ON LINEAR FORMS WITH COEFFICIENTS IN $\mathbb{N}\zeta(1+\mathbb{N})$

The detailed version, **Part 4**

Leonid A. Gutnik

In memory of Professor Alexander Buchstab

Content

§0. Foreword

Here I have collected the auxiliary results, which may be obtained by means of the elementary considerations described in [25].

§1. For the left hand.

For a prime number p, let v_p denotes the p-adic valuation on $\mathbb{Q}$.
Lemma 4.1.1. *Let p is a prime number. Let*

$$d \in \mathbb{N}-1, r \in \mathbb{N}-1, d_1 \in \mathbb{N}-1, d_2 \in \mathbb{N}-1, r_1 \in \mathbb{N}-1, r_2 \in \mathbb{N}-1,$$

and

$$max(r_1, r_2) < p. \tag{4.1.1}$$

Then

$$p^{-d}(dp+r)! \in (-1)^d d!r! + p\mathbb{Z} \tag{4.1.2}$$

and

$$\binom{(d_1+d_2)p+r_1+r_2}{d_1p+r_1} \in \binom{d_1+d_2}{d_1}\binom{r_1+r_2}{r_1} + p\mathbb{Z}. \tag{4.1.3}$$

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Proof may be found in [25], Lemma 9.

Lemma 4.1.2. *Let p is a prime number. Let*

$$d \in \mathbb{N}, r \in \mathbb{N}, r < p, d^{\sim} \in \mathbb{N} - 1,$$

and $d^{\sim} < d$. Then

$$\binom{dp}{d^{\sim}p + r} \in d\binom{d-1}{d^{\sim}}\binom{p}{r} + p^2\mathbb{Z}. \tag{4.1.4}$$

Proof may be found in [25], Lemma 10.

Let $l \in \mathbb{N}$ and p is arbitrary prime number in $(5, +\infty)$.

Since, according to (1.2.16), (i.e. Part 1, (2.16)),

$$\alpha^*_{\nu,k} = (-1)^{\nu+\nu\delta+k}\binom{\nu\delta}{k}^3\binom{\nu\delta+k}{\nu\delta-\nu}^3, \tag{4.1.5}$$

where $k = 0, \dots, \nu\delta$, it follows that, in view of the Lemma 4.1.1, if $v_p(k) = 0$, then

$$v_p(\alpha^*_{pl,k}) \geq 3. \tag{4.1.6}$$

In view of the Lemma 4.1.1,

$$v_p(\alpha^*_{pl,pd} - \alpha^*_{l,d}) \geq 1, \tag{4.1.7}$$

where $d = 0, \dots, l\Delta$, and therefore

$$v_p(\alpha^*(z, pl) - \alpha^*(z^p, l) \geq 1. \tag{4.1.8}$$

If $a \in \mathbb{Z}, b \in \mathbb{N}, |a| \geq b, z = a/b, p > |a| + b$, then

$$v_p(z^pm - z^m) \geq 1 \tag{4.1.9}$$

for $m \in \mathbb{Z}$ and therefore

$$v_p(\alpha^*(z, pl) - \alpha^*(z, l) \geq 1. \tag{4.1.10}$$

According to (1.2.17) and (1.2.18),

$$\beta^*_{\nu,k} = 3\alpha^*_{\nu,k}\left(-\left(\sum_{\kappa=\nu+k+1}^{\nu\Delta+k} 1/\kappa\right) - \left(\sum_{\kappa=1}^{\nu\Delta-k} 1/\kappa\right) + \left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right), \tag{4.1.11}$$

$$2\gamma^*_{\nu,k} = 9\alpha^*_{\nu,k}\left(-\left(\sum_{\kappa=\nu+k+1}^{\nu\Delta+k} 1/\kappa\right)-\left(\sum_{\kappa=1}^{\nu\Delta-k} 1/\kappa\right)+\left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right)^2 - \tag{4.1.12}$$
$$3\alpha^*_{\nu,k}\left(\left(\sum_{\kappa=\nu+k+1}^{\nu\Delta+k} 1/\kappa^2\right)-\left(\sum_{\kappa=1}^{\nu\Delta-k} 1/\kappa^2\right)-\left(\sum_{\kappa=1}^{k} 1/\kappa^2\right)\right),$$

where $k = 0,\dots,\nu\Delta$. Let $p > 2l\Delta$. In view of (4.1.6), (4.1.11) and (4.1.12), if $k \in [0, pl\Delta] \cap \mathbb{Z}$ and $v_p(k) = 0$, then

$$v_p(\beta^*_{pl,k}) \geq 2, v_p(\gamma^*_{pl,k}) \geq 1. \tag{4.1.13}$$

Let $\mathfrak{O}_p$ is the ring of all the $p-$ integers of the field $\mathbb{Q}$. Then

$$\left(-\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} 1/\kappa\right)-\left(\sum_{\kappa=1}^{pl\Delta-k} 1/\kappa\right)+\left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right) - \tag{4.1.14}$$
$$\left(-\left(\sum_{\substack{\kappa=pl+k+1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta+k} 1/\kappa\right)-\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta-k} 1/\kappa\right)+\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{k} 1/\kappa\right)\right) \in \mathfrak{O}_p,$$

$$\left(-\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} 1/\kappa\right)-\left(\sum_{\kappa=1}^{pl\Delta-k} 1/\kappa\right)+\left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right)^2 - \tag{4.1.15}$$
$$\left(-\left(\sum_{\substack{\kappa=pl+k+1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta+k} 1/\kappa\right)-\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta-k} 1/\kappa\right)+\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{k} 1/\kappa\right)\right)^2 \in \mathfrak{p}^{-1}\mathfrak{O}_p,$$

$$\left(\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} 1/\kappa^2\right)-\left(\sum_{\kappa=1}^{pl\Delta-k} 1/\kappa^2\right)-\left(\sum_{\kappa=1}^{k} 1/\kappa^2\right)\right) - \tag{4.1.16}$$
$$\left(\left(\sum_{\substack{\kappa=pl+k+1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta+k} 1/\kappa^2\right)-\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta-k} 1/\kappa^2\right)-\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{k} 1/\kappa^2\right)\right) \in \mathfrak{O}_p,$$

and, if $k = pd$, with $d = 0\,, \ldots\,, l\Delta$, then

$$\left(-\left(\sum_{\substack{\kappa=pl+pd+1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta+pd} 1/\kappa\right)-\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta-pd} 1/\kappa\right)+\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{pd} 1/\kappa\right)\right)= \tag{4.1.17}$$

$$\left(-\left(\sum_{\delta=l+d+1}^{l\Delta+d} 1/(p\delta)\right)-\left(\sum_{\delta=1}^{l\Delta-d} 1/(p\delta)\right)+\left(\sum_{\kappa=1}^{d} 1/(p\delta)\right)\right)=$$

$$p^{-1}\left(-\left(\sum_{\delta=l+d+1}^{l\Delta+d} 1/\delta\right)-\left(\sum_{\delta=1}^{l\Delta-d} 1/\delta\right)+\left(\sum_{\delta=1}^{d} 1/\delta\right)\right),$$

$$\left(-\left(\sum_{\substack{\kappa=pl+pd+1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta+pd} 1/\kappa\right)-\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta-pd} 1/\kappa\right)+\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{pd} 1/\kappa\right)\right)^2= \tag{4.1.18}$$

$$p^{-2}\left(-\left(\sum_{\delta=l+d+1}^{l\Delta+d} 1/\delta\right)-\left(\sum_{\delta=1}^{l\Delta-d} 1/\delta\right)+\left(\sum_{\delta=1}^{d} 1/\delta\right)\right)^2,$$

$$\left(\left(\sum_{\substack{\kappa=pl+pd+1\\ \kappa\equiv 0\,(mod\,p)}}^{pl\Delta+pd} 1/\kappa^2\right)-\left(\sum_{\substack{\kappa=1\\ \kappa^2\equiv 0\,(mod\,p)}}^{pl\Delta-pd} 1/\kappa\right)-\left(\sum_{\substack{\kappa=1\\ \kappa\equiv 0\,(mod\,p)}}^{pd} 1/\kappa^2\right)\right)= \tag{4.1.19}$$

$$p^{-2}\left(\left(\sum_{\delta=l+d+1}^{l\Delta+d} 1/\delta^2\right)-\left(\sum_{\delta=1}^{l\Delta-d} 1/\delta^2\right)-\left(\sum_{\delta=1}^{d} 1/\delta^2\right)\right).$$

In view of (4.1.13) - (4.1.18), if $p > 2l\Delta$, then

$$\left(-\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} 1/\kappa\right)-\left(\sum_{\kappa=1}^{pl\Delta-k} 1/\kappa\right)+\left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right)\in p^{-1}\mathfrak{O}_p, \tag{4.1.20}$$

$$\left(-\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} 1/\kappa\right)-\left(\sum_{\kappa=1}^{pl\Delta-k} 1/\kappa\right)+\left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right)^2\in p^{-2}\mathfrak{O}_p, \tag{4.1.21}$$

$$\left(\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} 1/\kappa^2\right) - \left(\sum_{\kappa=1}^{pl\Delta-k} 1/\kappa^2\right) + \left(\sum_{\kappa=1}^{k} 1/\kappa^2\right)\right) \in p^{-2}\mathfrak{O}_p. \tag{4.1.22}$$

