Rates of asymptotic regularity for Halpern iterations of nonexpansive mappings *

Laurenţiu Leuştean

Department of Mathematics, Technische Universität Darmstadt,
Schlossgartenstrasse 7, 64289 Darmstadt, Germany

and

Institute of Mathematics "Simion Stoilow" of the Romanian Academy,
Calea Griviţei 21, P.O. Box 1-462, Bucharest, Romania

E-mail: leustean@mathematik.tu-darmstadt.de

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Abstract

In this paper we obtain new effective results on the Halpern iterations of nonexpansive mappings using methods from mathematical logic or, more specifically, proof-theoretic techniques. We give effective rates of asymptotic regularity for the Halpern iterations of nonexpansive self-mappings of nonempty convex sets in normed spaces. The paper presents another case study in the project of *proof mining*, which is concerned with the extraction of effective uniform bounds from (prima-facie) ineffective proofs.

1 Introduction

This paper presents another case study in the project of *proof mining*, by which we mean the logical analysis of mathematical proofs with the aim of extracting new numerically relevant information hidden in the proofs.

General logical metatheorems were obtained (using proof-theoretic methods) in [10] and [4] for various classes of spaces in functional analysis and metric geometry, such as metric, hyperbolic spaces in the sense of Reich/Kirk/ Kohlenbach, CAT(0), (uniformly convex) normed and inner product spaces. Further examples ($\mathbb{R}$-trees, hyperbolic spaces in the sense of Gromov and uniformly convex hyperbolic spaces) are discussed in [14]. These metatheorems guarantee a priorly, under very general logical conditions, the extractability of effective

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bounds from large classes of proofs in functional analysis, and moreover they provide algorithms for actually extracting the bounds. The bounds are uniform for all parameters meeting very weak local boundedness conditions. We refer to Kohlenbach's forthcoming book for details [11].

In this paper we apply proof mining to metric fixed point theory, more specifically to the (approximate) fixed point theory of nonexpansive mappings, one of the most active branches of nonlinear functional analysis. We refer to [7] for an extensive account of metric fixed point theory.

In the following, $(X, \|\cdot\|)$ is a normed space and C is a nonempty convex subset of X. A mapping $T : C \to C$ is called *nonexpansive* if for all $x, y \in C$,

$$\|Tx - Ty\| \leq \|x - y\|.$$

The usual Picard iterations are not the proper iterations for nonexpansive mappings and that's why other iterations were considered in this case. The *Krasnoselski-Mann iteration* [18, 13, 5] star-ting with $x \in C$ is defined by:

$$x_0 := x, \quad x_{n+1} := (1 - \lambda_n)x_n + \lambda_n T x_n \quad \text{for } n \geq 0, \tag{1}$$

where $(\lambda_n)_{n\geq 0}$ is a sequence in $[0, 1]$.

One of the most important notions in fixed point theory is the *asymptotic regularity*, defined in [2], but already implicit in [13, 19, 3]. A mapping $T : C \to C$ is called *asymptotically regular* if for all $x \in C$,

$$\lim_{n\to\infty} \|T^n(x) - T^{n+1}(x)\| = 0.$$

For constant $\lambda_n = \lambda \in [0, 1]$, the asymptotic regularity of the averaged mapping $T_\lambda := (1 - \lambda)I + \lambda T$ is equivalent to the fact that $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x \in C$. Therefore, for general (λ_n) in $[0, 1]$, a nonexpansive mapping T is λ_n*-asymptotically regular* [1] if for all $x \in C$,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{2}$$

Methods of proof mining were applied in [8, 9, 12, 15] to obtain effective rates of as-ymptotic regularity for the Krasnoselski-Mann iterations of nonexpansive mappings in normed and CAT(0)-spaces or even in the more general class of (uniformly convex) hyperbolic spaces.

