu})$ when two of the λ_i vanish:

Proposition 6. *Let* $\boldsymbol{\lambda}=(0,0,\lambda_3)$ *and* $\boldsymbol{\mu}=(\mu_1,\mu_2,\mu_3)\ge\boldsymbol{\lambda}$. *Then*

$$C(\boldsymbol{\lambda};\boldsymbol{\mu}) = \frac{(-1)^{\lambda_3}(\mu_1+\mu_2+\lambda_3+s)!(\mu_1+\mu_3+s)!(\mu_2+\mu_3+s)!}{\lambda_3!\mu_1!\mu_2!(\mu_3-\lambda_3)!(\mu_1+\lambda_3+s)!(\mu_2+\lambda_3+s)!(\mu_3-\lambda_3+s)!}\,.$$

Note that this is again a simple closed multiplicative formula, whereas using the generating function directly would have given a formula for $C(0,0,\lambda_3;\boldsymbol{\mu})$ as a double sum.

We next consider the $C(\boldsymbol{\lambda};\boldsymbol{\mu})$ where only one of the λ_i vanishes. Here the formula which we will obtain is no longer multiplicative, but it is a simple sum of products of binomial coefficients rather than the four-fold sum that we would get if we simply expanded the generating function (69).

Equation (80) is equivalent by induction over λ_2 to the formula

$$C(0,\lambda_2,\lambda_3;\boldsymbol{\mu}) = \frac{(-1)^{\lambda_2}\,(\mu_1+\mu_3+\lambda_2+s)!}{\mu_1!\,\mu_3!\,(\mu_1+\mu_3+2\lambda_2+s)!}\sum_{n=\max(0,\lambda_2-\lambda_3)}^{\min(\lambda_2,\mu_2)}\frac{(\mu_3+n)!(\mu_1+2\lambda_2-n)!}{n!\,(\lambda_2-n)!}$$
$$\times\, C(0,0,\lambda_3-\lambda_2+n;\mu_1+2\lambda_2-n,\mu_2-n,\mu_3+n)\,.$$

Substituting into this the formula from Proposition 6, we find the expression

$$C(0,\lambda_2,\lambda_3;\boldsymbol{\mu}) \;=\; = (-1)^{\lambda_3}\frac{(\mu_2+\mu_3+s)!\,(\mu_1+\mu_2+\lambda_3+s)!(\mu_1+\mu_3+\lambda_2+s)!}{\mu_1!\,(\nu_2+s)!\,(\nu_3+s)!}$$
$$\times\, S(-\mu_3;0,\lambda_2-\lambda_3;\lambda_2,\mu_2;\mu_1+\mu_2+\lambda_2+\lambda_3+s), \tag{84}$$

where $S(a;b,c;d,e;f)$ for integers $a\le b,c\le d,e\le f$ is defined by

$$S(a;b,c;d,e;f) = \frac{1}{(b-a)!(c-a)!(f-d)!(f-e)!}\sum_{n=\max(b,c)}^{\min(d,e)}\frac{(-1)^n\,(n-a)!\,(f-n)!}{(n-b)!\,(n-c)!\,(d-n)!\,(e-n)!}.$$

Formula (84) expresses $C(\boldsymbol{\lambda};\boldsymbol{\mu})$ when $\lambda_1=0$ as a simple sum of multinomial coefficients as opposed to the quadruple sum expression which would have been obtained by using the generating function (69). On the other hand, it is not symmetric in the indices "2" and "3". In looking for a symmetric expression, we discover the remarkable **symmetry property**

$$(-1)^{d+e}S(a;b,c;d,e;f) = S(c;d,e;f,d+e-b;f-a+c) \tag{85}$$

(where the right-hand side is zero if $f-a<d+e-b-c$). The proof is amusing and the reader may enjoy looking for it. Applying (85) to (84) gives the new expression

$$C(0,\lambda_2,\lambda_3;\boldsymbol{\mu}) \;=\; \frac{(\mu_2+\mu_3+s)!}{\mu_1!\,\lambda_2!\,\lambda_3!\,(\nu_2+s)!\,(\nu_3+s)!}$$
$$\times\sum_{n=\max(\lambda_2,\lambda_3)}^{\min(\mu_3+\lambda_2,\mu_2+\lambda_3)}\frac{(-1)^n\,n!\,(\nu_1+\nu_2+\nu_3+s-n)!}{(n-\lambda_2)!\,(n-\lambda_2)!\,(\mu_3+\lambda_2-n)!\,(\mu_2+\lambda_3-n)!}\,, \tag{86}$$

which is again a simple sum of multinomial coefficients but is now symmetric in indices 2 and 3.

We now look for an explanation of the symmetry property (85). Combining (85) with the trivial symmetries under $b\leftrightarrow c$ and $d\leftrightarrow e$, $(a;b,c;d,e;f)\mapsto(a+k;b+k,c+k;d+k,e+k;f+k)$ ($k\in\mathbb{Z}$, and up to sign $(-1)^k$) and $(a;b,c;d,e;f)\mapsto(k-f;k-e,k-d;k-c,k-b;k-a)$ (again up to sign), we find that there are 9 essentially different sums $\pm S$ which are equal by (85). The nine inequalities on the variables a through f can be made uniform by writing $(a;b,c;d,e;f)$ as $(a_1+a_2-b_3;a_1,a_2;b_1,b_2;b_1+b_2-a_3)$ where $b_i\ge a_j$ for all i and j. The formula which makes all symmetries evident is then

$$S(a_1+a_2-b_3;a_1,a_2;b_1,b_2;b_1+b_2-a_3)$$
$$= \sum_{n=\max(a_1,a_2,a_3)}^{\min(b_1,b_2,b_3)}\frac{(-1)^n}{(n-a_1)!(n-a_2)!(n-a_3)!(b_1-n)!(b_2-n)!(b_3-n)!}, \tag{87}$$

whose proof we omit. Inserting this into (84) gives the following formula, which is again symmetric in the indices "2" and "3" but in general has fewer terms than (84) or (86) (for instance, it reduces to a single term if any of the seven integers λ_2, λ_3, μ_2, μ_3, ν_1, ν_2 or ν_3 is 0):

Proposition 7. *Let* $\boldsymbol{\lambda} = (0, \lambda_2, \lambda_3)$ *and* $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ *subject to the three inequalities* $\nu_i \geq 0$, *where* $\nu_1 = \mu_1 + \lambda_2 + \lambda_3$, $\nu_2 = \mu_2 - \lambda_2 + \lambda_3$, $\nu_3 = \mu_3 + \lambda_2 - \lambda_3$. *Then*

$$\begin{aligned} C(0, \lambda_2, \lambda_3; \boldsymbol{\mu}) \;&=\; (-1)^{\lambda_3} \, \frac{(\mu_2 + \mu_3 + s)!\,(\mu_1 + \mu_2 + \lambda_3 + s)!\,(\mu_1 + \mu_3 + \lambda_2 + s)!}{\mu_1!\,(\nu_2 + s)!\,(\nu_3 + s)!} \times \\ &\sum_{n=\max(0,\lambda_2-\mu_2,\lambda_3-\mu_3)}^{\min(\lambda_2,\lambda_3)} \frac{(-1)^n}{(\lambda_2 - n)!(\lambda_3 - n)!(\nu_1 - n + s)!n!(\mu_2 - \lambda_2 + n)!(\mu_3 - \lambda_3 + n)!}\,. \end{aligned}$$

Finally, we come to the general coefficient $C(\boldsymbol{\lambda}; \boldsymbol{\mu})$. Equation (79) and induction on λ_1 give

$$\begin{aligned} C(\boldsymbol{\lambda}; \boldsymbol{\mu}) = \frac{(-1)^{\lambda_1}\,(\mu_2 + \mu_3 + \lambda_1 + s)!}{\mu_2!\,\mu_3!\,(\mu_2 + \mu_3 + 2\lambda_1 + s)!} &\sum_{\substack{a,b,c\geq 0 \\ a+b+c=\lambda_1}} \frac{(\mu_2 + a + 2b)!\,(\mu_3 + a + 2c)!}{a!\,b!\,c!} \\ &\times\, C(0, \lambda_2 - c, \lambda_3 - b; \mu_1 - a, \mu_2 + a + 2b, \mu_3 + a + 2c)\,. \end{aligned}$$

Substituting into this any of the three formulas obtained above for $C(0, \lambda_2, \lambda_3; \boldsymbol{\mu})$ gives a formula for the general coefficient $C(\boldsymbol{\lambda}; \boldsymbol{\mu})$ as a triple sum rather than an octuple one, which is what we would get from the generating function (69). For instance, the expression obtained using the formula in Proposition 7 is

$$\begin{aligned} C(\boldsymbol{\lambda}; \boldsymbol{\mu}) = \frac{(\mu_2 + \mu_3 + \lambda_1 + s)!}{\mu_2!\mu_3!(\nu_2 + s)!(\nu_3 + s)!} &\sum_{\substack{a,b,c\geq 0 \\ \lambda_1-\mu_1\leq b+c\leq\lambda_1 \\ \lambda_2-\nu_3\leq a+c\leq\lambda_2 \\ \lambda_3-\nu_2\leq a+b\leq\lambda_3}} \frac{(-1)^{\lambda_1+a}}{a!\,b!\,c!\,(\lambda_1 - b - c)!\,(\lambda_2 - a - c)!\,(\lambda_3 - a - b)!} \\ \times\, \frac{(\mu_1 + \mu_2 + \lambda_3 + b + s)!(\mu_1 + \mu_3 + \lambda_2 + c + s)!(\mu_2 + \lambda_1 + b - c)!(\mu_3 + \lambda_1 - b + c)!}{(\nu_1 - a + s)!\,(\mu_1 - \lambda_1 + b + c)!\,(\nu_2 - \lambda_3 + a + b)!\,(\nu_3 - \lambda_2 + a + c)!}&\,. \end{aligned}$$

This formula is symmetric in the indices "2" and "3", but not in all three indices. Despite a fair amount of effort we were not able to simplify it or to find an expression for $C(\boldsymbol{\lambda}; \boldsymbol{\mu})$ as a triple sum which is symmetric in all three indices. The formula for $C(0, \lambda_2, \lambda_3; \boldsymbol{\mu})$ in Proposition 7 can be written as

$$\begin{aligned} C(0, \lambda_2, \lambda_3; \boldsymbol{\mu}) = &\frac{(\mu_2 + \mu_3 + s)!\,(\mu_1 + \mu_2 + \lambda_3 + s)!\,(\mu_1 + \mu_3 + \lambda_2 + s)!}{\mu_1!\,\mu_2!\,\mu_3!\,(\nu_1 + s)!\,(\nu_2 + s)!\,(\nu_3 + s)!} \\ &\times\, \mathrm{Coeff}_{Y^{\lambda_2}Z^{\lambda_3}}\big((1 - Y)^{\mu_2}(1 - Z)^{\mu_3}(1 - YZ)^{\nu_1+s}\big)\,, \end{aligned}$$

which has a obvious generalization to

$$C(\boldsymbol{\lambda}; \boldsymbol{\mu}) \overset{?}{=} \prod_{i=1}^{3} \frac{(\lambda_i + \mu_{i+1} + \mu_{i+1} + s)!}{\mu_i!\,(\nu_i + s)!} \times \mathrm{Coeff}_{X_1^{\lambda_1}X_2^{\lambda_2}X_3^{\lambda_3}}\left(\prod_{i=1}^{3}(1 - X_i)^{\mu_i}(1 - X_{i+1}X_{i+2})^{\nu_i+s}\right).$$

but unfortunately this formula, which is symmetric and relatively simple, is not correct when all three λ_i are strictly positive; for instance, for $C(1, 1, 1; 0, 0, 0)$ it gives 0 rather than the correct value of $4(s + 2)$.

We remark that the numbers $S(a; b, c; d, e; f)$ are, up to simple normalizing factors, equal to the well-known Wigner $3j$-symbols or Clebsch-Gordan coefficients occurring in the theory of angular momentum in quantum mechanics, and that an equality equivalent to (87) can be found in the physics literature. It is quite likely that the general coefficients $C(\boldsymbol{\lambda}; \boldsymbol{\mu})$ can also be expressed in terms of the more complicated $6j$-symbols (or Racah coefficients) and $9j$-symbols of quantum mechanics, but we were not able to check whether this is the case.

E. Inhomogeneous coordinates and angular coordinates. As we already discussed at the end of §1, the homogeneity of the higher spherical polynomials makes it natural to consider them also as polynomials on the space $\mathcal{S}_n^1$ of $n \times n$ symmetric matrices with 1's on the diagonal, or on the open subset $\mathcal{S}_n^{1,+}$ of this space consisting of positive definite matrices. For $n = 3$, this latter set can be identified by mapping $\mathbf{t}$ to the matrix $T = T(\mathbf{t})$ defined in (7) with the semi-algebraic set

$$\mathcal{T} \;=\; \bigl\{\mathbf{t} = (t_1, t_2, t_3) \,\in\, \mathbb{R}^3 \;\bigm|\; |t_i| < 1, \quad \Delta(\mathbf{t}) > 0\bigr\} \tag{88}$$

shown in Figure 1 below, where $\Delta : \mathbb{R}^3 \to \mathbb{R}$ is the polynomial

$$\Delta(\mathbf{t}) \;=\; \det T(\mathbf{t}) \;=\; 1 - t_1^2 - t_2^2 - t_3^2 + 2t_1t_2t_3 \,. \tag{89}$$

(To see that $\mathbf{t} \mapsto T(\mathbf{t})$ is surjective, recall that a symmetric matrix is positive definite if and only if its principal minors are positive. In fact only the top left minors are needed, so to define $\mathcal{T}$ it suffices to require $\Delta(\mathbf{t}) > 0$ and $|t_3| < 1$.) By Remark 4 after Theorem 2 in §3, we know that the higher spherical polynomials $P_{\boldsymbol{\nu}}(T) = P_{\boldsymbol{\nu},d}(T)$ for a fixed value of $d = 2s + 4 > 2$ then form an orthogonal Hilbert space basis of $L^2(\mathcal{T},\, \Delta^s dt_1 dt_2 dt_3)$. (Compare eq. 20). In this final subsection of §8 we wish to discuss a few further properties of the set $\mathbf{T}$ (alternative coordinates, symmetry, ...). These properties will also play a role in [12], where we will study the differential equation (8) and its non-polynomial solutions.

We begin by mentioning two simple algebraic properties. The first is that the adjoint of the matrix $T = T(\mathbf{t})$ (i.e., the matrix T^* such that $TT^* = \Delta \cdot 1_3$) is given by

$$T^* \;=\; \begin{pmatrix} 1 - t_1^2 & \Delta_3 & \Delta_2 \\ \Delta_3 & 1 - t_2^2 & \Delta_1 \\ \Delta_2 & \Delta_1 & 1 - t_3^2 \end{pmatrix}, \quad \text{where} \quad \Delta_i \;:=\; \frac{1}{2}\,\frac{\partial \Delta}{\partial t_i} \;=\; t_j t_k - t_i\,. \tag{90}$$

(Here and in the rest of this subsection, whenever we write indices i, j and k in the same formula we mean that $\{i, j, k\} = \{1, 2, 3\}$.) The other is that the determinant Δ has three algebraic factorizations

$$\Delta \;=\; \Delta_1^+\,\Delta_1^- \;=\; \Delta_2^+\,\Delta_2^- \;=\; \Delta_3^+\,\Delta_3^-\,, \tag{91}$$

where the quantities $\Delta_i^\pm$ are defined, using the convention and quantities just introduced, by

$$\Delta_i^\pm \;=\; \sqrt{\bigl(1 - t_j^2\bigr)\bigl(1 - t_k^2\bigr)} \,\pm\, \Delta_i\,. \tag{92}$$

We will give an explanation and refinement of these factorizations below.

We now consider the symmetries of the set $\mathcal{T}$, which involve a small surprise for which we have no real explanation. The polynomial Δ defined in (89) and the set $\mathcal{T}$ defined by (88) have a symmetry group G of order 24, given by the six permutations of the t_i together with the four changes of sign of the t_i preserving their product, i.e. $G = \mathfrak{S}_3 \ltimes E$ where

$$E \;=\; \bigl\{\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3) \,\in\, \{\pm 1\}^3 \bigm| \varepsilon_1\varepsilon_2\varepsilon_3 = 1\bigr\}$$

(Klein group). This group can be identified with $\mathfrak{S}_4$ by identifying the set of indices $\{1, 2, 3\}$ with the three possibile partitions of $\{1, 2, 3, 4\}$ into two disjoint 2-tuples via $i \leftrightarrow \{\{j, k\},\, \{i, 4\}\}$ for $\{i, j, k\} = \{1, 2, 3\}$. The surprise is that the set $\mathcal{T}$ itself can be rewritten in a visibly

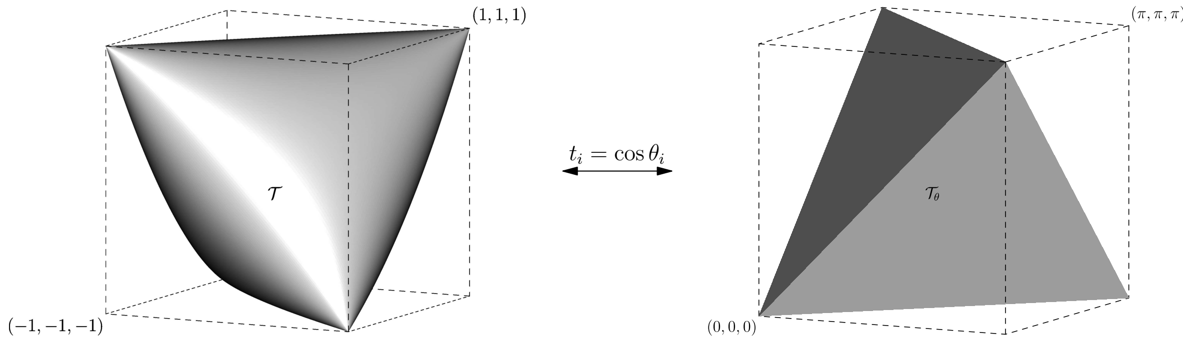

Figure 1: The semi-algebraic set $\mathcal{T}$ and the tetrahedron $\mathcal{T}_\theta$

$\mathfrak{S}_4$-symmetric manner. We describe this here algebraically, and below in terms of angular coordinates. Let $\widehat{\mathcal{S}}_4^1$ denote the set of 4×4 real symmetric matrices that have 1's on the diagonal and whose (i,j)- and (k,l)-entries sum to 0 whenever $\{i,j,k,l\} = \{1,2,3,4\}$. The group $\mathfrak{S}_4$ acts on $\widehat{\mathcal{S}}_4^1$ by simultaneous permutations of the rows and columns, and if we identify G with $\mathfrak{S}_4$ as explained above, then we have an $\mathfrak{S}_4$-equivariant isomorphism $\mathcal{S}_3^1 \cong \widehat{\mathcal{S}}_4^1$ given by

$$T = \begin{pmatrix} 1 & t_3 & t_2 \\ t_3 & 1 & t_1 \\ t_2 & t_1 & 1 \end{pmatrix} \quad \leftrightarrow \quad \widehat{T} \;=\; \begin{pmatrix} 1 & t_3 & t_2 & -t_1 \\ t_3 & 1 & t_1 & -t_2 \\ t_2 & t_1 & 1 & -t_3 \\ -t_1 & -t_2 & -t_3 & 1 \end{pmatrix} . \tag{93}$$

The principal 2×2 minors of $\widehat{T}$ are just the numbers $1 - t_i^2$ and the principal 3×3 minors of $\widehat{T}$ are all equal to Δ, so the inequalities defining the set $\mathcal{T}$ say precisely that the principal 2×2 and 3×3 minors of $\widehat{T}$ are positive. This is an $\mathfrak{S}_4$-invariant condition and hence explains the $\mathfrak{S}_4$-symmetry of $\mathcal{T}$ (except that we cannot really explain the origin of the isomorphism (93)). Note that for the positive definiteness of the matrix $\widehat{T}$ (rather than just of its submatrix T) we need the positivity of *all* its principal minors, so that $\widehat{T}$ is positive definite if and only if T is and the determinant of $\widehat{T}$ is positive. This determinant, unlike that of T, has a rational factorization, as $\prod_{\boldsymbol{\varepsilon} \in E}(1 + \boldsymbol{\varepsilon} \cdot \mathbf{t})$, the factors being the eigenvalues of the constant eigenvectors $\{(\varepsilon_1, \varepsilon_2, \varepsilon_3, -1)\}_{\boldsymbol{\varepsilon} \in E}$ of $\widehat{T}$. Finally, we mention that the adjoint of $\widehat{T}$ is a matrix of the same form as $\widehat{T}$ but with the diagonal entries 1 replaced by Δ and the off-diagonal entries t_i replaced by the expressions

$$\widehat{t_i} \;=\; t_i \left(1 + t_j^2 + t_k^2 - t_i^2\right) \;-\; 2t_j t_k \;=\; t_i\, \Delta \;-\; 2\Delta_j \Delta_k \,. \tag{94}$$

These quantities will occur again in [12] in the construction of non-polynomal solutions of (8).

Our next topic, which again involves a surprise for which we have no real explanation, concerns angular coordinates. In the case of 2×2 symmetric matrices, corresponding to classical spherical polynomials, one frequently makes the change of variables $t = \cos\theta$ to identify the space $\mathcal{S}_2^{1,+} = \left\{ \left(\begin{smallmatrix} 1 & t \\ t & 1 \end{smallmatrix}\right) \mid -1 < t < 1 \right\}$ with the interval $(0, \pi)$. Here we make the corresponding substitution $t_i = \cos\theta_i$ with $\theta_i \in (0, \pi)$. The surprise is that the semi-algebraic set $\mathcal{T}$, which is shaped like a rounded tetrahedron, is then mapped isomorphically onto the *linear* tetrahedron

$$\mathcal{T}_\theta \;=\; \left\{ (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3 \;\middle|\; \theta_i < \theta_j + \theta_k < 2\pi - \theta_i \quad \text{for} \quad \{i,j,k\} = \{1,2,3\} \right\} , \tag{95}$$

which is the interior of the convex hull of the four vertices $(0,0,0)$, $(0,\pi,\pi)$, $(\pi,0,\pi)$ and $(\pi,\pi,0)$ of the cube $[0,\pi]^3$. (See Figure 1.) This gives another explanation of the isomorphism between the symmetry group of $\mathcal{T}$ and $\mathfrak{S}_4$, identified with the group of permutations of these four vertices. But we can do even better: if we introduce new coordinates δ_ν ($\nu = 1, 2, 3, 4$) by

$$\delta_i \;=\; \frac{\theta_j + \theta_k - \theta_i}{2} \qquad (1 \le i \le 3), \qquad \delta_4 \;=\; \pi \,-\, \frac{\theta_1 + \theta_2 + \theta_3}{2}\,, \tag{96}$$

then the angles θ_i can be expressed in terms of the δ_ν by

$$\theta_i \;=\; \delta_j \,+\, \delta_k\,, \qquad \pi \,-\, \theta_i \;=\; \delta_i \,+\, \delta_4 \qquad\qquad (1 \le i \le 3), \tag{97}$$

the tetrahedron $\mathcal{T}_\theta$ is identified with the simplex $\{(\delta_1, \delta_2, \delta_3, \delta_4) \in \mathbb{R}^4_{>0} \mid \delta_1 + \delta_2 + \delta_3 + \delta_4 = \pi\}$, and G acts simply by permuting the δ's. The six numbers $\{\pm t_i\}_{1 \le i \le 3}$ are the six numbers $\{\cos(\delta_\mu + \delta_\nu)\}_{1 \le \mu < \nu \le 4}$, these being precisely the non-diagonal entries of the matrix $\widehat{T}$ defined in (93) and with the same numbering. Finally, the three factorizations of Δ given in (91) can be replaced by the single fourfold factorization

$$\Delta \;=\; 4 \prod_{\nu=1}^{4} \sin \delta_\nu\,, \tag{98}$$

which explains and refines (91) because the functions Δ_i^- and Δ_i^+ are equal to $2 \sin \delta_i \sin \delta_4$ and $2 \sin \delta_j \sin \delta_k$, respectively. This equation also makes it clear that the condition $\Delta > 0$ is equivalent to the inequalities $\delta_\nu > 0$ and hence explains why the sets $\mathcal{T}$ and $\mathcal{T}_\theta$ correspond.

§9. A universal generating function for the descending basis

In Remark 1 in §8.**A** we observed that the generating function of Theorem 9 could be expressed in terms of only three quantities, namely the coefficients of the characteristic polynomial of the product of T with a 3×3 matrix of "dummy" variables X_j. That remark will now be generalized to construct a generating function for all n whose coefficients give a basis of the space $\mathcal{P}^{(n)}(d)$ for all n and d, and in fact a basis that coincides, up to scalar factors, with the "descending" basis of $\mathcal{P}^{(n)}(d)$ defined in Chapter I.

We fix n (at least initially), but take d to be generic. Equivalently, we work over the field $\mathcal{K} = \mathbb{Q}(d)$ rather than thinking of d as a specific complex number. (Compare the discussion following Theorem 6 in §5.) We would like to construct a generating function over $\mathcal{K}[\mathcal{S}_n]$ whose coefficients $P_{\boldsymbol{\nu}}(T)$ are multiples of the descending basis, i.e., which satisfy

$$D_{ij}\, P_{\boldsymbol{\nu}}(T) \doteq \mathcal{P}_{\boldsymbol{\nu} - \mathbf{e}_{ij}}(T) \tag{99}$$

for all $\boldsymbol{\nu} \in \mathcal{N}_0$ and all $i,\, j = 1, \ldots, n$. Here $\doteq$ denotes equality up to a non-zero constant and we are also using the convention that $P_{\boldsymbol{\nu}}{=}0$ if some entry of $\boldsymbol{\nu}$ is negative, so (99) includes the fact that each $P_{\boldsymbol{\nu}}$ belongs to $\mathcal{P}^{(n)}(d)$. Generalizing the observation about Theorem 9 quoted above, we make the following Ansatz for the form of this generating function:

$$\sum_{\boldsymbol{\nu} \in \mathcal{N}_0} P_{\boldsymbol{\nu}}(T)\, X^{\boldsymbol{\nu}} \;=\; G^{(n)}(\sigma_1(XT),\, \sigma_2(XT), \ldots, \sigma_n(XT))\,, \tag{100}$$

where $G^{(n)}$ is some power series in n variables, σ_a of a square matrix denotes the ath elementary symmetric function of its eigenvalues, X is a symmetric $n \times n$ matrix of "dummy" variables $x_{ij} = x_{ji}$ with $x_{ii} = 0$ for all i, and $X^{\boldsymbol{\nu}} = \prod_{i \ne j} x_{ij}^{\nu_{ij}/2} = \prod_{i<j} x_{ij}^{\nu_{ij}}$. Our object is to show that

a function $G^{(n)}(\sigma_1,\dots,\sigma_n)$ can be found such that the coefficients $P_{\boldsymbol\nu}$ defined by (100) satisfy the property (99). By the Remark cited above, we already know the answer for $n=3$, namely

$$G^{(3)}(\sigma_1,\sigma_2,\sigma_3) \;=\; \frac{1}{\sqrt{\Delta_0^2-8\sigma_3}}\left(\frac{\Delta_0\,+\,\sqrt{\Delta_0^2-8\sigma_3}}{2}\right)^{-\frac{d-4}{2}}, \qquad \Delta_0 \;:=\; \bigl(1-\frac{\sigma_1}{2}\bigr)^2-\sigma_2\,. \quad (101)$$

It will turn out that the functions $G^{(n)}$ are actually independent of n in the sense that there is a single power series $G(\sigma_1,\sigma_2,\dots)$ in infinitely many variables such that each $G^{(n)}(\sigma_1,\dots,\sigma_n)$ is simply $G(\sigma_1,\dots,\sigma_n,0,0,\dots)$. (Notice that such a power series makes sense because we can introduce a grading by $|\sigma_a|=a$ and then the space of monomials of any fixed degree is finite-dimensional.) The power series G turns out not to be quite unique, but to be uniquely determined by $G^{(1)}(\sigma_1)=G(\sigma_1,0,0,\dots)$, which is arbitrary. The choice corresponding to (101) in the case $n=3$ and to the classical generating function of Gegenbauer polynomials in the case $n=2$ is $G^{(1)}(\sigma_1)=(1-\sigma_1/2)^{2-d}$, and this will be our standard choice, but there exist at least two other special choices which have some nice properties, as will be discussed briefly in Subsection **C**.