In view of (4.1.7),(4.1.11),(4.1.12), (4.1.14) - (4.1.22),

$$\begin{gathered}\beta^*_{pl,pd} - p^{-1}\beta^*_{l,d} = \\ 3(\alpha^*_{pl,pd} - \alpha^*_{l,d}) \left(-\left(\sum_{\kappa=pl+k+1}^{pl\Delta+k} 1/\kappa\right) - \left(\sum_{\kappa=1}^{pl\Delta-k} 1/\kappa\right) + \left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right) + \\ 3\alpha^*_{l,d}\left(\left(-\left(\sum_{\kappa=pl+pd+1}^{pl\Delta+pd} 1/\kappa\right) - \left(\sum_{\kappa=1}^{pl\Delta-pd} 1/\kappa\right) + \left(\sum_{\kappa=1}^{pd} 1/\kappa\right)\right) - \right. \\ \left. p^{-1}\left(-\left(\sum_{\delta=l+d+1}^{l\Delta+\delta} 1/\delta\right) - \left(\sum_{\delta=1}^{l\Delta-d} 1/\delta\right) + \left(\sum_{\kappa=1}^{d} 1/\delta\right)\right)\right) \in \mathfrak{O}_p,\end{gathered} \tag{4.1.23}$$

$$\begin{gathered}2\gamma^*_{pl,pd} - p^{-2}\gamma^*_{l,d} = \\ 9(\alpha^*_{pl,pd} - \alpha^*_{l,d}) \left(-\left(\sum_{\kappa=pl+pd+1}^{pl\Delta+pd} 1/\kappa\right) - \left(\sum_{\kappa=1}^{pl\Delta-pd} 1/\kappa\right) + \left(\sum_{\kappa=1}^{pd} 1/\kappa\right)\right)^2 - \\ 3(\alpha^*_{pl,pd} - \alpha^*_{l,d}) \left(\left(\sum_{\kappa=pl+pd+1}^{pl\Delta+pd} 1/\kappa^2\right) - \left(\sum_{\kappa=1}^{pl\Delta-pd} 1/\kappa^2\right) - \left(\sum_{\kappa=1}^{pd} 1/\kappa^2\right)\right) + \\ 9\alpha^*_{l,d}\left(\left(-\left(\sum_{\kappa=pl+pd+1}^{pl\Delta+pd} 1/\kappa\right) - \left(\sum_{\kappa=1}^{pl\Delta-pd} 1/\kappa\right) + \left(\sum_{\kappa=1}^{pd} 1/\kappa\right)\right)^2 - \right. \\ \left. p^{-2}\left(-\left(\sum_{\delta=l+d+1}^{l\Delta+\delta} 1/\delta\right) - \left(\sum_{\delta=1}^{l\Delta-d} 1/\delta\right) + \left(\sum_{\kappa=1}^{d} 1/\delta\right)\right)^2\right) - \\ 3\alpha^*_{l,d}\left(\left(\left(\sum_{\kappa=pl+pd+1}^{pl\Delta+pd} 1/\kappa^2\right) - \left(\sum_{\kappa=1}^{pl\Delta-pd} 1/\kappa^2\right) + \left(\sum_{\kappa=1}^{pd} 1/\kappa^2\right)\right) - \right. \\ \left. p^{-2}\left(-\left(\sum_{\delta=l+d+1}^{l\Delta+\delta} 1/\delta^2\right) - \left(\sum_{\delta=1}^{l\Delta-d} 1/\delta^2\right) - \left(\sum_{\delta=1}^{d} 1/\delta^2\right)\right)\right) \in p^{-1}\mathfrak{O}_p.\end{gathered} \tag{4.1.24}$$

In view of (4.1.13), (4.1.23), (4.1.24),

(4.1.25) $$v_p(\beta^*(z;pl)-p^{-1}\beta^*(z^p;l))\geq 0, v_p(\gamma^*(z,\ pl)-p^{-2}\gamma^*(z^p,\ l))\geq -1$$

and, if $a\in\mathbb{Z}, b\in\mathbb{N}, |a|>0, z=a/b, p>2l\Delta+|a|+b$, in view of (4.1.9),

(4.1.26) $$v_p(\beta^*(z;pl)-p^{-1}\beta^*(z;l))\geq 0, v_p(\gamma^*(z,\ pl)-p^{-2}\gamma^*(z,\ l))\geq -1.$$

According to (1.2.24), (1.2.26) and (1.2.28),

(4.1.27) $$\phi^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-3}+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-2}+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-1}=$$

$$=(-z)^\nu\sum_{t=1}^{\nu}z^{-t}\big(t^{-3}\alpha^*(z;\nu)+t^{-2}\beta^*(z;\nu)+t^{-1}\gamma^*(z;\nu)\big)+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-3}+(-z)^\nu\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-2}+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-1}=$$

$$=(-z)^\nu\sum_{t=1}^{\nu}z^{-t}\big(t^{-3}\alpha^*(z;\nu)+t^{-2}\beta^*(z;\nu)+t^{-1}\gamma^*(z;\nu)\big)+$$

$$(-z)^\nu\sum_{t=1+\nu}^{\nu\Delta+\nu}z^{-t}t^{-3}\sum_{k=t-\nu}^{\nu\Delta}\alpha^*_{\nu,k}z^k+(-z)^\nu\sum_{t=1+\nu}^{\nu\Delta+\nu}z^{-t}t^{-2}\sum_{k=t-\nu}^{\nu\Delta}\beta^*_{\nu,k}z^k+$$

$$(-z)^\nu\sum_{t=1+\nu}^{\nu\Delta+\nu}z^{-t}t^{-1}\sum_{k=t-\nu}^{\nu\Delta}\gamma^*_{\nu,k}z^k;$$

(4.1.28) $$\psi^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}3\alpha^*_{\nu,k}\,z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-4}+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}2\beta^*_{\nu,k}\,z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-3}+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}\,z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-2}=$$

$$= (-z)^\nu \sum_{t=1}^{\nu} z^{-t}\big(t^{-4}3\alpha^*(z;\nu) + t^{-3}2\beta^*(z;\nu) + t^{-2}\gamma^*(z;\nu)\big) +$$

$$+ (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\alpha^*_{\nu,k} z^k \sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-4} + (-z)^\nu \sum_{k=0}^{\nu\Delta} 2\beta^*_{\nu,k} z^k \sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-3} +$$

$$+ (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-2} =$$

$$= (-z)^\nu \sum_{t=1}^{\nu} z^{-t}\big(t^{-4}3\alpha^*(z;\nu) + t^{-3}2\beta^*(z;\nu) + t^{-2}\gamma^*(z;\nu)\big) +$$

$$(-z)^\nu \sum_{t=1+\nu}^{\nu\Delta+\nu} z^{-t}t^{-4} \sum_{k=t-\nu}^{\nu\Delta} 3\alpha^*_{\nu,k} z^k + (-z)^\nu \sum_{t=1+\nu}^{\nu\Delta+\nu} z^{-t}t^{-3} \sum_{k=t-\nu}^{\nu\Delta} 2\beta^*_{\nu,k} z^k +$$

$$(-z)^\nu \sum_{t=1+\nu}^{\nu\Delta+\nu} z^{-t}t^{-2} \sum_{k=t-\nu}^{\nu\Delta} \gamma^*_{\nu,k} z^k;$$

$$\xi^*(z;\nu) = (-z)^\nu \sum_{k=0}^{\nu\Delta} 6\alpha^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t}t^{-5} + \tag{4.1.29}$$

$$+ (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\beta^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t}t^{-4} + (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1}^{k+\nu} z^{-t}t^{-3} =$$

$$= (-z)^\nu \sum_{t=1}^{\nu} z^{-t}\big(6t^{-5}\alpha^*(z;\nu) + 3t^{-4}\beta^*(z;\nu) + t^{-3}\gamma^*(z;\nu)\big) +$$

$$+ (-z)^\nu \sum_{k=0}^{\nu\Delta} 6\alpha^*_{\nu,k} z^k \sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-5} + (-z)^\nu \sum_{k=0}^{\nu\Delta} 3\beta^*_{\nu,k} z^k \sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-4} +$$

$$+ (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma^*_{\nu,k} z^k \sum_{t=1+\nu}^{k+\nu} z^{-t}t^{-3} =$$

$$= (-z)^\nu \sum_{t=1}^{\nu} z^{-t}\big(t^{-5}6\alpha^*(z;\nu) + t^{-4}3\beta^*(z;\nu) + t^{-3}\gamma^*(z;\nu)\big) +$$

$$(-z)^\nu \sum_{t=1+\nu}^{\nu\Delta+\nu} z^{-t}t^{-5} \sum_{k=t-\nu}^{\nu\Delta} 6\alpha^*_{\nu,k} z^k + (-z)^\nu \sum_{t=1+\nu}^{\nu\Delta+\nu} z^{-t}t^{-4} \sum_{k=t-\nu}^{\nu\Delta} 3\beta^*_{\nu,k} z^k +$$

$$(-z)^{\nu}\sum_{t=1+\nu}^{\nu\Delta+\nu} z^{-t}t^{-3}\sum_{k=t-\nu}^{\nu\Delta}\gamma^*_{\nu,k}z^k.$$

According to (4.1.10), (4.1.25) - (4.1.29), if $a\in\mathbb{Z}, b\in\mathbb{N}, |a|>0, z=a/b, p>2l\Delta+|a|+b$, then

$$v_p(\phi^*(z;pl)-p^{-3}\phi^*(z;l))\geq -2, \tag{4.1.30}$$

$$v_p(\psi^*(z;pl)-p^{-4}\psi^*(z;l))\geq -3, \tag{4.1.31}$$

$$v_p(\xi^*(z;pl)-p^{-5}\xi^*(z;l))\geq -4. \tag{4.1.32}$$

Let

$$\alpha^*_0(z;\nu)=\alpha^*(z;\nu), \alpha^*_1(z;\nu)=\beta^*(z;\nu), \alpha^*_2(z;\nu)=\gamma^*(z;\nu),$$

(4.1.33)

$$\alpha^*_3(z;\nu)=\phi^*(z;\nu), \alpha^*_4(z;\nu)=\psi^*(z;\nu), \alpha^*_5(z;\nu)=\xi^*(z;\nu),$$

where $\nu\in\mathbb{N}$. Then (4.1.10), (4.1.26), (4.1.30) - (4.1.32) may be rewritten in the form

$$v_p(\alpha^*_j(z;pl)-p^{-j}\alpha^*_j(z;l))\geq 1-j, \tag{4.1.33}$$

where $j=0,\dots,5$.

