EXCEPTIONAL DEL PEZZO HYPERSURFACES

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Abstract. We classify weakly exceptional quasismooth well-formed del Pezzo weighted hypersurfaces in $\mathbb{P}(a_1, a_2, a_3, a_4)$, and we compute their global log canonical thresholds.

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Part 1. Introduction

1.1. Background

The multiplicity of a nonzero polynomial $f \in \mathbb{C}[z_1, \cdots, z_n]$ at a point $P \in \mathbb{C}^n$ is the non-negative integer m such that $f \in \mathfrak{m}_P^m \setminus \mathfrak{m}_P^{m+1}$, where $\mathfrak{m}_P$ is the maximal ideal of polynomials vanishing at the point P in $\mathbb{C}[z_1, \cdots, z_n]$. It can be also defined by derivatives. The multiplicity of f at the point P is the number

$$\mathrm{mult}_P(f) = \min\left\{ m \;\middle|\; \frac{\partial^m f}{\partial^{m_1} z_1 \partial^{m_2} z_2 \cdots \partial^{m_n} z_n}(P) \neq 0 \right\}.$$

On the other hand, we have a similar invariant that is defined by integrations. This invariant, which is called the complex singularity exponent of f at the point P, is given by

$$c_P(f) = \sup\Big\{c \;\Big|\; |f|^{-c} \text{ is locally } L^2 \text{ near the point } P \in \mathbb{C}^n\Big\}.$$

It is hard to calculate it in general. However for some cases there are easy ways to calculate it.

Example 1.1.1. Let f be a polynomial in $\mathbb{C}[z_1, z_2]$. Suppose that the polynomial defines an irreducible curve passing through the origin O in $\mathbb{C}^2$. We then have

$$c_O(f) = \min\left(1, \frac{1}{m} + \frac{1}{n}\right),$$

where (m, n) is the first pair of Puiseux exponents of f (see [32]). In particular, we have

$$c_O\left(z_1^{n_1} z_2^{n_2}\left(z_1^{km_1} + z_2^{km_2}\right)\right) = \min\left(\frac{1}{n_1}, \frac{1}{n_2}, \frac{\frac{1}{m_1} + \frac{1}{m_2}}{k + \frac{n_1}{m_1} + \frac{n_2}{m_2}}\right),$$

where n_1, n_2, m_1, m_2, k are non-negative integers.

Example 1.1.2. Let $m_1, \ldots, m_n$ be positive integers. Then

$$\min\left(1, \sum_{i=1}^{n} \frac{1}{m_i}\right) = c_O\left(\sum_{i=1}^{n} z_i^{m_i}\right) \geqslant c_O\left(\prod_{i=1}^{n} z_i^{m_i}\right) = \min\left(\frac{1}{m_1}, \frac{1}{m_2}, \ldots, \frac{1}{m_n}\right).$$

Let X be a variety[1] with at most log canonical singularities (see [28]), let $Z \subseteq X$ be a closed subvariety, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on the variety X. Then the number

$$\mathrm{lct}_Z\big(X, D\big) = \sup\Big\{\lambda \in \mathbb{Q} \;\Big|\; \text{the log pair } \big(X, \lambda D\big) \text{ is log canonical along } Z\Big\} \in \mathbb{Q} \cup \big\{+\infty\big\}$$

is called a log canonical threshold of the divisor D along Z. It follows from [28] that for a polynomial f in n variables over $\mathbb{C}$

$$\mathrm{lct}_O\Big(\mathbb{C}^n, (f = 0)\Big) = c_O(f),$$

so that the log canonical threshold $\mathrm{lct}_Z(X, D)$ is an algebraic counterpart of the complex singularity exponent $c_O(f)$. We can define the log canonical threshold of D on X by

$$\begin{aligned}\mathrm{lct}_X\big(X, D\big) &= \inf\Big\{\mathrm{lct}_P\big(X, D\big) \;\Big|\; P \in X\Big\}\\ &= \sup\Big\{\lambda \in \mathbb{Q} \;\Big|\; \text{the log pair } \big(X, \lambda D\big) \text{ is log canonical}\Big\},\end{aligned}$$

and, for simplicity, we put $\mathrm{lct}(X, D) = \mathrm{lct}_X(X, D)$.

Example 1.1.3. Suppose that $X = \mathbb{P}^2$ and $D \in |\mathcal{O}_{\mathbb{P}^2}(3)|$. Then

$$\mathrm{lct}\big(X, D\big) = \begin{cases} 1 \text{ if } D \text{ is a smooth curve,}\\ 1 \text{ if } D \text{ is a curve with ordinary double points,}\\ \frac{5}{6} \text{ if } D \text{ is a curve with one cuspidal point,}\\ \frac{3}{4} \text{ if } D \text{ consists of a conic and a line that are tangent,}\\ \frac{2}{3} \text{ if } D \text{ consists of three lines intersecting at one point,}\\ \frac{1}{2} \text{ if } \mathrm{Supp}\big(D\big) \text{ consists of two lines,}\\ \frac{1}{3} \text{ if } \mathrm{Supp}\big(D\big) \text{ consists of one line.}\end{cases}$$

Now we suppose that X is a Fano variety with at most log terminal singularities (see [24]).

[1] All varieties are assumed to be complex, algebraic, projective and normal unless otherwise stated.

Definition 1.1.4. The global log canonical threshold of the Fano variety X is the number defined by

$$\mathrm{lct}\big(X\big)=\inf\Big\{\mathrm{lct}\big(X,D\big)\ \Big|\ D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ such that } D\ \sim_{\mathbb{Q}} -K_X\Big\}.$$

The number $\mathrm{lct}(X)$ is an algebraic counterpart of the α-invariant of Tian (see [15], [48]).

The group $\mathrm{Pic}(X)$ is torsion free because X is rationally connected (see [53]). Therefore, we have

$$\mathrm{lct}\big(X\big)=\sup\left\{\lambda\in\mathbb{Q}\ \left|\ \begin{array}{l}\text{the log pair } \Big(X,\lambda D\Big) \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D\equiv -K_X\end{array}\right.\right\}.$$

It immediately follows from Definition 1.1.4 that

$$\mathrm{lct}\big(X\big)=\sup\left\{\varepsilon\in\mathbb{Q}\ \left|\ \begin{array}{l}\text{the log pair } \Big(X,\dfrac{\varepsilon}{n}D\Big) \text{ is log canonical for}\\ \text{every divisor } D\in\big|-nK_X\big| \text{ and every } n\in\mathbb{N}\end{array}\right.\right\}.$$

Example 1.1.5. Suppose that $\mathbb{P}(a_0,a_1,\ldots,a_n)$ is a well-formed weighted projective space (see [23]). Then

$$\mathrm{lct}\Big(\mathbb{P}\big(a_0,a_1,\ldots,a_n\big)\Big)=\frac{a_0}{\sum_{i=0}^{n}a_i}.$$

Example 1.1.6. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $m\leqslant n$. The paper [6] shows that

$$\mathrm{lct}\big(X\big)=\frac{1}{n+1-m}$$

if $m<n$. For the case $m=n\geqslant 2$ it also shows that

$$1-\frac{1}{n}\leqslant\mathrm{lct}(X)\leqslant 1$$

and that $\mathrm{lct}(X)=1-\frac{1}{n}$ if X contains a cone of dimension $n-2$. Meanwhile, the papers [14] and [41] show that

$$1\geqslant\mathrm{lct}\big(X\big)\geqslant\begin{cases}1 \text{ if } n\geqslant 6,\\ \dfrac{22}{25} \text{ if } n=5,\\ \dfrac{16}{21} \text{ if } n=4,\\ \dfrac{3}{4} \text{ if } n=3,\end{cases}$$

if X is general.

Example 1.1.7. Let X be a smooth hypersurface in the weighted projective space $\mathbb{P}(1^{n+1},d)$ of degree $2d\geqslant 4$. Then

$$\mathrm{lct}\big(X\big)=\frac{1}{n+1-d}$$

in the case when $d<n$ (see [8, Proposition 20]). Suppose that $d=n$. Then the inequalities

$$\frac{2n-1}{2n}\leqslant\mathrm{lct}\big(X\big)\leqslant 1$$

hold (see [14]). But $\mathrm{lct}(X)=1$ if X is general and $n\geqslant 3$. Furthermore for the case $n=3$ the papers [14] and [41] prove that

$$\mathrm{lct}\big(X\big)\in\left\{\frac{5}{6},\frac{43}{50},\frac{13}{15},\frac{33}{38},\frac{7}{8},\frac{33}{38},\frac{8}{9},\frac{9}{10},\frac{11}{12},\frac{13}{14},\frac{15}{16},\frac{17}{18},\frac{19}{20},\frac{21}{22},\frac{29}{30},1\right\}$$

and all these values can be attained. For instance, if the hypersurface X is given by

$$w^2=x^6+y^6+z^6+t^6+x^2y^2zt\subset\mathbb{P}\big(1,1,1,1,3\big)\cong\mathrm{Proj}\Big(\mathbb{C}\big[x,y,z,t,w\big]\Big),$$

where $\mathrm{wt}(x)=\mathrm{wt}(y)=\mathrm{wt}(z)=\mathrm{wt}(t)=1$ and $\mathrm{wt}(w)=3$, then $\mathrm{lct}(X)=1$ (see [14]).

Example 1.1.8. Let X be a rational homogeneous space such that $-K_X \sim rD$ and

$$\mathrm{Pic}\big(X\big) = \mathbb{Z}\big[D\big],$$

where D is an ample Cartier divisor and $r \in \mathbb{Z}_{>0}$. Then $\mathrm{lct}(X) = \frac{1}{r}$ (see [22]).

Example 1.1.9. Let X be a quasismooth well-formed (see [23]) hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $\sum_{i=1}^{4} a_i$ with terminal singularities (see [28]), where $a_1 \leqslant \ldots \leqslant a_4$. Then

- there are exactly 95 possibilities for the quadruple (a_1, a_2, a_3, a_4) (see [23], [26]),
- if $X \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ is general, then it follows from [7], [9], [10] and [14] that

$$1 \geqslant \mathrm{lct}\big(X\big) \geqslant \begin{cases} \dfrac{16}{21} & \text{if } a_1 = a_2 = a_3 = a_4 = 1,\\ \dfrac{7}{9} & \text{if } (a_1, a_2, a_3, a_4) = (1,1,1,2),\\ \dfrac{4}{5} & \text{if } (a_1, a_2, a_3, a_4) = (1,1,2,2),\\ \dfrac{6}{7} & \text{if } (a_1, a_2, a_3, a_4) = (1,1,2,3),\\ 1 & \text{in the remaining cases,} \end{cases}$$

- the global log canonical threshold of the hypersurface

$$w^2 = t^3 + z^9 + y^{18} + x^{18} \subset \mathbb{P}\big(1,1,2,6,9\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t, w\big]\Big)$$

  is equal to $\frac{17}{18}$ (see [7]), where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 9$.

Example 1.1.10. Let X be a singular cubic surface in $\mathbb{P}^3$ such that X has at most canonical singularities. The possible singularities of X are listed in [5]. It follows from [12] that

$$\mathrm{lct}\big(X\big) = \begin{cases} \dfrac{2}{3} & \text{if } \mathrm{Sing}\big(X\big) = \big\{\mathbb{A}_1\big\},\\ \dfrac{1}{3} & \text{if } \mathrm{Sing}\big(X\big) \supseteq \big\{\mathbb{A}_4\big\},\ \mathrm{Sing}\big(X\big) = \big\{\mathbb{D}_4\big\} \text{ or } \mathrm{Sing}\big(X\big) \supseteq \big\{\mathbb{A}_2, \mathbb{A}_2\big\},\\ \dfrac{1}{4} & \text{if } \mathrm{Sing}\big(X\big) \supseteq \big\{\mathbb{A}_5\big\} \text{ or } \mathrm{Sing}\big(X\big) = \big\{\mathbb{D}_5\big\},\\ \dfrac{1}{6} & \text{if } \mathrm{Sing}\big(X\big) = \big\{\mathbb{E}_6\big\},\\ \dfrac{1}{2} & \text{in the remaining cases.} \end{cases}$$

So far we have not seen any single variety whose global log canonical threshold is irrational. In general, it is unknown whether $\mathrm{lct}(X)$ is a rational number or not[2] (cf. Question 1 in [50]). However, we expect more than this as follows.

Conjecture 1.1.11. There is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ on the variety X such that

$$\mathrm{lct}\big(X\big) = \mathrm{lct}\big(X, D\big) \in \mathbb{Q}.$$

The following definition is due to [46] (cf. [25], [31], [34], [40]).

Definition 1.1.12. The variety X is exceptional (resp. weakly exceptional, strongly exceptional) if for every effective $\mathbb{Q}$-divisor D on the variety X such that $D \equiv -K_X$, the pair (X, D) is log terminal (resp. $\mathrm{lct}(X) \geqslant 1$, $\mathrm{lct}(X) > 1$).

It is easy to see the implications

$$\text{strongly exceptional} \implies \text{exceptional} \implies \text{weakly exceptional}.$$

However, if Conjecture 1.1.11 holds for X, then we see that X is exceptional if and only if X is strongly exceptional.

[2]Even for a del Pezzo surfaces with log terminal singularities the rationality of the global log canonical threshold is unknown.

Exceptional del Pezzo surfaces, which are called del Pezzo surfaces without tigers in [29], lie in finitely many families (see [46], [40]). We expect that strongly exceptional Fano varieties with quotient singularities enjoy very interesting geometrical properties (cf. [44, Theorem 3.3], [38, Theorem 1]).

The global log canonical threshold plays important roles both in birational geometry and in complex geometry.

Example 1.1.13. Let $X_1, \ldots, X_r$ be threefolds satisfying hypotheses of Example 1.1.9. Then

- the threefolds $X_1, \ldots, X_r$ are non-rational (see [16]),
- for every $i = 1, \ldots, r$, there is no rational dominant map $\rho\colon X_i \dashrightarrow Y$ such that
  - general fiber of the map ρ is rationally connected,
  - the inequality $\dim(Y) \geqslant 1$ holds,
- there is no non-biregular birational map $\rho\colon X_i \dashrightarrow Y$ such that
  - the variety Y has terminal $\mathbb{Q}$-factorial singularities,
  - the equality $\operatorname{rk}\operatorname{Pic}(Y) = 1$ holds.
- the structures of the groups $\operatorname{Bir}(X_1), \ldots, \operatorname{Bir}(X_r)$ are completely described in [16] and [13],
- if the equality $\operatorname{lct}(X_1) = \operatorname{lct}(X_2) = \ldots = \operatorname{lct}(X_r) = 1$ holds, then
  - the variety $X_1 \times \ldots \times X_r$ is non-rational and

$$\operatorname{Bir}\Big(X_1 \times \ldots \times X_r\Big) = \Big\langle \prod_{i=1}^{r} \operatorname{Bir}(X_i),\ \operatorname{Aut}\Big(X_1 \times \ldots \times X_r\Big)\Big\rangle.$$

  - for every dominant map $\rho\colon X_1 \times \ldots \times X_r \dashrightarrow Y$ whose general fiber is rationally connected, there is a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$ and a commutative diagram

$$\begin{array}{ccc} X_1 \times \ldots \times X_r & \overset{\sigma}{\dashrightarrow} & X_1 \times \ldots \times X_r \\ \downarrow{\scriptstyle\pi} & & \searrow{\scriptstyle\rho} \\ X_{i_1} \times \ldots \times X_{i_k} & \underset{\xi}{\dashrightarrow} & Y, \end{array}$$

  where ξ and σ are birational maps, and π is a projection (see [7], [41]).

The following result was proved in [17], [37], [48] (see [15, Appendix A]).

Theorem 1.1.14. Suppose that X is a Fano variety with at most quotient singularities. Then X admits an orbifold Kähler–Einstein metric if

$$\operatorname{lct}\big(X\big) > \frac{\dim\big(X\big)}{\dim\big(X\big) + 1}.$$

Examples 1.1.6, 1.1.7 and 1.1.9 are good examples to which we may apply Theorem 1.1.14.

There are many known obstructions for the existence of orbifold Kähler–Einstein metrics on Fano varieties with quotient singularities (see [18], [20], [33], [36], [43], [51]).

Example 1.1.15. Let X be a quasismooth hypersurface in $\mathbb{P}(a_0, \ldots, a_n)$ of degree $d < \sum_{i=0}^{n} a_i$, where $a_0 \leqslant \ldots \leqslant a_n$. Suppose that X is well-formed and has a Kähler–Einstein metric. Then

$$d\left(\sum_{i=0}^{n} a_i - d\right)^{n} \leqslant n^n \prod_{i=0}^{n} a_i,$$

and $\sum_{i=0}^{n} a_i \leqslant d + n a_0$ by [21] (see [2], [47]).

The problem of existence of Kähler–Einstein metrics on smooth del Pezzo surfaces is completely solved by [49].

Theorem 1.1.16. If X is a smooth del Pezzo surface, then the following conditions are equivalent:

- the automorphism group $\operatorname{Aut}(X)$ is reductive;
- the surface X admits a Kähler–Einstein metric;
- the surface X is not a blow up of $\mathbb{P}^2$ at one or two points.

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1.2. Notation

We reserve the following notation that will be used throughout the paper:

- $\mathbb{P}(a_0, a_1, a_2, a_3)$ denotes the weighted projective space $\mathrm{Proj}(\mathbb{C}[x, y, z, t])$ with weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, where we always assume $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$.
- O_x is the point in $\mathbb{P}(a_0, a_1, a_2, a_3)$ defined by $y = z = t = 0$. The points O_y, O_z and O_t are defined in the similar way.
- X denotes a quasismooth and well-formed hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ (see Definitions 6.3 and 6.9 in [23], respectively).
- C_x is the curve on X cut by the equation $x = 0$. The curves C_y, C_z and C_t are defined by the similar way.
- L_{xy} is the one-dimensional strata on $\mathbb{P}(a_0, a_1, a_2, a_3)$ defined by $x = y = 0$ and the other one-dimensional stratum are labeled in the same way.
- Let D be a divisor on X and $P \in X$. Choose an orbifold chart $\pi : \tilde{U} \to U$ for some neighborhood $P \in U \subset X$. We put $\mathrm{mult}_P(D) = \mathrm{mult}_P(\pi^* D)$ and refer to this quantity as the multiplicity of D at P.

1.3. Results

Let X be a hypersurface in $\mathbb{P} = \mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d. Then X is given by a quasihomogeneous polynomial equation $f(x, y, z, t) = 0$ of degree d. The quasihomogeneous equation

$$f\big(x, y, z, t\big) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

defines an isolated quasihomogeneous singularity (V, O) with the Milnor number $\prod_{i=0}^{n}(\frac{d}{a_i} - 1)$, where O is the origin of $\mathbb{C}^4$. It follows from the adjunction formula that

$$K_X \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(a_0, a_1, a_2, a_3)}\Big(d - \sum_{i=0}^{3} a_i\Big),$$

and it follows from [19], [28, Proposition 8.14], [42] that the following conditions are equivalent:

- the inequality $d \leqslant \sum_{i=0}^{3} a_i - 1$ holds;
- the surface X is a del Pezzo surface;
- the singularity (V, O) is rational;
- the singularity (V, O) is canonical.

Blowing up $\mathbb{C}^4$ at the origin O with weights (a_0, a_1, a_2, a_3), we get a purely log terminal blow up of the singularity (V, O) (see [30], [39]). The paper [39] shows that the following conditions are equivalent:

- the surface X is exceptional (weakly exceptional, respectively);
- the singularity (V, O) is exceptional[3] (weakly exceptional, respectively).

From now on we suppose that $d \leqslant \sum_{i=0}^{3} a_i - 1$. Then X is a del Pezzo surface. Put $I = \sum_{i=0}^{3} a_i - d$. The set of possible values of (a_0, a_1, a_2, a_3, d) can be obtained from [52]. The list of possible values of (a_0, a_1, a_2, a_3, d) with $2I < 3a_0$ can be found in [4]. If the equality $I = 1$ holds, then it follows from [27] that

- either the surface X is smooth and
$$\big(a_0, a_1, a_2, a_3\big) \in \Big\{\big(1, 1, 1, 1\big),\ \big(1, 1, 1, 2\big),\ \big(1, 1, 2, 3\big)\Big\},$$
- or the surface X is singular and
  - either $(a_0, a_1, a_2, a_3) = (2, 2n + 1, 2n + 1, 4n + 1)$, where $n \in \mathbb{Z}_{>0}$,

[3] For notions of exceptional and weakly exceptional singularities see [39, Definition 4.1], [46], [25].

– or the quadruple (a_0,a_1,a_2,a_3) lies in the set

$$\left\{\begin{array}{l}(1,2,3,5),(1,3,5,7),(1,3,5,8),(2,3,5,9)\\ (3,3,5,5),(3,5,7,11),(3,5,7,14),(3,5,11,18)\\ (5,14,17,21),(5,19,27,31),(5,19,27,50),(7,11,27,37)\\ (7,11,27,44),(9,15,17,20),(9,15,23,23),(11,29,39,49)\\ (11,49,69,128),(13,23,35,57),(13,35,81,128)\end{array}\right\}.$$

The global log canonical thresholds of such del Pezzo surfaces have been considered either implicitly or explicitly in [1], [3], [11], [17], [27]. For example, the papers [1], [3], [17] and [27] gives us lower bounds for global log canonical thresholds of singular del Pezzo surfaces with $I=1$.

Theorem 1.3.1. Suppose that $I=1$ and X is singular. Then

$$\mathrm{lct}\big(X\big)\geqslant\left\{\begin{array}{l}1\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(2,2n+1,2n+1,4n+1\big),\ \text{where } n\geqslant 2,\\ \dfrac{33}{38}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(2,3,3,5\big),\\ \dfrac{7}{10}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(1,2,3,5\big),\\ 1\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(1,3,5,7\big) \text{ and } X \text{ is general},\\ \dfrac{11}{16}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(1,3,5,8\big) \text{ and } X \text{ is general},\\ 1\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(2,3,5,9\big),\\ 1\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(3,3,5,5\big),\\ \dfrac{21}{25}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(3,5,7,11\big),\\ \dfrac{3}{4}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(3,5,7,14\big),\\ \dfrac{11}{12}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(3,5,11,18\big),\\ \dfrac{5}{4}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(5,14,17,21\big),\\ \dfrac{5}{3}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(5,19,27,31\big),\\ \dfrac{27}{20}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(5,19,27,50\big),\\ \dfrac{7}{3}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(7,11,27,37\big),\\ \dfrac{189}{88}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(7,11,27,44\big),\\ \dfrac{51}{20}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(9,15,17,20\big),\\ 3\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(9,15,23,23\big),\\ \dfrac{429}{127}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(11,29,39,49\big),\\ \dfrac{759}{256}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(11,49,69,128\big),\\ \dfrac{455}{127}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(13,23,35,57\big),\\ \dfrac{1053}{256}\quad\text{if } \big(a_0,a_1,a_2,a_3\big)=\big(13,35,81,128\big).\end{array}\right.$$

Meanwhile, the paper [11] deals with the exact values log the global log canonical thresholds of smooth del Pezzo surfaces with $I=1$.

Theorem 1.3.2. Suppose that $I=1$ and X is smooth. Then

$$\mathrm{lct}(X)=\begin{cases} 1 & \text{if } (a_0,a_1,a_2,a_3)=(1,1,2,3) \text{ and } |-K_X| \text{ contains no cuspidal curves,}\\ \frac{5}{6} & \text{if } (a_0,a_1,a_2,a_3)=(1,1,2,3) \text{ and } |-K_X| \text{ contains a cuspidal curve,}\\ \frac{5}{6} & \text{if } (a_0,a_1,a_2,a_3)=(1,1,1,2) \text{ and } |-K_X| \text{ contains no tacnodal curves,}\\ \frac{3}{4} & \text{if } (a_0,a_1,a_2,a_3)=(1,1,1,2) \text{ and } |-K_X| \text{ contains a tacnodal curve,}\\ \frac{3}{4} & \text{if } X \text{ is a cubic in } \mathbb{P}^3 \text{ with no Eckardt points,}\\ \frac{2}{3} & \text{if either } X \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point.} \end{cases}$$

A singular del Pezzo hypersurface X must satisfy exclusively one of the following properties:

(1) $2I \geqslant 3a_0$;
(2) $2I < 3a_0$ and

$$\big(a_0,a_1,a_2,a_3,d\big)=\big(I-k, I+k, a, a+k, 2a+k+I\big)$$

for some $\mathbb{Z}_{>0} \ni a \geqslant I+k$ and $I > k \in \mathbb{Z}_{\geqslant 0}$;
(3) $2I < 3a_0$ but

$$\big(a_0,a_1,a_2,a_3,d\big)\neq\big(I-k, I+k, a, a+k, 2a+k+I\big)$$

for any $\mathbb{Z}_{>0} \ni a \geqslant I+k$ and $I > k \in \mathbb{Z}_{\geqslant 0}$.

For the first two cases it is easy to see $\mathrm{lct}(X,\frac{I}{a_0}C_x)\leq\frac{2}{3}$ and hence $\mathrm{lct}(X)\leq\frac{2}{3}$ (for instance, see [4]). All the values of (a_0,a_1,a_2,a_3,d) whose hypersurface X satisfies the last condition are listed in Table 4 (see [4]).

We already know the global log canonical thresholds of smooth del Pezzo surfaces. For del Pezzo surfaces corresponding to the first two conditions, their global log canonical thresholds are relatively too small to enjoy the condition of Theorem 1.1.14. However, the global log canonical thresholds of del Pezzo surfaces corresponding to the last condition have not been investigated sufficiently. In the present paper we compute all of them and then we obtain the following result.

Theorem 1.3.3. Let X be a del Pezzo surface that appears in Table 4. Then

$$\mathrm{lct}(X)=\min\Big\{\mathrm{lct}\Big(X,\frac{I}{a_0}C_x\Big),\ \mathrm{lct}\Big(X,\frac{I}{a_1}C_y\Big),\ \mathrm{lct}\Big(X,\frac{I}{a_2}C_z\Big)\Big\}.$$

In particular, we obtain the value of $\mathrm{lct}(X)$ for every quintuple (a_0,a_1,a_2,a_3,d) listed in Table 4. As a result, we obtain the following corollaries.

Corollary 1.3.4. Suppose that $I=1$. Then X is exceptional if and only if $K_X^2\leqslant\frac{1}{15}$.

Corollary 1.3.5. The following assertions are equivalent:

- the surface X is exceptional;
- $\mathrm{lct}(X) > 1$;

- the quintuple (a_0, a_1, a_2, a_3, d) lies in the set

$$\left\{\begin{array}{l}(2,3,5,9,18),(3,3,5,5,15),(3,5,7,11,25),(3,5,7,14,28)\\(3,5,11,18,36),(5,14,17,21,56),(5,19,27,31,81),(5,19,27,50,100)\\(7,11,27,37,81),(7,11,27,44,88),(9,15,17,20,20),(9,15,23,23,69)\\(11,29,39,49,127),(11,49,69,128,256),(13,23,35,57,127)\\(13,35,81,128,256),(3,4,5,10,20),(3,4,10,15,30),(5,13,19,22,57)\\(5,13,19,35,70),(6,9,10,13,36),(7,8,19,25,57),(7,8,19,32,64)\\(9,12,13,16,48),(9,12,19,19,57),(9,19,24,31,81),(10,19,35,43,105)\\(11,21,28,47,105),(11,25,32,41,107),(11,25,34,43,111),(11,43,61,113,226)\\(13,18,45,61,135),(13,20,29,47,107),(13,20,31,49,111),(13,31,71,113,226)\\(14,17,29,41,99),(5,7,11,13,33),(5,7,11,20,40),(11,21,29,37,95)\\(11,37,53,98,196),(13,17,27,41,95),(13,27,61,98,196),15,19,43,74,148)\\(9,11,12,17,45),(10,13,25,31,75),(11,17,20,27,71),(11,17,24,31,79)\\(13,14,19,29,71),(13,14,23,33,79),(13,23,51,83,166),(11,13,19,25,63)\\(11,31,45,83,83),(11,25,37,68,136),(13,19,41,68,136)\\(11,19,29,53,106),(13,15,31,53,106),(11,13,21,38,76)\end{array}\right\}.$$

Corollary 1.3.6. The following assertions are equivalent:

- the surface X is weakly exceptional and not exceptional;
- $\mathrm{lct}(X) = 1$;
- one of the following holds
  - the quintuple (a_0, a_1, a_2, a_3, d) lies in the set

$$\left\{\begin{array}{l}(2,2n+1,2n+1,4n+1,8n+4),(4,2n+3,2n+3,4n+4,8n+12)\\(3,3n+1,6n+1,9n+3,18n+6),(3,3n+1,6n+1,9n,18n+3)\\(3,3n,3n+1,3n+1,9n+3),(3,3n+1,3n+2,3n+2,9n+6)\\(4,2n+1,4n+2,6n+1,12n+6),(6,6n+3,6n+5,6n+5,18n+15)\\(6,6n+5,12n+8,18n+9,36n+24)\\(6,6n+5,12n+8,18n+15,36n+30)\\(8,4n+5,4n+7,4n+9,12n+23)\\(9,3n+8,3n+11,6n+13,12n+35)\\(1,3,5,8,16),(2,3,4,7,14),(3,7,8,13,29)\\(3,10,11,19,41),(5,6,8,9,24),(5,6,8,15,30)\end{array}\right\},$$

    where $n \in \mathbb{Z}_{>0}$,
  - $(a_0, a_1, a_2, a_3, d) = (1, 2, 3, 5, 10)$ and C_x has an ordinary double point,
  - $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the defining equation of X contains yzt,
  - $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ and the defining equation of X contains yzt.

Corollary 1.3.7. The del Pezzo surface X has an orbifold Kähler-Einstein metric unless one of the following holds

- the quintuple (a_0, a_1, a_2, a_3, d) lies in the set

$$\left\{\begin{array}{l}(7,10,15,19,45),(7,18,27,37,81),(7,15,19,32,64)\\(7,19,25,41,82),(7,26,39,55,117)\end{array}\right\},$$

- $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the defining equation of X does not contain yzt,
- $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ and the defining equation of X does not contain yzt.

Theorem 1.3.3 shows that Conjecture 1.1.11 holds for del Pezzo surfaces described in Table 4.

Let Y be a variety with log terminal singularities. Let us consider an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor

$$B_Y = \sum_{i=1}^{r} a_i B_i$$

on Y, where B_i is a prime Weil divisor. Let $\pi\colon \bar{Y} \to Y$ be a birational morphism of a smooth variety $\bar{Y}$. Put

$$B_{\bar{Y}} = \sum_{i=1}^{r} a_i \bar{B}_i,$$

where $\bar{B}_i$ is the proper transform of the divisor B_i on the variety $\bar{Y}$. Then

$$K_{\bar{Y}} + B_{\bar{Y}} = \pi^*\Big(K_Y + B_Y\Big) + \sum_{i=1}^{n} c_i E_i,$$

where $c_i \in \mathbb{Q}$ and E_i is an exceptional divisor of the morphism π. Suppose that the divisor

$$\sum_{i=1}^{r} \bar{B}_i + \sum_{i=1}^{n} E_i$$

is simple normal crossing and put

$$B^{\bar{Y}} = B_{\bar{Y}} - \sum_{i=1}^{n} c_i E_i.$$

The singularities of (Y, B_Y) are log canonical (resp. log terminal) if $a_i \leqslant 1$ (resp. $a_i < 1$) and $c_j \geqslant -1$ (resp. $c_j > -1$) for every $i = 1, \ldots, r$ and $j = 1, \ldots, n$. The locus of log canonical singularities of the pair (Y, B_Y), denoted by $\mathrm{LCS}(Y, B_Y)$, is defined by the set

$$\mathrm{LCS}\Big(Y, B_Y\Big) = \left(\bigcup_{a_i \geqslant 1} B_i\right) \bigcup \left(\bigcup_{c_i \leqslant -1} \pi\big(E_i\big)\right) \subsetneq Y.$$

A proper irreducible subvariety $Z \subsetneq Y$ is said to be a center of log canonical singularities of the log pair (Y, B_Y) if either $Z = B_i$ with $a_i \geqslant 1$ or $Z = \pi(E_i)$ with $c_i \leqslant -1$ for some choice of the birational morphism $\pi\colon \bar{Y} \to Y$. The set of all centers of log canonical singularities of (Y, B_Y) is denoted by $\mathbb{LCS}(Y, B_Y)$. Every member of $\mathbb{LCS}(Y, B_Y)$ is contained in $\mathrm{LCS}(Y, B_Y)$. We see that the set $\mathrm{LCS}(Y, B_Y)$ is empty, equivalently the set $\mathbb{LCS}(Y, B_Y)$ is empty, if and only if the log pair (Y, B_Y) is log terminal.

Let $\mathcal{H}$ be a base point free linear system on Y and let H be a sufficiently general divisor in the linear system $\mathcal{H}$. For an irreducible proper subvariety W of Y put

$$W\Big|_H = \sum_{i=1}^{m} Z_i,$$

where $Z_i \subset H$ is an irreducible subvariety. It follows that the subvariety W belongs to $\mathbb{LCS}(Y,\ B_Y)$ if and only if the set $\{Z_1, \ldots, Z_m\}$ is contained in $\mathbb{LCS}(H,\ B_Y\big|_H)$ (cf. Theorem 1.4.5).

Example 1.4.1. Let $\alpha\colon V \to Y$ be the blow up at a smooth point $O \in Y$. Then

$$B_V = \alpha^*\big(B_Y\big) - \mathrm{mult}_O\big(B_Y\big)E$$

where $\mathrm{mult}_O(B_Y) \in \mathbb{Q}$ and E is the exceptional divisor of the blow up α. Then

$$\mathrm{mult}_O\big(B_Y\big) > 1$$

if the log pair (Y, B_Y) is not log canonical at the point O. Put

$$B^V = B_V + \Big(\mathrm{mult}_O\big(B_Y\big) - \dim\big(Y\big) + 1\Big)E,$$

and suppose that $\mathrm{mult}_O(B_Y) \geqslant \dim(Y) - 1$. Then $O \in \mathbb{LCS}(Y, B_Y)$ if and only if

- either $E \in \mathbb{LCS}(V, B^V)$ (equivalently, $\mathrm{mult}_O(B_Y) \geqslant \dim(Y)$)
- or there is a subvariety $Z \subsetneq E$ such that $Z \in \mathbb{LCS}(V, B^V)$.

The locus $\mathrm{LCS}(Y, B_Y) \subset Y$ can be equipped with a scheme structure (see [37], [45]). The ideal sheaf defined by

$$\mathcal{I}\Big(Y, B_Y\Big) = \pi_* \mathcal{O}_{\bar{Y}}\Big(\sum_{i=1}^{n} \lceil c_i \rceil E_i - \sum_{i=1}^{r} \lfloor a_i \rfloor \bar{B}_i\Big),$$

is called the multiplier ideal sheaf of (Y, B_Y). The subscheme $\mathcal{L}(Y, B_Y)$ corresponding to the multiplier ideal sheaf $\mathcal{I}(Y, B_Y)$ is called the subscheme of log canonical singularities of (Y, B_Y). It follows from the construction of the subscheme $\mathcal{L}(Y, B_Y)$ that

$$\mathrm{Supp}\Big(\mathcal{L}(Y, B_Y)\Big) = \mathrm{LCS}\Big(Y, B_Y\Big) \subset Y.$$

The following result is called the Nadel–Shokurov vanishing theorem (see [37], [45]).

Theorem 1.4.2. Let H be a nef and big $\mathbb{Q}$-divisor on Y such that

$$K_Y + B_Y + H \equiv D$$

for some Cartier divisor D on the variety Y. Then for every $i \geqslant 1$

$$H^i\Big(Y,\ \mathcal{I}(Y, B_Y) \otimes \mathcal{O}_Y(D)\Big) = 0.$$

Proof. It follows from the Kawamata–Viehweg vanishing theorem (see [28]) that

$$R^i\pi_*\Big(\pi^*\mathcal{O}_Y\big(K_Y + B_Y + H\big) \otimes \mathcal{O}_{\bar{Y}}\big(\sum_{i=1}^{n} \lceil c_i \rceil E_i - \sum_{i=1}^{r} \lfloor a_i \rfloor \bar{B}_i\big)\Big) = 0$$

for every $i > 0$. It follows from the equality of sheaves

$$\pi_*\Big(\pi^*\mathcal{O}_Y\big(K_Y + B_Y + H\big) \otimes \mathcal{O}_{\bar{Y}}\big(\sum_{i=1}^{n} \lceil c_i \rceil E_i - \sum_{i=1}^{r} \lfloor a_i \rfloor \bar{B}_i\big)\Big) = \mathcal{I}\big(Y, B_Y\big) \otimes \mathcal{O}_Y\big(D\big)$$

and from the degeneration of a local-to-global spectral sequence that

$$H^i\Big(Y, \mathcal{I}(Y, B_Y) \otimes \mathcal{O}_Y(D)\Big) = H^i\Big(\bar{Y}, \pi^*\mathcal{O}_Y\big(K_Y B_Y + H\big) \otimes \mathcal{O}_{\bar{Y}}\big(\sum_{i=1}^{n} \lceil c_i \rceil E_i - \sum_{i=1}^{r} \lfloor a_i \rfloor \bar{B}_i\big)\Big),$$

for every $i \geqslant 0$. But for $i > 0$, the cohomology group

$$H^i\Big(\bar{Y}, \pi^*\mathcal{O}_Y\big(K_Y B_Y + H\big) \otimes \mathcal{O}_{\bar{Y}}\big(\sum_{i=1}^{n} \lceil c_i \rceil E_i - \sum_{i=1}^{r} \lfloor a_i \rfloor \bar{B}_i\big)\Big),$$

is trivial by the Kawamata–Viehweg vanishing theorem (see [28]). □

For every Cartier divisor D on the variety Y, let us consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}\big(Y, B_Y\big) \otimes \mathcal{O}_Y\big(D\big) \longrightarrow \mathcal{O}_Y\big(D\big) \longrightarrow \mathcal{O}_{\mathcal{L}(Y, B_Y)}\big(D\big) \longrightarrow 0.$$

We have the corresponding exact sequence of cohomology groups

$$H^0\Big(Y, \mathcal{O}_Y\big(D\big)\Big) \longrightarrow H^0\Big(\mathcal{L}(Y,\ B_Y), \mathcal{O}_{\mathcal{L}(Y, B_Y)}\big(D\big)\Big) \longrightarrow H^1\Big(Y, \mathcal{I}\big(Y, B_Y\big) \otimes \mathcal{O}_Y\big(D\big)\Big).$$

Theorem 1.4.3. Suppose that $-(K_Y + B_Y)$ is nef and big. Then $\mathrm{LCS}(Y, B_Y)$ is connected.

Proof. Put $D = 0$. Then it follows from Theorem 1.4.2 that the sequence

$$\mathbb{C} = H^0\Big(Y, \mathcal{O}_Y\Big) \longrightarrow H^0\Big(\mathcal{L}(Y,\ B_Y), \mathcal{O}_{\mathcal{L}(Y, B_Y)}\Big) \longrightarrow H^1\Big(Y, \mathcal{I}\big(Y, B_Y\big)\Big) = 0$$

is exact. Thus, the locus

$$\mathrm{LCS}\Big(Y,\ B_Y\Big) = \mathrm{Supp}\Big(\mathcal{L}\big(Y,\ B_Y\big)\Big)$$

is connected. □

One can generalize Theorem 1.4.3 in the following way (see [45, Lemma 5.7]).

Theorem 1.4.4. Let $\psi\colon Y \to Z$ be a morphism. Then the set

$$\mathrm{LCS}\Big(\bar{Y},\ B^{\bar{Y}}\Big)$$

is connected in a neighborhood of every fiber of the morphism $\psi \circ \pi\colon \bar{Y} \to Z$ in the case when

- the morphism ψ is surjective and has connected fibers,
- the divisor $-(K_Y + B_Y)$ is nef and big with respect to ψ.

Let us consider one important application of Theorem 1.4.4.

Theorem 1.4.5. Suppose that B_1 is a Cartier divisor, $a_1 = 1$, and B_1 has at most log terminal singularities. Then the following assertions are equivalent:

- the log pair (Y, B_Y) is log canonical in a neighborhood of the divisor B_1;
- the singularities of the log pair $(B_1, \sum_{i=2}^r a_i B_i|_{B_1})$ are log canonical.

Proof. Suppose that the singularities of the log pair (Y, B_Y) are not log canonical in a neighborhood of the divisor $B_1 \subset Y$. Let us show that $(B_1, \sum_{i=2}^r a_i B_i|_{B_1})$ is not log canonical.

In the case when $a_m > 1$ and $B_m \cap B_1 \neq \varnothing$ for some $m \geqslant 2$, the log pair

$$\Big(B_1,\ \sum_{i=2}^r a_i B_i\Big|_{B_1}\Big)$$

is not log canonical. Thus, we may assume that $a_i \leqslant 1$ for every i. Then

$$\Big(Y,\ B_1 + \sum_{i=2}^r \lambda a_i B_i\Big)$$

is not log canonical as well for some rational number $\lambda < 1$. Then

$$K_{\bar{Y}} + \bar{B}_1 + \sum_{i=2}^r \lambda a_i \bar{B}_i = \pi^*\Big(K_Y + B_1 + \sum_{i=2}^r \lambda a_i B_i\Big) + \sum_{i=1}^n d_i E_i$$

for some rational numbers $d_1, \ldots, d_n$. It follows from Theorem 1.4.4 that

$$\bar{B}_1 \cap E_k \neq \varnothing$$

and the inequality $d_k \leqslant -1$ holds for some k. But

$$K_{\bar{B}_1} + \sum_{i=2}^r \lambda a_i \bar{B}_i\Big|_{B_1} = \phi^*\Big(K_{B_1} + \sum_{i=2}^r \lambda a_i B_i\Big|_{B_1}\Big) + \sum_{i=1}^n d_i E_i\Big|_{B_1},$$

where $\phi\colon \bar{B}_1 \to B_1$ is a birational morphism that is induced by π.

Thus, the log pair $(B_1, \sum_{i=2}^r \lambda a_i B_i|_{B_1})$ is not log terminal. Then the log pair

$$\Big(B_1,\ \sum_{i=2}^r a_i B_i\Big|_{B_1}\Big)$$

is not log canonical. The rest of the proof is similar (see the proof of [28, Theorem 7.5]). □

The simplest application of Theorem 1.4.5 is a non-obvious result.

Lemma 1.4.6. Suppose that $\dim(Y) = 2$ and $a_1 \leqslant 1$. Then

$$\Big(\sum_{i=2}^r a_i B_i\Big) \cdot B_1 > 1$$

whenever (Y, B_Y) is not log canonical at a point $O \in B_1$ such that $O \notin \mathrm{Sing}(Y) \cup \mathrm{Sing}(B_1)$.

Proof. Suppose that (Y, B_Y) is not log canonical at a point $O \in B_1$. By Theorem 1.4.5, the pair $(B_1, \sum_{i=2}^r a_i B_i|_{B_1})$ is not log canonical at the point O. Therefore,

$$\Big(\sum_{i=2}^r a_i B_i\Big) \cdot B_1 \geqslant \mathrm{mult}_O\Big(\sum_{i=2}^r a_i B_i\Big|_{B_1}\Big) > 1$$

if $O \notin \mathrm{Sing}(Y) \cup \mathrm{Sing}(B_1)$. □

Let P be a point in Y. Let us consider an effective divisor

$$\Delta = \sum_{i=1}^{r} \varepsilon_i B_i \sim_{\mathbb{Q}} B_Y,$$

where ε_i is a non-negative rational number. Suppose that

- the divisor Δ is a $\mathbb{Q}$-Cartier divisor,
- the log pair (Y, Δ) is log canonical at the point $P \in X$.

Remark 1.4.7. Suppose that (Y, B_Y) is not log canonical in the point $P \in Y$. Put

$$\alpha = \min \left\{ \frac{a_i}{\varepsilon_i} \;\middle|\; \varepsilon_i \neq 0 \right\},$$

where α is well defined, because there is $\varepsilon_i \neq 0$. Then $\alpha < 1$, the log pair

$$\left(Y, \sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha} B_i \right)$$

is not log canonical in the point $P \in Y$, the equivalence

$$\sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha} B_i \sim_{\mathbb{Q}} B_X \sim_{\mathbb{Q}} \Delta$$

holds, and at least one irreducible component of the divisor $\mathrm{Supp}(\Delta)$ is not contained in

$$\mathrm{Supp} \left(\sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha} B_i \right).$$

Suppose that X is a hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d.

Lemma 1.4.8. Let C be a reduced and irreducible curve on X and D be an ample effective $\mathbb{Q}$-divisor on X. Suppose that for a given positive rational number λ we have $\lambda \mathrm{mult}_C D \leqslant 1$. If $\lambda(C \cdot D - (\mathrm{mult}_C D) C^2) \leqslant 1$, then the pair $(X, \lambda D)$ is log canonical at each smooth point P of C not in $\mathrm{Sing}(X)$. Furthermore, if the point P of C is a singular point of X of type $\frac{1}{r}(a, b)$ and $r\lambda(C \cdot D - (\mathrm{mult}_C D) C^2) \leqslant 1$, then the pair $(X, \lambda D)$ is log canonical at P.

Proof. We may write $D = mC + \Omega$, where Ω is an effective divisor whose support does not contain the curve C. Suppose that the pair $(X, \lambda D)$ is not log canonical at a smooth point P of C not in $\mathrm{Sing}(X)$. Since $\lambda m \leqslant 1$, the pair $(X, C + \lambda\Omega)$ is not log canonical at the point P. Then by Lemma 1.4.6 we obtain an absurd inequality

$$1 < \lambda\Omega \cdot C = \lambda C \cdot (D - mC) \leqslant 1.$$

Also, if the point P is a singular point of X, then we have

$$\frac{1}{r} < \lambda\Omega \cdot C = \lambda C \cdot (D - mC) \leqslant \frac{1}{r}.$$

This proves the second statement. □

Let D be an effective $\mathbb{Q}$-divisor on X such that

$$D \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(a_0, a_1, a_2, a_3)}(1).$$

Lemma 1.4.9. Let l be a positive integer such that the linear system

$$\left| \mathcal{O}_{\mathbb{P}(a_0, a_1, a_2, a_3)}(l) \right|$$

contains effective divisors that are given by the vanishing of

- at least two different monomials of the form $x^\alpha y^\beta$,
- at least two different monomials of the form $x^\gamma z^\delta$,
- at least two different monomials of the form $x^\mu t^\nu$,

where $\alpha, \beta, \gamma, \delta, \mu, \nu$ are non-negative integers. Let P be a point in $X \setminus (\mathrm{Sing}(X) \cup C_x)$. Then

$$\mathrm{mult}_P(D) \leqslant \frac{ld}{a_0 a_1 a_2 a_3}.$$

Proof. The required assertion follows from [1, Lemma 3.3]. ☐

Let $\psi\colon X \dashrightarrow \mathbb{P}(a_0, a_1, a_2)$ be a projection.

Lemma 1.4.10. Let l be a positive integer such that the linear system

$$\Big|\mathcal{O}_{\mathbb{P}(a_0,\,a_1,\,a_2,\,a_3)}(l)\Big|$$

contains effective divisors that are given by the vanishing of

- at least two different monomials of the form $x^{\alpha}y^{\beta}$,
- at least two different monomials of the form $x^{\gamma}z^{\delta}$,

where $\alpha, \beta, \gamma, \delta$ are non-negative integers. Let P be a point in $X \setminus (\mathrm{Sing}(X) \cup C_x)$. Then

$$\mathrm{mult}_P\big(D\big) \leqslant \frac{ld}{a_0a_1a_2a_3}$$

in the case when P is not contained in any curve that is contracted by ψ.

Proof. Arguing as in the proof of [1, Corollary 3.4], we obtain the required assertion. ☐

The following result is [4, Corollary 5.3] (cf. [27, Proposition 11]).

Lemma 1.4.11. Suppose that X is given by a quasihomogeneous equation

$$f\big(x, y, z, t\big) = 0 \subset \mathbb{P}\big(a_0, a_1, a_2, a_3\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

where $\mathrm{wt}(x) = a_0, \mathrm{wt}(y) = a_1, \mathrm{wt}(z) = a_2, \mathrm{wt}(t) = a_3$. Then

$$\mathrm{lct}\big(X\big) \geqslant \begin{cases} \dfrac{a_0a_1}{dI}, \\ \dfrac{a_0a_2}{dI} \text{ if } f(0,0,z,t) \neq 0, \\ \dfrac{a_0a_3}{dI} \text{ if } f(0,0,0,t) \neq 0, \end{cases}$$

Lemma 1.4.12. Suppose that C_x is irreducible and reduced, and $C_x \not\subset \mathrm{Supp}(D)$. Then

$$\mathrm{lct}\big(X, D\big) \geqslant \begin{cases} \dfrac{a_1a_2}{d}, \\ \dfrac{a_1a_3}{d} \text{ if } f(0,0,0,t) \neq 0. \end{cases}$$

Proof. Arguing as in the proof of [27, Proposition 11], we obtain the required assertion. ☐

Thus, using Remark 1.4.7, we obtain the following result.

Corollary 1.4.13. Suppose that C_x is irreducible and reduced, and $d < \sum_{i=0}^{3} a_i$. Then

$$\mathrm{lct}\big(X\big) \geqslant \begin{cases} \min\left(\dfrac{a_1a_2}{dI},\ \mathrm{lct}\left(X, \dfrac{I}{a_0}C_x\right)\right), \\ \min\left(\dfrac{a_1a_3}{dI},\ \mathrm{lct}\left(X, \dfrac{I}{a_0}C_x\right)\right) \text{ if } f(0,0,0,t) \neq 0, \end{cases}$$

where $I = \sum_{i=0}^{3} a_i - d$.

Part 2. Infinite series

2.1. Infinite series with $I = 1$

Lemma 2.1.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (2, 2n+1, 2n+1, 4n+1, 8n+4)$ for $n \in \mathbb{Z}_{>0}$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X is singular at the point O_t, which is a singular point of type $\frac{1}{4n+1}(1,1)$ on the surface X. But X has also 4 singular points O_1, O_2, O_3, O_4, which are cut out on X by the equations $x=t=0$. Then O_i is a singular point of type $\frac{1}{2n+1}(1,2n)$ on the surface X.

The curve C_x is reducible. Namely, we have

$$C_x = L_1 + L_2 + L_3 + L_4,$$

where L_i is an irreducible reduced smooth rational curves such that

$$-K_X \cdot L_i = \frac{1}{(2n+1)(4n+1)},$$

and $L_1 \cap L_2 \cap L_3 \cap L_4 = O_t$. Then $L_i \cdot L_j = 1/(4n+1)$ for $i \neq j$. The subadjunction formula implies that

$$L_i \cdot L_i = \frac{1}{(2n+1)(4n+1)} - \frac{1}{2n+1} - \frac{1}{4n+1} = -\frac{6n+1}{(2n+1)(4n+1)}.$$

Note that $\mathrm{lct}(X, C_x) = 1/2$, which implies that $\mathrm{lct}(X) \leqslant 1$. Suppose that $\mathrm{lct}(X) < 1$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Suppose that $P \notin C_x$. Then P is a smooth point of the surface X. Then

$$1 < \mathrm{mult}_P\big(D\big) \leqslant \frac{(4n+2)(8n+4)}{2(2n+2)^2(4n+1)} = \frac{4}{4n+1} < 1$$

by Lemma 1.4.10. We see that $P \in C_x$. It follows from Remark 1.4.7 that we may assume that $L_i \not\subset \mathrm{Supp}(D)$ for some $i = 1, \ldots, 4$.

Suppose that $P = O_t$. Then

$$\frac{1}{(2n+1)(4n+1)} = -K_X \cdot L_i = D \cdot L_i \geqslant \frac{\mathrm{mult}_{O_t}\big(D\big)}{4n+1} > \frac{1}{4n+1},$$

which is a contradiction. Thus, we see that $P \neq O_t$. Then either $P = O_1$, or $P \in X \setminus \mathrm{Sing}(X)$.

Without loss of generality, we may assume that $P \in L_1$. Put $D = mL_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{(2n+1)(4n+1)} = -K_X \cdot L_i = D \cdot L_i = \big(mL_1 + \Omega\big) \cdot L_i \geqslant mL_1 \cdot L_i = \frac{m}{4n+1},$$

which implies that $m \leqslant 1/(2k+1)$. Then it follows from Lemma 1.4.6 that

$$\frac{1+m(6n+1)}{(2n+1)(4n+1)} = \big(-K_X - mL_1\big) \cdot L_1 = \Omega \cdot L_1 > \begin{cases} 1 \text{ if } P \neq O_1, \\ \dfrac{1}{2n+1} \text{ if } P = O_1, \end{cases}$$

which implies, in particular, that $m > 4n/(6n+1)$. But we already proved that $m \leqslant 1/(2k+1)$. The obtained contradiction completes the proof. □

2.2. Infinite series with $I = 2$

Lemma 2.2.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (4, 2n+3, 2n+3, 4n+4, 8n+12)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_t of index $4n+4$, two singular points O_{xt}^i, $i = 1, 2$, of index 4 on the stratum $y = z = 0$, and four singular points O_{yz}^i, $i = 1, \ldots, 4$, of index $2n+3$ on the stratum $x = t = 0$.

The curve C_x is reduced and splits into four irreducible components $L_1, \ldots, L_4$ (L_i passing through O_{yz}^i) that intersect at O_t. One can easily see that $\mathrm{lct}(X, C_x) = 1/2$, which implies $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Suppose that $P = O_t$. By Remark 1.4.7 we may assume that one of the curves L_i (say, L_1) is not contained in $\mathrm{Supp}\,(D)$. One has

$$\frac{1}{(2n+2)(2n+3)} = L_1 \cdot D \geqslant \frac{\mathrm{mult}_P(L_1)\mathrm{mult}_P(D)}{4n+4} > \frac{1}{4n+4} > \frac{1}{(2n+2)(2n+3)}$$

for all $n \geqslant 1$, which is a contradiction.

