Nonexistence of weakly almost complex structures on Grassmannians

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Abstract

In this paper we prove that, for $2 \leq k \leq n/2$, the unoriented Grassmann manifold $G_k(\mathbf{R}^n)$ admits a weakly almost complex structure if and only if $n = 2k = 4$ or 6; for $3 \leq k \leq n/2$, none of the oriented Grassmann manifolds $\widetilde{G_k}(\mathbf{R}^n)$ - except $\widetilde{G_3}(\mathbf{R}^6)$, and a few as yet undecided ones - admits a weakly almost complex structure.

1. Introduction

For $1 \leq k < n$, let $\widetilde{G_k}(\mathbf{R}^n)$ ($G_k(\mathbf{R}^n)$ resp.) denote the oriented (unoriented) Grassmann manifold of oriented (unoriented) k-dimensional vector subspace of $\mathbf{R}^n$. $\widetilde{G_k}(\mathbf{R}^n)$ $(G_k(\mathbf{R}^n))$ is a smooth manifold of dimension $k(n-k)$. Note that $\widetilde{G_1}(\mathbf{R}^n) \cong S^{n-1}\left(G_1(\mathbf{R}^n) \cong RP^{n-1}\right)$, the $(n-1)$-sphere (real projective space), and that $\widetilde{G_k}(\mathbf{R}^n) \cong \widetilde{G_{n-k}}(\mathbf{R}^n)$ $\left(G_k(\mathbf{R}^n) \cong G_{n-k}(\mathbf{R}^n)\right)$ under the diffeomorphism that sends a k-plane V to its orthogonal complement $V^{\perp}$.

Recall that a smooth manifold M is a said to be (weakly) almost complex if its tangent bundle τM is (stably) isomorphic to the realification of a complex vector bundle over M. For example, $\widetilde{G_1}(\mathbf{R}^n) \cong S^{n-1}$ is weakly almost complex for all n, but is almost complex only when $n = 3$ or 7 ([1]); $G_1(\mathbf{R}^n) \cong RP^{n-1}$ is weakly almost complex only when n even. It is a classical result that $\widetilde{G_2}(\mathbf{R}^n) \cong SO(n)/(SO(2) \times SO(n-2))$ is an Hermitian symmetric space, and is therefore almost complex for all n. Our main results are

Theorem 1.1 *Let* $2 \leq k \leq n/2$. *Then* $G_k(\mathbf{R}^n)$ *is weakly almost complex if and only if* $n = 2k = 4$ *or 6.*

Theorem 1.2 *Let* $3 \leq k \leq n/2$. *Then* $\widetilde{G_k}(\mathbf{R}^n)$ *is not weakly almost complex if* n *is odd or if* $(n-k) \geq 8$.

Our results are sharper than that in [6]. Note that $\widetilde{G_3}(\mathbf{R}^6)$ is weakly almost complex ([6]). The unsolved cases for weakly complexility of $\widetilde{G_k}(\mathbf{R}^n)$ are: $\widetilde{G_4}(\mathbf{R}^8)$, $\widetilde{G_5}(\mathbf{R}^{10})$, $\widetilde{G_6}(\mathbf{R}^{12})$, $\widetilde{G_7}(\mathbf{R}^{14})$, $\widetilde{G_3}(\mathbf{R}^8)$, $\widetilde{G_4}(\mathbf{R}^{10})$, $\widetilde{G_5}(\mathbf{R}^{12})$ and $\widetilde{G_3}(\mathbf{R}^{10})$. Let $\widetilde{\gamma_{n,k}}(\gamma_{n,k})$ denote the canonical k-plane bundle over $\widetilde{G_k}(\mathbf{R}^n)$ $(G_k(\mathbf{R}^n))$, and let $\widetilde{\beta_{n,k}}(\beta_{n,k})$ be its orthogonal complement, whose fiber over a $V \in \widetilde{G_k}(\mathbf{R}^n)$ $(G_k(\mathbf{R}^n))$ is the vector space $V^{\perp} \subset \mathbf{R}^n$. We have bundle equivalence

$$\widetilde{\gamma_{n,k}} \oplus \widetilde{\beta_{n,k}} \cong n\varepsilon \left(\gamma_{n,k} \oplus \beta_{n,k} \cong n\varepsilon\right), \tag{1.3}$$

where ε denotes a trivial line bundle.

It is well known that the tangent bundle $\tau\widetilde{G_k}(\mathbf{R}^n)$ $(\tau G_k(\mathbf{R}_n))$ of $\widetilde{G_k}(\mathbf{R}^n)$ $(G_k(\mathbf{R}^n))$ has the following description (see [4])

$$\begin{gathered} \tau\widetilde{G_k}(\mathbf{R}^n) \cong \widetilde{\gamma_{n,k}} \otimes \widetilde{\beta_{n,k}} \\ \left(\tau G_k(\mathbf{R}^n) \cong \gamma_{n,k} \otimes \beta_{n,k}\right). \end{gathered} \tag{1.4}$$

Using (1.3) and (1.4), we obtain

$$\begin{gathered} \tau\widetilde{G_k}(\mathbf{R}^n) \oplus \left(\widetilde{\gamma_{n,k}} \otimes \widetilde{\gamma_{n,k}}\right) \cong n\widetilde{\gamma_{n,k}} \\ \left(\tau G_k(\mathbf{R}^n) \oplus \left(\gamma_{n,k} \otimes \gamma_{n,k}\right) \cong n\gamma_{n,k}\right). \end{gathered} \tag{1.5}$$

For a CW complex X, let $r : K(X) \to KO(X)$ denote the homomorphism of Abelian groups gotten by restriction of scalars to $\mathbf{R}$, and let $c : KO(X) \to K(X)$ denote the complexification, $c[\xi] = [\xi \otimes_{\mathbf{R}} \mathbb{C}]$, which is a ring homomorphism.