Since this section is again quite long, we have divided it into subsections. In Subsection **A** we show how to characterize the power series $G^{(n)}$ such that the Taylor coefficients $P_{\boldsymbol\nu}$ in (100) satisfy (99), and prove that for each k the space of the degree k parts of such power series is one-dimensional, which implies the desired existence and uniqueness results. The proofs of two of the propositions needed for this are quite long and are given separately in Subsection **B**, which can be omitted without loss of continuity. One consequence of the proof is that if $G^{(1)}$ (which is arbitrary) is chosen to be holomorphic in d, then the coefficients of the power series $G^{(n)}$ have poles only at integral values of d. This implies the result, left open in §5, that the scalar product on $\mathcal{P}^{(n)}(d)$ is non-degenerate for all $d\in\mathbb{C}\smallsetminus\mathbb{Z}_{<n}$. Subsection **A** also contains an explicit inductive construction of the power series $G^{(n)}$ from an arbitrary initial value of $G^{(1)}$ by applying suitable differential operators. Finally, in Subsection **C** we give a discussion of the various good choices of $G^{(1)}$ and a number of examples and partial results about the coefficients of the power series $G^{(n)}$ for the standard choice. We have not been able to find a complete formula for the coefficients in general.

A. Existence of the power series $G^{(n)}$. We fix the notation

$$V \;=\; \mathcal{K}[[\sigma_1,\sigma_2,\sigma_3,\,\dots]], \qquad V_k \;=\; V_{\deg=k}\,,$$

where $\mathcal{K}=\mathbb{Q}(d)$ as before and the degrees are determined by $|\sigma_a|=a$. Thus V_k is finite-dimensional, of dimension $p(k)$ (number of partitions of k), and $V\cong\prod_{k\ge0}V_k$. We have a homomorphism $V\to\mathcal{K}[\mathcal{S}_n][[X]]$ defined by $F\mapsto\widetilde F$, where

$$\widetilde F(T,X) \;:=\; F(\sigma_1(XT),\,\sigma_2(XT),\dots,\sigma_n(XT),0,0,\dots)\,.$$

Our goal is to find a $G^{(n)}$ such that the coefficients $P_{\boldsymbol\nu}$ in (100) satisfy (99), so the first thing we need is to compute the action of all D_{ij} on $\widetilde F$ for arbitrary F. (Note that the operators D_{ij} do not involve the X-variables, so $D_{ij}(G^{(n)})=\sum(D_{ij}P_{\boldsymbol\nu})X^{\boldsymbol\nu}$.) The image of the map $F\mapsto\widetilde F$ is not stable under D_{ij}, but the following proposition, whose proof will be given in Subsection **B**, calculates $D_{ij}(\widetilde F)$ for all $F\in V$ in terms of only the quantities $\partial_{ij}(\sigma_a)$ and certain differential operators $\mathcal{L}_p$ that do preserve V. For convenience we set $\partial_a=\partial/\partial\sigma_a$, while ∂_{ij} retains its usual meaning $(1+\delta_{ij})\,\partial/\partial t_{ij}$.

Proposition 1. *For $p \geq 1$, define a second order differential operator $\mathcal{L}_p : V \to V$ of degree $-p$ (i.e., $\mathcal{L}_p(V_k) \subseteq V_{k-p}$) by*

$$\mathcal{L}_p \;=\; \bigl(d+1-p\bigr)\,\partial_p \,+\, 2\Biggl(\sum_{a,\,b\geq p} \,-\, \sum_{0<a,\,b<p}\Biggr)\sigma_{a+b-p}\,\partial_a\,\partial_b\,, \tag{102}$$

where σ_q is taken to be 1 if $q=0$ and 0 if $q<0$. Then for all i and j we have

$$D_{ij} \;=\; \sum_{p=1}^{\infty}\partial_{ij}(\sigma_p)\,\mathcal{L}_p\,. \tag{103}$$

The more precise meaning of (103) for a given value of n is that

$$D_{ij}\bigl(\widetilde{F}\bigr) \;=\; \sum_{p=1}^{n}\partial_{ij}(\sigma_p(XT))\,\widetilde{\mathcal{L}_p(F)} \qquad (1\leq i,j\leq n)\,,$$

where the sum over p can be extended to infinity if one likes since $\sigma_p(XT)=0$ for $p>n$. Note that here one could replace the sums in (102) by the subsums with $1\leq a,b\leq n$ and $a+b-p\leq n$, since if $a+b-p>n$ then σ_{a+b-p} vanishes, and if $a+b-p\leq n$ then a and b are $\leq n$ automatically in the first sum and also in the second sum whenever $p\leq n$. Similar remarks apply to all later formulas, which we will write in a way not explicitly mentioning n, but remembering that, when we apply them to $\sigma_i=\sigma_i(XT)$ for matrices X and T of a fixed size n, then $\sigma_a=0$ for all $a>n$.

Corollary. *Let $W = \bigcap_{p\geq 2}\mathrm{Ker}(\mathcal{L}_p) \subset V$. Then for any $F\in W$ we have $D_{ii}(\widetilde{F})=0$ and $D_{ij}(\widetilde{F}) = 2x_{ij}\,\widetilde{\mathcal{L}_1(F)}$ for $i\neq j$.*

Proof. This follows immediately from (103), since $\partial_{ii}(\sigma_1(XT))=0$ and $\partial_{ij}(\sigma_1(XT))=2\,x_{ij}$. $\square$

The following proposition, whose proof will also be postponed to Subsection **B**, is crucial.

Proposition 2. *The operators $\mathcal{L}_p$ satisfy the commutation formula*

$$\bigl[\mathcal{L}_{p_1},\,\mathcal{L}_{p_2}\bigr] \;=\; -2\sum_{p_1<p\leq p_2}\partial_{p_1+p_2-p}\,\mathcal{L}_p \qquad (0<p_1<p_2)\,. \tag{104}$$

Corollary. *The subspace W of V is mapped into itself by $\mathcal{L}_1$. More generally, the space $W\langle m\rangle := \bigcap_{p>m}\mathrm{Ker}(\mathcal{L}_p)\subset V$ is mapped into itself by $\mathcal{L}_m$ for all $m\geq 1$.*

Proof. This follows immediately from (104). In fact, we need only that the product $\mathcal{L}_{p_2}\mathcal{L}_{p_1}$ is in the left ideal generated by $\mathcal{L}_p$'s with $p>p_1$. $\square$

We now use this to deduce the main result of this subsection:

Theorem 10. *The space $W_k = W\cap V_k$ is one-dimensional for each $k\geq 0$. If G_k is a non-zero element of W_k for every $k\geq 0$, then $\mathcal{L}_1(G_k)$ is a non-zero multiple of G_{k-1} for all $k\geq 1$ and if we set $G=\sum_k G_k\in W$ then the polynomials $P_{\boldsymbol{\nu}}\in\mathcal{K}[\mathcal{S}_n]$ defined by $G=\sum P_{\boldsymbol{\nu}}(T)X^{\boldsymbol{\nu}}$ satisfy* (99).

Proof. For each $m\geq 1$ we denote by $V^{(m)}$ the ring $\mathcal{K}[\sigma_1,\dots,\sigma_m]$ and by $\rho_m : V\to V^{(m)}$ the restriction map $G\mapsto G(\sigma_1,\dots,\sigma_m,0,0,\dots)$. We will show by downwards induction on m that for each $m\geq 0$ the composite map

$$\rho_m\big|_{W\langle m\rangle}:\qquad W\langle m\rangle \longrightarrow V \xrightarrow{\;\rho_m\;} V^{(m)} \tag{105}$$

is an isomorphism. The case $m = 1$ gives the first assertion of the theorem, since $V^{(1)} = \mathcal{K}[\sigma_1]$ is 1-dimensional over $\mathcal{K}$ in each degree. Here "downward induction" makes sense because both maps (105) preserves degree and are obviously isomorphisms in degree k when $m > k$, so that the "initial step" of the induction is automatically satisfied.

We will show below that the map (105) is injective. For the surjectivity we use a dimension argument. Denote by $W_k\langle m\rangle$ and $V_k^{(m)}$ the degree k parts of $W\langle m\rangle$ and $V^{(m)}$, respectively. The dimension of $V_k^{(m)}$ is the number $p_m(k)$ of partitions of k into parts $\leq m$. These numbers are given by the generating function

$$\sum_{k=0}^{\infty} p_m(k)\, t^k \;=\; \frac{1}{(1-t)(1-t^2)\cdots(1-t^m)}\,,$$

and from this or directly from the definition one sees that they satisfy the recursion

$$p_m(k) \;=\; p_m(k-m) \,+\, p_{m-1}(k) \tag{106}$$

(with the initial condition $p_0(k) = \delta_{k0}$ and the convention $p_m(k) = 0$ for $k < 0$). On the other hand, $\mathcal{L}_m$ acts on $W\langle m\rangle$ by the corollary to Proposition 2, clearly with kernel $W\langle m-1\rangle$, so since $\mathcal{L}_m$ has degree $-m$ we have an exact sequence

$$0 \longrightarrow W_k\langle m-1\rangle \longrightarrow W_k\langle m\rangle \xrightarrow{\mathcal{L}_m} W_{k-m}\langle m\rangle \tag{107}$$

for every m and k (with the convention that the last space is $\{0\}$ if $m > k$). If we assume that the map (105) is an isomorphism for m, then it follows from (107) and (106) that $\dim W_k\langle m-1\rangle \geq p_{m-1}(k) = \dim V_k^{(m-1)}$, and hence, assuming that the injectivity is known in general, that the map (105) with m replaced by $m-1$ is an isomorphism, completing the induction.

It remains to prove the injectivity in (105). To do this we will show that

$$\mathrm{Ker}(\mathcal{L}_p) \,\cap\, \mathrm{Ker}(\rho_{p-1}) \;\subseteq\; \mathrm{Ker}(\rho_p) \tag{108}$$

for all p. Then for $F \in W\langle m\rangle = \mathrm{Ker}(\mathcal{L}_{m+1}) \cap \mathrm{Ker}(\mathcal{L}_{m+2}) \cap \cdots$ we can apply (108) successively with $p = m+1,\, m+2,\, \ldots$ to get $\rho_m(F) = 0 \Rightarrow \rho_{m+1}(F) = 0 \Rightarrow \rho_{m+2}(F) = 0 \Rightarrow \cdots$ and hence $F{=}0$, proving the injectivity of ρ_m.

To prove (108), write the expansion of an arbitrary element $F \in V$ as

$$F(\sigma_1, \sigma_2, \ldots) \;=\; \sum_{\mathbf{r}\geq\mathbf{0}} A(\mathbf{r})\, \frac{\sigma^{\mathbf{r}}}{\mathbf{r!}} \;=\; \sum_{r_1, r_2, \cdots \geq 0} A(r_1, r_2, \ldots)\, \frac{\sigma_1^{r_1}}{r_1!}\, \frac{\sigma_2^{r_2}}{r_2!}\; \cdots\,, \tag{109}$$

where $\mathbf{r}$ runs over ∞-tuples $(r_1, r_2, \ldots)$ with $r_i \geq 0$ for all i and $r_i = 0$ for all but finitely many i. (The coefficient $A(\mathbf{r})$ is an element of $\mathcal{K} \subset \mathbb{Q}(d)$ and will be denoted $A(\mathbf{r}; d)$ when we want to make the dependence on d explicit.) We also make the convention

$$r_0 \,=\, 1\,, \qquad r_{-1} \,=\, r_{-2} \,=\, \cdots \,=\, 0\,, \tag{110}$$

and will also write simply $A(r_1, \ldots, r_p)$ for $A(r_1, \ldots, r_p, 0, 0, \ldots)$ if $r_i = 0$ for all $i > p$. Studying $\rho_p(F)$ means looking only at the coeficients $A(r_1, \ldots, r_p)$ in the expansion (109). If $F \in \mathrm{Ker}(\mathcal{L}_p)$, then by computing the coefficient of $\frac{\sigma_1^{r_1}}{r_1!} \cdots \frac{\sigma_{p-1}^{r_{p-1}}}{r_{p-1}!}\, \frac{\sigma_p^{r_p - 1}}{(r_p - 1)!}$ in $\mathcal{L}_p(F)$ we find

$$A(\mathbf{r}) \;=\; \frac{2}{d-p+2r_p-1} \sum_{\substack{0<a,\, b<p \\ a+b\geq p}} r_{a+b-p}\, A(\mathbf{r} \,-\, \mathbf{e}_{a+b-p} \,+\, \mathbf{e}_a \,+\, \mathbf{e}_b \,-\, \mathbf{e}_p) \tag{111}$$

for all $\mathbf{r} = (r_1, \dots, r_p)$ with $r_p > 0$, where $\mathbf{e}_i$ as usual denotes the vector with a 1 in the ith place and 0's elsewhere (so $\mathbf{e}_0 = 0$) and $r_0 = 1$ by the convention (110). From this formula it follows inductively that all coefficients $A(r_1, \dots, r_p)$ are determined by the coefficients $A(r_1, \dots, r_{p-1})$ and hence that $\rho_{p-1}(F) = 0 \;\Rightarrow\; \rho_p(F) = 0$, as was to be proved.

This completes the proof of the isomorphy of the map (105) and hence of the first statement of Theorem 10. The other two statements are now easy. Choose a basis $\{G_k\}$ of the 1-dimensional space W_k for each $k \geq 0$. From the corollary to Proposition 2 we see that $\mathcal{L}_1(G_k)$ is a multiple of G_{k-1} for all $k \geq 1$, and this multiple is non-zero because of (107), since $W_k\langle 0\rangle \cong V_k^{(0)} = \{0\}$ for $k > 0$. Write $\mathcal{L}_1(G_k) = c_k G_{k-1}$ and write $G = \sum_{k=0}^{\infty} G_k$ as $\sum_{\boldsymbol{\nu}} P_{\boldsymbol{\nu}}(T) X^{\boldsymbol{\nu}}$. By the corollary to Proposition 1 we have $D_i(\widetilde{G}_k) = 0$ and $D_{ij}(\widetilde{G}_k) = 2c_k x_{ij} \widetilde{G}_{k-1}$, and comparing the coefficients of $X^{\boldsymbol{\nu}}$ on both sides of this equality we find that $D_i(P_{\boldsymbol{\nu}}) = 0$ and $D_{ij}(P_{\boldsymbol{\nu}}) = 2c_{\|\boldsymbol{\nu}\|} P_{\boldsymbol{\nu}-\mathbf{e}_{ij}}$ for all $\boldsymbol{\nu} \neq \mathbf{0}$, where $\|\boldsymbol{\nu}\| = \sum_{i<j} \nu_{ij}$. $\square$

Before finishing this subsection, we mention two consequences of the explicit recursion (111). First, in this formula, the only denominators that occur are of the form $d - d_0$ with $d_0 \in \mathbb{Z}$. It follows that, if we choose our normalization of the functions $G_k \in W_k$ (or equivalently, of the function $G^{(1)}$) to have coefficients in the subring

$$\mathcal{R} \;=\; \mathbb{Q}\bigl[d,\, \frac{1}{d},\, \frac{1}{d \pm 1},\, \frac{1}{d \pm 2},\, \dots\bigr] \tag{112}$$

of $\mathcal{K}$ consisting of rational functions of d having poles only at integers, then the whole function G has coefficients in this ring. The coefficients $c_k \in \mathcal{K}$ relating $\mathcal{L}_1(G_k)$ and G_{k-1} also have no zeros or poles in d outside of $\mathbb{Z}$, and it follows that the descending basis elements $P_{\boldsymbol{\nu}}^D(T)$ that we constructed in §5 also have coefficients in the ring $\mathcal{R}$. In particular, these polynomials exist for any $d \in \mathbb{C} \smallsetminus \mathbb{Z}$. It then follows from the duality between the bases $\{P_{\boldsymbol{\nu}}^D\}$ and $\{P_{\boldsymbol{\nu}}^M\}$ (Theorem 6 and the remarks immediately preceding and following it) that the scalar product in $\mathcal{P}^{(n)}(d)$ is non-degenerate for these values of d. But we already know that the scalar product is non-degenerate for real $d > n - 1$, because it is defined by a convergent integral with strictly positive integrand (cf. Theorem 2), so we deduce the following result, already mentioned in §5:

Theorem 11. *The scalar product in $\mathcal{P}^{(n)}(d)$ is non-degenerate for all $d \in \mathbb{C} \smallsetminus \mathbb{Z}_{<n}$.*

We remark that the non-degeneracy for $d \in \mathbb{Z}_{\geq n}$ can also be obtained purely algebraically, without using the positivity of the scalar product, since the denominator $d - d_0$ in (111) always satisfies $d_0 < p$ (because $r_p \geq 1$) and since $p \leq n$ for the coefficients $A(\mathbf{r})$ occurring in $G^{(n)}$.

The second observation is that the recursion (111) leads relatively easily to a "closed formula" for the generating series $G(\sigma_1, \sigma_2, \dots)$. Indeed, for each n, let us expand the generating function $G^{(n)} = \rho_n(G)$ $(n > 1)$ as a power series

$$G^{(n)}(\sigma_1, \dots, \sigma_n) \;=\; \sum_{r=0}^{\infty} g_r^{(n)}(\sigma_1, \dots, \sigma_{n-1}) \, \frac{\sigma_n^r}{r!} \qquad \bigl(g_0^{(n)} = G^{(n-1)}\bigr)$$

in its last variable σ_n. Applied to any function involving only $\sigma_1, \dots, \sigma_n$, the operator $\mathcal{L}_n$ equals $(d + 1 - n)\partial_n + 2\sigma_n \partial_n^2 - 2\mathcal{M}_n$, where $\mathcal{M}_n$ is the second order differential operator

$$\mathcal{M}_n \;=\; \sum_{\substack{0<a,\, b<n \\ a+b \geq n}} \sigma_{a+b-n} \, \partial_a \, \partial_b \,, \tag{113}$$

with $\sigma_0 = 1$ as usual. Equating the coefficient of σ_n^r in $\mathcal{L}_n(G^{(n)})$ to 0 therefore gives

$$\frac{d - n + 2r + 1}{2} \, g_{r+1}^{(n)} \;=\; \mathcal{M}_n\bigl(g_r^{(n)}\bigr) \qquad (r \geq 0) \tag{114}$$

and hence, by induction on r and using the initial condition $g_0^{(n)} = G^{(n-1)}$, the general formula

$$g_r^{(n)} \;=\; \frac{1}{(\frac{1}{2}(d-n+1))_r}\,\mathcal{M}_n^r\big(G^{(n-1)}\big) \qquad (r \ge 0)\,.$$

Since the operators $\mathcal{M}_n$ and (multiplication by) σ_n commute, this implies that

$$G^{(n)}(\sigma_1,\dots,\sigma_n) \;=\; \mathbb{J}_{\frac{d-n-1}{2}}(\sigma_n\mathcal{M}_n)\big(G^{(n-1)}(\sigma_1,\dots,\sigma_{n-1})\big)\,, \tag{115}$$

where $\mathbb{J}_\nu(x)$ is the power series

$$\mathbb{J}_\nu(x) \;=\; \sum_{r=0}^{\infty}\frac{x^r}{r!\,(\nu+1)_r} \;=\; 1 \,+\, \frac{x}{\nu+1} \,+\, \frac{x^2}{2(\nu+1)(\nu+2)} \,+\, \cdots\,, \tag{116}$$

related to the J-Bessel function $J_\nu(x)$ by $J_\nu(x) = \frac{(x/2)^\nu}{\Gamma(\nu+1)}\mathbb{J}_\nu(-x^2/4)$. Now induction on n gives

$$G^{(n)}(\sigma_1,\dots,\sigma_n) \;=\; \mathbb{J}_{\frac{d-n-1}{2}}(\sigma_n\mathcal{M}_n)\,\mathbb{J}_{\frac{d-n}{2}}(\sigma_{n-1}\mathcal{M}_{n-1})\,\cdots\,\mathbb{J}_{\frac{d-3}{2}}(\sigma_2\mathcal{M}_2)\big(G^{(1)}(\sigma_1)\big)\,, \tag{117}$$

the desired "closed formula" for $G^{(n)}$ (or even, if we extend the product of operators infinitely to the left, for the whole generating function G). The form of (117) makes it clear that the power series $G^{(1)}(\sigma_1)$ can be chosen arbitrarily but then determines all of the higher power series $G^{(n)}(\sigma_1,\dots,\sigma_n)$. Summarizing, we have:

Theorem 12. *The universal generating function $G^{(n)}$ can be obtained for all $n \ge 1$ from an arbitrary choice of $G^{(1)}$ by* (117)*, where $\mathbb{J}_\nu(x)$ and $\mathcal{M}_n$ are the power series and second order differential operator defined by* (116) *and* (113)*, respectively.*

The above argument described the generating function $G^{(n)}$ for generic values of d in terms of the initial choice $G^{(1)}$, the key point being that $G^{(n)}$ is determined in terms of $G^{(n-1)}$ by equation (115) as long as $(d-n-1)/2$ is not a negative integer. We end this subsection by saying something about the exceptional cases where $G(\sigma_1,\dots,\sigma_n)$ is *not* determined by its specialization to $\sigma_n = 0$. For clarity we write $W^{(n)}(d)$ for the space of functions in $V^{(n)} = \mathcal{K}[[\sigma_1,\dots,\sigma_n]]$ annihilated by $\mathcal{L}_2,\dots,\mathcal{L}_n$, including the dependence on d in the notation. Then we have:

Proposition 3. *Suppose that $G(\sigma_1,\dots,\sigma_n) \in W^{(n)}(d)\smallsetminus\{0\}$ with $G(\sigma_1,\dots,\sigma_{n-1},0) \equiv 0$. Then $d = n+1-2m$ for some integer $m \ge 1$ and G has the form*

$$G(\sigma_1,\dots,\sigma_n) \;=\; \sum_{\nu=0}^{\infty}\mathcal{M}_n^\nu(g)(\sigma_1,\dots,\sigma_{n-1})\,\frac{\sigma_n^{m+\nu}}{\nu!\,(m+\nu)!} \tag{118}$$

for some function $g \in W^{(n-1)}(d+4m) = W^{(n-1)}(n+2m+1)$. Conversely, for any $m \in \mathbb{Z}_{\ge 1}$ and any $g \in W^{(n-1)}(n+2m+1)$, the function G defined by (118) *belongs to $W^{(n)}(n-2m+1)$.*

Proof. Write G as $\sum_{r\ge0} g_r(\sigma_1,\dots,\sigma_{n-1})\,\sigma_n^r/r!$, and let $m \ge 1$ be the smallest integer for which $g_m \not\equiv 0$. From $\mathcal{L}_n^{(n,d)}(G) = 0$ we get $\frac{1}{2}(d-n+2r+1)g_{r+1} = \mathcal{M}_n(g_r)$ for all $r \ge 0$ (equation (114)). Applying this with $r = m-1$ gives $d = n-2m+1$, and then applying it inductively for $r = m+\nu$ gives $g_{m+\nu} = \mathcal{M}_n^\nu(g_m)/\nu!$ for all $\nu \ge 0$, so G has an expansion as in (118) with $g = g_m$. To prove that the function g belongs to $W^{(n-1)}(d+4m)$, we must show that $\mathcal{L}_p^{(n-1,d+4m)}g = 0$ for all $2 \le p < n$, where for clarity we write $\mathcal{L}_p^{(n,d)}$ for the operator $\mathcal{L}_p$ in $V^{(n)}$ with a given value of d. From the definition (102) we get

$$\mathcal{L}_p^{(n,d)} \;=\; \mathcal{L}_p^{(n-1,d+4m)} \,+\, 4\Big(\sigma_n\frac{\partial}{\partial\sigma_n} \,-\, m\Big)\,\frac{\partial}{\partial\sigma_p} \,+\, 2\sigma_n\sum_{p<i<n}\frac{\partial^2}{\partial\sigma_i\,\partial\sigma_{n+p-i}} \qquad (2 \le p < n),$$

and the claim then follows by computing the coefficient of σ_n^m in the equation $\mathcal{L}_p^{(n,d)}(G) = 0$. Conversely, if $g \in W^{(n-1)}(n+2m+1)$, then by reversing the above calculations we find that the function G defined by (118) satisfies $G = \mathrm{O}(\sigma_n^m)$, $G \in \mathrm{Ker}\big(\mathcal{L}_n^{(n,d)}\big)$, and $\mathcal{L}_p^{(n,d)}(G) = \mathrm{O}(\sigma_n^{m+1})$ for $2 \le p < n$. But by the corollary to Proposition 2, $\mathcal{L}_p^{(n,d)}(G)$ for $2 \le p < n$ also belongs to $\mathrm{Ker}\big(\mathcal{L}_n^{(n,d)}\big)$, and then it must vanish since the first part of the proof shows that any element of this kernel for $d = n+1-2m$ is determined by its σ_n^m term. $\square$

B. Proofs of the closed formula and commutation relation for $\mathcal{L}_p$. In this subsection we prove Propositions 1 and 2. Both proofs are rather lengthy and, as already mentioned, can be omitted without loss of continuity. To simplify them somewhat, we define coefficients $\varepsilon(m)$, $\varepsilon^{\pm}_{a,b}(m) \in \{0, \pm 1, \pm 2\}$ for a, b, $m \in \mathbb{Z}$ as follows:

$$\begin{aligned}
\varepsilon(m) \;&=\; \mathrm{sign}(m+\tfrac12) = \begin{cases} +1 & \text{if } m \ge 0, \\ -1 & \text{if } m < 0, \end{cases} \\
\varepsilon^{+}_{a,b}(m) \;&=\; \varepsilon(a-m) + \varepsilon(b-m) = \begin{cases} +2 & \text{if } a,\, b \ge m, \\ -2 & \text{if } a,\, b < m, \\ \;\;0 & \text{otherwise,} \end{cases} \\
\varepsilon^{-}_{a,b}(m) \;&=\; \varepsilon(a-m) - \varepsilon(b-m) = \begin{cases} +2 & \text{if } b < m \le a, \\ -2 & \text{if } a < m \le b, \\ \;\;0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Then the definition (102) of the operators $\mathcal{L}_p$ on V can be written more compactly as

$$\mathcal{L}_p \;=\; (d+1-p)\,\partial_p \;+\; \sum_{a,b\ge 1} \varepsilon^{+}_{a,b}(p)\,\sigma_{a+b-p}\,\partial_a\,\partial_b \qquad (p \ge 1) \tag{119}$$

(with $\partial_a = \partial/\partial\sigma_a$, and with the convention that $\sigma_0 = 1$ and $\sigma_a = 0$ if $a < 0$), and the commutation relation (104) can be written as

$$[\mathcal{L}_p,\, \mathcal{L}_q] \;=\; \sum_{\substack{a,\,b \ge 1 \\ a+b=p+q}} \varepsilon^{-}_{p,q}(b)\,\partial_a\,\mathcal{L}_b \qquad (p,\, q \ge 1). \tag{120}$$

Proof of Proposition 1. Since both the left- and right-hand sides of (103) are second order differential operators, it suffices to prove their equality on monomials in $\mathcal{K}[\sigma_1, \sigma_2, \dots]$ of degree ≤ 2, i.e., for 1, σ_a and $\sigma_a\sigma_b$ $(a,\, b \ge 1)$. The first case is trivial (both sides are zero), the second case can be rewritten (since $\mathcal{L}_p(\sigma_a) = (d+1-p)\delta_{ap}$) in the form

$$D_{ij}(\sigma_p) \;=\; (d+1-p)\,\partial_{ij}(\sigma_p) \qquad (1 \le i,\, j \le n, \quad 1 \le p \le n)\,, \tag{121}$$

and the third case is equivalent, by virtue of (119) and (121), to the identity

$$\begin{aligned}
D_{ij}(\sigma_a\sigma_b) \;&=\; \sum_{p=1}^{\infty} \partial_{ij}(\sigma_p)\,\big[(d+1-p)(\delta_{ap}\sigma_b + \delta_{bp}\sigma_a) \;+\; 2\,\varepsilon^{+}_{a,b}(p)\sigma_{a+b-p}\big] \\
&=\; D_{ij}(\sigma_a)\,\sigma_b \;+\; \sigma_a\,D_{ij}(b) \;+\; 2\sum_{p\ge 1} \varepsilon^{+}_{a,b}(p)\,\partial_{ij}(\sigma_p)\,\sigma_{a+b-p}\,,
\end{aligned} \tag{122}$$

It therefore suffices to prove the formulas (121) and (122).