Suppose that $P = O^1_{xt}$. By a coordinate change we may assume that $P = O_x$. The curve C_t is reduced and splits into four irreducible components $L'_1, \ldots, L'_4$ (L'_i passing through O^i_{yz}) that intersect at O_x. One can easily see that the log pair $(X, \frac{1}{2} \cdot \frac{4}{4n+4} C_t)$ is log canonical at least for $n \geqslant 1$ since $\mathrm{mult}_P(C_t) = 4$. By Remark 1.4.7 we may assume that one of the curves L'_i (say, L'_1) is not contained in $\mathrm{Supp}\,(D)$. One has

$$\frac{1}{2(2n+3)} = L'_1 \cdot D \geqslant \frac{\mathrm{mult}_P(L'_1)\mathrm{mult}_P(D)}{4} > \frac{1}{4} > \frac{1}{2(2n+3)}$$

for all $n \geqslant 1$, which is a contradiction. The point O^1_{xt} is excluded in a similar way.

Suppose that $P = O^1_{yz}$. Put $D = \mu L_1 + \Omega$, where Ω is an effective divisor such that $L_1 \not\subset \mathrm{Supp}\,(\Omega)$. We claim that

$$\mu \leqslant \frac{1}{2n+3}.$$

Indeed, if the inequality fails, by Remark 1.4.7 we may assume that one of the curves L_2, L_3 and L_4 (say, L_2) is not contained in $\mathrm{Supp}\,(D)$. Then

$$\frac{\mu}{4n+4} = \mu L_1 \cdot L_2 \leqslant D \cdot L_2 = \frac{1}{2(n+1)(2n+3)},$$

which is a contradiction. Note that

$$L_1^2 = -\frac{6n+5}{4(n+1)(2n+3)}.$$

By Lemma 1.4.6 one has

$$\frac{1}{2n+3} < \Omega \cdot L_1 = \frac{2+(6n+5)\mu}{4(n+1)(2n+3)} < \frac{1}{2n+3}$$

for all $n \geqslant 1$, which is a contradiction. The points O^i_{zt}, $i = 2, 3, 4$, are excluded in a similar way. So are the smooth points on C_x, which are excluded by this argument for $n = 1$ as well.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.4.10 (which is possible since the projection of X from O_t has finite fibers outside C_x), we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2 \cdot 4(n+1)(8n+12)}{2(2n+3)(2n+3) \cdot 4(n+1)} < 1$$

for $n \geqslant 1$, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(8n+12))$ contains x^{2n+3}, y^4 and z^4. The obtained contradiction completes the proof. □

Lemma 2.2.2. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 4, 7, 12, 24)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^3t + xz^3 + x^4t + \epsilon_1 y^6 + \epsilon_2 x^2yz^2 + \epsilon_3 x^3y^2z + \epsilon_4 x^4y^3 + \epsilon_5 x^8 = 0,$$

where $\epsilon_i \in \mathbb{C}$. The surface X is singular at the point O_z. It is also singular at two points P_1 and P_2 that are cut out on X by the equations $y = z = 0$. It is also singular at two points Q_1 and Q_2 that are cut out on X by the equations $x = z = 0$.

The curve C_x is reducible. We have $C_x = L_1 + L_2$, where L_1 and L_2 are irreducible and reduced curves such that $Q_1 \in L_1$ and $Q_2 \in L_2$. We have

$$L_1 \cdot L_1 = L_2 \cdot L_2 = \frac{-9}{28},\ L_1 \cdot L_2 = \frac{3}{7},$$

and $L_1 \cap L_2 = O_z$. The curve C_y is irreducible and

$$1 = \mathrm{lct}\left(X, \frac{2}{3}C_y\right) < \mathrm{lct}\left(X, \frac{2}{4}C_y\right) = 2,$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair (X, D) is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, without loss of generality we may assume that $L_2 \not\subseteq \mathrm{Supp}(D)$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(21))$ contains x^7, x^3y^3 and z^3, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = P_1$. Then

$$\frac{4}{21} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{4} > \frac{1}{4},$$

which is a contradiction. We see that $P \neq P_1$. Similarly, we see that $P \neq P_2$. Then $P \in C_x$.

Suppose that $P \in L_2$. Then

$$\frac{1}{14} = D \cdot L_2 > \begin{cases} 1 \text{ if } P \neq O_z \text{ and } P \neq Q_2, \\ \dfrac{1}{7} \text{ if } P = O_z, \\ \dfrac{1}{4} \text{ if } P = Q_2, \end{cases}$$

which is a contradiction. The obtained contradiction shows that $P \notin L_2$.

We see that $P \neq O_z$ and $P \in L_1$. Put $D = mL_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subseteq \mathrm{Supp}(\Omega)$. Then

$$\frac{1}{14} = D \cdot L_2 = \big(mL_1 + \Omega\big) \cdot L_2 \geqslant mL_1 \cdot L_2 = \frac{3m}{7},$$

which implies that $m \leqslant 1/6$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 9m}{28} = \big(-K_X - mL_1\big) \cdot L_1 = \Omega \cdot L_1 > \begin{cases} 1 \text{ if } P \neq Q_1, \\ \dfrac{1}{4} \text{ if } P = Q_1, \end{cases}$$

which implies that $m > 5/9$. But we already proved that $m \leqslant 1/6$. The obtained contradiction completes the proof. □

Lemma 2.2.3. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 3n+1, 6n+1, 9n+3, 18n+6)$ for $n \geqslant 2$. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_z of index $6n+1$, two singular points O_{xt}^i, $i = 1, 2$, of index 3 on the stratum $y = z = 0$, and two singular points O_{yt}^i, $i = 1, 2$, of index $3n+1$ on the stratum $x = z = 0$.

The curve C_x is reduced and splits into two components L_1 and L_2 that intersect at O_z. It is easy to see that $\mathrm{lct}(X, C_x) = 2/3$, which implies $\mathrm{lct}(X) \leqslant 1$.

The curve C_y is reduced and splits into two components L_1' and L_2' that intersect at O_z. It is easy to see that the log pair $(X, \frac{2}{3} \cdot \frac{3}{3n+1} C_y)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Note that

$$L_1 \cdot L_2 = (L_1 \cdot L_2)_{O_z} = \frac{3}{6n+1} \text{ and } L_i^2 = \frac{3 - 9n}{(3n+1)(6n+1)}.$$

Suppose that $P = O_z$. Put $D = \mu L_1 + \Omega$, where $L_1 \not\subset \mathrm{Supp}\,(\Omega)$. If $\mu > 0$, then by Remark 1.4.7 one can assume that $L_2 \not\subset \mathrm{Supp}\,(D)$, and hence

$$\frac{2}{(3n+1)(6n+1)} = D \cdot L_2 \geqslant \frac{3\mu}{(3n+1)(6n+1)},$$

so that

$$\mu \leqslant \frac{2}{3(3n+1)}.$$

Since (X,D) is not log canonical at O_z, by Theorem 1.4.5 one has

$$\frac{1}{6n+1} \leqslant \Omega\cdot L_1 = \frac{2+\mu(9n-3)}{(3n+1)(6n+1)} < \frac{4}{(3n+1)(6n+1)}$$

which is impossible for all $n \geqslant 1$. The points $P = O_{yt}^i \in L_i$ and the smooth points $P \in C_x$ are excluded in a similar way.

Suppose that $P = O_{xt}^1 \in L_1'$. Note that

$$L_1'\cdot L_2' = (L_1'\cdot L_2')_{O_z} = \frac{3n+1}{6n+1} \text{ and } (L_i')^2 = \frac{-2(3n+1)}{3(6n+1)}.$$

Put $D = \mu L_1' + \Omega$, where $L_1' \not\subset \mathrm{Supp}\,(\Omega)$. If $\mu > 0$, then by Remark 1.4.7 one can assume that $L_2' \not\subset \mathrm{Supp}\,(D)$, and hence

$$\mu \leqslant \frac{2}{3(3n+1)}.$$

Since (X,D) is not log canonical at O_z, by Theorem 1.4.5 one has

$$\frac{1}{3n+1} \leqslant \Omega\cdot L_1 = \frac{2+2\mu(3n+1)}{3(6n+1)} \leqslant \frac{10}{9(6n+1)}$$

which is impossible for all $n \geqslant 1$. The point $P = O_{xt}^2 \in L_2'$ is excluded in a similar way.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.4.10 (which is possible since the projection of X from O_t has finite fibers), we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2(18n+6)^2}{3(3n+1)(6n+1)(9n+3)} < 1$$

for all $n \geqslant 2$, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(18n+6))$ contains x^{6n+2}, y^6 and z^3x. The obtained contradiction completes the proof. ◻

Lemma 2.2.4. Suppose that $(a_0,a_1,a_2,a_3,d) = (3, 3n+1, 6n+1, 9n, 18n+3)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_y of index $3n+1$, a singular point O_t of index $9n$, and two singular points O_{xt}^i, $i = 1, 2$, of index 3 on the stratum $y = z = 0$.

The curve C_x is reduced and irreducible and has the only singularity (of multiplicity 3) at O_t. It is easy to see that $\mathrm{lct}(X, C_x) = 2/3$, which implies $\mathrm{lct}(X) \leqslant 1$.

The curve C_y is quasismooth. It is easy to see that the log pair $(X, \frac{2}{3}\cdot\frac{3}{3n+1}C_y)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X,D) is not log canonical at some point $P \in X$. By Remark 1.4.7 we may assume that neither C_x nor C_y is contained in $\mathrm{Supp}\,(D)$.

Suppose that $P = O_t$. One has

$$\frac{2}{3n(3n+1)} = C_x\cdot D \geqslant \frac{\mathrm{mult}_P(C_x)\mathrm{mult}_P(D)}{9n} > \frac{3}{9n} > \frac{2}{3n(3n+1)},$$

for all $n \geqslant 1$, which is a contradiction.

Suppose that $P = O_y$. One has

$$\frac{2}{(3n+1)} = C_x\cdot D \geqslant \frac{\mathrm{mult}_P(C_x)\mathrm{mult}_P(D)}{3n} > \frac{1}{3n} > \frac{2}{3n(3n+1)}$$

for all $n \geqslant 1$, which is a contradiction. The smooth points on C_x are excluded in a similar way.

Suppose that $P = O_{xt}^1$. One has

$$\frac{2}{9n} = C_y\cdot D \geqslant \frac{\mathrm{mult}_P(D)}{3n+1} > \frac{1}{3n+1} > \frac{2}{9n}$$

for all $n \geqslant 1$, which is a contradiction.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.4.10 (which is possible since the projection of X from O_t has finite fibers outside of C_x), we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2(18n+3)^2}{3(3n+1)(6n+1)\cdot 9n} < 1$$

for all $n \geqslant 2$, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(18n+3))$ contains x^{6n+1}, y^3x^{3n} and z^3.

Thus, we see that P is a smooth point of $X \setminus C_x$ and $n = 1$. Applying Lemma 1.4.10, we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{24}{3 \cdot 4 \cdot 7 \cdot 9} < 1,$$

because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(12))$ contains x^4, y^3 and xt. The obtained contradiction completes the proof. □

Lemma 2.2.5. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 3, 4, 4, 12)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$\prod_{i=1}^{4}(\alpha_i x + \beta_i y) = \prod_{i=1}^{3}(\gamma_i z + \delta_i t),$$

where $(\alpha_i, \beta_i) \in \mathbb{P}^1 \ni (\gamma_i, \delta_i)$.

Let P_i be a point in X that is given by $z = t = \alpha_i x + \beta_i y = 0$, where $i = 1, \ldots, 4$. Then P_i is a singular point of X of type $\frac{1}{3}(1,1)$.

Let Q_i be a point in X that is given by $x = y = \gamma_i z + \delta_i t = 0$, where $i = 1, \ldots, 3$. Then Q_i is a singular point of X of type $\frac{1}{4}(1,1)$.

Let L_{ij} be a curve in X that is given by $\alpha_i x + \beta_i y = \gamma_j z + \delta_j t = 0$, where $i = 1, \ldots, 4$ and $j = 1, \ldots, 3$. Then

$$\frac{L_{i1} + L_{i2} + L_{i3}}{3} \sim_{\mathbb{Q}} \frac{L_{1j} + L_{2j} + L_{3j} + L_{4j}}{4} \sim_{\mathbb{Q}} -\frac{1}{2}K_X,$$

and $L_{i1} \cap L_{i2} \cap L_{i3} = P_i$ and $L_{1j} \cap L_{2j} \cap L_{3j} \cap L_{4j} = Q_j$. We have

$$\mathrm{lct}\left(X, \frac{2}{3}\Big(L_{i1} + L_{i2} + L_{i3}\Big)\right) = \mathrm{lct}\left(X, \frac{2}{4}\Big(L_{1j} + L_{2j} + L_{3j} + L_{4j}\Big)\right) = 1,$$

which implies that $\mathrm{lct}(X) \leqslant 3/2$. We have $L_{ij} \cdot L_{ik} = 1/3$ and $L_{ji} \cdot L_{ki} = 1/4$ if $k \neq j$. But $L_{ij}^2 = -5/12$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair (X, D) is not log canonical at some point P. For every $i = 1, ldots, 4$, we may assume that the support of the divisor D does not contain at least one curve among L_{i1}, L_{i2}, L_{i3}. For every $j = 1, \ldots, 3$, we may assume that the support of the divisor D does not contain at least one curve among $L_{1j}, L_{2j}, L_{3j}, L_{4j}$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Suppose that $P = P_1$. If $L_{1k} \not\subseteq \mathrm{Supp}\,(D)$, then

$$\frac{1}{6} = D \cdot L_{1k} \geqslant \frac{\mathrm{mult}_P(D)}{4} > \frac{1}{4} > \frac{1}{6},$$

which implies that $P \neq P_1$. Similarly, we see that $P \notin \mathrm{Sing}(X)$.

Suppose that $P \in L_{11}$. Put $D = \mu L_{11} + \Omega$, where Ω is an effective divisor such that $L_{11} \not\subset \mathrm{Supp}\,(\Omega)$. If $\mu > 0$, then $\mu \leqslant 1/2$, because either $L_{12} \cdot \Omega \geqslant 0$ or $L_{13} \cdot \Omega \geqslant 0$ in the case when $\mu > 0$. Thus, by Lemma 1.4.6 one has

$$1 < \Omega \cdot L_{11} = \frac{2 + 5\mu}{12},$$

which implies that $m > 1/2$. But we know that $\mu \leqslant 1/2$. Thus, we see that $P \notin L_{11}$. Similarly, we see that

$$P \notin \bigcup_{i=1}^{4}\bigcup_{j=1}^{3} L_{ij}.$$

There is a unique curve $C \subset X$ such that $P \in C$ and C is cut out on X by $\lambda x + \mu y = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$. Then C is irreducible and quasismooth. Thus, we may assume that C is not contained in the support of D. Then

$$\frac{1}{2} = D \cdot C \geqslant \mathrm{mult}_P(D) > 1,$$

which is a contradiction. The obtained contradiction completes the proof. □

Lemma 2.2.6. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 3n, 3n+1, 3n+1, 9n+3)$ for $n \geqslant 2$. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_y of index $3n$, three singular points O_{xy}^i, $i = 1,2,3$, of index 3 on the stratum $z = t = 0$, and three singular points O_{zt}^i, $i = 1,2,3$, of index $3n+1$ on the stratum $x = y = 0$.

The curve C_x is reduced and splits into three irreducible components L_1, L_2 and L_3 (L_i passing through O_{zt}^i) that intersect at O_y. One can easily check that $\mathrm{lct}(X, C_x) = 2/3$, which implies $\mathrm{lct}(X) \leqslant 1$.

The curve C_y is quasismooth. One can easily see that the log pair $(X, \frac{2}{3} \cdot \frac{3}{3n} C_y)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Suppose that $P = O_y$. By Remark 1.4.7 we may assume that one of the curves L_i (say, L_1) is not contained in $\mathrm{Supp}\,(D)$. One has

$$\frac{2}{3n(3n+1)} = L_1 \cdot D \geqslant \frac{\mathrm{mult}_P(L_1)\mathrm{mult}_P(D)}{3n} > \frac{1}{3n} > \frac{2}{3n(3n+1)}$$

for all $n \geqslant 1$, which is a contradiction.

Suppose that $P = O_{zt}^1$. Put $D = \mu L_1 + \Omega$, where Ω is an effective divisor such that $L_1 \not\subset \mathrm{Supp}\,(\Omega)$. We claim that

$$\mu \leqslant \frac{2}{3n+1}.$$

Indeed, if the inequality fails, by Remark 1.4.7 we may assume that one of the curves L_2 and L_3 (say, L_2) is not contained in $\mathrm{Supp}\,(D)$. Then

$$\frac{\mu}{3n} = \mu L_1 \cdot L_2 \leqslant D \cdot L_2 = \frac{2}{3n(3n+1)},$$

which is a contradiction. Note that

$$L_1^2 = -\frac{6n-1}{3n(3n+1)}.$$

By Lemma 1.4.6 one has

$$\frac{1}{3n+1} < \Omega \cdot L_1 = \frac{2+(6n-1)\mu}{3n(3n+1)} < \frac{1}{3n+1}$$

for all $n \geqslant 2$, which is a contradiction. The points O_{zt}^2 and O_{zt}^3 are excluded in a similar way. So are the smooth points on C_x, which are excluded by this argument for $n = 1$ as well.

Suppose that $P = O_{xy}^1$. By Remark 1.4.7 we may assume that C_y is not contained in $\mathrm{Supp}\,(D)$. One has

$$\frac{2}{3n+1} = C_y \cdot D \geqslant \frac{\mathrm{mult}_P(C_y)\mathrm{mult}_P(D)}{3} > \frac{1}{3} > \frac{2}{3n+1}$$

for all $n \geqslant 2$, which is a contradiction. The points O_{xy}^2 and O_{xy}^3 are excluded in a similar way.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.4.10 (which is possible since the projection of X from O_t has finite fibers), we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2(9n+3) \cdot 12n}{3 \cdot 3n(3n+1)(3n+1)} < 1$$

for $n \geqslant 2$, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(12n))$ contains x^{4n}, y^4 and z^3x^{n-1}. The obtained contradiction completes the proof. □

Lemma 2.2.7. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 3n+1, 3n+2, 3n+2, 9n+6)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_y of index $3n+1$, and three singular points O_{zt}^i, $i = 1,2,3$, of index $3n+2$ on the stratum $x = y = 0$.

The curve C_x is reduced and reducible. We have $C_x = L_1+L_2+L_3$, where L_i is an irreducible curve such that $O^i_{zt} \in L_i$. Then $L_1 \cap L_2 \cap L_3 = O_y$. One can easily see that $\mathrm{lct}(X, C_x) = 2/3$, which implies $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. By Remark 1.4.7 we may assume that L_1 is not contained in $\mathrm{Supp}\,(D)$.

Suppose that $P \in L_1$. Then

$$\frac{2}{(3n+1)(3n+2)} = L_1 \cdot D \geqslant \begin{cases} 1 \text{ if } P \neq O^1_{zt}, \\ \dfrac{\mathrm{mult}_P(D)}{3n+2} \text{ if } P = O^1_{zt}, \end{cases} > \frac{1}{3n+2} > \frac{2}{(3n+1)(3n+2)}$$

for all $n \geqslant 1$, which is a contradiction. Thus, we see that $P \notin L_1$. In particular, we see that $P \neq O_y$.

Suppose that $P \in L_2$. Put $D = \mu L_2 + \Omega$, where Ω is an effective divisor such that $L_2 \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{\mu}{3n+1} = \mu L_1 \cdot L_2 \leqslant D \cdot L_1 = \frac{2}{(3n+1)(3n+2)},$$

which implies that $\mu \leqslant 2/(3n+2)$. Note that the inequality

$$L_1^2 = -\frac{6n+1}{(3n+1)(3n+2)}$$

holds. Therefore, by Lemma 1.4.6 one has

$$\frac{2+(6n+1)\mu}{(3n+1)(3n+2)} = \Omega \cdot L_2 > \begin{cases} 1 \text{ if } P \neq O^2_{zt}, \\ \dfrac{1}{3n+2} \text{ if } P = O^2_{zt}, \end{cases}$$

which implies that $n = 1$ and $P = O^2_{zt}$, because $\mu \leqslant 2/(3n+2)$.

Let R_2 be a unique curve in the pencil $|\mathcal{O}_{\mathbb{P}}(3n+2)|_X|$ that passes through the point O^2_{zt}. Then $R_2 = L_2 + Z_2$, where Z_2 is an irreducible reduced curve that is singular at the point O^2_{zt}. Moreover, the log pair $(X, \frac{2}{5}(L_2 + R_2)$ is log canonical at the point O^2_{zt}. By Remark 1.4.7, we may assume that $R_2 \not\subseteq \mathrm{Supp}\,(D)$. Then

$$\frac{2}{5} < \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(R_2)}{5} \leqslant D \cdot R_2 = \frac{2}{5}$$

which is a contradiction. Thus, we see that $P \notin L_2$. Similarly, we see that $P \notin L_3$.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.4.10, we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2(9n+6) \cdot 3(3n+2)}{3(3n+1)(3n+2)(3n+2)} < 1$$

for $n \geqslant 2$, because because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3(3n+2)))$ contains x^{3n+2}, y^3x and z^3. Therefore, we see that $n = 1$.

Let R_P be a unique curve in the pencil $|\mathcal{O}_{\mathbb{P}}(5)|_X|$ that passes through the point P. The log pair $(X, \frac{2}{5}R_P)$ is log terminal at the point P. By Remark 1.4.7, we may assume that $\mathrm{Supp}\,(D)$ does not contain at least one irreducible component of R_P. Note that either R_P is irreducible or $O^k_{zt} \in R_P$ for some $k = 1, 2, 3$.

Suppose that R_P is irreducible. Then

$$1 < \mathrm{mult}_P(D) \leqslant D \cdot R_P = \frac{1}{2} < 1$$

which is contradiction. Thus, we see that $O^k_{zt} \in R_P$. Then $R_P = L_k + Z$, where Z is an irreducible curve such that $P \in Z$. We have

$$L_k \cdot L_k = \frac{-7}{20},\ L_k \cdot Z = \frac{3}{5},\ Z \cdot Z = \frac{2}{5}.$$

Put $D = mZ + \Delta$, where Δ is an effective divisor such that $Z \not\subset \mathrm{Supp}\,(\Delta)$. If $m > 0$, then

$$\frac{3m}{5} = mL_k \cdot Z \leqslant D \cdot L_k = \frac{1}{10},$$

which implies that $\mu \leqslant 1/6$. Therefore, by Lemma 1.4.6 one has

$$\frac{2-2m}{5} = \Delta \cdot Z > 1$$

which is a contradiction. The obtained contradiction completes the proof. ☐

Lemma 2.2.8. Suppose that $(a_0, a_1, a_2, a_3, d) = (4, 2n+1, 4n+2, 6n+1, 12n+6)$ for $n \in \mathbb{Z}_{>0}$. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_x of index 4, a singular point O_t of index $6n+1$, a singular point O_{xz} of index 2 on the stratum $y = t = 0$, and three singular points O_{yz}^i, $i = 1, 2, 3$, of index $2n+1$ on the stratum $x = t = 0$.

The curve C_x is reduced and splits into three irreducible components L_1, L_2 and L_3 (L_i passing through O_{yz}^i) that intersect at O_t. One can easily see that $\mathrm{lct}(X, C_x) = 1/2$, which implies $\mathrm{lct}(X) \leqslant 1$.

The curve C_y is quasismooth. One can easily see that the log pair $(X, \frac{1}{2} \cdot \frac{4}{2n+1} C_y)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Suppose that $P = O_t$. By Remark 1.4.7 we may assume that one of the curves L_i (say, L_1) is not contained in $\mathrm{Supp}\,(D)$. One has

$$\frac{2}{(2n+1)(6n+1)} = L_1 \cdot D \geqslant \frac{\mathrm{mult}_P(L_1)\mathrm{mult}_P(D)}{6n+1} > \frac{1}{6n+1} > \frac{2}{(2n+1)(6n+1)}$$

for all $n \geqslant 1$, which is a contradiction.

Suppose that $P = O_{yz}^1$. Put $D = \mu L_1 + \Omega$, where Ω is an effective divisor such that $L_1 \not\subset \mathrm{Supp}\,(\Omega)$. We claim that

$$\mu \leqslant \frac{1}{2n+1}.$$

Indeed, if the inequality fails, by Remark 1.4.7 we may assume that one of the curves L_2 and L_3 (say, L_2) is not contained in $\mathrm{Supp}\,(D)$. Then

$$\frac{2\mu}{6n+1} = \mu L_1 \cdot L_2 \leqslant D \cdot L_2 = \frac{2}{(2n+1)(6n+1)},$$

which is a contradiction. Note that

$$L_1^2 = -\frac{8n}{(2n+1)(6n+1)}.$$

By Lemma 1.4.6 one has

$$\frac{1}{2n+1} < \Omega \cdot L_1 = \frac{2+8n\mu}{(2n+1)(6n+1)} < \frac{2}{(2n+1)^2} < \frac{1}{2n+1}$$

for all $n \geqslant 1$, which is a contradiction. The points O_{yz}^2 and O_{yz}^3 are excluded in a similar way, and so are the smooth points on C_x.

Suppose that $P = O_x$. By Remark 1.4.7 we may assume that C_y is not contained in $\mathrm{Supp}\,(D)$. One has

$$\frac{3}{6n+1} = C_y \cdot D \geqslant \frac{\mathrm{mult}_P(C_y)\mathrm{mult}_P(D)}{4} > \frac{1}{4} > \frac{3}{6n+1}$$

for all $n \geqslant 2$, which is a contradiction. The point O_{xz} is excluded in a similar way.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.4.10 (which is possible since the projection of X from O_t has finite fibers outside C_x), we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2(12n+6) \cdot 12n}{2(2n+1)(4n+2)(6n+1)} < 1$$

for $n \geqslant 2$, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(12n))$ contains x^{3n}, y^4x^{n-1} and z^2x^{n-1}. The obtained contradiction completes the proof. ☐

Lemma 2.3.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (6, 6n+3, 6n+5, 6n+5, 18n+15)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_x of index 6, a singular point O_y of index $6n+3$, a singular point O_{xy} of index 3 on the stratum $z = t = 0$, and three singular points O_{zt}^i, $i = 1, 2, 3$, of index $6n+5$ on the stratum $x = y = 0$.

The curve C_x is reduced and splits into three irreducible components L_1, L_2 and L_3 (L_i passing through O_{zt}^i) that intersect at O_y. One can easily check that $\mathrm{lct}(X, C_x) = 2/3$, which implies $\mathrm{lct}(X) \leqslant 1$.

The curve C_y is reduced and splits into three irreducible components L_1', L_2' and L_3' (L_i' passing through O_{zt}^i) that intersect at O_x. One can easily see that the log pair $(X, \frac{2}{3} \cdot \frac{6}{6n+3} C_y)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. By Remark 1.4.7 we may assume that L_1 and L_1' are not contained in $\mathrm{Supp}\,(D)$.

Suppose that $P = O_x$. Then

$$\frac{4}{6(6n+5)} = L_1' \cdot D \geqslant \frac{\mathrm{mult}_P(L_1')\mathrm{mult}_P(D)}{6} > \frac{1}{6} > \frac{4}{6(6n+5)}$$

for all $n \geqslant 1$, which is a contradiction.

Suppose that $P = O_{xy}$. Let R be a general curve in the pencil $|\mathcal{O}_{\mathbb{P}}(6n+5)|_X|$. Then

$$\frac{1}{3} < \frac{\mathrm{mult}_P(D)}{3} \leqslant D \cdot R = \frac{4(18n+15) \cdot (6n+5)}{6(6n+3)(6n+5)(6n+5)} < 1$$

for all $n \geqslant 1$, which is a contradiction. Thus, we see that $P \neq O_{xy}$.

Suppose that $P \in L_1$. Then

$$\frac{4}{(6n+3)(6n+5)} = L_1 \cdot D \geqslant \begin{cases} 1 \text{ if } P \neq O_{zt}^1 \text{ and } P \neq O_y, \\ \dfrac{\mathrm{mult}_P(D)}{6n+3} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{6n+5} \text{ if } P = O_{zt}^1, \end{cases} > \frac{1}{6n+5} > \frac{4}{(6n+3)(6n+5)}$$

for all $n \geqslant 1$, which is a contradiction. Thus, we see that $P \notin L_1$. In particular, we see that $P \neq O_y$.

Suppose that $P \in L_2$. Put $D = \mu L_2 + \Omega$, where Ω is an effective divisor such that $L_2 \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{\mu}{6n+3} = \mu L_1 \cdot L_2 \leqslant D \cdot L_1 = \frac{4}{(6n+3)(6n+5)},$$

which implies that $\mu \leqslant 4/(6n+5)$. Note that the inequality

$$L_2^2 = -\frac{12n+4}{(6n+3)(6n+5)}$$

holds. Therefore, by Lemma 1.4.6 one has

$$\frac{4 + (12n+4)\mu}{(6n+3)(6n+5)} = \Omega \cdot L_2 > \begin{cases} 1 \text{ if } P \neq O_{zt}^2, \\ \dfrac{1}{6n+5} \text{ if } P = O_{zt}^2, \end{cases}$$

which implies that $n = 1$ and $P = O_{zt}^2$, because $\mu \leqslant 4/(6n+5)$.

Let R_2 be a unique curve in the pencil $|\mathcal{O}_{\mathbb{P}}(6n+5)|_X|$ that passes through the point O_{zt}^2. Then $R_2 = L_2 + Z_2$, where Z_2 is an irreducible reduced curve that is singular at the point O_{zt}^2. Moreover, the log pair $(X, \frac{4}{11}(L_2 + R_2)$ is log canonical at the point O_{zt}^2. By Remark 1.4.7, we may assume that $R_2 \not\subseteq \mathrm{Supp}\,(D)$. Then

$$\frac{2}{11} < \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(R_2)}{11} \leqslant D \cdot R_2 = \frac{2}{11}$$

which is a contradiction. Thus, we see that $P \notin L_2$. Similarly, we see that $P \notin L_3$.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.4.10, we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{4(18n+15)\cdot 6(6n+5)}{6(6n+3)(6n+5)(6n+5)} < 1$$

for $n \geqslant 2$, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6(6n+5))$ contains x^{6n+5}, y^6x^2 and z^6. Therefore, we see that $n = 1$.

Let R_P be a unique curve in the pencil $|\mathcal{O}_{\mathbb{P}}(11)|_X|$ that passes through the point P. The log pair $(X, \frac{4}{11}R_P)$ is log terminal at the point P. By Remark 1.4.7, we may assume that $\mathrm{Supp}\,(D)$ does not contain at least one irreducible component of R_P. Note that either R_P is irreducible or $O_{zt}^k \in R_P$ for some $k = 1, 2, 3$.

Suppose that R_P is irreducible. Then

$$1 < \mathrm{mult}_P(D) \leqslant D \cdot R_P = \frac{2}{9} < 1$$

which is contradiction. Thus, we see that $O_{zt}^k \in R_P$. Then $R_P = L_k + Z$, where Z is an irreducible curve such that $P \in Z$. We have

$$L_k \cdot L_k = \frac{-16}{99},\ L_k \cdot Z = \frac{3}{11},\ Z \cdot Z = \frac{5}{22}.$$

Put $D = mZ + \Delta$, where Δ is an effective divisor such that $Z \not\subset \mathrm{Supp}\,(\Delta)$. If $m > 0$, then

$$\frac{3m}{11} = mL_k \cdot Z \leqslant D \cdot L_k = \frac{4}{99},$$

which implies that $\mu \leqslant 4/27$. Therefore, by Lemma 1.4.6 one has

$$\frac{4-5m}{22} = \Delta \cdot Z > 1$$

which is a contradiction. The obtained contradiction completes the proof. □

Lemma 2.3.2. Suppose that $(a_0, a_1, a_2, a_3, d) = (6, 6n+5, 12n+8, 18n+9, 36n+24)$ for $n \in \mathbb{Z}_{>0}$. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_y of index $6n + 5$, a singular point O_t of index $18n + 9$, and a singular point O_{xt} of index 3 on the stratum $y = z = 0$.

The curve C_x is reduced and irreducible and has the only singularity (of multiplicity 3) at O_t. It is easy to see that $\mathrm{lct}(X, C_x) = 2/3$, which implies $\mathrm{lct}(X) \leqslant 1$.

The curve C_y is quasismooth. It is easy to see that the log pair $(X, \frac{2}{3}\cdot\frac{6}{6n+5}C_y)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. By Remark 1.4.7 we may assume that neither C_x nor C_y is contained in $\mathrm{Supp}\,(D)$.

Suppose that $P = O_t$. One has

$$\frac{4}{(6n+3)(6n+5)} = C_x \cdot D \geqslant \frac{\mathrm{mult}_P(C_x)\mathrm{mult}_P(D)}{18n+9} > \frac{3}{18n+9} > \frac{4}{(6n+3)(6n+5)},$$

which is a contradiction.

Suppose that $P = O_y$. One has

$$\frac{4}{(6n+3)(6n+5)} = C_x \cdot D \geqslant \frac{\mathrm{mult}_P(C_x)\mathrm{mult}_P(D)}{6n+5} > \frac{1}{6n+5} > \frac{4}{(6n+3)(6n+5)},$$

which is a contradiction. The smooth points on C_x are excluded in a similar way.

Suppose that $P = O_{xt}$. One has

$$\frac{2}{3(6n+3)} = C_y \cdot D \geqslant \frac{\mathrm{mult}_P(D)}{3} > \frac{1}{3} > \frac{2}{3(6n+3)},$$

which is a contradiction.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.4.10 (which is possible since the projection of X from O_t has finite fibers), we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{4(36n+24)(36n+30)}{6(6n+5)(12n+8)(18n+9)} < 1,$$

because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(36n+30))$ contains x^{6n+5}, y^6 and z^3x. The obtained contradiction completes the proof. □

Lemma 2.3.3. Suppose that $(a_0, a_1, a_2, a_3, d) = (6, 6n+5, 12n+8, 18n+15, 36n+30)$ for $n \in \mathbb{Z}_{>0}$. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_z of index $12n+8$, a singular point O_{xz} of index 2 on the stratum $y = t = 0$, a singular point O_{xt} of index 3 on the stratum $y = z = 0$, and two singular points O_{yt}^i, $i = 1, 2$, of index $6n+5$ on the stratum $x = z = 0$.

The curve C_x is reduced and splits into two irreducible components L_1 and L_2 (L_i passing through O_{yt}^i) that are tangent to order 2 at (the preimage of) the point O_z. One can easily check that $\mathrm{lct}(X, C_x) = 2/3$, which implies $\mathrm{lct}(X) \leqslant 1$.

The curve C_y is quasismooth. It is easy to see that the log pair $(X, \frac{2}{3} \cdot \frac{6}{6n+5} C_y)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Suppose that $P = O_z$. By Remark 1.4.7 we may assume that one of the curves L_1 and L_2 (say, L_1) is not contained in $\mathrm{Supp}\,(D)$. One has

$$\frac{1}{(3n+2)(6n+5)} = L_1 \cdot D \geqslant \frac{\mathrm{mult}_P(L_1)\mathrm{mult}_P(D)}{12n+8} > \frac{1}{12n+8} > \frac{1}{(3n+2)(6n+5)},$$

which is a contradiction.

Suppose that $P = O_{xt}$. By Remark 1.4.7 we may assume that C_y is not contained in $\mathrm{Supp}\,(D)$. One has

$$\frac{1}{3(3n+2)} = C_y \cdot D \geqslant \frac{\mathrm{mult}_P(D)}{3} > \frac{1}{3}\frac{1}{3(3n+2)},$$

which is a contradiction. The point O_{xz} is excluded in a similar way.

Suppose that $P = O_{yt}^1$. Put $D = \mu L_1 + \Omega$, where Ω is an effective divisor such that $L_1 \not\subset \mathrm{Supp}\,(\Omega)$. We claim that

$$\mu \leqslant \frac{4}{3(6n+5)}.$$

Indeed, if the inequality fails, by Remark 1.4.7 we may assume that L_2 is not contained in $\mathrm{Supp}\,(D)$. Then

$$\frac{3\mu}{12n+8} = \mu L_1 \cdot L_2 \leqslant D \cdot L_2 = \frac{1}{(3n+2)(6n+5)},$$

which is a contradiction. Note that

$$L_1^2 = -\frac{18n+9}{(12n+8)(6n+5)}.$$

By Lemma 1.4.6 one has

$$\frac{1}{6n+5} < \Omega \cdot L_1 = \frac{4 + (18n+9)\mu}{(12n+8)(6n+5)} < \frac{1}{6n+5},$$

which is a contradiction. The points O_{yt}^2 and the smooth points on C_x are excluded in a similar way.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.4.10 (which is possible since the projection of X from O_t has finite fibers), we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{4(36n+30)(3(12n+8)+6)}{6(6n+5)(12n+8)(18n+15)} < 1,$$

because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3(12n+8)+6))$ contains x^{12n+9}, y^6 and z^3x. The obtained contradiction completes the proof. □

Lemma 2.4.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (8, 4n+5, 4n+7, 4n+9, 12n+13)$ for $n \geqslant 2$. Then $\operatorname{lct}(X) = 1$.

Proof. The surface X can be given by the equation

$$z^2t + yt^2 + xy^3 + x^{n+2}z = 0,$$

and the only singularities of X are O_x, O_y, O_z and O_t.

The curve C_x is reduced and splits into a union of the stratum L_{xt} and a residual curve M_x intersecting at O_y. One can easily see that $\operatorname{lct}(X, C_x) = 3/4$, which implies $\operatorname{lct}(X) \leqslant 1$.

The curve C_y is reduced and splits into a union of the stratum L_{yz} and a residual curve M_y intersecting at O_t. One can easily see that $\operatorname{lct}(X, C_y) = \frac{n+3}{2n+4}$, and hence the log pair $(X, \frac{4n+5}{6}C_y)$ is log canonical for $n \geqslant 1$.

The curve C_z is reduced and splits into a union of the stratum L_{yz} and a residual curve M_z intersecting at O_x. One can easily see that $\operatorname{lct}(X, C_z) = 2/3$, and hence the log pair $(X, \frac{4n+7}{6}C_z)$ is log terminal for $n \geqslant 1$.

The curve C_t is reduced and splits into a union of the stratum L_{xt} and a residual curve M_t intersecting at O_z. One can easily see that $\operatorname{lct}(X, C_t) = \frac{2n-1}{5(n-1)}$, and hence the log pair $(X, \frac{4n+9}{6}C_t)$ is log terminal for $n \geqslant 1$.

One has the following intersection numbers.

$$L_{xt} \cdot D = \frac{6}{(4n+5)(4n+7)}, L_{xt} \cdot M_x = \frac{2}{4n+5}, L_{xt} \cdot M_t = \frac{3}{4n+7},$$

$$L_{xt}^2 = -\frac{8n+6}{(4n+5)(4n+7)},$$

$$M_x \cdot D = \frac{12}{(4n+5)(4n+9)}, M_t \cdot D = \frac{18}{8(4n+7)},$$

$$M_x^2 = -\frac{8n+2}{(4n+5)(4n+9)}, M_t^2 = -\frac{4n-3}{8(4n+7)},$$

$$L_{yz} \cdot D = \frac{6}{8(4n+9)}, L_{yz} \cdot M_y = \frac{n+2}{4n+9}, L_{yz} \cdot M_z = \frac{1}{4},$$

$$L_{yz}^2 = -\frac{4n+11}{8(4n+9)},$$

$$M_y \cdot D = \frac{6(n+2)}{(4n+7)(4n+9)}, M_z \cdot D = \frac{12}{8(4n+5)},$$

$$M_y^2 = -\frac{2n+4}{(4n+7)(4n+9)}, M_z^2 = -\frac{4n+1}{8(4n+5)}.$$

Suppose that $\operatorname{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Suppose that $P = O_x$. Assume that $L_{yz} \not\subset \operatorname{Supp}(D)$. Then

$$\frac{6}{8(4n+9)} = L_{yz} \cdot D > \frac{1}{8},$$

which is a contradiction for all $n \geqslant 1$. Hence $L_{yz} \subset \operatorname{Supp}(D)$. By Remark 1.4.7 we may assume that $M_y \not\subset \operatorname{Supp}(D)$. Put $D = \mu L_{yz} + \Omega$, where $L_{yz} \not\subset \operatorname{Supp}(\Omega)$. By Theorem 1.4.5 one has

$$\frac{1}{8} < \Omega \cdot L_{yz} = \frac{6 + (4n+11)\mu}{8(4n+9)},$$

and hence $\mu > (4n+3)(4n+11)$. On the other hand,

$$\frac{6(n+2)}{(4n+7)(4n+9)} = D \cdot M_y \geqslant \mu L_{yz} \cdot M_y + \frac{\operatorname{mult}_{O_x}(D) - \mu}{8} > \frac{\mu(n+2)}{4n+9} + \frac{1-\mu}{8},$$

which is a contradiction for $n \geqslant 1$, because $\mu > (4n+3)(4n+11)$.

Suppose that $P = O_y$. Assume that $L_{xt} \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6}{(4n+5)(4n+7)} = L_{xt} \cdot D > \frac{1}{4n+5},$$

which is a contradiction for all $n \geqslant 1$. Hence $L_{xt} \subset \mathrm{Supp}\,(D)$, and by Remark 1.4.7 we may assume that $M_x \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{xt} + \Omega$, where $L_{xt} \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{12}{(4n+5)(4n+9)} = D \cdot M_x < \frac{2\mu}{4n+5},$$

which gives $\mu \leqslant 6/(4n+9)$. By Theorem 1.4.5 one has

$$\frac{1}{4n+5} < \Omega \cdot L_{xt} = \frac{6+(8n+6)\mu}{(4n+5)(4n+7)},$$

which is a contradiction for $n \geqslant 2$.

Suppose that $P = O_z$. Assume that $L_{xt} \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6}{(4n+5)(4n+7)} = L_{xt} \cdot D > \frac{1}{4n+7},$$

which is a contradiction for $n \geqslant 1$. Hence $L_{xt} \subset \mathrm{Supp}\,(D)$, and by Remark 1.4.7 we may assume that $M_x \not\subset \mathrm{Supp}\,(D) \not\supset M_t$. Then $\mu \leqslant 6/(4n+9)$ as above, and by Theorem 1.4.5 one has

$$\frac{1}{4n+7} < \Omega \cdot L_{xt} = \frac{6+(8n+6)\mu}{(4n+5)(4n+7)} \leqslant \frac{18}{(4n+7)(4n+9)},$$

which is a contradiction for $n \geqslant 3$. If $n = 2$, then

$$\frac{18}{8 \cdot 15} = M_t \cdot D \geqslant \frac{\mathrm{mult}_{O_z}(D)\mathrm{mult}_{O_z}(M_t)}{17} = \frac{3\mathrm{mult}_{O_z}(D)}{17} > \frac{3}{17},$$

which is a contradiction.

Suppose that $P = O_t$. Assume that $M_x \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{12}{(4n+5)(4n+9)} = M_x \cdot D > \frac{1}{4n+9},$$

which is a contradiction for $n \geqslant 2$. Hence $M_x \subset \mathrm{Supp}\,(D)$, and by Remark 1.4.7 we may assume that $L_{xt} \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu M_x + \Omega$, where $M_x \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{6}{(4n+5)(4n+7)} = L_{xt} \cdot D < \frac{2\mu}{4n+5},$$

which implies that $\mu \leqslant 3/(4n+7)$. By Theorem 1.4.5 one has

$$\frac{1}{4n+9} < \Omega \cdot M_x = \frac{12+(8n+2)\mu}{(4n+5)(4n+9)} \leqslant \frac{18}{(4n+7)(4n+9)},$$

which is a contradiction for $n \geqslant 2$.

Suppose that P is a smooth point on L_{xt}. Assume that $L_{xt} \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6}{(4n+5)(4n+7)} = L_{xt} \cdot D > 1,$$

which is a contradiction for all $n \geqslant 1$. Hence $L_{xt} \subset \mathrm{Supp}\,(D)$, and by Remark 1.4.7 we may assume that $M_x \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{xt} + \Omega$, where $L_{xt} \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$1 < \Omega \cdot L_{xt} = \frac{6+(8n+6)\mu}{(4n+5)(4n+7)} \leqslant \frac{18}{(4n+7)(4n+9)}$$

by Theorem 1.4.5, because $\mu \leqslant 6/(4n+9)$, which is a contradiction for all $n \geqslant 1$.

Suppose that P is a smooth point on M_x. Assume that $M_x \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{12}{(4n+5)(4n+9)} = M_x \cdot D > 1,$$

which is a contradiction for all $n \geqslant 1$. Hence $M_x \subset \mathrm{Supp}\,(D)$. By Remark 1.4.7 we may assume that $L_{xt} \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu M_x + \Omega$, where $M_x \not\subset \mathrm{Supp}\,(\Omega)$. By Theorem 1.4.5 one has

$$1 < \Omega \cdot M_x = \frac{12 + (8n+2)\mu}{(4n+5)(4n+9)} \leqslant \frac{18}{(4n+7)(4n+9)},$$

which is a contradiction for all $n \geqslant 1$, because $\mu \leqslant 3/(4n+7)$.

Suppose that P is a smooth point on L_{yz}. Assume that $L_{yz} \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6}{8(4n+9)} = L_{yz} \cdot D > 1,$$

which is a contradiction for all $n \geqslant 1$. Hence $L_{yz} \subset \mathrm{Supp}\,(D)$. By Remark 1.4.7 we may assume that $M_y \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{yz} + \Omega$, where $L_{yz} \not\subset \mathrm{Supp}\,(\Omega)$. By Theorem 1.4.5 one has

$$1 < \Omega \cdot L_{yz} = \frac{6 + (4n+11)\mu}{8(4n+9)} \leqslant \frac{3}{2(4n+7)},$$

which is a contradiction for all $n \geqslant 1$, because $\mu \leqslant 6/(4n+7)$.

Suppose that P is a smooth point on M_y. Assume that $M_y \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6(n+2)}{(4n+7)(4n+9)} = M_y \cdot D > 1,$$

which is a contradiction for all $n \geqslant 1$. Hence $M_y \subset \mathrm{Supp}\,(D)$, and by Remark 1.4.7 we may assume that $L_{yz} \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu M_y + \Omega$, where $M_y \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{6}{8(4n+9)} = L_{yz} \cdot D < \frac{\mu(n+2)}{4n+9},$$

which implies that $\mu \leqslant 6/(8n+16)$. By Theorem 1.4.5 one has

$$1 < \Omega \cdot M_y = \frac{12 + (8n+2)\mu}{(4n+5)(4n+9)} \leqslant \frac{6(24n+34)}{8(n+2)(4n+5)(4n+9)},$$

which is a contradiction for all $n \geqslant 1$.

Hence P is a smooth point of $X \setminus (C_x \cup C_y)$. Applying Lemma 1.4.10 (which is possible since the projection of X from O_t has finite fibers outside L_{yz}) we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{6(12n+23) \cdot 8(4n+7)}{8(4n+5)(4n+7)(4n+9)} < 1,$$

for $n \geqslant 3$, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(8(4n+7)))$ contains x^{2n+4}, y^8x^2 and z^8. Arguing as in the end of the proof of Lemma 2.4.3, we see that $n \neq 2$. □

Lemma 2.4.2. Suppose that $(a_0, a_1, a_2, a_3, d) = (8, 9, 11, 13, 35)$. Then $\mathrm{lct}(X) = 1$.

Proof. We have $I = 6$. Let us use the notations and assumptions of the proof of Lemma 2.4.1, where $n = 2$. Then it follows from the proof of Lemma 2.4.3 that either $P = O_z$ or O_t.

Suppose that $P = O_z$. Then $L_{xt} \subset \mathrm{Supp}(D)$, since otherwise we have

$$\frac{6}{9 \cdot 11} = D \cdot L_{xt} > \frac{1}{11} > \frac{6}{9 \cdot 11},$$

which is a contradiction. We may assume that $M_t \not\subset \mathrm{Supp}(D)$ by Remark 1.4.7. Put

$$D = mL_{xt} + cM_y + \Omega,$$

where $m > 0$ and $c \geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor such that $L_{xt} \not\subset \mathrm{Supp}(\Omega) \not\supset M_y$. Then

$$\frac{18}{8 \cdot 11} = D \cdot M_t = \big(mL_{xt} + cM_y + \Omega\big) \cdot M_t \geqslant \frac{3m}{11} + \frac{\mathrm{mult}_{O_z}(D) - m}{33} > \frac{m+1}{11},$$

which implies that $m < 1/4$. Then it follows from Lemma 1.4.6 that

$$\frac{6 + 14m}{9 \cdot 11} = \big(-K_X - mL_{xt}\big) \cdot L_{xt} = (\Omega + cM_y) \cdot L_{xt} > \frac{1}{11},$$

which implies that $m > 3/14$. On the other hand, if $c > 0$, then

$$\frac{6}{8 \cdot 13} = D \cdot L_{yz} = \big(mL_{xt} + cM_y + \Omega\big) \cdot L_{yz} \geqslant \frac{3c}{13},$$

which implies that $c \leqslant 1/4$.

Let $\pi\colon \bar{X} \to X$ be a weighted blow up of O_z with weights $(3,2)$, let E be the exceptional curve of π, let $\bar{\Omega}$, $\bar{L}_{xt}$ and $\bar{M}_y$ be the proper transforms of Ω, L_{xt} and M_y, respectively. Then

$$K_{\bar{X}} \equiv \pi^*(K_X) - \frac{6}{11}E,\ \bar{L}_{xt} \equiv \pi^*(L_{xt}) - \frac{3}{11}E,\ \bar{M}_y \equiv \pi^*(M_y) - \frac{2}{11}E,\ \bar{\Omega} \equiv \pi^*(\Omega) - \frac{a}{11}E.$$

where a is a positive rational number a.

The curve E contains two singular points Q_2 and Q_3 of $\bar{X}$ such that Q_2 is a singular point of type $\frac{1}{2}(1,1)$, and Q_3 is a singular point of type $\frac{1}{2}(1,2)$. Then

$$\bar{L}_{xt} \not\ni Q_3 \in \bar{M}_y \not\ni Q_2 \in \bar{L}_{xt},$$

and $\bar{L}_{xt} \cap \bar{M}_y = \varnothing$. The log pull back of the log pair (X, D) is the log pair

$$\left(\bar{X},\ \bar{\Omega} + m\bar{L}_{xt} + c\bar{M}_y + \frac{6+a+3m+2c}{11}E\right),$$

which must have non-log canonical singularity at some point $Q \in E$. We have

$$\frac{18+6c}{11\cdot 13} - \frac{m}{11} - \frac{a}{33} = \bar{\Omega}\cdot\bar{M}_y \geqslant 0 \leqslant \bar{\Omega}\cdot\bar{L}_{xt} = \frac{6+14m}{9\cdot 11} - \frac{c}{11} - \frac{a}{22},$$

hence $a \leqslant (12+28m)/9 \leqslant 19/9$, because $m \leqslant 1/4$. Then $6+a+3m+2c < 11$, because $c \leqslant 1/4$.

Suppose that $Q \neq Q_2$ and $Q \neq Q_3$. Then $Q \notin \bar{L}_{xt} \cup \bar{M}_y$. By Lemma 1.4.6, we have

$$\frac{a}{2\cdot 3} = -\frac{a}{11}E^2 = \bar{\Omega}\cdot E > 1,$$

which implies that $a > 6$, which is impossible, because $a < 19/9$.

Therefore, we see that either $Q = Q_2$ or $Q = Q_3$.

Suppose that $Q = Q_2$. Then $Q \notin \bar{M}_y$. Hence, it follows from Lemma 1.4.6 that

$$\frac{6+14m}{9\cdot 11} - \frac{c}{11} - \frac{a}{22} + \frac{6+a+3m+2c}{22} = \left(\bar{\Omega} + \frac{6+a+3m+2c}{11}E\right)\cdot\bar{L}_{xt} > \frac{1}{2},$$

which implies that $m > 68/55$. But $m < 1/4$, which is a contradiction.

Thus, we see that $Q = Q_3$. Then $Q \notin \bar{L}_{xt}$, and it follows from Lemma 1.4.6 that

$$\frac{18+6c}{11\cdot 13} - \frac{m}{11} - \frac{a}{33} + \frac{6+a+3m+2c}{33} = \left(\bar{\Omega} + \frac{6+a+3m+2c}{11}E\right)\cdot\bar{M}_y > \frac{1}{3},$$

which implies that $c > 1/4$. But $c \leqslant 1/4$. The obtained contradiction shows that $P \neq O_z$.

We see that $P = O_t$. Then $L_{yz} \not\subset \mathrm{Supp}(D)$, since otherwise we have

$$\frac{6}{8\cdot 13} = D\cdot L_{yz} > \frac{1}{13} > \frac{6}{8\cdot 13},$$

which is a contradiction. By Remark 1.4.7, we may assume that $M_y \not\subset \mathrm{Supp}(D)$. Put

$$D = mL_{yz} + cM_x + \Omega,$$

where $m > 0$ and $c \geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor such that $L_{yz} \not\subset \mathrm{Supp}(\Omega) \not\supset M_x$. Then

$$\frac{8}{11\cdot 13} = D\cdot M_y = \big(mL_{yz} + cM_x + \Omega\big)\cdot M_y \geqslant \frac{3m}{13} + \frac{\mathrm{mult}_{O_t}(D) - m}{13} > \frac{2m+1}{13},$$

which implies that $m < 7/22$. Then it follows from Lemma 1.4.6 that

$$\frac{6+15m}{8\cdot 13} = \big(-K_X - mL_{yz}\big)\cdot L_{yz} = (\Omega + cM_x)\cdot L_{yz} > \frac{1}{13},$$

which implies that $m > 2/15$. On the other hand, if $c > 0$, then

$$\frac{6}{9\cdot 11} = D\cdot L_{xt} = \big(mL_{yz} + cM_x + \Omega\big)\cdot L_{yt} = \big(cM_x + \Omega\big)\cdot L_{yt} \geqslant \frac{3c}{11},$$

which implies that $c \leqslant 3/11$.

Let $\pi\colon \bar{X} \to X$ be a weighted blow up of O_t with weights $(5,2)$, let E be the exceptional curve of π, let $\bar{\Omega}$, $\bar{L}_{yz}$ and $\bar{M}_x$ be the proper transforms of Ω, L_{yz} and M_x, respectively. Then

$$K_{\bar{X}} \equiv \pi^*(K_X) + \frac{6}{13}E,\ \bar{L}_{yz} \equiv \pi^*(L_{yz}) - \frac{2}{13}E,\ \bar{M}_x \equiv \pi^*(M_x) - \frac{5}{13}E,\ \bar{\Omega} \equiv \pi^*(\Omega) - \frac{a}{13}E,$$

where a is a positive rational number.