In this paper, we consider other iterations, introduced in [6]. For $x \in C$ and $(\lambda_n)_{n\geq 1}$ in $[0, 1]$, the *Halpern iteration* starting with x is defined as:

$$x_0 := x, \quad x_{n+1} := \lambda_{n+1}x + (1 - \lambda_{n+1})Tx_n \quad \text{for } n \geq 0. \tag{3}$$

As Wittmann remarked [20, 21], if T is linear and $\lambda_n := \dfrac{1}{n+1}$, then $x_n = \dfrac{1}{n+1}\sum\limits_{i=0}^{n} T^i x$, so the Halpern iterations could be regarded as nonlinear generalizations of the usual Cesaro averages.

One of the earliest and most important results on the convergence of Halpern iterations is the following one.

Theorem 1.1. *[21, Theorem 2]*
Let X be a Hilbert space, $C \subseteq X$ a nonempty closed convex subset, $T : C \to C$ nonexpansive and $(\lambda_n)_{n\geq 1}$ be a sequence in $\in [0,1]$ satisfying the following conditions: $\lim\limits_{n\to\infty} \lambda_n = 0$, $\sum\limits_{n=1}^{\infty} \lambda_n$ is divergent and $\sum\limits_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ is convergent. Assume moreover that the set $Fix(T)$ of fixed points of T is nonempty .

Then for any $x \in C$, the Halpern iteration $(x_n)_{n\geq 1}$ is norm convergent to the unique fixed point Px of T with $\|x - Px\| \leq \|x - y\|$ for any $y \in Fix(T)$.

Generalizations of this theorem to the Banach space case and different conditions on (λ_n) were considered in numerous papers. We refer to [24] for a nice exposition.

In the following, we consider the important problem of asymptotic regularity, this time associated to the Halpern iterations: $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$, where $(x_n)_{n\geq 1}$ is defined by (3). By inspecting the proof of Theorem 1.1 (and its generalizations), it is easy to see that the first step is to obtain asymptotic regularity, and that this can be done in a much more general setting.

Thus, the following theorem, essentially contained in [21, 22, 23], can be proved.

Theorem 1.2. *Let $(X, \|\cdot\|)$ be a normed space, $C \subseteq X$ a nonempty convex subset and $T : C \to C$ be nonexpansive.*

Assume that $(\lambda_n)_{n\geq 1}$ is a sequence in $[0,1]$ such that $\lim\limits_{n\to\infty} \lambda_n = 0$, $\sum\limits_{n=1}^{\infty} \lambda_n$ is divergent and $\sum\limits_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ is convergent.

Let $x \in C$ be such that (x_n) is bounded.

Then

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This theorem is our point of departure. By a logical analysis of its proof, we shall obtain a quantitative version (Theorem 2.1), providing for the first time effective rates of asymptotic regularity for the Halpern iterates, that is rates of convergence of $\big(\|x_n - Tx_n\|\big)$ towards 0.

2 Main results

Before stating our main theorem, let us recall some terminology.

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. If the series $\sum\limits_{n=1}^{\infty} a_n$ is divergent, then a function $\gamma : \mathbb{N}^* \to \mathbb{N}^*$ is called a *rate of divergence* of $\sum\limits_{n=1}^{\infty} a_n$ if

$$\forall n \in \mathbb{N}^* \left(\sum_{i=1}^{\gamma(n)} a_i \geq n \right). \tag{4}$$

If $(a_n)_{n\geq 1}$ is convergent, then a function $\gamma : (0, \infty) \to \mathbb{N}^*$ is called a *Cauchy modulus* of (a_n) if

$$\forall \varepsilon > 0 \, \forall n \in \mathbb{N}^* \left(a_{\gamma(\varepsilon)+n} - a_{\gamma(\varepsilon)} < \varepsilon \right). \tag{5}$$

If $\lim\limits_{n\to\infty} a_n = a$, then a function $\gamma : (0, \infty) \to \mathbb{N}^*$ is called a *rate of convergence* of (a_n) if

$$\forall \varepsilon > 0 \, \forall n \geq \gamma(\varepsilon) \left(|a_n - a| < \varepsilon \right). \tag{6}$$