We have the following identity:

$$rc(x) = 2x \; \forall \, x \in KO(X) \tag{1.6}$$

2. The unoriented Grassmannians

<u>**Lemma 2.1**</u> $G_2(\mathbf{R}^6)$ is not weakly almost complex.

Proof: It is well known that

$$H^*(G_2(\mathbf{R}^6); \mathbf{Z}_2) \cong \mathbf{Z}_2[w_1, w_2, \overline{w}_1, \overline{w}_2, \overline{w}_3, \overline{w}_4] \text{ modulo the}$$

relation $(1 + w_1 + w_2)(1 + \overline{w}_1 + \overline{w}_2 + \overline{w}_3 + \overline{w}_4) = 1$, so

$$H^*(G_2(\mathbf{R}^6); \mathbf{Z}_2) \cong \mathbf{Z}_2[w_1, w_2] / \langle w_1^5 + w_1 w_2^2, w_1^2 w_2^2 + w_1^4 w_2 + w_2^3 \rangle.$$

The fact $H^8(G_2(\mathbf{R}^6); \mathbf{Z}_2) \cong \mathbf{Z}_2$ implies $w_2^4 \neq 0$. By (1.5), the total Stiefel-Whitney classes of $G_2(\mathbf{R}^6)$ are given by

$$\begin{aligned} w(G_2(\mathbf{R}^6)) &= (1 + w_1 + w_2)^6 / (1 + w_1^2) \\ &= 1 + (w_1^4 + w_2^2) + w_1^2 w_2^2 + w_2^4. \end{aligned}$$

This gives

$$w_2(G_2(\mathbf{R}^6)) = 0, \; w_8(G_2(\mathbf{R}^6)) = w_2^4 \neq 0.$$

The following results follows immediately from Wu's formula $sq^1 w_2 = w_1 w_2$ ([5]):

$$\begin{aligned} sq(w_1^6) &= w_1^6, \; sq(w_1^4 w_2) = w_1^4 w_2 + w_1^5 w_2, \\ sq(w_1^2 w_2^2) &= w_1^2 w_2^2, \; sq(w_2^3) = w_2^3 + w_2^3 w_1. \end{aligned}$$

Therefore, $sq^2 : H^6(G_2(\mathbf{R}^6); \mathbf{Z}_2) \to H^8(G_2(\mathbf{R}^6); \mathbf{Z}_2)$ is zero. Hence, $w_8(G_2(\mathbf{R}^6))$ is not in the image of $H^6(G_2(\mathbf{R}^6); \mathbf{Z})$ under the homomorphism sq^2. Our lemma immedaitely follows from the following criterion ([3]): M^8 admits a weakly almost complex structure iff $\delta w_2(M) = 0$ and $w_2(M) \in sq^2 H^6(M; \mathbf{Z})$.

<u>**Lemma 2.2**</u> If $G_k(\mathbf{R}^n)$ is weakly almost complex, then so are $G_{k-1}(\mathbf{R}^{n-2})$ and $G_k(\mathbf{R}^{n-2})$.

Proof: Let us consider the maps

$$G_{k-1}(\mathbf{R}^{n-2}) \xrightarrow{i} G_{k-1}(\mathbf{R}^{n-1}) \xrightarrow{j} G_k(\mathbf{R}^n)$$

where i regards a V in $\mathbf{R}^{n-2}$ as a V in $\mathbf{R}^{n-1}$, j sends a V to $V \oplus \mathbf{R}$.

It is easy to see that

$$\begin{aligned} i^*(\gamma_{n-1,k-1}) &\cong \gamma_{n-2,k-1}, \; i^*(\beta_{n-1,k-1}) \cong \beta_{n-2,k-1} \oplus \varepsilon \\ j^*(\gamma_{n,k}) &\cong \gamma_{n-1,k-1} \oplus \varepsilon, \; j^*(\beta_{n,k}) \cong \beta_{n-1,k-1}\,. \end{aligned} \tag{2.3}$$

So we have

$$\begin{aligned}(j \circ i)^* \tau G_k(\mathbf{R}^n) &\cong i^* \circ j^* \left(\gamma_{n,k} \otimes \beta_{n,k}\right) \\ &\cong i^* \left(\gamma_{n-1,k-1} \oplus \varepsilon\right) \otimes i^* \left(\beta_{n-1,k-1}\right) \\ &\cong \left(\gamma_{n-2,k-1} \oplus \varepsilon\right) \otimes \left(\beta_{n-2,k-1} \oplus \varepsilon\right) \\ &\cong \gamma_{n-2,k-1} \otimes \beta_{n-2,k-1} \oplus \gamma_{n-2,k-1} \oplus \beta_{n-2,k-1} \oplus \varepsilon \\ &\cong \tau G_{k-1}\left(\mathbf{R}^{n-2}\right) \oplus (n-1)\varepsilon.\end{aligned}$$

So the conclusion for $G_{k-1}\left(\mathbf{R}^{n-2}\right)$ is true.