The curve E contains two singular points Q_5 and Q_2 of $\bar{X}$ such that Q_5 is a singular point of type $\frac{1}{5}(1,1)$, and Q_2 is a singular point of type $\frac{1}{2}(1,1)$. Then

$$\bar{L}_{yz} \not\ni Q_2 \in \bar{M}_x \not\ni Q_5 \in \bar{L}_{yz},$$

and $\bar{L}_{yz} \cap \bar{M}_x = \varnothing$. The log pull back of the log pair (X, D) is the log pair

$$\left(\bar{X},\ \bar{\Omega} + m\bar{L}_{yz} + c\bar{M}_y + \frac{6+a+2m+5c}{13}E\right),$$

which must have non-log canonical singularity at some point $Q \in E$. Then

$$\frac{12+10c}{9\cdot 13} - \frac{m}{13} - \frac{a}{26} = \bar{\Omega}\cdot\bar{M}_x \geqslant 0 \leqslant \bar{\Omega}\cdot\bar{L}_{yz} = \frac{6+15m}{8\cdot 13} - \frac{c}{13} - \frac{a}{65},$$

which implies that $30+75m \geqslant 40c+8a$ and $24+20c \geqslant 18m+9a$. In particular, we see that $a \leqslant 36/11$. Then $6+a+2m+5c < 13$, because $c \leqslant 3/11$ and $m \leqslant 7/22$.

Suppose that $Q \neq Q_2$ and $Q \neq Q_5$. Then $Q \notin \bar{L}_{yz} \cup \bar{M}_x$. By Lemma 1.4.6, we have

$$\frac{a}{10} = -\frac{a}{13}E^2 = \bar{\Omega}\cdot E > 1,$$

which implies that $a > 10$, which is impossible, because $a < 36/11$. Therefore, we see that either $Q = Q_2$ or $Q = Q_5$.

Suppose that $Q = Q_2$. Then $Q \notin \bar{L}_{yz}$. Hence, it follows from Lemma 1.4.6 that

$$\frac{12+10c}{9\cdot 13} - \frac{m}{13} - \frac{a}{26} + \frac{6+a+2m+5c}{26} = \left(\bar{\Omega} + \frac{6+a+2m+5c}{13}E\right)\cdot\bar{M}_x > \frac{1}{2},$$

which implies that $c > 3/5$. But $c \leqslant 3/11$, which is a contradiction.

Thus, we see that $Q = Q_5$. Then $Q \notin \bar{M}_x$, and it follows from Lemma 1.4.6 that

$$\frac{6+15m}{8\cdot 13} + \frac{6+2m}{65} = \left(\bar{\Omega} + \frac{6+a+2m+5c}{13}E\right)\cdot\bar{L}_{yz} > \frac{1}{5} < \left(\bar{\Omega} + m\bar{L}_{yz}\right)\cdot E = \frac{a}{10} + \frac{m}{5},$$

which implies that $m > 2/7$ and $a + 2m > 2$. But we have no contradiction here.

Let $\psi\colon \tilde{X} \to \bar{X}$ be a weighted blow up of Q_5 with weights $(1,1)$, let G be the exceptional curve of ψ, let $\tilde{\Omega}$, $\tilde{L}_{yz}$, $\tilde{M}_x$ and $\tilde{E}$ be the proper transforms of Ω, L_{yz}, M_x and E, respectively. Then

$$K_{\tilde{X}} \equiv \psi^*(K_{\bar{X}}) - \frac{3}{5}G,\ \tilde{L}_{yz} \equiv \psi^*(\bar{L}_{yz}) - \frac{1}{5}G,\ \tilde{E} \equiv \psi^*(E) - \frac{1}{5}G,\ \tilde{\Omega} \equiv \psi^*(\bar{\Omega}) - \frac{b}{5}G,$$

where b is a positive rational number.

The surface is smooth along G. The log pull back of (X, D) is the log pair

$$\left(\tilde{X},\ \tilde{\Omega} + m\tilde{L}_{yz} + c\tilde{M}_x + \frac{6+a+2m+5c}{13}\tilde{E} + \theta G\right),$$

where $\theta = 3m/13 + c/13 + a/65 + b/5 + 9/13$. Then the log pull back of the log pair (X, D) is not log canonical at some point $O \in G$. We have

$$\frac{a}{10} - \frac{b}{5} = \tilde{E}\cdot\tilde{\Omega} \geqslant 0 \leqslant \tilde{L}_{yz}\cdot\tilde{\Omega} = \frac{6+15m}{8\cdot 13} - \frac{c}{13} - \frac{a}{65} - \frac{b}{5},$$

which implies that $30+75m \geqslant 4-c+8a+104b$ and $a \geqslant 2b$. The system of inequalities

$$\begin{cases} 30+75m \geqslant 40c+8a+104b, \\ 3m+c+a/5+13b/5+9 \geqslant 13, \\ 7/22 \geqslant m, \end{cases}$$

is inconsistent. Thus, we see that $\theta < 1$.

Suppose that $O \notin \tilde{E} \cup \tilde{L}_{yz}$. Then it follows from Lemma 1.4.6 that

$$b = -\frac{b}{5}G^2 = \tilde{\Omega}\cdot G > 1,$$

which implies that $b > 1$. But the system of inequalities

$$\begin{cases} 30 + 75m \geqslant 40c + 8a + 104b, \\ a \geqslant 2b > 1, \\ 3/11 \geqslant c, \\ 24 + 12c \geqslant 18m + 9a, \end{cases}$$

is inconsistent. Therefore, we see that $O \notin \tilde{E} \cup \tilde{L}_{yz}$. Note that $\tilde{E} \cap \tilde{L}_{yz} = \varnothing$.

Suppose that $O \in \tilde{L}_{yz}$. Then it follows from Lemma 1.4.6 that

$$b + m = \big(\tilde{\Omega} + m\tilde{L}_{yz}\big) \cdot G > 1 < \big(\tilde{\Omega} + \theta G\big) \cdot \tilde{L}_{yz} = \frac{6 + 15m}{8 \cdot 13} - \frac{c}{13} - \frac{a}{65} - \frac{b}{5} + \theta,$$

which implies that $b + m > 1$ and $m > 2/3$. But $m < 7/22$, which is a contradiction.

Thus, we see that $O \in \tilde{E}$. Hence, it follows from Lemma 1.4.6 that

$$b + \frac{6 + a + 2m + 5c}{13} = \left(\tilde{\Omega} + \frac{6 + a + 2m + 5c}{13}\tilde{E}\right) \cdot G > 1 < \big(\tilde{\Omega} + \theta G\big) \cdot \tilde{E} = \frac{a}{10} - \frac{b}{5} + \theta,$$

which implies that which implies that $130a + 845m + 1820c > 1312$. Applying Lemma 1.4.6 again, we see that

$$\frac{65}{32}\frac{b}{13 \cdot 14} = \frac{65}{32}\tilde{\Omega} \cdot G > \frac{37}{462} - \frac{1495m}{14784} - \frac{65c}{1056} - \frac{65a}{14784},$$

which implies that $13b + a + 2m + 5c > 7$ and $3a + 2c + 6m > 8$.

Let $\phi\colon \hat{X} \to \tilde{X}$ be a blow up of the point O, let F be the exceptional curve of ϕ, let $\hat{\Omega}$, $\hat{L}_{yz}$, $\hat{M}_x$, $\hat{E}$ and $\hat{G}$ be the proper transforms of Ω, L_{yz}, M_x, E and G, respectively. Then

$$K_{\hat{X}} \equiv \phi^*(K_{\tilde{X}}) + F,\ \hat{G} \equiv \phi^*(G) - F,\ \hat{E} \equiv \phi^*(\tilde{E}) - F,\ \hat{\Omega} \equiv \phi^*(\tilde{\Omega}) - dF,$$

where d is a positive rational number. The log pull back of (X, D) is the log pair

$$\left(\hat{X},\ \hat{\Omega} + m\hat{L}_{yz} + c\hat{M}_x + \frac{6 + a + 2m + 5c}{13}\hat{E} + \theta\hat{G} + \nu F\right),$$

where $\nu = d + 5m/13 + 6a/65 + 6c/13 + b/5 + 2/13$. Then the log pull back of the log pair (X, D) is not log canonical at some point $A \in F$. We have

$$\frac{a}{10} - \frac{b}{5} - d = \hat{E} \cdot \hat{\Omega} \geqslant 0 \leqslant \hat{G} \cdot \hat{\Omega} = b - d,$$

which implies that $b \geqslant d$ and $a \geqslant 2b + 10d$. The system of inequalities

$$\begin{cases} 30 + 75m \geqslant 40c + 8a + 104b, \\ 13d + 5m + 6a/5 + 6c + 13b/5 \geqslant 11, \\ b \geqslant d, \\ 7/22 \geqslant m, \end{cases}$$

is inconsistent. Thus, we see that $\nu < 1$.

Suppose that $A \notin \hat{E} \cup \hat{G}$. Then t follows from Lemma 1.4.6 that

$$d = \hat{\Omega} \cdot F > 1,$$

which is impossible, because the system of inequalities

$$\begin{cases} 30 + 75m \geqslant 40c + 8a + 104b, \\ 24 + 20c \geqslant 18m + 9a, \\ a \geqslant 2b + 10d, \\ 7/22 \geqslant m, \\ b \geqslant d > 1, \end{cases}$$

is inconsistent. Thus, we see that $A \in \hat{E} \cup \hat{G}$. Note that $\hat{E} \cap \hat{G} = \varnothing$.

Suppose that $A \in \hat{E}$. Then it follows from Lemma 1.4.6 that

$$\frac{a}{10} - \frac{b}{5} - d + \nu = \big(\hat{\Omega} + \nu F\big) \cdot \hat{E} > 1,$$

which implies that $5a + 10m + 12c > 22$. But the system of inequalities

$$\begin{cases} 5a + 10m + 12c > 22, \\ 24 + 12c \geqslant 18m + 9a, \\ 3/11 \geqslant c, \end{cases}$$

is inconsistent. Thus, we see that $A \notin \hat{E}$. Then $A \in \hat{G}$. By Lemma 1.4.6, we see that

$$b - d + \nu = \big(\hat{\Omega} + \nu F\big) \cdot \hat{G} > 1,$$

which implies that $6a + 25m + 30c + 78b > 55$. But the system of inequalities

$$\begin{cases} 6a + 25m + 30c + 78b > 55, \\ 30 + 75m \geqslant 40c + 8a + 104b, \\ 7/22 \geqslant m, \end{cases}$$

is inconsistent. The obtained contradiction completes the proof. □

Lemma 2.4.3. Suppose that $(a_0, a_1, a_2, a_3, d) = (9, 3n+8, 3n+11, 6n+13, 12n+35)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be given by the equation

$$z^2t + y^3z + xt^2 + x^{n+3}y = 0,$$

and the only singularities of X are O_x, O_y, O_z and O_t.

The curve C_x is reduced and splits into a union of the stratum L_{xz} and a residual curve M_x intersecting at O_t. One can easily see that $\mathrm{lct}(X, C_x) = 2/3$, which implies $\mathrm{lct}(X) \leqslant 1$.

The curve C_y is reduced and splits into a union of the stratum L_{yt} and a residual curve M_y intersecting at O_x. One can easily see that $\mathrm{lct}(X, C_y) = 3/4$, and hence the log pair $(X, \frac{3n+8}{6}C_y)$ is log canonical for $n \geqslant 1$.

The curve C_z is reduced and splits into a union of the stratum L_{xz} and a residual curve M_z intersecting at O_y. One can easily see that $\mathrm{lct}(X, C_z) = \frac{2n+3}{4n+4}$, and hence the log pair $(X, \frac{3n+11}{6}C_z)$ is log terminal for $n \geqslant 1$.

The curve C_t is reduced and splits into a union of the stratum L_{yt} and a residual curve M_t intersecting at O_z. One can easily see that $\mathrm{lct}(X, C_t) = \frac{2n+5}{4n+9}$, and hence the log pair $(X, \frac{6n+13}{6}C_t)$ is log terminal for $n \geqslant 1$.

One has the following intersection numbers.

$$L_{xz} \cdot D = \frac{6}{(3n+8)(6n+13)},\ L_{xz} \cdot M_x = \frac{3}{6n+13}, L_{xz} \cdot M_z = \frac{2}{3n+8},$$

$$L_{xz}^2 = -\frac{9n15}{(3n+8)(6n+13)},$$

$$M_x \cdot D = \frac{18}{(3n+11)(6n+13)}, M_z \cdot D = \frac{12}{9(3n+8)},$$

$$M_x^2 = -\frac{9n+6}{(3n+11)(6n+13)}, M_z^2 = -\frac{3n+5}{9(3n+8)},$$

$$L_{yt} \cdot D = \frac{6}{9(3n+11)}, L_{yt} \cdot M_y = \frac{2}{9}, L_{yt} \cdot M_t = \frac{n+3}{3n+11}, L_{yt}^2 = -\frac{3n+14}{9(3n+11)},$$

$$M_y \cdot D = \frac{12}{9(6n+13)}, M_t \cdot D = \frac{6(n+3)}{(3n+8)(3n+11)},$$

$$M_y^2 = -\frac{6n+10}{9(6n+13)}, M_t^2 = -\frac{1}{(3n+8)(3n+11)}.$$

Now we suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Suppose that $P = O_x$. Assume that $L_{yt} \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6}{9(3n+11)} = L_{yt} \cdot D > \frac{1}{9},$$

which is a contradiction for all $n \geqslant 1$. Hence $L_{yt} \subset \mathrm{Supp}\,(D)$. By Remark 1.4.7 we may assume that $M_y \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{yt} + \Omega$, where $L_{yt} \not\subset \mathrm{Supp}\,(\Omega)$. By Theorem 1.4.5 one has

$$\frac{1}{9} < \Omega \cdot L_{yt} = \frac{6+(3n+14)\mu}{9(3n+11)},$$

and hence $\mu > (3n+5)/(3n+14)$. On the other hand,

$$\frac{12}{9(6n+13)} = D \cdot M_y \geqslant \mu L_{yt} \cdot M_y + \frac{\mathrm{mult}_{O_x}(D) - \mu}{9} > \frac{2\mu}{9} + \frac{1-\mu}{9} > \frac{6n+19}{9(3n+14)},$$

which is a contradiction for $n \geqslant 1$.

Suppose that $P = O_y$. Assume that $L_{xz} \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6}{(3n+8)(6n+13)} = L_{xz} \cdot D > \frac{1}{3n+8},$$

which is a contradiction for all $n \geqslant 1$. Hence $L_{xz} \subset \mathrm{Supp}\,(D)$, and by Remark 1.4.7 we may assume that $M_x, M_z \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{xz} + \Omega$, where $L_{xz} \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{18}{(3n+11)(6n+13)} = D \cdot M_x < \frac{3\mu}{6n+13},$$

which implies that $\mu \leqslant 6/(3n+11)$. By Theorem 1.4.5 one has

$$\frac{1}{3n+8} < \Omega \cdot L_{xz} = \frac{6+(9n+15)\mu}{(3n+8)(6n+13)},$$

which contradicts the inequality $\mu \leqslant 6/(3n+11)$ for $n \geqslant 1$.

Suppose that $P = O_z$. Assume that $L_{yt} \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6}{9(3n+11)} = L_{yt} \cdot D > \frac{1}{3n+11},$$

which is a contradiction for $n \geqslant 1$. Hence $L_{yt} \subset \mathrm{Supp}\,(D)$. By Remark 1.4.7 we may assume that $M_t \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{yt} + \Omega$, where $L_{yt} \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{6(n+3)}{(3n+8)(3n+11)} = M_t \cdot D \geqslant \mu L_{yt} \cdot M_t + \frac{(\mathrm{mult}_{O_z})(D) - \mu)\mathrm{mult}_{O_z}(M_t)}{3n+11} > \frac{\mu(n+3)}{3n+11} + \frac{2(1-\mu)}{3n+11},$$

which implies that $\mu < 2/((3n+8)(n+1))$ for $n \geqslant 1$. By Theorem 1.4.5 one has

$$\frac{6}{9(3n+11)} = D \cdot L_{yt} = -\mu\frac{3n+14}{9(3n+11)} + \Omega \cdot L_{yz} > -\mu\frac{3n+14}{9(3n+11)} + \frac{1}{3n+11},$$

which gives $\mu > 3/(3n+14)$, which is impossible for $n \geqslant 1$.

Suppose that $P = O_t$. Assume that $L_{xz} \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6}{(3n+8)(6n+13)} = L_{xz} \cdot D > \frac{1}{6n+13},$$

which is a contradiction for all $n \geqslant 1$. Hence $L_{xz} \subset \mathrm{Supp}\,(D)$. By Remark 1.4.7 we may assume that $M_x \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{xz} + \Omega$, where $L_{xz} \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{18}{(3n+11)(6n+13)} = D \cdot M_x \geqslant \mu L_{xz} \cdot M_x + \frac{\mathrm{mult}_{O_t}(D) - \mu}{6n+13} > \frac{1+2\mu}{6n+13},$$

but arguing as above, we get $\mu > (6n+7)/(9n+15)$, which is a contradiction for $n \geqslant 1$.

Suppose that P is a smooth point on L_{xz}. Assume that $L_{xz} \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{6}{(3n+8)(6n+13)} = L_{xz} \cdot D > 1,$$

which is a contradiction for all $n \geqslant 1$. Hence $L_{xz} \subset \mathrm{Supp}\,(D)$, and by Remark 1.4.7 we may assume that $M_x \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{xz} + \Omega$, where $L_{xz} \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$1 < \Omega \cdot L_{xz} = \frac{6 + (3n+3)\mu}{(3n+8)(6n+13)} \leqslant \frac{6(6n+14)}{(3n+8)(3n+11)(6n+13)},$$

by Theorem 1.4.5, because $\mu \leqslant 6/(3n+11)$. Thus, we have a contradiction here for all $n \geqslant 1$.

Suppose that P is a smooth point on M_x. Assume that $M_x \not\subset \mathrm{Supp}\,(D)$. Then

$$\frac{18}{(3n+11)(6n+13)} = M_x \cdot D > 1,$$

which is a contradiction for all $n \geqslant 1$. Hence $M_x \subset \mathrm{Supp}\,(D)$, and by Remark 1.4.7 we may assume that $L_{xz} \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu M_x + \Omega$, where $M_x \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\mu \leqslant \frac{3n+11}{3(3n+8)}$$

as above. On the other hand, by Theorem 1.4.5 one has

$$1 < \Omega \cdot M_x = \frac{18 + (9n+6)\mu}{(3n+11)(6n+13)},$$

which is a contradiction for all $n \geqslant 1$. Hence $P \notin C_x$. Similarly, we see that $P \notin C_y \cup C_z \cup C_t$.

Applying Lemma 1.4.10, we see that $n \leqslant 3$, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(9(3n+11)))$ contains x^{3n+11}, y^9x^3 and z^9. Thus, either $n = 4$ or $n = 3$.

There is a unique curve $Z_\alpha \subset X$ that is cut out by

$$xt + \alpha z^2 = 0$$

such that $P \in Z$, where $0 \neq \alpha \in \mathbb{C}$. The curve Z_α is always reducible. Indeed, one can easily check that $Z_\alpha = C_\alpha + L_{xz}$ where C_α is a reduced curve whose support contains no L_{xz}.

The open subset $Z_\alpha \setminus (Z_\alpha \cap C_x)$ of the curve Z_α is a $\mathbb{Z}_9$-quotient of the affine curve

$$t + \alpha z^2 = 0 = z^2t + y^3z + t^2 + y = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

which is isomorphic to a plane affine quartic curve that is given by the equation

$$\alpha(\alpha - 1)z^4 + y + y^3z = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big),$$

which implies that the curve C_α is irreducible and $\mathrm{mult}_P(C_\alpha) \leqslant 3$ if $\alpha \neq 1$.

The case $\alpha = 1$ is special. Namely, if $\alpha = 1$, then $C_1 = R_1 + M_y$, where R_1 is a reduced curve whose support does not contain the curve C_1. Arguing as in the case $\alpha \neq 1$, we see that R_1 is irreducible and R_1 is smooth at the point P.

By Remark 1.4.7, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible components of the curve Z_α.

Suppose that $\alpha \neq 1$. Then elementary calculations imply that

$$C_\alpha \cdot L_{xz} = \frac{9n+25}{(3n+8)(6n+13)},\ C_\alpha \cdot C_\alpha = \frac{144(n+2)^2 + 237(n+2) + 67}{9(3n+8)(6n+13)},\ D \cdot C_\alpha = \frac{6(24n+61)}{9(3n+8)(6n+13)},$$

and we can put $D = \epsilon C_\alpha + \Xi$, where Ξ is an effective $\mathbb{Q}$-divisor such that $C_\alpha \not\subset \mathrm{Supp}(\Xi)$. Then

$$\frac{6}{(3n+8)(6n+13)} = D \cdot L_{xz} = \epsilon C_\alpha \cdot L_{xz} + \Xi \cdot L_{xz} \geqslant \epsilon \frac{9n+25}{(3n+8)(6n+13)},$$

if $\epsilon > 0$. Thus, we see that $\epsilon \leqslant 6/(9n+25)$. But

$$\begin{aligned} \frac{6(24n+61)}{9(3n+8)(6n+13)} &= D \cdot C_\alpha \\ &= \epsilon C_\alpha^2 + \Xi \cdot C_\alpha \\ &\geqslant \epsilon C_\alpha^2 + \mathrm{mult}_P\big(\Xi\big) \\ &= \epsilon C_\alpha^2 + \mathrm{mult}_P\big(D\big) - \epsilon \mathrm{mult}_P\big(C_\alpha\big) \\ &> \epsilon C_\alpha^2 + 1 - 3\epsilon, \end{aligned}$$

which implies that $6/(9n+25) \geqslant \epsilon > (162(n+2)^2-9(n+2)-60)/(342(n+2)^2+168(n+2)-13)$. The latter is impossible for $n \geqslant 1$.

Thus, we see that $\alpha = 1$. Then elementary calculations imply that

$$R_1 \cdot L_{xz} = \frac{6n+17}{(3n+8)(6n+13)},\ R_1 \cdot R_1 = \frac{6(n+2)^2+13(n+2)+3}{(3n+8)(6n+13)},$$

$$M_y \cdot R_1 = \frac{2n+5}{6n+13},\ D \cdot R_1 = \frac{6(2n+5)}{(3n+8)(6n+13)},$$

and we can put $D = \epsilon_1 R_1 + \Xi_1$, where Ξ_1 is an effective $\mathbb{Q}$-divisor such that $R_1 \not\subset \mathrm{Supp}(\Xi_1)$. Now we obtain the inequality $\epsilon_1 \leqslant 1$, because either $\epsilon_1 = 0$, or $L_{xy} \cdot \Xi_1 \geqslant 0$ or $M_z \cdot \Xi_1 \geqslant 0$. By Lemma 1.4.6, we see that

$$\frac{6(2n+5) - \epsilon_1(6(n+2)^2+13(n+2)+3)}{(3n+8)(6n+13)} = \Xi_1 \cdot R_1 > 1,$$

which is impossible for $n \geqslant 1$. The obtained contradiction completes the proof. □

Part 3. Sporadic cases

3.1. Sporadic cases with $I = 1$

Lemma 3.1.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (1, 2, 3, 5, 10)$. Then

$$\mathrm{lct}(X) = \begin{cases} 1 \text{ if } C_x \text{ has an ordinary double point,} \\ 7/10 \text{ if } C_x \text{ has a non-ordinary double point.} \end{cases}$$

Proof. The curve C_x is reduced and irreducible. Moreover, we have

$$\mathrm{lct}(X, C_x) = \begin{cases} 1 \text{ if the curve } C_x \text{ has an ordinary double point at the point } O_z, \\ 7/10 \text{ if the curve } C_x \text{ has a non-ordinary double point at the point } O_z. \end{cases}$$

Let D be an arbitrary effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that $C_x \not\subset \mathrm{Supp}(D)$, and the log pair (X, D) is not log canonical at some point $P \in X$. Then $P \in C_x$ by Lemma 1.4.10. Then

$$\frac{1}{3} = D \cdot C_x \geqslant \begin{cases} \mathrm{mult}_P(D)\mathrm{mult}_P(C_x) \text{ if } P \neq O_z, \\ \dfrac{\mathrm{mult}_P(D)\mathrm{mult}_P(C_x)}{3} \text{ if } P = O_z, \end{cases} > \begin{cases} 1 \text{ if } P \neq O_z, \\ \dfrac{2}{3} \text{ if } P = O_z, \end{cases}$$

because the curve C_x is singular at the point O_z. The obtained contradiction completes the proof due to Remark 1.4.7. □

Lemma 3.1.2. Suppose that $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$. Then

$$\mathrm{lct}(X) = \begin{cases} 1 \text{ if } f(x,y,z,t) \text{ contains } yzt, \\ 8/15 \text{ if } f(x,y,z,t) \text{ does not contain } yzt, \end{cases}$$

Proof. The curve C_x is reduced and irreducible. Moreover, we have

$$\mathrm{lct}(X, C_x) = \begin{cases} 1 \text{ if } f(x,y,z,t) \text{ contains } yzt, \\ 8/15 \text{ if } f(x,y,z,t) \text{ does not contain } yzt, \end{cases}$$

Let D be an arbitrary effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that $C_x \not\subset \mathrm{Supp}(D)$, and the log pair (X, D) is not log canonical at some point $P \in X$. Then $P \in C_x$ by Lemma 1.4.10. Hence, we have

$$\frac{1}{7} = D \cdot C_x \geqslant \begin{cases} \mathrm{mult}_P(D) \text{ if } P \neq O_t, \\ \dfrac{\mathrm{mult}_P(D)}{7} \text{ if } P = O_t, \end{cases} > \begin{cases} 1 \text{ if } P \neq O_t, \\ \dfrac{1}{7} \text{ if } P = O_t, \end{cases}$$

which is a contradiction. The obtained contradiction completes the proof due to Remark 1.4.7. □

Lemma 3.1.3. Suppose that $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 8, 16)$. Then $\mathrm{lct}(X) = 1$.

Proof. We have $d = 16$. The surface X is singular at the point O_y, which is a singular point of type $\frac{1}{3}(1,1)$ on the surface X. The surface X is singular at the point O_z, which is a singular point of type $\frac{1}{5}(1,1)$ on the surface X.

It follows from the quasismoothness of X that the curve C_x is reduced. Then C_x is reducible. Namely, we have $C_x = L_1 + L_2$, where L_1 and L_2 are irreducible reduced smooth rational curves such that

$$-K_X \cdot L_1 = -K_X \cdot L_2 = \frac{1}{15},$$

and $L_1 \cap L_2 = O_y \cup O_z$. Then

$$L_1 \cdot L_1 = L_2 \cdot L_2 = -\frac{7}{15}$$

and $L_1 \cdot L_2 = 8/15$. Moreover, we have $\mathrm{lct}(X, C_x) = 1$.

Let D be an arbitrary effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. Suppose that $\mathrm{Supp}(D)$ does not contain the curve L_1. Then $P \in C_x$ by Lemma 1.4.10.

Suppose that $P \in L_1$. Then

$$\frac{1}{15} = D \cdot L_1 \geqslant \begin{cases} \dfrac{\mathrm{mult}_P\big(D\big)}{3} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P\big(D\big)}{5} \text{ if } P = O_z, \\ \mathrm{mult}_P\big(D\big) \text{ if } P \neq O_y \text{ and } P \neq O_z, \end{cases} > \begin{cases} \dfrac{1}{3} \text{ if } P = O_y, \\ \dfrac{1}{5} \text{ if } P = O_z, \\ 1 \text{ if } P \neq O_y \text{ and } P \neq O_z, \end{cases}$$

which is a contradiction. Thus, we see that $P \in L_2$ and $P \in X \setminus \mathrm{Sing}(X)$. Put

$$D = mL_2 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $L_2 \not\subset \mathrm{Supp}(\Omega)$. Then

$$\frac{1}{15} = D \cdot L_2 = \big(mL_2 + \Omega\big) \cdot L_1 \geqslant mL_1 \cdot L_2 = \frac{m8}{15},$$

which implies that $m \leqslant 1/8$. Thus, it follows from Lemma 1.4.6 that

$$\frac{1 + 7m}{15} = \big(-K_X - mL_2\big) \cdot L_2 = \Omega \cdot L_2 > 1,$$

which implies that $m > 2$. But $m \leqslant 1/8$. The obtained contradiction completes the proof due to Remark 1.4.7. □

Lemma 3.1.4. Suppose that $(a_0, a_1, a_2, a_3) = (2, 3, 5, 9, 18)$. Then

$$\mathrm{lct}\big(X\big) = \begin{cases} 2 \text{ if } C_y \text{ has a tacknodal point,} \\ 11/6 \text{ if } C_y \text{ has no tacknodal points.} \end{cases}$$

Proof. We have $d = 18$. The surface X is singular at the point O_z, which is a singular point of type $\frac{1}{5}(1,2)$ on the surface X. The surface X also has 2 singular points O_1 and O_2, which are cut out on X by the equations $x = z = 0$. The points O_1 and O_2 are singular points of type $\frac{1}{3}(1,1)$ on the surface X.

The curves C_x and C_y are irreducible, $\mathrm{lct}(X, C_x) = 1$, and

$$\mathrm{lct}\big(X, C_y\big) = \begin{cases} \dfrac{3}{4} \text{ if } C_y \text{ has a tacknodal singularity at the point } O_z, \\ \dfrac{11}{18} \text{ if } C_y \text{ has a non-tacknodal singularity at the point } O_z, \end{cases}$$

If C_y has a tacknodal point, put $\epsilon = 2$. Otherwise put $e = 11/6$. Then $\mathrm{lct}(X) \leqslant \epsilon$. Suppose that $\mathrm{lct}(X) < \epsilon$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the log pair $(X, \epsilon D)$ is not log canonical at some point $P \in X$. Then it follows from Remark 1.4.7 that we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Suppose that $P \notin C_x \cup C_y$. Then $P \in X \setminus \mathrm{Sing}(X)$ and there is a unique curve C in the pencil $|-5K_X|$ such that $P \in C$. The curve C is a hypersurface in $\mathbb{P}(1, 2, 3)$ of degree 6 such that the natural projection

$$C \longrightarrow \mathbb{P}(1, 2) \cong \mathbb{P}^1$$

is a double cover. Thus, we have $\mathrm{mult}_P(C) \leqslant 2$. In particular, the log pair $(X, \frac{\epsilon}{5}C)$ is log canonical. Thus, it follows from Remark 1.4.7 that we may assume that the support of the divisor D does not contain one of the irreducible components of the curve C. Then

$$\frac{1}{3} = D \cdot C \geqslant \mathrm{mult}_P\big(D\big) > \frac{1}{2}$$

in the case when C is irreducible (but possibly non-reduced). Therefore, the curve C must be reducible and reduced. Then

$$C = C_1 + C_2,$$

where C_1 and C_2 are irreducible and reduced smooth rational curves such that

$$C_1 \cdot C_1 = C_2 \cdot C_2 = -\frac{7}{6}$$

and $C_1 \cdot C_2 = 2$ on the surface X. Without loss of generality we may assume that $P \in R_1$. Put

$$D = mR_1 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $R_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then $R_2 \not\subset \mathrm{Supp}(\Omega)$ and

$$\frac{1}{6} = D \cdot R_2 = \big(mR_1 + \Omega\big) \cdot R_2 \geqslant mR_1 \cdot R_2 = 2m,$$

which implies that $m \leqslant 1/6$. Thus, it follows from Lemma 1.4.6 that

$$\frac{1+7m}{6} = \big(-K_X - mR_1\big) \cdot R_1 = \Omega \cdot R_1 > \frac{1}{\epsilon} \geqslant \frac{1}{2}$$

which implies, in particular, that $m > 2/7$. But $m \leqslant 1/6$. The obtained contradiction implies that $P \in C_x \cup C_y$.

Suppose that $P \in C_x$. Then

$$\frac{2}{15} = D \cdot C_x \geqslant \begin{cases} \mathrm{mult}_P\big(D\big) \text{ if } P \in X \setminus \mathrm{Sing}(X), \\ \dfrac{\mathrm{mult}_P\big(D\big)}{3} \text{ if } P = O_1 \text{ or } P = O_2, \\ \dfrac{\mathrm{mult}_P\big(D\big)}{5} \text{ if } P = O_z, \end{cases} > \begin{cases} \dfrac{1}{2} \text{ if } P \in X \setminus \mathrm{Sing}(X), \\ \dfrac{1}{6} \text{ if } P = O_1 \text{ or } P = O_2, \\ \dfrac{1}{10} \text{ if } P = O_z, \end{cases}$$

which implies that $P = O_z$. Then

$$\frac{1}{5} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(C_y\big)}{5} = \frac{2\mathrm{mult}_P\big(D\big)}{5} > \frac{2}{5\epsilon} \geqslant \frac{1}{5},$$

which is a contradiction. Thus, we see that $P \notin C_x$. Then $P \in C_y$ and $P \in X \setminus \mathrm{Sing}(X)$, which implies that

$$\frac{1}{5} = D \cdot C_y \geqslant \mathrm{mult}_P\big(D\big) > \frac{1}{\epsilon} \geqslant \frac{1}{2},$$

which is a contradiction. The obtained contradiction completes the proof. □

Lemma 3.1.5. Suppose that $(a_0, a_1, a_2, a_3) = (3, 3, 5, 5, 15)$. Then $\mathrm{lct}(X) = 2$.

Proof. We have $d = 15$. The surface X has 5 singular points $O_1, \ldots, O_5$ of type $\frac{1}{3}(1,1)$, which are cut out on X by the equations $z = t = 0$. The surface X has 3 singular points Q_1, Q_2, Q_3 of type $\frac{1}{5}(1,1)$, which are cut out on X by the equations $x = y = 0$. The surface X is exceptional by [25].

Let C_i be a curve in the pencil $|-3K_X|$ such that $O_i \in C_i$, where $i = 1, \ldots, 5$. Then

$$C_i = L_1^i + L_2^i + L_3^i,$$

where L_j^i is an irreducible reduced smooth rational curve such that

$$-K_X \cdot L_j^i = \frac{1}{15},$$

and $Q_j \in L_j^i$. Then $L_1^i \cap L_2^i \cap L_3^i = O_i$ and $L_j^i \cdot L_k^i = 1/3$ if $j \neq k$. It follows from the subadjunction formula that

$$L_1^i \cdot L_1^i = L_2^i \cdot L_2^i = L_3^i \cdot L_3^i = -\frac{7}{15}.$$

Note that $\mathrm{lct}(X, C_i) = 2/3$, which implies that $\mathrm{lct}(X) \leqslant 2$. Suppose that $\mathrm{lct}(X) < 2$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the log pair $(X, 2D)$ is not log canonical at some point $P \in X$.

Suppose that $P \notin C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$. Then $P \in X \setminus \mathrm{Sing}(X)$ and there is a unique curve $C \in |-3K_X|$ such that $P \in C$. Then C is different from the curves $C_1, \ldots, C_5$, which implies that C is irreducible and (X, C) is log canonical. Thus, it follows from Remark 1.4.7 that we may assume that $C \not\subseteq \mathrm{Supp}(D)$. Then

$$\frac{1}{5} = D \cdot C \geqslant \mathrm{mult}_P\big(D\big) > \frac{1}{2},$$

because $(X, 2D)$ is not log canonical at the point P. The obtained contradiction implies that $P \in C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$. Without loss of generality, we may assume that $P \in C_1$.

It follows from Remark 1.4.7 that we may assume that $L_i^1 \not\subseteq \mathrm{Supp}(D)$ for some $i = 1, 2, 3$.

Suppose that $P = O_1$. Then

$$\frac{1}{15} = D \cdot L_i^1 \geqslant \frac{\mathrm{mult}_{O_1}\big(D\big)}{3} > \frac{1}{6},$$

because $(X, 2D)$ is not log canonical at the point P. The obtained contradiction implies that $P \neq O_1$.

Without loss of generality, we may assume that $P \in L_1^1$. Then either $P = Q_1$, or $P \in X \setminus \mathrm{Sing}(X)$.

Suppose that $P = Q_1$. Let Z be a curve in the pencil $|-5K_X|$ such that $Q_1 \in Z$. Then

$$Z = Z_1 + Z_2 + Z_3 + Z_4 + Z_5,$$

where Z_i is an irreducible reduced smooth rational curve such that

$$-K_X \cdot Z_i = \frac{1}{15},$$

and $O_i \in Z_i$. Then $Z_1 \cap Z_2 \cap Z_3 \cap Z_4 \cap Z_5 = Q_1$ and $\mathrm{lct}(X, Z) = 2/5$. Thus, it follows from Remark 1.4.7 that we may assume that $Z_k \not\subseteq \mathrm{Supp}(D)$ for some $k = 1, \ldots, 5$. Then

$$\frac{1}{15} = D \cdot Z_k \geqslant \frac{\mathrm{mult}_{Q_1}\big(D\big)}{5} > \frac{1}{10},$$

because $(X, 2D)$ is not log canonical at the point P. The obtained contradiction implies that $P \neq Q_1$.

Thus, we see that $P \in L_1^1$ and $P \in X \setminus \mathrm{Sing}(X)$. Put

$$D = mL_1^1 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $L_1^1 \not\subseteq \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{15} = D \cdot L_i^1 = \big(mL_1^1 + \Omega\big) \cdot L_i^1 \geqslant mL_1^1 \cdot L_i^1 = \frac{m}{3},$$

which implies that $m \leqslant 1/5$. Then it follows from Lemma 1.4.6 that

$$\frac{1 + 7m}{15} = \big(-K_X - mL_1^1\big) \cdot L_1^1 = \Omega \cdot L_1^1 > \frac{1}{2},$$

which implies that $m > 13/14$. But $m \leqslant 1/5$. The obtained contradiction completes the proof. □

Lemma 3.1.6. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 5, 7, 11, 25)$. Then $\mathrm{lct}(X) = 21/10$.

Proof. By the quasismoothness of X, the curve $C_x = X \cap \{x = 0\}$ is irreducible and reduced. It is easy to see that $\mathrm{lct}(X, \frac{1}{3}C_x) = 21/10$, which implies that $\mathrm{lct} X \leqslant 21/10$.

Suppose that $\mathrm{lct} X < 21/10$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the log pair $(X, \frac{21}{10}D)$ is not log canonical at some point $P \in X$. We may assume that the support of D does not contain the curve C_x by Remark 1.4.7.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(21))$ contains x^7, x^2y^3, z^3, we have

$$\frac{10}{21} < \mathrm{mult}_P(D) \leqslant \frac{21 \cdot 25}{3 \cdot 5 \cdot 7 \cdot 11} < \frac{10}{21}$$

in the case when $P \in X \setminus C_x$ or $P \neq O_x$. Thus, we see that either $P \in C_x \cup O_x$.

Since C_x is smooth outside of the singular locus of X, we have

$$\frac{5}{77} = D \cdot C_x \geqslant \begin{cases} \mathrm{mult}_P(D)\mathrm{mult}_P(C_x) \text{ if } P \in X \setminus \mathrm{Sing}(X), \\ \dfrac{\mathrm{mult}_P(D)\mathrm{mult}_P(C_x)}{7} \text{ if } P = O_z, \\ \dfrac{\mathrm{mult}_P(D)\mathrm{mult}_P(C_x)}{11} \text{ if } P = O_t, \end{cases} > \begin{cases} \dfrac{10}{21} \text{ if } P \in X \setminus \mathrm{Sing}(X), \\ \dfrac{10}{147} \text{ if } P = O_z, \\ \dfrac{20}{231} \text{ if } P = O_t, \end{cases}$$

in the case when $P \in C_x$. Therefore, we see that $P = O_x$.

Since the curve C_y is irreducible and the log pair $(X, \frac{1}{5}C_y)$ is log canonical at the point O_x, we may assume that the support of D does not contain the curve C_y. Then

$$\frac{10}{63} < \frac{\mathrm{mult}_{O_x}(D)}{3} \leqslant D \cdot C_y = \frac{25}{231} < \frac{10}{63},$$

which is a contradiction. The obtained contradiction completes the proof. □

Lemma 3.1.7. Suppose that $(a_0, a_1, a_2, a_3) = (3, 5, 7, 14, 28)$. Then $\mathrm{lct}(X) = 9/4$.

Proof. We have $d = 28$. The surface X is singular at the point O_x, which is a singular point of type $\frac{1}{3}(1,1)$ on the surface X. The surface X is singular at the point O_y, which is a singular point of type $\frac{1}{5}(1,2)$ on the surface X. But X has also 2 singular points O_1 and O_2, which are cut out on X by the equations $x = y = 0$. The points O_1 and O_2 are singular points of type $\frac{1}{7}(3,5)$ on the surface X.

We have $C_x = L_1 + L_2$, where L_i is an irreducible reduced smooth rational curve such that

$$-K_X \cdot L_i = \frac{1}{35},$$

and $L_1 \cap L_2 = O_y$. Then $L_1 \cdot L_2 = 2/5$ and

$$L_1 \cdot L_1 = L_2 \cdot L_2 = -\frac{11}{35}.$$

Without loss of generality, we may assume that $O_1 \in L_1$ and $O_2 \in L_2$.

Note that $\mathrm{lct}(X, C_x) = 3/4$, which implies that $\mathrm{lct}(X) \leqslant 9/4$. Suppose that $\mathrm{lct}(X) < 9/4$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the log pair $(X, \frac{9}{4}D)$ is not log canonical at some point $P \in X$.

Suppose that $P \notin C_x$ and $P \in X \setminus \mathrm{Sing}(X)$. Then

$$\mathrm{mult}_P\big(D\big) \leqslant \frac{588}{1470}$$

by Lemma 1.4.10, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(21))$ contains x^7, z^3, x^2y^3. On the other hand, we have $\mathrm{mult}_P(D) > 4/9 > 588/1470$, because $(X, \frac{9}{4}D)$ is not log canonical at the point P. We see that either $P \in C_x$ or $P = O_x$.

It follows from Remark 1.4.7 that we may assume that $L_i \not\subset \mathrm{Supp}(D)$ for some $i = 1, 2$. Similarly, we may assume that $C_y \not\subset \mathrm{Supp}(D)$, because $(X, \frac{9}{4}C_y)$ is log canonical and the curve C_y is irreducible.

Suppose that $P = O_x$. Then

$$\frac{2}{21} = D \cdot C_y \geqslant \frac{\mathrm{mult}_{O_x}}{3}\big(D\big) > \frac{4}{27},$$

which is a contradiction. Thus, we see that $P \neq O_x$. Then $P \in C_x$.

Suppose that $P = O_y$. Then

$$\frac{1}{35} = D \cdot L_i \geqslant \frac{\mathrm{mult}_{O_y}\big(D\big)}{5} > \frac{4}{45},$$

which is a contradiction. Thus, we see that $P \neq O_y$.

Without loss of generality, we may assume that $P \in L_1$. Put $D = mL_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{35} = D \cdot L_i = \big(mL_1 + \Omega\big) \cdot L_i \geqslant mL_1 \cdot L_i = \frac{2m}{5},$$

which implies that $m \leqslant 1/14$. Then it follows from Lemma 1.4.6 that

$$\frac{1+11m}{35} = \big(-K_X - mL_1\big)\cdot L_1 = \Omega \cdot L_1 > \begin{cases} \dfrac{4}{9} \text{ if } P \neq O_1, \\ \dfrac{4}{63} \text{ if } P = O_1, \end{cases}$$

which implies that $m > 1/9$. But $m \leqslant 1/14$. The obtained contradiction completes the proof. □

Lemma 3.1.8. Suppose that $(a_0, a_1, a_2, a_3, d) = \mathbb{P}(3, 5, 11, 18, 36)$. Then $\mathrm{lct}(X) = 21/10$.

Proof. The surface X is singular at the points O_y and O_z. It is also singular at two points P_1 and P_2 on the curve defined by $y = z = 0$. By the quasismoothness of X, the curve C_x is irreducible and reduced. It is easy to see that $\mathrm{lct}(X, \frac{1}{3}C_x) = 21/10$. Also, the curve C_y is always irreducible and the pair $(X, \frac{21}{5\cdot 10}C_y)$ is log canonical.

We see that $\mathrm{lct}X \leqslant 21/10$. Suppose that $\mathrm{lct}X < 21/10$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{21}{10}D)$ is not log canonical at some point $P \in X$. By Remark 1.4.7, we may assume that the support of D contain neither the curve C_x nor C_y.

If $P \in C_x$ and $P \in X \setminus \mathrm{Sing}(X)$, then

$$\frac{10}{21} < \mathrm{mult}_P(D) \leqslant D \cdot C_x = \frac{36}{5\cdot 11\cdot 18} < \frac{10}{21},$$

which is a contradiction. Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(39))$ contains x^{13}, x^3y^6, x^2z^3, we have

$$\frac{10}{21} < \mathrm{mult}_P(D) \leqslant \frac{36\cdot 39}{3\cdot 5\cdot 11\cdot 18} < \frac{10}{21}$$

in the case when $P \notin C_x$ and $P \in X \setminus \mathrm{Sing}(X)$. Thus, we see that $P \in \mathrm{Sing}(X)$. Then

$$\frac{10}{105} < \frac{\mathrm{mult}_{O_y}(D)}{5} \leqslant D \cdot C_x = \frac{3\cdot 36}{3\cdot 5\cdot 11\cdot 18} < \frac{10}{105}$$

in the case when $P = O_y$. Similarly, we have

$$\frac{10}{231} < \frac{\mathrm{mult}_{O_z}(D)}{21} \leqslant D \cdot C_x = \frac{3\cdot 36}{3\cdot 5\cdot 11\cdot 18} < \frac{10}{231}$$

in the case when $P = O_z$. Thus, we see that $P = P_i$. Then

$$\frac{10}{63} < \frac{\mathrm{mult}_{P_i}(D)}{3} \leqslant D \cdot C_y = \frac{5\cdot 36}{3\cdot 5\cdot 11\cdot 18} < \frac{10}{63},$$

which is a contradiction. The obtained contradiction completes the proof. □

Lemma 3.1.9. Suppose that $(a_0, a_1, a_2, a_3) = (5, 14, 17, 21, 56)$. Then $\mathrm{lct}(X) = 25/8$.

Proof. We have $d = 56$. The surface X is singular at the point O_x, which is a singular point of type $\frac{1}{5}(2,1)$ on the surface X, the surface X is singular at the point O_z, which is a singular point of type $\frac{1}{17}(7,2)$ on the surface X, the surface X is singular at the point O_t, which is a singular point of type $\frac{1}{21}(5,17)$ on the surface X. The surface X also one singular point O of type $\frac{1}{7}(5,3)$ such that the points O and O_t are cut out on the surface X by the equations $x = z = 0$.

The curves C_x and C_y are reducible. Namely, we have $C_x = L + Z_x$ and $C_y = L + Z_y$, where L, Z_x and Z_y are irreducible curves such that the curve L is cut out on X by the equations $x = y = 0$. Easy calculations imply that

$$L\cdot L = -\frac{37}{357},\ L\cdot Z_x = \frac{2}{17},\ Z_x\cdot Z_x = -\frac{9}{119},\ L\cdot Z_y = \frac{1}{7},\ Z_y\cdot Z_y = \frac{9}{35},$$

the curve Z_x is singular at the point O_z, the curve Z_y is singular at the point O_t. Moreover, we have $Z_x \cap L = O_z$ and $Z_y \cap L = O_t$.

We have $\mathrm{lct}(X, C_x) = 5/8$ and $\mathrm{lct}(X, C_y) = 3/7$, which implies that $\mathrm{lct}(X) \leqslant 25/8$. Suppose that $\mathrm{lct}(X) < 25/8$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the log pair $(X, \frac{25}{8}D)$ is not log canonical at some point $P \in X$. Then it follows from Remark 1.4.7 that we may assume that the support of the divisor D does not contain either the curve L, or both curves Z_x and Z_y.

Suppose that $P \notin C_x \cup C_y$. Then $P \in X \setminus \mathrm{Sing}(X)$ and

$$\mathrm{mult}_P\big(D\big) \leqslant \frac{340}{3570} < \frac{8}{25}$$

by Lemma 1.4.10, because the natural projection $X \dashrightarrow \mathbb{P}(5,14,17)$ is a finite morphism outside of the curve C_y, and $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(85))$ contains monomials x^{17}, z^5, x^3y^5. On the other hand, we have $\mathrm{mult}_P(D) > 8/25$, because $(X, \frac{25}{8}D)$ is not log canonical at the point P. Thus, we see that $P \in C_x \cup C_y$.

Suppose that $P \in L$. Put $D = mL + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{119} = D \cdot Z_x = \big(mL + \Omega\big) \cdot Z_x \geqslant mL \cdot Z_x = \frac{2m}{17},$$

which implies that $m \leqslant 1/14$. Then it follows from Lemma 1.4.6 that

$$\frac{1+37m}{357} = \big(-K_X - mL\big) \cdot L = \Omega \cdot L > \begin{cases} \dfrac{8}{525} \text{ if } P = O_t,\\ \dfrac{8}{425} \text{ if } P = O_z,\\ \dfrac{8}{25} \text{ if } P \neq O_z \text{ and } P \neq O_t,\end{cases}$$

which implies, in particular, that $m > 3/25$. But $m \leqslant 1/14$. The obtained contradiction implies that $P \notin L$.

Suppose that $P \in Z_x$. Put $D = aZ_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $Z_x \not\subset \mathrm{Supp}(\Upsilon)$. If $a \neq 0$, then

$$\frac{1}{357} = D \cdot L = \big(aZ_x + \Upsilon\big) \cdot L \geqslant aL \cdot Z_x = \frac{2a}{17},$$

which implies that $a \leqslant 1/42$. Then it follows from Lemma 1.4.6 that

$$\frac{1+9a}{119} = \big(-K_X - aZ_x\big) \cdot Z_x = \Upsilon \cdot Z_x > \begin{cases} \dfrac{8}{175} \text{ if } P = O,\\ \dfrac{8}{25} \text{ if } P \neq O,\end{cases}$$

which is impossible, because $a \leqslant 1/42$. Thus, we see that $P \notin C_x$.

Suppose that $P = O_x$. The curve C_z is irreducible and $(X, \frac{25}{8}C_z)$ is log canonical. Thus, it follows from the Remark 1.4.7 that we may assume that $C_z \not\subset \mathrm{Supp}(D)$. Then

$$\frac{4}{105} = D \cdot C_z \geqslant \frac{\mathrm{mult}_{O_x}\big(D\big)}{5} > \frac{8}{125},$$

which is a contradiction. Hence, we see that $P \neq O_x$.

We see that $P \in Z_y$ and $P \in X \setminus \mathrm{Sing}(X)$. Put $D = bZ_y + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_y \not\subset \mathrm{Supp}(\Delta)$. If $b \neq 0$, then

$$\frac{1}{357} = D \cdot L = \big(bZ_y + \Delta\big) \cdot L \geqslant bL \cdot Z_y = \frac{b}{7},$$

which implies that $b \leqslant 1/51$. Then it follows from Lemma 1.4.6 that

$$\frac{1+9b}{35} = \big(-K_X - bZ_y\big) \cdot Z_y = \Delta \cdot Z_y > \frac{8}{25}$$

which is impossible, because $b \leqslant 1/51$. The obtained contradiction completes the proof. □

Lemma 3.1.10. Suppose that $(a_0, a_1, a_2, a_3, d) = (5, 19, 27, 31, 81)$. Then $\mathrm{lct}(X) = 25/6$.

Proof. By the quasismoothness of X, the curve C_x is irreducible and reduced. Moreover, the curve C_x is smooth outside of the singular locus of the surface X. It is easy to see that $\mathrm{lct}(X, \frac{1}{5}C_x) = 25/6$. Hence, we have $\mathrm{lct}(X) \leqslant 25/6$.

Suppose that $\mathrm{lct}(X) < \frac{25}{6}$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{25}{6}D)$ is not log canonical at some point $P \in X$. We may assume that the support of D does not contain the curve C_x by Remark 1.4.7.

Suppose that $P \notin C_x \cup O_x$. Then

$$\frac{6}{25} < \mathrm{mult}_P(D) \leqslant \frac{190 \cdot 81}{5 \cdot 19 \cdot 27 \cdot 31} < \frac{6}{25}$$

by Lemma 1.4.10, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(190))$ contains $x^{38}, x^{11}z, y^{10}$. Thus, we see that $P \in C_x \cup O_x$.

Suppose that $P \in X \setminus \mathrm{Sing}(X)$. Then $P \in C_x$ and

$$\frac{6}{25} < \mathrm{mult}_P(D) \leqslant D \cdot C_x = \frac{81}{19 \cdot 27 \cdot 31} < \frac{6}{25},$$

because $(X, \frac{25}{6}D)$ is not log canonical at the point $P \in X$.

We see that $P \in \mathrm{Sing}(X)$. Suppose that $P = O_y$. Then

$$\frac{6}{475} < \frac{\mathrm{mult}_{O_y}(D)}{19} \leqslant D \cdot C_x = \frac{5 \cdot 81}{5 \cdot 19 \cdot 27 \cdot 31} < \frac{6}{475}$$

which is a contradiction. Hence, we see that $P \neq O_y$. Suppose that $P = O_t$. Then

$$\frac{6}{775} < \frac{\mathrm{mult}_{O_t}(D)}{31} \leqslant D \cdot C_x = \frac{5 \cdot 81}{5 \cdot 19 \cdot 27 \cdot 31} < \frac{6}{775}$$

which is a contradiction. Hence, we see that $P = O_x$.

Since the curve C_y is irreducible and the log pair $(X, \frac{1}{19}C_y)$ is log canonical at the point O_x, we may assume that the support of D does not contain the curve C_y by Remark 1.4.7. Then

$$\frac{6}{125} < \frac{\mathrm{mult}_{O_x}(D)}{5} \leqslant D \cdot C_y = \frac{19 \cdot 81}{5 \cdot 19 \cdot 27 \cdot 31} < \frac{6}{125},$$

which is a contradiction. The obtained contradiction completes the proof. □

Lemma 3.1.11. Suppose that $(a_0, a_1, a_2, a_3, d) = (5, 19, 27, 50, 100)$. Then $\mathrm{lct}(X) = 25/6$.