Lemma 4.1.3. *Let $k_j\in\mathbb{N}-1$, where $j=0,\dots,5, \sum\_j=1^5>0, J=\{j\in\{0,\dots,5\}\colon k_j>0\}$ and $\prod_{j\in J}\alpha^*_j(z;l_j)\neq 0$ for some $z\in\mathbb{Q}\setminus\{0\}$ and $l_j)\in\mathbb{N}$, with $j\in J$. Then, for each $m\in\mathbb{N}$ the sequences*

$$\alpha^*_j(z;m+1),\dots,\alpha^*_j(z;m+\mu),\dots \tag{4.1.34}$$

with $j\in J$ compose a linearly independent system over $\mathbb{C}$.

Proof. Since $z\in\mathbb{Q}$, it follows that $\alpha^*_j(z;l_j)\in\mathbb{Q}$ for $j\in J$ and there exists $d\in\mathbb{N}$, such that $d\alpha^*_j(z;l_j)\in\mathbb{N}$ for $j\in J$. Since $\alpha^*_j(z;\nu)\in\mathbb{Q}$ for $j\in J$ and $\nu\in\mathbb{N}$, it follows that it is sufficient to prove the linear independence of the system (4.1.34) over $\mathbb{Q}$. The contrary means the existence $b_j\in\mathbb{Z}$, where $j\in J$, such that

$$\sum_{j\in J}|b_j|>0, \sum_{j\in J}b_j\alpha^*_j(z;\nu)=0, \tag{4.1.35}$$

where $\nu \in m + \mathbb{N}$. Let $k = \sup\{j \in J : b_j \neq 0\}$ and let p is the prime number such that

$$p > \sum_{j \in J} d|b_j \alpha_j^*(z;\nu)| + m + 1 + d.$$

We take now in (4.1.33) $\nu = pl_k$. Then according (4.1.35), (4.1.33),

$$-k = v_p(\alpha_j^*(z;pl_k)) = v_p\left(\sum_{j \in J \setminus \{k\}} b_j \alpha_j^*(z;\nu)\right) \geq 1 - k.$$

■

Lemma 4.1.4. *Let Δ is equal to the prime number $p > 5$. If $z = 1$, then condition of the Lemma 4.1.3 is fulfilled for $J = \{0,1,2,4,5\}$. If $z = -1$, then condition of the Lemma 4.1.3 is fulfilled for $J = \{0,1,2,3,4,5\}$.*

Proof. In view of (4.1.5),

$$\alpha_{1,0}^* = p^3. \tag{4.1.36}$$

and,for $k = 1, \dots, p$ there exist $x_k \in \mathfrak{O}_p$, such that

$$\alpha_{1,p-1}^* = p^3 \left(\prod_{k=1}^{p-1}(1+\frac{p}{k}\right)^3 \in p^3 + p^5\mathfrak{O}_p, \tag{4.1.37}$$

$$\alpha_{1,p}^* = -8p^3 \left(\prod_{k=1}^{p-1}(1+\frac{p}{k}\right)^3 /(1+p)^3 \in -8p^3 + 24p^4 + p^5\mathfrak{O}_p, \tag{4.1.38}$$

$$\alpha_{1,k}^* \in p^6\mathfrak{O}_p, \tag{4.1.39}$$

where $k = 1, \dots, p-2$. According to (1.2.21),

$$\alpha^*(z;\nu) = (-z)^\nu \sum_{k=0}^{\nu\Delta} \alpha_{\nu,k}^* z^k, \beta^*(z;\nu) = (-z)^\nu \sum_{k=0}^{\nu\Delta} \beta_{\nu,k}^* z^k, \tag{4.1.40}$$
$$\gamma^*(z;\nu) = (-z)^\nu \sum_{k=0}^{\nu\Delta} \gamma_{\nu,k}^* z^k,$$

where $\nu \in \mathbb{N} - 1$. In view of (4.1.40), (4.1.36) - (4.1.39),

$$\alpha^*(1;1) \in 6p^3 - 24p^4 + p^5\mathfrak{O}_p, \alpha^*(-1;1) \in 10p^3 - 24p^4 + p^5\mathfrak{O}_p. \tag{4.1.41}$$

In view of (4.1.11), (4.1.12), (4.1.36) - (4.1.39),

$$\beta^*_{1,0} = 3\alpha^*_{1,0}\left(-\left(\sum_{\kappa=2}^{p} 1/\kappa\right) - \left(\sum_{\kappa=1}^{p} 1/\kappa\right)\right) \in \tag{4.1.42}$$

$$3p^3(-\frac{2}{p} + 1 + p\mathfrak{O}_p) = 3p^2(-2 + p + p^2\mathfrak{O}_p,$$

$$-\left(\sum_{\kappa=p+1}^{2p-1} 1/\kappa\right) - 1 + \left(\sum_{\kappa=1}^{p-1} 1/\kappa\right) =$$

$$-1 + p\sum_{\kappa=1}^{p-1}(\frac{1}{\kappa(p+\kappa)} \in -1 - p^2\mathfrak{O}_p,$$

$$\beta^*_{1,p-1} = 3\alpha^*_{1,p-1}\left(-\left(\sum_{\kappa=p+1}^{2p-1} 1/\kappa\right) - 1 + \left(\sum_{\kappa=1}^{p-1} 1/\kappa\right)\right) \in \tag{4.1.43}$$

$$-3p^3(1 + p^2\mathfrak{O}_p)(1 + p^2\mathfrak{O}_p) = -3p^3(1 + p^2\mathfrak{O}_p),$$

$$-\left(\sum_{\kappa=p+2}^{2p} 1/\kappa\right) + \left(\sum_{\kappa=1}^{p} 1/\kappa\right) =$$

$$\frac{1}{2p} + \frac{1}{p+1} + p\left(\sum_{\kappa=1}^{p-1} \frac{1}{\kappa(p+\kappa)}\right) \in \frac{1}{2p} + 1 - p + p^2\mathfrak{O}_p = \frac{1}{2p}(1 + 2p - 2p^2 + p^3\mathfrak{O}_p),$$

$$\beta^*_{1,p} = 3\alpha^*_{1,p}\left(-\left(\sum_{\kappa=p+2}^{2p} 1/\kappa\right) + \left(\sum_{\kappa=1}^{p} 1/\kappa\right)\right) \in \tag{4.1.44}$$

$$-12p^2(1 - 3p + p^2\mathfrak{O}_p)(1 + 2p - p^2 + p^3\mathfrak{O}_p)) = -12p^2(1 - p + p^2\mathfrak{O}_p),$$

$$\beta^*_{1,k} = 3\alpha^*_{1,k}\left(-\left(\sum_{\kappa=k+2}^{p+k} 1/\kappa\right) - \left(\sum_{\kappa=1}^{p-k} 1/\kappa\right) + \left(\sum_{\kappa=1}^{k} 1/\kappa\right)\right) \in p^5\mathfrak{O}_p, \tag{4.1.45}$$

where $k = 1, \dots, p-2$,

$$2\gamma^*_{1,0} = 9\alpha^*_{1,0}\Bigg(-\left(\sum_{\kappa=2}^{p} 1/\kappa\right) - \left(\sum_{\kappa=1}^{p} 1/\kappa\right)\Bigg)^2 - \tag{4.1.46}$$

$$3\alpha^*_{1,0}\left(\sum_{\kappa=2}^{p} 1/\kappa^2\right) - \left(\sum_{\kappa=1}^{p} 1/\kappa^2\right)\Bigg) \in$$

$$9p(-2+p+p^2\mathfrak{O}_p)^2-3p^3\subset 36p(1-p+p^2\mathfrak{O}_p),$$

$$\left(\sum_{\kappa=p+1}^{2p-1}1/\kappa^2\right)-\left(\sum_{\kappa=1}^{p-1}1/\kappa^2\right)=$$

$$-p\left(\sum_{\kappa=1}^{p-1}\frac{2\kappa+p}{\kappa^2(\kappa+p)^2}\right)\in p^2\mathfrak{O}_p,$$

$$2\gamma^*_{1,p-1}=9\alpha^*_{1,p-1}\left(-\left(\sum_{\kappa=p+1}^{2p-1}1/\kappa\right)-1+\left(\sum_{\kappa=1}^{p-1}1/\kappa\right)\right)^2-
3\alpha^*_{1,p-1}\left(\left(\sum_{\kappa=p+1}^{2p-1}1/\kappa^2\right)-1-\left(\sum_{\kappa=1}^{p-1}1/\kappa^2\right)\right)\in \tag{4.1.47}$$

$$9p^3(1+p^2\mathfrak{O}_p)(1+p^2\mathfrak{O}_p)^2-3p^3(1+p^2\mathfrak{O}_p)(-1+p^2\mathfrak{O}_p)=12p^3(1+p^2\mathfrak{O}_p),$$

$$\left(\sum_{\kappa=p+2}^{2p}1/\kappa^2\right)-\left(\sum_{\kappa=1}^{p}1/\kappa^2\right)=$$