The following quantitative version of Theorem 1.2 is the main result of our paper.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a normed space, $C \subseteq X$ a nonempty convex subset and $T : C \to C$ be nonexpansive.*

Assume that $(\lambda_n)_{n\geq 1}$ is a sequence in $[0, 1]$ such that $\lim\limits_{n\to\infty} \lambda_n = 0$, $\sum\limits_{n=1}^{\infty} \lambda_n$ is divergent and $\sum\limits_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n|$ is convergent. Moreover, let $\alpha : (0, \infty) \to \mathbb{N}^*$ be a rate of convergence of (λ_n), $\beta : (0, \infty) \to \mathbb{N}^*$ be a Cauchy modulus of $s_n := \sum\limits_{i=1}^{n} |\lambda_{i+1} - \lambda_i|$ and $\theta : \mathbb{N}^* \to \mathbb{N}^*$ be a rate of divergence of $\sum\limits_{n=1}^{\infty} \lambda_n$.

Let $x \in C$ be such that (x_n) is bounded.

Then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ and moreover

$$\forall \varepsilon \in (0, 2) \forall n \geq \Phi(\alpha, \beta, \theta, M, \varepsilon) \, \big(\|x_n - Tx_n\| < \varepsilon \big),$$

where

$$\Phi(\alpha, \beta, \theta, M, \varepsilon) = \max \left\{ \theta \left(\beta \left(\frac{\varepsilon}{8M} \right) + 1 + \left\lceil \ln \left(\frac{8M}{\varepsilon} \right) \right\rceil \right), \ \alpha \left(\frac{\varepsilon}{4M} \right) \right\},$$

$M \in \mathbb{N}^*$ is such that $M \geq \|x_n\| + \|x\| + \|Tx\|$ for all $n \geq 1$.

We shall give the proof of the above theorem in the last section of our paper. We derive now some further consequences.

Corollary 2.2. *Let $(X, \|\cdot\|)$ be a normed space, $C \subseteq X$ a nonempty convex bounded subset with finite diameter d_C and $T : C \to C$ be nonexpansive.*

Assume that $(\lambda_n)_{n\geq 1}$ satisfies the hypotheses of Theorem 2.1.

Then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x \in C$ and moreover

$$\forall \varepsilon \in (0, 2) \forall n \geq \Phi(\alpha, \beta, \theta, d_C, \varepsilon) \, \big(\|x_n - Tx_n\| < \varepsilon \big),$$

where

$$\Phi(\alpha,\beta,\theta,d_C,\varepsilon)=\max\left\{\theta\left(\beta\left(\frac{\varepsilon}{8M}\right)+1+\left\lceil\ln\left(\frac{8M}{\varepsilon}\right)\right\rceil\right),\ \alpha\left(\frac{\varepsilon}{4M}\right)\right\},$$

$M\in\mathbb{N}^*$ *is such that* $M\geq 3d_C$.

Proof. Since C is bounded, it has a finite diameter $d_C:=\sup\{\|x\|\mid x\in C\}$. Moreover, for all $x\in C$, (x_n) is bounded and $\|x_n\|+\|x\|+\|Tx\|\leq 3d_C$. Apply now Theorem 2.1. □ □

Thus, for bounded C, we get asymptotic regularity for general (λ_n) and an explicit rate of asymptotic regularity $\Phi(\alpha,\beta,\theta,d_C,\varepsilon)$ which depends only on the error ε, on the diameter d_C of C, and on (λ_n) via α,β,θ, but not on the nonexpansive mapping T, the starting point $x\in C$ of the Halpern iteration or other data related with C and X.