Proof. By the quasismoothness of X, the curve C_x is irreducible and reduced. It is easy to see that $\mathrm{lct}(X, \frac{1}{5}C_x) = 25/6$, which implies that $\mathrm{lct}(X) \leqslant 25/6$. Suppose that $\mathrm{lct}(X) < 25/6$. Then it follows from Remark 1.4.7 that there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that $C_x \not\subset \mathrm{Supp}(D)$, and the pair $(X, \frac{25}{6}D)$ is not log canonical at some point $P \in X$.

Suppose that $P \in X \setminus \mathrm{Sing}(X)$ and $P \notin C_x$. Then

$$\frac{6}{25} < \mathrm{mult}_P(D) \leqslant \frac{270 \cdot 100}{5 \cdot 19 \cdot 27 \cdot 50} < \frac{6}{25}$$

by Lemma 1.4.10, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(270))$ contains $x^{54}, x^{16}y^{10}, z^{10}$. Thus, we see that either $P \in \mathrm{Sing}(X)$ or $P \in C_x$.

Suppose that $P \in X \setminus \mathrm{Sing}(X)$ and $P \in C_x$. Then

$$\frac{6}{25} < \mathrm{mult}_P(D) \leqslant D \cdot C_x = \frac{100}{19 \cdot 27 \cdot 50} < \frac{6}{25},$$

because $C_x \not\subset \mathrm{Supp}(D)$. Thus, we see that $P \in \mathrm{Sing}(X)$.

Note that X is singular at O_y and O_z. The surface X is also singular at two points P_1 and P_2 on the curve defined by $y = z = 0$.

Suppose that $P = O_y$. Then it follows from $C_x \not\subset \mathrm{Supp}(D)$ that

$$\frac{6}{475} < \frac{\mathrm{mult}_{O_y}(D)}{19} \leqslant D \cdot C_x = \frac{5 \cdot 100}{5 \cdot 19 \cdot 27 \cdot 50} < \frac{6}{475},$$

which is a contradiction. Suppose that $P = O_z$. Then

$$\frac{6}{675} < \frac{\mathrm{mult}_{O_z}(D)}{27} \leqslant D \cdot C_x = \frac{5 \cdot 100}{5 \cdot 19 \cdot 27 \cdot 50} < \frac{6}{675},$$

which is a contradiction. Thus, we see that $P = P_i$.

The curve C_z is irreducible, and the log pair $(X, \frac{25}{6 \cdot 27}C_z)$ is log canonical. By Remark 1.4.7, we may assume that the support of D does not contain the curve C_z. Then

$$\frac{6}{125} < \frac{\mathrm{mult}_{P_i}(D)}{5} \leqslant D \cdot C_z = \frac{27 \cdot 100}{5 \cdot 19 \cdot 27 \cdot 50} < \frac{6}{125},$$

which is a contradiction. □

Lemma 3.1.12. Suppose that $(a_0, a_1, a_2, a_3, d) = (7, 11, 27, 37, 81)$. Then $\mathrm{lct}(X) = 49/12$.

Proof. The curve C_x is irreducible and reduced, because X is quasismooth. It is easy to see that $\mathrm{lct}(X, \frac{1}{7}C_x) = 49/12$, which implies that $\mathrm{lct}(X) \leqslant 49/12$.

Suppose that $\mathrm{lct}(X) < 49/12$. By Remark 1.4.7, there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the support of D does not contain the curve C_x, and the log pair $(X, \frac{49}{12}D)$ is not log canonical at some point $P \in X$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(189))$ contains $x^{27}, x^{16}y^7, z^7$, it follows from Lemma 1.4.10 that

$$\frac{12}{49} < \mathrm{mult}_P(D) \leqslant \frac{189 \cdot 81}{7 \cdot 11 \cdot 27 \cdot 37} < \frac{12}{49}$$

in the case when $P \in X \setminus \mathrm{Sing}(X)$ and $P \in X \setminus C_x$. On the other hand, we have

$$\frac{12}{49} < \mathrm{mult}_P(D) \leqslant D \cdot C_x = \frac{81}{11 \cdot 27 \cdot 37} < \frac{12}{49}$$

if $P \in X \setminus \mathrm{Sing}(X)$ and $P \in C_x$. Thus, we see that $P \in \mathrm{Sing}(X)$.

Either $\mathrm{mult}_{O_x}(D) > 12/49$, $\mathrm{mult}_{O_y}(D) > 12/49$ or $\mathrm{mult}_{O_t}(D) > 12/49$. In the former case we have

$$\frac{12}{539} < \frac{\mathrm{mult}_{O_y}(D)}{11} \leqslant D \cdot C_x = \frac{7 \cdot 81}{7 \cdot 11 \cdot 27 \cdot 37} < \frac{12}{539},$$

which is a contradiction. If $\mathrm{mult}_{O_t}(D) > 12/49$, then

$$\frac{36}{1813} < \frac{\mathrm{mult}_{O_t}(D)\mathrm{mult}_{O_t}(C_x)}{37} \leqslant D \cdot C_x = \frac{7 \cdot 81}{3 \cdot 7 \cdot 11 \cdot 27 \cdot 37} < \frac{12}{1813},$$

which is a contradiction. Therefore, we must have $\mathrm{mult}_{O_x}(D) > 12/49$. Since the curve C_y is irreducible and the log pair $(X, \frac{49}{11 \cdot 12}C_y)$ is log canonical at the point O_x, we may assume that the support of D does not contain the curve C_y. Then, we obtain

$$\frac{12}{343} < \frac{\mathrm{mult}_{O_x}(D)}{7} \leqslant D \cdot C_y = \frac{11 \cdot 81}{7 \cdot 11 \cdot 27 \cdot 37} < \frac{12}{343},$$

which is a contradiction. □

Lemma 3.1.13. Suppose that $(a_0, a_1, a_2, a_3) = (7, 11, 27, 44, 88)$. Then $\mathrm{lct}(X) = 35/8$.

Proof. We have $d = 88$. The surface X is singular at the point O_x, which is a singular point of type $\frac{1}{7}(3, 1)$ on the surface X. The surface X is singular at the point O_z, which is a singular point of type $\frac{1}{27}(11, 17)$ on the surface X. The surface X has 2 singular points O_1 and O_2 of type $\frac{1}{11}(7, 5)$ that are cut out on the surface X by the equations $x = z = 0$.

The curve C_x is irreducible. Namely, we have $C_x = L_1 + L_2$, where L_1 and L_2 are smooth irreducible rational curves such that $O_1 \in L_1$ and $O_2 \in L_2$. Then

$$L_1 \cdot L_1 = L_2 \cdot L_2 = -\frac{5}{99}, L_1 \cdot L_2 = \frac{2}{27},$$

and $L_1 \cap L_2 = O_z$.

We have $\mathrm{lct}(X, C_x) = 5/8$, which implies that $\mathrm{lct}(X) \leqslant 35/8$. Suppose that $\mathrm{lct}(X) < 35/8$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the log pair $(X, \frac{35}{8}D)$ is not log canonical at some point $P \in X$. Then it follows from Remark 1.4.7 that we may assume that $L_i \not\subset \mathrm{Supp}(D)$ for some $i = 1, 2$.

Suppose that $P \notin C_x$ and $P \neq O_x$. Then

$$\mathrm{mult}_P\big(D\big) \leqslant \frac{2}{11} < \frac{8}{35}$$

by Lemma 1.4.10, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(189))$ contains monomials $x^{27}, z^7, x^{16}y^7$. Thus, we see that $P \in C_x \cup O_x$.

Suppose that $P = O_z$. Then

$$\frac{1}{297} = D \cdot L_i \geqslant \frac{\mathrm{mult}_{O_z}\big(D\big)}{27} > \frac{8}{945},$$

which is a contradiction. Thus, we see that $P \neq O_z$.

Suppose that $P=O_x$. The curve C_y is irreducible and $(X,\frac{35}{8}C_y)$ is log canonical. Thus, we may assume that $C_y\not\subset\mathrm{Supp}(D)$ by Remark 1.4.7. Then

$$\frac{2}{189}=D\cdot C_y\geqslant\frac{\mathrm{mult}_{O_x}(D)\mathrm{mult}_{O_x}(C_y)}{7}=\frac{2\mathrm{mult}_{O_x}(D)}{7}>\frac{16}{245},$$

which is a contradiction. Hence, we see that $P\neq O_x$. In particular, we see that $P\in C_x$.

Without loss of generality we may assume that $P\in L_1$. Put

$$D=mL_1+\Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $L_1\not\subset\mathrm{Supp}(\Omega)$. If $m\neq 0$, then

$$\frac{1}{297}=D\cdot L_i=\big(mL_1+\Omega\big)\cdot L_i\geqslant mL_1\cdot L_i=\frac{2m}{27},$$

which implies that $m\leqslant 1/22$. Then it follows from Lemma 1.4.6 that

$$\frac{1+15m}{297}=\big(-K_X-mL_1\big)\cdot L_1=\Omega\cdot L_1>\begin{cases}\dfrac{8}{275} & \text{if } P=O_1,\\ \dfrac{8}{25} & \text{if } P\neq O_1,\end{cases}$$

which implies, in particular, that $m>191/375$. But $m\leqslant 1/22$, which is a contradiction. The obtained contradiction completes the proof. □

Lemma 3.1.14. Suppose that $(a_0,a_1,a_2,a_3,d)=(9,15,17,20,60)$. Then $\mathrm{lct}(X)=21/4$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$xz^3+x^5y+y^4+t^3=0.$$

Note that X is singular at O_x and O_z. It is also singular at a point P_1 on the curve defined by $z=t=0$ and at a point P_2 on the curve defined by $x=z=0$. The point P_1 is different from the point O_x.

The curves C_x, C_y, and C_z are irreducible. We have

$$\mathrm{lct}(X,\frac{1}{9}C_x)=\frac{21}{4},\ \mathrm{lct}(X,\frac{1}{15}C_y)=\frac{2\cdot 15}{3},\ \mathrm{lct}(X,\frac{1}{17}C_z)=\frac{6\cdot 17}{15},$$

which implies, in particular, that $\mathrm{lct}(X)\leqslant 21/4$.

Suppose that $\mathrm{lct}(X)<21/4$. Then there is a $\mathbb{Q}$-effective divisor $D\equiv -K_X$ such that the pair $(X,\frac{21}{4}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of D contains none of the curves C_x, C_y, C_z.

Suppose that $P\in C_x$ and $P\notin\mathrm{Sing}(X)$. Then

$$\frac{4}{21}<\mathrm{mult}_P(D)\leqslant D\cdot C_x=\frac{60}{15\cdot 17\cdot 20}<\frac{4}{21},$$

which is a contradiction. Suppose that $P\in C_y$ and $P\notin\mathrm{Sing}(X)$. Then

$$\frac{4}{21}<\mathrm{mult}_P(D)\leqslant D\cdot C_y=\frac{60}{9\cdot 17\cdot 20}<\frac{4}{21},$$

which is a contradiction. Suppose that $P\in C_z$ and $P\notin\mathrm{Sing}(X)$. Then

$$\frac{4}{21}<\mathrm{mult}_P(D)\leqslant D\cdot C_z=\frac{60}{5\cdot 15\cdot 20}<\frac{4}{21},$$

which is a contradiction. Suppose that $P=O_x$. Then

$$\frac{4}{21}<\mathrm{mult}_{O_x}(D)\leqslant 9D\cdot C_y=\frac{9\cdot 15\cdot 60}{9\cdot 15\cdot 17\cdot 20}<\frac{4}{21},$$

which is a contradiction. Suppose that $P=O_z$. Then

$$\frac{4}{21}<\mathrm{mult}_{O_z}(D)\leqslant\frac{17}{3}D\cdot C_x=\frac{17\cdot 9\cdot 60}{3\cdot 9\cdot 15\cdot 17\cdot 20}<\frac{4}{21},$$

which is a contradiction. Suppose that $P=P_1$. Then

$$\frac{4}{21}<\mathrm{mult}_{P_1}(D)\leqslant 3D\cdot C_z=\frac{3\cdot 17\cdot 60}{9\cdot 15\cdot 17\cdot 20}<\frac{4}{21},$$

which is a contradiction. Suppose that $P = P_2$. Then
$$\frac{4}{21} < \mathrm{mult}_{P_2}(D) \leqslant 5D\cdot C_x = \frac{5\cdot 9\cdot 60}{9\cdot 15\cdot 17\cdot 20} < \frac{4}{21}.$$
which is a contradiction. Thus, there is a point $Q \in X\setminus \mathrm{Sing}(X)$ such that $P \notin C_x\cup C_y\cup C_z$ and $\mathrm{mult}_Q(D) > 4/21$.

Let $\mathcal{L}$ be the pencil on X that is cut out by the pencil
$$\lambda z^3 + \mu x^4 y = 0,$$
where $[\lambda : \mu] \in \mathbb{P}^1$. Then the base locus of the pencil $\mathcal{L}$ consists of the points P_2 and O_x.

Let C be the unique curve in $\mathcal{L}$ that passes through the point Q. Then C is cut out on X by an equation
$$x^4y = \alpha z^3,$$
where α is a non-zero constant. The curve C is smooth outside of the points P_2 and O_x by the Bertini theorem, because C is isomorphic to a general curve in the pencil $\mathcal{L}$ unless $\alpha = -1$. In the case when $\alpha = -1$, the curve C is smooth outside the points P_2 and O_x as well.

We claim that the curve C is irreducible. If so, then we may assume that the support of D does not contain the curve C and hence we obtain
$$\frac{4}{21} < \mathrm{mult}_Q(D) \leqslant D\cdot C = \frac{51\cdot 60}{9\cdot 15\cdot 17\cdot 20} < \frac{4}{21},$$
which is a contradiction.

For the irreducibility of the curve C, we may consider the curve C as a surface in $\mathbb{A}^4$ defined by the equations $t^3 + y^4 + (1+\alpha)xz^3 = 0$ and $x^4y = \alpha z^3$. Then the surface is isomorphic to the surface in $\mathbb{A}^4$ defined by the equations $t^3 + y^4 + \beta xz^3 = 0$ and $x^4y = z^3$, where $\beta = 1$ or 0. Then, we consider the surface in $\mathbb{P}^4$ defined by the equations $t^3w + y^4 + \beta xz^3 = 0$ and $x^4y = z^3w^2$. We then take the affine piece defined by $t \neq 1$. Then, the affine piece is isomorphic to the surface defined by the equation $x^4y + z^3(y^4 + \beta xz^3)^2 = 0$ in $\mathbb{A}^3$. If $\beta = 1$, the surface is irreducible. If $\beta = 0$, then it has an extra component defined by $y = 0$. However, this component originates from the hyperplane $w = 0$ in $\mathbb{P}^4$. Therefore, the surface in $\mathbb{A}^4$ defined by the equations $t^3 + y^4 = 0$ and $x^4y = z^3$ is also irreducible. □

Lemma 3.1.15. Suppose that $(a_0, a_1, a_2, a_3, d) = (9, 15, 23, 23, 69)$. Then $\mathrm{lct}(X) = 6$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation
$$zt(z-t) + xy^4 + x^6y = 0,$$
which implies that X is singular at three distinct points O_x, O_y, P_1 on the curve defined by $z = t = 0$. Also, the surface X is singular at three distinct points O_z, O_t, Q_1 on the curve defined by $x = y = 0$.

Note that $\mathrm{lct}(X, \frac{1}{9}C_x) = 6$, which implies that $\mathrm{lct}(X) \leqslant 6$. Suppose that $\mathrm{lct}(X) < 6$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, 6D)$ is not log canonical at some point $P \in X$.

The curve C_x consists of three distinct curves $L_1 = \{x = z = 0\}$, $L_2 = \{x = t = 0\}$ and $L_3 = \{x = z - t = 0\}$ that intersect altogether at the point O_y. Similarly, the curve C_y consists of three curves $L_1' = \{y = z = 0\}$, $L_2' = \{y = t = 0\}$ and $L_3' = \{y = z - t = 0\}$ that intersect altogether at the point O_x.

The pairs $(X, \frac{6}{9}C_x)$ and $(X, \frac{6}{15}C_y)$ are log canonical. By Remark 1.4.7, we may assume that the support of D does not contain at least one component, say L_1', of C_y. Also, we may assume that the support of D does not contain at least one component, say L_1, of C_x. Then
$$\mathrm{mult}_{O_x}(D) \leqslant 9D\cdot L_1' = \frac{9\cdot 23\cdot 15}{9\cdot 15\cdot 23\cdot 23} < \frac{1}{6} > \frac{15\cdot 23\cdot 9}{9\cdot 15\cdot 23\cdot 23} = 15D\cdot L_1 \geqslant \mathrm{mult}_{O_y}(D),$$
which imply that $P \neq O_x$ and $P \neq O_y$.

The curve C_z consists of three distinct curves L_1, L_1' and $C = \{z = y^3 + x^5 = 0\}$. It is easy to see $\mathrm{lct}(X, \frac{1}{23}C_z) = 8$. Therefore, we may assume that the support of D does not contain at least one component of C_z by Remark 1.4.7. Then the equalities
$$D\cdot L_1 = \frac{1}{15\cdot 23} < \frac{1}{6\cdot 23},\quad D\cdot L_1' = \frac{1}{9\cdot\cdot 23} < \frac{1}{6\cdot 23},\quad \frac{1D\cdot C}{3} = \frac{1}{9\cdot 23} < \frac{1}{6\cdot 23}$$

show that $\mathrm{mult}_{O_t}(D) < 1/6$. Thus, we see that $P \neq O_t$. By the same way, one can show that $P \neq O_z$ and $P \neq Q_1$.

Suppose that $P = P_1$. Put $D = mC + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $C \not\subset \mathrm{Supp}(\Omega)$. Then $m \leqslant 1/6$, because $(X, 6D)$ is log canonical at O_z. We have

$$C \cdot (L_1 + L_1') = \frac{5+3}{23} = \frac{8}{23},\ C \cdot C_z = \frac{1}{3},$$

which implies that $C^2 = C \cdot (C_z - L_1 - L_1') = -1/69$. Hence, it follows from Lemma 1.4.6 that

$$\frac{1}{3 \cdot 6} < \Omega \cdot C = D \cdot C - mC^2 = \frac{1+m}{3 \cdot 23} \leqslant \frac{7}{6 \cdot 3 \cdot 23} < \frac{1}{3 \cdot 6},$$

which is absurd. Thus, we see that P is a smooth point of the surface X.

Suppose that P is not contained in $C_z \cup C_t \cup \{z - t = 0\}$. Let E be the unique curve on X such that E is given by the equation $z = \lambda t$ and $P \in E$, where λ is a non-zero constant different from 1. Then E is quasismooth and hence irreducible. Therefore, we may assume that the support of D does not contain the curve E. Then

$$\mathrm{mult}_P(D) \leqslant D \cdot E = \frac{23 \cdot 69}{9 \cdot 15 \cdot 23 \cdot 23} < \frac{1}{6},$$

which is a contradiction. Thus, we see that $P \in C_z \cup C_t \cup \{z - t = 0\}$.

Suppose that $P \in L_1$. Put $D = aL_1 + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor, whose support does not contain the curve L_1. Then $a \leqslant 1/6$. Hence, it follows from Lemma 1.4.6 that

$$1 < 6\Omega \cdot L_1 = 6(D \cdot L_1 - aL_1^2) = \frac{6 \cdot (1 + 37a)}{345} \leqslant \frac{6 + 37}{345} < 1,$$

because $L_i^2 = -37/345$. Thus, we see that $P \notin L_1$. Similarly, we see that $P \notin L_1'$ and $P \notin C$. Thus, we see that $P \notin C_z$. By the same way, one can see that P is not contained in the curves C_t and $\{z - t = 0\}$. The obtained contradiction completes the proof. □

Lemma 3.1.16. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 29, 39, 49, 127)$. Then $\mathrm{lct}(X) = 33/4$.

Proof. The hypersurface X is unique, it can be given by the equation

$$z^2t + yt^2 + xy^4 + x^8z = 0,$$

and the singularities of X consist of a singular point of type $1/11(7,5)$ at O_x, a singular point of type $1/29(1,2)$ at O_y, a singular point of type $1/39(11,29)$ at O_z, and a singular point of type $1/49(11,39)$ at O_t.

The curve C_x is reduced and reducible. We have $C_x = L_{xt} + M_x$, where L_{xt} and M_x are irreducible curves such that L_{xt} is given by the equations $x = t = 0$, and M_x is given by the equations $x = z^2 + yt = 0$. Note that $O_y \in C_x$ and C_x is smooth outside of the point O_y. We have $\mathrm{lct}(X, 1/11C_x) = 33/4$, which implies that $\mathrm{lct}(X) \leqslant 33/4$.

The curve C_y is reduced and reducible. We have $C_y = L_{yz} + M_y$, where L_{yz} and M_y are irreducible curves such that L_{yz} is given by the equations $y = z = 0$, and M_y is given by the equations $y = x^8 + zt = 0$. The only singular point of the curve C_y is O_t. It is easy to see that the log pair $(X, \frac{33}{4 \cdot 29}C_y)$ is log terminal.

The curve C_z is reduced and reducible. We have $C_z = L_{yz} + M_z$, where M_z is an irreducible curve that is given by the equations $z = t^2 + xy^3 = 0$. The only singular point of C_z is O_x. It is easy to see that the log pair $(X, \frac{33}{4 \cdot 39}C_z)$ is log terminal.

The curve C_t is reduced and reducible. We have $C_t = L_{xt} + M_t$, where M_t is an irreducible curve that is given by the equations $t = y^4 + x^7z = 0$. The only singular point of C_t is O_z. It is easy to see that the log pair $(X, \frac{33}{4 \cdot 49}C_t)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 33/4$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, 33/4D)$ is not log canonical at some point $P \in X$.

Suppose tat $P = O_y$. Let us show that this assumption leads to a contradiction. One has

$$C_x \cdot D = \frac{127}{29 \cdot 39 \cdot 49},\ L_{xt} \cdot D = \frac{1}{29 \cdot 39},\ M_x \cdot D = \frac{2}{29 \cdot 49},$$

and we may assume that either $L_{xt} \not\subseteq \mathrm{Supp}(D)$ or $M_x \not\subseteq \mathrm{Supp}(D)$ by Remark 1.4.7. If $L_{xt} \not\subset \mathrm{Supp}(D)$, then

$$\frac{1}{29 \cdot 39} = L_{xt} \cdot D \geqslant \frac{\mathrm{mult}_{O_y}(D)}{29} > \frac{4}{29 \cdot 33} > \frac{1}{29 \cdot 39},$$

which is a contradiction. Thus, we see that $M_x \subseteq \mathrm{Supp}\,(D)$. Then

$$\frac{2}{29 \cdot 49} = M_x \cdot D \geqslant \frac{\mathrm{mult}_{O_y}(D)}{29} > \frac{4}{29 \cdot 33} > \frac{2}{29 \cdot 49},$$

which gives a contradiction. Thus, we see that $P \neq O_y$.

Suppose that $P = O_x$. Let us show that this assumption leads to a contradiction. One has

$$C_z \cdot D = \frac{127}{11 \cdot 29 \cdot 49},\ L_{yz} \cdot D = \frac{1}{11 \cdot 49},\ M_z \cdot D = \frac{2}{11 \cdot 29},$$

and we may assume that either $L_{yz} \not\subseteq \mathrm{Supp}(D)$ or $M_z \not\subseteq \mathrm{Supp}(D)$ by Remark 1.4.7. If $L_{yz} \not\subset \mathrm{Supp}(D)$, then

$$\frac{1}{11 \cdot 49} = L_{yz} \cdot D \geqslant \frac{\mathrm{mult}_{O_x}(D)}{11} > \frac{4}{11 \cdot 33} > \frac{1}{11 \cdot 49},$$

which is a contradiction. Thus, we see that $M_z \subseteq \mathrm{Supp}\,(D)$. Then

$$\frac{2}{11 \cdot 29} = M_z \cdot D \geqslant \frac{\mathrm{mult}_{O_x}(D)\mathrm{mult}_{O_x}(M_z)}{11} > \frac{2}{11} \cdot \frac{4}{33} > \frac{2}{11 \cdot 29},$$

because M_z is singular at the point O_x. The obtained contradiction shows that $P \neq O_x$.

Suppose that $P = O_z$. Let us show that this assumption leads to a contradiction. One has

$$C_t \cdot D = \frac{127}{11 \cdot 29 \cdot 39},\ M_t \cdot D = \frac{4}{11 \cdot 39},$$

and we may assume that either $L_{xt} \not\subseteq \mathrm{Supp}(D)$ or $M_t \not\subseteq \mathrm{Supp}(D)$ by Remark 1.4.7. If $L_{xt} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{1}{29 \cdot 39} = L_{xt} \cdot D \geqslant \frac{\mathrm{mult}_{O_z}(D)}{39} > \frac{4}{39 \cdot 33} > \frac{1}{29 \cdot 39},$$

which is a contradiction. Thus, we see that $M_t \subseteq \mathrm{Supp}\,(D)$. Then

$$\frac{4}{11 \cdot 39} = M_t \cdot D \geqslant \frac{\mathrm{mult}_{O_z}(D)\mathrm{mult}_{O_z}(M_t)}{39} > \frac{4}{39} \cdot \frac{4}{33} > \frac{4}{11 \cdot 39},$$

because M_t is singular at the point O_z. The obtained contradiction shows that $P \neq O_t$.

Suppose that $P = O_t$. Let us show that this assumption leads to a contradiction. By Remark 1.4.7 we may assume that either $L_{xt} \not\subseteq \mathrm{Supp}\,(D)$ or $M_{xt} \not\subseteq \mathrm{Supp}\,(D)$. Note that

$$M_x \cdot L_{xt} = 2/29,$$

which implies that $M_x^2 = -76/1421$ and $L_{xt}^2 = -67/1131$. Put

$$D = \mu M_x + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $M_x \not\subset \mathrm{Supp}\,(\Omega)$. If $\mu > 0$, then

$$\frac{2}{29}\mu = \mu M_x \cdot L_{xt} \leqslant D \cdot L_{xt} = \frac{1}{29 \cdot 39},$$

which implies that $\mu \leqslant 1/78$. Then

$$\frac{1}{49} \cdot \frac{4}{33} < \Omega \cdot M_x = D \cdot M_x - \mu M_x^2 = \frac{2 + 76\mu}{29 \cdot 49} < \frac{1}{49} \cdot \frac{4}{33},$$

by Lemma 1.4.6. The obtained contradiction shows that $P \neq O_t$.

Therefore, we see that P is a smooth point of the surface X.

Suppose that $P \in L_{xt}$. Put $D = \epsilon L_{xt} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $L_{xt} \not\subset \mathrm{Supp}\,(\Delta)$. Then $\epsilon \leqslant 4/33$, because $(X, \frac{33}{4}D)$ is log canonical at the point $O_y \in L_{xt}$. Thus, it follows from Lemma 1.4.6 that

$$\frac{4}{33} < \Delta \cdot L_{xt} = D \cdot L_{xt} - \epsilon L_{xt}^2 = \frac{1 + 67\epsilon}{29 \cdot 39} < \frac{4}{33},$$

which is a contradiction. We see that $P \notin L_{xt}$.

Suppose that $P \in M_x$. Put $D = \omega M_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $M_x \not\subset \mathrm{Supp}(\Upsilon)$. Then $\omega \leqslant 4/33$, because $(X, \frac{33}{4}D)$ is log canonical at the point $O_y \in M_x$. Hence, it follows from Lemma 1.4.6 that

$$\frac{4}{33} < \Upsilon \cdot M_x = D \cdot M_x - \omega M_x^2 = \frac{2 + 76\omega}{29 \cdot 49} < \frac{4}{33},$$

which is a contradiction. We see that $P \notin M_x$.

We see that P is a smooth point of X such that P is not contained in C_x. Then it follows from Lemma 1.4.9 that

$$\frac{4}{33} < \mathrm{mult}_P(D) \leqslant \frac{539 \cdot 127}{11 \cdot 29 \cdot 39 \cdot 49} < \frac{4}{33},$$

because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(190))$ contains $x^{20}y^{11}$, x^{49}, $x^{10}z^{11}$ and t^{11}. The obtained contradiction completes the proof. □

Lemma 3.1.17. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 49, 69, 128, 256)$. Then $\mathrm{lct}(X) = 55/6$.

Proof. By the quasismoothness of X, the curve C_x is irreducible and reduced. Moreover, it is easy to see that $\mathrm{lct}(X, \frac{1}{11}C_x) = 55/6$, which implies that $\mathrm{lct}(X) \leqslant 55/6$.

Suppose that $\mathrm{lct}(X) < 55/6$. By Remark 1.4.7, there is an effective $\mathbb{Q}$divisor $D \equiv -K_X$ such that $C_x \not\subset \mathrm{Supp}(D)$, and the log pair $(X, \frac{55}{6}D)$ is not log canonical at some point $P \in X$.

Suppose that $P \in X \setminus \mathrm{Sing}(X)$ and $P \in X \setminus C_x$. Then

$$\frac{6}{55} < \mathrm{mult}_P(D) \leqslant \frac{759 \cdot 256}{11 \cdot 49 \cdot 69 \cdot 128} < \frac{6}{55}$$

by Lemma 1.4.10, because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(759))$ contains $x^{69}, x^{20}y^{11}, z^{11}$. But

$$\frac{6}{55} < \mathrm{mult}_P(D) \leqslant D \cdot C_x = \frac{256}{49 \cdot 69 \cdot 128} < \frac{6}{55}$$

if $P \in X \setminus \mathrm{Sing}(X)$ and $P \in C_x$. Thus, we see that $P \in \mathrm{Sing}(X)$.

Suppose that $P = O_y$. Then

$$\frac{6}{55} < \mathrm{mult}_{O_y}(D) \leqslant 49 D \cdot C_x = \frac{49 \cdot 11 \cdot 256}{11 \cdot 49 \cdot 69 \cdot 128} < \frac{6}{55},$$

which is a contradiction. Suppose that $P = O_z$. Then

$$\frac{6}{55} < \mathrm{mult}_{O_z}(D) \leqslant 69 D \cdot C_x = \frac{69 \cdot 11 \cdot 256}{11 \cdot 49 \cdot 69 \cdot 128} < \frac{6}{55},$$

which is a contradiction. Therefore, we see that $P = O_x$.

Since the curve C_y is irreducible and the log pair $(X, \frac{1}{49}C_y)$ is log canonical at the point O_x, we may assume that the support of D does not contain the curve C_y due to Remark 1.4.7. Then

$$\frac{6}{55} < \mathrm{mult}_{O_x}(D) \leqslant 11 D \cdot C_y = \frac{11 \cdot 49 \cdot 256}{11 \cdot 49 \cdot 69 \cdot 128} < \frac{6}{55},$$

which is a contradiction. The obtained contradiction completes the proof. □

Lemma 3.1.18. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 23, 35, 57, 127)$. Then $\mathrm{lct}(X) = 65/8$.

Proof. The only singularities of X are a singular point of type $1/13(9, 5)$ at O_x, a singular point of type $1/23(13, 11)$ at O_y, a singular point of type $1/35(13, 23)$ at O_z, and a singular point of type $1/57(23, 35)$ at O_t. Note that the hypersurface X is unique and can is given by an equation

$$z^2t + y^4z + xt^2 + x^8y = 0.$$

The curve C_x is reduced and reducible. We have $C_x = L_{xz} + M_x$, where L_{xz} and M_x are irreducible curves such that L_{xz} is given by the equations $x = z = 0$, and M_x is given by the equations $x = zt + y^4 = 0$. Note that the only singular point of the curve C_x is the point $O_t \in C_x$. The inequality $\mathrm{lct}(X, C_x) = 5/8$ holds, which implies that $\mathrm{lct}(X) \leqslant 65/8$.

The curve C_y is reduced and reducible. We have $C_y = L_{yt} + M_y$, where L_{yt} and M_y are irreducible curves such that L_{yt} is given by the equations $y = t = 0$, and M_y is given by the equations $y = z^2 + xt = 0$. The only singular point of C_y is O_x. It is easy to see that the log pair $(X, \frac{65}{8 \cdot 23}C_y)$ is log terminal.

The curve C_z is reduced and reducible. We have $C_z = L_{xz} + M_z$, where M_z is an irreducible curve that is given by the equations $z = t^2 + x^7y = 0$. The only singular point of C_z is O_y. It is easy to see that the log pair $(X, \frac{65}{8\cdot 35}C_z)$ is log terminal.

The curve C_t is reduced and reducible. We have $C_t = L_{yt} + M_t$, where M_t is an irreducible curve that is given by the equations $t = y^3z + x^8 = 0$. The only singular point of C_t is O_z. It is easy to see that the log pair $(X, \frac{65}{8\cdot 57}C_t)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 65/8$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, 65/8D)$ is not log canonical at some point $P \in X$.

Suppose that $P = O_t$. Then $L_{xz} \subseteq \mathrm{Supp}\,(D)$, because

$$\frac{1}{23\cdot 57} = L_{xz}\cdot D \geqslant \frac{\mathrm{mult}_{O_t}(D)}{57} > \frac{8}{57\cdot 65} > \frac{1}{23\cdot 57},$$

if $L_{xz} \not\subseteq \mathrm{Supp}\,(D)$. By Remark 1.4.7 we may assume that $M_x \not\subseteq \mathrm{Supp}\,(D)$. Then

$$\frac{4}{35\cdot 57} = M_x\cdot D \geqslant \frac{\mathrm{mult}_{O_t}(D)}{57} > \frac{8}{57\cdot 65} > \frac{4}{35\cdot 57},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P = O_z$. Then $L_{yt} \subseteq \mathrm{Supp}\,(D)$, because

$$\frac{1}{13\cdot 35} = L_{yt}\cdot D \geqslant \frac{\mathrm{mult}_{O_z}(D)}{35} > \frac{8}{35\cdot 65} > \frac{1}{13\cdot 35},$$

if $L_{yt} \not\subseteq \mathrm{Supp}\,(D)$. By Remark 1.4.7 we may assume that $M_t \not\subseteq \mathrm{Supp}\,(D)$. Then

$$\frac{8}{23\cdot 35} = M_t\cdot D \geqslant \frac{\mathrm{mult}_{O_z}(D)\mathrm{mult}_{O_z}(M_t)}{35} > \frac{24}{35\cdot 65} > \frac{8}{23\cdot 35},$$

because M_t is singular at O_t. The obtained contradiction shows that $P \neq O_z$.

Suppose that $P = O_y$. Then $L_{xz} \subseteq \mathrm{Supp}\,(D)$, because

$$\frac{1}{23\cdot 57} = L_{xz}\cdot D \geqslant \frac{\mathrm{mult}_{O_y}(D)}{23} > \frac{8}{23\cdot 65} > \frac{1}{23\cdot 57},$$

if $L_{xz} \not\subseteq \mathrm{Supp}\,(D)$. By Remark 1.4.7 we may assume that $M_z \not\subseteq \mathrm{Supp}\,(D)$. Then

$$\frac{2}{13\cdot 23} = M_z\cdot D \geqslant \frac{\mathrm{mult}_{O_y}(D)\mathrm{mult}_{O_y}(M_z)}{23} > \frac{16}{23\cdot 65} > \frac{2}{13\cdot 23},$$

because M_z is singular at O_y. The obtained contradiction shows that $P \neq O_y$.

Suppose that $P = O_x$. Then $L_{yt} \subseteq \mathrm{Supp}\,(D)$, because

$$\frac{1}{13\cdot 35} = L_{xz}\cdot D \geqslant \frac{\mathrm{mult}_{O_x}(D)}{13} > \frac{8}{13\cdot 65} > \frac{1}{13\cdot 35},$$

if $L_{yt} \not\subseteq \mathrm{Supp}\,(D)$. By Remark 1.4.7 we may assume that $M_y \not\subseteq \mathrm{Supp}\,(D)$. Then

$$\frac{2}{13\cdot 57} = M_y\cdot D \geqslant \frac{\mathrm{mult}_{O_x}(D)}{13} > \frac{8}{13\cdot 65} > \frac{2}{13\cdot 57},$$

which is a contradiction. Thus, we see that $P \neq O_x$.

Therefore, we see that P is a smooth point of the surface X. Note that

$$L_{xz}^2 = -\frac{79}{23\cdot 57},\ M_x^2 = -\frac{88}{35\cdot 57}.$$

Suppose that $P \in L_{xz}$. Put $D = \mu L_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subset \mathrm{Supp}\,(\Omega)$. Then $\mu \leqslant 8/65$, because the log pair $(X, \frac{65}{8}D)$ is log canonical at the point $O_t \in L_{xz}$. Hence, it follows from Lemma 1.4.6 that

$$1 < \frac{65}{8}\Omega\cdot L_{xz} = \frac{65}{8}(D\cdot L_{xz} - \mu L_{xz}^2) = \frac{65}{8}\cdot\frac{1+79\mu}{23\cdot 57} < 1,$$

which is a contradiction. We see that $P \notin L_{xz}$.

Suppose that $P \in M_x$. Put $D = \epsilon M_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $M_x \not\subset \mathrm{Supp}\,(\Delta)$. Then $\epsilon \leqslant 8/65$, because the log pair $(X, \frac{65}{8}D)$ is log canonical at the point $O_t \in M_x$. So, it follows from Lemma 1.4.6 that

$$1 < \frac{65}{8}\Delta\cdot M_x = \frac{65}{8}(D\cdot M_x - \epsilon M_x^2) = \frac{65}{8}\cdot\frac{4+88\epsilon}{35\cdot 57} < 1,$$

which is a contradiction. We see that $P \notin C_x$.

Applying Lemma 1.4.9, we see that

$$\frac{8}{65} < \mathrm{mult}_P(D) \leqslant \frac{741 \cdot 127}{13 \cdot 23 \cdot 35 \cdot 57} < \frac{8}{65},$$

because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(741))$ contains $x^{11}y^{26}, x^{34}y^{13}, x^{57}, x^{22}z^{13}, t^{13}$. The obtained contradiction completes the proof. □

Lemma 3.1.19. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 35, 81, 128, 256)$. Then $\mathrm{lct}(X) = 91/10$.

Proof. The only singularities of X are a singular point of type $1/13(3, 11)$ at O_x, a singular point of type $1/35(13, 23)$ at O_y, and a singular point of type $1/81(35, 47)$ at O_z. In fact, the hypersurface X is unique and can be given by an equation

$$t^2 + y^5t + xz^3 + x^{17}y = 0.$$

The curve C_x is reduced and irreducible. One can easily check that $\mathrm{lct}(X, C_x) = 7/10$, which implies $\mathrm{lct}(X) \leqslant 91/10$.

The curve C_y is reduced and irreducible. The only singular point of C_y is O_x. Moreover, elementary calculations imply that the log pair $(X, \frac{91}{10 \cdot 35}C_y)$ is log terminal.

Suppose that $\mathrm{lct}(X) < 91/10$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{91}{10}D)$ is not log canonical at some point $P \in X$. By Remark 1.4.7 we may assume neither C_x nor C_y is contained in $\mathrm{Supp}\,(D)$.

Suppose that $P = O_z$. Then

$$\frac{2}{35 \cdot 81} = C_x \cdot D \geqslant \frac{\mathrm{mult}_P(C_x)\mathrm{mult}_P(D)}{81} = \frac{2\mathrm{mult}_P(D)}{81} > \frac{2}{81} \cdot \frac{10}{91} > \frac{2}{35 \cdot 81},$$

which is a contradiction. Suppose that $P = O_y$. Then

$$\frac{2}{35 \cdot 81} = C_x \cdot D \geqslant \frac{\mathrm{mult}_P(D)}{35} > \frac{1}{35} \cdot \frac{10}{91} > \frac{2}{35 \cdot 81},$$

which is a contradiction. Suppose that $P = O_x$. Then

$$\frac{2}{13 \cdot 81} = C_y \cdot D \geqslant \frac{\mathrm{mult}_P(C_y)\mathrm{mult}_P(D)}{13} > \frac{2}{13} \cdot \frac{10}{91} > \frac{2}{13 \cdot 81},$$

which is a contradiction. Hence, we see that $P \notin \mathrm{Sing}(X)$.

We see that P is a smooth point of the surface X. Suppose that $P \in C_x$. Then

$$\frac{2}{35 \cdot 81} = C_x \cdot D \geqslant \mathrm{mult}_P(D) > \frac{10}{91} > \frac{2}{35 \cdot 81},$$

which is a contradiction. Thus, we see that $P \notin C_x$.

Applying Lemma 1.4.10, we see that

$$\mathrm{mult}_P(D) \leqslant \frac{1053 \cdot 256}{13 \cdot 35 \cdot 81 \cdot 128} < \frac{10}{91},$$

because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1053))$ contains x^{81}, $x^{11}y^{26}$ and z^{13}. The obtained contradiction completes the proof. □

3.2. Sporadic cases with $I = 2$

Lemma 3.2.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$. Then $\mathrm{lct}(X) = 1$ if X contains the term yzt. And $\mathrm{lct}(X) = \frac{7}{12}$ if it contains no yzt.

Proof. We may assume that X is defined by the quasihomogenous equation

$$z(z - x^2)(z - \epsilon x^2) + y^4 + xt^2 + ayzt + bxy^2z + cx^2yt + dx^3y^2,$$

where ϵ ($\neq 0, 1$), a, b, c, d are constants. Note that X is singular at the point O_t and three points $Q_1 = [1 : 0 : 0 : 0]$, $Q_2 = [1 : 0 : 1 : 0]$, $Q_3 = [1 : 0 : \epsilon : 0]$.

First, we consider the case where $a = 0$. The curve C_x is irreducible and reduced. Also we have $\mathrm{lct}(X, C_x) = \frac{7}{12}$. Suppose that $\mathrm{lct}(X) < \frac{7}{12}$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$

such that the log pair $(X, \frac{7}{12}D)$ is not log canonical at some point $P \in X$. Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6))$ contains x^3, y^2, and xz, Lemma 1.4.10 implies that for a smooth point $O \in X \setminus C_x$

$$\mathrm{mult}_O D < \frac{2 \cdot 12 \cdot 6}{2 \cdot 3 \cdot 4 \cdot 5} < \frac{12}{7}.$$

Therefore, the point P cannot be a smooth point in $X \setminus C_x$. Since the curve C_x is irreducible we may assume that the support of D does not contain the curve C_x. The inequality

$$\frac{5}{3}D \cdot C_x = \frac{5 \cdot 2 \cdot 2 \cdot 12}{3 \cdot 2 \cdot 3 \cdot 4 \cdot 5} < \frac{12}{7}$$

implies that the point P is located in the outside of C_x, i.e., the point P must be one of the point Q_1, Q_2, Q_3. The curve C_y is quasismooth. Therefore, we may assume that the support of D does not contain the curve C_y. Then the inequality

$$\mathrm{mult}_{Q_i} D \geqslant 2D \cdot C_y = \frac{2 \cdot 2 \cdot 3 \cdot 12}{2 \cdot 3 \cdot 4 \cdot 5} < \frac{12}{7}$$

gives us a contradiction.

From now we consider the case where $a \neq 0$. In this case,the curve C_x is also irreducible and reduced. However, we have $\mathrm{lct}(X, C_x) = 1$. Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. Since

$$\frac{5}{2}D \cdot C_x = \frac{5 \cdot 2 \cdot 2 \cdot 12}{2 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 1$$

the point P is located in the outside of C_x.

The curve C_z is irreducible and the log pair $(X, \frac{1}{2}C_z)$ is log canonical. Therefore, we may assume that the support of D does not contain the curve C_z. The curve C_z is singular at the point Q_1. The inequality

$$\mathrm{mult}_{Q_1} D \geqslant D \cdot C_z = \frac{2 \cdot 4 \cdot 12}{2 \cdot 3 \cdot 4 \cdot 5} < 1$$

implies that P cannot be the point Q_1. We consider the curves C_{z-x^2} defined by $z = x^2$ and $C_{z-\epsilon x^2}$ defined by $z = \epsilon x^2$. Then by coordinate changes we can see they have the same properties as that of C_z. Moreover, we can see that the point P can be neither Q_2 nor Q_3. Therefore, the point P must be located in the outside of $C_x \cup C_z \cup C_{z-x^2} \cup C_{z-\epsilon x^2}$.

Let $\mathcal{L}$ be the pencil on X defined by $\lambda x^2 + \mu z = 0$, where $[\lambda : \mu] \in \mathbb{P}^1$. Let C the curve in $\mathcal{L}$ that passes through the point P. Then it is cut by $z = \alpha x^2$, where $\alpha \neq 0, 1, \epsilon$. The curve C is isomorphic to the curve in $\mathbb{P}(2, 3, 5)$ defined by

$$x^6 + y^4 + xt^2 + \beta x^2 yt = 0,$$

where β is a constant. We can easily see that the curve C is irreducible. Since

$$D \cdot C = \frac{2 \cdot 4 \cdot 12}{2 \cdot 3 \cdot 4 \cdot 5} < 1$$

it is enough to show that $(X, \frac{1}{4}C)$ is log canonical. If $\beta \neq \zeta 2\sqrt{2}$, where ζ is a forth root of unity, then the curve C is quasismooth and hence the pair is log canonical at the point P. If $\beta = \zeta 2\sqrt{2}$, then the curve C is singular at $[1 : \zeta : -\zeta^2\sqrt{2}]$. However, elementary calculation shows the pair $(X, \frac{1}{4}C)$ is log canonical. □

Lemma 3.2.2. Suppose that $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 7, 14)$. Then $\mathrm{lct}(X) = 1$.

Proof. We may assume that X is defined by the quasihomogenous equation

$$t^2 - y^2z^2 + x(z - \beta_1 x^2)(z - \beta_2 x^2)(z - \beta_3 x^2) + \epsilon xy^2(y^2 - \gamma x^3)$$

where ϵ $(\neq 0)$, β_1, β_2, β_3, γ are constants. Note that X is singular at the points O_y, O_z and three points $Q_1 = [1 : 0 : \beta_1 : 0]$, $Q_2 = [1 : 0 : \beta_2 : 0]$, $Q_3 = [1 : 0 : \beta_3 : 0]$. The constants β_1, β_2 and β_3 are distinct since X is quasismooth. The curve C_x consists of two irreducible reduced curves C_- and C_+. However, the curves C_y and C_z are irreducible. We can easily see that $\mathrm{lct}(X, C_x) = 1$, $\mathrm{lct}(X, \frac{2}{3}C_y) = \frac{3}{2}$ and $\mathrm{lct}(X, \frac{1}{2}C_z) > 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6))$ contains x^3, y^2 and

xz, Lemma 1.4.10 implies that the point P is either a singular point of X or a point of C_x. Furthermore, since C_y is irreducible and hence we may assume that the support of D does not contain the curve C_y the equality

$$2C_y \cdot D = \frac{2 \cdot 3 \cdot 2 \cdot 14}{2 \cdot 3 \cdot 4 \cdot 5} = 1$$

implies that $P \neq Q_i$ for each $i = 1, 2, 3$. In particular, the point must belong to C_x.

We have the following intersection numbers:

$$C_x \cdot C_- = C_x \cdot C_+ = \frac{1}{6}, \quad C_- \cdot C_+ = \frac{7}{12}, C_-^2 = C_+^2 = -\frac{5}{12}.$$

We may assume that the support of D cannot contain both C_- and C_+. If D does not contain the curve C_+, then we obtain

$$\mathrm{mult}_{O_y} D, \quad \mathrm{mult}_{O_z} D \geqslant 4D \cdot C_+ = \frac{2}{3} < 1.$$

On the other hand, if D does not contain the curve C_-, then we obtain

$$\mathrm{mult}_{O_y} D, \quad \mathrm{mult}_{O_z} D \geqslant 4D \cdot C_- = \frac{2}{3} < 1.$$

Therefore, the point P must be in $C_x \setminus \mathrm{Sing}(X)$.

We write $D = mC_+ + \Omega$, where the support of Ω does not contain the curve C_+. Then $m \geqslant \frac{2}{7}$ since $D \cdot C_- \geq mC_+ \cdot C_-$. Then we see $C_+ \cdot D - mC_+^2 < 1$. By the same way, we also obtain $C_- \cdot D - mC_-^2 < 1$. Then Lemma 1.4.8 completes the proof. ☐

Lemma 3.2.3. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 4, 5, 10, 20)$. Then $\mathrm{lct}(X) = 3/2$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 = z^4 + y^5 + x^5 z + \epsilon_1 xy^3 z + \epsilon_2 x^2 yz^2 + \epsilon_3 x^4 y^2 = 0,$$

where $\epsilon_i \in \mathbb{C}$. Note that X is singular at the point O_x. Note that X is also singular at a point O that is cut out on X by the equations $x = z = 0$, and X is also singular at points P_1 and P_2 that are cut out on X by the equations $x = y = 0$.

The curves C_x, C_y and C_z are irreducible. Moreover, we have

$$\frac{3}{2} = \mathrm{lct}(X, \frac{2}{3}C_x) < \mathrm{lct}(X, \frac{2}{5}C_z) = \frac{7}{4} < \mathrm{lct}(X, \frac{2}{4}C_y) = 2,$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 3/2$.

Suppose that $\mathrm{lct}(X) < 3/2$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{3}{2}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x, C_y and C_z.

Suppose that $P \notin C_x \cup C_y \cup C_z$. Then there is a unique (possibly reducible or non-reduced) curve $Z \subset X$ that is cut out by

$$\alpha y^2 = zx$$

such that $P \in Z$, where $0 \neq \alpha \in \mathbb{C}$. There is a natural double cover $\omega \colon Z \to C$, where C is a curve in $\mathbb{P}(3, 4, 5)$ that is given by the equations

$$\alpha y^2 = zx \subset \mathbb{P}(3, 4, 5) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, z, y\big]\Big),$$

where $\mathrm{wt}(x) = 3$, $\mathrm{wt}(y) = 4$ and $\mathrm{wt}(z) = 5$. The curve C is quasismooth, and $\omega(P)$ is a smooth point of $\mathbb{P}(3, 4, 5)$. Thus, we see that $\mathrm{mult}_P(Z) \leqslant 2$, the curve Z consists of at most 2 components, each component of Z is a smooth rational curve.

We may assume that $\mathrm{Supp}\,(D)$ does not contain at least one irreducible component of Z. Thus, if Z is irreducible, then

$$\frac{8}{15} = D \cdot C \geqslant \mathrm{mult}_P(D)\mathrm{mult}_P(C) \geqslant \frac{2}{3} > \frac{8}{15},$$

which is a contradiction. So, we see that $C = C_1 + C_2$, where C_1 and C_2 are smooth irreducible rational curves. Then

$$C_1 \cdot C_1 = C_2 \cdot C_2 = -\frac{4}{5}, \ C_1 \cdot C_2 = \frac{4}{3}.$$

Without loss of generality we may assume that $P \in C_1$. Put $D = \delta C_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $C_1 \not\subset \mathrm{Supp}(\Omega)$. If $\delta \neq 0$, then

$$\frac{4}{15} = D \cdot C_2 = \big(\delta C_1 + \Omega\big) \cdot C_2 \geqslant \delta C_1 \cdot C_2 = \frac{4\delta}{3},$$

which implies that $\delta \leqslant 1/5$. Then it follows from Lemma 1.4.6 that

$$\frac{4+4\delta}{15} = \big(-K_X - \delta C_1\big) \cdot C_1 = \Omega \cdot C_1 > \frac{2}{3},$$

which implies that $\delta > 3/2$. But $\delta \leqslant 1/5$. The obtained contradiction show that $P \in C_x \cup C_y \cup C_z$.

Suppose that $P \in C_x$ and $P \notin \mathrm{Sing}(X)$. Then

$$\frac{1}{5} = D \cdot C_x \geqslant \mathrm{mult}_P(D) \geqslant \frac{2}{3} > \frac{1}{5},$$

which is a contradiction. Suppose that $P \in C_y$ and $P \notin \mathrm{Sing}(X)$. Then

$$\frac{4}{15} = D \cdot C_y \geqslant \mathrm{mult}_P(D) \geqslant \frac{2}{3} > \frac{4}{15},$$

which is a contradiction. Suppose that $P \in C_z$ and $P \notin \mathrm{Sing}(X)$. Then

$$\frac{1}{3} = D \cdot C_z \geqslant \mathrm{mult}_P(D) \geqslant \frac{2}{3} > \frac{1}{3},$$

which is a contradiction. Thus we see that $P \in \mathrm{Sing}(X)$.

Suppose that $P = O_x$. The curve C_z is singular at the point O_x. Thus, we have

$$\frac{1}{3} = D \cdot C_z \geqslant \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(C_z)}{3} \geqslant \frac{4}{9} > \frac{1}{3},$$

which is a contradiction. Suppose that $P = O$. Then

$$\frac{1}{5} = D \cdot C_x \geqslant \frac{\mathrm{mult}_P(D)}{2} \geqslant \frac{1}{3} > \frac{1}{5},$$

which is a contradiction. Hence, without loss of generality we may assume that $P = P_1$. Note that $C_x \cap C_y = \{P_1, P_2\}$.

Let $\pi\colon \bar{X} \to X$ be a weighted blow up of the point P_1 with weights $(3,4)$, let E be the exceptional curve of π, let $\bar{D}$, $\bar{C}_x$ and $\bar{C}_y$ be the proper transforms of D, C_x and C_y, respectively. Then

$$K_{\bar{X}} \equiv \pi^*(K_X) + \frac{2}{5}E,\ \bar{C}_x \equiv \pi^*(C_x) - \frac{3}{5}E,\ \bar{C}_y \equiv \pi^*(C_y) - \frac{4}{5}E,\ \bar{D} \equiv \pi^*(D) - \frac{a}{5}E,$$

where a is a positive rational. The curve E contains two singular points Q_3 and Q_4 of the surface $\bar{X}$ such that Q_3 is a singular point of type $\frac{1}{3}(1,1)$, and Q_4 is a singular point of type $\frac{1}{4}(1,1)$. Then

$$\bar{C}_x \not\ni Q_3 \in \bar{C}_y \not\ni Q_4 \in \bar{C}_x,$$

and the intersection $\bar{C}_x \cap \bar{C}_y$ consists of the single point that dominates the point P_2.

The log pull back of the log pair $(X, \frac{3}{2}D)$ is the log pair

$$\left(\bar{X}, \frac{3}{2}\bar{D} + \frac{\frac{3a}{2} - 2}{5}E\right),$$

which is not log canonical at some point $Q \in E$. We have $E^2 = 5/12$. Then

$$0 \leqslant \bar{C}_x \cdot \bar{D} = C_x \cdot D - \frac{a}{5}E \cdot \bar{C}_x = C_x \cdot D + \frac{3a}{25}E^2 = \frac{1}{5} - \frac{a}{20},$$

which implies that $a \leqslant 4$. Hence, we see that

$$\frac{\frac{3a}{2} - 2}{5} \leqslant \frac{4}{5} < 1,$$

which implies that the log pull back of the log pair $(X, \frac{3}{2}D)$ is log canonical in a punctured neighborhood of the point Q.