$$-\frac{3}{4p^2}-\frac{1}{(p+1)^2}-p\left(\sum_{\kappa=1}^{p-1}\frac{p+2\kappa}{\kappa^2(p+\kappa)^2}\right)=$$

$$-\frac{3}{4p^2}-1+2p-p^2x_{2p+8}\in-\frac{1}{4p^2}(3+4p^2-8p^3+4p^4\mathfrak{O}_p),$$

$$2\gamma^*_{1,p}=9\alpha^*_{1,p}\left(-\left(\sum_{\kappa=p+2}^{2p}1/\kappa\right)+\left(\sum_{\kappa=1}^{p}1/\kappa\right)\right)^2-
3\alpha^*_{1,p}\left(\left(\sum_{\kappa=p+2}^{2p}1/\kappa^2\right)-\left(\sum_{\kappa=1}^{p}1/\kappa^2\right)\right)\in \tag{4.1.48}$$

$$-18p(1-3p+p^2\mathfrak{O}_p)(1+2p-2p^2+p^3\mathfrak{O}_p)^2-
6p(1-3p+p^2x_p)(3+4p^2-8p^3+4p^4x_{2p+8})=p(-36+36p+p^2\mathfrak{O}_p),$$

$$2\gamma^*_{1,k}=9\alpha^*_{1,k}\left(-\left(\sum_{\kappa=2+k}^{p+k}1/\kappa\right)-\left(\sum_{\kappa=1}^{p-k}1/\kappa\right)+\left(\sum_{\kappa=1}^{k}1/\kappa\right)\right)^2- \tag{4.1.49}$$

$$3\alpha^*_{\nu,k}\left(\left(\sum_{\kappa=2+k}^{p+k}1/\kappa^2\right)-\left(\sum_{\kappa=1}^{p-k}1/\kappa^2\right)-\left(\sum_{\kappa=1}^{k}1/\kappa^2\right)\right)\in p^4\mathfrak{O}_p,$$

where $k=1\,,\ldots\,,p-2$.

In view of (4.1.40), (4.1.42) - (4.1.45),

$$\beta^*(1,1)\in 18p^2-12p^3+p^4\mathfrak{O}_p, \tag{4.1.50}$$

$$\beta^*(-1,1)\in 6p^2-12p^3+p^4\mathfrak{O}_p. \tag{4.1.51}$$

In view of (4.1.46) - (4.1.49),

$$2\gamma^*(\varepsilon,1)\in 36(1-\varepsilon)p(1-p)+p^3\mathfrak{O}_p, \tag{4.1.52}$$

where $\varepsilon^2=1$. In view of (4.1.27) - (4.1.29), (4.1.36) - (4.1.39), (4.1.41), (4.1.50) - (4.1.52), if $\varepsilon^2=1$, then

$$\phi^*(\varepsilon;1)=-\big(\alpha^*(\varepsilon;1)\beta^*(\varepsilon;1)+\gamma^*(\varepsilon;1)\big)+$$

$$(-\varepsilon)\sum_{t=2}^{p+1}\varepsilon^{-t}t^{-3}\sum_{k=t-1}^{p}\alpha^*_{1,k}\varepsilon^k+(-\varepsilon)\sum_{t=1+2}^{p+1}\varepsilon^{-t}t^{-2}\sum_{k=t-1}^{p}\beta^*_{1,k}\varepsilon^k+$$

$$(-\varepsilon)\sum_{t=2}^{p+1}\varepsilon^{-t}t^{-1}\sum_{k=t-1}^{p}\gamma^*_{1,k}\varepsilon^k\in$$

$$-p^{-3}(\alpha^*_{1,p-1}+\alpha^*_{1,p}\varepsilon)-p^{-2}(\beta^*_{1,p-1}+\beta^*_{1,p}\varepsilon)-p^{-1}(\gamma^*_{1,p-1}+\gamma^*_{1,p}\varepsilon)+p\mathfrak{O}_p\subset$$

$$-1+38\varepsilon+p\mathfrak{O}_p,$$

$$\psi^*(\varepsilon;1)\in$$

$$-p^{-4}(3\alpha^*_{1,p-1}+3\alpha^*_{1,p}\varepsilon)-p^{-3}(2\beta^*_{1,p-1}+2\beta^*_{1,p}\varepsilon)-p^{-2}(\gamma^*_{1,p-1}+\gamma^*_{1,p}\varepsilon)+p\mathfrak{O}_p\subset$$

$$p^{-1}(-3+66\varepsilon)+\mathfrak{O}_p,$$

$$\xi^*(\varepsilon;1)\in$$

$$-p^{-5}(6\alpha^*_{1,p-1}+6\alpha^*_{1,p}\varepsilon)-p^{-4}(3\beta^*_{1,p-1}+3\beta^*_{1,p}\varepsilon)-p^{-3}(\gamma^*_{1,p-1}+\gamma^*_{1,p}\varepsilon)+p\mathfrak{O}_p\subset$$

$$p^{-2}(-6+102\varepsilon)+p^{-1}\mathfrak{O}_p.$$

■

According to the Lemmata 4.1.3 and 4.1.4, for each $m \in \mathbb{N}$ the sequences of numbers

$$\alpha_j^*(-1; m+1), \dots, \alpha_j^*(-1; m+\mu), \dots \tag{4.1.53}$$

where $j = 0, 1, 2, 3, 4, 5$, compose a linear independent system over $\mathbb{C}$. Consequently, for each $m \in \mathbb{N}$ the sequences of polynomials

$$\alpha_j^*(z; m+1), \dots, \alpha_j^*(z; m+\mu), \dots \tag{4.1.54}$$

with $j = 0, 1, 2, 3, 4, 5$ compose a linear independent system over $\mathbb{C}(z)$. Therefore for the equation (3.1.106) we have the inequality

$$q_6(z; \nu^{-1}) q_0(z; \nu^{-1}) \in \mathbb{C}[z] \diagdown 0. \tag{4.1.55}$$

§2. On one string.

Lemma 4.2.1. *If $z \geq 1$, then*

$$\lim_{\nu \to \infty} (|f_2^*(z, \nu)|)^{1/\nu} = -h^{\sim}(\eta_1^{\wedge}(z, \pi, \delta_0)),$$

$$\lim_{\nu \to \infty} f_{2+2k}^*(z, \nu) / f_2^*(z, \nu) = 2((log z)/2)^k,$$

where $k = 1, 2$, and $h^{\sim}(\eta)$ is defined in (2.4.55).

Proof Together with the function

$$R_0(t, \nu) = \frac{\prod\limits_{\kappa=1+\nu}^{\nu\Delta} (t-\kappa)}{\prod\limits_{\kappa=0}^{\nu\Delta} (t+\kappa)} \frac{(\nu\Delta)!}{(\nu d_1)!} \tag{4.2.1}$$

introduced in (1.2.6) we consider also the rational functions

$$R_1(t, \nu) = R_0(t, \nu)(t + \nu\Delta), R_2(t, \nu) = (t + \nu\Delta)^{-1},$$

where $\nu \in \mathbb{N} - 1$. Then

$$R_0(t, \nu) = R_1(t, \nu) R_2(t, \nu), \tag{4.2.2}$$

where $\nu \in \mathbb{N} - 1$. If $z \geq 1$, then, in view of (1.2.6), (1.2.15)

$$|f_2^*(z, \nu)| = z^\nu \sum_{t=1+\nu\Delta}^{+\infty} z^{-t} R_0(t; \nu)^3, \tag{4.2.3}$$

$$|f_2^*(z,\nu)| \le z^\nu \left(\sum_{t=1+\nu\Delta}^{+\infty} R_2(t;\nu)^3\right) \times \tag{4.2.4}$$

$$\sup\{z^{-t}R_1(t;\nu)^3 \colon t \in \mathbb{Z}, t \ge 1+\nu\Delta\} \le$$

$$z^\nu \left(\int_{2\nu\Delta}^{+\infty} z^{-t}t^{-3}\right) \times$$

$$\sup\{z^{-t}R_1(t;\nu)^3 \colon t \in \mathbb{Z}, t \ge 1+\nu\Delta\} =$$

$$z^\nu 2^{-3}(\nu\Delta)^{-2} \sup\{z^{-t}R_1(t;\nu)^3 \colon t \in \mathbb{Z}, t \ge 1+\nu\Delta\}.$$

Let $t = \nu\tau\Delta$. Since

$$z^{-t-1}R_1(t+1;\nu)^3 = z^{-t}R_1(t;\nu)^3 z^{-1}t^3(t-\nu)^3(t-\nu\Delta)^{-3}(t+\nu\Delta)^{-3}, \tag{4.2.5}$$

it follows that

$$z^{-t-1}R_1(t+1;\nu)^3 = z^{-t}R_1(t;\nu)^3 z^{-1}\tau^3(\tau-\delta_0)^3(\tau-1)^{-3}(\tau+1)^{-3}, \tag{4.2.6}$$

where, as in Part 1, paragraph 2, $\delta_0 = 1/\Delta$. Let

$$g(z,\tau) = z^{-1}\tau^3(\tau-\delta_0)^3(\tau-1)^{-3}(\tau+1)^{-3} - 1, \tag{4.2.7}$$

where $1 \le z, 1 \le \tau$. The substitution

$$\tau = (1+\eta)/(1-\eta) \tag{4.2.8}$$

transforms the function (4.2.7) into the function

$$h(z,\eta) = g(z,(1+\eta)/(1-\eta)) = \tag{4.2.9}$$

$$(1+\delta_0)^3 2^{-6} z^{-1}((\eta+1)^3(\eta+\gamma_1)^3 - 2^3(1+\gamma_1)^3\eta^3 z) =$$

$$(1+\delta_0)^3 2^{-6} z^{-1} D^\wedge(z,\eta),$$

where, as in Part 1, paragraph 2, $\gamma_1 = (1-\delta_0)/(1+\delta_0)$ and $D^\wedge(z,\eta)$, as in (3.0.1). Clearly, for fixed $z = |z| = r^3 \ge 1$, the polynomial $D^\wedge(z,\eta)$, has only two real roots $\eta_k^\wedge(r,\pi,\delta_0$, where $k = 0,1$, and, since we consider now $\delta_0 \in (0,1)$, according to (3.1.55), it follows that

$$0 < \eta_1^\wedge(r,\pi,\delta_0) = \gamma_1/\eta_0(r,\pi,\delta_0 \le \gamma_1 < 1 \le \eta_0(r,\pi,\delta_0, \tag{4.2.10}$$