Corollary 2.3. *Let* $(X,\|\cdot\|)$ *be a normed space,* $C\subseteq X$ *be a nonempty convex subset and* $T:C\to C$ *nonexpansive.*
Assume that $(\lambda_n)_{n\geq 1}$ *is a decreasing sequence in* $[0,1]$ *such that* $\lim\limits_{n\to\infty}\lambda_n=0$, $\sum\limits_{n=1}^{\infty}\lambda_n$ *is divergent and let* $\alpha:(0,\infty)\to\mathbb{N}^*$ *be a rate of convergence of* (λ_n) *and* $\theta:\mathbb{N}^*\to\mathbb{N}^*$ *be a rate of divergence of* $\sum\limits_{n=1}^{\infty}\lambda_n$.
Let $x\in C$ *be such that* (x_n) *is bounded.*
Then $\lim\|x_n-Tx_n\|=0$ *and moreover*

$$\forall\varepsilon\in(0,2)\forall n\geq\Psi(\alpha,\theta,M,\varepsilon)\ \big(\|x_n-Tx_n\|<\varepsilon\big),$$

where

$$\Psi(\alpha,\theta,M,\varepsilon)=\max\left\{\theta\left(\alpha\left(\frac{\varepsilon}{8M}\right)+1+\left\lceil\ln\left(\frac{8M}{\varepsilon}\right)\right\rceil\right),\ \alpha\left(\frac{\varepsilon}{4M}\right)\right\},$$

$M\in\mathbb{N}^*$ *is such that* $M\geq\|x_n\|+\|x\|+\|Tx\|$ *for all* $n\geq 1$.

Proof. Remark that (λ_n) decreasing implies that

$$s_n:=\sum_{i=1}^{n}|\lambda_{i+1}-\lambda_i|=\sum_{i=1}^{n}(\lambda_i-\lambda_{i+1})=\lambda_1-\lambda_{n+1}.$$

Since $\lim\limits_{n\to\infty}\lambda_n=0$, it follows that $\sum\limits_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|=\lambda_1$, that is it is convergent. Moreover, for all $\varepsilon>0, n\in\mathbb{N}^*$,

$$\begin{aligned}s_{\alpha(\varepsilon)+n}-s_{\alpha(\varepsilon)} &= (\lambda_1-\lambda_{\alpha(\varepsilon)+n+1})-(\lambda_1-\lambda_{\alpha(\varepsilon)+1})=\lambda_{\alpha(\varepsilon)+1}-\lambda_{\alpha(\varepsilon)+n+1}\\ &\leq\lambda_{\alpha(\varepsilon)+1}\leq\lambda_{\alpha(\varepsilon)}<\varepsilon,\end{aligned}$$

since α is a rate of convergence of (λ_n). Thus, α is a Cauchy modulus of (s_n), so we can apply now Theorem 2.1 with $\beta:=\alpha$. □ □

The rate of asymptotic regularity can be further simplified for $\lambda_n = 1/n$.

Corollary 2.4. *Let $(X, \|\cdot\|)$ be a normed space, $C \subseteq X$ a nonempty convex bounded subset with finite diameter d_C and $T : C \to C$ be nonexpansive.*

Assume that $\lambda_n = \dfrac{1}{n}$ for all $n \geq 1$.

Then $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ for all $x \in C$ and moreover

$$\forall \varepsilon \in (0,2) \forall n \geq \Phi(d_C, \varepsilon) \ \big(\|x_n - Tx_n\| < \varepsilon\big),$$

where

$$\Phi(d_C, \varepsilon) = \exp\left(\ln 4 \cdot \left(\frac{16M}{\varepsilon} + 3\right)\right),$$
$$M \in \mathbb{N}^* \text{ is such that } M \geq 3d_C.$$

Proof. Obviously, $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ with a rate of convergence

$$\alpha : (0, \infty) \to \mathbb{N}^*, \quad \alpha(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil + 1.$$