Note that the log pull back of the the log pair $(X, \frac{3}{2}D)$ is effective if and only if $a \geqslant 4/3$. On the other hand, if $a \leqslant 4/3$, then the log pair $(\bar{X}, \frac{3}{2}\bar{D})$ is not log canonical at Q as well, which implies that

$$\frac{a}{12} = \frac{a}{5}E^2 = \bar{D} \cdot E > \begin{cases} \frac{2}{3} \text{ if } Q \neq Q_3 \text{ and } Q \neq Q_4, \\ \frac{2}{3}\frac{1}{3} \text{ if } Q = Q_3, \\ \frac{2}{3}\frac{1}{4} \text{ if } Q = Q_4, \end{cases}$$

which implies, in particular, that $a > 2$, which is a contradiction. Hence, we see that $a > 4/3$ and the log pull back of the the log pair $(X, \frac{3}{2}D)$ is always effective. Then

$$\text{mult}_P(D) > \frac{2}{3}\left(1 - \frac{\frac{3a}{2} - 2}{5}\right) = \frac{7\frac{2}{3} - a}{15}.$$

Suppose that $Q \neq Q_3$ and $Q \neq Q_4$. Then it follows from Lemma 1.4.6 that

$$\frac{a}{12} = \frac{a}{5}E^2 = \bar{D} \cdot E > \frac{2}{3},$$

which is a contradiction. Therefore, we see that either $Q = Q_3$ or $Q = Q_4$.

Suppose that $Q = Q_4$. Then

$$\frac{1}{5} - \frac{a}{20} = \bar{D} \cdot \bar{C}_x \geqslant \frac{\text{mult}_{Q_4}(D)}{4} > \frac{7\frac{2}{3} - a}{20},$$

which immediately leads to a contradiction. Thus, we see that $Q = Q_3$. Then

$$\frac{4}{15} - \frac{a}{15} = \bar{D} \cdot \bar{C}_y \geqslant \frac{\text{mult}_{Q_3}(D)}{3} > \frac{7\frac{2}{3} - a}{15},$$

which immediately leads to a contradiction. □

Lemma 3.2.4. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 4, 6, 7, 18)$. Then $\text{lct}(X) = 1$.

Proof. The surface X can be defined by the the quasihomogenous equation

$$t^2y + y^3z + (z - \beta_1x^2)(z - \beta_2x^2)(z - \beta_3x^2)$$

where β_1, β_2, β_3 are distinct nonzero constants. Note that X is singular at the points O_y, O_t and three points $P_1 = [1 : 0 : \beta_1 : 0]$, $P_2 = [1 : 0 : \beta_2 : 0]$, $P_3 = [1 : 0 : \beta_3 : 0]$ and one point $Q = [0 : -1 : 1 : 0]$.

The curve C_y is reducible. We have $C_y = L_1 + L_2 + L_3$, where L_i is an irreducible and reduced curve such that $P_i \in L_i$. We have

$$L_1 \cdot L_1 = L_2 \cdot L_2 = L_3 \cdot L_3 = -\frac{8}{21},\ L_1 \cdot L_2 = L_1 \cdot L_3 = L_2 \cdot L_3 = \frac{2}{7},$$

and $L_1 \cap L_2 \cap L_3 = O_t$. The curve C_x is irreducible and

$$1 = \text{lct}\left(X, \frac{2}{4}C_y\right) < \text{lct}\left(X, \frac{2}{3}C_x\right) = \frac{3}{2},$$

which implies, in particular, that $\text{lct}(X) \leqslant 1$.

Suppose that $\text{lct}(X) < 1$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair (X, D) is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curve C_x. Similarly, we may assume that $L_k \not\subseteq \text{Supp}(D)$ for some $k = 1, 2, 3$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(12))$ contains x^4, y^3 and z^2, it follows from Lemma 1.4.10 that $P \in C_x \cup C_y$.

Suppose that $P = O_t$. Then

$$\frac{2}{21} = D \cdot L_k \geqslant \frac{\text{mult}_P(D)}{7} > \frac{1}{7} > \frac{2}{21},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P \in C_x$. Then

$$\frac{3}{14} = D\cdot C_x > \begin{cases} 1 \text{ if } P \neq O_y \text{ and } P \neq Q,\\ \dfrac{1}{4} \text{ if } P = O_y,\\ \dfrac{1}{2} \text{ if } P = Q,\end{cases}$$

because $P \neq O_t$. The obtained contradiction shows that $P \notin C_x$.

Without loss of generality we may assume that $P \in L_1$. Put $D = mL_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{2}{21} = D\cdot L_k = \big(mL_1 + \Omega\big)\cdot L_k \geqslant mL_1\cdot L_k = \frac{2m}{7},$$

which implies that $m \leqslant 1/3$. Then it follows from Lemma 1.4.6 that

$$\frac{2+8m}{21} = \big(-K_X - mL_1\big)\cdot L_1 = \Omega\cdot L_1 > \begin{cases} 1 \text{ if } P \neq P_1,\\ \dfrac{1}{3} \text{ if } P = P_1,\end{cases}$$

which implies that $m > 5/8$. But we already proved that $m \leqslant 1/3$. The obtained contradiction completes the proof. □

Lemma 3.2.5. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 4, 10, 15, 30)$. Then $\mathrm{lct}(X) = 3/2$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 = z^3 + y^5z + x^{10} + \epsilon_1x^2yz^2 + \epsilon_2x^2y^6 + \epsilon_3x^4y^2z + \epsilon_4x^6y^3,$$

where $\epsilon_i \in \mathbb{C}$. The surface X is singular at the point O_y. Note that X is also singular at a point O_2 that is cut out on X by the equations $x = t = 0$, the surface X is also singular at a point O_5 that is cut out on X by the equations $x = y = 0$, and X is also singular at points P_1 and P_2 that are cut out on X by the equations $y = z = 0$.

The curves C_x and C_y are irreducible. Moreover, we have

$$\frac{3}{2} = \mathrm{lct}\left(X, \frac{2}{3}C_x\right) > \mathrm{lct}\left(X, \frac{2}{4}C_y\right) = 2,$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 3/2$.

Suppose that $\mathrm{lct}(X) < 3/2$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{3}{2}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(20))$ contains y^5, y^2x^4, z^2, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_y$.

Suppose that $P \in C_y$ and $P \notin \mathrm{Sing}(X)$. Then

$$\frac{2}{15} = D\cdot C_y \geqslant \mathrm{mult}_P(D) > \frac{2}{3} > \frac{2}{15},$$

which is a contradiction. Suppose that $P = P_1$. Then

$$\frac{2}{15} = D\cdot C_y \geqslant \mathrm{mult}_P(D) > \frac{2}{9} > \frac{2}{15},$$

which is a contradiction. Similarly, we see that $P \neq P_2$.

Thus, we see that $P \in C_x \cap \mathrm{Sing}(X)$. Then

$$\frac{1}{10} = D\cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{2} \text{ if } P = O_2,\\ \dfrac{\mathrm{mult}_P(D)}{4} \text{ if } P = O_y,\\ \dfrac{\mathrm{mult}_P(D)}{5} \text{ if } P = O_5,\end{cases} > \begin{cases} \dfrac{2}{6} \text{ if } P = O_2,\\ \dfrac{2}{12} \text{ if } P = O_y,\\ \dfrac{2}{15} \text{ if } P = O_5,\end{cases} > \frac{1}{10}$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 3/2$. □

Lemma 3.2.6. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 7, 8, 13, 29)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be given by the equation

$$z^2t + y^3z + xt^2 + x^7z + \epsilon_1 x^2yz^2 + \epsilon_2 x^3yt + \epsilon_2 x^5y^2 = 0,$$

where $\epsilon_i \in \mathbb{C}$. The surface X is singular at the point O_x, O_y, O_z and O_t.

The curves C_x is reducible. Namely, we have $C_x = L + Z$, where L and Z are irreducible curves such that the curve L is cut out on X by the equations $x = z = 0$. Easy calculations imply that

$$L \cdot L = -\frac{18}{91},\ L \cdot Z = \frac{3}{13},\ Z \cdot Z = -\frac{15}{104},$$

the curve Z contains the points $O_z$and O_t, the curve L contains the points O_y and O_t, and $L \cap Z = O_t$. We have $\mathrm{lct}(X, C_x) = 2/3$, which implies that $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. Then it follows from Remark 1.4.7 that we may assume that the support of the divisor D does not contain either the curve L or the curve Z.

The curve C_y is irreducible and $(X, \frac{2}{7}C)$ is log canonical. Thus, it follows from Remark 1.4.7 that we may assume that the support of the divisor D does not contain the curve C_y as well.

Suppose that $P \notin C_x \cup C_y$. Then $P \in X \setminus \mathrm{Sing}(X)$ and

$$1 < \mathrm{mult}_P\big(D\big) \leqslant \frac{91}{58} < 1$$

by Lemma 1.4.10, because the natural projection $X \dashrightarrow \mathbb{P}(3, 7, 8)$ is a finite morphism outside of the curve C_x, and $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(24))$ contains monomials x^8, z^3, xy^3. Thus, we see that $P \in C_x \cup C_y$.

Suppose that $P \in C_y$ and $P \notin \mathrm{Sing}(X)$. Then

$$1 < \mathrm{mult}_P(D) \leqslant D \cdot C_y = \frac{29}{156} < 1,$$

which is a contradiction. Suppose that $P = O_x$. Then

$$\frac{1}{3} < \frac{\mathrm{mult}_{O_x}(D)}{3} \leqslant D \cdot C_y = \frac{29}{156} < \frac{1}{3},$$

which is a contradiction. Thus, we see that $P \in C_x$.

Suppose that $P = O_t$ and $L \not\subset \mathrm{Supp}(D)$. Then

$$\frac{1}{13} < \frac{\mathrm{mult}_{O_t}(D)}{13} \leqslant D \cdot L = \frac{2}{91} < \frac{1}{13},$$

which is a contradiction. Suppose that $P = O_t$ and $M \not\subset \mathrm{Supp}(D)$. Then

$$\frac{1}{13} < \frac{\mathrm{mult}_{O_t}(D)}{13} \leqslant D \cdot M = \frac{3}{52} < \frac{1}{13},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P \in L$. Put $D = mL + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{3}{52} = D \cdot Z = \big(mL + \Omega\big) \cdot Z \geqslant mL \cdot Z = \frac{3m}{13},$$

which implies that $m \leqslant 1/4$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 18m}{91} = \big(-K_X - mL\big) \cdot L = \Omega \cdot L > \begin{cases} \frac{1}{7} \text{ if } P = O_y, \\ 1 \text{if } P \neq O_y, \end{cases}$$

because $P \neq O_t$. Therefore, we see that $m > 11/18$. But $m \leqslant 1/4$. The obtained contradiction implies that $P \notin L$.

Suppose that $P \in Z$. Put $D = aZ + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $Z \not\subset \mathrm{Supp}(\Upsilon)$. If $a \neq 0$, then

$$\frac{2}{91} = D \cdot L = \big(aZ + \Upsilon\big) \cdot L \geqslant aL \cdot Z = \frac{3a}{13},$$

which implies that $a \leqslant 2/21$. Then it follows from Lemma 1.4.6 that

$$\frac{6+15a}{104} = \big(-K_X - aZ\big)\cdot Z = \Upsilon \cdot Z > \begin{cases} \frac{1}{8} \text{ if } P = O_z, \\ 1 \text{ if } P \neq O_z, \end{cases}$$

which implies, in particular, that $a > 7/15$. But $a \leqslant 2/21$. The obtained contradiction completes the proof. ∎

Lemma 3.2.7. Suppose that $(a_0, a_1, a_2, a_3, d) = (3, 10, 11, 19, 41)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^2t + y^3z + xt^2 + x^{10}z + \epsilon_1 x^3yz^2 + \epsilon_2 x^4yt + \epsilon_3 x^7y^2 = 0,$$

where $\epsilon_i \in \mathbb{C}$. The surface X is singular at the point O_x, O_y and O_z.

The curve C_x is reducible. We have $C_x = L_{xz} + Z_x$, where L_{xz} and Z_x are irreducible and reduced curves such that L_{xz} is given by the equations $x = z = 0$, and Z_x is given by the equations $x = tz + y^3 = 0$. Then

$$L_{xz}\cdot L_{xz} = \frac{-27}{10\cdot 19},\ Z_x \cdot Z_x = \frac{-21}{11\cdot 19},\ L_{xz}\cdot Z_x = \frac{3}{19},$$

and $L_{xz} \cap Z_x = O_t$. The curve C_y is irreducible and

$$1 = \mathrm{lct}\left(X, \frac{2}{3}C_x\right) < \mathrm{lct}\left(X, \frac{2}{10}C_y\right) = 5,$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair (X, D) is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, we may assume that either $L_{xz} \not\subseteq \mathrm{Supp}(D)$, or $Z_x \not\subseteq \mathrm{Supp}(D)$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(60))$ contains x^{20}, y^6 and x^6z^2, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = O_t$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{1}{5\cdot 19} = D\cdot L_{xz} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{19} > \frac{1}{19} > \frac{1}{5\cdot 19},$$

which is a contradiction. If $Z_x \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{8}{11\cdot 19} = D\cdot Z_x \geqslant \frac{\mathrm{mult}_P\big(D\big)}{19} > \frac{1}{19} > \frac{8}{11\cdot 19},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P \in L_{xz}$. Put $D = mL_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{8}{11\cdot 19} = -K_X\cdot Z_x = D\cdot Z_x = \big(mL_{xz} + \Omega\big)\cdot Z_x \geqslant mL_{xz}\cdot Z_x = \frac{3m}{19},$$

which implies that $m \leqslant 8/33$. Then it follows from Lemma 1.4.6 that

$$\frac{2+27m}{190} = \big(-K_X - mL_{xz}\big)\cdot L_{xz} = \Omega\cdot L_{xz} > \begin{cases} 1 \text{ if } P \neq O_y, \\ \frac{1}{10} \text{ if } P = O_y, \end{cases}$$

which implies that $m > 17/27$. But we already proved that $m \leqslant 8/33$. Thus, we see that $P \notin L_{xz}$.

Suppose that $P \in Z_x$. Put $D = \epsilon Z_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_x \not\subset \mathrm{Supp}(\Delta)$. If $\epsilon \neq 0$, then

$$\frac{2}{190} = -K_X\cdot L_{xz} = D\cdot L_{xz} = \big(\epsilon Z_x + \Delta\big)\cdot L_{xz} \geqslant \epsilon L_{xz}\cdot Z_x = \frac{3\epsilon}{19},$$

which implies that $\epsilon \leqslant 1/15$. Then it follows from Lemma 1.4.6 that

$$\frac{8+21\epsilon}{11\cdot 19} = \big(-K_X - \epsilon Z_x\big)\cdot Z_x = \Delta \cdot Z_x > \begin{cases} 1 \text{ if } P \neq O_z, \\ \dfrac{1}{11} \text{ if } P = O_z, \end{cases}$$

which implies that $\epsilon > 11/21$. But we already proved that $\epsilon \leqslant 1/15$. Thus, we see that $P \notin Z_x$.

We see that $P \notin C_x$ and $P \in \mathrm{Sing}(X)$. Then $P = O_x$. We have

$$\frac{82}{627} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{3} > \frac{1}{3} > \frac{82}{627},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 1$. □

Lemma 3.2.8. Suppose that $(a_0, a_1, a_2, a_3, d) = (5, 13, 19, 22, 57)$. Then $\mathrm{lct}(X) = 25/12$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + yt^2 + xy^4 + x^7t + \epsilon x^5yz = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular at the points O_x, O_y and O_t.

The curves C_x and C_y are irreducible. Moreover, we have

$$\frac{25}{12} = \mathrm{lct}\left(X, \frac{2}{5}C_x\right) > \mathrm{lct}\left(X, \frac{2}{13}C_y\right) = \frac{65}{21},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 25/12$.

Suppose that $\mathrm{lct}(X) < 25/12$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{25}{12}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(110))$ contains x^9y^5 x^{22} and t^5, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = O_x$. Then

$$\frac{3}{55} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{5} > \frac{12}{125} > \frac{3}{55},$$

which is a contradiction. Thus, we see that $P \in C_x$. Then

$$\frac{3}{143} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{13} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{22} \text{ if } P = O_t, \\ \mathrm{mult}_P(D) \text{ if } P \notin O_y \text{ and } P \notin O_t, \end{cases} > \frac{12}{25\cdot 22} > \frac{3}{143}$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 25/12$. □

Lemma 3.2.9. Suppose that $(a_0, a_1, a_2, a_3, d) = (5, 13, 19, 35, 70)$. Then $\mathrm{lct}(X) = 25/12$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^5 + x^{14} + \epsilon x^5y^2z = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular at the points O_y and O_z. It is also singular at two points P_1 and P_2 that are cut out on X by the equations $y = z = 0$.

The curves C_x and C_y are irreducible. Moreover, we have

$$\frac{25}{12} = \mathrm{lct}\left(X, \frac{2}{5}C_x\right) > \mathrm{lct}\left(X, \frac{2}{13}C_y\right) = \frac{26}{7},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 25/12$.

Suppose that $\mathrm{lct}(X) < 25/12$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{25}{12}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(95))$ contains x^6y^5, x^{19}, z^5, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = P_1$. Then

$$\frac{4}{95} = D \cdot C_y \geqslant \frac{\mathrm{mult}_{P_1}(D)}{5} > \frac{12}{125} > \frac{4}{95},$$

which is a contradiction. We see that $P \neq P_1$. Similarly, we see that $P \neq P_2$. Then $P \in C_x$ and

$$\frac{4}{247} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{13} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{19} \text{ if } P = O_z, \\ \mathrm{mult}_P(D) \text{ if } P \notin O_y \text{ and } P \notin O_z, \end{cases} > \frac{12}{25 \cdot 19} > \frac{4}{247}$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 25/12$. □

Lemma 3.2.10. Suppose that $(a_0, a_1, a_2, a_3, d) = (6, 9, 10, 13, 36)$. Then $\mathrm{lct}(X) = 25/12$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$zt^2 + y^4 + xz^3 + x^6 + \epsilon x^3 y^2 = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular at the points O_z and O_t. It is also singular at two points P_1 and P_2 that are cut out on X by the equations $z = t = 0$. The surface X is also singular at two points Q_1 and Q_2 that are cut out on X by the equations $y = t = 0$.

The curve C_z is reducible. We have $C_z = C_1 + C_2$, where C_1 and C_2 are irreducible and reduced curves on X such that

$$C_1 \cdot C_1 = C_2 \cdot C_2 = -\frac{8}{39},\ C_1 \cdot C_2 = \frac{6}{13},$$

and $Q_1 \in C_1 \not\ni Q_2 \in C_2 \not\ni Q_1$. The curves C_x and C_y are irreducible. Then

$$\frac{25}{12} = \mathrm{lct}\left(X, \frac{2}{10} C_z\right) > \frac{9}{4} = \mathrm{lct}\left(X, \frac{2}{6} C_x\right) > \frac{9}{2} = \mathrm{lct}\left(X, \frac{2}{9} C_y\right),$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 25/12$.

Suppose that $\mathrm{lct}(X) < 25/12$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{25}{12} D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y, and the support of the divisor D does not contain either C_1 or C_2.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(30))$ contains x^2t^2, x^5, z^3, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x \cup C_z$.

Suppose that $P \in C_1$. Put $D = mC_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $C_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{2}{39} = -K_X \cdot C_2 = D \cdot C_2 = \big(mC_1 + \Omega\big) \cdot C_2 \geqslant mC_1 \cdot C_2 = \frac{6m}{13},$$

which implies that $m \leqslant 1/9$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + m8}{39} = \big(-K_X - mC_1\big) \cdot C_1 = \Omega \cdot C_1 > \begin{cases} \dfrac{12}{25} \text{ if } P \neq Q_1, \\ \dfrac{12}{25}\dfrac{1}{2} \text{ if } P = W_1, \end{cases} \geqslant \frac{6}{25},$$

which contradicts the inequality $m \leqslant 1/9$. Thus, we see that $P \notin C_1$. Similarly, we see that $P \notin C_2$.

Suppose that $P = P_1$. Then

$$\frac{6}{65} = D \cdot C_y \geqslant \frac{\mathrm{mult}_{P_1}(D)}{3} \geqslant \frac{12}{75} > \frac{6}{65},$$

which is a contradiction. We see that $P \neq P_1$. Similarly, we see that $P \neq P_2$. Then $P \in C_x$ and

$$\frac{4}{65} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{10} \text{ if } P = O_z, \\ \dfrac{\mathrm{mult}_P(D)}{13} \text{ if } P = O_t, \\ \mathrm{mult}_P(D) \text{ if } P \notin O_z \text{ and } P \notin O_t, \end{cases} > \frac{12}{25 \cdot 13} > \frac{4}{65}$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 25/12$. □

Lemma 3.2.11. Suppose that $(a_0, a_1, a_2, a_3, d) = (7, 8, 19, 25, 57)$. Then $\mathrm{lct}(X) = 49/24$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + y^4t + xt^2 + x^7y + \epsilon x^2y^3z = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular at the point O_x, O_y and O_t. The curves C_x, C_y and C_z are irreducible. We have

$$\frac{49}{24} = \mathrm{lct}\left(X, \frac{2}{7}C_x\right) < \mathrm{lct}\left(X, \frac{2}{8}C_y\right) = \frac{10}{3} < \mathrm{lct}\left(X, \frac{2}{19}C_z\right) = \frac{19}{2},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 49/24$.

Suppose that $\mathrm{lct}(X) < 49/24$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{49}{24}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x, C_y and C_z.

The point P is not contained in the curve $P \in C_x$, because otherwise we have

$$\frac{3}{200} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{8} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{25} \text{ if } P = O_t, \\ \mathrm{mult}_P(D) \text{ if } P \neq O_y \text{ and } P \neq O_t, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 24/49$. Similarly, we see that $P \neq C_y \cup C_z$. Then there is a unique curve $Z \subset X$ that is cut out by

$$zy^2 = \alpha x^5$$

such that $P \in Z$, where $0 \neq \alpha \in \mathbb{C}$. We see that $C_y \not\subset \mathrm{Supp}(Z)$. But the open subset $Z \setminus (Z \cap C_y)$ of the curve Z is a $\mathbb{Z}_8$-quotient of the affine curve

$$z - \alpha x^5 = z^3 + t + xt^2 + x^7 + \epsilon x^2z = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, z, t\big]\Big),$$

which is isomorphic to a plane affine curve that is given by the equation

$$\alpha^3x^{15} + t + xt^2 + x^7 + \epsilon\alpha x^7 = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, t\big]\Big),$$

which is easily seen to be irreducible. In particular, the curve Z is irreducible and $\mathrm{mult}_P(Z) \leqslant 14$. Thus, we may assume that $\mathrm{Supp}(D)$ does not contain the curve Z by Remark 1.4.7. Then

$$\frac{3}{40} = D \cdot Z \geqslant \mathrm{mult}_P(D) > \frac{24}{49},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 49/24$. □

Lemma 3.2.12. Suppose that $(a_0, a_1, a_2, a_3, d) = (7, 8, 19, 32, 64)$. Then $\mathrm{lct}(X) = 35/16$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^8 + xz^3 + x^8y + \epsilon x^3y^3z,$$

where $\epsilon \in \mathbb{C}$. Note that X is singular at the points O_x and O_z. The surface X also has two singular points P_1 and P_2 of type $\frac{1}{8}(7, 3)$ that are cut out on X by the equations $x = z = 0$.

The curve C_x is reducible. We have $C_x = C_1 + C_2$, where C_1 and C_2 are irreducible reduced curves such that

$$C_1 \cdot C_1 = C_2 \cdot C_2 = -\frac{25}{8 \cdot 19}, \ C_1 \cdot C_2 = \frac{4}{19},$$

and $P_1 \in C_1$, $P_2 \in C_2$. Then $C_1 \cap C_2 = O_z$. The curve C_y is irreducible. We have

$$\mathrm{lct}\left(X, \frac{2}{7}C_x\right) = \frac{35}{16} < \mathrm{lct}\left(X, \frac{2}{8}C_y\right) = \frac{10}{3},$$

which implies that $\mathrm{lct}(X) \leqslant 35/16$.

Suppose that $\mathrm{lct}(X) < 35/16$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{35}{16}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of D does not contain the curve C_y. Moreover, we may assume that the support of D does not contain either the curve C_1 or the curve C_2.

Suppose that $P = O_z$. We know that $C_i \not\subset \mathrm{Supp}(D)$ for some $i = 1, 2$. Then

$$\frac{16}{35}\frac{1}{19} < \frac{\mathrm{mult}_{O_z}(D)}{19} \leqslant D \cdot C_i = \frac{1}{4 \cdot 19},$$

which is a contradiction. Therefore, we see that $P \neq O_z$.

Suppose that $P \in C_1$. Put $D = mC_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $C_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{4 \cdot 19} = -K_X \cdot C_2 = D \cdot C_2 = \big(mC_1 + \Omega\big) \cdot C_2 \geqslant mC_1 \cdot C_2 = \frac{4m}{19},$$

which implies that $m \leqslant 1/16$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 25m}{8 \cdot 19} = \big(-K_X - mC_1\big) \cdot C_1 = \Omega \cdot C_1 > \begin{cases} \dfrac{16}{35} \text{ if } P \neq P_1, \\ \dfrac{16}{35}\dfrac{1}{8} \text{ if } P = P_1, \end{cases}$$

which is impoassible, because $m \leqslant 1/16$. Thus, we see that $P \notin C_1$. Similarly, we see that $P \notin C_2$.

Suppose that $P \in C_x$. Then

$$\frac{4}{7 \cdot 19} = D \cdot C_y \geqslant \begin{cases} \mathrm{mult}_P\big(D\big) \text{ if } P \neq O_x, \\ \dfrac{\mathrm{mult}_{O_y}\big(D\big)}{7} \text{ if } P = O_x, \end{cases}$$

which leads to a contradiction, because $\mathrm{mult}_P(D) > 16/35$. Thus, we see that $P \notin C_x$.

Thus, we see that $P \in X \setminus \mathrm{Sing}(X)$ and $P \notin C_x \cup C_y$. But $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(64))$ contains monomials y^8, x^8y, y^4t and t^2, which is impossible by Lemma 1.4.10. The obtained contradiction completes the proof. ☐

Lemma 3.2.13. Suppose that $(a_0, a_1, a_2, a_3, d) = (9, 12, 13, 16, 48)$. Then $\mathrm{lct}(X) = 63/24$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^3 + y^4 + xz^3 + x^4y = 0,$$

the surface X is singular at the point O_x and O_z. The surface X is also singular at a point Q_4 that is cut out on X by the equations $z = x = 0$. The surface X is also singular at a point Q_3 such that $Q_3 \neq O_x$ and the points Q_3 and Q_x are cut out on X by the equations $z = t = 0$.

The curves C_x, C_y, C_z and C_t are irreducible. We have

$$\frac{63}{24} = \mathrm{lct}\left(X, \frac{2}{9}C_x\right) < \mathrm{lct}\left(X, \frac{2}{12}C_y\right) = 4 < \mathrm{lct}\left(X, \frac{2}{13}C_z\right) = \frac{13}{2} < \mathrm{lct}\left(X, \frac{2}{16}C_t\right) = \frac{16}{2},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 63/24$.

Suppose that $\mathrm{lct}(X) < 63/24$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{63}{24}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x, C_y, C_z and C_t.

The point P is not contained in the curve C_x, because otherwise we have

$$\frac{9}{18\cdot 13}=D\cdot C_x\geqslant\begin{cases}\dfrac{\mathrm{mult}_P\big(D\big)}{13} & \text{if } P=O_z,\\ \dfrac{\mathrm{mult}_P\big(D\big)}{4} & \text{if } P=Q_4,\\ \mathrm{mult}_P\big(D\big) & \text{if } P\neq O_z \text{ and } P\neq Q_4,\end{cases}$$

which is impossible, because $\mathrm{mult}_P(D)>24/63$. Similarly, we see that $P\neq C_y\cup C_z\cup C_t$. Then there is a unique curve $Z\subset X$ that is cut out by

$$xt=\alpha yz$$

such that $P\in Z$, where $0\neq\alpha\in\mathbb{C}$. We see that $C_x\not\subset\mathrm{Supp}(Z)$. But the open subset $Z\setminus(Z\cap C_x)$ of the curve Z is a $\mathbb{Z}_9$-quotient of the affine curve

$$t-\alpha yz=t^3+y^4+z^3+y=0\subset\mathbb{C}^3\cong\mathrm{Spec}\Big(\mathbb{C}\big[y,z,t\big]\Big),$$

which is isomorphic to a plane affine quartic curve that is given by the equation

$$\alpha^2y^2z^2+y^4+z^3=0\subset\mathbb{C}^2\cong\mathrm{Spec}\Big(\mathbb{C}\big[y,z\big]\Big),$$

which is easily seen to be irreducible. In particular, the curve Z is irreducible and $\mathrm{mult}_P(Z)\leqslant 3$. Thus, we may assume that $\mathrm{Supp}(D)$ does not contain the curve Z by Remark 1.4.7. Then

$$\frac{25}{18\cdot 13}=D\cdot Z\geqslant\mathrm{mult}_P\big(D\big)>\frac{24}{63},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X)=63/24$. □

Lemma 3.2.14. Suppose that $(a_0,a_1,a_2,a_3,d)=(9,12,19,19,57)$. Then $\mathrm{lct}(X)=3$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$zt(z-t)+xy^4+x^5y=0,$$

which implies that X is singular at three distinct points O_x, O_y, P_1 on the curve defined by $z=t=0$. Also, the surface X is singular at three distinct points O_z, O_t, Q_1 on the curve defined by $x=y=0$, where O_z is cut out by $x=y=z=0$, the point O_t is cut out by $x=y=t=0$, and Q_1 is cut out by $x=y=z-t=0$.

Note that $\mathrm{lct}(X,\frac{2}{9}C_x)=3$, which implies that $\mathrm{lct}(X)\leqslant 3$. Suppose that $\mathrm{lct}(X)<3$. Then there is a $\mathbb{Q}$-effective divisor $D\equiv -K_X$ such that the pair $(X,3D)$ is not log canonical at some point $P\in X$.

The curve C_x consists of three distinct curves $L_1=\{x=z=0\}$, $L_2=\{x=t=0\}$ and $L_3=\{x=z-t=0\}$ that intersect altogether at the point O_y. We have

$$L_1^2=L_2^2=L_2^2=\frac{-29}{19\cdot 12},\ L_1\cdot L_2=L_1\cdot L_3=L_3\cdot L_3=\frac{1}{12},$$

and $D\cdot L_1=D\cdot L_2=D\cdot L_3=1/114$. Similarly, the curve C_y consists of three curves $L_1'=\{y=z=0\}$, $L_2'=\{y=t=0\}$ and $L_3'=\{y=z-t=0\}$ that intersect altogether at the point O_x. We have

$$L_1'^2=L_2'^2=L_2'^2=\frac{-26}{19\cdot 9},\ L_1'\cdot L_2'=L_1'\cdot L_3'=L_3'\cdot L_3'=\frac{1}{9},$$

and $D\cdot L_1'=D\cdot L_2'=D\cdot L_3'=2/171$.

The pairs $(X,\frac{6}{9}C_x)$ and $(X,\frac{6}{12}C_y)$ are log canonical. By Remark 1.4.7, we may assume that the support of D does not contain at least one component of C_y. Also, we may assume that the support of D does not contain at least one component of C_x. Then arguing as in the proof of Lemma 3.1.15, we see that $P\neq O_x$ and $P\neq O_y$.

The curve C_z consists of three distinct curves L_1, L_1' and M_z, where M_z is an irreducible reduced curve that is cut out by the equations $z=y^3+x^4=0$. The curve C_t consists of three distinct curves L_2, L_2' and M_t, where M_t is an irreducible reduced curve that is cut out by the equations $t=y^3+x^4=0$.

Let C_1 be the curve that is cut out on X by $z-t$. Then C_1 consists of three distinct curves L_3, L_3' and M_1, where M_1 is an irreducible reduced curve that is cut out by the equations $z-t=y^3+x^4=0$. We have

$$\mathrm{lct}\left(X, \frac{2}{19}C_z\right)=\mathrm{lct}\left(X, \frac{2}{19}C_t\right)=\mathrm{lct}\left(X, \frac{2}{19}C_1\right)=\frac{7}{2},$$

and $D\cdot M_z=D\cdot M_t=D\cdot M_1=2/57$. By Remark 1.4.7, we may assume that the support of D does not contain at least one component of every curve C_z, C_t and C_1. Arguing as in the proof of Lemma 3.1.15, we see that $P\neq O_t$, $P\neq O_z$ and $P\neq Q_1$.

Suppose that $P=P_1$. We have $P_1=M_z\cap M_t\cap M_z$, the log pair

$$\left(X,\ \frac{3}{18}\Big(M_z+M_t+M_z\Big)\right)$$

is log canonical at P_1, and $M_z+M_t+M_z\sim -18K_X$. By Remark 1.4.7, we may assume that the support of D does not contain at least one curve among M_z, M_t and M_1. Without loss of generality, we may assume that the support of D does not contain the curve M_z. Then

$$\frac{2}{57}=D\cdot M_z\geqslant\frac{\mathrm{mult}_P(D)}{3}>\frac{1}{9},$$

which is a contradiction. Thus, we see that $P\neq P_1$. Then $P\not\in\mathrm{Sing}(X)$.

Arguing Arguing as in the proof of Lemma 3.1.15, we see that $P\not\in C_z\cup C_t\cup C_1$. Then there is a quasismooth irreducible curve $E\subset X$ such that E is given by the equation $z=\lambda t$ and $P\in E$, where λ is a non-zero constant different from 1. By Remark 1.4.7, we may assume that the support of D does not contain the curve E. Then

$$\frac{1}{3}<\mathrm{mult}_P(D)\leqslant D\cdot E=\frac{1}{18},$$

which is a contradiction. The obtained contradiction completes the proof. □

Lemma 3.2.15. Suppose that $(a_0,a_1,a_2,a_3,d)=(9,19,24,31,81)$. Then $\mathrm{lct}(X)=77/30$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$yt^2+y^3z+xz^3+x^9=0,$$

and X is singular at the point O_y, O_z and O_t. The surface X is also singular at a point Q such that $Q\neq O_z$ and the points Q and Q_z are cut out on X by the equations $y=t=0$.

The curve C_x is reducible. We have $C_x=L_{xy}+Z_x$, where L_{xy} and Z_x are irreducible and reduced curves such that L_{xy} is given by the equations $x=y=0$, and Z_x is given by the equations $x=t^2+y^2z=0$. Then

$$L_{xy}\cdot L_{xy}=\frac{-53}{24\cdot 31},\ Z_x\cdot Z_x=\frac{-20}{19\cdot 24},\ L_{xy}\cdot Z_x=\frac{2}{24},$$

and $L_{xy}\cap Z_x=O_z$. The curve C_y is also reducible. We have $C_x=L_{xy}+Z_y$, where Z_y is an irreducible and reduced curve that is given by the equations $y=z^3+x^8=0$. Then

$$Z_y\cdot Z_y=\frac{10}{3\cdot 31},\ L_{xy}\cdot Z_y=\frac{3}{31},\ D\cdot Z_y=\frac{2}{3\cdot 31},\ D\cdot Z_x=\frac{4}{19\cdot 24},\ D\cdot L_{xy}=\frac{2}{24\cdot 31}$$

and $L_{xy}\cap Z_y=O_t$. The curve C_z is irreducible. We see that $\mathrm{lct}(X)\leqslant 3$, because

$$3=\mathrm{lct}\left(X,\frac{2}{9}C_x\right)<\mathrm{lct}\left(X,\frac{2}{21}C_y\right)=\frac{209}{54}<\mathrm{lct}\left(X,\frac{2}{24}C_z\right)=\frac{22}{3}.$$

Suppose that $\mathrm{lct}(X)<3$. Then there is a $\mathbb{Q}$-effective divisor $D\equiv -K_X$ such that the pair $(X,3D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that $C_z\not\in\mathrm{Supp}(D)$, and either $L_{xy}\not\subseteq\mathrm{Supp}(D)$, or $Z_x\not\subseteq\mathrm{Supp}(D)\not\supset Z_y$.

Since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(171))$ contains y^9, x^{19}, x^3z^6, $x^{11}z^3$, it follows from Lemma 1.4.9 that $P\in\mathrm{Sing}(X)\cup C_x\cup C_y$.

Suppose that $P=O_t$. If $L_{xy}\not\subseteq\mathrm{Supp}(D)$, then

$$\frac{2}{24\cdot 31}=D\cdot L_{xy}\geqslant\frac{\mathrm{mult}_P(D)}{31}>\frac{1}{3\cdot 31}>\frac{2}{24\cdot 31},$$

which is a contradiction. If $Z_y \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{3\cdot 31} = D\cdot Z_y \geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(Z_y\big)}{31} = \frac{3\mathrm{mult}_P\big(D\big)}{31} > \frac{1}{31} > \frac{2}{3\cdot 31},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P = O_z$. If $L_{xy} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{24\cdot 31} = D\cdot L_{xy} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{24} > \frac{1}{3\cdot 24} > \frac{2}{24\cdot 31},$$

which is a contradiction. If $Z_x \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{4}{19\cdot 24} = D\cdot Z_x \geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(Z_x\big)}{24} = \frac{2\mathrm{mult}_P\big(D\big)}{24} > \frac{2}{3\cdot 24} > \frac{4}{19\cdot 24},$$

which is a contradiction. Thus, we see that $P \neq O_z$.

Suppose that $P = O_y$. Then

$$\frac{18}{19\cdot 31} = D\cdot C_z \geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(C_z\big)}{19} = \frac{2\mathrm{mult}_P\big(D\big)}{19} > \frac{2}{3\cdot 19} > \frac{18}{19\cdot 31},$$

which is a contradiction. Thus, we see that $P \neq O_y$.

Suppose that $P \in L_{xy}$. Put $D = mL_{xy} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xy} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{2}{19\cdot 12} = -K_X\cdot Z_x = D\cdot Z_x = \big(mL_{xy}+\Omega\big)\cdot Z_x \geqslant mL_{xy}\cdot Z_x = \frac{m}{12},$$

which implies that $m \leqslant 2/19$. Then it follows from Lemma 1.4.6 that

$$\frac{2+53m}{24\cdot 31} = \big(-K_X - mL_{xy}\big)\cdot L_{xy} = \Omega\cdot L_{xy} > \frac{1}{3},$$

which is impossible, because $m \leqslant 2/19$. Thus, we see that $P \notin L_{xy}$.

Suppose that $P \in Z_y$. Put $D = \epsilon Z_y + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_y \not\subset \mathrm{Supp}(\Delta)$. If $\epsilon \neq 0$, then

$$\frac{2}{24\cdot 31} = -K_X\cdot L_{xy} = D\cdot L_{xy} = \big(\epsilon Z_y+\Delta\big)\cdot L_{xy} \geqslant \epsilon L_{xy}\cdot Z_y = \frac{3\epsilon}{31},$$

which implies that $\epsilon \leqslant 1/36$. Then it follows from Lemma 1.4.6 that

$$\frac{6-30\epsilon}{9\cdot 31} = \big(-K_X - \epsilon Z_y\big)\cdot Z_y = \Delta\cdot Z_x > \begin{cases} \dfrac{1}{3} \text{ if } P \neq Q,\\ \dfrac{1}{9} \text{ if } P = Q,\end{cases}$$

which is impossible, because $\epsilon \leqslant 1/36$. Thus, we see that $P \notin Z_y$. Then $P \in Z_x$.

Put $D = \delta Z_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $Z_x \not\subset \mathrm{Supp}(\Upsilon)$. If $\epsilon \neq 0$, then

$$\frac{1}{12\cdot 31} = -K_X\cdot L_{xy} = D\cdot L_{xy} = \big(\delta Z_x+\Upsilon\big)\cdot L_{xy} \geqslant \delta L_{xy}\cdot Z_x = \frac{1\delta}{12},$$

which implies that $\delta \leqslant 1/31$. Then it follows from Lemma 1.4.6 that

$$\frac{4+20\delta}{19\cdot 24} = \big(-K_X - \delta Z_x\big)\cdot Z_x = \Upsilon\cdot Z_x > \frac{1}{3},$$

which is impossible, because $\delta \leqslant 1/31$. The obtained contradiction shows that $\mathrm{lct}(X) = 3$. □

Lemma 3.2.16. Suppose that $(a_0, a_1, a_2, a_3, d) = (10, 19, 35, 43, 105)$. Then $\mathrm{lct}(X) = 57/14$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + yt^2 + xy^5 + x^7z = 0,$$

and X is singular at the point O_x, O_y and O_t. The surface X is also singular at a point Q such that $Q \neq O_x$ and the points Q and Q_x are cut out on X by the equations $y = t = 0$.

The curve C_y is reducible. We have $C_y = L_{yz} + Z_y$, where L_{yz} and Z_y are irreducible and reduced curves such that L_{yz} is given by the equations $y = z = 0$, and Z_y is given by the equations $y = z^2 + x^7 = 0$. Then

$$L_{yz} \cdot L_{yz} = \frac{-51}{10 \cdot 43},\ Z_y \cdot Z_y = \frac{-32}{10 \cdot 43},\ L_{yz} \cdot Z_y = \frac{7}{43},$$

and $L_{yz} \cap Z_y = O_t$. The curve C_x is irreducible and

$$\frac{57}{14} = \mathrm{lct}\left(X, \frac{2}{19}C_y\right) < \mathrm{lct}\left(X, \frac{2}{10}C_x\right) = \frac{25}{6},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 57/14$.

Suppose that $\mathrm{lct}(X) < 57/14$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{57}{14}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curve C_x. Similarly, we may assume that either $L_{yz} \not\subseteq \mathrm{Supp}(D)$, or $Z_y \not\subseteq \mathrm{Supp}(D)$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(190))$ contains x^{19}, y^{10}, x^5z^4 and $x^{12}z^2$, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x \cup C_y$.

Suppose that $P = O_t$. If $L_{yz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{10 \cdot 43} = D \cdot L_{yz} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{43} > \frac{14}{57 \cdot 43} > \frac{2}{10 \cdot 43},$$

which is a contradiction. If $Z_y \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{4}{10 \cdot 43} = D \cdot Z_y \geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(Z_y\big)}{43} = \frac{2\mathrm{mult}_P\big(D\big)}{43} > \frac{28}{57 \cdot 43} > \frac{4}{10 \cdot 43},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P \in L_{yz}$. Put $D = mL_{yz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{yz} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{4}{10 \cdot 43} = -K_X \cdot Z_y = D \cdot Z_y = \big(mL_{yz} + \Omega\big) \cdot Z_y \geqslant mL_{yz} \cdot Z_y = \frac{7m}{43},$$

which implies that $m \leqslant 4/70$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 51m}{430} = \big(-K_X - mL_{yz}\big) \cdot L_{yz} = \Omega \cdot L_{yz} > \begin{cases} \dfrac{14}{57} \text{ if } P \neq O_x, \\ \dfrac{14}{57 \cdot 10} \text{ if } P = O_x, \end{cases}$$

which is impossible, because $m \leqslant 4/70$. Thus, we see that $P \notin L_{yz}$.

Suppose that $P \in Z_y$. Put $D = \epsilon Z_y + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_y \not\subset \mathrm{Supp}(\Delta)$. If $\epsilon \neq 0$, then

$$\frac{2}{430} = -K_X \cdot L_{yz} = D \cdot L_{yz} = \big(\epsilon Z_y + \Delta\big) \cdot L_{yz} \geqslant \epsilon L_{yz} \cdot Z_y = \frac{7\epsilon}{43},$$

which implies that $\epsilon \leqslant 2/70$. Then it follows from Lemma 1.4.6 that

$$\frac{4 + 32\epsilon}{430} = \big(-K_X - \epsilon Z_y\big) \cdot Z_y = \Delta \cdot Z_y > \begin{cases} \dfrac{14}{57} \text{ if } P \neq Q, \\ \dfrac{14}{57 \cdot 5} \text{ if } P = Q, \end{cases}$$

which is impossible, because $\epsilon \leqslant 2/70$. Thus, we see that $P \notin Z_y$.

We see that $P \in C_x$ and $P \notin C_y$. Then have

$$\frac{6}{19 \cdot 43} = D \cdot C_x \geqslant \begin{cases} \dfrac{14}{57} \text{ if } P \neq O_y, \\ \dfrac{14}{57 \cdot 19} \text{ if } P = O_y, \end{cases}$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 57/14$. □

Lemma 3.2.17. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 21, 28, 47, 105)$. Then $\mathrm{lct}(X) = 77/30$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$yz^3 + y^5 + xt^2 + x^7z = 0,$$

and X is singular at the point O_x, O_z and O_t. The surface X is also singular at a point Q such that $Q \neq O_z$ and the points Q and Q_z are cut out on X by the equations $x = t = 0$.

The curve C_x is reducible. We have $C_x = L_{xy} + Z_x$, where L_{xy} and Z_x are irreducible and reduced curves such that L_{xy} is given by the equations $x = y = 0$, and Z_x is given by the equations $x = z^3 + y^4 = 0$. Then

$$L_{xy} \cdot L_{xy} = \frac{-73}{28 \cdot 47},\ Z_x \cdot Z_x = \frac{-10}{7 \cdot 47},\ L_{xy} \cdot Z_x = \frac{3}{47},$$

and $L_{xy} \cap Z_x = O_t$. The curve C_y is also reducible. We have $C_x = L_{xy} + Z_y$, where Z_y is an irreducible and reduced curve that is given by the equations $y = t^2 + x^6z = 0$. Then

$$Z_y \cdot Z_y = \frac{20}{11 \cdot 28},\ L_{xy} \cdot Z_y = \frac{2}{28},\ D \cdot Z_y = \frac{4}{11 \cdot 28},\ D \cdot Z_x = \frac{2}{11 \cdot 47},\ D \cdot L_{xy} = \frac{2}{28 \cdot 47}$$

and $L_{xy} \cap Z_y = O_z$. We see that $\mathrm{lct}(X) \leqslant 77/30$, because

$$\frac{77}{30} = \mathrm{lct}\left(X, \frac{2}{11}C_x\right) < \mathrm{lct}\left(X, \frac{2}{21}C_y\right) = 6.$$

Suppose that $\mathrm{lct}(X) < 77/30$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{77}{30}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that either $L_{xy} \not\subseteq \mathrm{Supp}(D)$, or $Z_x \not\subseteq \mathrm{Supp}(D) \not\supset Z_y$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(517))$ contains x^5y^{22}, $x^{26}y^{11}$, x^{47}, $x^{19}z^{11}$, x^{47}, t^{11}, it follows from Lemma 1.4.9 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = O_t$. If $L_{xy} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{28 \cdot 47} = D \cdot L_{xy} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{47} > \frac{30}{77 \cdot 47} > \frac{2}{28 \cdot 47},$$

which is a contradiction. If $Z_x \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{7 \cdot 47} = D \cdot Z_x \geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(Z_x\big)}{47} = \frac{3\mathrm{mult}_P\big(D\big)}{47} > \frac{90}{91 \cdot 47} > \frac{2}{7 \cdot 47},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P = O_z$. If $L_{xy} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{28 \cdot 47} = D \cdot L_{xy} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{28} > \frac{30}{77 \cdot 28} > \frac{2}{28 \cdot 47},$$

which is a contradiction. If $Z_y \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{4}{11 \cdot 28} = D \cdot Z_y \geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(Z_y\big)}{28} = \frac{2\mathrm{mult}_P\big(D\big)}{28} > \frac{60}{91 \cdot 28} > \frac{4}{11 \cdot 28},$$

which is a contradiction. Thus, we see that $P \neq O_z$.

Suppose that $P \in L_{xy}$. Put $D = mL_{xy} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xy} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{2}{7 \cdot 47} = -K_X \cdot Z_x = D \cdot Z_x = \big(mL_{xy} + \Omega\big) \cdot Z_x \geqslant mL_{xy} \cdot Z_x = \frac{3m}{47},$$

which implies that $m \leqslant 2/21$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 73m}{28 \cdot 47} = \big(-K_X - mL_{xy}\big) \cdot L_{xy} = \Omega \cdot L_{xy} > \frac{30}{77},$$

which is impossible, because $m \leqslant 2/21$. Thus, we see that $P \notin L_{xy}$.

Suppose that $P \in Z_x$. Put $D = \epsilon Z_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_x \not\subset \mathrm{Supp}(\Delta)$. If $\epsilon \neq 0$, then

$$\frac{2}{28 \cdot 47} = -K_X \cdot L_{xy} = D \cdot L_{xy} = \big(\epsilon Z_x + \Delta\big) \cdot L_{xy} \geqslant \epsilon L_{xy} \cdot Z_x = \frac{3\epsilon}{47},$$

which implies that $\epsilon \leqslant 1/42$. Then it follows from Lemma 1.4.6 that

$$\frac{2+10\epsilon}{7\cdot 47} = \big(-K_X-\epsilon Z_x\big)\cdot Z_x = \Delta\cdot Z_x > \begin{cases} \dfrac{30}{77} \text{ if } P\neq Q,\\ \dfrac{30}{77\cdot 7} \text{ if } P=Q,\end{cases}$$

which is impossible, because $\epsilon \leqslant 1/42$. Thus, we see that $P\notin Z_x$. Then $P=O_x$.

Put $D=\delta Z_y+\Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $Z_y\not\subset \mathrm{Supp}(\Upsilon)$. If $\epsilon\neq 0$, then

$$\frac{2}{28\cdot 47} = -K_X\cdot L_{xy} = D\cdot L_{xy} = \big(\delta Z_y+\Upsilon\big)\cdot L_{xy} \geqslant \delta L_{xy}\cdot Z_y = \frac{2\delta}{28},$$

which implies that $\delta\leqslant 1/47$. Then it follows from Lemma 1.4.6 that

$$\frac{4-20\delta}{11\cdot 28} = \big(-K_X-\delta Z_y\big)\cdot Z_y = \Upsilon\cdot Z_y > \frac{30}{77\cdot 11},$$

which is impossible, because $\delta\leqslant 1/47$. The obtained contradiction shows that $\mathrm{lct}(X)=77/30$.

□

Lemma 3.2.18. Suppose that $(a_0,a_1,a_2,a_3,d)=(11,25,32,41,107)$. Then $\mathrm{lct}(X)=11/3$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$yt^2+y^3z+xz^3+x^6t=0,$$

and X is singular at the point O_x, O_y, O_z and O_t.

The curve C_x is reducible. We have $C_x=L_{xy}+M_x$, where L_{xy} and M_x are irreducible and reduced curves such that L_{xy} is given by the equations $x=y=0$, and M_x is given by the equations $x=t^2+y^2z=0$. Then

$$L_{xy}\cdot L_{xy}=\frac{-71}{32\cdot 41},\ M_x\cdot M_x=\frac{-28}{25\cdot 32},\ L_{xy}\cdot M_x=\frac{3}{32},$$

and $L_{xy}\cap M_x=O_z$. The curve C_y is also reducible. We have $C_y=L_{xy}+M_y$, where M_y is an irreducible and reduced curve that is given by the equations $y=z^3+x^5t=0$. Then

$$M_y\cdot M_y=\frac{42}{11\cdot 41},\ L_{xy}\cdot M_y=\frac{3}{41},\ D\cdot M_y=\frac{6}{11\cdot 41},\ D\cdot M_x=\frac{3}{11\cdot 32},\ D\cdot L_{xy}=\frac{2}{32\cdot 41}$$

and $L_{xy}\cap M_y=O_t$. The curve C_z is also reducible. We have $C_z=L_{zt}+M_z$, where L_{zt} and M_z are irreducible and reduced curves such that L_{zt} is given by the equations $z=t=0$, and M_z is given by the equations $z=x^6+ty=0$. Then

$$L_{zt}\cdot L_{zt}=\frac{-34}{11\cdot 25},\ L_{zt}\cdot M_z=\frac{6}{25},\ D\cdot L_{zt}=\frac{2}{11\cdot 25},\ D\cdot M_z=\frac{12}{25\cdot 41}$$

and $L_{zt}\cap M_z=O_y$. The curve C_t is also reducible. We have $C_t=L_{zt}+M_t$, where M_t is an irreducible and reduced curve that is given by the equations $t=y^3+xz^2=0$. Then $\mathrm{lct}(X)\leqslant 11/3$, because

$$\frac{11}{3}=\mathrm{lct}\left(X,\frac{2}{11}C_x\right)<\frac{50}{9}=\mathrm{lct}\left(X,\frac{2}{25}C_y\right)<\frac{28}{3}=\mathrm{lct}\left(X,\frac{2}{32}C_z\right)<\frac{205}{18}=\mathrm{lct}\left(X,\frac{2}{41}C_t\right).$$

Suppose that $\mathrm{lct}(X)<11/3$. Then there is a $\mathbb{Q}$-effective divisor $D\equiv -K_X$ such that the pair $(X,\frac{11}{3}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that either $\mathrm{Supp}(D)$ does not contain at least one irreducible component of C_x, C_y, C_z and C_t.

Since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(352))$ contains x^7y^{11}, x^{32} and z^{11}, it follows from Lemma 1.4.9 that $P\in \mathrm{Sing}(X)\cup C_x\cup C_y$.

Suppose that $P=O_t$. If $L_{xy}\not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{32\cdot 41}=D\cdot L_{xy}\geqslant \frac{\mathrm{mult}_P\big(D\big)}{41}>\frac{3}{11\cdot 41}>\frac{2}{32\cdot 41},$$

which is a contradiction. If $M_y\not\subseteq \mathrm{Supp}(D)$, then

$$\frac{6}{11\cdot 41}=D\cdot M_y\geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(M_y\big)}{41}=\frac{3\mathrm{mult}_P\big(D\big)}{41}>\frac{9}{11\cdot 41}>\frac{6}{11\cdot 41},$$

which is a contradiction. Thus, we see that $P\neq O_t$.