$h(z,\eta) > 0$, if $\eta \in (0, \eta_1^\wedge(r,\pi,\delta_0)$, and $h(z,\eta) < 0$, if $\eta \in (\eta_1^\wedge(r,\pi,\delta_0), 1)$. Therefore the defined in (4.2.7) function $g(z,\tau)$ has onto $(0,+\infty)$ the single root

(4.2.11) $$\tau_1 = \tau_1(r,\delta_0) = (1+\eta_1^\wedge(r,\pi,\delta_0))/(1-\eta_1^\wedge(r,\pi,\delta_0)),$$

where $r = z^{1/3} \geq 1, \delta_0 \in (0,1)$; clearly $\delta_0 < 1 < \tau_1$; moreover $g(z,\tau) > 0$, if $\tau \in (0, \tau_1(r,\delta_0))$ and $g(z,\tau) < 0$, if $\tau \in (\tau_1(r,\delta_0), +\infty)$. Consequently,

$$(R_1(t+1,\nu))^3 z^{-t-1} \leq (R_1(t,\nu))^3 z^{-t}$$

for $t \geq \nu\Delta\tau_1 > \nu$ and

$$R_1(t+1,\nu)|z|^{-t-1} \geq R_1(t,\nu)|z|^{-t}$$

for $\nu\Delta + 1 \leq t \leq \nu\Delta\tau_1$; these inequalities are fulfilled for $\nu\Delta(\tau_1 - 1) \geq 1$, i.e

(4.2.12) $$\nu \geq (\Delta(\tau_1 - 1))^{-1},$$

and we consider further only such large ν. Therefore

(4.2.13) $$\sup\{z^{-t}R_1(t;\nu)^3 : t \in \mathbb{Z}, t \geq 1+\nu\Delta\} = z^{-\nu\Delta\tau_1 - \theta_\nu} R_1(\nu\Delta\tau_1 + \theta_\nu; \nu)^3,$$

where $\nu\Delta\tau_1 + \theta_\nu \in \mathbb{N}, O \leq \theta_\nu < 1$. Since all the summands of the sum (4.2.3) are not less than 0, in view of (4.2.4) and (4.2.13),it follows that

(4.2.14) $$z^\nu z^{-\nu\Delta\tau_1 - \theta_\nu}(\nu\Delta\tau_1 + \theta_\nu + \nu\Delta)^{-3}(R_1(\nu\Delta\tau_1 + \theta_\nu; \nu))^3 =$$

$$z^\nu z^{[-\nu\Delta\tau_1]}(-[-\nu\Delta\tau_1] + \nu\Delta)^{-3} R_1(-[-\nu\Delta\tau_1]; \nu)^3 \leq$$

$$z^\nu \sum_{t=1+\nu\Delta}^{+\infty} z^{-t} R_0(t;\nu)^3 = |f_2^*(z,\nu)| \leq$$

$$z^\nu 2^{-3}(\nu\Delta)^{-2} z^{-\nu\Delta\tau_1 - \theta_\nu}(R_1(\nu\Delta\tau_1 + \theta_\nu; \nu))^3$$

In order to study the value

$$z^{-t}(R_1(t))^3 = z^{-t}\left(\frac{\Gamma(t+\Delta\nu)\Gamma(t-\nu)}{\Gamma(t-\Delta\nu)\Gamma(t)}\frac{(\nu\Delta)!}{(\nu d_1)!}\right)^3$$

we use the Stirling's formula ([19], chapter IV, 4.4.2). Then we see that

(4.2.15) $$\log(z^{-t}(R_1(t;\nu))^3) = 3((\nu\tau\Delta - 1/2)\log(\nu\tau\Delta) - \nu\tau\Delta) -$$

$$3((\nu(\tau-1)\Delta - 1/2)\log(\nu(\tau-1)\Delta) - \nu(\tau-1)\Delta) -$$

$$3((\nu(\tau+1)\Delta-1/2)\log(\nu(\tau+1)\Delta)-\nu(\tau+1)\Delta)+$$
$$3((\nu(\tau-\delta_0)\Delta-1/2)\log(\nu(\tau-\delta_0)\Delta)-\nu(\tau-\delta_0)\Delta)+3((\nu\Delta+1/2)\log(\nu\Delta)-\nu\Delta)-$$
$$3((\nu(1-\delta_0)\Delta-1/2)\log(\nu(1-\delta_0)\Delta)-\nu(1-\delta_0)\Delta)-(\nu\tau\Delta)\log(z)+3\theta_1^*(\Delta,\tau,\nu)/4=$$
$$\left(\nu\tau\Delta-\frac{1}{2}-2\left(\nu\tau\Delta-\frac{1}{2}\right)+\nu(\tau-\delta_0)\Delta-\frac{1}{2}+\nu\Delta+\frac{1}{2}-(\nu(1-\delta_0)\Delta-\frac{1}{2}\right)\times$$
$$3(\log(\nu\Delta))+3(\nu\tau\Delta-1/2)\log(\tau)-3((\nu(\tau-1)\Delta-1/2)\log(\tau-1)-$$
$$3((\nu(\tau+1)\Delta-1/2)\log(\tau+1)+3((\nu(\tau-\delta_0)\Delta-1/2)\log(\tau-\delta_0)-$$
$$\nu(-\delta_0)\Delta)+3(-\nu\Delta)-$$
$$3((\nu(1-\delta_0)\Delta-1/2)\log(1-\delta_0)-\nu(1-\delta_0)\Delta)-(\nu\tau\Delta)\log(z)+3\theta_1^*(\Delta,\tau,\nu)/4=$$
$$3\nu\tau\Delta\log(\tau)-3\nu(\tau-1)\Delta\log(\tau-1)-$$
$$3\nu(\tau+1)\Delta\log(\tau+1)+3(\nu(\tau-\delta_0)\Delta\log(\tau-\delta_0)-$$
$$3(\nu(1-\delta_0)\Delta\log(1-\delta_0))-(\nu\tau\Delta)\log(z)-$$
$$(3/2)\log((\tau^2-\tau\delta_0)(1-\delta_0)/(\tau^2-1))+3\theta_1^*(\Delta,\tau,\nu)/4,$$

where $\nu(\tau-1)\Delta\geq 1$ and $|\theta_1^*(\Delta,\tau,\nu)|\leq 1$. Let

$$u(z,\tau)=3\tau\Delta\log(\tau)-3(\tau-1)\Delta\log(\tau-1)- \tag{4.2.16}$$
$$3(\tau+1)\Delta\log(\tau+1)+3(\tau-\delta_0)\Delta\log(\tau-\delta_0)-$$
$$3((1-\delta_0)\Delta\log(1-\delta_0))-\tau\Delta\log(z),$$

where $\tau>1$ and $z\geq 1$. In view of (4.2.15) and (4.2.16)

$$\log(z^{-\nu\tau\Delta}(R_1(\nu\tau\Delta;\nu))^3)=u(z,\tau)\nu- \tag{4.2.17}$$
$$(3/2)\log((\tau^2-\tau\delta_0)(1-\delta_0)/(\tau^2-1))+3\theta_1^*(\Delta,\tau,\nu)/4,$$

where $\nu(\tau-1)\Delta\geq 1$ and $|\theta_1^*(\Delta,\tau,\nu)|\leq 1$. Let

$$w_1(\Delta,z)=u(z,\tau_1)=3\tau_1\Delta\log(\tau_1)-3(\tau_1-1)\Delta\log(\tau_1-1)- \tag{4.2.18}$$
$$3(\tau_1+1)\Delta\log(\tau_1+1)+3(\tau_1-\delta_0)\Delta\log(\tau_1-\delta_0)-$$
$$3((1-\delta_0)\Delta\log(1-\delta_0))-\tau_1\Delta\log(z),$$

where $z\geq 1$. If $t=\nu\tau\Delta, \tau=\tau_1+\theta_\nu/(\nu\Delta)$ then

$$\log(z^{-\tau_1+\theta_\nu/(\nu\Delta)}(R_1(\tau_1+\theta_\nu/(\nu\Delta)))^3)= \tag{4.2.19}$$

$$3\nu(\tau_1+\theta_\nu/(\nu\Delta))\Delta\log(\tau_1+\theta_\nu/(\nu\Delta))-$$
$$3\nu(\tau_1+\theta_\nu/(\nu\Delta)-1)\Delta\log(\tau_1+\theta_\nu/(\nu\Delta)-1)-$$
$$3\nu(\tau_1+\theta_\nu/(\nu\Delta)+1)\Delta\log(\tau_1+\theta_\nu/(\nu\Delta)+1)+$$
$$3(\nu(\tau_1+\theta_\nu/(\nu\Delta)-\delta_0)\Delta\log(\tau_1+\theta_\nu/(\nu\Delta)-\delta_0)-$$
$$3(\nu(1-\delta_0)\Delta\log(1-\delta_0))-(\nu(\tau_1+\theta_\nu/(\nu\Delta))\Delta)\log(z)-$$
$$-(3/2)\log((\tau^2-\tau\delta_0)(1-\delta_0)/(\tau^2-1))+3\theta_1^*(\Delta,\tau,\nu)/4=$$
$$3\nu(\tau_1)\Delta\log(\tau_1)-3\nu(\tau_1-1)\Delta\log(\tau_1-1)-$$
$$3\nu(\tau_1+1)\Delta\log(\tau_1+1)+3(\nu(\tau_1-\delta_0)\Delta\log(\tau_1-\delta_0)-$$
$$3(\nu(1-\delta_0)\Delta\log(1-\delta_0))-(\nu\tau_1\Delta)\log(z)+O(1)=w_1(\Delta,z)\nu+O(1),$$

when $\nu\to\infty$. Since $g(z,\tau_1)=0$, it follows that, according to (4.2.7),

$$-\log(z)+3\log(\tau_1)+3\log(\tau_1-\delta_0)-3\log(\tau_1-1)-3\log(\tau_1+1)=0. \tag{4.2.20}$$