Moreover, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent with a rate of divergence given by

$$\theta : \mathbb{N}^* \to \mathbb{N}^*, \quad \theta(n) = 4^n.$$

Since, furthermore, $\left(\dfrac{1}{n}\right)$ is decreasing, we can apply Corollaries 2.3 and 2.2 to get that $\lim \|x_n - Tx_n\| = 0$ for all $x \in C$ and moreover

$$\forall \varepsilon \in (0,2) \forall n \geq \Psi(\alpha, \theta, M, \varepsilon) \ \big(\|x_n - Tx_n\| < \varepsilon\big),$$

where

$$\begin{aligned}
\Psi(\alpha, \theta, M, \varepsilon) &= \max\left\{\theta\left(\alpha\left(\frac{\varepsilon}{8M}\right) + 1 + \left\lceil \ln\left(\frac{8M}{\varepsilon}\right)\right\rceil\right),\ \alpha\left(\frac{\varepsilon}{4M}\right)\right\} \\
&= \theta\left(\alpha\left(\frac{\varepsilon}{8M}\right) + 1 + \left\lceil \ln\left(\frac{8M}{\varepsilon}\right)\right\rceil\right)
\end{aligned}$$

and $M \in \mathbb{N}^*$ is such that $M \geq 3d_C$. Using that $\lceil a \rceil < a + 1$ and $1 + \ln a \leq a$ for all $a > 0$, we get that

$$\begin{aligned}
\alpha\left(\frac{\varepsilon}{8M}\right) + 1 + \left\lceil \ln\left(\frac{8M}{\varepsilon}\right)\right\rceil &< \alpha\left(\frac{\varepsilon}{8M}\right) + 2 + \ln\left(\frac{8M}{\varepsilon}\right) \leq \alpha\left(\frac{\varepsilon}{8M}\right) + 1 + \frac{8M}{\varepsilon} \\
&= \left\lceil \frac{8M}{\varepsilon} \right\rceil + 2 + \frac{8M}{\varepsilon} < \frac{16M}{\varepsilon} + 3,
\end{aligned}$$

we get that

$$\Psi(\alpha, \theta, M, \varepsilon) \quad < \quad 4^{\frac{16M}{\varepsilon} + 3} = \exp\left(\ln 4 \cdot \left(\frac{16M}{\varepsilon} + 3\right)\right) = \Phi(d_c, \varepsilon).$$

The conclusion follows now immediately. $\square$ $\square$

Hence, we get an exponential (in $1/\varepsilon$) rate of asymptotic regularity in the case $\lambda_n = 1/n$.

3 Some technical lemmas

The following lemma collects some useful properties of Halpern iterations and it is essentially contained in [22, 23]. In order to make the paper self-contained, we still give the proof.

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a normed space, $C \subseteq X$ be a nonempty convex subset, $T : C \to C$ nonexpansive and $(\lambda_n)_{n\geq 1}$ be a sequence in $[0,1]$. Assume that $(x_n)_{n\geq 1}$ is the Halpern iteration starting with $x \in C$. Then*

1. *For all $n \geq 1$,*

$$
\begin{aligned}
\|Tx_n\| &\leq \|x_n\| + \|x\| + \|Tx\|, \\
\|Tx_n - x_n\| &\leq \|x_{n+1} - x_n\| + \lambda_{n+1}\|x - Tx_n\|, \\
\|x_{n+1} - x_n\| &\leq (1-\lambda_{n+1})\|x_n - x_{n-1}\| + |\lambda_{n+1} - \lambda_n| \cdot \|x - Tx_{n-1}\|.
\end{aligned}
$$

2. *If (x_n) is bounded, then (Tx_n) is also bounded. Moreover, if $M \geq \|x_n\|, \|Tx_n\|$ for all $n \geq 1$, then*

$$
\begin{aligned}
\|Tx_n - x_n\| &\leq \|x_{n+1} - x_n\| + 2M\lambda_{n+1}, \\
\|x_{n+1} - x_n\| &\leq (1-\lambda_{n+1})\|x_n - x_{n-1}\| + 2M|\lambda_{n+1} - \lambda_n|
\end{aligned}
$$

for all $n \geq 1$.