Suppose that $P = O_z$. If $L_{xy} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{32 \cdot 41} = D \cdot L_{xy} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{32} > \frac{3}{11 \cdot 32} > \frac{2}{32 \cdot 41},$$

which is a contradiction. If $M_x \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{4}{25 \cdot 32} = D \cdot M_x \geqslant \frac{\mathrm{mult}_P\big(D\big)}{32} > \frac{3}{11 \cdot 32} > \frac{4}{25 \cdot 32},$$

which is a contradiction. Thus, we see that $P \neq O_z$.

Suppose that $P = O_x$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{11 \cdot 25} = D \cdot L_{zt} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{11} > \frac{3}{11 \cdot 11} > \frac{2}{11 \cdot 25},$$

which is a contradiction. If $M_t \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{6}{11 \cdot 32} = D \cdot M_t \geqslant \frac{\mathrm{mult}_P\big(D\big)}{11} > \frac{3}{11 \cdot 11} > \frac{6}{11 \cdot 32},$$

which is a contradiction. Thus, we see that $P \neq O_x$.

Suppose that $P = O_y$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{11 \cdot 25} = D \cdot L_{zt} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{25} > \frac{2}{11 \cdot 25},$$

which is a contradiction. Thus, we see that $M_z \not\subseteq \mathrm{Supp}(D)$. Put $D = \epsilon L_{zt} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $L_{zt} \not\subset \mathrm{Supp}(\Omega)$. If $\epsilon \neq 0$, then

$$\frac{12}{25 \cdot 41} = D\cdot M_z = \big(\epsilon L_{zt}+\Delta\big)\cdot M_z \geqslant \epsilon L_{zt}\cdot M_z+\frac{\mathrm{mult}_{O_y}(D) - \epsilon}{25} > \epsilon L_{zt}\cdot M_z+\frac{3/11 - \epsilon}{25} = \frac{6\epsilon}{25}+\frac{3/11 - \epsilon}{25},$$

which implies that $\epsilon < 9/2255$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 34\epsilon}{11 \cdot 25} = \big(- K_X - \epsilon L_{zt}\big) \cdot L_{zt} = \Omega \cdot L_{zt} > \frac{3}{11 \cdot 25},$$

which implies that $\epsilon > 1/34$. But $\epsilon < 9/2255$. Thus, we see that $P \neq O_y$. Then $P \notin \mathrm{Sing}(X)$.

Suppose that $P \in L_{xy}$. Put $D = mL_{xy} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xy} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{4}{25 \cdot 32} = -K_X \cdot M_x = D \cdot M_x = \big(mL_{xy} + \Omega\big) \cdot M_x \geqslant mL_{xy} \cdot M_x = \frac{2m}{32},$$

which implies that $m \leqslant 2/25$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 71m}{32 \cdot 41} = \big(- K_X - mL_{xy}\big) \cdot L_{xy} = \Omega \cdot L_{xy} > \frac{3}{11},$$

which is impossible, because $m \leqslant 2/25$. Thus, we see that $P \notin L_{xy}$.

Suppose that $P \in M_x$. Put $D = \delta M_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $M_x \not\subset \mathrm{Supp}(\Upsilon)$. If $\epsilon \neq 0$, then

$$\frac{2}{32 \cdot 41} = -K_X \cdot L_{xy} = D \cdot L_{xy} = \big(\delta M_x + \Upsilon\big) \cdot L_{xy} \geqslant \delta L_{xy} \cdot M_x = \frac{2\delta}{32},$$

which implies that $\delta \leqslant 1/41$. Then it follows from Lemma 1.4.6 that

$$\frac{4 + 28\delta}{25 \cdot 32} = \big(- K_X - \delta M_x\big) \cdot M_x = \Upsilon \cdot M_x > \frac{3}{11},$$

which contradicts to $\delta \leqslant 1/41$. Similarly, we see that $P \notin M_y$, which is a contradiction. □

Lemma 3.2.19. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 25, 34, 43, 111)$. Then $\mathrm{lct}(X) = 33/8$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y + tz^2 + xy^4 + x^7z = 0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of $L_{xt} = \{x = t = 0\}$ (resp. $L_{yz} = \{y = z = 0\}$, L_{yz}, L_{xt}) and $R_x = \{x = yt + z^2 = 0\}$ (resp.

$R_y = \{y = zt + x^7 = 0\}$, $R_z = \{z = xy^3 + t^2 = 0\}$, $R_t = \{t = y^4 + x^6 z = 0\}$). Also, we see that

$$L_{xt} \cap R_x = \{O_y\},\ L_{yz} \cap R_y = \{O_t\},\ L_{yz} \cap R_z = \{O_x\},\ L_{xt} \cap R_t = \{O_z\}.$$

We can easily see that

$$\mathrm{lct}(X, \frac{2}{11}C_x) = \frac{33}{8} < \mathrm{lct}(X, \frac{2}{25}C_y),\ \ \mathrm{lct}(X, \frac{2}{34}C_z),\ \ \mathrm{lct}(X, \frac{2}{43}C_t).$$

Therefore, $\mathrm{lct}(X) \leq \frac{33}{8}$. Suppose $\mathrm{lct}(X) < \frac{33}{8}$. Then, there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the log pair $(X, \frac{33}{8}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xt}, L_{yz}, R_x, R_y, R_z, R_t are as follows:

$$D \cdot L_{xt} = \frac{1}{17 \cdot 25},\quad D \cdot R_x = \frac{4}{25 \cdot 43},\quad D \cdot R_y = \frac{14}{34 \cdot 43},$$
$$D \cdot L_{yz} = \frac{2}{11 \cdot 43},\quad D \cdot R_z = \frac{4}{11 \cdot 25},\quad D \cdot R_t = \frac{8}{11 \cdot 34},$$
$$L_{xt} \cdot R_x = \frac{2}{25},\quad L_{yz} \cdot R_y = \frac{7}{43},\quad L_{yz} \cdot R_z = \frac{2}{11},\quad L_{xt} \cdot R_t = \frac{4}{34},$$
$$L_{xt}^2 = -\frac{57}{2 \cdot 17 \cdot 25},\quad R_x^2 = -\frac{64}{25 \cdot 43},\quad R_y^2 = -\frac{63}{34 \cdot 43},$$
$$L_{yz}^2 = -\frac{52}{11 \cdot 43},\quad R_z^2 = \frac{18}{11 \cdot 25},\quad R_t^2 = \frac{64}{11 \cdot 17}.$$

By Remark 1.4.7 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$25D \cdot L_{xt} = \frac{1}{17} < \frac{8}{33},\quad 25D \cdot R_x = \frac{4}{43} < \frac{8}{33}$$

imply $P \neq O_y$. The inequalities

$$11D \cdot L_{yz} = \frac{2}{43} < \frac{8}{33},\quad 11D \cdot R_z = \frac{4}{25} < \frac{8}{33}$$

imply $P \neq O_x$. The inequalities

$$34D \cdot L_{xt} = \frac{34}{17 \cdot 25} < \frac{8}{33},\quad \frac{34}{4}D \cdot R_t = \frac{2}{11} < \frac{8}{33}$$

imply $P \neq O_z$. The curve R_t is singular at the point O_z.

We write $D = a_1 L_{xt} + a_2 L_{yz} + a_3 R_x + a_4 R_y + a_5 R_z + a_6 R_t + \Omega$, where Ω is an effective divisor whose support contains none of the curves L_{xt}, L_{yz}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{33}{8}D)$ is log canonical at the points O_x, O_y, O_z, the numbers a_i are at most $\frac{8}{33}$. Then by Lemma 1.4.8 the following inequalities enable us to conclude that either the point P is in the outside of $C_x \cup C_y \cup C_z \cup C_t$ or $P = O_t$:

$$\frac{33}{8}D \cdot L_{xt} - L_{xt}^2 = \frac{261}{8 \cdot 17 \cdot 25} < 1,\quad \frac{33}{8}D \cdot L_{yz} - L_{xt}^2 = \frac{241}{4 \cdot 11 \cdot 43} < 1,$$
$$\frac{33}{8}D \cdot R_x - R_x^2 = \frac{161}{2 \cdot 25 \cdot 43} < 1,\quad \frac{33}{8}D \cdot R_y - R_y^2 = \frac{483}{4 \cdot 34 \cdot 43} < 1,$$
$$\frac{33}{8}D \cdot R_z - R_z^2 \leq \frac{33}{8}D \cdot R_z = \frac{11}{2 \cdot 25} < 1,\quad \frac{33}{8}D \cdot R_t - R_t^2 \leq \frac{33}{8}D \cdot R_t = \frac{3}{34} < 1.$$

Suppose that $P \neq O_t$. Then we consider the pencil $\mathcal{L}$ defined by $\lambda yt + \mu z^2 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. The base locus of the pencil consists of the curve L_{yz} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P \notin C_x \cup C_y \cup C_z \cup C_t$, the divisor E is defined by the equation $z^2 = \alpha yt$, where $\alpha \neq 0$.

Suppose that $\alpha \neq -1$. Then the curve E is isomorphic to the curve defined by the equations $yt = z^2$ and $t^2 y + xy^4 + x^7 z = 0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t \neq 0$ is the curve given by

$z(z^3+xz^7+x^7)=0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{yz} and C. We have the intersection numbers

$$D\cdot C = D\cdot E - D\cdot L_{yz} = \frac{394}{11\cdot 25\cdot 43},\quad C\cdot L_{yz} = E\cdot L_{yz} - L_{yz}^2 = \frac{120}{11\cdot 43}.$$

Also, we see

$$C^2 = E\cdot C - C\cdot L_{yz} > 0.$$

By Lemma 1.4.8 the inequality $D\cdot C < \frac{8}{33}$ gives us a contradiction.

Suppose that $\alpha=-1$. Then divisor E consists of three irreducible and reduced curves L_{yz}, R_x, and M. Note that the curve M is different from the curves R_y and L_{xt}. Also, it is smooth at the point P. We have

$$D\cdot M = D\cdot E - D\cdot L_{yz} - D\cdot R_x = \frac{14}{11\cdot 43},$$

$$M^2 = E\cdot M - L_{yz}\cdot M - R_x\cdot M \geq E\cdot M - C_y\cdot M - C_x\cdot M > 0.$$

By Lemma 1.4.8 the inequality $D\cdot M < \frac{8}{33}$ gives us a contradiction. Therefore, $P=O_t$.

Put $D=bR_x+\Delta$, where Δ is an effective divisor whose support contains neither R_x. By Remark 1.4.7, we may assume that $R_x\not\subseteq \mathrm{Supp}(\Delta)$ if $b>0$. Thus, if $b>0$, then

$$\frac{2}{25\cdot 34} = D\cdot L_{xt} \geqslant bR_x\cdot L_{xt} = \frac{2b}{25},$$

which implies that $b\leqslant 1/34$. On the other hand, it follows from Lemma 1.4.6 that

$$\frac{4+64a}{25\cdot 43} = \Delta\cdot R_x > \frac{8}{33\cdot 43},$$

which implies that $b>17/528$. But $17/528>1/34$, which is a contradiction. □

Lemma 3.2.20. Suppose that $(a_0,a_1,a_2,a_3,d)=(11,43,61,113,226)$. Then $\mathrm{lct}(X)=55/12$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2+yz^3+xy^5+x^{15}z=0,$$

the surface X is singular at the point O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{55}{12} = \mathrm{lct}\left(X, \frac{2}{11}C_x\right) < \mathrm{lct}\left(X, \frac{2}{43}C_y\right) = \frac{17\cdot 43}{60},$$

which implies, in particular, that $\mathrm{lct}(X)\leqslant 55/12$.

Suppose that $\mathrm{lct}(X)<55/12$. Then there is a $\mathbb{Q}$-effective divisor $D\equiv -K_X$ such that the pair $(X,\frac{55}{12}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(671))$ contains $x^{18}y^{11}$, x^{61} and z^{11}, it follows from Lemma 1.4.10 that $P\in \mathrm{Sing}(X)\cup C_x$.

Suppose that $P\in C_x$. Then

$$\frac{4}{43\cdot 61} = D\cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{43} & \text{if } P=O_y,\\ \dfrac{\mathrm{mult}_P(D)}{61} & \text{if } P=O_z,\\ \mathrm{mult}_P(D) & \text{if } P\neq O_y \text{ and } P\neq O_z,\end{cases}$$

which is impossible, because $\mathrm{mult}_P(D)>12/55$. Thus, we see that $P=O_x$. Then

$$\frac{4}{11\cdot 61} = D\cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{11} > \frac{12}{55\cdot 11} > \frac{4}{11\cdot 61},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X)=55/12$. □

Lemma 3.2.21. Suppose that $(a_0,a_1,a_2,a_3,d)=(13,18,45,61,135)$. Then $\mathrm{lct}(X)=91/30$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + y^5 z + xt^2 + x^9 y = 0,$$

and X is singular at the point O_x, O_y and O_t. The surface X is also singular at a point Q such that $Q \neq O_y$ and the points Q and Q_y are cut out on X by the equations $x = t = 0$.

The curve C_x is reducible. We have $C_x = L_{xz} + Z_x$, where L_{xz} and Z_x are irreducible and reduced curves such that L_{xz} is given by the equations $x = z = 0$, and Z_x is given by the equations $x = z^2 + y^5 = 0$. Then

$$L_{xz} \cdot L_{xz} = \frac{-77}{18 \cdot 61},\ Z_x \cdot Z_x = \frac{-32}{9 \cdot 61},\ L_{xz} \cdot Z_x = \frac{5}{61},$$

and $L_{xz} \cap Z_x = O_t$. The curve C_y is irreducible and

$$\frac{91}{30} = \mathrm{lct}\left(X, \frac{2}{13} C_x\right) < \mathrm{lct}\left(X, \frac{2}{18} C_y\right) = \frac{15}{2},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 91/30$.

Suppose that $\mathrm{lct}(X) < 91/30$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{91}{30} D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, we may assume that either $L_{xz} \not\subseteq \mathrm{Supp}(D)$, or $Z_x \not\subseteq \mathrm{Supp}(D)$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(793))$ contains $x^7 y^{39}$, $x^{25} y^{26}$, $x^{43} y^{13}$, x^{61}, $x^{16} z^{13}$, t^{13}, it follows from Lemma 1.4.9 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = O_t$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{18 \cdot 61} = D \cdot L_{xz} \geqslant \frac{\mathrm{mult}_P(D)}{61} > \frac{30}{91 \cdot 61} > \frac{2}{18 \cdot 61},$$

which is a contradiction. If $Z_x \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{4}{18 \cdot 61} = D \cdot Z_x \geqslant \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(Z_x)}{61} = \frac{2\mathrm{mult}_P(D)}{61} > \frac{60}{91 \cdot 61} > \frac{4}{18 \cdot 61},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P \in L_{xz}$. Put $D = mL_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{4}{18 \cdot 61} = -K_X \cdot Z_x = D \cdot Z_x = \big(mL_{xz} + \Omega\big) \cdot Z_x \geqslant mL_{xz} \cdot Z_x = \frac{5m}{61},$$

which implies that $m \leqslant 2/45$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 77m}{18 \cdot 61} = \big(-K_X - mL_{xz}\big) \cdot L_{xz} = \Omega \cdot L_{xz} > \begin{cases} \dfrac{30}{91} \text{ if } P \neq O_y, \\ \dfrac{30}{91 \cdot 18} \text{ if } P = O_y, \end{cases}$$

which is impossible, because $m \leqslant 2/45$. Thus, we see that $P \notin L_{xz}$.

Suppose that $P \in Z_x$. Put $D = \epsilon Z_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_x \not\subset \mathrm{Supp}(\Delta)$. If $\epsilon \neq 0$, then

$$\frac{2}{18 \cdot 61} = -K_X \cdot L_{xz} = D \cdot L_{xz} = \big(\epsilon Z_x + \Delta\big) \cdot L_{xz} \geqslant \epsilon L_{xz} \cdot Z_x = \frac{5\epsilon}{61},$$

which implies that $\epsilon \leqslant 1/45$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 32\epsilon}{9 \cdot 61} = \big(-K_X - \epsilon Z_x\big) \cdot Z_x = \Delta \cdot Z_x > \begin{cases} \dfrac{30}{91} \text{ if } P \neq Q, \\ \dfrac{30}{91 \cdot 9} \text{ if } P = Q, \end{cases}$$

which is impossible, because $\epsilon \leqslant 1/45$. Thus, we see that $P \notin Z_x$. Then $P = O_x$. We have

$$\frac{6}{13 \cdot 61} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{13} > \frac{30}{91 \cdot 13} > \frac{6}{13 \cdot 61},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 91/30$. □

Lemma 3.2.22. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 20, 29, 47, 107)$. Then $\mathrm{lct}(X) = 65/18$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$yz^3 + y^3t + xt^2 + x^6z = 0,$$

and X is singular at the point O_x, O_y, O_z and O_t.

The curve C_x is reducible. We have $C_x = L_{xy} + M_x$, where L_{xy} and M_x are irreducible and reduced curves such that L_{xy} is given by the equations $x = y = 0$, and M_x is given by the equations $x = z^3 + y^2t = 0$. Then

$$L_{xy} \cdot L_{xy} = \frac{-74}{29 \cdot 47},\ M_x \cdot M_x = \frac{-21}{20 \cdot 47},\ L_{xy} \cdot M_x = \frac{3}{47},$$

and $L_{xy} \cap M_x = O_t$. The curve C_y is also reducible. We have $C_y = L_{xy} + M_y$, where M_y is an irreducible and reduced curve that is given by the equations $y = t^2 + x^5z = 0$. and $L_{xy} \cap M_y = O_t$. The curve C_z is also reducible. We have $C_z = L_{zt} + M_z$, where L_{zt} and M_z are irreducible and reduced curves such that L_{zt} is given by the equations $z = t = 0$, and M_z is given by the equations $z = y^3 + xt^2 = 0$. Then $L_{zt} \cap M_z = O_x$. The curve C_t is also reducible. We have $C_t = L_{zt} + M_t$, where M_t is an irreducible and reduced curve that is given by the equations $t = x^6 + yz^2 = 0$. Then

$$D \cdot L_{xy} = \frac{2}{29 \cdot 47},\ D \cdot L_{zt} = \frac{2}{13 \cdot 20},\ D \cdot M_x = \frac{6}{20 \cdot 47},$$
$$D \cdot M_y = \frac{4}{13 \cdot 19},\ D \cdot M_z = \frac{6}{13 \cdot 47},\ D \cdot M_t = \frac{12}{20 \cdot 29},$$

and the inequality then $\mathrm{lct}(X) \leqslant 65/18$ holds, because

$$\frac{65}{18} = \mathrm{lct}\left(X, \frac{2}{13}C_x\right) < \frac{70}{12} = \mathrm{lct}\left(X, \frac{2}{20}C_y\right) < \frac{145}{18} = \mathrm{lct}\left(X, \frac{2}{29}C_z\right) < \frac{82}{9} = \mathrm{lct}\left(X, \frac{2}{47}C_t\right).$$

Suppose that $\mathrm{lct}(X) < 65/18$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{65}{18}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that either $\mathrm{Supp}(D)$ does not contain at least one irreducible component of C_x, C_y, C_z and C_t.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(377))$ contains x^9y^{13}, x^{29} and z^{13}, it follows from Lemma 1.4.9 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = O_t$. If $L_{xy} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{29 \cdot 47} = D \cdot L_{xy} \geqslant \frac{\mathrm{mult}_P(D)}{47} > \frac{18}{65 \cdot 47} > \frac{2}{29 \cdot 47},$$

which is a contradiction. If $M_x \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{6}{29 \cdot 47} = D \cdot M_x \geqslant \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(M_x)}{47} = \frac{2\mathrm{mult}_P(D)}{47} > \frac{36}{65 \cdot 47} > \frac{6}{29 \cdot 47},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P = O_z$. If $L_{xy} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{6}{29 \cdot 47} = D \cdot L_{xy} \geqslant \frac{\mathrm{mult}_P(D)}{32} > \frac{18}{65 \cdot 29} > \frac{6}{29 \cdot 47},$$

which is a contradiction. If $M_y \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{4}{13 \cdot 29} = D \cdot M_y \geqslant \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(M_y)}{29} = \frac{2\mathrm{mult}_P(D)}{29} > \frac{36}{65 \cdot 29} > \frac{4}{13 \cdot 29},$$

which is a contradiction. Thus, we see that $P \neq O_z$.

Suppose that $P = O_y$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{13 \cdot 20} = D \cdot L_{zt} \geqslant \frac{\mathrm{mult}_P(D)}{20} > \frac{18}{65 \cdot 20} > \frac{2}{13 \cdot 20},$$

which is a contradiction. If $M_t \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{12}{20 \cdot 29} = D \cdot M_t \geqslant \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(M_t)}{20} = \frac{2\mathrm{mult}_P(D)}{20} > \frac{36}{65 \cdot 20} > \frac{12}{20 \cdot 29},$$

which is a contradiction. Thus, we see that $P \neq O_y$.

Suppose that $P = O_y$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{13\cdot 20} = D\cdot L_{zt} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{20} > \frac{18}{65\cdot 13} > \frac{2}{13\cdot 20},$$

which is a contradiction. If $M_z \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{6}{13\cdot 47} = D\cdot M_z \geqslant \frac{\mathrm{mult}_P\big(D\big)}{13} > \frac{18}{65\cdot 13} > \frac{6}{13\cdot 47},$$

which is a contradiction. Thus, we see that $P \neq O_x$. Then $P \notin \mathrm{Sing}(X)$.

Suppose that $P \in L_{xy}$. Put $D = mL_{xy} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xy} \not\subseteq \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{3}{10\cdot 47} = -K_X\cdot M_x = D\cdot M_x = \big(mL_{xy} + \Omega\big)\cdot M_x \geqslant mL_{xy}\cdot M_x = \frac{3m}{47},$$

which implies that $m \leqslant 1/10$. Then it follows from Lemma 1.4.6 that

$$\frac{2+74m}{29\cdot 47} = \big(-K_X - mL_{xy}\big)\cdot L_{xy} = \Omega\cdot L_{xy} > \frac{18}{65},$$

which is impossible, because $m \leqslant 1/10$. Thus, we see that $P \notin L_{xy}$.

Put $D = \delta M_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $M_x \not\subseteq \mathrm{Supp}(\Upsilon)$. If $\epsilon \neq 0$, then

$$\frac{2}{29\cdot 47} = -K_X\cdot L_{xy} = D\cdot L_{xy} = \big(\delta M_x + \Upsilon\big)\cdot L_{xy} \geqslant \delta L_{xy}\cdot M_x = \frac{3\delta}{47},$$

which implies that $\delta \leqslant 2/87$. Then it follows from Lemma 1.4.6 that

$$\frac{6+21\delta}{20\cdot 47} = \big(-K_X - \delta M_x\big)\cdot M_x = \Upsilon\cdot M_x > \frac{18}{65},$$

which contradicts to $\delta \leqslant 2/87$. The obtained contradiction shows that $\mathrm{lct}(X) = 65/18$. □

Lemma 3.2.23. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 20, 31, 49, 111)$. Then $\mathrm{lct}(X) = 65/16$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^2t + y^4z + xt^2 + x^7y = 0,$$

and X is singular at the point O_x, O_y, O_z and O_t.

The curve C_x is reducible. We have $C_x = L_{xz} + M_x$, where L_{xz} and M_x are irreducible reduced curves such that L_{xz} is given by the equations $x = z = 0$, and M_x is given by the equations $x = y^4 + zt = 0$. Then

$$L_{xz}\cdot L_{xz} = \frac{-67}{20\cdot 49},\ M_x\cdot M_x = \frac{-72}{31\cdot 49},\ L_{xz}\cdot M_x = \frac{4}{49},\ D\cdot L_{xz} = \frac{2}{20\cdot 49},\ D\cdot M_x = \frac{8}{31\cdot 49},$$

and $L_{xz}\cap M_x = O_t$. The curves C_y, C_z and C_t are also reducible. We have $C_y = L_{yt} + M_y$, where L_{yt} and M_y are irreducible reduced curves such that L_{yt} is given by the equations $y = t = 0$, and M_y is given by the equations $y = z^2 + xt = 0$. We have $C_z = L_{xz} + M_z$ and $C_t = L_{yt} + M_t$, where M_z and M_t are irreducible reduced curves such that M_z is given by the equations $z =^2 + x^6y = 0$, and M_t is given by the equations $t = x^7 + zy^3 = 0$. Then the equalities

$$D\cdot L_{yt} = \frac{2}{13\cdot 31},\ D\cdot M_y = \frac{4}{13\cdot 49},\ D\cdot M_z = \frac{4}{13\cdot 20},\ D\cdot M_t = \frac{14}{20\cdot 31},$$

holds. We have $L_{yt}\cap M_y = O_x$, $L_{xz}\cap M_z = O_y$ and $L_{yt}\cap M_t = O_z$. Then $\mathrm{lct}(X) \leqslant 65/16$, because

$$\frac{65}{16} = \mathrm{lct}\left(X, \frac{2}{13}C_x\right) < \frac{30}{4} = \mathrm{lct}\left(X, \frac{2}{20}C_y\right) < \frac{245}{28} = \mathrm{lct}\left(X, \frac{2}{49}C_t\right) < \frac{62}{7} = \mathrm{lct}\left(X, \frac{2}{31}C_z\right).$$

Suppose that $\mathrm{lct}(X) < 65/16$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{65}{16}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that either $\mathrm{Supp}(D)$ does not contain at least one irreducible component of C_x, C_y, C_z and C_t.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(403))$ contains $x^{11}y^{13}$, x^{31} and z^{13}, it follows from Lemma 1.4.9 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = O_x$. If $L_{yt} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{13 \cdot 31} = D \cdot L_{yt} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{13} > \frac{16}{65 \cdot 13} > \frac{2}{13 \cdot 31} =,$$

which is a contradiction. If $M_y \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{4}{13 \cdot 49} = D \cdot M_y \geqslant \frac{\mathrm{mult}_P\big(D\big)}{13} > \frac{16}{65 \cdot 13} > \frac{4}{13 \cdot 49},$$

which is a contradiction. Thus, we see that $P \neq O_x$.

Suppose that $P = O_y$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{20 \cdot 49} = D \cdot L_{xz} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{20} > \frac{16}{65 \cdot 20} > \frac{2}{20 \cdot 49},$$

which is a contradiction. If $M_z \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{4}{13 \cdot 20} = D \cdot M_x \geqslant\geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(M_z\big)}{20} = \frac{2\mathrm{mult}_P\big(D\big)}{20} > \frac{32}{65 \cdot 20} > \frac{4}{13 \cdot 20},$$

which is a contradiction. Thus, we see that $P \neq O_y$.

Suppose that $P = O_z$. If $L_{yt} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{13 \cdot 31} = D \cdot L_{yt} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{31} > \frac{16}{65 \cdot 31} > \frac{2}{13 \cdot 31},$$

which is a contradiction. If $M_t \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{14}{20 \cdot 31} = D \cdot M_t \geqslant\geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P\big(M_t\big)}{20} = \frac{3\mathrm{mult}_P\big(D\big)}{31} > \frac{48}{65 \cdot 20} > \frac{14}{20 \cdot 31},$$

which is a contradiction. Thus, we see that $P \neq O_z$.

Suppose that $P \in M_x \setminus O_t$. Put $D = \delta M_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $M_x \not\subseteq \mathrm{Supp}(\Upsilon)$. If $\epsilon \neq 0$, then

$$\frac{2}{20 \cdot 49} = -K_X \cdot L_{xz} = D \cdot L_{xz} = \big(\delta M_x + \Upsilon\big) \cdot L_{xz} \geqslant \delta L_{xz} \cdot M_x = \frac{4\delta}{49},$$

which implies that $\delta \leqslant 1/40$. Then it follows from Lemma 1.4.6 that

$$\frac{8 + 72\delta}{31 \cdot 49} = \big(-K_X - \delta M_x\big) \cdot M_x = \Upsilon \cdot M_x > \frac{16}{65},$$

because $P \neq O_z$. But $\delta \leqslant 1/40$. Thus, we see that $M \notin M_x \setminus O_t$.

We see that $P \in L_{xz}$ and $P \neq O_y$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{2}{20 \cdot 49} = D \cdot L_{xz} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{49} > \frac{16}{65 \cdot 49} > \frac{2}{20 \cdot 49},$$

which is a contradiction. Thus, we see that $M_x \not\subseteq \mathrm{Supp}(D)$. Put $D = \epsilon L_{xz} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subseteq \mathrm{Supp}(\Omega)$. Then

$$\frac{8}{31 \cdot 49} = D \cdot M_x = \big(\epsilon L_{xz} + \Delta\big) \cdot M_x \geqslant \epsilon L_{xz} \cdot M_x = \frac{4\epsilon}{49},$$

which implies that $\epsilon \leqslant 2/31$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 67\epsilon}{20 \cdot 49} = \big(-K_X - \epsilon L_{xz}\big) \cdot L_{xz} = \Omega \cdot L_{xz} >> \begin{cases} \dfrac{16}{65} \text{ if } P \neq O_t, \\ \dfrac{16}{65 \cdot 49} \text{ if } P = O_t, \end{cases}$$

which implies that $\epsilon > 38/871$ and $P = O_t$, because $\epsilon \leqslant 2/31$. Then

$$\frac{8}{31 \cdot 49} = D \cdot M_x = \big(\epsilon L_{xz} + \Delta\big) \cdot M_x \geqslant \epsilon L_{xz} \cdot M_x + \frac{\mathrm{mult}_{O_t}(D) - \epsilon}{49} > \epsilon L_{xz} \cdot M_x + \frac{16/65 - \epsilon}{49} = \frac{4\epsilon}{49} + \frac{16/65 - \epsilon}{49},$$

which implies that $\epsilon < 8/2015$. But $\epsilon > 38/871 > 8/2015$, which is a contradiction. □

Lemma 3.2.24. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 31, 71, 113, 226)$. Then $\mathrm{lct}(X) = 91/20$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^5 z + xz^3 + x^{15} y = 0,$$

the surface X is singular at the point O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{91}{20} = \mathrm{lct}\left(X, \frac{2}{13}C_x\right) < \mathrm{lct}\left(X, \frac{2}{31}C_y\right) = \frac{17\cdot 71}{60},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 91/20$.

Suppose that $\mathrm{lct}(X) < 91/20$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{91}{20}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(923))$ contains x^{71}, $y^{26}x^9$, $y^{13}x^{40}$ and z^{13}, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P \in C_x$. Then

$$\frac{4}{31\cdot 71} = D\cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{31} & \text{if } P = O_y,\\ \dfrac{\mathrm{mult}_P(D)}{71} & \text{if } P = O_z,\\ \mathrm{mult}_P(D) & \text{if } P \neq O_y \text{ and } P \neq O_z,\end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 20/91$. Thus, we see that $P = O_x$. Then

$$\frac{4}{13\cdot 71} = D\cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{13} > \frac{20}{91\cdot 13} > \frac{4}{13\cdot 71},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 91/20$. □

Lemma 3.2.25. Suppose that $(a_0, a_1, a_2, a_3, d) = (14, 17, 29, 41, 99)$. Then $\mathrm{lct}(X) = 21/4$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2 y + tz^2 + xy^5 + x^5 z = 0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of $L_{xt} = \{x = t = 0\}$ (resp. $L_{yz} = \{y = z = 0\}$, L_{yz}, L_{xt}) and $R_x = \{x = yt + z^2 = 0\}$ (resp. $R_y = \{y = zt + x^5 = 0\}$, $R_z = \{z = xy^4 + t^2 = 0\}$, $R_t = \{t = y^5 + x^4 z = 0\}$). Also, we see that

$$L_{xt} \cap R_x = \{O_y\},\ L_{yz} \cap R_y = \{O_t\},\ L_{yz} \cap R_z = \{O_x\},\ L_{xt} \cap R_t = \{O_z\}.$$

We can easily see that

$$\mathrm{lct}(X, \frac{2}{14}C_x) = \frac{21}{4} < \mathrm{lct}(X, \frac{2}{17}C_y),\ \ \mathrm{lct}(X, \frac{2}{29}C_z),\ \ \mathrm{lct}(X, \frac{2}{41}C_t).$$

Therefore, $\mathrm{lct}(X) \leq \frac{21}{4}$. Suppose $\mathrm{lct}(X) < \frac{21}{4}$. Then, there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the log pair $(X, \frac{21}{4}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xt}, L_{yz}, R_x, R_y, R_z, R_t are as follows:

$$D\cdot L_{xt} = \frac{2}{17\cdot 29},\quad D\cdot R_x = \frac{4}{17\cdot 41},\quad D\cdot R_y = \frac{10}{29\cdot 41},$$

$$D\cdot L_{yz} = \frac{1}{7\cdot 41},\quad D\cdot R_z = \frac{2}{7\cdot 17},\quad D\cdot R_t = \frac{5}{7\cdot 29},$$

$$L_{xt}\cdot R_x = \frac{2}{17},\quad L_{yz}\cdot R_y = \frac{5}{41},\quad L_{yz}\cdot R_z = \frac{1}{7},\quad L_{xt}\cdot R_t = \frac{5}{29},$$

$$L_{xt}^2 = -\frac{44}{17\cdot 29},\quad R_x^2 = -\frac{54}{17\cdot 41},\quad R_y^2 = -\frac{60}{29\cdot 41},$$

$$L_{yz}^2 = -\frac{53}{14\cdot 41},\quad R_z^2 = \frac{12}{7\cdot 17},\quad R_t^2 = \frac{135}{14\cdot 29}.$$

By Remark 1.4.7 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$17D\cdot L_{xt}=\frac{2}{29}<\frac{4}{21},\quad 17D\cdot R_x=\frac{4}{41}<\frac{4}{21}$$

imply $P\neq O_y$. The inequalities

$$14D\cdot L_{yz}=\frac{2}{41}<\frac{4}{21},\quad 7D\cdot R_z=\frac{2}{17}<\frac{4}{21}$$

imply $P\neq O_x$. The curve R_z is singular at the point O_x. The inequalities

$$29D\cdot L_{xt}=\frac{2}{17}<\frac{4}{21},\quad \frac{29}{4}D\cdot R_t=\frac{5}{28}<\frac{4}{21}$$

imply $P\neq O_z$. The curve R_t is singular at the point O_z.

We write $D=a_1L_{xt}+a_2L_{yz}+a_3R_x+a_4R_y+a_5R_z+a_6R_t+\Omega$, where Ω is an effective divisor whose support contains none of the curves L_{xt}, L_{yz}, R_x, R_y, R_z, R_t. Since the pair $(X,\frac{21}{4}D)$ is log canonical at the points O_x, O_y, O_z, the numbers a_i are at most $\frac{4}{21}$. Then by Lemma 1.4.8 the following inequalities enable us to conclude that either the point P is in the outside of $C_x\cup C_y\cup C_z\cup C_t$ or $P=O_t$:

$$\frac{21}{4}D\cdot L_{xt}-L_{xt}^2=\frac{109}{2\cdot 17\cdot 29}<1,\quad \frac{21}{4}D\cdot L_{yz}-L_{xt}^2=\frac{127}{4\cdot 7\cdot 41}<1,$$

$$\frac{21}{4}D\cdot R_x-R_x^2=\frac{75}{17\cdot 41}<1,\quad \frac{21}{4}D\cdot R_y-R_y^2=\frac{225}{2\cdot 29\cdot 41}<1,$$

$$\frac{21}{4}D\cdot R_z-R_z^2\leq\frac{21}{4}D\cdot R_z=\frac{3}{2\cdot 17}<1,\quad \frac{21}{4}D\cdot R_t-R_t^2\leq\frac{21}{4}D\cdot R_t=\frac{15}{4\cdot 29}<1.$$

Suppose that $P\neq O_t$. Then we consider the pencil $\mathcal{L}$ defined by $\lambda yt+\mu z^2=0$, $[\lambda:\mu]\in\mathbb{P}^1$. The base locus of the pencil consists of the curve L_{yz} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P\notin C_x\cup C_y\cup C_z\cup C_t$, the divisor E is defined by the equation $z^2=\alpha yt$, where $\alpha\neq 0$.

Suppose that $\alpha\neq -1$. Then the curve E is isomorphic to the curve defined by the equations $yt=z^2$ and $t^2y+xy^5+x^5z=0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t\neq 0$ is the curve given by $z(z+xz^9+x^5)=0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{yz} and C. We have the intersection number

$$D\cdot C=D\cdot E-D\cdot L_{yz}=\frac{181}{7\cdot 17\cdot 41}.$$

Also, we see

$$C^2=E\cdot C-C\cdot L_{yz}\geq E\cdot C-C_y\cdot C>0$$

since C is different from R_y. By Lemma 1.4.8 the inequality $D\cdot C<\frac{4}{21}$ gives us a contradiction.

Suppose that $\alpha=-1$. Then divisor E consists of three irreducible and reduced curves L_{yz}, R_x, and M. Note that the curve M is different from the curves R_y and L_{xt}. Also, it is smooth at the point P. We have

$$D\cdot M=D\cdot E-D\cdot L_{yz}-D\cdot R_x=\frac{153}{7\cdot 17\cdot 41},$$

$$M^2=E\cdot M-L_{yz}\cdot M-R_x\cdot M\geq E\cdot M-C_y\cdot M-C_x\cdot M>0.$$

By Lemma 1.4.8 the inequality $D\cdot M<\frac{4}{21}$ gives us a contradiction. Therefore, $P=O_t$.

Put $D=aL_{yz}+bR_x+\Delta$, where Δ is an effective divisor whose support contains neither L_{yz} nor R_x. Then $a>0$, because otherwise

$$\frac{2}{14\cdot 41}=D\cdot L_{yz}=\geqslant \mathrm{mult}_P(D)41>\frac{4}{21\cdot 41}>\frac{2}{14\cdot 41},$$

which is a contradiction. Therefore, we may assume that $R_y\not\subseteq\mathrm{Supp}(\Delta)$ by Remark 1.4.7. Similarly, we may assume that $L_{xt}\not\subseteq\mathrm{Supp}(\Delta)$ if $b>0$.

Let us find upper bounds for a and b. If $b > 0$, then

$$\frac{2}{17\cdot 29} = D\cdot L_{xt} \geqslant bR_x\cdot L_{xt} = \frac{2b}{17},$$

which implies that $b \leqslant 1/29$. Similarly, we have

$$\frac{10}{29\cdot 41} = D\cdot R_y \geqslant \frac{7a}{41} + \frac{b}{41} + \frac{\mathrm{mult}_{O_t}(D) - a - b}{41} > \frac{6a + \frac{4}{21}}{41},$$

which implies that $a < 47/1827$. On the other hand, it follows from Lemma 1.4.6 that

$$\frac{2+53a}{14\cdot 41} = \Delta\cdot L_{yz} > \frac{4/21 - b}{41},$$

which implies that $a > 2/159$.

Let $\pi\colon \bar{X} \to X$ be the weighted blow up of the point O_t with weight $(9,4)$, and let F be the exceptional curve of the morphism π. Then F contains two singular points Q_9 and Q_4 such that Q_9 is a singular point of type $\frac{1}{9}(1,1)$, and Q_4 is a singular point of type $\frac{1}{4}(1,3)$. Then

$$K_{\bar{X}} = \pi^*(K_X) - \frac{38}{41}F,\ \bar{L}_{yz} = \pi^*(L_{yz}) - \frac{4}{41}F,\ \bar{R}_x = \pi^*(R_x) - \frac{9}{41}F,\ \bar{R}_y = \pi^*(R_y) - \frac{4}{41}F,\ \bar{\Delta} = \pi^*(\Delta) - \frac{c}{41}F,$$

where $\bar{L}_{yz}$, $\bar{R}_x$, $\bar{R}_y$ and $\bar{\Delta}$ are the proper transforms of L_{yz}, R_x, R_y and Δ by π, respectively, and c is a non-negative rational number c. Note that $F\cap \bar{R}_x = Q_4$ and $F\cap \bar{L}_{yz} = Q_9$.

The log pull-back of the log pair $(X, \frac{21}{4}D)$ by π is the log pair

$$\left(\bar{X},\ \frac{21a}{4}\bar{L}_{yz} + \frac{21b}{4}\bar{R}_x + \frac{21}{4}\bar{\Delta} + \theta_1 F\right),$$

which is not log canonical at some point $Q\in F$, where $\theta_1 = (21(c+4a+9b)/4 + 28)/41$. We have

$$\frac{2+53a}{14\cdot 41} - \frac{b}{41} - \frac{c}{9\cdot 41} = \bar{\Delta}\cdot\bar{L}_{yz} \geqslant 0 \leqslant \bar{\Delta}\cdot\bar{R}_x = \frac{4+54b}{17\cdot 41} - \frac{a}{41} - \frac{c}{4\cdot 41},$$

which implies that $\theta_1 < 1$, because $b < 1/29$. Similarly, we see that

$$0 \leqslant \bar{\Delta}\cdot\bar{R}_y = \frac{10}{29\cdot 41} - \frac{7a}{41} - \frac{b}{41} - \frac{c}{9\cdot 41}.$$

Suppose that $Q \notin \bar{R}_x \cup \bar{L}_{yz}$. Then

$$\frac{21c}{16\cdot 9} = \frac{21}{4}\bar{\Delta}\cdot F > 1$$

by Lemma 1.4.6. Thus, we see that $c > 48/7$. But the system of inequalities

$$\begin{cases} \dfrac{2+53a}{14\cdot 41} - \dfrac{b}{41} - \dfrac{c}{9\cdot 41} \geqslant 0, \\ \dfrac{4+54b}{17\cdot 41} - \dfrac{a}{41} - \dfrac{c}{4\cdot 41} \geqslant 0, \quad b \leqslant 1/29, \\ c > 48/7, \end{cases}$$

is inconsistent. Thus, we see that $Q \in \bar{R}_x \cup \bar{L}_{yz}$.

Suppose that $Q \in \bar{M}_x$. Then $Q = Q_4$, and it follows from Lemma 1.4.6 that

$$\frac{21}{4}\left(\frac{4+54b}{17\cdot 41} - \frac{a}{41} - \frac{c}{4\cdot 41}\right) + \frac{\theta_1}{4} = \left(\frac{21}{4}\bar{\Delta} + \theta_1 F\right)\cdot\bar{M}_x > \frac{1}{4} < \left(\frac{21}{4}\bar{\Delta} + \frac{21b}{4}\bar{M}_x\right)\cdot F = \frac{21}{4}\left(\frac{c}{4\cdot 9} + \frac{b}{4}\right)$$

which implies that $b > 548/7749$. But $b < 1/29$, which is a contradiction.

We see that $Q = Q_9$. Then it follows from Lemma 1.4.6 that

$$\frac{21}{4}\left(\frac{2+53a}{14\cdot 41} - \frac{b}{41} - \frac{c}{9\cdot 41}\right) + \frac{\theta_1}{9} = \left(\frac{21}{4}\bar{\Delta} + \theta_1 F\right)\cdot\bar{L}_{yz} > \frac{1}{9} < \left(\frac{21}{4}\bar{\Delta} + \frac{21a}{4}\bar{L}_{yz}\right)\cdot F = \frac{21}{4}\left(\frac{c}{4\cdot 9} + \frac{a}{9}\right),$$

which leads to a contradiction, because the system of inequalities

$$\begin{cases}\frac{21}{4}\Big(\frac{c}{4\cdot 9}+\frac{a}{9}\Big)>\frac{1}{9},\\ \frac{21}{4}\Big(\frac{2+53a}{14\cdot 41}-\frac{b}{41}-\frac{c}{9\cdot 41}\Big)+\frac{\theta_1}{9}>\frac{1}{9},\\ \frac{2+53a}{14\cdot 41}-\frac{b}{41}-\frac{c}{9\cdot 41}\geqslant 0,\\ \frac{4+54b}{17\cdot 41}-\frac{a}{41}-\frac{c}{4\cdot 41}\geqslant 0,\\ a<47/1827,\\ b\leqslant 1/29,\end{cases}$$

is inconsistent. The obtained contradiction completes the proof. □

3.3. Sporadic cases with $I = 3$

Lemma 3.3.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (5, 7, 11, 13, 33)$. Then $\mathrm{lct}(X) = 49/36$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + yt^2 + xy^4 + x^4t + \epsilon x^3yz = 0,$$

where $\epsilon \in \mathbb{C}$. Note that X is singular at O_x, O_y and O_t.

The curves C_x and C_y are irreducible. Moreover, we have

$$\frac{25}{18} = \mathrm{lct}(X, \frac{3}{5}C_x) > \mathrm{lct}(X, \frac{3}{7}C_y) = \frac{49}{36},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 49/36$.

Suppose that $\mathrm{lct}(X) < 49/36$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{49}{36}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of D does not contain the curves C_x and C_y.

Suppose that $P \in C_x$ and $P \notin \mathrm{Sing}(X)$. Then

$$\frac{36}{49} < \mathrm{mult}_P(D) \leqslant D \cdot C_x = \frac{9}{91} < \frac{36}{49},$$

which is a contradiction. Suppose that $P \in C_y$ and $P \notin \mathrm{Sing}(X)$. Then

$$\frac{36}{49} < \mathrm{mult}_P(D) \leqslant D \cdot C_y = \frac{9}{65} < \frac{36}{49},$$

which is a contradiction. Suppose that $P = O_x$. Then

$$\frac{36}{49}\frac{1}{5} < \frac{\mathrm{mult}_{O_x}(D)}{5} \leqslant D \cdot C_y = \frac{9}{65} < \frac{36}{49}\frac{1}{5},$$

which is a contradiction. Suppose that $P = O_t$. Then

$$\frac{36}{49}\frac{3}{13} < \frac{3\mathrm{mult}_{O_t}(D)}{13} = \frac{\mathrm{mult}_{O_t}(D)\mathrm{mult}_{O_t}(C_y)}{13} \leqslant D \cdot C_y = \frac{9}{65} < \frac{36}{49}\frac{3}{13},$$

which is a contradiction. Suppose that $P = O_y$. Then

$$\frac{36}{49}\frac{1}{7} < \frac{\mathrm{mult}_{O_y}(D)}{7} \leqslant D \cdot C_x = \frac{9}{91} < \frac{36}{49}\frac{1}{7},$$

which is a contradiction. Thus, we see that $P \in X \setminus \mathrm{Sing}(X)$ and $P \notin C_x \cup C_y$.

Let $\mathcal{L}$ be the pencil on X that is cut out by the pencil

$$\lambda x^7 + \mu y^5 = 0,$$

where $[\lambda : \mu] \in \mathbb{P}^1$. Then the base locus of the pencil $\mathcal{L}$ consists of the point O_t.

Let C be the unique curve in $\mathcal{L}$ that passes through the point P. Suppose that C is irreducible and reduced. Then $\mathrm{mult}_P(C) \leqslant 3$, because C is a triple cover of the curve

$$\lambda x^7 + \mu y^5 = 0 \subset \mathbb{P}(5, 7, 13) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, t\big]\Big)$$

such that $\lambda \neq 0$ and $\mu \neq 0$. In particular, the log pair $(X, \frac{3}{35}C)$ is log canonical. Thus, we may assume that the support of D does not contain the curve C and hence we obtain

$$\frac{10}{13} < \mathrm{mult}_P(D) \leqslant D \cdot C = \frac{9}{13} < \frac{10}{13},$$

which is a contradiction. Thus, to conclude the proof we must prove that C is irreducible and reduced.

Let $S \subset \mathbb{C}^4$ be an affine subscheme that is given by the equations

$$y^5 - \alpha x^7 = z^3 + yt^2 + xy^4 + x^4 t + \epsilon x^3 yz = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

where $\epsilon \in \mathbb{C}$ and $\alpha \in \mathbb{C}^4$ such that $\alpha \neq 0$. To conclude the proof, it is enough to prove that the subscheme S is an irreducible. For simplicity, we treat S as a surface in $\mathbb{C}^4$.

Let $\bar{S} \subset \mathbb{P}^4$ be a natural compactification of the surface $S \subset \mathbb{C}^4$ that is given by the equations

$$\bar{y}^5\bar{w}^2 - \alpha\bar{x}^7 = \bar{z}^3\bar{w}^2 + \bar{y}\bar{t}^2\bar{w}^2 + \bar{x}\bar{y}^4 + \bar{x}^4\bar{t} + \epsilon\bar{x}^3\bar{y}\bar{z} = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\Big(\mathbb{C}\big[\bar{x}, \bar{y}, \bar{z}, \bar{t}, \bar{w}\big]\Big),$$

and let $\bar{H}$ be a surface in $\mathbb{P}^4$ that is given by the equations $\bar{x} = \bar{w} = 0$. Then

$$\mathrm{Supp}(\bar{S}) = \mathrm{Supp}(\bar{S}') \cup \bar{H},$$

where $\bar{S}'$ is another compactification of the affine surface S. Then S is irreducible $\iff$ $\bar{S}'$ is irreducible.

Let $\bar{T}$ be be a hyperplane in $\mathbb{P}^4$ that is given by the equation $\bar{y} = 0$. Then the intersection $\bar{T} \cap \bar{S}$ is one-dimensional. Consider an affine open subset $U = \mathbb{P}^4 \setminus \bar{T} \subset \mathbb{P}^4$. Put $\breve{S}' = U \cap \bar{S}'$, $\breve{S} = U \cap \bar{S}$ and $\breve{H} = U \cap \bar{H}$. Then S is irreducible $\iff$ $\breve{S}'$ is irreducible.

The surface $\breve{S}$ can be given by the equations

$$\breve{w}^2 - \alpha\breve{x}^7 = \breve{z}^3\breve{w}^2 + \breve{t}^2\breve{w}^2 + \breve{x} + \breve{x}^4\breve{t} + \epsilon\breve{x}^3\breve{z} = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[\breve{x}, \breve{z}, \breve{t}, \breve{w}\big]\Big),$$

where $\breve{H}$ is given by $\breve{x} = \breve{w} = 0$. Therefore, the surface $\breve{S}$ is isomorphic to an affine hypersurface

$$\alpha\breve{x}^7\breve{z}^3 + \alpha\breve{x}^7\breve{t}^2 + \breve{x} + \breve{x}^4\breve{t} + \epsilon\breve{x}^3\breve{z} = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[\breve{x}, \breve{z}, \breve{t}\big]\Big),$$

where $\breve{H}$ is given by $\breve{x} = 0$. Thus, we see that the surface $\breve{S}'$ is a hypersurface in $\mathbb{C}^3$ that is given by the zeroes of the polynomial

$$f(\breve{x}, \breve{z}, \breve{t}) = \alpha\breve{x}^6\breve{z}^3 + \alpha\breve{x}^6\breve{t}^2 + 1 + \breve{x}^3\breve{t} + \epsilon\breve{x}^2\breve{z},$$

which implies that S is irreducible $\iff$ the polynomial $f(\breve{x}, \breve{z}, \breve{t})$ is irreducible. But elementary calculations imply that the polynomial $f(\breve{x}, \breve{z}, \breve{t})$ is irreducible. □

Lemma 3.3.2. Suppose that $(a_0, a_1, a_2, a_3, d) = (5, 7, 11, 20, 40)$. Then $\mathrm{lct}(X) = 25/18$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^5 + x^4 t + x^8 + \epsilon x^3 y^2 z,$$

where $\epsilon \in \mathbb{C}$. Note that X is singular at the points O_y and O_z. The surface X also has two singular points P_1 and P_2 of type $\frac{1}{5}(2, 1)$ that are cut out on X by the equations $y = z = 0$.

The curve C_x is irreducible. We have

$$\mathrm{lct}(X, \frac{3}{5}C_x) = \frac{25}{18},$$

which implies that $\mathrm{lct}(X) \leqslant 49/36$. The curve C_y is reducible. We have $C_y = C_1 + C_2$, where C_1 and C_2 are irreducible reduced curves such that

$$C_1 \cdot C_1 = C_2 \cdot C_2 = -\frac{13}{55},\ C_1 \cdot C_2 = \frac{4}{11},$$

and $P_1 \in C_1$, $P_2 \in C_2$. Then $C_1 \cap C_2 = O_z$.

Suppose that $\mathrm{lct}(X) < 25/18$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{25}{18}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of D does not contain the curve C_x. Moreover, we may assume that the support of D does not contain either the curve C_1 or the curve C_2, because

$$\mathrm{lct}(X, \frac{3}{7}C_x) = \frac{35}{24} > \frac{25}{18}.$$

Suppose that $P \in C_x$. Then

$$\frac{18}{25} > \frac{18}{25}\frac{1}{7} > \frac{6}{77} = D \cdot C_x \geqslant \begin{cases} \mathrm{mult}_P\big(D\big) \text{ if } P \in X \setminus \mathrm{Sing}(X), \\ \dfrac{\mathrm{mult}_{O_y}\big(D\big)}{7} \text{ if } P = O_y, \end{cases} > \begin{cases} \dfrac{18}{25} \text{ if } P \in X \setminus \mathrm{Sing}(X), \\ \dfrac{18}{25}\dfrac{1}{7} \text{ if } P = O_y, \end{cases}$$

which is a contradiction. Thus, we see that $P \notin C_x$.

Suppose that $P = O_z$. We know that $C_i \not\subset \mathrm{Supp}(D)$ for some $i = 1, 2$. Then

$$\frac{18}{25}\frac{1}{11} < \frac{\mathrm{mult}_{O_z}(D)}{11} \leqslant D \cdot C_i = \frac{3}{55} < \frac{18}{25}\frac{1}{11},$$

which is a contradiction. Therefore, we see that $P \neq O_z$.

Suppose that $P \in C_1$. Put $D = mC_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $C_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{3}{55} = -K_X \cdot C_2 = D \cdot C_2 = \big(mC_1 + \Omega\big) \cdot C_2 \geqslant mC_1 \cdot C_2 = \frac{4m}{11},$$

which implies that $m \leqslant 3/20$. Then it follows from Lemma 1.4.6 that

$$\frac{3 + m13}{55} = \big(-K_X - mC_1\big) \cdot C_1 = \Omega \cdot C_1 > \begin{cases} \dfrac{18}{25} \text{ if } P \neq P_1, \\ \dfrac{18}{25}\dfrac{1}{5} \text{ if } P = P_1, \end{cases}$$

because $P \neq O_z$. Thus, we see that $m > 123/325$, which is impossible, because $m \leqslant 3/20$.