For this purpose, it will be convenient to use generating functions. Let λ be an independent variable and set

$$P \;=\; P_\lambda \;=\; X^{-1} + \lambda T\,, \qquad \Delta \;=\; \Delta_\lambda \;=\; \det(P_\lambda) \;=\; \det(X)^{-1} \sum_{p=0}^{n} \sigma_p\, \lambda^p\,.$$

The subscript "λ" will be omitted when no confusion can result. Since the factor $\det(X)^{-1}$ is independent of λ and the T-variables, (121) can be rewritten in the form

$$D^0_{ij}(\Delta) \;=\; \bigl(1 \,-\, \lambda\,\frac{d}{d\lambda}\bigr)\,\partial_{ij}(\Delta)\,, \tag{123}$$

where $D^0_{ij} := D_{ij} - d\partial_{ij}$ denotes the purely second degree part of the differential operator D_{ij}. (Cf. the definition of D_{ij} in eq. (31).) Since the (i,j)-entry P_{ij} of P equals $\lambda t_{ij} + (X^{-1})_{ij}$, we see from the expansion of $\det(P)$ as a sum of products that

$$\partial_{ij}(\Delta) \;=\; 2\,\lambda\,\Delta^{i;j}\,, \tag{124}$$

where $\Delta^{i;j}$ is defined as $(-1)^{i+j}$ times the determinant of the $(n-1)\times(n-1)$ matrix $P^{i;j}$ obtained from P by omitting its ith row and jth column. (The factor 2 in (124) arises for $i \neq j$ because P is symmetric and contains t_{ij} twice, and for $i = j$ because $\partial_{ii} = 2\,\partial/\partial t_{ii}$.) Since each entry of the matrix $P^{i;j} = \lambda T^{i;j} + (X^{-1})^{i;j}$ contains λ multiplied by the corresponding entry of $T^{i;j}$, which is some $t_{k\ell}$ with $k \neq i$, $\ell \neq j$, we have

$$\frac{\partial}{\partial\lambda}\bigl(\Delta^{i;j}\bigr) \;=\; \sum_{k,\,\ell=1}^{n} t_{k\ell}\,\Delta^{i,k;j,\ell}\,, \tag{125}$$

where $\Delta^{i,k;j,\ell}$ is the determinant of the $(n-2)\times(n-2)$ matrix obtained from P by omitting its ith and kth rows and jth and ℓth columns, multiplied by $(-1)^{i+k+j+\ell}$ and by an additional factor $\operatorname{sgn}(i-k)\operatorname{sgn}(j-\ell)$ to take into account the antisymmetry of the determinant (and $\Delta^{i,k;j,\ell} = 0$ if $i = k$ or $j = \ell$). Combining (124) and (125) gives

$$\bigl(1 \,-\, \lambda\,\frac{d}{d\lambda}\bigr)\,\partial_{ij}(\Delta) \;=\; -2\,\lambda^2\,\frac{\partial}{\partial\lambda}\bigl(\Delta^{i;j}\bigr) \;=\; -2\,\lambda^2 \sum_{k,\,\ell} t_{k\ell}\,\Delta^{i,k;j,\ell}\,. \tag{126}$$

On the other hand, using (124) with (i,j) replaced by (j,ℓ) and computing ∂_{ik} of it by the same argument, we find

$$\partial_{ik}\partial_{j\ell}(\Delta) \;=\; 2\,\lambda\,\partial_{ik}\bigl(\Delta^{j;\ell}\bigr) \;=\; 2\lambda^2\,\bigl[\Delta^{i,j;k,\ell} \,+\, \Delta^{k,j;i,\ell}\bigr]\,. \tag{127}$$

(Here there are two terms on the right-hand side rather than a factor of 2 as in (124), because $P^{j;\ell}$ is no longer symmetric.) Multiplying this by $t_{k\ell}$ and summing over all $1 \le k, \ell, \le n$ gives

$$D^0_{ij}(\Delta) \;=\; \sum_{k,\,\ell} t_{k\ell}\,\partial_{ik}\partial_{j\ell}(\Delta) \;=\; 2\lambda^2 \sum_{k,\ell} t_{k\ell}\,\Delta^{k,j;i,\ell}\,. \tag{128}$$

(The whole contribution from the first term on the right-hand side of (127) drops out because it is antisymmetric in k and ℓ while $t_{k\ell}$ is symmetric.) Interchanging i and j in this formula and using the antisymmetry once again, we see that the right-hand sides of (126) and (128) agree, completing the proof of (123) and hence of (121).

We now turn to equation (122). Again we use the generating function $\Delta_\lambda = \det(P_\lambda)$, but this time for two independent variables λ and μ (which we now cannot omit from the notations). From the definition of D_{ij} together with equation (124), which by the standard formula for the inverse of a matrix can be written in the form $\partial_{ij}(\Delta_\lambda) = 2\lambda\Delta_\lambda(P_\lambda^{-1})_{ij}$, we find

$$\begin{aligned} D_{ij}(\Delta_\lambda\Delta_\mu) \,-\, D_{ij}(\Delta_\lambda)\,\Delta_\mu \,-\, \Delta_\mu\, D_{ij}(\Delta_l) \,&=\, \sum_{k,\,\ell} t_{k\ell}\,\bigl(\partial_{ik}(\Delta_l)\partial_{j\ell}(\Delta_\mu) \,+\, \partial_{ik}(\Delta_\mu)\partial_{j\ell}(\Delta_\lambda)\bigr) \\ &=\, 4\,\lambda\,\mu\,\Delta_\lambda\,\Delta_\mu \sum_{k,\,\ell}\bigl((P_\lambda^{-1})_{ik}\,t_{k\ell}\,(P_\mu^{-1})_{\ell j} \,+\, (P_\mu^{-1})_{ik}\,t_{k\ell}\,(P_\lambda^{-1})_{\ell j}\bigr) \\ &=\, 4\,\lambda\,\mu\,\Delta_\lambda\,\Delta_\mu\,\bigl(P_\lambda^{-1}\,T\,P_\mu^{-1} \,+\, P_\mu^{-1}\,T\,P_\lambda^{-1}\bigr)_{ij} \\ &=\, -8\,\lambda\,\mu\,\Delta_\lambda\,\Delta_\mu\left(\frac{P_\lambda^{-1} \,-\, P_\mu^{-1}}{\lambda \,-\, \mu}\right)_{ij} \qquad \left(\text{because } T = \frac{P_\lambda - P_\mu}{\lambda - \mu}\right) \\ &=\, -\frac{4}{\lambda-\mu}\Bigl(\mu\,\Delta_\mu\,\partial_{ij}(\Delta_\lambda) \,-\, \lambda\,\Delta_\lambda\,\partial_{ij}(\Delta_\mu)\Bigr) \\ &=\, \frac{4}{\det(X)^2}\sum_{p\ge 0,\,q\ge 0}\sigma_q\,\partial_{ij}(\sigma_p)\,\frac{\lambda^{q+1}\mu^p \,-\, \mu^{q+1}\lambda^p}{\lambda \,-\, \mu}\,, \end{aligned}$$

and now expanding the fraction in the last line as a geometric series and comparing the coefficients of $\lambda^a\mu^b$ on both sides we obtain the desired equation (122). □

Proof of Proposition 2. Any element of the Weyl algebra $\mathbb{Q}[\sigma_a, \sigma_2, \ldots, \partial_1, \partial_2, \ldots]$ can be written uniquely using the commutation relation $[\partial_a, \sigma_b] = \delta_{ab}$ as a sum of monomials $\sigma_1^*\sigma_2^*\cdots\partial_1^*\partial_2^*\cdots$, so it suffices to put the left- and right-hand sides of equation (120) into this form and compare coefficients. We claim that both sides of (120) are equal to

$$2\sum_{\substack{i,\,j\ge 1\\ i+j=p+q}}(d+1-i)\,\varepsilon_{p,q}^-(i)\,\partial_i\,\partial_j \;+\; \sum_{i,\,j,\,k\ge 1}\varepsilon_{p,q}^-(p+q-k)\,\varepsilon_{i,j}^+(p+q-k)\,\sigma_{i+j+k-p-q}\,\partial_i\partial_j\partial_k\,.$$

For the left side we use the commutation relation to find in succession

$$\begin{aligned} [\partial_p,\,\mathcal{L}_q] \,&=\, \sum_{a+b=p+q}\varepsilon_{a,b}^+(q)\,\partial_a\partial_b\,, \\ [\sigma_i\partial_a\partial_b,\,\sigma_I\partial_A\partial_B] \,&=\, \sigma_i\,(\delta_{b,I}\partial_a + \delta_{a,I}\partial_b)\partial_A\partial_B \,-\, \sigma_I\,(\delta_{B,i}\partial_A + \delta_{A,i}\partial_B)\partial_a\partial_b\,, \\ [\mathcal{L}_p,\,\mathcal{L}_q] \,&=\, \sum_{i+j=p+q}\bigl[(d+1-p)\varepsilon_{i,j}^+(q) \,-\, (d+1-q)\varepsilon_{i,j}^+(p)\bigr]\,\partial_i\partial_j \\ &\quad+\, \sum_{i,\,j,\,k>0}\Bigl(\varepsilon_{i,j}^+(q)\,\varepsilon_{k,i+j-q}^+(p) \,-\, \varepsilon_{i,j}^+(p)\,\varepsilon_{k,i+j-p}^+(q)\Bigr)\,\sigma_{i+j+k-p-q}\,\partial_i\partial_j\partial_k\,. \end{aligned}$$

The final expression gives $[\mathcal{L}_p,\,\mathcal{L}_q]$ as a sum of canonically ordered monomials in the Weyl algebra and can be compared with the formula given at the beginning of the proof. The coefficients of $\partial_i\partial_j$ and $\sigma_{i+j+k-p-q}\partial_i\partial_j\partial_k$ in the two formulas do not agree as they stand, but if we symmetrize with respect to i and j (resp. to i, j and k), they do. (The somewhat tedious verification of this can be done either directly, by checking the various cases according to the inequalities satisfied by the variables, or else by forming generating functions out of the coefficients in question and comparing their values.) Similarly, if we use the commutation relations to calculate the right-hand side of (120), then after symmetrization we find the expression stated at the beginning of the proof except that the coefficient of $\partial_i\partial_j$ in the first sum has an extra term $\sum_{1\le h\le p+q}\varepsilon_{p,q}^-(h)\varepsilon_{i,j}^+(h)$, but this sum vanishes since $\varepsilon_{p,q}^-(h)$ is invariant and $\varepsilon_{i,j}^+(h)$ anti-invariant under $h \mapsto p+q+1-h$ $(=\, i+j+1-h)$. Again we omit the details of the computation. □

C. Special cases and explicit formulas. In Subsection **A**, we gave the proof of Theorem 10 in an algebraic form that led to an explicit recursion for the coefficients, and also recast it as an explicit inductive construction of the power series $G^{(n)} := \rho_n(G)$ via differential operators (eqs. (111) and (117)). Either of these can be used to make numerical calculations in any desired special case. In this final subsection we describe some of the results obtained, and a few supplementary results.

1. We start by looking at some small values of n. For the case **n=1** it might seem that nothing needs to be said, since, as we have seen, the function $G^{(1)}(\sigma_1)$ can be chosen arbitrarily. However, in fact not all choices are equally good, as we will see in a moment when we consider the next two values $n = 2$ and $n = 3$. Furthermore, there is a direct connection between the choice of $G^{(1)}$ and the constants of proportionality relating various normalizations of our canonical bases of higher spherical polynomials, and we discuss this point briefly here because it casts some light on the calculations done in §8 for $n = 3$. Write the generic choice of power series $G^{(1)}(\sigma_1)$ as

$$G^{(1)}(\sigma_1) \;=\; \sum_{k=0}^{\infty} \frac{\gamma_k}{k!}\,\sigma_1^k \tag{129}$$

with $\gamma_0 = 1$, where γ_k in general may depend on d. Our standard choice, dictated by the generating function (72) for $n = 2$ or (101) for $n = 3$, will be

$$\gamma_k \;=\; 2^{-k}\,(d-2)_k\,, \qquad G^{(1)}(\sigma_1) = (1-\sigma_1/2)^{2-d} \tag{130}$$

(or by an appropriate limiting value if $d = 2$). The polynomials corresponding to this normalization will be denoted simply by $P_{\boldsymbol\nu}(T)$ (or $P_{\boldsymbol\nu,d}(T)$). However, we will also consider various other possible choices. The simplest and most natural one, which will occur again in points **2.** and **6.** below, is

$$\underline{\gamma}_k \;=\; 1\,, \qquad \underline{G}^{(1)}(\sigma_1) = e^{\sigma_1}\,. \tag{131}$$

We will denote the corresponding generating function by $\underline{G}(\sigma_1, \sigma_2, \dots)$ and its coefficients by $\underline{A}(r_1, r_2, \dots)$, and notice that any other choice is related to this one by $G_k = \gamma_k\,\underline{G}_k$ and $A(\mathbf{r}) = \gamma_k\,\underline{A}(\mathbf{r})$, where $k = \sum i r_i$. Similarly, the higher spherical polynomials $\underline{P}_{\boldsymbol\nu}(T)$ by

$$\sum_{\boldsymbol\nu\in\mathcal{N}_0} \underline{P}_{\boldsymbol\nu}(T)\,X^{\boldsymbol\nu} \;=\; \underline{G}^{(n)}(\sigma_1(XT),\, \sigma_2(XT), \dots, \sigma_n(XT))\,, \tag{132}$$

i.e., by (100) with $G^{(n)}$ replaced by $\underline{G}^{(n)}$, are related to the standard polynomials by the formula

$$P_{\boldsymbol\nu}(T) \;=\; 2^{-k}(d-2)_k\,\underline{P}_{\boldsymbol\nu}(T) \qquad \bigl(k = \|\boldsymbol\nu\| := \sum_{i<j} \nu_{ij} = \tfrac12\,\mathbf{1}^t\cdot\boldsymbol\nu\cdot\mathbf{1}\bigr)\,. \tag{133}$$

Let us look how the higher spherical polynomials defined by these various choices are related to the descending basis defined in §5. Begin with an arbitrary choice of $G^{(1)}$ as in (129). In Theorem 10 we showed that $\mathcal{L}_1(G_k^{(n)}) = c_k\,G_{k-1}^{(n)}$ for certain constants c_k independent of n. Applying this to $n = 1$, with $G_k^{(1)}(\sigma_1) = \gamma_k\sigma_1^k/k!$, and observing that the operator $\mathcal{L}_1$ equals simply $d\,\partial_1 + 2\sigma_1^2\partial_1$ for $n = 1$, so that $\mathcal{L}_1(\sigma_1^k) = k(d+2-2)\sigma_1^{k-1}$, we find that the constants c_k are related to the coefficients γ_k in (129) by $c_k = (d+2k-2)\,\gamma_k/\gamma_{k-1}$. In particular, for the simplest generating functions $\underline{G}_k$ we have simply $\mathcal{L}_1(\underline{G}_k^{(n)}) = (d+2k-2)\,\underline{G}_{k-1}^{(n)}$ for all n, $k \ge 1$, so from (132) and the corollary to Proposition 1 we get $D_{ij}(\underline{P}_{\boldsymbol\nu}(T)) = 2(d+2k-2)\underline{P}_{\boldsymbol\nu-\mathbf{e}_{ij}}(T)$, where $k = \|\boldsymbol\nu\|$. Comparing this with the defining property (55) of the descending basis, we obtain:

Proposition 4. *The higher spherical polynomials* $\underline{P}_{\boldsymbol{\nu}}(T)$ *defined by the generating series* (132) *are related to the descending basis by*

$$\underline{P}_{\boldsymbol{\nu}}(T) \;=\; 2^{2k}\,(d/2)_k\,P^D_{\boldsymbol{\nu}}(T) \qquad (\boldsymbol{\nu}\in\mathcal{N}_0,\;\; k=\|\boldsymbol{\nu}\|)\,. \tag{134}$$

Combining this with equation (133), we find that we have also proved the formula

$$P_{\boldsymbol{\nu}}(T) \;=\; 2^{k}\,(d/2)_k\,(d-2)_k\,P^D_{\boldsymbol{\nu}}(T) \qquad (\boldsymbol{\nu}\in\mathcal{N}_0,\;\; k=\|\boldsymbol{\nu}\|)\,. \tag{135}$$

relating our standard basis to the descending basis. This generalizes the result already proved in Proposition 3 of §8 for the case $n=3$, and clarifies the origins of the two Pochhammer symbols occurring in equation (77).

2. To see which choices of γ_k have good properties, we look first at the case **n=2**, where, as we already mentioned in §8.**A**, the polynomials $P_{\boldsymbol{\nu}}(T)$ are called Gegenbauer polynomials in general and Legendre (resp. Chebyshev) polynomials in the special case $d=3$ (resp. $d=2$ or 4). With the normalization fixed in (71) they are given by the generating function (72), which corresponds in our notation (since the elementary symmetric functions of the eigenvalues of $\left(\begin{smallmatrix}0&X\\X&0\end{smallmatrix}\right)\left(\begin{smallmatrix}m_1&r/2\\r/2&m_2\end{smallmatrix}\right)$ are $\sigma_1=rX$ and $\sigma_2=(r^2/4-m_1m_2)X^2$) to

$$G^{(2)}(\sigma_1,\sigma_2) \;=\; \bigl((1-\sigma_1/2)^2-\sigma_2\bigr)^{-s-1} \qquad (d=2s+4) \tag{136}$$

and hence to the normalization given in (130). As already stated, this will be our default choice of normalization from now on, but it seems worth mentioning that there are at least two other special choices that have special properties for $n=2$ and therefore should also be considered. If we look in a standard reference work like [1], F22.9, we find, as well as the generating function (72), two other generating functions for Gegenbauer polynomials, namely, in our notations,

$$\sum_{a=0}^{\infty}\frac{2^a}{(2s+2)_a}\,P_{a,d}\left(\begin{smallmatrix}m_1&r/2\\r/2&m_2\end{smallmatrix}\right)X^a \;=\; e^{rX}\,\mathbb{J}_{s+1/2}\bigl((r^2-4m_1m_2)X^2\bigr)\,, \tag{137}$$

$$\sum_{a=0}^{\infty}\frac{(s+3/2)_a}{(2s+2)_a}\,P_{a,d}\left(\begin{smallmatrix}m_1&r/2\\r/2&m_2\end{smallmatrix}\right)X^a \;=\; \frac{\left(\dfrac{1-rX/2+\sqrt{1-rX+m_1m_2X^2}}{2}\right)^{-s-1/2}}{\sqrt{1-rX+m_1m_2X^2}}\,, \tag{138}$$

where $\mathbb{J}_\nu(x)$ is the modified Bessel function (116). We can write these in σ_p-coordinates as

$$\begin{aligned}\underline{G}^{(2)}(\sigma_1,\sigma_2) \;&=\; e^{\sigma_1}\,\mathbb{J}_{s+1/2}(4\sigma_2)\,,\\ \widehat{G}^{(2)}(\sigma_1,\sigma_2) \;&=\; \frac{1}{\sqrt{(1-\sigma_1/2)^2-\sigma_2}}\left(\frac{1-\sigma_1/2+\sqrt{(1-\sigma_1/2)^2-\sigma_2}}{2}\right)^{-s-1/2},\end{aligned} \tag{139}$$

corresponding to the two $n=1$ initializations (131) and

$$\widehat{\gamma}_k \;=\; (s+1/2)_k\,, \qquad \widehat{G}^{(1)}(\sigma_1) \;=\; (1-\sigma_1/2)^{-s-3/2}\,. \tag{140}$$

Notice that the generating function (138), like the standard generating function (72), is algebraic when d is rational. One can ask whether there are any other generating functions that are algebraic, at least for special values of s. In this connection we can observe that the $n=1$ specializations of all three generating series in (136) and (139), namely the functions

$G^{(1)}$, $\widehat{G}^{(1)}$ and $\underline{G}^{(1)}$ given in equations (130), (131) and (140), belong to a single one-parameter family $(1-\sigma_1)^{-K}$ (up to trivial rescalings $\sigma_1 \mapsto \lambda\sigma_1$, with the function $\underline{G}^{(1)}$ corresponding to the degenerate case $K = \lambda^{-1} \to \infty$). The corresponding $n = 2$ generating series are hypergeometric, and using Schwarz's theorem we can check that none of them is algebraic except when s is rational and K equals either $2s + 2$ or $s + 3/2$ (corresponding to G or $\widehat{G}$) or when s is an integer, in which case they are algebraic for every value of K.

3. We next look at the case **n=3**, where we already know that the answer for $G^{(n)}$ (in the standard normalization (130)) is the function given in (101). The coefficients of $P_{\boldsymbol{\nu}}(T)$ as polynomials in the coefficients of T were studied in §8.**D** and were extremely complicated expressions that we could not evaluate completely in closed form in all cases. The coefficients of $G^{(3)}$, on the other hand, turn out to be very simple. For $n = 3$ the space $W^{(3)} \subset \mathcal{K}[[\sigma_1, \sigma_2, \sigma_3]]$ is given by the vanishing of only two operators

$$\mathcal{L}_2 \;=\; (d-1)\,\partial_2 \,+\, 2\sigma_2\partial_2^2 \,+\, 4\sigma_3\partial_2\partial_3 \,-\, 2\partial_1^2\,, \qquad \mathcal{L}_3 \;=\; (d-2)\,\partial_3 \,+\, 2\sigma_3\partial_3^2 \,-\, 4\partial_1\partial_2 \,-\, 2\sigma_1\partial_2^2\,.$$

(The other terms in (102) can be omitted by virtue of the remarks following the statement of Proposition 1.) If we write $G^{(3)}$ as $\sum_{\alpha,\beta,\gamma\ge 0} A_{\alpha,\beta,\gamma}\frac{\sigma_1^\alpha}{\alpha!}\frac{\sigma_2^\beta}{\beta!}\frac{\sigma_3^\gamma}{\gamma!}$, so that $A_{\alpha,\beta,\gamma} = A(\alpha,\beta,\gamma)$ in the notation of (109), then the conditions $\mathcal{L}_2(G^{(3)}) = \mathcal{L}_3(G^{(3)}) = 0$ translate into the recursions

$$(d{+}2\beta{+}4\gamma{-}1)A_{\alpha,\beta+1,\gamma} \;=\; 2A_{\alpha+2,\beta,\gamma}\,, \qquad (d{+}2\gamma{-}2)A_{\alpha,\beta,\gamma+1} \;=\; 4A_{\alpha+1,\beta+1,\gamma} + 2\alpha A_{\alpha-1,\beta+2,\gamma}\,.$$

The right-hand side of the second of these equations can be replaced using the first equation by the simpler expression $2(\alpha+2\beta+4\gamma+d+1)A_{\alpha-1,\beta+2,\gamma}$, and using this one finds easily that the solution of these recursions, with the initial conditions $A_{\alpha,0,0} = 2^{-\alpha}(2s+2)_\alpha$, is given by

$$A_{\alpha,\beta,\gamma} = 2^{\gamma-\alpha}\,(2\beta+4\gamma+2s+2)_\alpha\,(2\gamma+s+1)_\beta\,(\gamma+s+1)_\gamma\,, \tag{141}$$

where $d = 2s+4$ and $(x)_m = x(x+1)\cdots(x+m-1)$ is the ascending Pochhammer symbol as usual. We thus obtain the following very simple description of $G^{(3)}$ as a power series.

Proposition 5. *The generating function $G^{(3)}$ is given by*

$$G^{(3)}(\sigma_1,\sigma_2,\sigma_3) \;=\; \sum_{\alpha,\,\beta,\,\gamma\,\ge\,0} 2^{\gamma-\alpha}\binom{\alpha+2\beta+4\gamma+2s+1}{\alpha}\binom{\beta+2\gamma+s}{\beta}\binom{2\gamma+s}{\gamma}\,\sigma_1^\alpha\sigma_2^\beta\sigma_3^\gamma\,,$$

(Note that we could also have obtained this expansion directly, using equation (70) and the binomial theorem, as the Taylor expansion of the algebraic function (101).)

We also mention that the $n = 3$ case allows us to reduce the freedom of choice that we still had for $n = 2$, since neither the function $\widehat{G}^{(3)}$ corresponding to the functions (137) or (140), nor the functions $n = 3$ corresponding to $G^{(1)}(\sigma_1) = (1-\sigma_1)^{-K}$ for s integral and other choices than $K = s + 3/2$ or $K = 2s+2$, turn out to be algebraic, leaving out standard normalization (130) as the only one (up to rescaling) that gives algebraic generating functions up to $n = 3$. From now on we always assume that we have made this choice.

4. For **n>3** the coefficients of the generating function $G^{(n)}$ no longer factor nicely into linear factors, and the generating function itself no longer seems to be algebraic in general for any choice of the initial function $G^{(1)}$. We will consider in detail the case **n=4**. Write $G^{(4)} = \sum_{\alpha,\beta,\gamma,\delta\ge 0} A_{\alpha,\beta,\gamma,\delta}\frac{\sigma_1^\alpha}{\alpha!}\frac{\sigma_2^\beta}{\beta!}\frac{\sigma_3^\gamma}{\gamma!}\frac{\sigma_4^\delta}{\delta!}$. Then the equation $\mathcal{L}_4(G^{(4)}) = 0$ gives the recursion

$$\frac{d+2\delta-3}{2}\,A_{\alpha,\beta,\gamma,\delta+1} \;=\; 2A_{\alpha+1,\beta,\gamma+1,\delta} \,+\, A_{\alpha,\beta+2,\gamma,\delta} \,+\, 2\alpha A_{\alpha-1,\beta+1,\gamma+1,\delta} \,+\, \beta A_{\alpha,\beta-1,\gamma+2,\delta}\,,$$

and this allows us to reduce the calculation of all $A_{\alpha,\beta,\gamma,\delta}$ to that of $A_{\alpha,\beta,\gamma,0} = A_{\alpha,\beta,\gamma}$, which is given by (141). Already for $\delta = 1$ the explicit formulas for the coefficients $A_{\alpha,\beta,\gamma,\delta}$ become complicated, but they can nevertheless be used to calculate as many terms of the power series $G^{(4)}$ as we want, or alternatively, we can use eq. (115) to compute $G^{(4)}$ directly from $G^{(3)}$. Carrying out these calculations for various small values of d ($d = 4, 5, \ldots, 10$) and up to fairly large degrees led to a number of conjectural statements, most of which we were then able to prove. These experiments, results, and proofs will be described in detail in §10. Here we mention only the highlights:

(a) For $d = 4$ the function $G^{(4)}$ is algebraic.

(b) For $d = 6$, 8 and 10 the function $G^{(4)}$ is not algebraic, but certain derivatives of it are. Specifically, the images of $G^{(4)}$ under $E+3$ for $d = 6$, under $(E+4)(E+5)(E+6)$ for $d = 8$, and under $(E+5)(E+6)(E+7)(E+8)(E+9)$ for $d = 10$, where E as usual denotes the Euler operator, all turned out experimentally to be algebraic functions, leading to the conjectural statement that $\binom{E+3s}{2s-1}G^{(4)}(\sigma_1,\sigma_2,\sigma_3,\sigma_4;2s+4)$ is algebraic for all positive integral values of s. This will be proved in §10.

(c) For $d = 5$, 7 and 9 we find numerically that the power series $G^{(4)}$ is not algebraic, but does have integral coefficients after rescaling by some simple factor (involving only the prime 2 for $d = 5$ and $d = 7$ and only the primes 2 and 3 for $d = 9$). We conjecture that this is true for all odd values of $d \geq 5$.

5. In points **1.–4.** above, we looked at the full generating series $G^{(n)}$ for small values of n. In a different direction, we can consider instead small values of the weight k. Here we can give a closed formula for any fixed value of k, the first examples being

$$\begin{aligned}
G_1 &= \frac{1}{2}(d-2)\,\sigma_1\,,\\
G_2 &= \frac{1}{4}\binom{d-1}{2}\sigma_1^2 + \frac{d-2}{2}\,\sigma_2\,,\\
G_3 &= \frac{1}{8}\binom{d}{3}\sigma_1^3 + \frac{d(d-2)}{4}\,\sigma_1\sigma_2 + d\,\sigma_3\,,\\
G_4 &= \frac{1}{16}\binom{d+1}{4}\sigma_1^4 + \frac{(d+1)d(d-2)}{16}\,\sigma_1^2\sigma_2 + \frac{d(d+2)}{2}\,\sigma_1\sigma_3 + \frac{d(d-2)}{8}\,\sigma_2^2 + \frac{d(5d+6)}{2(d-3)}\,\sigma_4\,.
\end{aligned}$$

Expressed differently, for each value of k there are only finitely many coefficients $A(r_1, r_2, \ldots)$ of weight k, corresponding to the partitions $k = r_1 + 2r_2 + \cdots$, and each of these coefficients is a rational function of s independent of n, e.g., the coefficient $A(3,2,0,1)$ of $\sigma_1^3\sigma_2^2\sigma_4/12$ (of total weight $k = 11$) is given by

$$A(3,2,0,1;d) \;=\; \frac{(s+3)(s+4)(s+5)(s+6)(10s^3+169s^2+935s+1634)}{2s+1}\,,$$

where we have written $d = 2s+4$ as usual. In general these rational functions are complicated, as this example shows, and it is unlikely that one give a general formula for them in closed form. However, we *can* give a complete formula for their asymptotic behavior as s (or d) tends to infinity, e.g. $A(3,2,0,1;d) \sim 5s^6$ as $s \to \infty$ in the above example.