Proof. 1.

$$
\begin{aligned}
\|Tx_n\| &\leq \|Tx_n - Tx\| + \|Tx\| \leq \|x_n - x\| + \|Tx\| \leq \|x_n\| + \|x\| + \|Tx\| \\
\|Tx_n - x_n\| &= \|(\lambda_{n+1}x + (1-\lambda_{n+1})Tx_n - \lambda_{n+1}(x - Tx_n)) - x_n\| \\
&= \|x_{n+1} - x_n - \lambda_{n+1}(x - Tx_n)\| \leq \|x_{n+1} - x_n\| + \lambda_{n+1}\|x - Tx_n\|
\end{aligned}
$$

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \|\lambda_{n+1}x + (1-\lambda_{n+1})Tx_n - \lambda_n x - (1-\lambda_n)Tx_{n-1}\| \\
&= \|(\lambda_{n+1} - \lambda_n)x + (1-\lambda_{n+1})(Tx_n - Tx_{n-1}) + (\lambda_n - \lambda_{n+1})Tx_{n-1}\| \\
&= \|(\lambda_{n+1} - \lambda_n)(x - Tx_{n-1}) + (1-\lambda_{n+1})(Tx_n - Tx_{n-1})\| \\
&\leq |\lambda_{n+1} - \lambda_n| \cdot \|x - Tx_{n-1}\| + (1-\lambda_{n+1})\|x_n - x_{n-1}\|, \\
&\quad \text{since } T \text{ is nonexpansive.}
\end{aligned}
$$

2. is an immediate consequence of 1.

∎

Lemma 3.2. *Let* $(\lambda_n)_{n\geq 1}$ *be a sequence in* $[0,1]$ *and* $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ *be sequences in* $\mathbb{R}_+$ *such that* $\sum_{n=1}^{\infty} b_n$ *is convergent and*

$$a_{n+1} \leq (1-\lambda_{n+1})a_n + b_n \quad \textit{for all } n \in \mathbb{N}^*.$$

Then

1. *for all* $m, n \in \mathbb{N}^*$,
$$a_{n+m} \leq \left[\prod_{j=n}^{n+m-1}(1-\lambda_{j+1})\right] a_n + \sum_{j=n}^{n+m-1} b_j \tag{7}$$

2. (a_n) *is bounded.*

Proof. 1. By an easy induction on m.

2. Applying (7) with $n := 1$, we get that for all $m \geq 1$,
$$0 \leq a_{m+1} \leq \left[\prod_{j=1}^{m}(1-\lambda_{j+1})\right] a_1 + \sum_{j=1}^{m} b_j \leq a_1 + \sum_{j=1}^{m} b_j \leq a_1 + \sum_{j=1}^{\infty} b_j < \infty,$$
since $\sum_{j=1}^{\infty} b_j < \infty$. Thus, (a_n) is bounded. □

□

The following lemma is a quantitative version of [17, Lemma 2].

Lemma 3.3.
Let $(\lambda_n)_{n\geq 1}$ *be a sequence in* $[0,1]$ *and* $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ *be sequences in* $\mathbb{R}_+$ *such that for all* $n \in \mathbb{N}^*$,

$$a_{n+1} \leq (1-\lambda_{n+1})a_n + b_n. \tag{8}$$

Assume moreover that $\sum_{n=1}^{\infty} \lambda_n$ *is divergent,* $\sum_{n=1}^{\infty} b_n$ *is convergent and let* $\delta : \mathbb{N}^* \to \mathbb{N}^*$ *be a rate of divergence of* $\sum_{n=1}^{\infty} \lambda_n$, $\gamma : (0,\infty) \to \mathbb{N}^*$ *be a Cauchy modulus of* $(s_m)_{m\geq 1}$, *where* $s_m := \sum_{i=1}^{m} b_i$.

Then $\lim_{n\to\infty} a_n = 0$ *and moreover*

$$\forall \varepsilon \in (0,2) \forall n \geq h(\gamma, \delta, D, \varepsilon) \ \big(a_n < \varepsilon\big), \tag{9}$$

where

$$h(\gamma,\delta,D,\varepsilon) = \delta\left(\gamma\left(\frac{\varepsilon}{2}\right) + 1 + \left\lceil \ln\left(\frac{2D}{\varepsilon}\right)\right\rceil\right),$$
$D \in \mathbb{N}^*$ *is an upper bound on* (a_n).