Thus, we see that $P \in X \setminus \mathrm{Sing}(X)$ and $P \notin C_x \cup C_y$. Then

$$\frac{18}{25} < \mathrm{mult}_P\big(D\big) \leqslant \frac{240}{385} < \frac{18}{25}$$

by Lemma 1.4.10, because the natural projection $X \dashrightarrow \mathbb{P}(5, 7, 20)$ is a finite morphism outside of the curve C_y, and $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(40))$ contains monomials x^8, xy^5, x^4t. The obtained contradiction completes the proof. □

Lemma 3.3.3. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 21, 29, 37, 95)$. Then $\mathrm{lct}(X) = 11/4$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y + tz^2 + xy^4 + x^6z = 0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of $L_{xt} = \{x = t = 0\}$ (resp. $L_{yz} = \{y = z = 0\}$, L_{yz}, L_{xt}) and $R_x = \{x = yt + z^2 = 0\}$ (resp. $R_y = \{y = zt + x^6 = 0\}$, $R_z = \{z = xy^3 + t^2 = 0\}$, $R_t = \{t = y^4 + x^5z = 0\}$). Also, we see that

$$L_{xt} \cap R_x = \{O_y\},\ L_{yz} \cap R_y = \{O_t\},\ L_{yz} \cap R_z = \{O_x\},\ L_{xt} \cap R_t = \{O_z\}.$$

We can easily see that

$$\mathrm{lct}(X, \frac{3}{11}C_x) = \frac{11}{4} < \mathrm{lct}(X, \frac{3}{21}C_y),\ \ \mathrm{lct}(X, \frac{3}{29}C_z),\ \ \mathrm{lct}(X, \frac{3}{37}C_t).$$

Therefore, $\mathrm{lct}(X) \leq \frac{11}{4}$. Suppose $\mathrm{lct}(X) < \frac{11}{4}$. Then, there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the log pair $(X, \frac{11}{4}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xt}, L_{yz}, R_x, R_y, R_z, R_t are as follows:

$$D \cdot L_{xt} = \frac{1}{7 \cdot 29},\quad D \cdot R_x = \frac{2}{7 \cdot 37},\quad D \cdot R_y = \frac{18}{29 \cdot 37},$$
$$D \cdot L_{yz} = \frac{3}{11 \cdot 37},\quad D \cdot R_z = \frac{2}{7 \cdot 11},\quad D \cdot R_t = \frac{12}{11 \cdot 29},$$

$$L_{xt}\cdot R_x=\frac{2}{21},\quad L_{yz}\cdot R_y=\frac{6}{37},\quad L_{yz}\cdot R_z=\frac{2}{11},\quad L_{xt}\cdot R_t=\frac{4}{29},$$

$$L_{xt}^2=-\frac{47}{21\cdot 29},\quad R_x^2=-\frac{52}{21\cdot 37},\quad R_y^2=-\frac{48}{29\cdot 37},$$

$$L_{yz}^2=-\frac{45}{11\cdot 37},\quad R_z^2=\frac{16}{11\cdot 21},\quad R_t^2=\frac{104}{11\cdot 29}.$$

By Remark 1.4.7 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$21D\cdot L_{xt}=\frac{3}{29}<\frac{4}{11},\quad 17D\cdot R_x=\frac{6}{37}<\frac{4}{11}$$

imply $P\neq O_y$. The inequalities

$$11D\cdot L_{yz}=\frac{3}{37}<\frac{4}{11},\quad \frac{11}{2}D\cdot R_z=\frac{1}{7}<\frac{4}{11}$$

imply $P\neq O_x$. The curve R_z is singular at the point O_x. The inequalities

$$29D\cdot L_{xt}=\frac{1}{7}<\frac{4}{11},\quad \frac{29}{4}D\cdot R_t=\frac{3}{11}<\frac{4}{11}$$

imply $P\neq O_z$. The curve R_t is singular at the point O_z.

We write $D=a_1L_{xt}+a_2L_{yz}+a_3R_x+a_4R_y+a_5R_z+a_6R_t+\Omega$, where Ω is an effective divisor whose support contains none of the curves L_{xt}, L_{yz}, R_x, R_y, R_z, R_t. Since the pair $(X,\frac{11}{4}D)$ is log canonical at the points O_x, O_y, O_z, the numbers a_i are at most $\frac{4}{11}$. Then by Lemma 1.4.8 the following inequalities enable us to conclude that either the point P is in the outside of $C_x\cup C_y\cup C_z\cup C_t$ or $P=O_t$:

$$\frac{11}{4}D\cdot L_{xt}-L_{xt}^2=\frac{221}{3\cdot 4\cdot 7\cdot 29}<1,\quad \frac{11}{4}D\cdot L_{yz}-L_{xt}^2=\frac{214}{4\cdot 11\cdot 37}<1,$$

$$\frac{11}{4}D\cdot R_x-R_x^2=\frac{137}{2\cdot 3\cdot 7\cdot 37}<1,\quad \frac{11}{4}D\cdot R_y-R_y^2=\frac{195}{2\cdot 29\cdot 37}<1,$$

$$\frac{11}{4}D\cdot R_z-R_z^2\leq\frac{11}{4}D\cdot R_z=\frac{1}{14}<1,\quad \frac{11}{4}D\cdot R_t-R_t^2\leq\frac{11}{4}D\cdot R_t=\frac{3}{29}<1.$$

Suppose that $P\neq O_t$. Then we consider the pencil $\mathcal{L}$ defined by $\lambda yt+\mu z^2=0$, $[\lambda:\mu]\in\mathbb{P}^1$. The base locus of the pencil consists of the curve L_{yz} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P\notin C_x\cup C_y\cup C_z\cup C_t$, the divisor E is defined by the equation $z^2=\alpha yt$, where $\alpha\neq 0$.

Suppose that $\alpha\neq -1$. Then the curve E is isomorphic to the curve defined by the equations $yt=z^2$ and $t^2y+xy^4+x^6z=0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t\neq 0$ is the curve given by $z(z+xz^7+x^6)=0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{yz} and C. We have the intersection number

$$D\cdot C=D\cdot E-D\cdot L_{yz}=\frac{169}{7\cdot 11\cdot 37}.$$

Also, we see

$$C^2=E\cdot C-C\cdot L_{yz}\geq E\cdot C-C_y\cdot C>0$$

since C is different from R_y. By Lemma 1.4.8 the inequality $D\cdot C<\frac{4}{11}$ gives us a contradiction.

Suppose that $\alpha=-1$. Then divisor E consists of three irreducible and reduced curves L_{yz}, R_x, and M. Note that the curve M is different from the curves R_y and L_{xt}. Also, it is smooth at the point P. We have

$$D\cdot M=D\cdot E-D\cdot L_{yz}-D\cdot R_x=\frac{147}{7\cdot 11\cdot 37},$$

$$M^2=E\cdot M-L_{yz}\cdot M-R_x\cdot M\geq E\cdot M-C_y\cdot M-C_x\cdot M>0.$$

By Lemma 1.4.8 the inequality $D\cdot M<\frac{4}{11}$ gives us a contradiction. Therefore, $P=O_t$.

Put $D = aL_{yz} + bR_x + \Delta$, where Δ is an effective divisor whose support contains neither L_{yz} nor R_x. Then $a > 0$, because otherwise

$$\frac{3}{11\cdot 37} = D\cdot L_{yz} =\geqslant \mathrm{mult}_P(D)37 > \frac{4}{11\cdot 37} > \frac{3}{11\cdot 37},$$

which is a contradiction. Therefore, we may assume that $R_y \not\subseteq \mathrm{Supp}(\Delta)$ by Remark 1.4.7. Similarly, we may assume that $L_{xt} \not\subseteq \mathrm{Supp}(\Delta)$ if $b > 0$.

Let us find upper bounds for a and b. If $b > 0$, then

$$\frac{3}{21\cdot 29} = D\cdot L_{xt} \geqslant bR_x\cdot L_{xt} = \frac{2b}{21},$$

which implies that $b \leqslant 3/42$. Similarly, we have

$$\frac{18}{29\cdot 37} = D\cdot R_y \geqslant \frac{6a}{37} + \frac{b}{37} + \frac{\mathrm{mult}_{O_t}(D) - a - b}{37} > \frac{5a+\frac{4}{11}}{37},$$

which implies that $a < 82/1595$. On the other hand, it follows from Lemma 1.4.6 that

$$\frac{3+45a}{11\cdot 37} = \Delta\cdot L_{yz} > \frac{4/11 - b}{37},$$

which implies that $a > 1/45$. Similarly, we see that

$$\frac{6+52b}{21\cdot 37} = \Delta\cdot R_x > \frac{4/11 - a}{37},$$

which implies that $b > 9/286$.

Let $\pi\colon \bar{X} \to X$ be the weighted blow up of the point O_t with weight $(13,4)$, and let F be the exceptional curve of the morphism π. Then F contains two singular points Q_{13} and Q_4 such that Q_{13} is a singular point of type $\frac{1}{13}(1,2)$, and Q_4 is a singular point of type $\frac{1}{4}(1,3)$. Then

$$K_{\bar{X}} = \pi^*(K_X) - \frac{20}{37}F,\ \bar{L}_{yz} = \pi^*(L_{yz}) - \frac{4}{37}F,\ \bar{R}_x = \pi^*(R_x) - \frac{13}{37}F,\ \bar{\Delta} = \pi^*(\Delta) - \frac{c}{37}F,$$

where $\bar{L}_{yz}$, $\bar{R}_x$ and $\bar{\Delta}$ are the proper transforms of L_{yz}, R_x and Δ by π, respectively, and c is a non-negative rational number c.

The log pull-back of the log pair $(X, \frac{11}{4}D)$ by π is the log pair

$$\left(\bar{X},\ \frac{11a}{4}\bar{L}_{yz} + \frac{11b}{4}\bar{R}_x + \frac{11}{4}\bar{\Delta} + \theta_1 F\right),$$

which is not log canonical at some point $Q \in F$, where $\theta_1 = (11(c+4a+13b)/4+20)/37)$. We have

$$\frac{3+45a}{11\cdot 37} - \frac{b}{37} - \frac{c}{13\cdot 37} = \bar{\Delta}\cdot\bar{L}_{yz} \geqslant 0 \leqslant \bar{\Delta}\cdot\bar{R}_x = \frac{6+52b}{21\cdot 37} - \frac{a}{37} - \frac{c}{4\cdot 37},$$

which implies that $\theta_1 < 1$, because $b < 3/42$. Note that $F\cap\bar{R}_x = Q_4$ and $F\cap\bar{L}_{yz} = Q_{13}$.

Suppose that $Q \notin \bar{R}_x \cup \bar{L}_{yz}$. Then

$$\frac{11c}{16\cdot 13} = \frac{11}{4}\bar{\Delta}\cdot F > 1$$

by Lemma 1.4.6. Thus, we see that $c > 208/11$. But the system of inequalities

$$\begin{cases} \dfrac{3+45a}{11\cdot 37} - \dfrac{b}{37} - \dfrac{c}{13\cdot 37} \geqslant 0, \\ \dfrac{6+52b}{21\cdot 37} - \dfrac{a}{37} - \dfrac{c}{4\cdot 37} \geqslant 0, \qquad b \leqslant 3/42, \\ c > 208/11, \end{cases}$$

is inconsistent. Thus, we see that $Q \in \bar{R}_x \cup \bar{L}_{yz}$.

Suppose that $Q \in \bar{M}_x$. Then $Q = Q_4$, and it follows from Lemma 1.4.6 that

$$\frac{11}{4}\left(\frac{6+52b}{21\cdot 37} - \frac{a}{37} - \frac{c}{4\cdot 37}\right) + \frac{\theta_1}{4} = \left(\frac{11}{4}\bar{\Delta} + \theta_1 F\right)\cdot\bar{M}_x > \frac{1}{4} < \left(\frac{11}{4}\bar{\Delta} + \frac{11b}{4}\bar{M}_x\right)\cdot F = \frac{11}{4}\left(\frac{c}{4\cdot 13} + \frac{b}{4}\right)$$

which implies that $b > 1164/5291$. But $b < 3/42$, which is a contradiction.

We see that $Q = Q_{13}$. Then it follows from Lemma 1.4.6 that

$$\frac{11}{4}\left(\frac{3+45a}{11\cdot 37}-\frac{b}{37}-\frac{c}{13\cdot 37}\right)+\frac{\theta_1}{13}=\left(\frac{11}{4}\bar{\Delta}+\theta_1 F\right)\cdot\bar{L}_{yz}>\frac{1}{13}<\left(\frac{11}{4}\bar{\Delta}+\frac{11a}{4}\bar{L}_{yz}\right)\cdot F=\frac{11}{4}\left(\frac{c}{4\cdot 13}+\frac{a}{13}\right)$$

Let $\phi\colon \tilde{X}\to\bar{X}$ be the weighted blow up at the point Q_{13} with weight $(1,2)$. Let G be the exceptional divisor of the morphism ϕ. Then G contains one singular point Q_2 of the surface $\tilde{X}$ that is a singular point of type $\frac{1}{2}(1,1)$. Let $\tilde{L}_{yz}$, $\tilde{R}_x$, $\tilde{\Delta}$ and $\tilde{F}$ be the proper transforms of L_{yz}, R_x, Δ and F by ϕ, respectively. We have

$$K_{\tilde{X}}=\phi^*(K_{\bar{X}})-\frac{10}{13}G,\ \tilde{L}_{yz}=\phi^*(\bar{L}_{yz})-\frac{2}{13}G,\ \tilde{F}=\phi^*(F)-\frac{1}{13}G,\ \tilde{\Delta}=\phi^*(\bar{\Delta})-\frac{d}{13}G,$$

where d is a positive rational number. The log pull-back of the log pair $(X,\frac{11}{4}D)$ via $\phi\circ\pi$ is

$$\left(\tilde{X},\frac{11a}{4}\tilde{L}_{yz}+\frac{11b}{4}\tilde{R}_x+\frac{11}{4}\tilde{\Delta}+\theta_1\tilde{F}+\theta_2 G\right),$$

where $\theta_2=33a/74+11c/1924+11b/148+11d/52+30/37$. This log pair is not log canonical at some point $O\in G$. We have

$$\frac{c}{13\cdot 4}-\frac{d}{13\cdot 2}=\tilde{\Delta}\cdot\tilde{F}\geqslant 0\leqslant\tilde{\Delta}\cdot\tilde{L}_{yz}=\frac{3+45a}{11\cdot 37}-\frac{b}{37}-\frac{c}{13\cdot 37}-\frac{d}{13},$$

which implies that $\theta_2<1$, because the system of inequalities

$$\begin{cases}\theta_2\geqslant 1,\\ \dfrac{3+45a}{11\cdot 37}-\dfrac{b}{37}-\dfrac{c}{13\cdot 37}-\dfrac{d}{13}\geqslant 0,\\ a\leqslant 82/1595,\end{cases}$$

is inconsistent. Note that $\tilde{F}\cap G=Q_2$ and $Q_2\notin\tilde{L}_{yz}$.

Suppose that $O\notin\tilde{F}\cup\tilde{L}_{yz}$. Applying Lemma 1.4.6, we get

$$1<\frac{11}{4}\tilde{\Delta}\cdot G=\frac{11d}{4\cdot 2},$$

which gives $d>8/11$. Hence, we obtain the system of inequalities

$$\begin{cases}\dfrac{3+45a}{11\cdot 37}-\dfrac{b}{37}-\dfrac{c}{13\cdot 37}-\dfrac{d}{13}\geqslant 0,\\ \dfrac{6+52b}{21\cdot 37}-\dfrac{a}{37}-\dfrac{c}{4\cdot 37}\geqslant 0,\\ \dfrac{c}{13\cdot 4}-\dfrac{d}{13\cdot 2}\geqslant 0,\\ d>8/11,\\ b\leqslant 3/42,\end{cases}$$

which is inconsistent. Thus, we see that $O\in\tilde{F}\cup\tilde{L}_{yz}$.

Suppose that $O\in\tilde{L}_{yz}$. Applying Lemma 1.4.6, we get

$$\frac{11}{4}\left(\frac{3+45a}{11\cdot 37}-\frac{b}{37}-\frac{c}{13\cdot 37}-\frac{d}{13}\right)+\theta_2=\left(\frac{11}{4}\tilde{\Delta}+\theta_2 G\right)\cdot\tilde{L}_{yz}>1<\left(\frac{11}{4}\tilde{\Delta}+\frac{11a}{4}\tilde{L}_{yz}\right)\cdot G=\frac{33}{16}\left(\frac{d}{2}+a\right),$$

which gives $a>25/11$. But $a<82/1595$, which is a contradiction. Thus, we see that $O\notin\tilde{L}_{yz}$.

We see that $O\in\tilde{F}$. Then $Q=Q_2$. Applying Lemma 1.4.6, we get

$$\frac{11}{4}\left(\frac{c}{4\cdot 13}-\frac{d}{2\cdot 13}\right)+\frac{\theta_2}{2}=\left(\frac{11}{4}\tilde{\Delta}+\theta_2 G\right)\cdot\tilde{F}>\frac{1}{2}<\left(\frac{11}{4}\tilde{\Delta}+\theta_1\tilde{F}\right)\cdot G=\frac{11d}{4\cdot 2}+\frac{\theta_1}{2}.$$

Let $\xi\colon\hat{X}\to\tilde{X}$ be the weighted blow up at the point Q_2 with weights $(1,1)$, let H be the exceptional divisor of ξ, let $\hat{L}_{yz}$, $\hat{R}_x$, $\hat{\Delta}$, $\hat{G}$, and $\hat{F}$ be the proper transforms of L_{yz}, R_x, Δ, G and F by ξ, respectively. Then $\bar{X}$ is smooth along H. We have

$$K_{\hat{X}}=\xi^*(K_{\tilde{X}})-\frac{1}{2}H,\ \hat{G}=\xi^*(G)-\frac{1}{2}H,\ \hat{F}=\xi^*(F)-\tilde{1}2G,\ \hat{\Delta}=\xi^*(\tilde{\Delta})-\frac{e}{2}G,$$

where e is a positive rational number. The log pull-back of the log pair $(X, \frac{11}{4}D)$ via $\phi \circ \pi$ is

$$\left(\hat{X}, \frac{11a}{4}\hat{L}_{yz} + \frac{11b}{4}\hat{R}_x + \frac{11}{4}\hat{\Delta} + \theta_1\hat{F} + \theta_2\hat{G} + \theta_3 H\right),$$

where $\theta_3 = (\theta_1 + \theta_2 + 11e/4)/2 = 55a/148 + 77b/148 + 77c/1924 + 11d/104 + 11/8e + 25/37$. This log pair is not log canonical at some point $A \in G$. We have

$$\frac{c}{13 \cdot 4} - \frac{d}{13 \cdot 2} - \frac{e}{2} = \hat{\Delta} \cdot \hat{F} \geqslant 0 \leqslant \tilde{\Delta} \cdot \hat{G} = \frac{d - e}{2},$$

which implies that $\theta_3 < 1$, because the system of inequalities

$$\begin{cases} \theta_3 \geqslant 1, \\ \dfrac{3 + 45a}{11 \cdot 37} - \dfrac{b}{37} - \dfrac{c}{13 \cdot 37} - \dfrac{d}{13} \geqslant 0, \\ d \geqslant e, \\ a \leqslant 82/1595, \end{cases}$$

is inconsistent. Note that $\hat{F} \cap \hat{G} = \varnothing$.

Suppose that $O \notin \hat{F} \cup \hat{G}$. Applying Lemma 1.4.6, we get

$$1 < \frac{11}{4}\hat{\Delta} \cdot H = \frac{11e}{4},$$

which gives $e > 4/11$. Hence, we obtain the system of inequalities

$$\begin{cases} \dfrac{3 + 45a}{11 \cdot 37} - \dfrac{b}{37} - \dfrac{c}{13 \cdot 37} - \dfrac{d}{13} \geqslant 0, \\ \dfrac{6 + 52b}{21 \cdot 37} - \dfrac{a}{37} - \dfrac{c}{4 \cdot 37} \geqslant 0, \\ \dfrac{c}{13 \cdot 4} - \dfrac{d}{13 \cdot 2} - \dfrac{e}{2} \geqslant 0, \\ d \geqslant e > 4/11, \\ a \leqslant 82/1595, \end{cases}$$

which is inconsistent. Thus, we see that $O \in \hat{F} \cup \hat{G}$.

Suppose that $O \in \hat{F}$. Applying Lemma 1.4.6, we get

$$\frac{11}{4}\left(\frac{c}{4 \cdot 13} - \frac{d}{2 \cdot 13} - \frac{e}{2}\right) + \theta_3 = \left(\frac{11}{4}\hat{\Delta} + \theta_3 H\right) \cdot \hat{F} > 1 < \left(\frac{11}{4}\hat{\Delta} + \theta_1\hat{F}\right) \cdot H = \frac{11e}{4} + \theta_1,$$

which leads to a contradiction, because the system of inequalities

$$\begin{cases} \dfrac{11}{4}\left(\dfrac{c}{4 \cdot 13} - \dfrac{d}{2 \cdot 13} - \dfrac{e}{2}\right) + \theta_3 > 1, \\ \dfrac{6 + 52b}{21 \cdot 37} - \dfrac{a}{37} - \dfrac{c}{4 \cdot 37} \geqslant 0, \\ b \leqslant 3/42, \end{cases}$$

is inconsistent. Thus, we see that $O \in \hat{F} \cup \hat{G}$. Then

$$\frac{11e}{4} + \theta_2 = \left(\frac{11}{4}\hat{\Delta} + \theta_2\hat{G}\right) \cdot H > 1 < \left(\frac{11}{4}\hat{\Delta} + \theta_3 H\right) \cdot \hat{G} = \frac{11}{4}\left(\frac{d}{2} - \frac{e}{2}\right) + \theta_3,$$

by Lemma 1.4.6. Thus, we obtain the system of inequalities

$$\begin{cases} \dfrac{11}{4}\left(\dfrac{d}{2} - \dfrac{e}{2}\right) + \theta_3 > 1, \\ \dfrac{3 + 45a}{11 \cdot 37} - \dfrac{b}{37} - \dfrac{c}{13 \cdot 37} - \dfrac{d}{13} \geqslant 0, \\ a \leqslant 82/1595, \end{cases}$$

is inconsistent. The obtained contradiction completes the proof. □

Lemma 3.3.4. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 37, 53, 98, 196)$. Then $\mathrm{lct}(X) = 55/18$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^5 + x^{13}z = 0,$$

the surface X is singular at the point O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{55}{18} = \mathrm{lct}\left(X, \frac{3}{11}C_x\right) < \mathrm{lct}\left(X, \frac{3}{37}C_y\right) = \frac{37 \cdot 5}{26},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 55/18$.

Suppose that $\mathrm{lct}(X) < 55/18$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{55}{3}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(583))$ contains x^{53}, $y^{11}x^{16}$ and z^{11}, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P \in C_x$. Then

$$\frac{6}{37 \cdot 53} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{37} & \text{if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{53} & \text{if } P = O_z, \\ \mathrm{mult}_P(D) & \text{if } P \neq O_y \text{ and } P \neq O_z, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 18/55$. Thus, we see that $P = O_x$. Then

$$\frac{6}{11 \cdot 53} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{11} > \frac{18}{55 \cdot 11} > \frac{6}{11 \cdot 53},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 55/18$. □

Lemma 3.3.5. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 17, 27, 41, 95)$. Then $\mathrm{lct}(X) = 65/24$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^2t + y^4z + xt^2 + x^6y = 0,$$

and X is singular at the point O_x, O_y, O_z and O_t.

The curve C_x is reducible. We have $C_x = L_{xz} + M_x$, where L_{xz} and M_x are irreducible and reduced curves such that L_{xz} is given by the equations $x = z = 0$, and M_x is given by the equations $x = y^4 + zt = 0$. Then

$$L_{xz} \cdot L_{xz} = \frac{-55}{17 \cdot 41},\ M_x \cdot M_x = \frac{-56}{27 \cdot 41},\ L_{xz} \cdot M_x = \frac{4}{41},\ D \cdot M_x = \frac{12}{27 \cdot 41},\ D \cdot L_{xz} = \frac{3}{17 \cdot 41}$$

and $L_{xz} \cap M_x = O_t$. The curve C_y is also reducible. We have $C_y = L_{yt} + M_y$, where L_{yt} and M_y are irreducible and reduced curves such that L_{yt} is given by the equations $y = t = 0$, and M_y is given by the equations $y = z^2 + xt = 0$. Then

$$L_{yt} \cdot M_{yt} = \frac{-37}{17 \cdot 41},\ M_y \cdot M_y = \frac{-48}{13 \cdot 41},\ L_{yt} \cdot M_y = \frac{2}{13},\ D \cdot M_y = \frac{6}{13 \cdot 41},\ D \cdot L_{yt} = \frac{3}{13 \cdot 27},$$

and $L_{yt} \cap M_y = O_x$. The curve C_z is also reducible. We have $C_z = L_{xz} + M_z$, where M_z is an irreducible and reduced curve that is given by the equations $z = t^2 + x^5y = 0$. Then

$$L_{xz} \cdot M_z = \frac{2}{17},\ L_{xz} \cdot M_z = \frac{-55}{17 \cdot 41},\ L_{xz} \cdot M_y = \frac{1}{41},\ D \cdot M_z = \frac{6}{13 \cdot 17}$$

and $L_{xz} \cap M_z = O_y$. The curve C_t is also reducible. We have $C_t = L_{yt} + M_t$, where M_t is an irreducible and reduced curve that is given by the equations $t = x^6 + zy^3 = 0$. Then

$$L_{yt} \cdot M_t = \frac{6}{27},\ M_t \cdot M_t = \frac{168}{13 \cdot 27},\ D \cdot M_t = \frac{18}{13 \cdot 27}$$

and $L_{yt} \cap M_t = O_z$. We have $\mathrm{lct}(X) \leqslant 65/24$, because

$$\frac{65}{24} = \mathrm{lct}\left(X, \frac{3}{13}C_x\right) < \frac{51}{12} = \mathrm{lct}\left(X, \frac{3}{17}C_y\right) < \frac{41}{8} = \mathrm{lct}\left(X, \frac{3}{41}C_t\right) < \frac{21}{4} = \mathrm{lct}\left(X, \frac{3}{27}C_z\right).$$

Suppose that $\mathrm{lct}(X) < 65/24$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{65}{24}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that either $\mathrm{Supp}(D)$ does not contain at least one irreducible component of C_x, C_y, C_z and C_t.

Suppose that $P \notin C_x \cup C_y \cup C_z \cup C_t$. Then there is a unique curve $Z_\alpha \subset X$ that is cut out by

$$xt + \alpha z^2 = 0$$

such that $P \in Z$, where $0 \neq \alpha \in \mathbb{C}$. The curve Z_α is reduced. But it is always reducible. Indeed, taking into account the geometry of the open subset $Z_\alpha \setminus (Z_\alpha \cap C_t)$, one can easily check that

$$Z_\alpha = C_\alpha + L_{xz}$$

for any $\alpha \neq 0$, where C_α is a curve whose support contains no L_{xy}. Let us prove that C_α is reduced and irreducible if $\alpha \neq 1$.

The open subset $Z_\alpha \setminus (Z_\alpha \cap C_x)$ of the curve Z_α is a $\mathbb{Z}_{13}$-quotient of the affine curve

$$t + \alpha z^2 = z^2 t + y^4 z + t^2 + y = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

which is isomorphic to a plane affine quartic curve that is given by the equation

$$\alpha(\alpha - 1)z^4 + y^4 z + y = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big),$$

which implies that the curve C_α is and irreducible reduced curve and $\mathrm{mult}_P(C_\alpha) \leqslant 3$ if $\alpha \neq 1$.

The case $\alpha = 1$ is special. Namely, if $\alpha = 1$, then

$$C_1 = R_1 + M_y,$$

where R_1 is a curve whose support contains no C_1. Arguing as in the case $\alpha \neq 1$, we see that R_1 is an irreducible reduced curve that is smooth at the point P.

By Remark 1.4.7, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible components of the curve Z_α.

Suppose that $\alpha \neq 1$. Then elementary calculations imply that

$$C_\alpha \cdot L_{xz} = \frac{109}{17 \cdot 41},\ C_\alpha \cdot C_\alpha = \frac{8141}{13 \cdot 17 \cdot 41},\ D \cdot C_\alpha = \frac{531}{13 \cdot 17 \cdot 41},$$

and we can put $D = \epsilon C_\alpha + \Delta_\alpha$, where Δ_α is an effective $\mathbb{Q}$-divisor such that $C_\alpha \not\subset \mathrm{Supp}(\Delta_\alpha)$. If $\epsilon \neq 0$, then

$$\frac{3}{17 \cdot 41} = D \cdot L_{xz} = \Big(\epsilon C_\alpha + \Delta_\alpha\Big) \cdot L_{xz} \geqslant \epsilon C_\alpha \cdot L_{xz} = \frac{109\epsilon}{17 \cdot 41},$$

which implies that $\epsilon \leqslant 3/109$. On the other hand, we see that

$$\frac{531}{13 \cdot 17 \cdot 41} = D\cdot C_\alpha = \epsilon C_\alpha^2 + \Delta_\alpha \cdot C_\alpha \geqslant \epsilon C^2 + \mathrm{mult}_P\big(\Delta_\alpha\big) = \epsilon C^2 + \mathrm{mult}_P\big(D\big) - \epsilon \mathrm{mult}_P\big(C_\alpha\big) > \epsilon C^2 + \frac{24}{65} - 3\epsilon,$$

which is impossible, because $\epsilon \leqslant 3/109$.

Thus, we see that $\alpha = 1$. We have

$$R_1 \cdot L_{xz} = \frac{92}{17 \cdot 41},\ R_1 \cdot R_1 = \frac{3177}{13 \cdot 17 \cdot 41},\ M_y \cdot R_1 = \frac{197}{13 \cdot 41},\ D \cdot R_1 = \frac{429}{13 \cdot 17 \cdot 41},$$

and we can put $D = \epsilon_1 R_1 + \Xi_1$, where Ξ_1 is an effective $\mathbb{Q}$-divisor such that $R_1 \not\subset \mathrm{Supp}(\Xi_1)$. Then $\epsilon_1 \leqslant 3/91$, because either $\epsilon_1 = 0$, or $L_{xz} \cdot \Xi_1 \geqslant 0$ or $M_y \cdot \Xi_1 \geqslant 0$. By Lemma 1.4.6, we see that

$$\frac{429 - 3177\epsilon_1}{13 \cdot 17 \cdot 41} = \Xi_1 \cdot R_1 > \frac{24}{65},$$

which is a contradiction. The obtained contradiction shows that $P \in C_x \cup C_y \cup C_z \cup C_t$.

Suppose that $P = O_t$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{3}{17 \cdot 41} = D \cdot L_{xz} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{41} > \frac{3}{11 \cdot 41} > \frac{24}{65 \cdot 41},$$

which is a contradiction. Thus, we see that $L_{xz} \not\subseteq \mathrm{Supp}(D) \supset M_x$. Put $D = \omega L_{xz} + \Psi$, where Ψ is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subset \mathrm{Supp}(\Psi)$, and $\omega > 0$. Then

$$\frac{12}{27 \cdot 41} = D\cdot M_x = \big(\omega L_{xz} + \Psi\big)\cdot M_x \geqslant \omega L_{xz}\cdot M_x + \frac{\mathrm{mult}_{O_t}(D) - \omega}{41} > \omega L_{xz}\cdot M_x + \frac{3/11 - \omega}{41} = \frac{4\omega}{41} + \frac{24/65 - \omega}{41},$$

which implies that $\omega 44/585$. Then it follows from Lemma 1.4.6 that

$$\frac{3+55\omega}{17\cdot 41} = \big(-K_X - \omega L_{xz}\big)\cdot L_{xz} = \Psi\cdot L_{xz} > \frac{24}{65\cdot 41},$$

which is impossible, because $\omega 44/585$. Thus, we see that $P\neq O_t$. Note, that applying similar arguments to $O_z = M_t\cap L_{yt}$, we do not see that $P\neq O_z$.

Suppose that $P = O_z$. Put $D = \epsilon M_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $M_x \not\subset \mathrm{Supp}(\Omega)$. If $\epsilon\neq 0$, then

$$\frac{3}{17\cdot 41} = D\cdot L_{xz} = \big(\epsilon M_x + \Delta\big)\cdot L_{xz} \geqslant \epsilon M_x\cdot L_{xz},$$

which implies that $\epsilon < 3/68$. Then it follows from Lemma 1.4.6 that

$$\frac{12+56\epsilon}{27\cdot 41} = \big(-K_X - \epsilon L_{xz}\big)\cdot L_{xz} = \Omega\cdot L_{xz} > \frac{22}{65\cdot 27},$$

which implies that $\epsilon > 51/910$. But $\epsilon < 3/68 < 51/910$. Thus, we see that $P\neq O_z$.

Suppose that $P = O_y$. If $L_{xz}\not\subseteq \mathrm{Supp}(D)$, then

$$\frac{3}{17\cdot 41} = D\cdot L_{xz} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{17} > \frac{24}{65\cdot 17} > \frac{3}{17\cdot 41},$$

which is a contradiction. If $M_z\not\subseteq \mathrm{Supp}(D)$,

$$\frac{6}{13\cdot 17} = D\cdot M_z \geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_{O_y}\big(M_z\big)}{17}\frac{2\mathrm{mult}_P\big(D\big)}{17} >> \frac{48}{65\cdot 17} > \frac{6}{13\cdot 17},$$

which is a contradiction. Thus, we see that $P\neq O_y$. Similarly, we see that $P\neq O_x = M_y\cap L_{yz}$. Then $P\notin \mathrm{Sing}(X)$.

Suppose that $P\in L_{xz}$. Put $D = mL_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz}\not\subset \mathrm{Supp}(\Omega)$. If $m\neq 0$, then

$$\frac{12}{27\cdot 41} = -K_X\cdot M_x = D\cdot M_x = \big(mL_{xz} + \Omega\big)\cdot M_x \geqslant mL_{xz}\cdot M_x = \frac{4m}{41},$$

which implies that $m\leqslant 3/27$. Then it follows from Lemma 1.4.6 that

$$\frac{3+55m}{17\cdot 41} = \big(-K_X - mL_{xz}\big)\cdot L_{xz} = \Omega\cdot L_{xz} > \frac{24}{65},$$

which is impossible, because $m\leqslant 3/27$. Thus, we see that $P\notin L_{xz}$. Similarly, we see that $P\notin L_{yt}$.

Suppose that $P\in M_x$. Put $D = \delta M_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $M_x\not\subset \mathrm{Supp}(\Upsilon)$. If $\epsilon\neq 0$, then

$$\frac{3}{17\cdot 41} = -K_X\cdot L_{xz} = D\cdot L_{xz} = \big(\delta M_x + \Upsilon\big)\cdot L_{xz} \geqslant \delta L_{xz}\cdot M_x = \frac{4\delta}{41},$$

which implies that $\delta\leqslant 3/68$. Then it follows from Lemma 1.4.6 that

$$\frac{12+56\delta}{27\cdot 41} = \big(-K_X - \delta M_x\big)\cdot M_x = \Upsilon\cdot M_x > \frac{24}{65},$$

which is impossible, because $\delta\leqslant 3/68$. Similarly, we see that $P\notin M_y\cup M_z\cup M_t$, which is a contradiction. □

Lemma 3.3.6. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 27, 61, 98, 196)$. Then $\mathrm{lct}(X) = 91/30$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^5 z + xz^3 + x^{13}y = 0,$$

the surface X is singular at the point O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{91}{30} = \mathrm{lct}\left(X, \frac{3}{13}C_x\right) < \mathrm{lct}\left(X, \frac{3}{27}C_y\right) = \frac{15}{2},$$

which implies, in particular, that $\mathrm{lct}(X)\leqslant 91/30$.

Suppose that $\mathrm{lct}(X) < 91/30$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{91}{30}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(793))$ contains x^{61}, $y^{26}x^7$, $y^{13}x^{34}$ and z^{13}, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P \in C_x$. Then

$$\frac{2}{9 \cdot 61} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{27} & \text{if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{61} & \text{if } P = O_z, \\ \mathrm{mult}_P(D) & \text{if } P \neq O_y \text{ and } P \neq O_z, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 30/91$. Thus, we see that $P = O_x$. Then

$$\frac{6}{13 \cdot 61} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{13} > \frac{30}{91 \cdot 13} > \frac{6}{13 \cdot 61},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 91/30$. □

Lemma 3.3.7. Suppose that $(a_0, a_1, a_2, a_3, d) = (15, 19, 43, 74, 148)$. Then $\mathrm{lct}(X) = 57/14$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^7 + x^7z = 0,$$

the surface X is singular at the point O_x, O_y and O_z, the curves C_x and C_y are irreducible, and

$$\frac{25}{6} = \mathrm{lct}\left(X, \frac{3}{15}C_x\right) > \mathrm{lct}\left(X, \frac{3}{19}C_y\right) = \frac{57}{14},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 57/14$.

Suppose that $\mathrm{lct}(X) < 57/14$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{57}{14}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(645))$ contains x^{43}, $y^{15}x^{24}$, $y^{30}x^5$ and z^{15}, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P \in C_x$. Then

$$\frac{6}{19 \cdot 43} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{19} & \text{if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{43} & \text{if } P = O_z, \\ \mathrm{mult}_P(D) & \text{if } P \neq O_y \text{ and } P \neq O_z, \end{cases}$$

which implies that $P = O_z$, because $\mathrm{mult}_P(D) > 14/57$. Then

$$\frac{6}{15 \cdot 43} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(C_y)}{43} > \frac{28}{57 \cdot 43} > \frac{6}{15 \cdot 43},$$

because $\mathrm{mult}_P(C_y) = 2$. Thus, we see that $P = O_x$. Then

$$\frac{6}{15 \cdot 43} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{15} > \frac{14}{57 \cdot 15} > \frac{6}{15 \cdot 43},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 57/14$. □

3.4. Sporadic cases with $I = 4$

Lemma 3.4.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (5, 6, 8, 9, 24)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + yt^2 + y^4 + \epsilon x^2 yz + x^3 t = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular at the point O_x and O_t. The surface X is also singular at a point Q_2 that is cut out on X by the equations $x = t = 0$. The surface X is also singular at a point Q_3 such that $Q_3 \neq O_t$ and the points Q_3 and Q_t are cut out on X by the equations $x = z = 0$.

The curves C_x, C_y, C_z and C_t are irreducible. We have

$$\mathrm{lct}\left(X, \frac{4}{9}C_t\right) > 1 = \mathrm{lct}\left(X, \frac{4}{6}C_y\right) < \mathrm{lct}\left(X, \frac{4}{5}C_x\right) = \frac{5}{4} < \mathrm{lct}\left(X, \frac{4}{8}C_z\right) = 2,$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair (X, D) is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x, C_y, C_z and C_t.

Suppose that $P \in C_y$. Then

$$\frac{12}{9} = D \cdot C_y \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{5} \text{ if } P = O_x, \\ \dfrac{\mathrm{mult}_P(D)\mathrm{mult}_{O_t}(C_y)}{9} \text{ if } P = O_t, \\ \mathrm{mult}_P(D) \text{ if } P \neq O_x \text{ and } P \neq O_t, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 1$ and $\mathrm{mult}_{O_t}(C_y) = 3$.

We see that $P \neq O_t$. Suppose that $P \in C_x$. Then

$$\frac{2}{9} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{2} \text{ if } P = Q_2, \\ \dfrac{\mathrm{mult}_P(D)}{3} \text{ if } P = Q_3, \\ \mathrm{mult}_P(D) \text{ if } P \neq Q_2 \text{ and } P \neq Q_3, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 1$. Thus, we see that $P \notin \mathrm{Sing}(X)$.

Let us show that $P \notin C_z$. Suppose that $P \in C_z$. Then

$$\frac{16}{45} = D \cdot C_z \geqslant \mathrm{mult}_P(D) > 1,$$

which is a contradiction. Similarly, we see that $P \notin C_t$.

We see that $P \notin C_x \cup C_y \cup C_z \cup C_t$. Then there is a unique curve $Z \subset X$ that is cut out by

$$xt = \alpha yz$$

such that $P \in Z$, where $0 \neq \alpha \in \mathbb{C}$. We see that $C_x \not\subset \mathrm{Supp}(Z)$. But the open subset $Z \setminus (Z \cap C_x)$ of the curve Z is a $\mathbb{Z}_5$-quotient of the affine curve

$$t - \alpha yz = z^3 + yt^2 + y^4 + \epsilon yz + t = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

which isomorphic to a plane affine quintic curve $R_x \subset \mathbb{C}^2$ that is given by the equation

$$z^3 + \alpha^2 y^3 z^2 + y^4 + (\epsilon + \alpha) yz = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big),$$

which is easily seen to be irreducible. In particular, the curve Z is irreducible.

The inequality $\mathrm{mult}_P(Z) \leqslant 3$ holds, because quintic R_x is singular at the origin. Thus, we may assume that $\mathrm{Supp}(D)$ does not contain the curve Z by Remark 1.4.7. Then

$$\frac{28}{45} = D \cdot Z \geqslant \mathrm{mult}_P(D) > 1,$$

which is a contradiction. The obtained contradiction completes the proof. ☐

Lemma 3.4.2. Suppose that $(a_0, a_1, a_2, a_3, d) = (5, 6, 8, 15, 30)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + y^5 + x^2y^2z + x^3t + x^6 = 0,$$

and X is singular at the point O_z. The surface X is also singular at points P_1 and P_2 that are cut out on X by the equations $y = z = 0$. The surface X is also singular at a point Q_3 that is cut out on X by the equations $x = z = 0$. The surface X is also singular at a point Q_2 such that $Q_2 \neq O_z$ and the points Q_2 and Q_z are cut out on X by the equations $x = t = 0$.

The curve C_y is reducible. We have $C_y = L_1 + L_2$, where L_1 and L_2 are irreducible and reduced curves such that $P_1 \in L_1$ and $P_2 \in L_2$. Then

$$L_1 \cdot L_1 = L_2 \cdot L_2 = \frac{-9}{40},\ L_1 \cdot L_2 = \frac{3}{8},$$

and $L_1 \cap L_2 = O_z$. The curve C_x is irreducible and

$$1 = \mathrm{lct}\left(X, \frac{4}{6}C_y\right) < \mathrm{lct}\left(X, \frac{4}{5}C_x\right) = \frac{5}{4},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair (X, D) is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curve C_x. Similarly, without loss of generality we may assume that $L_1 \not\subseteq \mathrm{Supp}(D)$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(30))$ contains y^5, yz^3 and t^2, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_y$.

Suppose that $P \in L_1$. Then

$$\frac{1}{10} = D \cdot L_1 \geqslant \begin{cases} 1 \text{ if } P \neq P_1 \text{ and } P \neq O_z, \\ \frac{1}{5} \text{ if } P = P_1, \\ \frac{1}{8} \text{ if } P = O_z, \end{cases}$$

which is a contradiction. Thus, we see that $P \notin L_1$. In particular, we see that $P \neq O_t$.

Suppose that $P \in L_2$. Put $D = mL_2 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_2 \not\subseteq \mathrm{Supp}(\Omega)$. Then

$$\frac{1}{10} = -K_X \cdot L_1 = D \cdot Z_x = \big(mL_2 + \Omega\big) \cdot L_1 \geqslant mL_2 \cdot L_1 = \frac{3m}{8},$$

which implies that $m \leqslant 4/15$. Then it follows from Lemma 1.4.6 that

$$\frac{2 + 9m}{40} = \big(-K_X - mL_2\big) \cdot L_2 = \Omega \cdot L_2 > \begin{cases} 1 \text{ if } P \neq P_2, \\ \frac{1}{5} \text{ if } P = P_2, \end{cases}$$

which implies that $m > 4/9$. But $m \leqslant 4/15$. Thus, we see that $P \notin L_1$.

Therefore, we see that either $P = Q_2$ or $P = Q_3$. Then

$$\frac{1}{6} = D \cdot C_x \geqslant \begin{cases} \frac{\mathrm{mult}_P\big(D\big)}{2} \text{ if } P = Q_2, \\ \frac{\mathrm{mult}_P\big(D\big)}{3} \text{ if } P = Q_3, \end{cases}$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 1$. □

Lemma 3.4.3. Suppose that $(a_0, a_1, a_2, a_3, d) = (9, 11, 12, 17, 45)$. Then $\mathrm{lct}(X) = 77/60$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y + y^3z + xz^3 + x^5 = 0.$$

Note that it is singular at the point O_y, O_z, O_t, and the point $Q = [1 : 0 : -1 : 0]$. The curve C_x consists of two irreducible and reduced curves $L_{xy} = \{x = y = 0\}$ and $R_x = \{x = t^2 + y^2z = 0\}$.

The curve C_y also consists of two irreducible and reduced curves L_{xy} and $R_y = \{y = z^3+x^4 = 0\}$. The curve C_z and C_t are irreducible and reduced. We have

$$\mathrm{lct}(X, \frac{4}{11}C_y) = \frac{77}{60} < \mathrm{lct}(X, \frac{4}{9}C_x), \quad \mathrm{lct}(X, \frac{4}{12}C_z), \quad \mathrm{lct}(X, \frac{4}{17}C_t).$$

Suppose that $\mathrm{lct}(X) < \frac{77}{60}$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $(X, \frac{77}{60}D)$ is not canonical at some point P. By Remark 1.4.7 we may assume that the support of D contains neither C_z nor C_t. The inequalities

$$D \cdot C_z = \frac{4 \cdot 12 \cdot 45}{9 \cdot 11 \cdot 12 \cdot 17} < \frac{60}{77},$$

$$D \cdot C_t = \frac{4 \cdot 17 \cdot 45}{9 \cdot 11 \cdot 12 \cdot 17} < \frac{60}{77}$$

imply $P \notin C_z \cup C_t \setminus \mathrm{Sing}(X)$. Moreover, we have

$$\mathrm{mult}_{O_y} D \leqslant \frac{11}{2} D \cdot C_z = \frac{10}{17} < \frac{60}{77},$$

$$\mathrm{mult}_{Q} D \leqslant 3 D \cdot C_t = \frac{5}{11} < \frac{60}{77}$$

and hence P can be neither the point O_y nor the point Q.

We can see that

$$L_{xy} \cdot D = \frac{1}{17 \cdot 3}, \quad R_x \cdot D = \frac{2}{33}, \quad R_y \cdot D = \frac{11}{9 \cdot 17}, \quad L_{xy} \cdot R_x = \frac{1}{6},$$

$$L_{xy} \cdot R_y = \frac{3}{17}, \quad L_{xy}^2 = -\frac{15}{4 \cdot 17}, \quad R_x^2 = -\frac{1}{33}, \quad R_y^2 = \frac{13}{4 \cdot 9 \cdot 17}.$$

By Remark 1.4.7 we may assume that the support of D does not contain both L_{xy} and R_x. If the support of D does not contain L_{xy}, then

$$\mathrm{mult}_{O_z} D \leqslant 12 D \cdot L_{xy} = \frac{4}{17} < \frac{60}{77}.$$

If the support of D does not contain R_x, then

$$\mathrm{mult}_{O_z} D \leqslant 12 D \cdot R_x = \frac{8}{11} < \frac{60}{77}.$$

Therefore, P cannot be O_z.

Also, we may assume that the support of D does not contain both L_{xy} and R_y. If the support of D does not contain L_{xy}, then

$$\mathrm{mult}_{O_t} D \leqslant 17 D \cdot L_{xy} = \frac{1}{3} < \frac{60}{77}.$$

If the support of D does not contain R_y, then

$$\mathrm{mult}_{O_t} D \leqslant \frac{17}{3} D \cdot R_y = \frac{11}{27} < \frac{60}{77}.$$

Therefore, P cannot be O_t.

By Remark 1.4.7 we may assume that the support of D does not contain both L_{xy} and R_x. If we write $D = nL_{xy} + \Delta$, where Δ does not contain the curve L_{xy}, then we can see $n \leqslant \frac{4}{11}$ since $D \cdot R_x \geqslant nR_x \cdot L_{xy}$. By Lemma 1.4.8 the inequality

$$\frac{77}{60}(L_{xy} \cdot D - mL_{xy}^2) \leqslant \frac{7 \cdot 14}{15 \cdot 3 \cdot 17} < 1$$

implies that the point P cannot belong to the curve L_{xy}. By the same method, we see the point P must be outside of R_x.

If we write $D = mR_y + \Omega$, where Ω does not contain the curve R_y, then we can see $0 \leqslant m \leqslant \frac{1}{9}$ since $D \cdot L_{xy} \geqslant mR_y \cdot L_{xy}$. By Lemma 1.4.8 the inequality

$$\frac{77}{60}(R_y \cdot D - mR_y^2) \leqslant \frac{77}{60} R_y \cdot D < 1$$

implies that the point P cannot belong to the curve R_y.

Now we consider the pencil $\mathcal{L}$ on X cut by $\lambda t^2 + \mu y^2 z = 0$. The base locus of the pencil consists of three points O_y, O_z, and Q. Let F be the member in $\mathcal{L}$ defined by $t^2 + y^2 z = 0$. The divisor F consists of two irreducible and reduced curves R_x and $E = \{t^2 + y^2 z = x^4 + z^3 = 0\}$. The Jacobian criterion shows us that the curve E is smooth in the outside of the base points. Also we have

$$F \cdot D = \frac{10}{33}, \quad R_x \cdot E = \frac{4}{11}, \quad E \cdot D = \frac{8}{3 \cdot 11}, \quad E^2 = \frac{4 \cdot 14}{3 \cdot 11}.$$

We write $D = lE + \Gamma$, where Γ does not contain the curve E. Since $(X, \frac{77}{60}D)$ is log canonical at the point O_y, the non-negative number l is at most $\frac{60}{77}$. By Lemma 1.4.8, the inequality shows

$$\frac{77}{60}(E \cdot D - lE^2) \leqslant \frac{77}{60} E \cdot D < 1$$

implies that the point P cannot belong to the curve E.

So far we have seen that the point P must lie in the outside of $C_x \cup C_y \cup C_z \cup C_t \cup E$. In particular, it is a smooth point. There is a unique member C in $\mathcal{L}$ which passes through the point P. Then the curve C is cut by $t^2 = \alpha y^2 z$ where α is a constant different from 0 and -1. The curve C is isomorphic to the curve defined by $y^3 z + xz^3 + x^5 = 0$ and $t^2 = y^2 z$. The curve C is smooth in the outside of the base points by the Bertini theorem, since it is isomorphic to a general curve in the pencil $\mathcal{L}$. We claim that the curve C is irreducible. If so then we may assume that the support of D does not contain the curve C hand hence we obtain

$$\mathrm{mult}_P D \leqslant C \cdot D = \frac{10}{33} < \frac{60}{77}.$$

This is a contradiction.

For the irreducibility of the curve C, we may consider the curve C as a surface in $\mathbb{A}^4$ defined by the equations $y^3 z + xz^3 + x^5 = 0$ and $t^2 = y^2 z$. Then, we consider the surface in $\mathbb{P}^4$ defined by the equations $y^3 zw + xz^3 w + x^5 = 0$ and $t^2 w = y^2 z$. We then take the affine piece defined by $t \neq 0$. Then, the affine piece is isomorphic to the surface defined by the equation $y^3 zw + xz^3 w + x^5 = 0$ and $w = y^2 z$ in $\mathbb{A}^4$. It is isomorphic the irreducible hypersurface $y^5 z^2 + xy^2 z^5 + x^5 = 0$ in $\mathbb{A}^3$. Therefore, the curve C must be irreducible.

□

Lemma 3.4.4. Suppose that and $(a_0, a_1, a_2, a_3, d) = (10, 13, 25, 31, 75)$. Then $\mathrm{lct}(X) = 91/60$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2 y + z^3 + xy^5 + x^5 z = 0.$$

It has singular points at O_x, O_y, O_t and $Q = [-1 : 0 : 1 : 0]$. The curve C_x and C_t are irreducible and reduced. The curve C_y consists of two irreducible reduced curves $L_{yz} = \{y = z = 0\}$ and $R_y = \{y = z^2 + x^5 = 0\}$. The curve C_z consists of two irreducible reduced curves L_{yz} and $R_z = \{y = t^2 + xy^4 = 0\}$. It is easy to see that

$$\mathrm{lct}(X, \frac{4}{13} C_y) = \frac{91}{60} < \mathrm{lct}(X, \frac{4}{10} C_x) < \mathrm{lct}(X, \frac{4}{25} C_z) < \mathrm{lct}(X, \frac{4}{31} C_t).$$

Also, we have the following intersection numbers:

$$C_x \cdot D = \frac{12}{13 \cdot 31}, \quad C_t \cdot D = \frac{6}{5 \cdot 13}, \quad L_{yz} \cdot D = \frac{2}{5 \cdot 31}, \quad R_y \cdot D = \frac{4}{5 \cdot 31}, \quad R_z \cdot D = \frac{4}{5 \cdot 13}$$

$$L_{yz} \cdot R_y = \frac{2}{31}, \quad L_{yz} \cdot R_z = \frac{1}{5}, \quad L_{yz}^2 = -\frac{7}{10 \cdot 31}, \quad R_y^2 = -\frac{3}{5 \cdot 31}, \quad R_z^2 = \frac{12}{5 \cdot 13}.$$

Suppose that $\mathrm{lct}(X) < \frac{91}{60}$. Then, there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the log pair $(X, \frac{91}{60} D)$ is not log canonical at some point $P \in X$. Since the curves C_x and C_t are irreducible we may assume that the support of D contains none of them. The inequalities

$$13 D \cdot C_x < \frac{60}{91}, \quad 5 D \cdot C_t < \frac{60}{91}$$

show that the point P must be in the outside of $C_x \cup C_t \setminus \{O_x, O_t\}$.

By Remark 1.4.7, we may assume that the support of D cannot contain both L_{yz} and R_y. If the support of D does not contain L_{yz}, then the inequality

$$31D\cdot L_{yz}=\frac{2}{5}<\frac{60}{91}$$

shows that the point P cannot be O_t. On the other hand, if the support of D does not contain R_y, then the inequality

$$\frac{31}{2}D\cdot R_y=\frac{2}{5}<\frac{60}{91}$$

shows that the point P cannot be O_t. We use the same method for R_z+L_{yz} so that we can see the point P cannot be O_x.

We write $D=mR_y+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve R_y. Then we see $m\leqslant\frac{1}{5}$ since the support of D cannot contain both L_{yz} and R_y and $D\cdot L_{yz}\geqslant mR_y\cdot L_{yz}$. Since $R_y\cdot D-mR_y^2<\frac{60}{91}$, Lemma 1.4.8 implies that the point P is located in the outside of R_y. Using the same argument for L_{yz} , we can also see that the point P is located in the outside of L_{yz}. Also, the same method shows that the point P is located in the outside of R_z Consequently, the point P must lie in the outside of $C_x\cup C_y\cup C_z\cup C_t$.