Multiplying (4.2.20) by the number $\Delta(\tau_1-\delta_0)$ and subtracting the obtied result from (4.2.18) we obtain the equality

$$w_1(\Delta,z)=u(z,\tau_1)=3\log(\tau_1)+3(\Delta-1)\log(\tau_1-1)- \tag{4.2.21}$$
$$3(\Delta+1)\log(\tau_1+1)-$$
$$3(\Delta-1)\log(1-\delta_0))-\log(z),$$

where $z\geq 1$. Let

$$w(z,\eta)=u(z,(1+\eta)/(1-\eta)), \tag{4.2.22}$$

where $0\leq\eta<1$. Then, in view of (4.2.21) and (4.2.11)

$$w_1(\Delta,z)=u(z,\tau_1)=w(z,\eta_1^\wedge(r,\pi,\delta_0))= \tag{4.2.23}$$
$$3\log\left(\frac{1+\eta_1^\wedge(r,\pi,\delta_0)}{1-\eta_1^\wedge(r,\pi,\delta_0)}\right)+3(\Delta-1)\log\left(\frac{1+\eta_1^\wedge(r,\pi,\delta_0)}{1-\eta_1^\wedge(r,\pi,\delta_0)}-1\right)-$$
$$3(\Delta+1)\log\left(\frac{1+\eta_1^\wedge(r,\pi,\delta_0)}{1-\eta_1^\wedge(r,\pi,\delta_0)}+1\right)-$$
$$3(\Delta-1)\log(1-\delta_0))-\log(z)=$$
$$3\log(1-(\eta_1^\wedge(r,\pi,\delta_0))^2)-6\log(2)+3(\Delta-1)\log(\eta_1^\wedge(r,\pi,\delta_0)-$$

$$3(\Delta-1)\log(1-\delta_0))-\log(z).$$

Therefore, in view of (2.4.55),

$$\exp(w_1(\Delta,z)) = \tag{4.2.24}$$

$$-z^{-1}(\eta_1^{\wedge}(r,\pi,\delta_0)^{3(\Delta-1)}((\eta_1^{\wedge}(r,\pi,\delta_0))^2-1)^3(1-\delta_0)^{-3(\Delta-1)}2^{-6} =$$

$$-z^{-1}h^{\sim}(\eta_1^{\wedge}(r,\pi,\delta_0)) = -z^{-1}P_1^{\sim}(\eta_1^{\wedge}(r,\pi,\delta_0))/Q_1^{\sim}(\eta_1^{\wedge}(r,\pi,\delta_0)).$$

According to (4.2.19) and (4.2.24), (in correspondence with (1.2.8) and (1.2.15))

$$z^{\nu}(R_1(\tau_1+\theta_\nu/)(\nu\Delta);\nu))^3 z^{-\tau_1+\theta_\nu/(\nu\Delta)} = (-h^{\sim}(\eta_1^{\wedge}(r,\pi,\delta_0)))^{\nu}\exp(O(1)), \tag{4.2.25}$$

where $z=r^3\geq 1$ and $\nu\to\infty$. If $z\geq 1$ and $\nu\in\mathbb{N}+2$, then, in view of (4.2.12),

$$|(3/2)\log((\tau^2-\tau\delta_0)(1-\delta_0)/(\tau^2-1))-3\theta_1^*(\Delta,\tau,\nu)/4| \leq \tag{4.2.26}$$

$$(3/2)\log\left(1+\frac{1+\delta_0}{\tau-\delta_0}\right)+(3/2)\log\left(1+\frac{1}{\tau-1}\right)-(3/2)\log(1-\delta_0)+3/4 \leq$$

$$(3/2)\log\left(1+\frac{1+\delta_0}{1-\delta_0}\right)+(3/2)\log(1+\Delta\nu)-(3/2)\log(1-\delta_0)+3/4 \leq$$
$$c_1(\Delta,z)(\log(\nu)),$$

where

$$c_1(\Delta,z) = ((3/2)\log\left(1+\frac{1+\delta_0}{1-\delta_0}\right)+(3/2)\log(1+\Delta)+1-(3/2)\log(1-\delta_0)+3/4.$$

In view of (4.2.17), (4.2.26),

$$\log(z^{-\nu\tau\Delta}(R_1(\nu\tau\Delta))^3) = \tag{4.2.27}$$

$$\nu w(z,\eta)+c_1(\Delta,z)\theta_2^*(\Delta,\tau,\nu)(\log(\nu)),$$

where $|\theta_2^*(\Delta,\tau,\nu)|\leq 1$. In view of (4.2.9), (4.2.10) and (4.2.22),

$$h(z,\eta_1^{\wedge}(r,\pi,\delta_0)) = 0, \tag{4.2.28}$$

$$h(z,\eta)(\eta-\eta_1^{\wedge}(r,\pi,\delta_0)) < 0, \tag{4.2.29}$$

where $z \geq 1, 0 \leq \eta < 1$ and $\eta \neq \eta_1^\wedge(r, \pi, \delta_0)$,

$$-\left(\frac{\partial}{\partial \eta} h\right)(z, \eta_1^\wedge(r, \pi, \delta_0) = \tag{4.2.30}$$

$$(1+\delta_0)^3 2^{-6} z^{-1} 24(1+\gamma_1)^2 (\eta_1^\wedge(r, \pi, \delta_0))^2 z^{2/3} \left(\frac{2z^{1/3}-1}{1+\delta} - \eta_1^\wedge(r, \pi, \delta_0\right) >$$

$$\frac{3}{2} z^{-1/3} (\eta_1^\wedge(r, \pi, \delta_0))^2 (2z^{1/3} - 2 + \delta_0) > 0,$$

$$\left(\frac{\partial}{\partial \tau} u\right)(z, \tau) = 3\Delta \log(\tau) + 3\Delta - 3\Delta \log(\tau - 1) - 3\Delta - \tag{4.2.31}$$

$$3\Delta \log(\tau + 1) - 3\Delta + 3\Delta \log(\tau - \delta_0) + 3\Delta -$$

$$\Delta \log(z) = \Delta \log(1 + g(z, \tau)),$$

where $\tau \geq 1, z \geq 1$,

$$\left(\frac{\partial}{\partial \eta} w\right)(z, \eta) = \tag{4.2.32}$$

$$2(1-\eta)^{-2} \left(\frac{\partial}{\partial \tau} u\right)(z, (1+\eta)/(1-\eta)) = 2(1-\eta)^{-2} \log(1 + h(z, \eta)),$$

where $0 \leq \eta < 1, z \geq 1$,

$$\left(\left(\frac{\partial}{\partial \eta}\right)^2 w\right)(z, \eta) = \tag{4.2.33}$$

$$2\Delta((1-\eta)^{-2}(1+h(z,\eta))^{-1} \left(\frac{\partial}{\partial \eta} h\right)(z, \eta) + 4(1-\eta)^{-3} \log(1 + h(z, \eta)),$$

where $0 \leq \eta < 1, z \geq 1$,

$$\left(\frac{\partial}{\partial \eta} w\right)(z, \eta_1^\wedge(r, \pi, \delta_0)) = 0, \tag{4.2.34}$$

$$\varepsilon_1(\Delta, z) = -2^{-1} \left(\left(\frac{\partial}{\partial \eta}\right)^2 w\right)(z, \eta_1^\wedge(r, \pi, \delta_0)) = \tag{4.2.35}$$

$$-2\Delta((1-\eta_1^\wedge(r,\pi,\delta_0))^{-2}(1+h(z,\eta_1^\wedge(r,\pi,\delta_0)))^{-1}\left(\frac{\partial}{\partial\eta}h\right)(z,\eta_1^\wedge(r,\pi,\delta_0))>0,$$

where $z\geq 1$. There exists $\delta_1(\Delta,z)\in(0,\eta_1^\wedge(r,\pi,\delta_0)/2)$, such that

$$|((\frac{\partial}{\partial\eta})^2w)(z,\eta)-(\frac{\partial}{\partial\eta})^2w)(z,\eta_1^\wedge(r,\pi,\delta_0)|\leq\varepsilon_1(\Delta,z), \tag{4.2.36}$$

where $z\geq 1, 0\leq\eta<1, |\eta-\eta_1^\wedge(r,\pi,\delta_0)|\leq\delta_1(\Delta,z)$. Let