Proof. By Lemma 3.2, (a_n) is bounded, so there exists $D \in \mathbb{N}^*$ such that $a_n \leq D$ for all $n \in \mathbb{N}^*$. Let $\varepsilon \in (0,2)$ and define

$$N := \gamma\left(\frac{\varepsilon}{2}\right) + 1. \tag{10}$$

Applying (7) with $n := N$, it follows that for all $m \in \mathbb{N}^*$

$$\begin{aligned}
a_{N+m} &\leq \left[\prod_{j=N}^{N+m-1}(1-\lambda_{j+1})\right] a_N + \sum_{j=N}^{N+m-1} b_j \\
&\leq \exp\left(-\sum_{j=N}^{N+m-1}\lambda_{j+1}\right) a_N + \sum_{j=N}^{N+m-1} b_j, \\
&\quad \text{since } 1-x \leq \exp(-x) \text{ for all } x \in [0,\infty) \\
&= \exp\left(-\sum_{j=N}^{N+m-1}\lambda_{j+1}\right) a_N + \left(s_{\gamma\left(\frac{\varepsilon}{2}\right)+m} - s_{\gamma\left(\frac{\varepsilon}{2}\right)}\right) \\
&< D\exp\left(-\sum_{j=N}^{N+m-1}\lambda_{j+1}\right) a_N + \frac{\varepsilon}{2}, \\
&\quad \text{since } \gamma \text{ is a Cauchy modulus of } (s_m).
\end{aligned}$$

For simplicity, let us denote $d_m := D\exp\left(-\sum_{j=N}^{N+m-1}\lambda_{j+1}\right)$. We have got then that for all $m \in \mathbb{N}^*$,

$$a_{N+m} < d_m + \frac{\varepsilon}{2}. \tag{11}$$

Let us note that

$$\begin{aligned}
d_m \leq \frac{\varepsilon}{2} &\Leftrightarrow \exp\left(-\sum_{j=N}^{N+m-1}\lambda_{j+1}\right) \leq \frac{\varepsilon}{2D} \Leftrightarrow -\sum_{j=N}^{N+m-1}\lambda_{j+1} \leq \ln\left(\frac{\varepsilon}{2D}\right) \\
&\Leftrightarrow \sum_{j=N}^{N+m-1}\lambda_{j+1} \geq -\ln\left(\frac{\varepsilon}{2D}\right) = \ln\left(\frac{2D}{\varepsilon}\right) \Leftrightarrow \sum_{i=N+1}^{N+m}\lambda_i \geq \ln\left(\frac{2D}{\varepsilon}\right) \\
&\Leftrightarrow \sum_{i=1}^{N+m}\lambda_i \geq \sum_{i=1}^{N}\lambda_i + \ln\left(\frac{2D}{\varepsilon}\right).
\end{aligned}$$

Let

$$M := \delta\left(N + \left\lceil \ln\left(\frac{2D}{\varepsilon}\right)\right\rceil\right) - N. \tag{12}$$

Since δ is a rate of divergence of $\sum_{n=1}^{\infty}\lambda_n$ and $\lambda_n \leq 1$, it is obvious that $\delta(n) \geq n$ for all $n \in \mathbb{N}^*$. Using also the fact that $\frac{2D}{\varepsilon} > D > 1$, so $\ln\left(\frac{2D}{\varepsilon}\right) > 0$, it is easy to see that $M \in \mathbb{N}^*$. Moreover, for $m \geq M$, we get that

$$\sum_{i=1}^{N+m} \lambda_i \geq \sum_{i=1}^{N+M} \lambda_i \geq N + \left\lceil \ln\left(\frac{2D}{\varepsilon}\right) \right\rceil \geq \sum_{i=1}^{N} \lambda_i + \ln\left(\frac{2D}{\varepsilon}\right).$$