Proposition 6. *For each fixed value of* $\mathbf{r} = (r_1, r_2, \ldots)$, *the coefficient* $A(\mathbf{r}; d = 2s+4)$ *of* $\prod_p \sigma_p^{r_p}/r_p!$ *in* $G(\sigma_1,\sigma_2,\ldots)$ *is a rational function of* s *with asymptotic behavior*

$$A(\mathbf{r};2s+4) \quad\sim\quad \left(1^{r_1}\,1^{r_2}\,2^{r_3}\,5^{r_4}\,\cdots\right)s^{r_1+r_2+r_3+\cdots}\,, \tag{142}$$

as $s \to \infty$, *where the coefficient is* $\prod_{i=1}^{\infty} C_{i-1}^{r_i}$ *with* $C_n = \frac{1}{n+1}\binom{2n}{n}$, *the* n*th Catalan number.*

Proof. We already know that $A(\mathbf{r};d)$ is a rational function of s (in fact, an element of the ring $\mathcal{R}$ defined in (112)) and that it is determined completely by the recursion (111) with the initial conditions $A(k,0,0,\dots;d)=\gamma_k$, with γ_k as in (130). In this recursion each coefficient $A(\mathbf{r})$ is expressed as a linear combination of coefficients $A(\mathbf{r'})$ with $r'_i = r_i - \delta_{i,p} + \delta_{i,a} + \delta_{i,b} - \delta_{i,a+b-p}$ for some integers a, $b>0$ with $a+b\ge p$. Since each $\mathbf{r'}$ is less than $\mathbf{r}$ in lexicographical ordering, we may assume by induction that (142) holds for $A(\mathbf{r'};d)$. (The initial value of the induction is correct, since $\gamma_k = 2^{-k}(d-2)_k \sim s^k$ as $s\to\infty$.) If $a+b>p$, then $\sum_{i\ge1} r'_i = \sum_{i\ge1} r_i$, so $A(\mathbf{r'};d)$ grows like the same power of s as in the as-yet-unproved formula for $A(\mathbf{r};d)$, and the factor $2/(d-p+2r_p-1)\sim 1/s$ in front of the sum in (111) implies that these terms give a negligible asymptotic contribution. If $a+b=p$, then $\sum_{i\ge1} r'_i$ is larger by 1 than $\sum_{i\ge1} r_i$, so the contributions of these terms in (130) have the same power of s as in (111). Also, the product $\prod_{i\ge1} C_{i-1}^{r'_i}$ is equal to $C_{a-1}C_{b-1}/C_{p-1}$ times the product $\prod_{i\ge1} C_{i-1}^{r_i}$, so the correctness of formula (142) follows from the identity $\sum_{a+b=p} C_{a-1}C_{b-1} = C_{p-1}$, which is the standard recursion relation for the Catalan numbers. □

6. In fact we can extend the idea of **5.** further by giving closed formulas for $A(\mathbf{r};d)$ when only r_2, $r_3,\dots$ are fixed and r_1 (or equivalently $k=r_1+2r_2+3r_3+\cdots$) as well as d is allowed to vary. For instance, we find that for every value of k and n the coefficient $A(k-2,1,0,\dots)$ of $\sigma_1^{k-2}\sigma_2/(k-2)!$ in $G_k^{(n)}$ is equal to $2^{1-k}(d-2)_k/(d-1)$. The factor of $2^{-k}(d-2)_k$ in this expression suggests that it would be better to work with the other normalization $\underline{A}(\mathbf{r})$ (cf. (133)), in which case the formula in question simplifies to just $\underline{A}(k-2,1,0,\dots)=2/(d-1)$. In this example the value is independent of k and is an element of the ring $\mathcal{R}$ defined in (112), but in general it turns out to be a polynomial in k with coefficients in $\mathcal{R}$. In other words, we have an expansion of the weight k part of $\underline{G}$ as

$$\underline{G}_k(\sigma_1,\sigma_2,\dots,\sigma_k) \;=\; \sum_{l=0}^{k} H_l(\sigma_2,\sigma_3,\dots,\sigma_l;k)\,\frac{\sigma_1^{k-l}}{(k-l)!} \tag{143}$$

where each H_l is a polynomial of (weighted) degree l in $\sigma_2,\dots,\sigma_l$ with coefficients in $\mathcal{R}[k]$, the first values being

$$H_0 = 1\,,\quad H_1 = 0\,,\quad H_2 = \frac{2}{d-1}\,\sigma_2\,,\quad H_3 = 8\,\frac{d+k-2}{(d+1)(d-1)(d-2)}\,\sigma_3\,,$$

$$H_4 = 8\,\frac{5d^2+(12k-27)d+8k^2-36k+34}{(d+3)(d+1)(d-1)(d-2)(d-3)}\,\sigma_4 + \frac{4}{(d+1)(d-1)}\,\frac{\sigma_2^2}{2!}\,,$$

$$H_5 = 32\,\frac{P_5}{(d+5)(d+3)(d+1)d(d-1)(d-2)(d-3)(d-4)}\,\sigma_5 + 16\frac{d+k-2}{(d+3)(d+1)(d-1)(d-2)}\,\sigma_2\sigma_3\,,$$

$$H_6 = 64\,\frac{P_6}{(d+7)(d+5)(d+3)(d+1)d(d-1)(d-2)(d-3)(d-4)(d-5)}\,\sigma_6 + 16\,\frac{5d^3+(12k-25)d^2+(8k^2-28k+22)d+8(k-1)(k-2)}{(d+5)(d+3)(d+1)d(d-1)(d-2)(d-3)}\,\sigma_2\sigma_4 + 64\,\frac{(d+k-1)(d+k-2)}{(d+5)(d+3)(d+1)d(d-1)(d-2)}\,\frac{\sigma_3^2}{2!} + \frac{8}{(d+3)(d+1)(d-1)}\,\frac{\sigma_2^3}{3!}$$

with

$$P_5 \;=\; 7d^4+(28k-72)d^3+(40k^2-206k+231)d^2+(20k^3-152k^2+338k-210)d \\ \quad +\; 4(k-1)(k-2)(k-5)\,,$$

$$P_6 \;=\; 21d^5+(120k-350)d^4+(270k^2-1574k+2047)d^3+(280k^3-2412k^2+6188k-4582)d^2 \\ \quad +\; (112k^4-1200k^3+4166k^2-5278k+1760)d+40(k-1)(k-2)(k-6)(2k-7)\,.$$

Here we should warn the reader that, although all the coefficients shown in this list have only simple poles in d, this property fails in general and in fact already in degree 7, where the coefficient of $\sigma_2\sigma_5\,\frac{\sigma_1^{k-7}}{(k-7)!}$ has a factor $(d+1)^2$ in its denominator. (That this example involves σ_5 is no accident, since it follows immediately from (115) that $A(r_1,r_2,r_3,r_4)$ has a denominator at most $(s+1/2)_{r_4}$ and hence has at most simple poles.) It is also perhaps worth mentioning that formula (143) for the weight k part $\underline{G}_k$ of the generating function can also be written in an equivalent form for the full function $\underline{G}=\sum_k \underline{G}_k$ as

$$\begin{aligned} e^{-\sigma_1}\,\underline{G}(\sigma_1,\sigma_2,\sigma_3,\dots) \;&=\; 1+\frac{2}{d-1}\,\sigma_2 \;+\; 8\,\frac{d+1+\sigma_1}{(d+1)(d-1)(d-2)}\,\sigma_3 \\ &\quad+\; 8\,\frac{5d^2+21d+18+12(d+3)\sigma_1+8\sigma_1^2}{(d+3)(d+1)(d-1)(d-2)(d-3)}\,\sigma_4 \;+\; \frac{4}{(d+1)(d-1)}\,\frac{\sigma_2^2}{2!} \;+\;\cdots \\ &\in\; \mathcal{R}[\sigma_1][[\sigma_2,\sigma_3,\dots]]\,. \end{aligned} \tag{144}$$

7. Our next point is that the new expansion (143) has an asymptotic property generalizing Proposition 6 to the case when k and s both go to infinity. Notice first that equation (142), when written in terms of the renormalized coefficients $\underline{A}(\mathbf{r};d)$, takes the form

$$\underline{A}(\mathbf{r};2s+4) \quad\sim\; \bigl(1^{r_2}\,2^{r_3}\,5^{r_4}\,\cdots\bigr)\,s^{-r_2-2r_3-3r_4-\cdots}\,,$$

in which r_1 does not occur at all. This is for k constant, but suggests now also looking at the asymptotics for $s\to\infty$ of the coefficients of $\underline{A}(\mathbf{r};2s+4)$ as polynomials in k. If we do this for the values listed in the table of H_l above, then we see a nice multiplicative property: the coefficient of σ_2 in H_2 is asymptotically equal to $\frac{1}{s}$ as $s\to\infty$, the coefficient of σ_4 in H_4 is asymptotically equal to $\frac{5s^2+6ks+2k^2}{s^5}$ as $s\to\infty$, and the coefficient of $\sigma_2\sigma_4$ in H_6 is asymptotically equal to $\frac{5s^2+6ks+2k^2}{s^6}$, which is the product of these two expressions. Similarly, the coefficient of $\sigma_3^2/2!$ in H_6 is asymptotically equal to $\frac{(2s+k)^2}{s^6}$ as $s\to\infty$, and this is the square of the asymptotic formula for the coefficient of σ_3 in H_3. In other words, we seem to have an asymptotic formula

$$\underline{A}(\mathbf{r};2s+4) \quad\sim\quad \prod_{p=1}^{\infty}\biggl(\frac{c_{p-1}(k/s)}{s^{p-1}}\biggr)^{r_p} \tag{145}$$

as s and k go simultaneously to infinity, where the $c_n(\lambda)$ are certain universal polynomials of $\lambda=k/s$ whose first few values are

$$c_0(\lambda)\;=\;1,\qquad c_1(\lambda)\;=\;1,\qquad c_2(\lambda)\;=\;\lambda+2,\qquad c_3(\lambda)\;=\;2\lambda^2+6\lambda+5,$$
$$c_4(\lambda)\;=\;5\lambda^3+20\lambda^2+28\lambda+14,\qquad c_5(\lambda)\;=\;14\lambda^4+70\lambda^3+135\lambda^2+120\lambda+42\,.$$

(In particular, the formula (145) remains unchanged if one replaces the product by one starting at $p=2$ instead of $p=1$.) To guess what these polynomials are, we first notice that compatibility with Proposition 6 requires that the constant term $c_n(0)$ equals C_n, the nth Catalan number, and that by inspection the highest term of $c_n(\lambda)$ for $n\ge 1$ is $C_{n-1}\lambda^{n-1}$. Since the generating function of $\sum_{n\ge0}C_nx^n$ is the reciprocal of a slightly simpler function, we look at the first few terms of the reciprocal of the series $\sum_{n\ge0}c_n(\lambda)x^n$ and immediately find that the coefficient of x^n in this reciprocal has the very simple form $-C_{n-1}(\lambda+1)^{n-1}$ for all $n\ge1$. This leads to the definition of the polynomials $c_n(\lambda)$ given in the following proposition, whose proof will be shorter than the discussion leading up to it.

Proposition 7. *For fixed values of $r_2, r_3, \dots \geq 0$, the coefficient $\underline{A}(r_1, r_2, r_3, \dots; d = 2s+4)$ of $\prod_{p\geq 1} \sigma_p^{r_p}/r_p!$ in $G_k(\sigma_1, \sigma_2, \dots)$ $(k = r_1 + 2r_2 + \cdots)$ is a polynomial in r_1 or in k of degree $\sum_{p\geq 3}(p-2)r_p$ with coefficients in the ring $\mathcal{R}$, and the asymptotic behavior of this coefficient as s and k tend to infinity is given by (145), where the polynomials $c_n(\lambda)$ are defined by the closed formula*

$$c_n(\lambda) \;=\; \begin{cases} 1 & \text{if } n = 0, \\ \dfrac{(2n)!}{n!(n-1)!} \displaystyle\sum_{j=0}^{n-1} \binom{n-1}{j} \frac{\lambda^j}{(n+j)(n+j+1)} & \text{if } n \geq 1 \end{cases} \tag{146}$$

or by the generating function

$$\sum_{n=0}^{\infty} c_n(\lambda)\, x^n \;=\; \frac{1 + 2\lambda \,-\, \sqrt{1 - 4(\lambda+1)x}}{2\,(\lambda + x)}\,. \tag{147}$$

Proof. Both the general form (143) of the generating function and the formula (145) for the asymptotic behavior of its coefficients follow from the recursion relation (111) for these coefficients (which is equally valid for the coefficients $\underline{A}(\mathbf{r})$ as for the standard coefficients $A(\mathbf{r})$, only the initial conditions being different), and the proof of Proposition 7 is almost identical to the proof of Proposition 6: the only difference is that the terms in (111) with $a + b = p + 1$ now also contribute to the leading term of the expansion (because the coefficient r_1 of $A(\mathbf{r'})$ for these terms is of the same order as s) and that the identity $C_{p-1} = \sum_{a+b=p} C_{a-1}C_{b-1}$ used in the proof of Proposition 6 must therefore be replaced by the more complicated identity

$$c_{p-1}(\lambda) \;=\; \sum_{\substack{a,\,b\geq 1 \\ a+b=p}} c_{a-1}(\lambda)\, c_{b-1}(\lambda) \;+\; \lambda \sum_{\substack{a,\,b\geq 2 \\ a+b=p+1}} c_{a-1}(\lambda)\, c_{b-1}(\lambda) \qquad (p \geq 2)\,,$$

which is in turn easily seen to be equivalent to the generating function (147). $\square$

8. Finally, we can use our generating functions to prove two statements about the determinant of the Gram matrices discussed in §5. We showed there (Theorem 6) that the monomial and descending bases are dual to one another with respect to the scalar product of §3, and hence that the two $N_0(\mathbf{a}) \times N_0(\mathbf{a})$ Gram matrices

$$\big(P^M_{\boldsymbol{\mu}}\,,\, P^M_{\boldsymbol{\nu}}\big)_{\boldsymbol{\mu},\,\boldsymbol{\nu}\in\mathcal{N}_0(\mathbf{a})} \qquad \text{and} \qquad \big(P^D_{\boldsymbol{\mu}}\,,\, P^D_{\boldsymbol{\nu}}\big)_{\boldsymbol{\mu},\,\boldsymbol{\nu}\in\mathcal{N}_0(\mathbf{a})}$$

are inverses of one another for generic values of d. From Theorem 11 we know that both matrices are regular and invertible away from integer values of $d \leq n-1$. It follows that the "Gram determinant" function $\mathrm{GD}^M_{\mathbf{a}}(d) \in \mathbb{Q}(d)$ defined by

$$\mathrm{GD}^M_{\mathbf{a}}(d) \;=\; \det\big(\big(P^M_{\boldsymbol{\mu}}\,,\, P^M_{\boldsymbol{\nu}}\big)_{\boldsymbol{\mu},\,\boldsymbol{\nu}\in\mathcal{N}_0(\mathbf{a})}\big)$$

is a rational function with zeros and poles only at integer values of its argument, i.e., that $\mathrm{GD}^M_{\mathbf{a}}(d)$ has the form $C\prod_m (d+m)^{e(m)}$ for some constant $C = C_{\mathbf{a}}$ and exponents $e(m) = e_{\mathbf{a}}(m)$ depending on $\mathbf{a}$, with $e(m) = 0$ for all but finitely many values of m, and $e(m) = 0$ for all $m \leq -n$. In the next proposition we will (1) give C and $\sum_m e(m)$ (or equivalently, the asymptotics of $\mathrm{GD}^M_{\mathbf{a}}(d)$ as $|d| \to \infty$) in all cases, and (2) give a complete formula for $\mathrm{GD}^M_{\mathbf{a}}(d)$ whenever $n > 3$ and $N_0(\mathbf{a}) = 1$ (so that $\mathrm{GD}^M_{\mathbf{a}}(d)$ is the determinant of a 1×1 matrix), in which case the exponents $e(m)$ turn out to depend only on the half-degree $k = \frac{1}{2}(a_1 + \cdots + a_n)$.

Proposition 8. (1) *As $d \to \infty$ with $\mathbf{a}$ fixed, we have*

$$GD^M_{\mathbf{a}}(d) \sim \prod_{\boldsymbol{\nu} \in \mathcal{N}_0(\mathbf{a})} \left(\boldsymbol{\nu}!\, d^k\right)$$

where $\boldsymbol{\nu}! := \prod_{i<j} \nu_{ij}!$.
(2) *Suppose that* $N_0(\mathbf{a}) = 1$ *for some* $\mathbf{a}$ *with* $n \geq 4$ *and* $a_i \geq 1$ *for all* i. *Then*

$$GD^M_{\mathbf{a}}(d) \;=\; \left(\prod_{i=1}^{n-1} a_i!\right)(d+2k-2)\,(d-1)_{k-1}\,.$$

Proof. (1) As in **5.** above, we denote by G_k the weight k part of the standard generating function. Then by (117), we have

$$G_k \;=\; \frac{(d-2)_k}{2^k k!}\sigma_1^k + O(d^{k-1})\,.$$

This is because the operator $\sigma_i \mathcal{M}_i$ with $2 \leq i \leq n$ on polynomials of weight k in $\sigma_1, \ldots, \sigma_{i-1}$ does not change the weight and the action of $((d-i+1)/2)_r(\sigma_i\mathcal{M}_i)^r$ decreases the order by d^r, so the dominant asymptotic term as $d \to \infty$ comes from the coefficient of σ_1^k. For any index $\boldsymbol{\nu}$ of weight k, the coefficient of $x^{\boldsymbol{\nu}}$ in σ_1^k is given by $(2^k k!/\boldsymbol{\nu}!)\,T^{\boldsymbol{\nu}}$, so by (135) we have

$$P^D_{\boldsymbol{\nu}} \;=\; \frac{1}{\boldsymbol{\nu}!\,(d/2)_k}T^{\boldsymbol{\nu}} \;+\; \mathrm{O}\big(d^{-k-1}\big) \;=\; \frac{1}{\boldsymbol{\nu}!\,(d/2)_k}P^M_{\boldsymbol{\nu}} \;+\; \mathrm{O}\big(d^{-k-1}\big)$$

and hence

$$\big(P^D_{\boldsymbol{\mu}},\, P^D_{\boldsymbol{\nu}}\,\big) \;\sim\; \frac{\delta_{\boldsymbol{\mu},\boldsymbol{\nu}}}{\boldsymbol{\nu}!\,d^k} \;+\mathrm{O}\big(d^{-k-1}\big)\,.$$

Since $\mathrm{GD}^M_{\mathbf{a}}(d) = \det((P^D_{\boldsymbol{\mu}}, P^D_{\boldsymbol{\nu}}))^{-1}$, we have the assertion.

(2) By symmetry, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$. Since $N_0(\mathbf{a}) = 1$, we have $a_n = a_1 + \cdots + a_{n-1}$ (cf. Remark **2.** in §2) and the corresponding index $\boldsymbol{\nu} = (\nu_{ij}) \in \mathcal{N}_0(\mathbf{a})$ satisfies $\nu_{in} = a_i$ for $i < n$ and $\nu_{ij} = 0$ for other (i,j). In order to calculate $\mathrm{GD}^M_{\mathbf{a}}(d)$, we only need the coefficient of $P^D_{\boldsymbol{\nu}}$ at $T^{\boldsymbol{\nu}}$. We see this coefficient by the generating function. To see this, since $\nu_{ij} = 0$ if $i < n$ and $j < n$, we may put $x_{ij} = 0$ for pairs (i,j) with $i < n$ and $j < n$ in the generating function. Under this substitution, we have $\sigma_l = 0$ for $l \geq 3$, $\sigma_1 = 2\sum_{i=1}^{n-1} t_{in}x_{in}$ and $\sigma_2 = \big(\sum_{i=1}^{n-1} t_{in}x_{in}\big)^2 = \sigma_1^2/4$. So the standard generating function reduces to

$$\big((1-\sigma_1/2)^2 - \sigma_2\big)^{-(d-2)/2} \;=\; (1-\sigma_1)^{-(d-2)/2} \;=\; \sum_{k=0}^{\infty} \frac{((d-2)/2)_k}{k!}\,\sigma_1^k\,.$$

Expanding σ_1^k, the coefficient of $x^{\boldsymbol{\nu}} = x_{1n}^{a_1} \cdots x_{n-1,n}^{a_{n-1}}$ is given by $2^k \cdot ((d-2)/2)_k / a_1! \cdots a_{n-1}!$. By (135), we have

$$\big(P^D_{\boldsymbol{\nu}},\, P^D_{\boldsymbol{\nu}}\big) \;=\; \frac{1}{(d-1)_{k-1}(d+2k-2)\,a_1!\,\cdots\,a_{n-1}!}\,.$$

This proves the assertion. $\square$

§10. Algebraic generating functions for the case $n=4$

As mentioned in point **C.4.** of §9, the formalism developed in that section permitted extensive numerical calculations of the generating function $G^{(4)}(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ that led to the formulation of a number of interesting properties, some of which we later proved and some of which remain conjectural. In this section give these results and their proofs, also including a brief description of the experiments that led up to them, without which the statements would be completely unmotivated. The first three subsections treat the case of even values of d, proving the algebraicity of $\binom{E+3s}{2s-1}G^{(4)}$ (where $d=2s+4$ as usual) that was already mentioned in §9.**C.4**. In §10.**D** we strengthen this result by proving that the generating series having these functions as coefficients is also algebraic. The last two subsections are devoted to a brief discussion of the case of odd d, for which the results are still conjectural, and to a final remark concerning the case when n is larger than 4.

A. Even values of d: experimental results. We look first at the case $d=4$ ($s=0$), since the generating function for $n=3$ was particularly simple for this value of d (§8.**A**). The expansion of $G^{(4)}(\sigma_1,\dots,\sigma_4)$ for $d=4$ begins

$$1+\sigma_1+\frac{3\sigma_1^2+4\sigma_2}{4}+\frac{\sigma_1^3+4\sigma_1\sigma_2+8\sigma_3}{2}+\frac{5\sigma_1^4+40\sigma_1^2\sigma_2+192\sigma_1\sigma_3+16\sigma_2^2+832\sigma_4}{16}+\cdots.$$

To simplify the coefficients of this and thus improve our chances of recognizing the function, we specialize the variables σ_p. Since we already know the $n=3$ function, we must not specialize σ_4 to 0, so the simplest choice is to take $\sigma_1=\sigma_2=\sigma_3=0$. Doing this, we find that the function $y=G^{(4)}(0,0,0,x/4)$ (where the factor $1/4$ is included just to reduce the size of the coefficients) has a power series expansion in x beginning

$$y\;=\;1\,+\,13\,x\,+\,321\,x^2\,+\,8989\,x^3\,+\,265729\,x^4\,+\,8097453\,x^5\,+\,\cdots.$$

Calculating this series to many terms and looking for linear relations among monomials of small degree in x and y, we find that within the accuracy of the calculation the function y satisfies the quadratic equation

$$(1-34x+x^2)^2\,y^4\,-\,(1+x)(1-34x+x^2)\,y^2\,+\,9x\;=\;0\,,$$

the solution of which is the explicit algebraic function

$$y\;=\;\sqrt{\frac{1+x+\sqrt{1-34x+x^2}}{2\,(1-34x+x^2)}}\;=\;\frac{1}{2\sqrt{1-6\sqrt{x}+x}}+\frac{1}{2\sqrt{1+6\sqrt{x}+x}}\quad. \tag{148}$$

Repeating the calculation for various other specializations of $G^{(4)}$ and interpolating the results obtained, we are led to conjecture the following result, whose proof will be given in §10.**B.**

Proposition 1. *The function $y_0=y_0(\sigma_1,\sigma_2,\sigma_3,\sigma_4)=G^{(4)}(\sigma_1,\sigma_2,\sigma_3,\sigma_4;d=4)$ is algebraic of degree 4 over $\mathbb{Q}(\sigma_1,\dots,\sigma_4)$ and is given explicitly by*

$$y_0\;=\;\sqrt{\frac{\Delta_1+\sqrt{\Delta_2}}{2\,\Delta_2}}\,, \tag{149}$$

where Δ_1 and Δ_2 are defined by

$$\Delta_0=(1-\sigma_1/2)^2-\sigma_2\,,\qquad \Delta_1=\Delta_0^2-8\sigma_3+4\sigma_4\,,\qquad \Delta_2\;=\;\Delta_1^2\,-\,16\sigma_4(4-2\sigma_1-\Delta_0)^2\,.$$

We remark that for $\sigma_4 = 0$ we have $\Delta_1 = \Delta_0^2 - 8\sigma_3$ and $\Delta_2 = \Delta_1^2$, so the right-hand side of (149) reduces in that case to $1/\sqrt{\Delta_0^2 - 8\sigma_3}$, in accordance with formula (101) when $d = 4$.

After this initial success we look at the case $d = 5$, but here there seems to be no algebraic formula. (We will return to this question in §10.**E**.) On the other hand, when we look at the following case $d = 6$, the situation is better. Here the expansion of $y = G^{(4)}(0, 0, 0, x/4)$ begins

$$1 \,+\, 9\,x \,+\, 153\,x^2 \,+\, \frac{16213}{5}\,x^3 \,+\, \cdots \,+\, \frac{240758592534924061536846814875 09}{29}\,x^{21} \,+\, \cdots .$$

Unlike the corresponding series for $d = 4$ and $d = 5$, which to the accuracy of the computation were in $\mathbb{Z}[[x]]$ and $\mathbb{Z}[[x/64]]$, respectively, this series appears to have infinitely many primes in its denominator, in which case it is definitely not algebraic since the Taylor coefficients of an algebraic function always belong to $\mathbb{Z}[1/N]$ for some integer N. On the other hand, looking more carefully, we find that the denominator of the coefficient of x^n is always a divisor of $4n+3$, which means that the power series $y_1 := 4xy' + 3y$ has integral coefficients, and now applying the same numerical procedure as before to y_1 leads to the experimental formula

$$y_1 \;=\; \sqrt{\frac{1 + x - \sqrt{1 - 34x + x^2}}{2\,x\,(1 - 34x + x^2)}} \;=\; \frac{1}{2\sqrt{x(1 - 6\sqrt{x} + x)}} - \frac{1}{2\sqrt{x(1 + 6\sqrt{x} + x)}} \quad , \tag{150}$$

very similar to the corresponding formula (148) for the $d = 4$ case. More generally, when we look at the Taylor expansion of $y = G^{(4)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4; 6)$ for general values of the σ_p, we find that the denominators in the coefficients are always divisors of $k + 3$, where k is the degree of the monomial in question, so that we must replace y by $y_1 = (E + 3)y$ to get integrality, where $E = \sum_p p\sigma_p$ is the Euler operator that multiplies a monomial of weight k in the σ_p by k. Just as before, by identifying the power series obtained for several other specializations as algebraic functions and interpolating, we are led to guess the following proposition, whose proof will also be given in Subsection **B** below.

Proposition 2. *The generating function $G^{(4)}$ is not algebraic for $d = 6$, but its image under $E + 3$, where E is the Euler operator defined above, is algebraic and is given by the formula*

$$y_1 \;=\; \sqrt{\frac{\Delta_1 - \sqrt{\Delta_2}}{8\,\sigma_4\,\Delta_2}}\;, \tag{151}$$

where Δ_1 and Δ_2 have the same meaning as in Proposition 1.

We observe that the product of y_1 and the function y_0 in (149) equals $(4 - 2\sigma_1 - \Delta_0)/\Delta_2$, which is a rational function in the initial variables σ_p, so that the function fields generated by the two algebraic functions y_0 and y_1 coincide.

Continuing to higher values, we find that for $d = 8$ $(s = 2)$ we have to apply the differential operator $\frac{1}{6}(E+4)(E+5)(E+6) = \binom{E+6}{3}$ to y to achieve integrality and that it is then algebraic (this case will also be discussed further in §10.**B**), while for $d = 10$ $(s = 3)$ we need to apply $\binom{E+9}{5}$ to get integrality of the Taylor coefficients, with no smaller polynomial in E sufficing. These special cases suggest the following theorem, whose proof will occupy most of this section.

Theorem 13. *For $s \in \mathbb{Z}_{\ge 0}$ define $y_s = y_s(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ by*

$$y_s \;=\; \begin{cases} G^{(4)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4; 4) & \text{if } s = 0, \\ \binom{E+3s}{2s-1} G^{(4)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4; 2s+4) & \text{if } s \ge 1. \end{cases} \tag{152}$$

Then each y_s is algebraic of degree 4 over $\mathbf{K} = \mathbb{Q}(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. More precisely, each y_s belongs to the 2-dimensional subspace $\mathbf{K}y_0 + \mathbf{K}y_1 = \mathbf{K}(\sqrt{\Delta_2})\, y_0$ of the degree 4 extension $\mathbf{K}(y_0) = \mathbf{K}(y_1)$ of $\mathbf{K}$, where y_0 and y_1 are given by (149) and (151), respectively.