$$\eta_{1+k}^*=\eta_1^\wedge(z^{1/3},\pi,\delta_0)+(-1)^k\delta_1(\Delta,z),$$

where $k=1,2$. Clearly, $\eta_{1+k}^*\in(0,1)$ for $k=1,2$. Let further

$$\tau_{1+k}^*=\tau_{1+k}^*(\Delta,z)=(1+\eta_{1+k}^*(r,\pi,\delta_0))/(1-\eta_{1+k}^*(r,\pi,\delta_0)),$$

where $k=1,2$. Clearly, $1<\tau_2^*<\tau_1^*<\tau_3^*$,

$$|\tau_{1+k}^*-\tau_1^*|=\frac{2\delta_1(\Delta,z)}{(1-\eta_1^\wedge(r,\pi,\delta_0))(1-\eta_{1+k}^*(r,\pi,\delta_0))}>2\delta_1(\Delta,z),$$

where $k=1,2$. If $z\geq 1,\tau_2^*\leq\tau$, then instead (4.2.26) we have more strong inequality

$$|(3/2)\log((\tau^2-\tau\delta_0)(1-\delta_0)/(\tau^2-1))-3\theta_1^*(\Delta,\tau,\nu)/4|\leq \tag{4.2.37}$$

$$(3/2)\log\left(1+\frac{1+\delta_0}{\tau-\delta_0}\right)+(3/2)\log\left(1+\frac{1}{\tau-1}\right)-(3/2)\log(1-\delta_0)+3/4\leq c_2(\Delta,z),$$

where

$$c_2(\Delta,z)=(3/2)\log\left(1+\frac{1+\delta_0}{\tau_2-\delta_0}\right)+(3/2)\log\left(1+\frac{1}{\tau_2-1}\right)-(3/2)\log(1-\delta_0)+3/4.$$

Therefore, if $z\geq 1,\tau_2^*\leq\tau$, then instead (4.2.27) we have equality

$$\log(z^{-\nu\tau\Delta}(R_1(\nu\tau\Delta))^3)=$$

$$\nu w_1(z,\eta)+c_2(\Delta,z)\theta_3^*(\Delta,\tau,\nu)(\log(\nu)), \tag{4.2.38}$$

where $|\theta_3^*(\Delta,\tau,\nu)|\leq 1$. In view of (4.2.32), (4.2.34), (4.2.35),(4.2.36), (4.2.38)

$$((\frac{\partial}{\partial\eta})^2w)(z,\eta))\leq-\varepsilon_1(\Delta,z),$$

where $z \geq 1, 0 \leq \eta < 1, |\eta - \eta_1^{\wedge}(r, \pi, \delta_0)| \leq \delta_1(\Delta, z)$,

$$\log(z^{-\nu\tau\Delta}(R_1(\nu\tau\Delta))^3) = \tag{4.2.39}$$

$$\nu w_1(z, \eta) + c_2(\Delta, z)\theta_3^*(\Delta, \tau, \nu)(\log(\nu)),$$

$$\nu w(z, \eta_1^{\wedge}(r, \pi, \delta_0)) + \nu(\frac{\partial}{\partial\eta}w)(z, \eta_1^{\wedge}(r, \pi, \delta_0)(\eta - \eta_1^{\wedge}(r, \pi, \delta_0) +$$

$$\nu((\partial\partial\eta)^2 w)(z, \eta^{\sim}))(\eta - \eta_1^{\wedge}(r, \pi, \delta_0)^2/2 + c_2(\Delta, z)\theta_3^*(\Delta, \tau, \nu) \leq$$

$$\nu w_1(\Delta, z) - \nu\varepsilon_1(\Delta, z)(\eta - \eta_1^{\wedge}(r, \pi, \delta_0))^2/2 + c_2(\Delta, z)\theta_3^*(\Delta, \tau, \nu)$$

where

$$\nu\Delta\tau \in \Delta + \mathbb{N}, 0 \leq \eta < 1, |\eta - \eta_1^{\wedge}(r, \pi, \delta_0| \leq \delta_1(\Delta, z),$$

$$|\eta^{\sim} - \eta_1^{\wedge}(r, \pi, \delta_0| \leq \delta_1(\Delta, z)| < |\eta - \eta_1^{\wedge}(r, \pi, \delta_0| \leq \delta_1(\Delta, z).$$

Therefore, since

$$\eta = (\tau - 1)/(\tau + 1), \tag{4.2.40}$$

it follows that

$$\log(z^{-\nu\tau\Delta}(R_1(\nu\tau\Delta; \nu))^3) \leq \tag{4.2.41}$$

$$\nu w_1(\Delta, z) - 2\nu\varepsilon_1(\Delta, z)\frac{(\tau - \tau_1)^2}{(\tau + 1)^2(\tau_1 + 1)^2}c_2(\Delta, z)\theta_3^*(\Delta, \tau, \nu),$$

where $\tau_2^* \leq \tau \leq \tau_3^*$. Let

$$M = \sup\left\{\left|\left(\frac{\partial}{\partial\tau}g\right)(z, \tau)\right| : \tau_2^* \leq \tau \leq \tau_3^*\right\}, \alpha^* \in (0, 1/2), \tag{4.2.42}$$

$$t_{1+k} = t_{1+k}(\Delta, z, \nu) \in \mathbb{N}, \nu_1(\Delta, z) \in \mathbb{N} + 2,$$

where $k = 1, 2, z \geq 1$, and let, if $\nu \in \nu_1(\Delta, z) + \mathbb{N} - 1, z \geq 1$, then

$$\nu^{-\alpha^*} < \delta_1(\Delta, z)/2, M\nu^{-\alpha^*} < 1, \nu(\tau_2^* - 1) > 1, \tag{4.2.43}$$

$$\nu\Delta < \nu\tau_2^* < \nu\tau_1 - \nu\Delta 2\delta_1(\Delta_1, z) < \nu\tau_1 - 2\nu\Delta\nu^{-\alpha^*} < t_2(z, \nu) < \tag{4.2.44}$$

$$\nu\tau_1 - \nu\Delta\nu^{-\alpha^*} < \nu\tau_1 + \nu\Delta\nu^{-\alpha^*} < t_3(z, \nu) < \nu\tau_1 + 2\nu\Delta\nu^{-\alpha^*} < \nu\tau_1 2\delta_1(\Delta_1, z) < \nu\tau_3^*.$$

In view of (4.2.41) and (4.2.44), if $z \geq 1, \nu \in \nu_1(\Delta, z) + \mathbb{N} - 1$,

$$t \in ([\nu\Delta + 1, \tau_2] \cup [\tau_3, \infty) \cap \mathbb{Z}, c_3(\Delta, z) = \exp(2c_2(\Delta, z)),$$

then

$$z^{-t}(R_1(t;\nu))^3) \leq \left(z^{-t_2}(R_1(t_2;\nu))^3 + z^{-t_3}(R_1(t_3;\nu))^3)\right) \leq \tag{4.2.45}$$

$$\exp(w_1(\Delta, z)\nu)\exp(-2\varepsilon_1(\Delta, z)(\tau_3 + 1)^{-4}\nu^{1-2\alpha^*})c_3(\Delta, z).$$

Therefore in view of (4.2.24) (and, as in (4.2.4),),

$$z^\nu\left(\left(\sum_{t=\nu\Delta+1}^{t_2(z,\nu)}(R_0(t,\nu))^3 z^{-t}\right) + \left(\sum_{t_3(z,\nu)}^{\infty}(R_0(t,\nu))^3 z^{-t}\right)\right) \leq \tag{4.2.46}$$

$$2^{-3}(\nu\Delta)^{-2}(-h^{\sim}(\eta_1^{\wedge}(r, \pi, \delta_0)))^\nu$$
$$\exp(-2\varepsilon_1(\Delta, z)(\tau_3 + 1)^{-4}\nu^{1-2\alpha^*})c_3(\Delta, z).$$

In view of (4.2.14) and (4.2.25),

$$(\nu)^{-3}(\Delta(\tau_1 + 1) + 1)^{-3}(-h^{\sim}(\eta_1^{\wedge}(r, \pi, \delta_0)))^\nu \exp(O(1)) \leq \tag{4.2.47}$$

$$z^\nu z^{-\nu\Delta\tau_1 - \theta_\nu}(\nu\Delta\tau_1 + \theta_\nu + \nu\Delta)^{-3}(R_1(\nu\Delta\tau_1 + \theta_\nu;\nu))^3 =$$
$$z^\nu z^{[-\nu\Delta\tau_1]}(-[-\nu\Delta\tau_1] + \nu\Delta)^{-3}R_1(-[-\nu\Delta\tau_1];\nu)^3 \leq$$
$$z^\nu \sum_{t=1+\nu\Delta}^{+\infty} z^{-t}R_0(t;\nu)^3 = |f_2^*(z,\nu)| \leq$$
$$z^\nu 2^{-3}(\nu\Delta)^{-2}z^{-\nu\Delta\tau_1-\theta_\nu}(R_1(\nu\Delta\tau_1 + \theta_\nu;\nu))^3 \leq$$
$$2^{-3}(\nu\Delta)^{-2}(-h^{\sim}(\eta_1^{\wedge}(r, \pi, \delta_0)))^\nu \exp(O(1)).$$