Hence, $d_m \leq \dfrac{\varepsilon}{2}$ for all $m \geq M$. Combining this with (11), we get that for all $m \geq M$, $a_{N+m} < \varepsilon$, that is

$$a_{N+M+n} < \varepsilon. \tag{13}$$

for all $n \in \mathbb{N}$. Define

$$h(\gamma, \delta, D, \varepsilon) := N + M = \delta\left(N + \left\lceil \ln\left(\frac{2D}{\varepsilon}\right) \right\rceil\right) \tag{14}$$

Then (9) follows. Thus, $\lim a_n = 0$ and $h(\gamma, \delta, D, \varepsilon)$ is a rate of convergence of (a_n) towards 0. $\square$

4 Proof of Theorem 2.1

By Lemma 3.1, we get that $M \geq \|x_n\|, \|Tx_n\|$ for all $n \geq 1$ and

$$\|x_{n+1} - x_n\| \leq (1 - \lambda_{n+1})\|x_n - x_{n-1}\| + 2M \cdot |\lambda_{n+1} - \lambda_n|. \tag{15}$$

Let us consider the sequences

$$a_n := \|x_n - x_{n-1}\|, \quad b_n := 2M|\lambda_{n+1} - \lambda_n|$$

and let $D := 2M$. Then D is a bound on (a_n) and, by (15), for all $n \geq 1$,

$$a_{n+1} \leq (1 - \lambda_{n+1})a_n + b_n.$$

Moreover, $\sum_{n=1}^{\infty} \lambda_n$ is divergent with rate of divergence θ and if we define

$$\gamma : (0, \infty) \to \mathbb{N}^*, \quad \gamma(\varepsilon) := \beta\left(\frac{\varepsilon}{2M}\right),$$

we get that for all $n \in \mathbb{N}^*$,

$$\begin{aligned}
\sum_{i=1}^{\gamma(\varepsilon)+n} b_i - \sum_{i=1}^{\gamma(\varepsilon)} b_i &= 2M\left(\sum_{i=1}^{\gamma(\varepsilon)+n} |\lambda_{i+1} - \lambda_i| - \sum_{i=1}^{\gamma(\varepsilon)} |\lambda_{i+1} - \lambda_i|\right) \\
&= 2M\left(s_{\beta\left(\frac{\varepsilon}{2M}\right)+n} - s_{\beta\left(\frac{\varepsilon}{2M}\right)}\right) \\
&< 2M \cdot \frac{\varepsilon}{2M} = \varepsilon,
\end{aligned}$$

so $\sum_{n=1}^{\infty} b_n$ is convergent and γ is a Cauchy modulus of $\left(\sum_{i=1}^{n} b_i\right)$.

Thus, the hypothesis of Lemma 3.3 are satisfied, so we can apply it to get that for all $\varepsilon \in (0,2)$ and for all $n \geq h_1(\beta, \theta, M, \varepsilon)$

$$\|x_n - x_{n-1}\| < \frac{\varepsilon}{2}, \tag{16}$$

where

$$h_1(\beta, \theta, M, \varepsilon) \quad := \quad \theta\left(\beta\left(\frac{\varepsilon}{8M}\right) + 1 + \left\lceil \ln\left(\frac{8M}{\varepsilon}\right)\right\rceil\right).$$

By Lemma 3.1.2, for all $n \geq 2$,

$$\|x_{n-1} - Tx_{n-1}\| \leq \|x_n - x_{n-1}\| + 2M\lambda_n. \tag{17}$$

Let $h_2(\alpha, M, \varepsilon) := \alpha\left(\frac{\varepsilon}{4M}\right)$. Then, using the fact that α is a rate of convergence of (λ_n) towards 0, we get that for all $n \geq h_2(\alpha, M, \varepsilon)$

$$2M\lambda_n < 2M\frac{\varepsilon}{4M} = \frac{\varepsilon}{2}. \tag{18}$$

Combining (16), (17) and (18), it follows that

$$\|x_{n-1} - Tx_{n-1}\| < \varepsilon$$

for all $n \geq \max\{h_1(\beta, \theta, M, \varepsilon), h_2(\alpha, M, \varepsilon)\}$, so the conclusion of the theorem follows with Φ defined by (7). $\square$

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