Remark. Applying a polynomial $P(E)$ in the Euler operator to the generating function $G^{(n)}$ is the same as multiplying the degree k terms by $P(k)$ for each $k \geq 0$, and hence is equivalent to working with a different initialization $G^{(1)}(\sigma_1) = \sum_{k\geq 0} P(k)(d-2)_k(\sigma_1/2)^k/k!$ of the $n = 1$ generating function. So our discussion can be summarized by saying that for $s \in \mathbb{Z}_{\geq 0}$ there is a normalization of the initial generating function $G^{(1)}$ which makes $G^{(n)}$ algebraic for all $n \leq 4$.

Finally, we mention an identity which is of interest in itself and will have an interpretation in terms of the generating series with respect to s studied in §10.**D**. For $s \in \mathbb{N}$, define

$$y_s^{\pm}(0) \;=\; y_s^{\pm}(\sigma_1, \sigma_2, \sigma_3, 0) \;=\; \binom{E + 3s}{2s - 1}\left(\frac{A_{\pm}^s}{\sqrt{\Delta_0^2 - 8\sigma_3}}\right), \tag{153}$$

where $A_{\pm} := \bigl(\Delta_0 \pm \sqrt{\Delta_0^2 - 8\sigma_3}\bigr)/2$, and

$$y_s^{\pm} \;=\; y_s^{\pm}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \;=\; \sum_{r=0}^{\infty} \mathcal{M}_4^r\bigl(y_s^{\pm}(\sigma_1, \sigma_2, \sigma_3, 0)\bigr)\, \frac{\sigma_4^r}{r!\,(s+\frac{1}{2})_r}, \tag{154}$$

so that (by (115)) $y_s = y_s^+$. The following identity will be proved in §10.**C**.

Theorem 14. *The functions y_s^+ and y_s^- coincide for $s \in \mathbb{N}$.*

B. Algebraicity and non-algebraicity proofs for $G^{(4)}$ when s is integral. In this section we give the proofs of Propositions 1 and 2 and begin the discussion of higher even values of d. These proofs involve the introduction of new coordinates for the ring $\mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \sigma_4]$ and thus suggest the problem, which may be an interesting direction for further research, of finding new coordinates for the whole ring $\mathbb{Q}[\sigma_1, \sigma_2, \ldots]$ that could help to clarify the structure of the generating function $G^{(n)}$ for higher values of n. (We return to this point briefly in §10.**F**.)

Motivated by the experimental formula (149) and a certain amount of trial and error, we make the (invertible) change of variables

$$\sigma_1 \;=\; 2u_1 + 2, \qquad \sigma_2 \;=\; u_2 + u_1^2 + 4u_1, \qquad \sigma_3 \;=\; u_3 + u_1u_2 + 2u_1^2. \tag{155}$$

In terms of these variables the operator $\mathcal{M}_4$ defined in (113) becomes simply

$$\mathcal{M}_4 \;=\; \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_1 \partial u_3}, \tag{156}$$

and, if we write g_3 and g_4 for the functions $G^{(3)}$ and $G^{(4)}$ expressed in the variables u_i and σ_4, then equations (101) and (115) take the form

$$g_3(u_1, u_2, u_3; d = 2s+4) \;=\; \frac{1}{\sqrt{u_2^2 - 8u_3}}\left(\frac{-4u_1 - u_2 + \sqrt{u_2^2 - 8u_3}}{2}\right)^{-s}, \tag{157}$$

$$g_4(u_1, u_2, u_3, \sigma_4; d) \;=\; \sum_{r=0}^{\infty}\left(\frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_1 \partial u_3}\right)^r \bigl(g_3(u_1, u_2, u_3; d)\bigr)\, \frac{\sigma_4^r}{r!\,(s+\frac{1}{2})_r}. \tag{158}$$

If $s=0$, then g_3 is independent of u_1 and (158) simplifies to $g_4 = \sum_{r=0}^{\infty} \frac{\partial^{2r} g_3}{\partial u_2^{2r}} \frac{\sigma_4^r}{r!\,(1/2)_r}$. This formula makes it very natural to make the further (non-invertible) change of variables

$$\sigma_4 \;=\; X^2/4\,, \tag{159}$$

in which case it can be written using Taylor's formula and the duplication formula for the gamma function as

$$\begin{aligned} g_4(u_1,u_2,u_3,X^2/4) \;&=\; \sum_{r=0}^{\infty} \frac{\partial^{2r} g_3(u_1,u_2,u_3)}{\partial u_2^{2r}} \frac{X^{2r}}{(2r)!} \;=\; \frac{g_3(u_1,u_2+X,u_3) \,+\, g_3(u_1,u_2-X,u_3)}{2} \\ &=\; \frac{1}{2}\left(\frac{1}{\sqrt{(u_2+X)^2-8u_3}} \,+\, \frac{1}{\sqrt{(u_2-X)^2-8u_3}}\right). \end{aligned} \tag{160}$$

Squaring this gives

$$g_4^2 \;=\; \frac{u_2^2+X^2-8u_3 \,+\, \sqrt{(u_2^2+X^2-8u_3)^2-4u_2^2X^2}}{2\big((u_2^2+X^2-8u_3)^2-4u_2^2X^2\big)} \;=\; \frac{\Delta_1+\sqrt{\Delta_2}}{2\Delta_2}$$

with Δ_1 and Δ_2 as in Proposition 1, completing the proof of equation (149).

The case $s=1$ ($d=6$) is only slightly more complicated. Write y^0 for the expression (101) with $s=1$ and y_1^0 for its image under $E+3$, where E denotes the Euler operator $\sum p\,\sigma_p\,\partial/\partial\sigma_p$. Because E is a derivation and $E(\sigma_3)=3\sigma_3$, we find

$$\begin{aligned} y_1^0 \;&=\; \big(E+3\big)\left[\frac{1}{4\sigma_3}\left(\frac{\Delta_0}{\sqrt{\Delta_0^2-8\sigma_3}}-1\right)\right] \;=\; \frac{1}{4\sigma_3}\,E\left(\frac{\Delta_0}{\sqrt{\Delta_0^2-8\sigma_3}}\right) \\ &=\; \frac{-2E(\Delta_0)+3\Delta_0}{(\Delta_0^2-8\sigma_3)^{3/2}} \;=\; \frac{u_2}{(u_2^2-8u_3)^{3/2}} \;=\; \frac{\partial}{\partial u_2}\left(\frac{-1}{\sqrt{u_2^2-8u_3}}\right), \end{aligned}$$

which (a) is again independent of u_1, and (b) can be expressed as a u_2-derivative. Now the relationship between $y_1 = (E+3)G^{(4)}$ and y_1^0 is the same as the relationship (115) between $y = G^{(4)}$ and y^0, because the operator $\sigma_4\mathcal{M}_4$ (or $\sigma_n\mathcal{M}_n$ for any n) is homogeneous of degree 0 and therefore commutes with E. We therefore get

$$\begin{aligned} y_1 \;&=\; \sum_{r=0}^{\infty} \frac{\partial^{2r}}{\partial u_2^{2r}}\big(y_1^0\big) \frac{\sigma_4^r}{r!\,(3/2)_r} \;=\; \frac{1}{X}\sum_{r=0}^{\infty} \frac{\partial^{2r+1}}{\partial u_2^{2r+1}}\left(\frac{-1}{\sqrt{u_2^2-8u_3}}\right)\frac{X^{2r+1}}{(2r+1)!} \\ &=\; \frac{1}{2X}\left(\frac{1}{\sqrt{(u_2-X)^2-8u_3}} \,-\, \frac{1}{\sqrt{(u_2+X)^2-8u_3}}\right), \end{aligned} \tag{161}$$

which is equivalent to formula (151) by a calculation similar to the one for $s=0$.

This completes the proof of the part of Proposition 2 saying that y_1 as an algebraic function. For the assertion that y itself is not algebraic in general, it suffices to prove the non-algebraicity of the specialization to $\sigma_1=\sigma_2=\sigma_3=0$, i.e., to prove that the integral of the function (150) divided by $x^{1/4}$ is not algebraic. We already know "morally" that it is not, because its first Taylor coefficients contain many different primes in the denominator, as was mentioned in the previous subsection, and this is also confirmed by asking a computer system like "Mathematica" to carry out the integral, obtaining as output the statement that there is no algebraic formula. A formal proof is as follows. We want to show that the integral $w = \int u\,dt$, where

$$u \;=\; \frac{1}{2\sqrt{P_-}} - \frac{1}{2\sqrt{P_+}}\,, \quad \text{with} \quad P_\pm \;=\; P_\pm(t) \;=\; 1 \pm 6t^2 + t^4\,,$$

is not algebraic. By a theorem of Liouville, if w is algebraic it must belong to the function field $\mathbb{Q}(t,u) = \mathbb{Q}(t) + \mathbb{Q}(t)\sqrt{P_+} + \mathbb{Q}(t)\sqrt{P_-} + \mathbb{Q}(t)\sqrt{P_+P_-}$. So we can assume that $w = a + b\sqrt{P_+} + c\sqrt{P_-} + d\sqrt{P_+P_-}$ where a, b, c, d are rational functions of t. From $w' = u$, we get $2b'P_- + bP'_- = 1$, $2c'P_+ + cP'_+ = -1$, and $a = d = 0$. By comparing the Laurent expansions around each $t = \alpha \in \mathbb{P}^1(\mathbb{C})$ (distinguishing the cases $P_-(\alpha) = 0$, $P_-(\alpha) \neq 0$ and $\alpha = \infty$), we see that b has no poles on the complex projective line and hence is a constant, which is obviously a contradiction. This completes the proof of Proposition 2. $\square$

Remark. We can recast the calculations just given for $s = 0$ and $s = 1$ into a slightly different and more suggestive form by using the differential equation $\mathcal{L}_4(G^{(4)}) = 0$ directly rather than the expansion of $G^{(4)}$ as an infinite linear combination of terms $\mathcal{M}_4^r(G^{(3)})\sigma_4^r$. For instance, if $s = 0$ then we have

$$\frac{1}{2}\,\mathcal{L}_4 \;=\; \frac{1}{2}\,\frac{\partial}{\partial\sigma_4} + \sigma_4\,\frac{\partial^2}{\partial\sigma_4^2} - \mathcal{M}_4 \;=\; \frac{\partial^2}{\partial X^2} - \mathcal{M}_4 \;=\; \Big(\frac{\partial}{\partial X} + \frac{\partial}{\partial u_2}\Big)\Big(\frac{\partial}{\partial X} - \frac{\partial}{\partial u_2}\Big) - \frac{\partial^2}{\partial u_1 \partial u_3}\,,$$

so the unique function of u_1, u_2, u_3 and X that is annihilated by $\mathcal{L}_4$, even in X, and equal to a given function $h(u_2,u_3)$ when $X = 0$ is simply $\frac{1}{2}\big(h(u_2+X,u_3) + h(u_2-X,u_3)\big)$, and we recover (160) without any need for using the infinite sum of differential operators occurring in eq. (115) and with almost no computation at all. Similarly, if $s = 1$ then we have

$$\frac{1}{2}\,\mathcal{L}_4 \;=\; X^{-1}\frac{\partial^2}{\partial X^2}X - \mathcal{M}_4 \;=\; X^{-1}\Big(\Big(\frac{\partial}{\partial X} + \frac{\partial}{\partial u_2}\Big)\Big(\frac{\partial}{\partial X} - \frac{\partial}{\partial u_2}\Big) - \frac{\partial^2}{\partial u_1\partial u_3}\Big)X\,,$$

so the unique function of u_1, u_2, u_3 and X that is annihilated by $\mathcal{L}_4$, even in X, and equal to $\partial h(u_2,u_3)/\partial u_2$ when $X = 0$ is simply $\frac{1}{2X}\big(h(u_2+X,u_3) - h(u_2-X,u_3)\big)$, and we recover (161) easily.

This completes our discussion of the cases $s = 0$ and $s = 1$. To understand how the general case should go, we look at the next case $s = 2$ ($d = 8$). Here by direct calculation we find

$$\begin{aligned} y_2^0 \;&:=\; \frac{1}{6}(E+4)(E+5)(E+6)G^{(3)}(\sigma_1,\sigma_2,\sigma_3;8) \\ &=\; -2\,\frac{3u_2(u_2^2-8u_3) + u_1u_2(u_2^2+12u_3) + 4u_3(u_2^2+2u_3)}{(u_2^2-8u_3)^{7/2}} \\ &=\; \frac{1}{6}\left(3\frac{\partial^2}{\partial u_2\partial u_3} + 2u_1\frac{\partial^3}{\partial u_2^3}\right)\left(\frac{1}{\sqrt{u_2^2-8u_3}}\right) + \frac{1}{12}\frac{\partial^3}{\partial u_2^3}\left(\frac{u_2}{\sqrt{u_2^2-8u_3}}\right) \end{aligned}$$

and hence

$$\begin{aligned} y_2 \;&=\; \sum_{r=0}^{\infty}\mathcal{M}_4^r(y_2^0)\,\frac{\sigma_4^r}{r!\,(5/2)_r} \;=\; \sum_{r=0}^{\infty}\Big(\frac{\partial^{2r}}{\partial u_2^{2r}} + r\,\frac{\partial^{2r}}{\partial u_1\partial u_2^{2r-2}\partial u_3} + \mathrm{O}\Big(\frac{\partial^2}{\partial u_1^2}\Big)\Big)(y_2^0)\,\frac{6(r+1)X^{2r}}{(2r+3)!} \\ &=\; \sum_{r=0}^{\infty}(r+1)\Big((2r+3)\frac{\partial^{2r+2}}{\partial u_2^{2r+1}\partial u_3} + 2u_1\frac{\partial^{2r+3}}{\partial u_2^{2r+3}}\Big)\left(\frac{1}{\sqrt{u_2^2-8u_3}}\right)\frac{X^{2r}}{(2r+3)!} \\ &\qquad + \frac{1}{2}\sum_{r=0}^{\infty}(r+1)\frac{\partial^{2r+3}}{\partial u_2^{2r+3}}\left(\frac{u_2}{\sqrt{u_2^2-8u_3}}\right)\frac{X^{2r}}{(2r+3)!} \\ &=\; \frac{2}{X}\sum_{r=0}^{\infty}\frac{\partial^{2r+1}}{\partial u_2^{2r+1}}\left(\frac{1}{(u_2^2-8u_3)^{3/2}}\right)\frac{X^{2r+1}}{(2r+1)!} \\ &\qquad + \frac{1}{4X}\,\frac{\partial}{\partial X}\,\frac{1}{X}\sum_{r=-1}^{\infty}\frac{\partial^{2r+3}}{\partial u_2^{2r+3}}\left(\frac{u_2+4u_1}{\sqrt{u_2^2-8u_3}}\right)\frac{X^{2r+3}}{(2r+3)!} \end{aligned}$$

$$= \; -\frac{1}{X}\left[\frac{1}{\big((u_2-X)^2-8u_3\big)^{3/2}} \;-\; \frac{1}{\big((u_2+X)^2-8u_3\big)^{3/2}}\right]$$
$$-\;\frac{1}{8X}\,\frac{\partial}{\partial X}\,\frac{1}{X}\left[\frac{u_2-X+4u_1}{\sqrt{(u_2-X)^2-8u_3}} \;-\; \frac{u_2+X+4u_1}{\sqrt{(u_2+X)^2-8u_3}}\right].$$

This is an explicit algebraic function of $\sigma_1,\ldots,\sigma_4$ and we can check directly that it belongs to the function field generated over $\mathbf{K}$ by the function y_0 or y_1 or, more precisely, to the subspace of this field anti-invariant under its Galois automorphism over its subfield $\mathbf{K}(\sqrt{\Delta_2})$. After a good deal of computation we find that y_2 is given with respect to the basis y_0, y_1 of this space by

$$y_2 \;=\; \frac{u_2\Delta_0\Delta_1-4\sigma_4\Delta_1-2u_2\widetilde{\Delta}_0}{16\sigma_4\Delta_2}\,y_0 \;-\; \frac{\Delta_0\Delta_1^2-8u_2\sigma_4(3u_2\Delta_0-4\sigma_4)+\widetilde{\Delta}_0\Delta_1}{16\sigma_4\Delta_2}\,y_1\,,$$

with $\widetilde{\Delta}_0 = 4\sigma_4(\Delta_0-u_2+8)$, confirming Theorem 13 in this case.

Looking at the three cases treated so far, we see that all we needed in these cases was that the initial values $y_s(0)=y_s(\sigma_1,\sigma_2,\sigma_3,0)$ of the functions defined in (152) had the form

$$y_0(0) \;=\; h_{0,0}\,,\qquad y_1(0) \;=\; \partial_2 h_{1,1}\,,\qquad y_2(0) \;=\; \big(\tfrac{3}{2}\partial_3+u_1\partial_2^2\big)\partial_2 h_{2,1} \;+\; \partial_2^3 h_{2,2}$$

for some functions $h_{s,i}$ depending on u_2 and u_3 but not on u_1, where $\partial_p = \partial/\partial u_p$. The same thing happens for the next case $s=3$, the required decomposition this time having the form

$$y_3(0) \;=\; \big(\tfrac{15}{4}\partial_3^2+5u_1\partial_2^2\partial_3+u_1^2\partial_2^4\big)\partial_2 h_{3,1} \;+\; \big(\tfrac{5}{2}\partial_3+u_1\partial_2^2\big)\partial_2^3 h_{3,2} \;+\; \partial_2^5 h_{3,3}$$

for appropriately chosen functions $h_{3,i}$ of u_2 and u_3. Continuing to higher s, we find each time that $y_s(0)$ has a decomposition

$$y_s(0) \;=\; \sum_{i=1}^{s} M_{s,i}\,h_{s,i} \qquad (s\ge 1) \tag{162}$$

where $M_{s,i}$ $(1\le i\le s)$ are the differential operators

$$M_{s,i} \;=\; \sum_{j=i}^{s}\binom{s-\frac12}{s-j}\frac{u_1^{j-i}}{(j-i)!}\,\partial_3^{s-j}\,\partial_2^{2j-1} \tag{163}$$

and $h_{s,i}$ $(1\le i\le s)$ the functions given by

$$h_{s,i} \;=\; \frac{(-1)^s 2^{s-i}(s-1)!}{(2s-1)!\,(i-1)!}\,\frac{1}{\sqrt{\Delta_0^2-8\sigma_3}}\left(\frac{u_2-\sqrt{\Delta_0^2-8\sigma_3}}{2}\right)^{i-1}. \tag{164}$$

We verified (162) by computer up to $s=16$, and will prove it for all $s\in\mathbb{N}$ in the next two subsections. Here we show that it implies the algebraicity assertion in Theorem 13. This is a special case of the following more general statement.

Proposition 4. *Let $M_{s,i}$ $(1\le i\le s)$ be the operator defined in* (163). *Then the function*

$$\sum_{r=0}^{\infty}\frac{\sigma_4^r}{r!\,(s+\frac12)_r}\,M_4^r\big(M_{s,i}h\big)$$

is algebraic for any algebraic function h of u_2 and u_3.

Proof. Since h is independent of u_1, we have (using (156)) the identity

$$\begin{aligned}
\mathcal{M}_4^r\big(M_{s,i}h\big) &= \Big(\sum_{l\ge 0}\binom{r}{l}\partial_2^{2r-2l}\partial_1^l\partial_3^l\Big)\Big(\sum_{k=i}^{s}\binom{s-\frac12}{s-k}\frac{u_1^{k-i}}{(k-i)!}\,\partial_3^{s-k}\,\partial_2^{2k-1}\Big)\,h \\
&= \sum_{j=i}^{s}\Big(\sum_{l=0}^{s-j}\binom{r}{l}\binom{s-\frac12}{s-j-l}\Big)\,\frac{u_1^{j-i}}{(j-i)!}\,\partial_3^{s-j}\,\partial_2^{2r+2j-1}h \\
&= \sum_{j=i}^{s}\binom{r+s-\frac12}{s-j}\frac{u_1^{j-i}}{(j-i)!}\,\partial_3^{s-j}\,\partial_2^{2r+2j-1}h\,. \qquad (165)
\end{aligned}$$

It follows that

$$\sum_{r=0}^{\infty}\frac{\sigma_4^r}{r!\,(s+\frac12)_r}\,\mathcal{M}_4^r\big(M_{s,i}h\big) \;=\; \sum_{j=i}^{s}\binom{s-\frac12}{s-j}\frac{u_1^{j-i}}{(j-i)!}\,\partial_3^{s-j}\,\Big(\sum_{r=0}^{\infty}\frac{\sigma_4^r}{r!\,(j+\frac12)_r}\,\partial_2^{2r+2j-1}h\Big)\,,$$

and this is algebraic because for each $j\ge 1$ we have the identity

$$\begin{aligned}
\sum_{r=0}^{\infty}\frac{\sigma_4^r\,\partial_2^{2r+2j-1}h}{r!\,(j+\frac12)_r} &= (3/2)_{j-1}\,\frac{\partial^{j-1}}{\partial\sigma_4^{j-1}}\sum_{r=1-j}^{\infty}\frac{(4\sigma_4)^{r+j-1}\,\partial_2^{2r+2j-1}h}{(2r+2j-1)!} \\
&= (3/2)_{j-1}\,\frac{\partial^{j-1}}{\partial\sigma_4^{j-1}}\Big(\frac{h(u_2+X,u_3)\,-\,h(u_2-X,u_3)}{2X}\Big) \qquad (166)
\end{aligned}$$

with $X=\sqrt{4\sigma_4}$ as in (159). $\square$

Remark. There is an interesting relation between the formalism here and certain differential operators from smooth functions of one variable to functions of two variables introduced in [14]. These operators[6] are defined by

$$D_nF(x,y) \;=\; \frac{(-1)^n}{n!}\sum_{i=0}^{n}\frac{(2n-i)!}{i!\,(n-i)!}\,\frac{(-1)^iF^{(i)}(x)\,-\,F^{(i)}(y)}{(x\,-\,y)^{2n-i+1}} \qquad (n=0,\,1,\,2,\dots)\,, \qquad (167)$$

the first three cases being

$$\begin{aligned}
D_0F(x,y) &= \frac{F(x)-F(y)}{x\,-\,y}\,, \\
D_1F(x,y) &= \frac{F'(x)+F'(y)}{(x\,-\,y)^2}\;-\;2\,\frac{F(x)-F(y)}{(x\,-\,y)^3}\,, \\
D_2F(x,y) &= \frac12\,\frac{F''(x)-F''(y)}{(x\,-\,y)^3}\;-\;3\,\frac{F'(x)+F'(y)}{(x\,-\,y)^4}+6\,\frac{F(x)-F(y)}{(x\,-\,y)^5}\,.
\end{aligned}$$

Though $D_nF(x,y)$ has an apparent pole of order $2n+1$ along the diagonal $x=y$, it is in fact smooth there. Other attractive properties of D_n proved in [14] are that it annihilates polynomials of degree $\le 2n$ (as one can verify by hand in the above formulas for $0\le n\le 2$), that it is given on monomials x^m of degree $\ge 2n+1$ by

$$D_n\big(x\mapsto x^m\big)(x,y) \;=\; \sum_{r+s=m-1}\binom{r}{n}\binom{s}{n}\,x^{r-n}y^{s-n}\,,$$

[6] with the slightly different notation and normalization $(\mathcal{D}_nF)(x,y)=n!\,(x-y)^{n+1}(D_nF)(x,y)$ in [14]

and that it satisfies the $SL(2)$-equivariance property $D_n(F|_{-2n}g) = D_n(F)|_{-n-1,-n-1}g$ for all $g \in SL(2,\mathbb{C})$, where $|_k$ (and similarly $|_{k,k'}$) denotes the usual "slash operator" $\big(f|_k\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big)\big)(x) = (cx+d)^{-k} f\big(\frac{ax+b}{cx+d}\big)$. All of these follow from the integral representation ([14], Prop. 4(v))

$$D_nF(x,y) \;=\; \frac{1}{n!^2\,(x-y)^{2n+1}} \int_y^x (z-y)^n (x-z)^n F^{(2n+1)}(z)\,dz$$
$$=\; \frac{1}{n!^2} \int_0^1 t^n\,(1-t)^n\,F^{(2n+1)}\big(tx+(1-t)y\big)\,dt\,.$$

We can also rewrite $D_nF(x,y)$ in terms of the derivatives of F only at $(x+y)/2$, though at the expense of changing the finite sum (167) to an infinite one. To do this, we combine the integral representation just given with the usual Taylor expansion to find

$$(D_nF)(u+X,u-X) \;=\; \frac{1}{2^{2n+1}n!^2} \int_{-1}^1 (1-t^2)^n\,F^{(2n+1)}(u+tX)\,dt$$
$$=\; \frac{1}{2^{2n+1}n!^2} \sum_{r=0}^{\infty} \biggl(\int_{-1}^1 t^{2r}(1-t^2)^n dt\biggr)\,F^{(2r+2n+1)}(u)\,\frac{X^{2r}}{(2r)!}\,.$$

Computing the coefficient in parentheses by the beta integral and the duplication formula as

$$\int_{-1}^1 t^{2r}(1-t^2)^n dt \;=\; \int_0^1 x^{r-\frac12}(1-x)^n dx \;=\; \frac{n!\,\Gamma(r+\frac12)}{\Gamma(n+r+\frac32)} \;=\; \frac{2^{2n+1}n!^2}{(2n+1)!}\,\frac{(2r)!}{2^{2r}r!\,(n+\frac32)_r}\,,$$

we find the following formula for the Taylor expansion of $(D_nF)(u+X,u-X)$:

$$(D_nF)(u+X,u-X) \;=\; \frac{1}{(2n+1)!} \sum_{r=0}^{\infty} F^{(2r+2n+1)}(u)\,\frac{(X/2)^{2r}}{r!\,(n+\frac32)_r}\,. \tag{168}$$

Together with (167) this gives another algebraic expression for the sum in eq. (166).

C. Properties of the generating function for fixed integral values of s. In the last subsection we reduced the proof of Theorem 13 to the proof of the experimentally discovered identity (162). In this subsection we will recast this identity in a nicer form and investigate some of its consequences, and will prove Theorem 14. The proof of the identity itself will follow from the generating function identities proved in §10.**D**.

Throughout this subsection we take as independent variables the variables u_i introduced in (155), and set $\partial_i = \partial/\partial u_i$. We also set

$$\Delta_0 \;=\; (1-\sigma_1/2)^2 - \sigma_2 \;=\; -u_2 - 4u_1\,, \qquad \Delta_1 \;=\; \Delta_0^2 - 8\sigma_3 \;=\; u_2^2 - 8u_3\,,$$

corresponding to the definitions in Propositions 1 and 2 at $\sigma_4 = 0$. With these notations the quantity $A_\pm$ occurring in (153) is given by $A_\pm = \frac{\Delta_0 \pm \sqrt{\Delta_1}}{2}$.

The first observation is that for a function h that does not depend on u_1 the operator (163) can be written in the more attractive form

$$M_{s,i}(h) \;=\; \mathcal{M}_4^{s-1/2}\biggl(\frac{u_1^{s-i}}{(s-i)!}\,h\biggr)\,. \tag{169}$$

Here the fractional power $\mathcal{M}_4^{s-1/2}$ of the differential operator (156) is not defined as an operator, but makes sense on functions of u_1, u_2, u_3 that are polynomials of degree $\le s-1$ with respect to u_1 if we use the binomial theorem to *define*

$$\mathcal{M}_4^{s-1/2} \;=\; \sum_{l=0}^{s-1} \binom{s-1/2}{l} \partial_2^{2s-2l-1}(\partial_1\partial_3)^l \tag{170}$$

for such functions. The observation (169), whose proof is immediate from (170) and the definition of $M_{s,i}$, also clarifies the identity (166) appearing in the proof of Proposition 4 above, since the left-hand side of this identity can be written as $\mathcal{M}_4^{r+s-1/2}\bigl(u_1^{s-i}h/(s-i)!\bigr)$ and the fractional power $\mathcal{M}_4^{r+s-1/2}$ then expanded by the binomial theorem in the same way as was done in (170).

Substituting (169) and (164) into (162) and using the binomial theorem, we find that the expression appearing on the right-hand side of (162) is simply $Y_s^+(0)$, where we define

$$Y_s^\pm(0) \;=\; Y_s^\pm(\sigma_1,\sigma_2,\sigma_3,0) \;=\; -\frac{1}{(2s-1)!}\,\mathcal{M}_4^{s-1/2}\left(\frac{A_\pm^{s-1}}{\sqrt{\Delta_1}}\right). \tag{171}$$

We will see in a moment that $Y_s^+(0) = Y_s^-(0)$, corresponding to (but much easier than) the corresponding assertion for $y_s^\pm(0)$ in Theorem 14. If, in analogy with (154), we also define

$$Y_s^\pm \;=\; Y_s^\pm(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \;=\; \sum_{r=0}^{\infty} \mathcal{M}_4^r\bigl(Y_s^\pm(0)\bigr)\,\frac{\sigma_4^r}{r!\,(s+\frac12)_r}\,, \tag{172}$$

then the identities to be proved are

$$y_s^+(\sigma_4) \;=\; y_s^-(\sigma_4) \;=\; Y_s^+(\sigma_4) \;=\; Y_s^-(\sigma_4)\,, \tag{173}$$

where we have suppressed the dependence on σ_1, σ_2 and σ_3. It clearly suffices to prove this for $\sigma_4 = 0$, since each of the four functions in (173) is obtained from its value at $\sigma_4 = 0$ by applying the operator $\sum_{r\ge0}(\sigma_4^r/r!(s+\frac12)_r)M_4^r$.