Let us consider the maps

$$G_k\left(\mathbf{R}^{n-2}\right) \xrightarrow{i_1} G_k\left(\mathbf{R}^{n-1}\right) \xrightarrow{i_2} G_k(\mathbf{R}^n).$$

By (2.3), we obtain

$$\begin{aligned}(i_2 \circ i_1)^* \tau G_k(\mathbf{R}^n) &\cong i_1^* \circ i_2^* \left(\gamma_{n,k} \otimes \beta_{n,k}\right) \\ &\cong i_1^* \left(\gamma_{n-1,k}\right) \otimes i_1^* \left(\beta_{n-1,k}\right) \oplus \varepsilon \\ &\cong \gamma_{n-2,k} \otimes \left(\beta_{n-2,k} \oplus \varepsilon \oplus \varepsilon\right) \\ &\cong \tau G_k\left(\mathbf{R}^{n-2}\right) \oplus 2\gamma_{n-2,k}\,.\end{aligned}$$

By (1.6), $2\gamma_{n-2,k}$ is in the image of $r : k\left(G_k\left(\mathbf{R}^{n-2}\right)\right) \to KO\left(G_k\left(\mathbf{R}^{n-2}\right)\right)$. These completes the proof.

Proof of theorem 1.1 The statement that $G_2\left(\mathbf{R}^4\right)$ and $G_3\left(\mathbf{R}^6\right)$ are weakly almost complex was obtained in [6].

We note that $G_2\left(\mathbf{R}^{2n+1}\right)$ is not weakly almost complex, since it is not orientable. The "only if" part of the theorem may be shown by using this fact, lemma 2.1 and lemma 2.2 repeatedly.

Remark: Borel and Hirzebruch [2, p. 526] proved that $G_2(\mathbf{R}^n)$ is not almost complex if $n \geq 5$. We extend their results.

3. The oriented Grassmannians

Proof of theorem 1.2 If n is odd, $3 \leq k \leq n/2$, then $\widetilde{G_k}(\mathbf{R}^n)$ is not weakly almost complex. The reason is that $w_3\left(\widetilde{G_k}(\mathbf{R}^n)\right) \neq 0$ ([6]).

By lemma 2.1, $G_2\left(\mathbf{R}^6\right)$ is not weakly almost complex. But $\tau G_2\left(\mathbf{R}^6\right) \oplus (\gamma_{6,2} \otimes \gamma_{6,2}) \cong 6\gamma_{6,2}$. So we see that the element $\gamma_{6,2} \otimes \gamma_{6,2}$ is not in the image of $r : K\left(G_2\left(\mathbf{R}^6\right)\right) \to KO\left(G_2\left(\mathbf{R}^6\right)\right)$.

Let ξ denote the line bundle whose $w_1(\xi)$ equals $w_1(\gamma_{6,2})$, then $\xi \oplus \gamma_{6,2}$ is an orientable 3-plane bundle with

$$(\xi \oplus \gamma_{6,2}) \otimes (\xi \oplus \gamma_{6,2}) \cong \gamma_{6,2} \otimes \gamma_{6,2} \oplus 2\gamma_{6,2} \oplus \varepsilon.$$

Then we have that

$$(\xi \oplus \gamma_{6,2})^2 \oplus \varepsilon \,\overline{\in}\, \operatorname{Im} r. \tag{3.1}$$

Now let n be even, $k \geq 3$, and $n - k \geq 8 = \dim G_2(\mathbf{R}^6)$. Since $\widetilde{G_k}(\mathbf{R}^n)$ is $(n-k)$-universal for orientable k-plane bundles, there exists a map $f : G_2(\mathbf{R}^6) \to \widetilde{G_k}(\mathbf{R}^n)$ such that $f^*(\gamma_{n,k}) \cong \xi \oplus \gamma_{6,2} \oplus m\varepsilon$, where $m = k - 3$. We have

$$f^*(\gamma_{n,k} \otimes \gamma_{n,k}) \cong (\xi \oplus \gamma_{6,2})^2 \oplus m^2\varepsilon \oplus 2m(\xi \oplus \gamma_{6,2})$$
$$f^*\tau\widetilde{G_k}(\mathbf{R}^n) \oplus (\xi \oplus T_{6,2})^2 \oplus m^2\varepsilon \oplus 2m(\xi \oplus \gamma_{6,2}) \cong nf^*(\gamma_{n,k}).$$

Using (3.1), (1.6), and the fact that n is even, we see that $\widetilde{G_k}(\mathbf{R}^n)$ is not weakly almost complex. This completes the proof of theorem.

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