Let $\gamma^* = \gamma^*(\Delta, z) = 2\varepsilon_1(\Delta, z)(\tau_3 + 1)^{-4}$. In view of (4.2.46) and (4.2.47),

$$(f_2^*(z,\nu))^{-1}\times \tag{4.2.48}$$

$$(-z)^\nu\left(\left(\sum_{t=\nu\Delta+1}^{t_2(z,\nu)}(R_0(t,\nu))^3 z^{-t}\right) + \left(\sum_{t_3(z,\nu)}^{\infty}(R_0(t,\nu))^3 z^{-t}\right)\right) =$$

$$O(1)\nu\exp(-\gamma^*\nu^{1-2\alpha^*}.$$

Therefore

$$(-z)^{\nu}\left(\sum_{t_2(z,\nu)+1}^{t_3(z,\nu)-1}(R_0(t,\nu))^3 z^{-t}\right) = \tag{4.2.49}$$

$$f_2^*(z,\nu)\left(1+O(1)\nu\exp(-\gamma^*\nu^{1-2\alpha^*}\right).$$

Since

$$\left(\left(\frac{\partial}{\partial t}\right)(R_0)^3\right)(t,\nu) = (R_0(t,\nu))^3\times \tag{4.2.50}$$

$$\left(\left(\sum_{\kappa=\nu+1}^{\nu\Delta}(t-\kappa)^{-1}\right)-\left(\sum_{\kappa=0}^{\nu\Delta}(t+\kappa)^{-1}\right)\right),$$

$$\left(\left(\frac{\partial}{\partial t}\right)^2(R_0)^3\right)(t,\nu) = (R_0(t,\nu))^3\times \tag{4.2.51}$$

$$\left(\left(\sum_{\kappa=\nu+1}^{\nu\Delta}(t-1/\kappa)^{-1}\right)-\left(\sum_{\kappa=0}^{\nu\Delta}(t+\kappa)^{-1}\right)\right)^2 -$$

$$(R_0(t,\nu))^3\times$$

$$\left(\left(\sum_{\kappa=\nu+1}^{\nu\Delta}(t-\kappa)^{-2}\right)-\left(\sum_{\kappa=0}^{\nu\Delta}(t+\kappa)^{-2}\right)\right),$$

where $\nu\in\mathbb{N}-1$, it follows that

$$\left|\left(\left(\frac{\partial}{\partial t}\right)(R_0)^3\right)(t,\nu)\right| \leq \tag{4.2.52}$$

$$3(R_0(t,\nu))^3\left(\left((t-\nu\Delta)^{-1}+\log\left(1+\frac{\nu(\Delta-1)-1}{t-\nu\Delta}\right)\right)\Bigg|_{t=\nu\Delta+1}\right)+$$

$$(R_0(t,\nu))^3\left(\left(t^{-1}+\log\left(1+\frac{\nu\Delta}{t}\right)\right)\Bigg|_{t=\nu\Delta+1}\right)\leq (R_0(t,\nu))^3 3\log(6\nu\Delta),$$

where $t\geq\nu\Delta+1$ and $\nu\in\mathbb{N}$,

$$\left|\left(\left(\frac{\partial}{\partial t}\right)(R_0)^3\right)(t,\nu)\right| \leq (R_0(t,\nu))^3(9(\log(6\nu\Delta))^2+5), \tag{4.2.53}$$

where $t \geq \nu\Delta + 1$ and $\nu \in \mathbb{N}$. If $t \geq \nu\Delta\tau_2^*, \nu \in \mathbb{N}$, then, using (4.2.50), (4.2.7), (and applying, for example, Sonin's formula, [20], questions in the end of the chapter II, the question 8) we obtain the equality

$$\left(\left(\frac{\partial}{\partial t}\right)(R_0)^3\right)(t,\nu) = 3(R_0(t,\nu))^3 \times \tag{4.2.54}$$

$$\left(\log\left(\frac{(t-\nu)t}{t^2-\nu^2\Delta^2}\right) + \frac{O(1)}{\nu\Delta(\tau_2^*-1)}\right) =$$

$$3(R_0(t,\nu))^3\left(\log\left(\frac{(\tau-\delta_0)\tau}{\tau^2-1}\right) + \frac{O(1)}{\nu\Delta(\tau_2^*-1)}\right) =$$

$$(R_0(t,\nu))^3\left(\log(z(1+g(z,\tau)) + \frac{O(1)}{\nu\Delta(\tau_2^*-1)}\right)$$

and

$$\left(\left(\frac{\partial}{\partial t}\right)^2(R_0)^3\right)(t,\nu) = \tag{4.2.55}$$

$$(R_0(t,\nu))^3\left(\left(\log(z(1+g(z,\tau)) + \frac{O(1)}{\nu\Delta(\tau_2^*-1)}\right)^2 + \frac{O(1)}{\nu\Delta(\tau_2^*-1)}\right).$$

If $t = \nu\Delta\tau \in [t_2, t_3]$, then, according to (4.2.44), $|\tau - \tau_1| < 2\nu^{-\alpha^*}$ and, since $g(z,\tau_1) = 0$, in view of (4.2.42)

$$\log(z(1+g(z,\tau) = O(nu^{-\alpha^*}). \tag{4.2.56}$$

Therefore,according to (4.2.54) and (4.2.55)

$$\left(\left(\frac{\partial}{\partial t}\right)(R_0)^3\right)(t,\nu) = (R_0(t,\nu))^3(log(z) + O(nu^{-\alpha^*}), \tag{4.2.57}$$

$$\left(\left(\frac{\partial}{\partial t}\right)^2(R_0)^3\right)(t,\nu) = (R_0(t,\nu))^3((log(z))^2 + (1+log(z))O(nu^{-\alpha^*}). \tag{4.2.58}$$

In view of (4.2.48), (4.2.52) and (4.2.53),

$$(f_2^*(z,\nu))^{-1}(-z)^\nu(\left(\sum_{t=\nu\Delta+1}^{t_2(z,\nu)}\left(\frac{\partial}{\partial t}(R_0)^3\right)(t,\nu)z^{-t}\right) + \tag{4.2.59}$$

$$(f_2^*(z,\nu))^{-1}(-z)^{\nu}\left(\sum_{t_3(z,\nu)}^{\infty}\left(\frac{\partial}{\partial t}(R_0)^3\right)(t,\nu)z^{-t}\right)=$$
$$O(1)\nu\exp(-\gamma^a st\nu^{1-2\alpha^a st}(\log(6\nu)),$$

$$(4.2.60)\qquad (f_2^*(z,\nu))^{-1}(-z^{\nu})(\left(\sum_{t=\nu\Delta+1}^{t_2(z,\nu)}\left(\left(\frac{\partial}{\partial t}\right)^2(R_0)^3\right)(t,\nu)z^{-t}\right)+$$
$$(f_2^*(z,\nu))^{-1}(-z)^{\nu}\left(\sum_{t_3(z,\nu)}^{\infty}\left(\left(\frac{\partial}{\partial t}\right)^2(R_0)^3\right)(t,\nu)z^{-t}\right)=$$
$$O(1)\nu\exp(-\gamma^a st\nu^{1-2\alpha^a st}(\log(6\nu))^2,$$

where $\nu\in\nu_1(\Delta,z)+\mathbb{N}-1$. In view of (4.2.57) and (4.2.58),

$$(4.2.61)\qquad (-z)^{\nu}\left(\sum_{t_2(z,\nu)+1}^{t_3(z,\nu)-1}\left(\frac{\partial}{\partial t}(R_0)^3\right)(t,\nu)z^{-t}\right)=$$
$$(-z)^{\nu}\left(\sum_{t_2(z,\nu)+1}^{t_3(z,\nu)-1}(R_0(t,\nu))^3z^{-t}\right)(log(z)+O(nu^{-\alpha^*}),$$

$$(4.2.62)\qquad (-z)^{\nu}\left(\sum_{t_2(z,\nu)+1}^{t_3(z,\nu)-1}\left(\left(\frac{\partial}{\partial t}\right)^2(R_0)^3\right)(t,\nu)z^{-t}\right)=$$
$$(-z)^{\nu}\left(\sum_{t_2(z,\nu)+1}^{t_3(z,\nu)-1}(R_0(t,\nu))^3z^{-t}\right)((log(z))^2+(1+log(z))O(nu^{-\alpha^*}),$$

where $\nu\in\nu_1(\Delta,z)+\mathbb{N}-1$, $z\ge 1$. In view of (4.2.49), (4.2.61) and (4.2.62),

$$(4.2.63)\qquad (-z)^{\nu}\left(\sum_{t_2(z,\nu)+1}^{t_3(z,\nu)-1}\left(\frac{\partial}{\partial t}(R_0)^3\right)(t,\nu)z^{-t}\right)=$$
$$f_2^*(z,\nu)\left(1+O(1)\nu\exp(-\gamma^a st\nu^{1-2\alpha^*}\right)(log(z)+O(\nu^{-\alpha^*}),$$

$$(4.2.64)\qquad (-z)^{\nu}\left(\sum_{t_2(z,\nu)+1}^{t_3(z,\nu)-1}\left(\left(\frac{\partial}{\partial t}\right)^2(R_0)^3\right)(t,\nu)z^{-t}\right)=$$
$$f_2^*(z,\nu)\left(1+O(1)\nu\exp(-\gamma^a st\nu^{1-2\alpha^a st}\right)((log(z))^2+(1+log(z))O(\nu^{-\alpha^*}),$$

where $\nu\in\nu_1(\Delta,z)+\mathbb{N}-1$, $z\ge 1$. The assertion of the Lemma follows from (4.2.47), (4.2.59), (4.2.63), (4.2.60), (4.2.64). ■

§3 A correction to [38].

In (2.3.16) of paper [38], there is a misprint.
The correct equality is:

$$\alpha_j = (log(r_j^{\wedge} R_1/R_2))/log(R_2)$$

(instead of the incorrect one printed there: $\alpha_j = (log(r_j^{\wedge} R_1))/log(R_2)$).

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Moscow State Institute
of Electronics and Mathematics
Bolshoi Trekhsvyatitelskij pereulok, 1-3/12, str.8
Moscow 109028
Russian Federation
E-mail: gutnik@gutnik.mccme.ru