Before proceeding we write the formula for $Y_s^\pm(\sigma_4)$ in a more explicit form. From the definition (170) (with s replaced by $r+s$) and the formula $\partial_1(A^\pm) = -2$ we get

$$\begin{aligned}\mathcal{M}_4^{r+s-\frac12}\left(\frac{A_\pm^{s-1}}{\sqrt{\Delta_1}}\right) \;&=\; \sum_{l=0}^{s-1}\binom{r+s-\frac12}{l}\partial_2^{2r+2s-2l-1}\partial_1^l\partial_3^l\left(\frac{A_\pm^{s-1}}{\sqrt{\Delta_1}}\right)\\ &=\; \sum_{l+n=s-1}\frac{(-1/2)^l(2s-1)!\,(s+\frac12)_r}{l!\,(2n+1)!\,(n+\frac32)_r}\,\partial_3^l\partial_2^{2r+2n+1}\left(\frac{A_\pm^n}{\sqrt{\Delta_1}}\right)\end{aligned}$$

and hence

$$Y_s^\pm(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \;=\; -\sum_{l+n=s-1}\frac{(-1/2)^l}{l!}\,\partial_3^l\left[\frac{1}{(2n+1)!}\sum_{r=0}^{\infty}\frac{\sigma_4^r}{r!\,(n+\frac32)_r}\,\partial_2^{2r+2n+1}\left(\frac{A_\pm^n}{\sqrt{\Delta_1}}\right)\right].$$

We can now use (168) to write the inner sum as $D_n\bigl(A_\pm^n/\sqrt{\Delta_1}\bigr)(u_2+X,u_2-X)$, giving an explicit expression for $Y_s^\pm$ as an algebraic function of its arguments and showing that it belongs to the space described in Theorem 13. We also see immediately that $Y_s^+ = Y_s^-$, because $(A_+^n - A_-^n)/\sqrt{\Delta_1}$ is a polynomial of degree at most $n-1$ in u_2 and hence is annihilated by

$\partial_2^{2r+2n+1}$ (or even by ∂_2^n). Finally, we can use simple binomial coefficient identities to rewrite the formula in a slightly different form, which will be used in §10**D**:

$$Y_s^{\pm}(\sigma_1,\sigma_2,\sigma_3,\sigma_4) \;=\; -\sum_{l+n=s-1} \frac{(-1/2)^l}{l!}\,\partial_3^l\left[\sum_{r=0}^{\infty}\frac{\binom{n+r}{r}\,(4\sigma_4)^r}{(2r+2n+1)!}\,\partial_2^{2r+2n+1}\left(\frac{A_{\pm}^n}{\sqrt{\Delta_1}}\right)\right]. \qquad (174)$$

We end the subsection by giving the proof of Theorem 14. As already noted, it suffices to show that $y_s^+(0)=y_s^-(0)$. We observe first that the form of the functions $y_s^{\pm}(0)$ can be simplified:

$$y_s^{\pm}(0) \;=\; \binom{E+3s}{2s-1}\frac{A_{\pm}^{-s}}{\sqrt{\Delta_1}} \;=\; \binom{E+3s}{2s-1}(2\sigma_3)^{-s}\,\frac{A_{\mp}^s}{\sqrt{\Delta_1}} \;=\; \frac{1}{(2\sigma_3)^s}\binom{E}{2s-1}\frac{A_{\mp}^s}{\sqrt{\Delta_1}}\,, \qquad (175)$$

where we have used $A_+A_-=2\sigma_3$ and $E(\sigma_3)=3\sigma_3$. The proof of the desired identity $y_s^+(0)=y_s^-(0)$ now depends on the following proposition.

Proposition 5. *Define polynomials* $B_k^{(s)}=B_k^{(s)}(\sigma_1,\sigma_2,\sigma_3)$ $(k\ge 0)$ *by the generating function*

$$\mathcal{B}^{(s)}(x) \;=\; \sum_{k=0}^{\infty} B_k^{(s)}x^k \;=\; \frac{1}{\sqrt{\widetilde{\Delta}_0^2+8\sigma_3x^3(1-x)}}\left(\frac{\widetilde{\Delta}_0+\sqrt{\widetilde{\Delta}_0^2\,+\,8\sigma_3x^3(1-x)}}{2}\right)^s, \qquad (176)$$

where $\widetilde{\Delta}_0=\widetilde{\Delta}_0(x)=1+2u_1x+\Delta_0x^2$. *Then we have the identities*

$$(E-k)B_k^{(s)}=(2s-k-1)B_{k-1}^{(s)} \qquad (177)$$

for all $k\ge 0$ *(with* $B_{-1}^{(s)}:=0$*) and*

$$B_{2s-2}^{(s)}=\frac{A_+^s-A_-^s}{\sqrt{\Delta_1}} \qquad (178)$$

Since the definition (176), discovered only after considerable experimentation, is not very enlightening, we give a few examples of the functions $B_k^{(s)}$ and properties (177) and (178). From $\mathcal{B}^{(s)}(x)\equiv R^{s-1} \pmod{x^3}$ we see that the first three values of $B_k^{(s)}$ are given by

$$B_0^{(s)}=1\,,\qquad B_1^{(s)}=2(s-1)u_1\,,\qquad B_2^{(s)}=2(s-1)(s-2)u_1^2+(s-1)\Delta_0\,.$$

From this, using $E(u_1)=u_1+1$ and $E(\Delta_0)=2\Delta_0+2u_1$, we find

$$\begin{aligned}E(B_0^{(s)}) \;&=\; E(1) \;=\; 0\,,\\(E-1)(B_1^{(s)}) \;&=\; 2(s-1)(E-1)(u_1) \;=\; (2s-2)B_0^{(s)},\\(E-2)(B_2^{(s)}) \;&=\; 2(s-1)(s-2)(2u_1)+(s-1)(2u_1) \;=\; (2s-3)B_1^{(s)}\,,\end{aligned}$$

verifying (177) for these three values, while for (178) we have

$$B_0 \;=\; 1 \;=\; \frac{A_+-A_-}{\sqrt{\Delta_1}}\,,\qquad B_2^{(2)} \;=\; \Delta_0 \;=\; \frac{A_+^2-A_-^2}{\sqrt{\Delta_1}}\,.$$

The proof that Proposition 5 implies Theorem 14 is almost immediate. Equation (177) and induction on n give $\binom{E-2s+1+n}{n}B_{2s-2}^{(s)}=B_{2s-2-n}^{(s)}$ for all $n\ge 0$. In particular, $\binom{E}{2s-1}B_{2s-2}^{(s)}$

vanishes, and in view of (178) and (175) this proves the theorem. It remains to prove the proposition.

Proof of (177): We have to show that

$$E(\mathcal{B}^{(s)}) \;=\; x\Big[(1-x)\frac{\partial\mathcal{B}^{(s)}}{\partial x} + (2s-2)\mathcal{B}^{(s)}\Big]\,.$$

Define $\widetilde{E} = \dfrac{1}{x}E - (1-x)\frac{\partial}{\partial x}$, which is still a derivation. Also set $Z = \sigma_3 x^3(1-x)$, so that

$$\mathcal{B}^{(s)} = \frac{1}{\sqrt{\widetilde{\Delta}_0^2 + 8Z}}\Big(\frac{\widetilde{\Delta}_0 + \sqrt{\widetilde{\Delta}_0^2 + 8Z}}{2}\Big)^s,$$

Then

$$\begin{aligned}\widetilde{E}(Z) &= \frac{3Z}{x} - (1-x)\sigma_3(3x^2 - 4x^3) = 4Z\\ \widetilde{E}(\widetilde{\Delta}_0) &= 2(u_1+1) + 2(u_1+\Delta_0)x - (1-x)(2u_1 + 2\Delta_0 x) = 2\widetilde{\Delta}_0\,,\end{aligned}$$

In other words, $\widetilde{\Delta}_0$ and Z are of degree 2 and 4 with respect to the modified Euler operator $\widetilde{E}$, so $\widetilde{E}(\mathcal{B}^{(s)}) = 2(s-1)\mathcal{B}^{(s)}$ as desired.

Proof of (178): Since $\widetilde{\Delta}_0 - \sqrt{\widetilde{\Delta}_0^2 + 8Z} = O(x^3)$, we can replace $\mathcal{B}^{(s)}$ by

$$\widetilde{\mathcal{B}}^{(s)} \;=\; \frac{1}{\sqrt{\widetilde{\Delta}_0^2 + 8Z}}\Big[\Big(\frac{\widetilde{\Delta}_0 + \sqrt{\widetilde{\Delta}_0^2 + 8Z}}{2}\Big)^s + \Big(\frac{\widetilde{\Delta}_0 - \sqrt{\widetilde{\Delta}_0^2 + 8Z}}{2}\Big)^s\Big]$$

without changing the value of the coefficient of x^{2s-2}. But $\widetilde{\mathcal{B}}^{(s)}$ is a polynomial, given explicitly by

$$\begin{aligned}2^{s-1}\widetilde{\mathcal{B}}^{(s)} &= \sum_{0\le 2j\le s-1}\binom{s}{2j+1}\widetilde{\Delta}_0^{s-2j-1}(\widetilde{\Delta}_0^2 + 8Z)^j\\ &= \sum_{0\le 2j\le s-1}\binom{s}{2j+1}(\Delta_0 x^2 + \cdots)^{s-2j-1}((\Delta_0^2 - 8\sigma_3)x^4 + \cdots)^j\\ &= x^{2s-2}\,\frac{(\Delta_0 + \sqrt{\Delta_1})^s - (\Delta_0 - \sqrt{\Delta_1})^s}{2\sqrt{\Delta_1}} + \cdots\,,\end{aligned}$$

where "$\cdots$" denotes lower order terms as $x \to \infty$. The result follows.

Remark. If we replace the expressions $\Delta_0 = (1-\sigma_1/2)^2 - \sigma_2$ and $\Delta_1 = \Delta_0^2 - 8\sigma_3$ by the corresponding homogeneous forms

$$\Delta_0(u,v) \;=\; (u - \sigma_1 v/2)^2 - \sigma_2 v^2\,, \qquad \Delta_1(u,v) = \Delta_0(u,v)^2 - 8\sigma_3 u v^3$$

of degrees 2 and 4, respectively, in u and v, then $\widetilde{\Delta}_0 = \Delta_0(1-x,x)$, $\widetilde{\Delta}_0^2 + 8Z = \Delta_1(1-x,x)$, and the strange-looking "modified Euler operator" $\widetilde{E}$ used in the proof above is simply the ordinary Euler operator $u\,\partial_u + v\,\partial_v$ with respect to the homogeneous variables u and v.

D. Generating function with respect to s. Since we have shown that both $y_s^\pm$ and $Y_s^\pm$ are independent of the choice of sign, we can write them simply as y_s and Y_s, respectively, and to obtain (173) it will suffice to prove that $Y_s(0) = y_s(0)$. We will not do this directly, but rather by proving (for $\sigma_4 = 0$) the equality of the two generating functions

$$\mathbf{Y}(T) \;=\; \mathbf{Y}(\sigma_1,\sigma_2,\sigma_3,\sigma_4;T) \;=\; \sum_{s=1}^{\infty} Y_s(\sigma_1,\sigma_2,\sigma_3,\sigma_4)\, T^{s-1}\,, \tag{179}$$

$$\mathbf{y}(T) \;=\; \mathbf{y}(\sigma_1,\sigma_2,\sigma_3,\sigma_4;T) \;=\; \sum_{s=1}^{\infty} y_s(\sigma_1,\sigma_2,\sigma_3,\sigma_4)\, T^{s-1} \tag{180}$$

with respect to the index s. This approach gives the additional dividend that the generating function $\mathbf{y}(T)$ turns out itself to be an algebraic function. This of course implies the algebraicity of each of its Taylor coefficients y_s, but it does not make superfluous the algebraic calculations of the last two subsections because, rather surprisingly, the degree *drops* under differentiation in T and specialization to $T=0$: the function $\mathbf{y}(T)$ will turn out to be algebraic of degree 6 over $\mathbb{Q}(\sigma_1,\ldots,\sigma_4,T)$, whereas Theorem 13 says that each individual $y_s(T)$ satisfies an algebraic equation of degree only 4.

Equation (174) simplifies considerably when we substitute it into the generating series (179), and we obtain:

$$\begin{aligned}
\mathbf{Y}(T) \;&=\; \sum_{l=0}^{\infty}\frac{(-T/2)^l}{l!}\,\partial_3^l\left[\sum_{m=0}^{\infty}\frac{\partial_2^{2m+1}}{(2m+1)!}\left(\sum_{n+r=m}\binom{m}{n}\frac{(4\sigma_4)^r(TA_\pm)^n}{\sqrt{\Delta_1}}\right)\right]\\
&=\; \sum_{m=0}^{\infty}\frac{\partial_2^{2m+1}}{(2m+1)!}\left[\sum_{l=0}^{\infty}\frac{(-T/2)^l}{l!}\,\partial_3^l\left(\frac{(4\sigma_4+TA_\pm)^m}{\sqrt{\Delta_1}}\right)\right]\\
&=\; \sum_{m=0}^{\infty}\frac{\partial_2^{2m+1}}{(2m+1)!}\left[\frac{1}{\sqrt{\Delta_1+4T}}\left(4\sigma_4\;+\;T\,\frac{\Delta_0\pm\sqrt{\Delta_1+4T}}{2}\right)^m\right],
\end{aligned} \tag{181}$$

where in the last line we have used the Taylor expansion and the fact that changing u_3 to $u_3 - T/2$ leaves Δ_0 unchanged but changes Δ_1 to $\Delta_1 + 4T$.

This expression can be further simplified by using Lagrange's theorem. Recall that this theorem is an identity generalizing Taylor's theorem which, in one of its forms, states that

$$\sum_{n=0}^{\infty}\frac{1}{n!}\,\frac{d^n}{dx^n}\bigl(f(x)\,b(x)^n\bigr) \;=\; \frac{f(z)}{1\,-\,b'(x)}\bigg|_{z\,=\,x+b(z)} \tag{182}$$

whenever the series makes sense (e.g., when everything is complex and absolutely convergent, or when the variables are power series in x with $b(x) = \mathrm{O}(x^2)$, or when b is a power series in some other variable with positive valuation). The corresponding "odd Lagrange theorem," obtained by replacing $b(x)$ and $f(x)$ by $\pm\sqrt{B(x)}$ and $F(x)/\sqrt{B(x)}$ and taking the difference of the two equations obtained, says that

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\,\frac{d^{2m+1}}{dx^{2m+1}}\bigl(F(x)\,B(x)^m\bigr) \;=\; \sum_{\pm}\frac{F(z)}{2(z-x)\;-\;B'(z)}\bigg|_{z\,=\,x\pm\sqrt{B(z)}}\,. \tag{183}$$

Applying this (with x and $z-x$ replaced by u_2 and w) to (181), we find after a short calculation that the two formulas (181) can be written in the form

$$\mathbf{Y}(T) \;=\; \frac{T}{P'(w_1)} \;+\; \frac{T}{P'(w_2)} \;=\; -\,\frac{T}{P'(w_3)} \;-\; \frac{T}{P'(w_4)}\,, \tag{184}$$

where w_1 and w_2 (resp. w_3 and w_4) are the solutions of

$$w_1 = \sqrt{4\sigma_4 + TA_+(u_1, u_2 + w_1, u_3 - T/2)}\,, \qquad w_2 = -\sqrt{4\sigma_4 + TA_+(u_1, u_2 + w_2, u_3 - T/2)}$$

(resp. of the corresponding equations with A_+ replaced by A_-) and where

$$\begin{aligned} P(w) &= (w^2 - 4\sigma_4 - T\Delta_0(u_1, u_2 + w, u_3 - T/2)/2)^2 \,-\, T^2\Delta_1(u_1, u_2 + w, u_3 - T/2)/4 \\ &= w^4 \,+\, Tw^3 \,-\, (8\sigma_4 + \Delta_0 T)w^2 \\ &\quad -\, (4\sigma_4 T - 2u_1 T^2)w \,+\, (16\sigma_4^2 + 4\Delta_0\sigma_4 T + 2\sigma_3 T^2 - T^3) \end{aligned} \tag{185}$$

is the monic quartic polynomial having $w_1, \ldots, w_4$ as its roots. The equality of the two expressions in (184), which appeared there because we had shown computationally that $Y_s^+ = Y_s^-$ for each s, now has a much more natural proof via the residue theorem, since

$$\sum_{i=1}^{4} \frac{1}{P'(w_i)} \;=\; \sum_{\alpha \in \mathbb{P}^1(\mathbb{C})} \mathrm{Res}_{w=\alpha}\Bigl(\frac{dw}{P(w)}\Bigr) \;=\; 0\,. \tag{186}$$

We now turn to the second generating function (180), which we will evaluate by using the odd Lagrange theorem again in combination with the formula (175). To achieve this, we introduce a "dummy" variable of homogeneity x by setting $\sigma_p = A_p x^P$ $(1 \le p \le 4)$, so that

$$\Delta_0 \;=\; (1 - A_1 x/2)^2 \,-\, A_2 x^2\,, \qquad \Delta_1 \;=\; \Delta_0^2 \,-\, 8A_3 x^3\,, \qquad A_\pm \;=\; (\Delta_0 \pm \sqrt{\Delta_1})/2$$

become functions of x and the Euler operator E becomes simply $x\,d/dx$. Then (175) gives

$$\mathbf{y}(0,T) \;=\; \sum_{s=1}^{\infty} \frac{T^{s-1}}{(2\sigma_3)^s} \binom{E}{2s-1} \frac{A_\pm^s}{\sqrt{\Delta_1}} \;=\; \frac{1}{2A_3 x^2} \sum_{m=0}^{\infty} \frac{(T/2A_3 x)^m}{(2m+1)!} \, \frac{d^{2m+1}}{dx^{2m+1}}\Bigl(\frac{A_\pm^{m+1}}{\sqrt{\Delta_1}}\Bigr)\,,$$

and applying (183) we find

$$\mathbf{y}(0,T) \;=\; \sum_{i=1}^{2} \frac{A_+(z_i)/\sqrt{\Delta_1(z_i)}}{2(z_i - x) \,-\, A_+'(z_i)} \;=\; -\sum_{i=3}^{4} \frac{A_-(z_i)/\sqrt{\Delta_1(z_i)}}{2(z_i - x) \,-\, A_-'(z_i)} \tag{187}$$

where z_1 and z_2 (resp. z_3 and z_4) are the solutions of the equations

$$z_1 \;=\; x + \sqrt{\frac{TA_+(z_1)}{2A_3 x}}\,, \qquad z_2 \;=\; x - \sqrt{\frac{TA_+(z_2)}{2A_3 x}} \tag{188}$$

(resp. of the same equations with A_+ replaced by A_-). Now substituting back $A_p = \sigma_p/x^p$ into the equations (188), we find that they are homogeneous (of degree 1) in z and x, so that only the ratio z/x matters, and making the change of variables $w = Tz/(z - X)$ we find that the four numbers z_i correspond exactly to the four numbers w_i defined above and that the right-hand side of (187) coincides exactly with that of (184) with σ_4 specialized to 0. This finally completes the proof of (173) and hence of Theorem 13,

We end by formulating a theorem describing the algebraic nature of the generating function(s) we have just computed.

Theorem 15. *The square of the function* $\mathbf{y}(\sigma_1,\dots,\sigma_4;T)$ *defined in* (180) *is algebraic of degree* 3 *over* $\mathbb{Q}(\sigma_1,\dots,\sigma_4,T)$.

Proof. This follows almost immediately from (184). Let $N = K(w_1,\dots,w_4)$ denote the splitting field of the polynomial $P(w)$ defined in (185) over the field $K = \mathbb{Q}(\sigma_1,\dots,\sigma_4,T)$. The fact that the trace of $h_i := T/P'(w_i)$ vanishes implies that the set $\{(h_1+h_2)^2,\,(h_1+h_3)^2,\,(h_1+h_4)^2\}$ is invariant under the Galois group of N over K, so the polynomial $Q(t) = \prod_{i=2}^4 (t-(h_1+h_i)^2)$ has coefficients in K, and equation (184) says that $\mathbf{Y}^2$ is a root of this polynomial. $\square$

We can get an explicit expression for $Q(t)$ using Ferrari's formula. If w is a root of the polynomial $P(w)$ defined in (185), then $h = T/P'(w)$ belongs to the field generated by w and hence also satisfies a fourth degree equation. This equation has a vanishing cubic term because of eq. (186), and by direct computation we find that it is given explicitly by

$$a_4h^4 \,+\, 0h^3 \,+\, a_2h^2 \,+\, a_1h \,+\, 1 \;=\; 0\,,$$

where

$$\begin{aligned} a_1 &= T(T-4u_2)\,,\\ a_2 &= 6(u_1-1)T^3 \,+\, \big(72u_1^2+4\Delta_0(7u_1-4)+\tfrac12\Delta_0^2-\tfrac32\Delta_1-12\sigma_4\big)\,T^2\\ &\quad -\,4\big(\Delta_0\Delta_1\,+\,4(8+4u_1+3\Delta_0)\sigma_4\big)\,T \,-\, 32\sigma_4(\Delta_1+4\sigma_4)\,,\\ a_4 &= \frac{\mathrm{disc}(P)}{T^4} \;=\; -27T^6+\cdots+\,256\sigma_4^2\left((\Delta_1+4\sigma_4)^2-16u_2^2\sigma_4\right)\,. \end{aligned}$$

Ferrari's formula tells us that the cubic polynomial having as its roots the three numbers

$$(h_1+h_2)^2=(h_3+h_4)^2,\quad (h_1+h_3)^2=(h_2+h_4)^2,\quad (h_1+h_4)^2=(h_2+h_3)^2$$

is given by

$$Q(t) \;=\; a_4^2\,t^3 \,+\, 2a_2a_4\,t^2 \,+\, (a_2^2-4a_4)\,t \,-\, a_1^2\,.$$

Then $Q(\mathbf{y}^2)=0$ is the sextic equation satisfied by our generating function $\mathbf{y}=\mathbf{y}(\sigma_1,\dots,\sigma_4,T)$.

E. Odd values of d**.** As already mentioned in **A.** above, for odd values of d we were not able to find any closed formula for the function $G^{(4)}$, either as an algebraic function or an integral of an algebraic function or as the solution of a linear differential equation with polynomial coefficients. On the other hand, in another respect these functions were actually *better* behaved than they were for even d. Namely, unlike the case of even d, where the coefficients of the power series for $d>4$ had infinitely many prime numbers occurring in their denominators and where each successive value of d required more differentiation of $G^{(4)}$ in order to achieve integrality, here the power series are apparently always integral after rescaling x by a fixed factor. For instance, for $d=5$ the same specialization $G^{(4)}(0,0,0,x/4)$ as we used in §10.**A** has a Taylor series beginning

$$y \;=\; 1+\frac{155}{16}\,x\,+\,\frac{194985}{1024}\,x^2\,+\,\frac{74509435}{16384}\,x^3\,+\,\frac{499522579465}{4194304}\,x^4\,+\,\cdots\,, \tag{189}$$

and within the limits of our calculation belongs to $\mathbb{Z}[[x/64]]$. Moreover, this integrality holds not only for this particular specialization, but (experimentally) for the full generating series $G^{(4)}(\sigma_1,\sigma_2,\sigma_3,\sigma_4;d=5)$, which begins

$$1+\frac{3\sigma_1}{2}+\frac{3\sigma_1^2+3\sigma_2}{2}+\frac{5\sigma_1^3+15\sigma_1\sigma_2+20\sigma_3}{4}+\frac{15\sigma_1^4+90\sigma_1^2\sigma_2+280\sigma_1\sigma_3+30\sigma_2^2+620\sigma_4}{16}+\cdots\,.$$

and in which the coefficient of degree k (up to $k = 194$, the limit of our computation) is always 2^{-k} times a polynomial in the σ_p with integral coefficients. (The specialized series (189) has an even stronger integrality property: up to a power of 2, the coefficient of x^m, corresponding to degree $4m$, seems always to be divisible by $4m+1$. The corresponding property for the full series, which would say that the terms of total degree k are divisible by $k+1$, is not always true. However, we do find that all the terms of total degree $p-1$ or $p-2$ are divisible by p if p is an odd prime number, as one can see in the case $p = 5$ in the expansion given above.) Similarly, for $d = 7$, 9, 11 and 13 we find that the terms of degree k in $G^{(4)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4; d)$ are integral after multiplication by $2^k D_d$ in the range tested, where $D_7 = 1$, $D_9 = 3$, $D_{11} = 5$ and $D_{13} = 35$. Based on these experimental results, we formulate:

Conjecture. *After rescaling $\sigma_p \mapsto 2^p \sigma_p$ $(1 \le p \le 4)$, the generating series $G^{(4)}(\sigma_1, \sigma_2, \sigma_3, \sigma_4; d)$ has a bounded denominator for every odd value of $d \ge 5$.*

In view of this conjecture, we can wonder whether the generating series for odd d are perhaps algebraic after all, even though we were not able to recognize even the special case (189) as an algebraic function. The following argument strongly suggests that they are not algebraic, and furthermore that—unlike what we saw in the case of even d—this remains true even after taking their images under any polynomial in the Euler operator. (This is a little surprising since the polynomial in E that we needed for d even, namely $\binom{E+3s}{2s-1}$, makes sense also for $d \ge 5$ odd, since $2s-1$ is then still a non-negative integer.) Denote by a_n the coefficient of x^n in the power series (189). Since both of the special functions (149) and (150), corresponding to $G^{(4)}$ and its derivative under $E+3$ for $d = 4$ and $d = 6$, respectively, had singularities at the root $x = (1+\sqrt{2})^{-4}$ of $1 - 34x + x^2 = 0$, we expect these coefficients to grow asymptotically like C^n, where $C = (1+\sqrt{2})^4 = 17 + 12\sqrt{2}$. In fact, applying the numerical extrapolation method that is described in [17] and in §5.1 of [6], we find the conjectural asymptotic expansion

$$a_n \overset{?}{\sim} \frac{A\,C^n}{n}\Bigl(1 - \frac{b_1}{n} + \frac{b_2}{n^2} - \cdots\Bigr) \tag{190}$$

with coefficients A, b_1, b_2, ... given numerically to high precision by

$$A \approx 0.39001742238656069394171 47\,,$$

$$b_1 \approx 0.35826456543968\,, \qquad b_2 \approx 0.1163559071350\,, \qquad \ldots\,.$$

We can recognize these numbers as

$$A \overset{?}{=} \frac{(1+\sqrt{2})^2}{2^{9/4}\,\pi}\,, \tag{191}$$

and

$$b_1 \overset{?}{=} 5\,\frac{6-\sqrt{2}}{2^6}\,, \qquad b_2 \overset{?}{=} 5\,\frac{183-62\sqrt{2}}{2^{12}}\,,$$
$$b_3 \overset{?}{=} 5\,\frac{5802-2773\sqrt{2}}{2^{18}}\,, \qquad b_4 \overset{?}{=} 5\,\frac{389055-211804\sqrt{2}}{2^{25}}\,,$$
$$b_5 \overset{?}{=} 5\,\frac{13542450-7562711\sqrt{2}}{2^{31}}\,, \qquad b_6 \overset{?}{=} 5\,\frac{469116531-283263874\sqrt{2}}{2^{37}}\,,$$

where we can assert with a very high degree of confidence that these numerically guessed values are in fact correct. (In the case of A, for example, which is the most important value, the extrapolation method was first applied using only half of the known coefficients a_n to give a

15-digit prediction; this was then recognized as the number given in (191); and then a new calculation using all of the known coefficients gave a 25-digit value for A whose first 24 digits agreed with the prediction.) Formula (190) says that the function y has a logarithmic singularity at $x = C^{-1}$, which is incompatible with its being an algebraic function but would not preclude its being the integral of an algebraic function, but (190) and (191) together, if they are true, imply that $P(E)\,y$ cannot be algebraic for any polyomial $P(E)$ in the Euler operator E with integral coefficients, since the leading coefficient in the Laurent expansion of this function around $x = 1/C$ would equal π times an algebraic number, and a function that is algebraic over $\mathbb{C}(x)$ and belongs to $\mathbb{Q}[[x]]$ has to be algebraic over $\mathbb{Q}(x)$ and hence must have algebraic Laurent coefficients in its expansion at any algebraic point.

F. Final remarks. We do not know how to generalize to higher values of n the results that we found in this section for $n = 4$. In particular, it is hard to see how to generalize the change of variables (155) from $n = 3$ to larger n, and even harder to see how the change of variables (155) and (159) can be generalized from $n = 4$ to larger n. However, we did find a change of variables that makes the differential operators $\mathcal{M}_n$ simpler for all n, and that seems worth mentioning. Define new variables τ_p ($p = 1, 2, \dots$) by the generating function equality

$$\sqrt{1 + \sigma_1 x + \sigma_2 x^2 + \sigma_3 x^3 + \cdots} \;=\; 1 + \tau_1 x + \tau_2 x^2 + \tau_3 x^3 + \cdots\,,$$

so that the first τ_p are given by

$$\begin{aligned}
\tau_1 &= \frac{1}{2}\sigma_1\,,\\
\tau_2 &= \frac{1}{2}\sigma_2 - \frac{1}{8}\sigma_1^2\,,\\
\tau_3 &= \frac{1}{2}\sigma_3 - \frac{1}{4}\sigma_1\sigma_2 + \frac{1}{16}\sigma_1^3\,,\\
\tau_4 &= \frac{1}{2}\sigma_4 - \frac{1}{4}\sigma_1\sigma_3 - \frac{1}{8}\sigma_2^2 + \frac{3}{16}\sigma_1^2\sigma_2 - \frac{5}{128}\sigma_1^4\,.
\end{aligned}$$

Then an easy calculation shows that

$$\mathcal{M}_n \;=\; 4\sum_{i+j=n}\frac{\partial^2}{\partial\tau_i\partial\tau_j}\,, \tag{192}$$

which is simpler than the original definition (113) in two respects:

- there are only $\mathrm{O}(n)$ rather than $\mathrm{O}(n^2)$ terms, and
- the terms belong to the commutative algebra $\mathbb{Q}(\partial/\partial\tau_p)_{p\ge1}$ rather than to the non-commutative algebra $\mathbb{Q}(\sigma_p,\,\partial/\partial\sigma_p)_{p\ge1}$.

Unfortunately, we have not been able to see how to use these new variables to make any of the generating function calculations simpler.

Table 1. Spherical polynomials for $n = 3$

We give a table of the polynomials $P_{\boldsymbol{\nu},d}(T)$ for all $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ with $|\boldsymbol{\nu}| \le 5$, extending the one given in the special case $d = 4$ after Proposition 1 of §8. As there, we give only one representative for each $\mathfrak{S}_3$-orbit of indices $\boldsymbol{\nu}$. As usual, we write s for $(d-4)/2$ and $(x)_r$ for the ascending Pochhammer symbol $x(x+1)\cdots(x+r-1)$.

$|\boldsymbol{\nu}| = 0$:

$$P_{000} = 1$$

$|\boldsymbol{\nu}| = 1$:

$$P_{100} = (s+1)\, r_1$$

$|\boldsymbol{\nu}| = 2$:

$$P_{200} = \frac{(s+1)_2}{2}\, r_1^2 - (s+1)\, m_2 m_3$$

$$P_{110} = (s+1)_2\, r_1 r_2 - (s+1)\, m_3 r_3$$

$|\boldsymbol{\nu}| = 3$:

$$P_{300} = \frac{(s+1)_3}{6}\, r_1^3 - (s+1)_2\, m_2 m_3 r_1$$

$$P_{210} = \frac{(s+1)_3}{2}\, r_1^2 r_2 - (s+1)_2\, m_3 (r_1 r_3 + m_2 r_2)$$

$$P_{111} = (s+2)^3\, r_1 r_2 r_3 - (s+2)^2 \left(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2\right) + 4(s+2)\, m_1 m_2 m_3$$

$|\boldsymbol{\nu}| = 4$:

$$P_{400} = \frac{(s+1)_4}{24}\, r_1^4 - \frac{(s+1)_3}{2}\, m_2 m_3 r_1^2 + \frac{(s+1)_2}{2}\, m_2^2 m_3^2$$

$$P_{310} = \frac{(s+1)_4}{6}\, r_1^3 r_2 - \frac{(s+1)_3}{2}\, m_3 r_1 (r_1 r_3 + 2 m_2 r_2) + (s+1)_2\, m_2 m_3^2 r_3$$

$$P_{220} = \frac{(s+1)_4}{4}\, r_1^2 r_2^2 - \frac{(s+1)_3}{2}\, m_3 (2 r_1 r_2 r_3 + m_1 r_1^2 + m_2 r_2^2) + \frac{(s+1)_2}{2}\, m_3^2 (r_3^2 + 2 m_1 m_2)$$

$$P_{211} = \frac{(s+2)^2 (s+3)^2}{2}\, r_1^2 r_2 r_3 - \frac{(s+2)(s+3)^2}{2}\, m_1 r_1^3 - (s+2)^2 (s+3)\, r_1 (m_2 r_2^2 + m_3 r_3^2) - (s+1)(s+2)^2\, m_2 m_3 r_2 r_3 + (s+2)(5s+13)\, m_1 m_2 m_3 r_1$$

$|\boldsymbol{\nu}| = 5$:

$$P_{500} = \frac{(s+1)_5}{120}\, r_1^5 - \frac{(s+1)_4}{6}\, m_2 m_3 r_1^3 + \frac{(s+1)_3}{2}\, m_2^2 m_3^2 r_1$$

$$P_{410} = \frac{(s+1)_5}{24}\, r_1^4 r_2 \;-\; \frac{(s+1)_4}{6}\, m_3 r_1^2 (r_1 r_3 + 3 m_2 r_2) \;+\; \frac{(s+1)_3}{2}\, m_2 m_3^2 (2 r_1 r_3 + m_2 r_2)$$

$$P_{320} \;=\; \frac{(s+1)_5}{12}\, r_1^3 r_2^2 \;-\; \frac{(s+1)_4}{6}\, m_3 r_1 (3 r_1 r_2 r_3 + m_1 r_1^2 + 3 m_2 r_2^2)$$
$$+\; \frac{(s+1)_3}{2}\, m_3^2\, (r_1 r_3^2 + 2 m_2 r_2 r_3 + 2 m_1 m_2 r_1)$$

$$P_{311} \;=\; \frac{(s+2)_3}{6}\, \Big((s+2)(s+4)\, r_1^3 r_2 r_3 \;-\; (s+4)\, m_1 r_1^4 \;-\; 3(s+2)\, r_1^2 (m_2 r_2^2 + m_3 r_3^2) \Big)$$
$$-\; (s+2)_2 \Big((s+2)^2\, m_2 m_3 r_1 r_2 r_3 \;-\; (3s+10) m_1 m_2 m_3 r_1^2$$
$$-\; (s+2) m_2 m_3 (m_2 r_2^2 + m_3 r_3^2) \;+\; 4\, m_1 m_2^2 m_3^2 \Big)$$

$$P_{221} \;=\; \frac{(s+2)(s+3)^2}{2} \Big(\frac{(s+3)(s+4)}{2}\, r_1^2 r_2^2 r_3 \;-\; (s+4)\, r_1 r_2 (m_1 r_1^2 + m_2 r_2^2)$$
$$-\; 2(s+3)\, m_3 r_1 r_2 r_3^2 \;-\; s\, m_3 r_3 (m_1 r_1^2 + m_2 r_2^2) \;+\; m_3^2 r_3^3$$
$$+\; 12\, m_1 m_2 m_3 r_1 r_2 \Big) \;+\; (s+2)(s+3)(s-3)\, m_1 m_2 m_3^2 r_3$$

$|\boldsymbol{\nu}| \;= 6$:

$$P_{600} \;=\; \frac{(s+1)_6}{720}\, r_1^6 \;-\; \frac{(s+1)_5}{24}\, m_2 m_3 r_1^4 \;+\; \frac{(s+1)_4}{4}\, m_2^2 m_3^2 r_1^2 - \frac{(s+1)_3}{6} m_2^3 m_3^3$$

$$P_{510} = \frac{(s+1)_6}{120}\, r_1^5 r_2 \;-\; \frac{(s+1)_5}{24}\, m_3 r_1^3 (r_1 r_3 + 4 m_2 r_2) \;+\; \frac{(s+1)_4}{2}\, m_2 m_3^2 r_1 (r_1 r_3 + m_2 r_2)$$
$$-\; \frac{(s+1)_3}{2} m_2^2 m_3^3 r_3$$

$$P_{420} \;=\; \frac{(s+1)_6}{48}\, r_1^4 r_2^2 \;-\; \frac{(s+1)_5}{24}\, m_3 r_1^2 (4 r_1 r_2 r_3 + m_1 r_1^2 + 6 m_2 r_2^2)$$
$$+\; \frac{(s+1)_4}{4}\, m_3^2\, (r_1^2 r_3^2 + 4 m_2 r_1 r_2 r_3 + 2 m_1 m_2 r_1^2 + m_2^2 r_2^2)$$
$$-\; \frac{(s+1)_3}{2} m_2 m_3^3 (m_1 m_2 + r_3^2)$$

$$P_{411} \;=\; \frac{(s+2)_4}{24}\, \Big((s+2)(s+5)\, r_1^4 r_2 r_3 \;-\; (s+5)\, m_1 r_1^5 \;-\; 4(s+2)\, r_1^3 (m_2 r_2^2 + m_3 r_3^2) \Big)$$
$$-\; \frac{(s+2)_3}{6}\, \Big(3(s+2)_2\, m_2 m_3 r_1^2 r_2 r_3 \;-\; (7s+29) m_1 m_2 m_3 r_1^3$$
$$-\; 6(s+2) m_2 m_3 r_1 (m_2 r_2^2 + m_3 r_3^2) \Big)$$
$$+\; \frac{(s+1)_3 (s+2)}{2} m_2^2 m_3^2 r_2 r_3 \;-\; \frac{3(s+2)_2 (3s+11)}{2} m_1 m_2^2 m_3^2 r_1$$

$$P_{330} \;=\; \frac{(s+1)_6}{36} r_1^3 r_2^3 \;-\; \frac{(s+1)_5}{12} m_3 r_1 r_2 (3 r_1 r_2 r_3 + 2 m_1 r_1^2 + 2 m_2 r_2^2)$$
$$+\; \frac{(s+1)_4}{2} m_3^2 (r_1 r_2 r_3^2 + m_1 r_1^2 r_3 + m_2 r_2^2 r_3 + 2 m_1 m_2 r_1 r_2)$$
$$-\; \frac{(s+1)_3}{6}\, m_3^3 r_3 (6 m_1 m_2 + r_3^2)$$

$$
\begin{aligned}
P_{321} \;=\;& \frac{(s+2)_4(s+3)_2}{12} r_1^3 r_2^2 r_3 \\
&- \frac{(s+2)_3}{12}\, r_1^2 r_2 \Big(6(s+3)_2 m_3 r_3^2 + 3(s+3)(s+5) m_2 r_2^2 + 2(s+4)_2 m_1 r_1^2\Big) \\
&- \frac{(s+2)_3}{6} m_3 r_1 r_3 \Big((s-1)(s+4) m_1 r_1^2 + 3(s+1)(s+3) m_2 r_2^2 - 3(s+3) m_3 r_3^2\Big) \\
&+ \frac{(s+2)_2}{2} m_2 m_3 r_2 ((s+4)(7s+25) m_1 r_1^2 + (s+3)_2 m_2 r_2^2 + (s+3)(2s+5) m_3 r_3^2) \\
&+ (s+2)_2 (s^2-13) m_1 m_2 m_3^2 r_1 r_3 - (s+2)_2 (5s+17) m_1 m_2^2 m_3^2 r_2 \\
P_{222} \;=\;& \frac{(s+3)_2^3}{8}\, r_1^2 r_2^2 r_3^2 \;-\; \frac{(s+3)^3(s+4)^2}{2} r_1 r_2 r_3 (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2) \\
&- \frac{(s-2)(s+3)^3(s+4)}{4} (m_1 m_2 r_1^2 r_2^2 + m_2 m_3 r_2^2 r_3^2 + m_1 m_3 r_1^2 r_3^2) \\
&+ \frac{(s+3)^2(s+4)^2}{4} (m_1^2 r_1^4 + m_2^2 r_2^4 + m_3^2 r_3^4) + (s+3)^3 (7s+22) m_1 m_2 m_3 r_1 r_2 r_3 \\
&+ \frac{(s+3)(s^2-5s-30)}{2}\, m_1 m_2 m_3 \Big((s+3)(m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2) - 2 m_1 m_2 m_3\Big)
\end{aligned}
$$

Table 2. Spherical polynomials and Gram matrices for $n = 4$ and $n = 5$

n = 4 .

First we repeat the example at the end of section 5. For multidegree $\mathbf{a} = (1\,1\,1\,1)$, we have $k\ \left(= \frac{1}{2}\sum_{i=1}^{4} a_i)\right) = 2$ and $\dim \mathcal{P}_{\mathbf{a}}(d) = 3$. For

$$
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{N}_0(\mathbf{a}),
$$

the monomial basis is given by $P_A^M = t_{12}t_{34}$, $P_B^M = t_{13}t_{24}$, $P_C^M = t_{14}t_{23}$. The Gram matrix $((P_{\boldsymbol{\nu}}^D, P_{\boldsymbol{\mu}}^D))_{\boldsymbol{\nu},\boldsymbol{\mu}\in\mathcal{N}_0(\mathbf{a})}$ also tells us the coefficients of the expansion of $P_{\boldsymbol{\nu}}^D$ in terms of the monomial basis $P_{\boldsymbol{\mu}}^M$, e.g. we have

$$
d(d-1)(d+2)\, P_A^D \;=\; (d+1)P_A^M \;-\; P_B^M \;-\; P_C^M \;.
$$

If we encode the multi-indices $\boldsymbol{\nu}$ of the descending basis by the corresponding monomials $x^{\boldsymbol{\nu}} = \prod_{i<j} x_{ij}^{\nu_{ij}}$, then this formula can be expressed in a visually clearer form by presenting the coefficients of the polynomial $d(d-1)(d+2)\, P_A^D$ in the form of a table

	$t_{12}t_{34}$	$t_{13}t_{24}$	$t_{14}t_{23}$
$x_{12}x_{34}$	$d+1$	-1	-1
$x_{13}x_{24}$	-1	$d+1$	-1
$x_{14}x_{23}$	-1	-1	$d+1$

and this is the form in which we will present the data for higher multi-indices.

For any polynomial $P(T) = P((t_{ij})) \in \mathbb{C}[\mathcal{S}_n]$ and a permutation σ on n letters, we write $P^\sigma(T) = P((t_{\sigma(i),\sigma(j)}))$. If $P(T) \in \mathcal{P}_{\mathbf{a}}(d)$, then we have $P^\sigma \in \mathcal{P}_{\sigma\mathbf{a}}(d)$ where $\sigma\mathbf{a} = (a_{\sigma(i)})$. It is obvious that $(P^\sigma, Q^\sigma)_d = (P,Q)_d$. So if $\sigma\mathbf{a} = \mathbf{a}$, then σ permutes the monomial basis, and also the descending basis. In the above example, we have $P_B^M = P_A^{(23)}$ and $P_C = P_A^{(24)}$, so the symmetry of the table is quite obvious.

$\mathbf{a} = (1\,1\,1\,3)$, $k=3$, $\dim \mathcal{P}_{\mathbf{a}}(d) = 1$.
Here, since the dimension is 1 we have to give only one monomial basis element $P_{\boldsymbol{\nu}}^M$. It is

$$t_{14}t_{24}t_{34} - \frac{t_{44}}{d+2}\,(t_{12}t_{34} + t_{13}t_{24} + t_{14}t_{23})\,,$$

where we do not need to specify the index $\boldsymbol{\nu}$ because the leading term (the only one that survives if we set all t_{ii} equal to 0) tells us us what it is. In this case the corresponding descending basis element $P_{\boldsymbol{\nu}}^D$ is simply $1/(d-1)d(d+4)$ times the monomial basis element, which we can express (rather unnecessarily, in this case) in our new tabular form by saying that the polynomial $(d-1)d(d+4)P_{\boldsymbol{\nu}}^D$ is given by the table

	$t_{14}t_{24}t_{34}$
$x_{14}x_{24}x_{34}$	1

From now on we list only the monomial basis polynomials (writing simply P_i, $1 \le i \le \dim$, rather than $P_{\boldsymbol{\nu}}^M$, since $\boldsymbol{\nu}$ can be read off from the leading term, and including symmetries as explained above), the common factor F by which the descending basis elements must be multiplied in order to make all coefficients in their representation with respect to the monomial basis polynomials with integral coefficients, and then a table of these coefficients.

$\mathbf{a} = (1\,1\,2\,2)$, $k=3$, $\dim \mathcal{P}_{\mathbf{a}}(d) = 3$.

$$\begin{aligned} P_1 &= t_{12}t_{34}^2 - \frac{1}{d}\,t_{12}t_{33}t_{44} \\ P_2 &= t_{13}t_{24}t_{34} - \frac{1}{d}\,(t_{14}t_{24}t_{33} + t_{13}t_{23}t_{44}) + \frac{1}{d^2}\,t_{12}t_{33}t_{44} \\ P_3 &= (P_2)^{(34)} \;=\; t_{14}t_{23}t_{34} - \frac{1}{d}\,(t_{14}t_{24}t_{33} + t_{13}t_{23}t_{44}) + \frac{1}{d^2}\,t_{12}t_{33}t_{44} \\ F &= 2(d-2)(d-1)d(d+2)(d+4) \end{aligned}$$

	$t_{12}t_{34}^2$	$t_{13}t_{24}t_{34}$	$t_{14}t_{23}t_{34}$
$x_{12}x_{34}^2$	d^2+2d-4	$-2d$	$-2d$
$x_{13}x_{24}x_{34}$	$-2d$	$2(d^2+d-4)$	$-2d+8$
$x_{14}x_{23}x_{34}$	$-2d$	$-2d+8$	$2(d^2+d-4)$

$\mathbf{a} = (1\,1\,2\,4)$: $k=4$, $\dim \mathcal{P}_{\mathbf{a}}(d) = 1$.

$$\begin{aligned} P_1 &= t_{14}t_{24}t_{34}^2 - \frac{t_{44}}{d+4}(t_{14}t_{24}t_{33} + 2t_{14}t_{23}t_{34} + 2t_{13}t_{24}t_{34} + t_{12}t_{34}^2) \\ &\quad + \frac{t_{44}^2}{(d+2)(d+4)}\,(2t_{13}t_{23} + t_{12}t_{33}) \\ F &= 2(d-1)d(d+1)(d+6) \quad \text{and} \quad P^D = F^{-1}P^M = F^{-1}P_1 \end{aligned}$$

$\mathbf{a} = (1\,1\,3\,3)$, $k = 4$, $\dim \mathcal{P}_{\mathbf{a}}(d) = 3$.

$$\begin{aligned}
P_1 &= t_{12}t_{34}^3 - \frac{3}{d+2}\, t_{12}t_{34}t_{33}t_{44} \\
P_2 &= t_{13}t_{24}t_{34}^2 - \frac{2}{d+2}(t_{13}t_{23}t_{34}t_{44} + t_{14}t_{24}t_{33}t_{34}) \\
&\quad - \frac{1}{(d+2)^2}\Big(d\, t_{13}t_{24}t_{33}t_{44} - 2\, t_{12}t_{33}t_{34}t_{44} - 2\, t_{14}t_{23}t_{33}t_{44}) \\
P_3 &= (P_2)^{(34)} \;=\; t_{14}t_{23}t_{34}^2 - \frac{2}{d+2}(t_{14}t_{24}t_{33}t_{34} + t_{13}t_{23}t_{34}t_{44}) \\
&\quad - \frac{1}{(d+2)^2}\Big(d\, t_{14}t_{23}t_{33}t_{44} - 2\, t_{12}t_{33}t_{34}t_{44} - 2\, t_{13}t_{24}t_{33}t_{44}\Big) \\
F &= 6(d-2)(d-1)d(d+1)(d+4)(d+6)
\end{aligned}$$

	$t_{12}t_{34}^3$	$t_{13}t_{24}t_{34}^2$	$t_{14}t_{23}t_{34}^2$
$x_{12}x_{34}^3$	d^2+5d-2	$-3(d+2)$	$-3(d+2)$
$x_{13}x_{24}x_{34}^2$	$-3(d+2)$	$3(d^2+3d-6)$	$-3(d-6)$
$x_{14}x_{23}x_{34}^2$	$-3(d+2)$	$-3(d-6)$	$3(d^2+3d-6)$

$\mathbf{a} = (1\,2\,2\,3)$, $k = 4$, $\dim \mathcal{P}_{\mathbf{a}}(d) = 3$.

$$\begin{aligned}
P_1 &= t_{12}t_{24}t_{34}^2 - \frac{1}{d}\, t_{14}t_{22}t_{34}^2 \\
&\quad + \frac{t_{44}}{d(d+2)}\Big(t_{14}t_{22}t_{33} + 2\, t_{13}t_{22}t_{34} - d\, t_{12}(t_{24}t_{33} + 2\, t_{23}t_{34})\Big) \\
P_2 &= P_1^{(23)} \;=\; t_{13}t_{24}^2t_{34} - \frac{1}{d}\, t_{14}t_{24}^2t_{33} \\
&\quad + \frac{t_{44}}{d(d+2)}\Big(t_{14}t_{22}t_{33} + 2\, t_{12}t_{24}t_{33} - d\, t_{13}(t_{22}t_{34} + 2\, t_{23}t_{24})\Big) \\
P_3 &= t_{14}t_{23}t_{24}t_{34} - \frac{1}{d}\,(t_{14}t_{24}^2t_{33} + t_{14}t_{22}t_{34}^2) \\
&\quad - \frac{t_{44}}{d(d+2)}\Big(d\, t_{23}(t_{14}t_{23} + t_{13}t_{24} + t_{12}t_{34}) - 2\,(t_{14}t_{22}t_{33} + t_{12}t_{24}t_{33} + t_{13}t_{22}t_{34})\Big) \\
F &= 2(d-2)(d-1)d(d+1)(d+4)(d+6)
\end{aligned}$$

	$t_{12}t_{24}t_{34}^2$	$t_{13}t_{24}^2t_{34}$	$t_{14}t_{23}t_{24}t_{34}$
$x_{12}x_{24}x_{34}^2$	d^2+4d-4	$-2(d+2)$	$-2(d-2)$
$x_{13}x_{24}^2x_{34}$	$-2(d+2)$	d^2+4d-4	$-2(d-2)$
$x_{14}x_{23}x_{24}x_{34}$	$-2(d-2)$	$-2(d-2)$	$2(d^2+3d-2)$

$\mathbf{a} = (2\,2\,2\,2)$, $k = 4$, $\dim \mathcal{P}_{\mathbf{a}}(d) = 6$.

$$
\begin{aligned}
P_1 &= t_{12}^2 t_{34}^2 - \frac{1}{d}(t_{12}^2 t_{33} t_{44} + t_{11} t_{22} t_{34}^2) + \frac{1}{d^2} t_{11} t_{22} t_{33} t_{44} \;=\; \frac{1}{d^2}(d t_{12}^2 - t_{11} t_{22})(d t_{34}^2 - t_{33} t_{44}) \\
P_2 &= (P_1)^{(23)} \;=\; t_{13}^2 t_{24}^2 - \frac{1}{d}(t_{13}^2 t_{22} t_{44} + t_{24}^2 t_{11} t_{33}) + \frac{1}{d^2} t_{11} t_{22} t_{33} t_{44} \\
P_3 &= (P_1)^{(24)} \;=\; t_{14}^2 t_{23}^2 - \frac{1}{d}(t_{14}^2 t_{22} t_{33} + t_{23}^2 t_{11} t_{44}) + \frac{1}{d^2} t_{11} t_{22} t_{33} t_{44} \\
P_4 &= t_{13} t_{14} t_{23} t_{24} - \frac{1}{d}\Big(t_{13} t_{14} t_{22} t_{34} + t_{12} t_{14} t_{24} t_{33} + t_{11} t_{23} t_{24} t_{34} + t_{12} t_{13} t_{23} t_{44}) \\
&\quad + \frac{1}{d^2}\Big(t_{14}^2 t_{22} t_{33} + t_{11} t_{24}^2 t_{33} + t_{11} t_{22} t_{34}^2 + t_{13}^2 t_{22} t_{44} + t_{11} t_{23}^2 t_{44} + t_{12}^2 t_{33} t_{44}\Big) - \frac{3}{d^3} t_{11} t_{22} t_{33} t_{44} \\
P_5 &= (P_4)^{(23)} \;=\; t_{12} t_{14} t_{23} t_{34} - \frac{1}{d}\Big(t_{13} t_{14} t_{22} t_{34} + t_{12} t_{14} t_{24} t_{33} + t_{11} t_{23} t_{24} t_{34} + t_{12} t_{13} t_{23} t_{44}\Big) \\
&\quad + \frac{1}{d^2}\Big(t_{14}^2 t_{22} t_{33} + t_{11} t_{24}^2 t_{33} + t_{11} t_{22} t_{34}^2 + t_{13}^2 t_{22} t_{44} + t_{11} t_{23}^2 t_{44} + t_{12}^2 t_{33} t_{44}\Big) - \frac{3}{d^3} t_{11} t_{22} t_{33} t_{44} \\
P_6 &= (P_4)^{(24)} \;=\; t_{12} t_{13} t_{24} t_{34} - \frac{1}{d}\Big(t_{13} t_{14} t_{22} t_{34} + t_{12} t_{14} t_{24} t_{33} + t_{11} t_{23} t_{24} t_{34} + t_{12} t_{13} t_{23} t_{44}\Big) \\
&\quad + \frac{1}{d^2}\Big(t_{14}^2 t_{22} t_{33} + t_{11} t_{24}^2 t_{33} + t_{11} t_{22} t_{34}^2 + t_{13}^2 t_{22} t_{44} + t_{11} t_{23}^2 t_{44} + t_{12}^2 t_{33} t_{44}\Big) - \frac{3}{d^3} t_{11} t_{22} t_{33} t_{44} \\
F &= 4(d-3)(d-2)(d-1)d(d+1)(d+2)(d+4)(d+6)
\end{aligned}
$$

	$t_{12}^2 t_{34}^2$	$t_{13}^2 t_{24}^2$	$t_{14}^2 t_{23}^2$
$x_{12}^2 x_{34}^2$	$d^4 + 5d^3 - 10d^2 - 36d + 24$	$2(d^2 + 12)$	$2(d^2 + 12)$
$x_{13}^2 x_{24}^2$	$2(d^2 + 12)$	$d^4 + 5d^3 - 10d^2 - 36d + 24$	$2(d^2 + 12)$
$x_{14}^2 x_{23}^2$	$2(d^2 + 12)$	$2(d^2 + 12)$	$d^4 + 5d^3 - 10d^2 - 36d + 24$
$x_{13} x_{14} x_{23} x_{24}$	$4(d-10)d$	$-4(d-2)d(d+4)$	$-4(d-2)d(d+4)$
$x_{12} x_{14} x_{23} x_{34}$	$-4(d-2)d(d+4)$	$4(d-10)d$	$-4(d-2)d(d+4)$
$x_{12} x_{13} x_{24} x_{34}$	$-4(d-2)d(d+4)$	$-4(d-2)d(d+4)$	$4(d-10)d$

	$t_{13} t_{14} t_{23} t_{24}$	$t_{12} t_{14} t_{23} t_{34}$	$t_{12} t_{13} t_{24} t_{34}$
$x_{12}^2 x_{34}^2$	$4(d-10)d$	$-4(d-2)d(d+4)$	$-4(d-2)d(d+4)$
$x_{13}^2 x_{24}^2$	$-4(d-2)d(d+4)$	$4(d-10)d$	$-4(d-2)d(d+4)$
$x_{14}^2 x_{23}^2$	$-4(d-2)d(d+4)$	$-4(d-2)d(d+4)$	$4(d-10)d$
$x_{13} x_{14} x_{23} x_{24}$	$4(d-2)^2(d+3)(d+4)$	$-4(d^3 - 7d^2 - 10d + 24)$	$-4(d^3 - 7d^2 - 10d + 24)$
$x_{12} x_{14} x_{23} x_{34}$	$-4(d^3 - 7d^2 - 10d + 24)$	$4(d-2)^2(d+3)(d+4)$	$-4(d^3 - 7d^2 - 10d + 24)$
$x_{12} x_{13} x_{24} x_{34}$	$-4(d^3 - 7d^2 - 10d + 24)$	$-4(d^3 - 7d^2 - 10d + 24)$	$4(d-2)^2(d+3)(d+4)$

n = 5 .

$\mathbf{a} = (1\,1\,1\,1\,2)$. $\dim \mathcal{P}_{\mathbf{a}}(d) = 6$.

In this case the symmetry group acts transitively on the monomial basis.

$$\begin{aligned}
P_1 &= t_{12}t_{35}t_{45} - \frac{1}{d}\,t_{12}t_{34}t_{55} \\
P_2 &= (P_1)^{(23)} \;=\; t_{13}t_{25}t_{45} - \frac{1}{d}\,t_{13}t_{24}t_{55} \\
P_3 &= (P_1)^{(24)} \;=\; t_{14}t_{25}t_{35} - \frac{1}{d}\,t_{14}t_{23}t_{55} \\
P_4 &= (P_1)^{(13)} \;=\; t_{15}t_{23}t_{45} - \frac{1}{d}\,t_{14}t_{23}t_{55} \\
P_5 &= (P_1)^{(14)} \;=\; t_{15}t_{24}t_{35} - \frac{1}{d}\,t_{13}t_{24}t_{55} \\
P_6 &= (P_1)^{(13)(24)} \;=\; t_{15}t_{25}t_{34} - \frac{1}{d}\,t_{12}t_{34}t_{55} \\
F &= (d-2)(d-1)d(d+2)(d+4)
\end{aligned}$$

	$t_{12}t_{35}t_{45}$	$t_{13}t_{25}t_{45}$	$t_{14}t_{25}t_{35}$	$t_{15}t_{23}t_{45}$	$t_{15}t_{24}t_{35}$	$t_{15}t_{25}t_{34}$
$x_{12}x_{35}x_{45}$	d^2+2d-4	$-d$	$-d$	$-d$	$-d$	4
$x_{13}x_{25}x_{45}$	$-d$	d^2+2d-4	$-d$	$-d$	4	$-d$
$x_{14}x_{25}x_{35}$	$-d$	$-d$	d^2+2d-4	4	$-d$	$-d$
$x_{15}x_{23}x_{45}$	$-d$	$-d$	4	d^2+2d-4	$-d$	$-d$
$x_{15}x_{24}x_{35}$	$-d$	4	$-d$	$-d$	d^2+2d-4	$-d$
$x_{15}x_{25}x_{34}$	4	$-d$	$-d$	$-d$	$-d$	d^2+2d-4

$\mathbf{a} = (1\,1\,1\,1\,4)$, $\dim \mathcal{P}_{\mathbf{a}}(d) = 1$.

$$\begin{aligned}
P_1 \;=\;& t_{15}t_{25}t_{35}t_{45} \;-\; \frac{t_{55}}{d+4}\Big(t_{12}t_{35}t_{45} + t_{13}t_{25}t_{45} + t_{14}t_{25}t_{35} + t_{15}(t_{25}t_{34} + t_{24}t_{35} + t_{23}t_{45})\Big) \\
&+ \frac{t_{55}^2}{(d+2)(d+4)}(t_{12}t_{34} + t_{13}t_{24} + t_{14}t_{23}) \\
F \;=\;& (d-1)d(d+1)(d+6) \quad \text{and} \quad P^D \;=\; F^{-1}P^M \;=\; F^{-1}P_1
\end{aligned}$$

$\mathbf{a} = (1\,1\,1\,2\,3)$, $\dim \mathcal{P}_{\mathbf{a}}(d) = 6$.

Here the monomial basis is given by P_1, $P_1^{(23)}$, $P_1^{(123)}$, P_2, $P_2^{(12)}$ and $P_2^{(13)}$, where

$$
\begin{aligned}
P_1 &= t_{12}t_{35}t_{45}^2 - \frac{1}{d+2}\, t_{12}t_{55}(t_{35}t_{44} + 2\, t_{34}t_{45}) \\
P_2 &= t_{14}t_{25}t_{35}t_{45} - \frac{1}{d} t_{15}t_{25}t_{35}t_{44} \\
&\quad + \frac{1}{d(d+2)}\, t_{55}\Big(t_{44}(t_{15}t_{23} + t_{13}t_{25} + t_{12}t_{35}) \,-\, d\, t_{14}(t_{23}t_{45} + t_{24}t_{35} + t_{25}t_{34})\Big)
\end{aligned}
$$

In this case we have

$$F \;=\; 2(d-2)(d-1)d(d+1)(d+4)(d+6)$$

and the product of F with the Gram matrix of the descending basis (ordered as above) is

	$t_{12}t_{35}t_{45}^2$	$t_{13}t_{25}t_{45}^2$	$t_{15}t_{23}t_{45}^2$
$x_{12}x_{35}x_{45}^2$	d^2+5d-2	$-(d+2)$	$-(d+2)$
$x_{13}x_{25}x_{45}^2$	$-(d+2)$	d^2+5d-2	$-(d+2)$
$x_{15}x_{23}x_{45}^2$	$-(d+2)$	$-(d+2)$	d^2+5d-2
$x_{14}x_{25}x_{35}x_{45}$	$-2(d+2)$	$-2(d+2)$	8
$x_{15}x_{24}x_{35}x_{45}$	$-2(d+2)$	8	$-2(d+2)$
$x_{15}x_{25}x_{34}x_{45}$	8	$-2(d+2)$	$-2(d+2)$

	$t_{14}t_{25}t_{35}t_{45}$	$t_{15}t_{24}t_{35}t_{45}$	$t_{15}t_{25}t_{34}t_{45}$
$x_{12}x_{35}x_{45}^2$	$-2(d+2)$	$-2(d+2)$	8
$x_{13}x_{25}x_{45}^2$	$-2(d+2)$	8	$-2(d+2)$
$x_{15}x_{23}x_{45}^2$	8	$-2(d+2)$	$-2(d+2)$
$x_{14}x_{25}x_{35}x_{45}$	$2(d^2+4d-4)$	$-2(d-2)$	$-2(d-2)$
$x_{15}x_{24}x_{35}x_{45}$	$-2(d-2)$	$2(d^2+4d-4)$	$-2(d-2)$
$x_{15}x_{25}x_{34}x_{45}$	$-2(d-2)$	$-2(d-2)$	$2(d^2+4d-4)$

$\mathbf{a} = (1\,1\,2\,2\,2)$, $\dim \mathcal{P}_{\mathbf{a}}(d) = 10$.

The monomial basis is P_1, P_2, $P_2^{(34)}$, $P_2^{(35)}$, P_3, $P_3^{(45)}$, $P_3^{(34)}$, $P_3^{(345)}$, $P_3^{(354)}$, $P_3^{(35)}$, with

$$\begin{aligned}
P_1 &= t_{12}t_{34}t_{35}t_{45} - \frac{t_{12}}{d}(t_{35}^2t_{44} + t_{33}t_{45}^2 + t_{34}^2t_{55}) + \frac{2}{d^2}t_{12}t_{33}t_{44}t_{55}, \\
P_2 &= t_{13}t_{23}t_{45}^2 - \frac{1}{d}(t_{13}t_{23}t_{44}t_{55} + t_{12}t_{33}t_{45}^2) + \frac{1}{d^2}t_{12}t_{33}t_{44}t_{55} \\
&= \frac{1}{d^2}(d\,t_{13}t_{23} - t_{12}t_{33})(d\,t_{45}^2 - t_{44}t_{55}), \\
P_3 &= t_{13}t_{24}t_{35}t_{45} - \frac{1}{d}\,(t_{13}t_{25}t_{35}t_{44} + t_{13}t_{24}t_{34}t_{55} + t_{15}t_{24}t_{33}t_{45}) \\
&\quad + \frac{1}{d^2}\,(t_{13}t_{23}t_{44}t_{55} + t_{14}t_{24}t_{33}t_{55} + t_{15}t_{25}t_{33}t_{44}) - \frac{1}{d^3}\,t_{12}t_{33}t_{44}t_{55} \\
F &= 2(d-3)(d-2)(d-1)d(d+1)(d+2)(d+4)(d+6)
\end{aligned}$$

In this case, the Gram matrix of the descending basis, multiplied by F, is a 10×10 matrix with rather complicated entries (for instance, its first two diagonal entries are $2(d-2)(d^3+6d^2+3d-6)$ and $d^4 + 5d^3 - 10d^2 - 36d + 24$), so we do not write it out. However, the Gram matrix of the monomial basis in this case is somewhat simpler, and has polynomial coefficients after being multiplied by $F^* = d^2/(d-1)(d+2)$. We give a part of this matrix, namely, the set of values $F^*(P_i, P_{\boldsymbol{\mu}}^M)$ for $1 \le i \le 3$ and all ten $\boldsymbol{\mu}$ (ordered as above), in the following table. The remaining 70 values, of course, follow from the symmetry.

	$t_{12}t_{34}t_{35}t_{45}$	$t_{13}t_{23}t_{45}^2$	$t_{14}t_{24}t_{35}^2$	$t_{15}t_{25}t_{34}^2$
P_1	$(d-2)d^2(d+4)$	0	0	0
P_2	0	$2(d-1)d^2(d+2)$	$4d^2$	$4d^2$
P_3	$(d-2)d(d+4)$	$2(d-1)d(d+2)$	$2(d-1)d(d+2)$	$2(d-2)d$

	$t_{13}t_{24}t_{35}t_{45}$	$t_{13}t_{25}t_{34}t_{45}$	$t_{14}t_{23}t_{35}t_{45}$
P_1	$(d-2)d(d+4)$	$(d-2)d(d+4)$	$(d-2)d(d+4)$
P_2	$2(d-1)d(d+2)$	$2(d-1)d(d+2)$	$2(d-1)d(d+2)$
P_3	$(d-1)^2(d+2)^2$	$(d-2)(d-1)(d+2)$	d^3+3d^2-4d+4

	$t_{14}t_{25}t_{34}t_{35}$	$t_{15}t_{23}t_{34}t_{45}$	$t_{15}t_{24}t_{34}t_{35}$
P_1	$(d-2)d(d+4)$	$(d-2)d(d+4)$	$(d-2)d(d+4)$
P_2	$2(d-2)d$	$2(d-1)d(d+2)$	$2(d-2)d$
P_3	$(d-2)^2$	$(d-2)^2$	$(d-2)(d-1)(d+2)$

Table 3. Determinants of Gram matrices

In §9.C.8. we explained that the determinant $\mathrm{GD}^M_{\mathbf{a}}(d)$ of the Gram matrix for the monomial basis has the form $C\prod_{m\geq 1-n}(d+m)^{e(m)}$ for some $C\in\mathbb{Q}^\times$ and $e(m)\in\mathbb{Z}$. We give a table of the exponents $e(m)$ and of the constant C for $n=4$ and all tuples $\mathbf{a}$ with $k\leq 8$ (written as $(a_1\,a_2\,a_3\,a_4)$ with $1\leq a_1\leq a_2\leq a_3\leq a_4$, and also $a_4\leq a_1+a_2+a_3$ since otherwise the dimension $N_0(\mathbf{a})$ is 0), and then a shorter table for $n=5$ and all tuples $\mathbf{a}$ with $k\leq 6$. The tabulated values of m are consecutive to the left of the vertical line and even to its right.

n = 4 .

k = 2, 3, 4, 5 .

k	$\dim \mathcal{P}_{\mathbf{a}}(d)$	$\mathbf{a}$	C	-3	-2	-1	0	1	2	4	6	8
2	3	(1111)	1			2	3	0	1			
3	1	(1113)	1			1	1	0	0	1		
	3	(1122)	2		1	3	1	0	3	1		
4	1	(1124)	2			1	1	1	0	0	1	
	3	(1133)	$2^3\cdot 3$		1	3	3	2	-1	3	1	
		(1223)	2^2		2	3	1	2	0	3	1	
	6	(2222)	2^6	1	3	6	1	3	6	3	1	
5	1	(1135)	$2\cdot 3$			1	1	1	1	0	0	1
		(1225)	2^2			1	1	1	1	0	0	1
	3	(1144)	$2^5\cdot 3^3$		1	3	3	3	2	-1	3	1
		(1234)	$2^4\cdot 3$		2	3	2	3	1	0	3	1
		(1333)	2^3		3	3	3	3	-4	3	3	1
		(2224)	2^3		3	3	0	3	2	0	3	1
	6	(2233)	$2^8\cdot 3$	1	4	6	4	5	0	6	3	1

k = 6 .

dim	$\mathbf{a}$	C	-3	-2	-1	0	1	2	3	4	6	8	10
1	(1146)	$2^3\cdot 3$			1	1	1	1	1	0	0	0	1
	(1236)	$2^2\cdot 3$			1	1	1	1	1	0	0	0	1
	(2226)	2^3			1	1	1	1	1	0	0	0	1
3	(1155)	$2^9\cdot 3^3\cdot 5$		1	3	3	3	3	2	0	-1	3	1
	(1245)	$2^6\cdot 3^3$		2	3	2	3	3	2	-1	0	3	1
	(1335)	$2^6\cdot 3^2$		2	3	3	3	1	2	0	0	3	1
	(1344)	$2^5\cdot 3$		3	3	4	3	-1	2	-3	3	3	1
	(2235)	$2^5\cdot 3$		3	3	1	3	2	2	0	0	3	1
6	(2244)	$2^{13}\cdot 3^3$	1	4	6	5	6	4	3	-3	6	3	1
	(2334)	$2^{10}\cdot 3^2$	1	5	6	6	6	-1	3	0	6	3	1
10	(3333)	$2^{18}\cdot 3^6$	3	7	10	11	9	-4	4	10	6	3	1

k = 7 .

dim	a	C	−3	−2	−1	0	1	2	3	4	6	8	10	12
1	⋆	(see below)			1	1	1	1	1	1	0	0	0	1
3	(1166)	$2^{10}\cdot 3^4\cdot 5^3$		1	3	3	3	3	3	2	0	−1	3	1
	(1256)	$2^{10}\cdot 3^3\cdot 5$		2	3	2	3	3	3	2	−1	0	3	1
	(1346)	$2^8\cdot 3^4$		2	3	3	3	2	3	1	0	0	3	1
	(1355)	$2^7\cdot 3^3$		3	3	4	3	0	3	1	−3	3	3	1
	(1445)	$2^7\cdot 3^2$		3	3	5	3	1	3	−4	0	3	3	1
	(2246)	$2^7\cdot 3^3$		3	3	1	3	3	3	1	0	0	3	1
	(2336)	$2^7\cdot 3^2$		3	3	2	3	1	3	2	0	0	3	1
6	(2255)	$2^{17}\cdot 3^6\cdot 5$	1	4	6	5	6	6	5	2	−3	6	3	1
	(2345)	$2^{15}\cdot 3^4$	1	5	6	7	6	2	5	0	0	6	3	1
	(2444)	$2^{12}\cdot 3^3$	1	6	6	8	6	3	5	−9	6	6	3	1
	(3335)	$2^{12}\cdot 3^3$	1	6	6	8	6	−3	5	3	0	6	3	1
10	(3344)	$2^{23}\cdot 3^8$	3	8	10	13	10	1	7	−2	10	6	3	1

Here ⋆ denotes the list {(1157), (1247), (1337), (2237)} of **a** with $\dim \mathcal{P}_{\mathbf{a}}(d) = 1$, the corresponding C-values being 120, 48, 36 and 24.

k = 8 .

dim	a	C	−3	−2	−1	0	1	2	3	4	5	6	8	10	12	14
1	⋆	(see below)			1	1	1	1	1	1	1	0	0	0	0	1
3	(1177)	$2^{12}\cdot 3^6\cdot 5^3\cdot 7$		1	3	3	3	3	3	3	2	0	0	−1	3	1
	(1267)	$2^{11}\cdot 3^4\cdot 5^3$		2	3	2	3	3	3	3	2	0	−1	0	3	1
	(1357)	$2^{12}\cdot 3^4\cdot 5$		2	3	3	3	2	3	3	2	−1	0	0	3	1
	(1366)	$2^{11}\cdot 3^3\cdot 5$		3	3	4	3	0	3	3	2	−1	−3	3	3	1
	(1447)	$2^{10}\cdot 3^6$		2	3	3	3	3	3	1	2	0	0	0	3	1
	(1456)	$2^9\cdot 3^4$		3	3	5	3	2	3	−1	2	−3	0	3	3	1
	(1555)	$2^9\cdot 3^3$		3	3	6	3	3	3	−3	2	−6	3	3	3	1
	(2257)	$2^{11}\cdot 3^3\cdot 5$		3	3	1	3	3	3	3	2	−1	0	0	3	1
	(2347)	$2^9\cdot 3^4$		3	3	2	3	2	3	2	2	0	0	0	3	1
	(3337)	$2^9\cdot 3^3$		3	3	3	3	0	3	3	2	0	0	0	3	1
6	(2266)	$2^{24}\cdot 3^7\cdot 5^3$	1	4	6	5	6	6	6	5	3	−1	−3	6	3	1
	(2356)	$2^{19}\cdot 3^7\cdot 5$	1	5	6	7	6	3	6	4	3	−3	0	6	3	1
	(2446)	$2^{20}\cdot 3^6$	1	5	6	8	6	4	6	−1	3	0	0	6	3	1
	(2455)	$2^{17}\cdot 3^5$	1	6	6	9	6	5	6	−4	3	−6	6	6	3	1
	(3346)	$2^{17}\cdot 3^5$	1	6	6	9	6	−1	6	2	3	0	0	6	3	1
10	(3445)	$2^{28}\cdot 3^{10}$	3	9	10	15	10	5	9	−5	4	0	10	6	3	1
	(3355)	$2^{31}\cdot 3^{11}\cdot 5$	3	8	10	14	10	4	9	4	4	−6	10	6	3	1
15	(4444)	$2^{48}\cdot 3^{18}$	6	12	16	21	15	10	12	−12	5	15	10	6	3	1

Here ⋆ denotes the list {(1168), (1258), (1348), (2248), (2338)} of **a** with $\dim \mathcal{P}_{\mathbf{a}}(d) = 1$, the corresponding C-values being 720, 240, 144, 96 and 72.

n = 5 .
k = 3, 4, 5, 6 .

k	$\dim \mathcal{P}_{\mathbf{a}}(d)$	$\mathbf{a}$	C	-4	-3	-2	-1	0	1	2	3	4	6	8	10
3	6	(11112)	1			2	6	3	0	6	0	1			
4	1	(11114)	1				1	1	1	0	0	0	1		
	6	(11123)	2^3			3	6	5	3	0	0	6	1		
	10	(11222)	2^3		1	7	10	1	4	10	0	6	1		
5	1	(11125)	2				1	1	1	1	0	0	0	1	
	6	(11134)	$2^6 \cdot 3^3$			3	6	6	6	2	0	0	6	1	
		(11224)	2^6			4	6	4	6	3	0	0	6	1	
	10	(11233)	$2^8 \cdot 3$		1	8	10	7	9	-2	0	10	6	1	
	15	(12223)	2^{12}		3	12	15	6	12	5	0	15	6	1	
	22	(22222)	2^{10}	1	6	22	22	-7	16	22	0	21	6	1	
6	1	(11136)	$2 \cdot 3$				1	1	1	1	1	0	0	0	1
	1	(11226)	2^2				1	1	1	1	1	0	0	0	1
	6	(11145)	$2^{12} \cdot 3^6$			3	6	6	6	6	3	-1	0	6	1
		(11235)	$2^9 \cdot 3^3$			4	6	5	6	5	3	0	0	6	1
		(12225)	2^9			5	6	3	6	6	3	0	0	6	1
	10	(11244)	$2^{14} \cdot 3^6$		1	8	10	8	10	8	4	-6	10	6	1
		(11334)	$2^{13} \cdot 3^2$		1	9	10	12	10	-3	4	0	10	6	1
	15	(12234)	$2^{19} \cdot 3^3$		3	13	15	11	15	6	5	0	15	6	1
	21	(22224)	2^{30}		6	17	21	12	21	15	6	0	21	6	1
		(12333)	$2^{27} \cdot 3^3$		6	18	21	21	20	-9	6	21	15	6	1
	29	(22233)	$2^{31} \cdot 3$	1	10	28	29	15	28	-1	7	29	21	6	1

Final remark. In these tables, non-zero exponents $e(m)$ occur only if $1 - n \le m \le k - 3$ or if $k - 2 \le m \le 2k - 2$ and m is even, the latter statement being highlighted by the vertical lines that we have inserted. This suggests that the dimension of the space $\mathcal{P}_{\mathbf{a}}(d)$ has its generic value $\mathcal{N}_0(\mathbf{a})$, and that both the generic monomial and descending basis elements give bases of this space, in any of the following four cases:

- $d \notin \mathbb{Z}$,
- $d \ge n$,
- $d \le -2k$,
- $d \le 2 - k$ and d odd.

This is in fact true, and follows by combining results given at various points of our paper, namely Theorem 3 of §3 (which tells us that the inner product of monomials is polynomial in d), Theorem 5 of §5 (which tells us how to get the monomial basis by iterating the "raising operator"), Theorem 6 of §5 (giving the duality between the monomial and descending bases), equation (117) in §9.**A** (which tells us how to get the generating function by repeated applications of appropriate differential operators), and equation (134) of §9.**C** (which relates the generating function basis to the descending basis). But there are still many mysteries, e.g., why the value of $e(2k - 2)$ should always be 1, as the tables suggest.

List of Notations

We list the principal notations, omitting those that are used only locally, and including section or equation numbers when this seems helpful.

$\mathcal{S}_n$	space of real symmetric $n \times n$ matrices, with coordinate $T = (t_{ij})$
$\mathbb{C}[\mathcal{S}_n]$, $\mathbb{C}[V]$	space of polynomials on $\mathcal{S}_n$ or on any vector space V
β_n	map $(\mathbb{R}^d)^n \to \mathcal{S}_n$ defined by $(x_1, \ldots, x_n) \mapsto (\,(x_i, x_j)\,)$ or $X \mapsto XX^t$
β_n^*	induced map $P \mapsto \widetilde{P} = P \circ \beta_n$ from $\mathbb{C}[\mathcal{S}_n]$ to $C[(\mathbb{R}^d)^n]$
$\mathcal{N}$	set of symmetric $n \times n$ matrices $\boldsymbol{\nu} = (\nu_{ij})$ with $\nu_{ij} \in \mathbb{Z}_{\geq 0}$, ν_{ii} even
$T^{\boldsymbol{\nu}} \quad (\boldsymbol{\nu} \in \mathcal{N})$	monomial $\prod t_{ij}^{\nu_{ij}/2}$ in $\mathbb{C}[\mathcal{S}_n]$
$\partial_{ij} \quad (1 \leq i,\, j \leq n)$	partial derivative w.r.t. t_{ij}, multiplied by 2 if $i = j$
$\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$	vectors consisting of n 0's, resp. 1's, resp. 2's
$\mathcal{N}_0$	set of $\boldsymbol{\nu} \in \mathcal{N}$ with $\nu_{ii} = 0$ for all i
$\mathcal{N}(\mathbf{a})$, $\mathcal{N}_0(\mathbf{a}) \quad (\mathbf{a} \in \mathbb{Z}_{\geq 0}^n)$	set of $\boldsymbol{\nu}$ in $\mathcal{N}$, resp. $\mathcal{N}_0$, with $\nu \cdot \mathbf{1} = \mathbf{a}$
$N(\mathbf{a})$, $N_0(\mathbf{a}) \quad (\mathbf{a} \in \mathbb{Z}_{\geq 0}^n)$	cardinality of $\mathcal{N}(\mathbf{a})$, resp. $\mathcal{N}_0(\mathbf{a})$
$\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \quad (\mathbf{a} \in \mathbb{Z}_{\geq 0}^n)$	subspace of $\mathbb{C}[\mathcal{S}_n]$ spanned by monomials $T^{\boldsymbol{\nu}}$ with $\boldsymbol{\nu} \in \mathcal{N}(\mathbf{a})$
$D_i,\ E_i \quad (1 \leq i \leq n)$	ith Laplace, resp. Euler differential operators on $\mathbb{C}[\mathcal{S}_n]$ [§1]
$\mathcal{P}^{(n)}(d)$, $\mathcal{P}_{\mathbf{a}}(d) \quad (\mathbf{a} \in \mathbb{Z}_{\geq 0}^n)$	subspace of $\mathbb{C}[\mathcal{S}_n]$, resp. $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}}$, annihilated by all D_i
$\mathcal{S}_n^0$	space of matrices $T \in \mathcal{S}_n$ with all t_{ii} equal to 0
Φ	the generically bijective map $\mathcal{P}^{(n)}(d) \subset \mathbb{C}[\mathcal{S}_n] \to \mathbb{C}[\mathcal{S}_n^0] \cong \mathbb{C}^{\mathcal{N}_0}$
$\Xi(\mathbf{a}) \quad (\mathbf{a} \in \mathbb{Z}_{\geq 0}^n)$	exceptional set $2\mathbb{Z} \cap \cup_{i=1}^n [4 - 2a_i, 2 - a_i]$ in Theorem 1 [§2]
$\delta(T)^{\mathbf{m}} \quad (\mathbf{m} \in \mathbb{Z}_{\geq 0}^n)$	diagonal monomial $\prod_{i=1}^n t_{ii}^n \in \mathbb{C}[\mathcal{S}_n]_{2\mathbf{m}}$
$P_{\boldsymbol{\nu}}^M = P_{\boldsymbol{\nu},d}^M$	monomial basis of $\mathcal{P}^{(n)}(d)$ [§2]
$\mathcal{H}(\mathbb{R}^d)$	space $\mathrm{Ker}(\Delta)$ of harmonic polynomials in $\mathbb{C}[\mathbb{R}^d]$
$(x)_n \quad (x \in \mathbb{C},\ n \in \mathbb{Z}_{\geq 0})$	increasing Pochhammer symbol $x(x+1)\cdots(x+n-1)$
$\pi^{(d)}$, $\Pi_{\mathbf{a}}^{(d)}$	projection map $\mathbb{C}[\mathbb{R}^d] \to \mathcal{H}(\mathbb{R}^d)$, resp. $\mathbb{C}[\mathcal{S}_n]_{\mathbf{a}} \to \mathcal{P}_{\mathbf{a}}(d)$ [§2]
$\mathcal{S}_n^1$	space of matrices $T \in \mathcal{S}_n$ with all t_{ii} equal to 1
$\mathcal{S}_n^+$, $\mathcal{S}_n^{1,+}$	space of positive definite matrices in $\mathcal{S}_n$, resp. $\mathcal{S}_n^1$
$\mathsf{E}_{\mathbb{R}^d},\ \mathsf{E}_{S^{d-1}},\ \mathsf{E}_{(\mathbb{R}^d)^n},\ \mathsf{E}_{(S^{d-1})^n}$	expectation value w.r.t. probability measures on these spaces [§3]
$(\ ,\ )_d,\ (\ ,\ )_d^1$	scalar products on $\mathbb{C}[\mathcal{S}_n]$, resp. $\mathbb{C}[\mathcal{S}_n^1]$ [§3]
E_d, E_d^1	the corresponding functionals $(\ ,1)_d$, $(\ ,1)_d^1$
$c_n(d)$, $c_n^1(d)$	normalizing constants in the definitions of the scalar products
$\varepsilon_a(d)$, $\varepsilon_{\mathbf{a}}(d)$	normalizing factors relating the two scalar products [eqs. (14), (18)]
$\mathfrak{L}_n^+$	space of lower triangular $n \times n$ matrices with positive diagonal entries
$\mathsf{E}_d^{\mathfrak{L}}$	functional on $\mathbb{C}[\mathfrak{L}_n]^{\mathrm{ev}}$ defined in (25)
$D_{ij},\ E_{ij},\ F_{ij} \quad (1 \leq i,\, j \leq n)$	mixed Laplace and Euler operators and multiplication by t_{ij} [§4]
$\mathfrak{g}$	the Lie algebra spanned by all D_{ij}, E_{ij}, F_{ij} [§4]
$R_{ij},\ C_{ij} \quad (1 \leq i,\, j \leq n)$	raising, resp. "mixed Casimir" operators on $\mathbb{C}[\mathcal{S}_n]$ [§5]
$P_{\boldsymbol{\nu}}^D = P_{\boldsymbol{\nu},d}^D$	descending basis of $\mathcal{P}^{(n)}(d)$ (§5)
$\mathcal{K}$	the field $\mathbb{Q}(d)$, with d considered as a variable [§5, §9]
$\mathcal{R}$	the subring $\mathbb{Q}[d,\, d^{-1},\, (d \pm 1)^{-1},\, (d \pm 2)^{-1}, \ldots]$ of $\mathcal{K}$ [§9, eq. (112)]
$\mathcal{L}_p$	the differential operator w.r.t. $\sigma_1, \ldots, \sigma_n$ [§9, eq. (102)]
$\mathcal{M}_p$	the "main" part of $\mathcal{L}_p$ [§9, eq. (113)]
$G^{(n)}$	the standard generating function [§9, eqs. (117) and (130)]
G_k	the weight k part of the generating function [§9]
$P_{\boldsymbol{\nu}}(T)$, $P_{\boldsymbol{\nu},d}(T)$	the standard spherical polynomial [§8.**A**, §9.**C**.1]

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