Now we consider the pencil $\mathcal{L}$ on X cut by $\lambda t^2+\mu xy^4=0$. The base locus of the pencil consists of three points O_x, O_y, and Q. Let F be the member in $\mathcal{L}$ defined by $t^2+xy^4=0$. The divisor F consists of two irreducible and reduced curves R_z and $E=\{t^2+xy^4=z^2+x^5=0\}$. The Jacobian criterion shows us that the curve E is smooth in the outside of $\mathrm{Sing}(X)$. Also we have

$$F\cdot D=\frac{12}{5\cdot 13},\quad R_z\cdot E=\frac{2}{13},\quad E\cdot D=\frac{8}{5\cdot 13},\quad E^2=\frac{2}{5\cdot 13}.$$

We write $D=lE+\Gamma$, where Γ does not contain the curve E. Since $(X,\frac{91}{60}D)$ is log canonical at the point O_y, the non-negative number l is at most $\frac{60}{91}$. By Lemma 1.4.8, the inequality shows

$$\frac{91}{60}(E\cdot D-lE^2)\leqslant\frac{91}{60}E\cdot D<1$$

implies that the point P cannot belong to the curve E.

So far we have seen that the point P must lie in the outside of $C_x\cup C_y\cup C_z\cup C_t\cup E$. In particular, it is a smooth point. There is a unique member C in $\mathcal{L}$ which passes through the point P. Then the curve C is cut by $t^2=\alpha xy^4$ where α is a constant different from 0 and -1. The curve C is isomorphic to the curve defined by $xy^5+z^3+x^5z=0$ and $t^2=xy^4$. The curve C is smooth in the outside of the base points by the Bertini theorem, since it is isomorphic to a general curve in the pencil $\mathcal{L}$. We claim that the curve C is irreducible. If so then we may assume that the support of D does not contain the curve C and hence we obtain

$$\mathrm{mult}_P D\leqslant C\cdot D=\frac{12}{5\cdot 13}<\frac{60}{91}.$$

This is a contradiction.

For the irreducibility of the curve C, we may consider the curve C as a surface in $\mathbb{A}^4$ defined by the equations $xy^5+z^3+x^5z=0$ and $t^2=xy^4$. Then, we consider the surface in $\mathbb{P}^4$ defined by the equations $xy^5+w^3z^3+x^5z=0$ and $t^2w^3=xy^4$. We then take the affine piece defined by $y\neq 0$. Then, the affine piece is isomorphic to the surface defined by the equation $x+w^3z^3+x^5z=0$ and $t^2w^3=x$ in $\mathbb{A}^4$. It is isomorphic the hypersurface defined by $t^2w^3+w^3z^3+t^{10}w^{15}z=0$ in $\mathbb{A}^3$. It has two irreducible components $w=0$ and $t^2+z^3+t^{10}w^{12}z=0$. The former component originates from the hyperplane at infinity in $\mathbb{P}^4$. Therefore, the curve C must be irreducible. □

Lemma 3.4.5. Suppose that $(a_0,a_1,a_2,a_3,d)=(11,17,20,27,71)$. Then $\mathrm{lct}(X)=11/6$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y+y^3z+xz^3+x^4t=0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of $L_{xy}=\{x=y=0\}$ (resp. $L_{xy}=\{x=y=0\}$, $L_{zt}=\{z=t=0\}$, $L_{zt}=\{z=t=0\}$)

and $R_x = \{x = y^2z + t^2 = 0\}$ (resp. $R_y = \{y = x^3t + z^3 = 0\}$, $R_z = \{z = x^4 + yt = 0\}$, $R_t = \{t = y^3 + xz^2 = 0\}$). Also, we see that

$$L_{xy} \cap R_x = \{O_z\},\ L_{xy} \cap R_y = \{O_t\},\ L_{zt} \cap R_z = \{O_y\},\ L_{zt} \cap R_t = \{O_x\}.$$

We can easily see that

$$\mathrm{lct}(X, \frac{11}{4}C_x) = \frac{11}{6} < \mathrm{lct}(X, \frac{17}{4}C_y),\quad \mathrm{lct}(X, \frac{20}{4}C_z),\quad \mathrm{lct}(X, \frac{27}{4}C_t).$$

Therefore, $\mathrm{lct}(X) \geqslant \frac{11}{6}$. Suppose $\mathrm{lct}(X) < \frac{11}{6}$. Then, there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the log pair $(X, \frac{11}{6}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xy}, L_{zt}, R_x, R_y, R_z, R_t are as follows:

$$D \cdot L_{xy} = \frac{1}{5 \cdot 27},\quad D \cdot R_x = \frac{2}{5 \cdot 17},\quad D \cdot R_y = \frac{4}{9 \cdot 11},$$
$$D \cdot L_{zt} = \frac{4}{11 \cdot 17},\quad D \cdot R_z = \frac{16}{17 \cdot 27},\quad D \cdot R_t = \frac{3}{5 \cdot 11},$$
$$L_{xy} \cdot R_x = \frac{1}{10},\quad L_{xy} \cdot R_y = \frac{1}{9},\quad L_{zt} \cdot R_z = \frac{4}{17},\quad L_{zt} \cdot R_t = \frac{3}{11},$$
$$L_{xy}^2 = -\frac{43}{20 \cdot 27},\quad R_x^2 = -\frac{3}{5 \cdot 17},\quad R_y^2 = \frac{2}{3 \cdot 11},$$
$$L_{zt}^2 = \frac{24}{11 \cdot 17},\quad R_z^2 = -\frac{28}{17 \cdot 27},\quad R_t^2 = \frac{21}{20 \cdot 11}.$$

By Remark 1.4.7 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$11D \cdot L_{zt} = \frac{4}{17} < \frac{6}{11},\quad \frac{11}{2}D \cdot R_t = \frac{3}{10} < \frac{6}{11}$$

imply $P \neq O_x$. Note that the curve R_t is singular at O_x. The inequalities

$$20D \cdot L_{xy} = \frac{4}{27} < \frac{6}{11},\quad 20D \cdot R_x = \frac{8}{17} < \frac{6}{11}$$

imply $P \neq O_z$. The inequalities

$$27D \cdot L_{xy} = \frac{1}{5} < \frac{6}{11},\quad \frac{27}{3}D \cdot R_y = \frac{4}{11} < \frac{6}{11}$$

imply $P \neq O_t$. The curve R_y is singular at the point O_t.

Since the pair $(X, \frac{11}{6}D)$ is log canonical at the point O_x, $\mathrm{mult}_{L_{zt}}D \geqslant \frac{6}{11}$. By Lemma 1.4.8 the inequality $D \cdot L_{zt} - (\mathrm{mult}_{L_{zt}}D)L_{zt}^2 \geqslant D \cdot L_{zt} = \frac{4}{11 \cdot 17} \geqslant \frac{6}{17 \cdot 11}$ implies $P \not\in L_{zt}$. In particular, $P \neq O_y$. We write $D = a_1L_{xy} + a_2R_x + a_3R_y + a_4R_z + a_5R_t + \Omega$, where Ω is an effective divisor whose support contains none of the curves L_{xy}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{11}{6}D)$ is log canonical at the points O_x, O_y, O_z, O_t, the numbers a_i are at most $\frac{6}{11}$. Then by Lemma 1.4.8 the following inequalities enable us to conclude that the point P is in the outside of $C_x \cup C_y \cup C_z \cup C_t$:

$$\frac{11}{6}D \cdot L_{xy} - L_{xy}^2 < 1,\quad \frac{11}{6}D \cdot R_x - R_x^2 < 1,\quad \frac{11}{6}D \cdot R_z - R_z^2 < 1,$$
$$\frac{11}{6}D \cdot R_y - R_y^2 \geqslant \frac{11}{6}D \cdot R_y < 1,\quad \frac{11}{6}D \cdot R_t - R_t^2 \geqslant \frac{11}{6}D \cdot R_t < 1.$$

We consider the pencil $\mathcal{L}$ defined by $\lambda ty + \mu x^4 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. The base locus of the pencil consists of the curve L_{xy} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P \not\in C_x \cup C_y \cup C_z \cup C_t$, the divisor E is defined by the equation $ty = \alpha x^4$, where $\alpha \neq 0$.

Suppose that $\alpha \neq -1$. Then the curve E is isomorphic to the curve defined by the equations $ty = x^4$ and $x^4t + y^3z + xz^3 = 0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t \neq 0$ is the curve given by

$x(x^2+x^{11}z+z^3)=0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{xy} and C. We have the intersection numbers

$$D\cdot C=D\cdot E-D\cdot L_{xy}=\frac{267}{5\cdot 17\cdot 27},\quad C\cdot L_{xy}=E\cdot L_{xy}-L_{xy}^2=\frac{87}{20\cdot 27}.$$

Also, we see

$$C^2=E\cdot C-C\cdot L_{xy}=\frac{10269}{17\cdot 20\cdot 27}.$$

By Lemma 1.4.8 the inequality $D\cdot C<\frac{6}{11}$ gives us a contradiction.

Suppose that $\alpha=-1$. Then divisor E consists of three irreducible and reduced curves L_{xy}, R_z, and M. Note that the curve M is different from the curves R_x and L_{zt}. Also, it is smooth at the point P. We have

$$D\cdot M=D\cdot E-D\cdot L_{xy}-D\cdot R_z=\frac{187}{5\cdot 17\cdot 27},$$

$$M^2=E\cdot M-L_{xy}\cdot M-R_z\cdot M\geq E\cdot M-C_x\cdot M-C_z\cdot M=\frac{13}{4}D\cdot M>0.$$

By Lemma 1.4.8 the inequality $D\cdot M<\frac{6}{11}$ gives us a contradiction. □

Lemma 3.4.6. Suppose that $(a_0,a_1,a_2,a_3,d)=(11,17,24,31,79)$. Then $\mathrm{lct}(X)=33/16$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y+tz^2+xy^4+x^5z=0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of $L_{xt}=\{x=t=0\}$ (resp. $L_{yz}=\{y=z=0\}$, L_{yz}, L_{xt}) and $R_x=\{x=yt+z^2=0\}$ (resp. $R_y=\{y=zt+x^5=0\}$, $R_z=\{z=xy^3+t^2=0\}$, $R_t=\{t=y^4+x^4z=0\}$). Also, we see that

$$L_{xt}\cap R_x=\{O_y\},\ L_{yz}\cap R_y=\{O_t\},\ L_{yz}\cap R_z=\{O_x\},\ L_{xt}\cap R_t=\{O_z\}.$$

We can easily see that

$$\mathrm{lct}(X,\frac{4}{11}C_x)=\frac{33}{16}<\mathrm{lct}(X,\frac{4}{17}C_y),\ \ \mathrm{lct}(X,\frac{4}{24}C_z),\ \ \mathrm{lct}(X,\frac{4}{31}C_t).$$

Therefore, $\mathrm{lct}(X)\geqslant\frac{33}{16}$. Suppose $\mathrm{lct}(X)<\frac{33}{16}$. Then, there is an effective $\mathbb{Q}$-divisor $D\equiv -K_X$ such that the log pair $(X,\frac{33}{16}D)$ is not log canonical at some point $P\in X$.

The intersection numbers among the divisors D, L_{xt}, L_{yz}, R_x, R_y, R_z, R_t are as follows:

$$D\cdot L_{xt}=\frac{1}{6\cdot 17},\quad D\cdot R_x=\frac{8}{17\cdot 31},\quad D\cdot R_y=\frac{5}{6\cdot 31},$$
$$D\cdot L_{yz}=\frac{4}{11\cdot 31},\quad D\cdot R_z=\frac{8}{11\cdot 17},\quad D\cdot R_t=\frac{2}{3\cdot 11},$$
$$L_{xt}\cdot R_x=\frac{2}{17},\quad L_{yz}\cdot R_y=\frac{5}{31},\quad L_{yz}\cdot R_z=\frac{2}{11},\quad L_{xt}\cdot R_t=\frac{1}{6},$$
$$L_{xt}^2=-\frac{37}{17\cdot 24},\quad R_x^2=-\frac{40}{17\cdot 31},\quad R_y^2=-\frac{35}{24\cdot 31},$$
$$L_{yz}^2=-\frac{38}{11\cdot 31},\quad R_z^2=\frac{14}{11\cdot 17},\quad R_t^2=\frac{10}{3\cdot 11}.$$

By Remark 1.4.7 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$17D\cdot L_{xt}=\frac{1}{6}<\frac{16}{33},\quad 17D\cdot R_x=\frac{8}{31}<\frac{16}{33}$$

imply $P\neq O_y$. The inequalities

$$11D\cdot L_{yz}=\frac{4}{31}<\frac{16}{33},\quad 11D\cdot R_z=\frac{8}{17}<\frac{16}{33}$$

imply $P \neq O_x$. The inequalities

$$24D \cdot L_{xt} = \frac{24}{6 \cdot 17} < \frac{16}{33}, \quad \frac{24}{4} D \cdot R_t = \frac{4}{11} < \frac{16}{33}$$

imply $P \neq O_z$. The curve R_t is singular at the point O_z.

We write $D = a_1 L_{xt} + a_2 L_{yz} + a_3 R_x + a_4 R_y + a_5 R_z + a_6 R_t + \Omega$, where Ω is an effective divisor whose support contains none of the curves L_{xt}, L_{yz}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{33}{16}D)$ is log canonical at the points O_x, O_y, O_z, the numbers a_i are at most $\frac{16}{33}$. Then by Lemma 1.4.8 the following inequalities enable us to conclude that either the point P is in the outside of $C_x \cup C_y \cup C_z \cup C_t$ or $P = O_t$:

$$\frac{33}{16} D \cdot L_{xt} - L_{xt}^2 = \frac{181}{3 \cdot 17 \cdot 32} < 1, \quad \frac{33}{16} D \cdot R_x - R_x^2 = \frac{113}{2 \cdot 17 \cdot 31} < 1, \quad \frac{33}{16} D \cdot R_y - R_y^2 = \frac{25}{3 \cdot 31} < 1,$$

$$\frac{33}{16} D \cdot L_{yz} - L_{xt}^2 = \frac{185}{4 \cdot 11 \cdot 31} < 1, \quad \frac{33}{16} D \cdot R_z - R_z^2 = \frac{5}{2 \cdot 11 \cdot 17} < 1, \quad \frac{33}{16} D \cdot R_t - R_t^2 = \frac{-47}{3 \cdot 8 \cdot 11} < 1.$$

Suppose that $P \neq O_t$. Then we consider the pencil $\mathcal{L}$ defined by $\lambda yt + \mu z^2 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. The base locus of the pencil consists of the curve L_{yz} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P \notin C_x \cup C_y \cup C_z \cup C_t$, the divisor E is defined by the equation $z^2 = \alpha yt$, where $\alpha \neq 0$.

Suppose that $\alpha \neq -1$. Then the curve E is isomorphic to the curve defined by the equations $yt = z^2$ and $t^2 y + xy^4 + x^5 z = 0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t \neq 0$ is the curve given by $z(z + xz^7 + x^5) = 0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{yz} and C. We have the intersection numbers

$$D \cdot C = D \cdot E - D \cdot L_{yz} = \frac{564}{11 \cdot 17 \cdot 31}, \quad C \cdot L_{yz} = E \cdot L_{yz} - L_{yz}^2 = \frac{2}{11}.$$

Also, we see

$$C^2 = E \cdot C - C \cdot L_{yz} > 0.$$

By Lemma 1.4.8 the inequality $D \cdot C < \frac{16}{33}$ gives us a contradiction.

Suppose that $\alpha = -1$. Then divisor E consists of three irreducible and reduced curves L_{yz}, R_x, and M. Note that the curve M is different from the curves R_y and L_{xt}. Also, it is smooth at the point P. We have

$$D \cdot M = D \cdot E - D \cdot L_{yz} - D \cdot R_x = \frac{4 \cdot 119}{11 \cdot 17 \cdot 31},$$

$$M^2 = E \cdot M - L_{yz} \cdot M - R_x \cdot M \geq E \cdot M - C_y \cdot M - C_x \cdot M = 5D \cdot M > 0.$$

By Lemma 1.4.8 the inequality $D \cdot M < \frac{16}{33}$ gives us a contradiction. Therefore, $P = O_t$.

We write $D = aL_{yz} + bR_x + \Delta$, where Δ is an effective divisor whose support contains neither L_{yz} nor R_x. Note that we already assumed that the support of D does not contain both L_{yz} and R_y. If the support of D contains R_y, then it does not contain L_{yz}. However, the inequality $31 D \cdot L_{yz} = \frac{4}{11} < \frac{16}{33}$ shows that $P \neq O_t$. Therefore, the support of D does not contain the curve R_y. The inequality $D \cdot L_{xt} \geq bR_x \cdot L_{xt}$ implies $b \leqslant \frac{1}{12}$. On the other hand, we have

$$\frac{5}{6 \cdot 31} = D \cdot R_y \geq \frac{5a}{31} + \frac{b}{31} + \frac{\mathrm{mult}_{O_t} D - a - b}{31} > \frac{4a + \frac{16}{33}}{31},$$

and hence $a < \frac{23}{4 \cdot 66}$.

We now consider the weighted blow up $\pi \colon \bar{X} \to X$ at the point O_t with weight $(11, 24)$. Its exceptional divisor F passes through two singular points Q_{11} of type $\frac{1}{11}(1,1)$ and Q_{24} of type $\frac{1}{24}(13,7)$. We have

$$K_{\bar{X}} = \pi^*(K_X) + \frac{4}{31} F, \quad \bar{L}_{yz} = \pi^*(L_{yz}) - \frac{24}{31} F, \quad \bar{R}_x = \pi^*(R_x) - \frac{11}{31} F, \quad \bar{R}_y = \pi^*(R_y) - \frac{24}{31} F,$$

where $\bar{L}_{yz}$, $\bar{R}_x$ and $\bar{R}_y$ are the proper transforms of L_{yz}, R_x and R_y by π, respectively. Also, we have a non-negative rational number c such that

$$\bar{\Delta} = \pi^*(\Delta) - \frac{c}{31} F,$$

where $\bar{\Delta}$ is the proper transform of Δ by π. From

$$0 \leqslant \bar{\Delta} \cdot \bar{R}_y = \Delta \cdot R_y - \frac{c}{11 \cdot 31} = (D - aL_{yz} - bR_x) \cdot R_y - \frac{c}{11 \cdot 31} = \frac{5}{6 \cdot 31} - \frac{5a}{31} - \frac{b}{31} - \frac{c}{11 \cdot 31}$$

we obtain $55a + 11b + c \leqslant \frac{55}{6}$. Also from

$$0 \leqslant \bar{\Delta} \cdot \bar{L}_{yz} = \Delta \cdot L_{yz} - \frac{c}{11 \cdot 31} = (D - aL_{yz} - bR_x) \cdot L_{yz} - \frac{c}{11 \cdot 31} = \frac{4}{11 \cdot 31} + \frac{38a}{11 \cdot 31} - \frac{b}{31} - \frac{c}{11 \cdot 31}$$

we get $11b + c \leqslant 4 + 38a$. Combining this with the previous inequality, we get

$$\frac{55(11b + c - 4)}{38} + 11b + c \leqslant \frac{55}{6} \quad \Rightarrow \quad (1 + \frac{55}{38})c \leqslant \frac{55}{6} + \frac{4 \cdot 55}{38} \quad \Rightarrow \quad c \leqslant \frac{55}{9}.$$

Now we consider the log pull-back of the divisor $K_X + \frac{33}{16}D$ by π

$$\pi^*(K_X + \frac{33}{16}D) = K_{\bar{X}} + \frac{33a}{16}\bar{L}_{yz} + \frac{33b}{16}\bar{R}_x + \frac{33}{16}\bar{\Delta} + \theta_1 F,$$

where

$$\theta_1 = \frac{1}{16 \cdot 31}(24 \cdot 33a + 11 \cdot 33b + 33c - 64) < \frac{2843}{12 \cdot 16 \cdot 31}.$$

There must be a point Q in F at which the pair

$$\left(\bar{X}, \frac{33a}{16}\bar{L}_{yz} + \frac{33b}{16}\bar{R}_x + \frac{33}{16}\bar{\Delta} + \theta_1 F\right)$$

is not log canonical. Note that $F \cap \bar{R}_y = F \cap \bar{L}_{yz} = \{Q_{11}\}$ and $F \cap \bar{R}_x = \{Q_{24}\}$. Therefore, the pair

$$\left(\bar{X}, \frac{33a}{16}\bar{L}_{yz} + \frac{33b}{16}\bar{R}_x + \frac{33}{16}\bar{\Delta} + F\right)$$

is not log canonical at the point Q. If the point Q is a smooth point of $\bar{X}$ then we obtain an absurd inequality

$$1 > \frac{55}{6 \cdot 128} > \frac{c}{128} = \frac{33}{16}\bar{\Delta} \cdot F > 1.$$

In order to apply Lemma 1.4.6, we must first check that $\theta_1 \geqslant 0$. Suppose that $\theta_1 \leqslant 0$. Then $24a + 11b + c \leqslant 64/33$, and the log pair

$$\left(\bar{X}, \frac{33a}{16}\bar{L}_{yz} + \frac{33b}{16}\bar{R}_x + \frac{33}{16}\bar{\Delta}\right)$$

is not log canonical at the point Q as well. Then

$$\frac{4}{11 \cdot 24} > \frac{33(24a + 11b + c)}{11 \cdot 24 \cdot 16} = \left(\frac{33a}{16}\bar{L}_{yz} + \frac{33b}{16}\bar{R}_x + \frac{33}{16}\bar{\Delta}\right) \cdot F > \begin{cases} 1 \text{ if } Q_{24} \neq Q \neq Q_{11}, \\ \frac{1}{11} \text{ if } Q = Q_{11}, \\ \frac{1}{24} \text{ if } Q = Q_{24}, \end{cases}$$

which is absurd. Thus, we see that $\theta_1 > 0$.

Suppose that $Q = Q_{11}$. Then we also obtain a contradictory inequality

$$\frac{1}{11} < \frac{33a}{16}\bar{L}_{yz} \cdot F + \frac{33}{16}\bar{\Delta} \cdot F = \frac{33a}{11 \cdot 16} + \frac{33c}{11 \cdot 16 \cdot 24} < \frac{33 \cdot 23}{4 \cdot 11 \cdot 16 \cdot 66} + \frac{33 \cdot 55}{6 \cdot 11 \cdot 16 \cdot 24} < \frac{1}{11},$$

which implies that $Q \neq Q_{11}$. Therefore, we see that $Q = Q_{24}$.

Let $\phi : \tilde{X} \to \bar{X}$ be the weighted blow up at the point Q_{24} with weight $(13, 7)$. The exceptional divisor G of the morphism ϕ contains two singular points Q_{13} and Q_7 of $\tilde{X}$. The point Q_{13} is of type $\frac{1}{13}(11, 6)$ and the point Q_7 is of type $\frac{1}{7}(1, 3)$. We have

$$K_{\tilde{X}} = \phi^*(K_{\bar{X}}) - \frac{1}{6}G, \quad \tilde{R}_x = \phi^*(\bar{R}_x) - \frac{13}{24}G, \quad \tilde{F} = \phi^*(F) - \frac{7}{24}G, \quad \tilde{\Delta} = \phi^*(\bar{\Delta}) - \frac{d}{24}G,$$

where d is a positive rational number. Then

$$\frac{c}{11 \cdot 24} - \frac{d}{13 \cdot 24} = \tilde{\Delta} \cdot \tilde{F} \geqslant 0 \leqslant \tilde{\Delta} \cdot \tilde{R}_x = \frac{8}{17 \cdot 31} - \frac{a}{31} + \frac{40b}{17 \cdot 31} - \frac{c}{24 \cdot 31} - \frac{d}{7 \cdot 24}$$

which implies that $1344 + 6720b \geqslant 2856a + 119c + 527d$ and $13c \geqslant 11d$.

The log pull-back of $(X, \frac{33}{16}D)$ via $\phi\circ\pi$ is

$$\left(\tilde{X}, \frac{33a}{16}\tilde{L}_{yz} + \frac{33b}{16}\tilde{R}_x + \frac{33}{16}\tilde{\Delta} + \theta_1\tilde{F} + \theta_2 G\right),$$

which is not log canonical at some point O in G, where $\theta_2 = 231a/496 + 165b/124 + 77c/3968 + 11d/128 + 4/31$. Then $\theta_2 < 1$, because the system of inequalities

$$\begin{cases} \theta_2 \geqslant 1, \\ 1344 + 6720b \geqslant 2856a + 119c + 527d, \\ 13c - 11d \geqslant 0, \\ 4 + 38a \geqslant 11b + c >= 0, \\ 55a + 11b + c \leqslant 55/6, \\ a \leqslant 23/264, \\ b \leqslant 1/12, \end{cases}$$

is inconsistent. Note that $\tilde{R}_x \cap G = \{Q_7\}$ and $\tilde{F} \cap G = \{Q_{13}\}$. But $\bar{L}_{yz}$ does not pass through the point Q_{24}.

Suppose that $O \neq Q_7$ and $O \neq Q_{13}$. Applying Lemma 1.4.6, we get

$$1 < \frac{33}{16}\tilde{\Delta}\cdot G = \frac{33d}{16\cdot 7\cdot 13},$$

which gives $d > 3536/33$. Hence, we obtain the system of inequalities

$$\begin{cases} d > 3536/33, \\ 1344 + 6720b \geqslant 2856a + 119c + 527d, \\ 13c - 11d \geqslant 0, \\ 4 + 38a \geqslant 11b + c >= 0, \\ 55a + 11b + c \leqslant 55/6, \\ a \leqslant 23/264, \\ b \leqslant 1/12, \end{cases}$$

which is inconsistent. Thus, we see that either $O = Q_7$ or $O = Q_{13}$.

Suppose that $O = Q_7$. Applying Lemma 1.4.6, we get

$$\frac{33}{16}\left(\frac{8+40b}{17\cdot 31} - \frac{a}{31} - \frac{c}{24\cdot 31} - \frac{d}{7\cdot 24}\right) + \frac{\theta_2}{7} = \left(\frac{33}{16}\tilde{\Delta} + \theta_2 G\right)\cdot\tilde{R}_x > \frac{1}{7} < \frac{33}{16}\left(\tilde{\Delta} + b\tilde{R}_x\right)\cdot G = \frac{33}{16}\left(\frac{d}{7\cdot 13} + \frac{b}{7}\right),$$

which gives $b > 458/1705$ and $33d + 429b > 208$. But $b \leqslant 1/12$, which is a contradiction. Thus, we see that $O \neq Q_7$.

Therefore, we see that $O = Q_{13}$. Applying Lemma 1.4.6, we get

$$\frac{33}{16}\left(\frac{c}{11\cdot 24} - \frac{d}{13\cdot 24}\right) + \frac{\theta_2}{13} = \left(\frac{33}{16}\tilde{\Delta} + \theta_2 G\right)\cdot\tilde{F} > \frac{1}{13} < \left(\frac{33}{16}\tilde{\Delta} + \theta_1\tilde{F}\right)\cdot G = \frac{33d}{16\cdot 7\cdot 13} + \frac{\theta_1}{13},$$

which leads to a contradiction, because $4 + 38a \geqslant 11b + c$ and $a \leqslant 23/264$. □

Lemma 3.4.7. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 31, 45, 83, 166)$. Then $\mathrm{lct}(X) = 55/24$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^5 + x^{11}z = 0,$$

the surface X is singular at the point O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{55}{24} = \mathrm{lct}\left(X, \frac{4}{11}C_x\right) < \mathrm{lct}\left(X, \frac{4}{31}C_y\right) = \frac{13\cdot 31}{88},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 55/24$.

Suppose that $\mathrm{lct}(X) < 55/24$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{55}{24}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(495))$ contains x^{45}, $y^{11}x^{14}$ and z^{11}, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P \in C_x$. Then

$$\frac{4}{31 \cdot 45} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{31} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{45} \text{ if } P = O_z, \\ \mathrm{mult}_P(D) \text{ if } P \neq O_y \text{ and } P \neq O_z, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 24/55$. Thus, we see that $P = O_x$. Then

$$\frac{4}{11 \cdot 45} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{13} > \frac{24}{55 \cdot 11} > \frac{4}{11 \cdot 45},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 55/24$. □

Lemma 3.4.8. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 14, 19, 29, 71)$. Then $\mathrm{lct}(X) = 65/36$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$ty^3 + yz^3 + xt^2 + x^4z = 0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of $L_{xy} = \{x = y = 0\}$ (resp. $L_{xy} = \{x = y = 0\}$, $L_{zt} = \{z = t = 0\}$, $L_{zt} = \{z = t = 0\}$) and $R_x = \{x = z^3 + ty^2 = 0\}$ (resp. $R_y = \{y = x^3z + t^2 = 0\}$, $R_z = \{z = y^3 + xt = 0\}$, $R_t = \{t = x^4 + yz^2 = 0\}$). Also, we see that

$$L_{xy} \cap R_x = \{O_t\},\ L_{xy} \cap R_y = \{O_z\},\ L_{zt} \cap R_z = \{O_x\},\ L_{zt} \cap R_t = \{O_y\}.$$

We can easily see that

$$\mathrm{lct}(X, \frac{13}{4}C_x) = \frac{65}{36} < \mathrm{lct}(X, \frac{14}{4}C_y),\ \ \mathrm{lct}(X, \frac{19}{4}C_z),\ \ \mathrm{lct}(X, \frac{29}{4}C_t).$$

Therefore, $\mathrm{lct}(X) \geqslant \frac{65}{36}$. Suppose $\mathrm{lct}(X) < \frac{65}{36}$. Then, there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the log pair $(X, \frac{65}{36}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xy}, L_{zt}, R_x, R_y, R_z, R_t are as follows:

$$D \cdot L_{xy} = \frac{4}{19 \cdot 29},\quad D \cdot R_x = \frac{6}{7 \cdot 29},\quad D \cdot R_y = \frac{8}{13 \cdot 19},$$
$$D \cdot L_{zt} = \frac{2}{7 \cdot 13},\quad D \cdot R_z = \frac{12}{13 \cdot 29},\quad D \cdot R_t = \frac{8}{7 \cdot 19},$$
$$L_{xy} \cdot R_x = \frac{3}{29},\quad L_{xy} \cdot R_y = \frac{2}{19},\quad L_{zt} \cdot R_z = \frac{3}{13},\quad L_{zt} \cdot R_t = \frac{2}{7},$$
$$L_{xy}^2 = -\frac{44}{19 \cdot 29},\quad R_x^2 = -\frac{3}{14 \cdot 29},\quad R_y^2 = \frac{2}{13 \cdot 19},$$
$$L_{zt}^2 = -\frac{23}{13 \cdot 14},\quad R_z^2 = -\frac{30}{13 \cdot 29},\quad R_t^2 = \frac{20}{7 \cdot 19}.$$

By Remark 1.4.7 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$13D \cdot L_{zt} = \frac{2}{7} < \frac{36}{65},\quad 13D \cdot R_z = \frac{12}{29} < \frac{36}{65}$$

imply $P \neq O_x$. The inequalities

$$14D \cdot L_{zt} = \frac{4}{13} < \frac{36}{65},\quad 7D \cdot R_t = \frac{8}{19} < \frac{36}{65}$$

imply $P \neq O_y$. Note that the curve R_t is singular at the point O_y. The inequalities

$$19D \cdot L_{xy} = \frac{4}{29} < \frac{36}{65},\quad \frac{19}{2}D \cdot R_y = \frac{4}{13} < \frac{36}{65}$$

imply $P \neq O_z$. The curve R_y is singular at O_z. The inequalities

$$29D \cdot L_{xy} = \frac{4}{19} < \frac{36}{65}, \quad \frac{29}{2} D \cdot R_x = \frac{3}{7} < \frac{36}{65}$$

imply $P \neq O_t$. The curve R_x is singular at the point O_t.

We write $D = a_1 L_{xy} + a_2 L_{zt} + a_3 R_x + a_4 R_y + a_5 R_z + a_6 R_t + \Omega$, where Ω is an effective divisor whose support contains none of the curves L_{xy}, L_{zt}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{65}{36}D)$ is log canonical at the points O_x, O_y, O_z, O_t, the numbers a_i are at most $\frac{36}{65}$. Then by Lemma 1.4.8 the following inequalities enable us to conclude that the point P must be located in the outside of $C_x \cup C_y \cup C_z \cup C_t$:

$$\frac{65}{36} D \cdot L_{xy} - L_{xy}^2 = \frac{461}{9 \cdot 19 \cdot 29} < 1, \quad \frac{65}{36} D \cdot R_x - R_x^2 = \frac{74}{6 \cdot 7 \cdot 29} < 1,$$

$$\frac{65}{36} D \cdot L_{zt} - L_{zt}^2 = \frac{249}{7 \cdot 13 \cdot 18} < 1, \quad \frac{65}{36} D \cdot R_z - R_z^2 = \frac{155}{3 \cdot 13 \cdot 18} < 1,$$

$$\frac{65}{36} D \cdot R_y - R_y^2 \geqslant \frac{65}{36} D \cdot R_y = \frac{65}{13 \cdot 18 \cdot 19} < 1, \quad \frac{65}{36} D \cdot R_t - R_t^2 < 1.$$

We consider the pencil $\mathcal{L}$ defined by $\lambda tx + \mu y^3 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. The base locus of the pencil consists of the curve L_{xy} and the point O_x. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P \notin C_x \cup C_y \cup C_z \cup C_t$, the divisor E is defined by the equation $tx = \alpha y^3$, where $\alpha \neq 0$.

Suppose that $\alpha \neq -1$. Then the curve E is isomorphic to the curve defined by the equations $tx = y^3$ and $xt^2 + yz^3 + x^4 z = 0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t \neq 0$ is the curve given by $y(y^2 + y^{11} z + z^3) = 0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{xy} and C. We have the intersection numbers

$$D \cdot C = D \cdot E - D \cdot L_{xy} = \frac{800}{13 \cdot 19 \cdot 29}, \quad C \cdot L_{xy} = E \cdot L_{xy} - L_{xy}^2 = \frac{86}{19 \cdot 29}.$$

Also, we see

$$C^2 = E \cdot C - C \cdot L_{xy} \geq E \cdot C - C_x \cdot C > 0.$$

By Lemma 1.4.8 the inequality $D \cdot C < \frac{36}{65}$ gives us a contradiction.

Suppose that $\alpha = -1$. Then divisor E consists of three irreducible and reduced curves L_{xy}, R_z, and M. Note that the curve M is different from the curves R_x and L_{zt}. Also, it is smooth at the point P. We have

$$D \cdot M = D \cdot E - D \cdot L_{xy} - D \cdot R_z = \frac{572}{13 \cdot 19 \cdot 29},$$

$$M^2 = E \cdot M - L_{xy} \cdot M - R_z \cdot M \geq E \cdot M - C_x \cdot M - C_z \cdot M = \frac{5}{2} D \cdot M > 0.$$

By Lemma 1.4.8 the inequality $D \cdot M < 36/65$ gives us a contradiction. □

Lemma 3.4.9. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 14, 23, 33, 79)$. Then $\mathrm{lct}(X) = 65/32$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^2 t + y^4 z + xt^2 + x^5 y = 0,$$

and X is singular at O_x, O_y, O_z and O_t. We have

$$\mathrm{lct}\left(X, \frac{4}{13} C_x\right) = \frac{65}{32} < \mathrm{lct}\left(X, \frac{4}{13} C_x\right) = \frac{21}{8} < \mathrm{lct}\left(X, \frac{5}{25} C_t\right) = \frac{33}{10} < \mathrm{lct}\left(X, \frac{4}{23} C_z\right) = \frac{69}{20},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 65/32$.

The curve C_x is reducible. We have $C_x = L_{xz} + M_x$, where L_{xz} and M_x are irreducible reduced curves such that L_{xz} is given by $x = z = 0$, and M_x is given by $x = tz + y^4 = 0$. Then

$$L_{xz} \cdot L_{xz} = \frac{-43}{14 \cdot 33}, \quad M_x \cdot M_x = \frac{-40}{23 \cdot 33}, \quad L_{xz} \cdot M_x = \frac{4}{33}, \quad D \cdot L_{xz} = \frac{4}{14 \cdot 33}, \quad D \cdot M_x = \frac{16}{23 \cdot 33},$$

and $L_{xz} \cap M_x = O_t$. The curves L_{xz} and M_x are smooth.

The curve C_y is reducible. We have $C_y = L_{yt} + M_y$, where L_{yt} and M_y are irreducible curves such that L_{yt} is given by $y = t = 0$, and M_y is given by $y = xt + z^2 = 0$. Then

$$L_{yt} \cdot L_{yt} = \frac{-32}{13 \cdot 23},\ M_y \cdot M_y = \frac{-38}{13 \cdot 33},\ \ L_{yt} \cdot M_y = \frac{2}{13},\ D \cdot L_{yt} = \frac{4}{13 \cdot 23},\ D \cdot M_y = \frac{8}{13 \cdot 33},$$

and $L_{yz} \cap M_y = O_x$. We have $M_y \cdot M_x = L_{xz} \cdot M_y = 1/33$, $M_x \cdot L_{yt} = 1/23$ and $L_{xz} \cdot L_{yt} = 0$.

The curve C_z is reducible. We have $C_z = L_{xz} + M_z$, where M_z is an irreducible curve that is given by the equations $z = t^2 + x^4 x = 0$. We have

$$M_z \cdot M_z = \frac{20}{13 \cdot 14},\ L_{xz} \cdot M_z = \frac{2}{14},\ D \cdot M_z = \frac{46}{13 \cdot 14},$$

and $M_z \cap L_{xz} = O_y$. The only singular point of the curve M_z is O_y.

The curve C_t is reducible. We have $C_t = L_{yt} + M_t$, where M_t is an irreducible curve that is given by the equations $t = y^3 z + x^5 = 0$. We have

$$M_t \cdot M_t = \frac{95}{14 \cdot 13},\ L_{yt} \cdot M_t = \frac{5}{23},\ D \cdot M_t = \frac{20}{14 \cdot 23},$$

and $M_t \cap L_{yt} = O_z$. The only singular point of the curve M_t is O_z

We suppose that $\mathrm{lct}(X) < 65/8$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{65}{32}D)$ is not log canonical at some point $P \in X$. Let us derive a contradiction.

Suppose that $P \notin C_x \cup C_y \cup C_z \cup C_t$. Then there is a unique curve $Z_\alpha \subset X$ that is cut out by

$$xt + \alpha z^2 = 0$$

such that $P \in Z$, where $0 \neq \alpha \in \mathbb{C}$. The curve Z_α is reduced. But it is always reducible. Indeed, one can easily check that

$$Z_\alpha = C_\alpha + L_{xz}$$

where C_α is a reduced curve whose support contains no L_{xy}. Let us prove that C_α is irreducible if $\alpha \neq 1$.

The open subset $Z_\alpha \setminus (Z_\alpha \cap C_x)$ of the curve Z_α is a $\mathbb{Z}_{13}$-quotient of the affine curve

$$t + \alpha z^2 = 0 = z^2 t + y^4 z + t^2 + y = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

which is isomorphic to a plane affine curve that is given by the equation

$$\alpha(\alpha - 1)z^4 + y^4 z + y = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big),$$

which implies that the curve C_α is irreducible and $\mathrm{mult}_P(C_\alpha) \leqslant 3$ if $\alpha \neq 1$.

The case $\alpha = 1$ is special. Namely, if $\alpha = 1$, then

$$C_1 = R_1 + M_y,$$

where R_1 is a reduced curve whose support contains no C_1. Arguing as in the case $\alpha \neq 1$, we see that R_1 is irreducible and R_1 is smooth at the point P.

By Remark 1.4.7, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible components of the curve Z_α.

Suppose that $\alpha \neq 1$. Then elementary calculations imply that

$$C_\alpha \cdot L_{xz} = \frac{2}{14},\ C_\alpha \cdot C_\alpha = \frac{20}{13 \cdot 14},\ D \cdot C_\alpha = \frac{8}{13 \cdot 14},$$

and we can put $D = \epsilon C_\alpha + \Delta_\alpha$, where Δ_α is an effective $\mathbb{Q}$-divisor such that $C_\alpha \not\subset \mathrm{Supp}(\Delta_\alpha)$. If $\epsilon \neq 0$, then

$$\frac{4}{13 \cdot 33} = D \cdot L_{xz} = \Big(\epsilon C_\alpha + \Delta_\alpha\Big) \cdot L_{xz} \geqslant \epsilon C_\alpha \cdot L_{xz} = \frac{2\epsilon}{14},$$

which implies that $\epsilon \leqslant 2/33$. On the other hand, we see that

$$\frac{8}{13 \cdot 14} = D \cdot C_\alpha = \epsilon C_\alpha^2 + \Delta_\alpha \cdot C_\alpha \geqslant \epsilon C^2 + \mathrm{mult}_P\big(\Delta_\alpha\big) = \epsilon C^2 + \mathrm{mult}_P\big(D\big) - \epsilon \mathrm{mult}_P\big(C_\alpha\big) > \epsilon C^2 + \frac{32}{65} - 3\epsilon,$$

which is impossible, because $\epsilon \leqslant 2/33$.

Thus, we see that $\alpha = 1$. We have

$$R_1 \cdot L_{xz} = \frac{52}{14 \cdot 33},\ R_1 \cdot R_1 = \frac{-398}{3003},\ M_y \cdot R_1 = \frac{71}{13 \cdot 33},\ D \cdot R_1 = \frac{152}{13 \cdot 14 \cdot 33},$$

and we can put $D = \epsilon_1 R_1 + \Xi_1$, where Ξ_1 is an effective $\mathbb{Q}$-divisor such that $R_1 \not\subset \mathrm{Supp}(\Xi_1)$. Then $\epsilon_1 \leqslant 8/71$, because either $\epsilon_1 = 0$, or $L_{xz} \cdot \Xi_1 \geqslant 0$ or $M_y \cdot \Xi_1 \geqslant 0$. By Lemma 1.4.6, we see that

$$\frac{152 + 796\epsilon_1}{13 \cdot 14 \cdot 33} = \Xi_1 \cdot R_1 > \frac{32}{65},$$

which implies that $\epsilon_1 > 3506/995$. But $\epsilon_1 \leqslant 8/71$. The obtained contradiction shows that $P \in C_x \cup C_y \cup C_z \cup C_t$.

It follows from Remark 1.4.7 that we may assume that $\mathrm{Supp}(D)$ does not contains are least one irreducible component of the curves C_x, C_y, C_z, C_t.

Suppose that $P \in M_x \setminus (O_t \cup O_y)$. Put $D = eM_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $M_x \not\subset \mathrm{Supp}(\Upsilon)$. If $e \neq 0$, then

$$\frac{4}{13 \cdot 33} = D \cdot L_{xz} = \big(eM_x + \Upsilon\big) \cdot L_{xz} \geqslant eL_{xz} \cdot M_x = \frac{4e}{33},$$

which implies that $e \leqslant 1/14$. Then it follows from Lemma 1.4.6 that

$$\frac{16 + 40e}{23 \cdot 33} = \big(-K_X - eM_x\big) \cdot M_x = \Upsilon \cdot M_x > \frac{32}{65},$$

because $P \notin \mathrm{Sing}(X)$. Thus, we see that $e > 2906/325$, which is impossible, because $e \leqslant 1/14$.

Thus, we see that $P \notin M_x \setminus (O_y \cup O_t)$. Similarly, we see that

$$P \notin M_y \cup M_z \cup M_z \cup M_t \setminus \Big(O_x \cup O_y \cup O_z \cup O_t\Big).$$

Suppose that $P \in L_{yt}$. Put $D = \delta L_{yt} + \Theta$, where Θ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{yt}. If $\delta \neq 0$, then

$$\frac{8}{13 \cdot 33} = D \cdot M_y = \big(\delta L_{yt} + \Theta\big) \cdot M_y \geqslant \delta L_{yt} \cdot M_y = \frac{2\delta}{13},$$

which implies that $\delta \leqslant 4/33$. Then it follows from Lemma 1.4.6 that

$$\frac{4 + 32\delta}{13 \cdot 23} = \big(-K_X - \delta L_{yz}\big) \cdot L_{yz} = \Theta \cdot L_{yz} > \begin{cases} \dfrac{32}{65} \text{ if } P \neq O_x \text{ and } P \neq O_z, \\ \dfrac{32}{65 \cdot 13} \text{ if } P = O_x, \\ \dfrac{32}{65 \cdot 23} \text{ if } P = O_z, \end{cases}$$

which implies that $P = O_z$ and $\delta > 3/40$. Then $M_t \not\subset \mathrm{Supp}(D)$. Hence, we have

$$\frac{20}{14 \cdot 23} = D \cdot M_t \geqslant \frac{\mathrm{mult}_{O_z}(D)\mathrm{mult}_{O_z}(M_t)}{23} = \frac{3\mathrm{mult}_{O_z}(D)}{23} > \frac{3 \cdot 32}{65 \cdot 23},$$

which is a contradiction. The obtained contradiction shows that $P \notin L_{yt}$.

We see that $P \in L_{xt}$. Arguing as above we see that $P = O_t$. Then

$$\frac{4}{14 \cdot 33} = D \cdot L_{xz} > \frac{32}{65 \cdot 33} > \frac{4}{14 \cdot 33}$$

whenever $L_{xz} \not\subset \mathrm{Supp}(D)$. Thus, we see that $L_{xz} \subset \mathrm{Supp}(D)$. Then $M_x \not\subset \mathrm{Supp}(D)$. Put

$$D = mL_{xz} + cM_y + \Omega,$$

where $m > 0$ and $c \geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subset \mathrm{Supp}(\Omega) \not\supset M_y$. Then

$$\frac{16}{23 \cdot 33} = D \cdot M_x = \big(mL_{xz} + cM_y + \Omega\big) \cdot M_x \geqslant \frac{4m}{33} + \frac{c}{33} + \frac{\mathrm{mult}_{O_t}(D) - m - c}{33} > \frac{3m + \frac{32}{65}}{33},$$

which implies that $m < 304/4485$. Then it follows from Lemma 1.4.6 that

$$\frac{4 + 43m}{14 \cdot 33} = \big(-K_X - mL_{xz}\big) \cdot L_{xz} = (\Omega + cM_y) \cdot L_{xz} > \frac{32}{65 \cdot 33},$$

which implies that $m > 88/2795$. On the other hand, if $c > 0$, then

$$\frac{4}{13 \cdot 23} = D \cdot L_{yt} = \big(mL_{xz} + cM_y + \Omega\big) \cdot L_{yt} \geqslant \frac{2c}{13},$$

which implies that $c \leqslant 2/23$. We will see later that $c > 0$.

Let $\pi\colon \bar{X}\to X$ be a weighted blow up of O_t with weights $(14,23)$, let E be the exceptional curve of π, let $\bar{\Omega}$, $\bar{L}_{xz}$, $\bar{M}_y$, $\bar{M}_x$ be the proper transforms of Ω, L_{xz}, M_y, M_x, respectively. Then

$$K_{\bar{X}}\equiv\pi^*(K_X)+\frac{4}{33}E,\ \bar{L}_{xz}\equiv\pi^*(L_{xz})-\frac{23}{33}E,\ \bar{M}_y\equiv\pi^*(M_y)-\frac{14}{33}E,\ \bar{M}_x\equiv\pi^*(M_x)-\frac{23}{33}E,$$

and there is a positive rational number a such that

$$\bar{\Omega}\equiv\pi^*(\Omega)-\frac{a}{33}E.$$

The curve E contains two singular points Q_{14} and Q_{23} of $\bar{X}$ such that Q_{14} is a singular point of type $\frac{1}{14}(13,1)$, and Q_{19} is a singular point of type $\frac{1}{23}(13,14)$. Then

$$\bar{L}_{xz}\cup\bar{M}_x\not\ni Q_{23}\in\bar{M}_y\not\ni Q_{14}=\bar{L}_{xz}\cap\bar{M}_x,$$

and $\bar{L}_{xz}\cap\bar{M}_y=\varnothing$. The log pull back of the log pair $(X,\frac{65}{32}D)$ is the log pair

$$\left(\bar{X},\ \frac{65}{32}\bar{\Omega}+\frac{65m}{32}\bar{L}_{xz}+\frac{65c}{32}\bar{M}_y+\left(\frac{1495m}{1056}+\frac{455c}{528}+\frac{65a}{1056}-\frac{4}{33}\right)E\right),$$

which must have non-log canonical singularity at some point $Q\in E$. We have

$$0\leqslant\bar{L}_{xz}\cdot\bar{\Omega}=\frac{4+43m-14c-a}{14\cdot 33},$$

which gives $a+14c\leqslant 4+43m$. Then $a<31012/4485$, because $m<304/4485$. We have

$$\left(\frac{1495m}{1056}+\frac{455c}{528}+\frac{65a}{1056}-\frac{4}{33}\right)<1,$$

because $a+14c\leqslant 4+43m$, $c\leqslant 2/23$ and $304/4485>m>88/2795$.

The log pull back of $(X,\frac{13}{8}D)$ has effective boundary if and only if the inequality

$$23m+14c+a\leqslant\frac{128}{65}$$

holds. On the other hand, if $23m+14c+a\leqslant 128/65$, then the log pair

$$\left(\bar{X},\ \frac{65}{32}\bar{\Omega}+\frac{65m}{32}\bar{L}_{xz}+\frac{65c}{32}\bar{M}_y\right)$$

is not log canonical at the point Q as well. Thus, if $23m+14c+a\leqslant 128/65$, then

$$\frac{128}{65\cdot 14\cdot 23}\geqslant\frac{a+23m+14c}{14\cdot 23}=\left(\bar{\Omega}+m\bar{L}_{yz}+c\bar{M}_x\right)\cdot E>\begin{cases}\dfrac{32}{65} & \text{if } Q_{14}\ne Q\ne Q_{23},\\ \dfrac{32}{65\cdot 14} & \text{if } Q=Q_{14},\\ \dfrac{32}{65\cdot 23} & \text{if } Q=Q_{23},\end{cases}$$

which is absurd. Thus, the boundary of the log pull back of the log pair $(X,\frac{65}{32}D)$ is effective.

Suppose that $Q\ne Q_{14}$ and $Q\ne Q_{23}$. Then $Q\not\in\bar{L}_{xz}\cup\bar{M}_y$. By Lemma 1.4.6, we have

$$\frac{a}{14\cdot 23}=-\frac{a}{33}E^2=\bar{\Omega}\cdot E>\frac{65}{32},$$

which implies that $a>10304/65$, which is impossible, because $a<31012/4485$.

Therefore, we see that either $Q=Q_{14}$ or $Q=Q_{23}$.

Suppose that $Q=Q_{11}$. Then $Q\not\in\bar{M}_y$. Hence, it follows from Lemma 1.4.6 that

$$\left(\frac{65}{32}\bar{\Omega}+\left(\frac{1495m}{1056}+\frac{455c}{528}+\frac{65a}{1056}-\frac{4}{33}\right)E\right)\cdot\bar{L}_{xz}>\frac{1}{14},$$

but $\bar{L}_{xz}\cdot E=1/14$ and $\bar{L}_{xz}\cdot\bar{M}_y=0$. Moreover, we have

$$\bar{\Omega}\cdot\bar{L}_{xz}=\left(\bar{\Omega}+c\bar{M}_y\right)\cdot\bar{L}_{xz}=\left(D-mL_{xz}\right)\cdot L_{xz}-\frac{a+14c}{14\cdot 33}=\frac{4+43m-14c-a}{14\cdot 25},$$

which immediately implies that $m>66/325$. But $m<304/4485$, which is a contradiction.

Thus, we see that $Q = Q_{23}$. Then $Q \notin \bar{L}_{xz}$, and it follows from Lemma 1.4.6 that

$$\left(\frac{65}{32}\bar{\Omega} + \left(\frac{1495m}{1056} + \frac{455c}{528} + \frac{65a}{1056} - \frac{4}{33}\right) E\right) \cdot \bar{M}_y > \frac{1}{23},$$

but we have $\bar{M}_y \cdot E = 1/23$. Applying Lemma 1.4.6 one more time, we see that

$$\left(\frac{65}{32}\bar{\Omega} + \frac{65c}{32}\bar{M}_y\right) \cdot E > \frac{1}{23},$$

which gives $a + 14c > 448/65$. On the other hand, we know that

$$0 \leqslant \bar{\Omega} \cdot \bar{M}_y = \Omega \cdot M_y - \frac{a}{33 \cdot 23} = D \cdot M_y - mL_{xz} \cdot M_y - cM_y \cdot M_y - \frac{a}{33 \cdot 23} = \frac{8 + 38c - 13m}{13 \cdot 33} - \frac{a}{33 \cdot 23},$$

which implies that $184 + 874c \geqslant 299m + 13a$ and $c > 1/20$. But we have no contradiction here.

Let $\psi\colon \tilde{X} \to \bar{X}$ be a weighted blow up of Q_{23} with weights $(13, 14)$, let G be the exceptional curve of ψ, let $\tilde{\Omega}$, $\tilde{L}_{xz}$, $\tilde{M}_y$, $\tilde{E}$ be the proper transforms of Ω, L_{xz}, M_y, E, respectively. Then

$$K_{\tilde{X}} \equiv \psi^*(K_{\bar{X}}) + \frac{4}{23}G,\ \tilde{M}_y \equiv \psi^*(\bar{M}_y) - \frac{14}{23}G,\ \tilde{E} \equiv \psi^*(E) - \frac{13}{23}G,\ \tilde{\Omega} \equiv \psi^*(\bar{\Omega}) - \frac{b}{23}G,$$

where b is a positive rational number.

The curve G contains two singular points O_{13} and O_{14} of $\tilde{X}$ such that O_{13} is a singular point of type $\frac{1}{13}(1,3)$, and O_{14} is a singular point of type $\frac{1}{14}(1,9)$. Then

$$\tilde{E} \not\ni O_{13} \in \tilde{M}_y \not\ni O_{14} \in \tilde{E},$$

where $\tilde{E} \cap \tilde{M}_y = \varnothing$. The log pull back of the log pair $(X, \frac{65}{32}D)$ is the log pair

$$\left(\tilde{X}, \frac{65}{32}\tilde{\Omega} + \frac{65m}{32}\tilde{L}_{xz} + \frac{65c}{32}\tilde{M}_y + \left(\frac{1495m}{1056} + \frac{455c}{528} + \frac{65a}{1056} - \frac{4}{33}\right)\tilde{E} + \theta G\right),$$

which must have non-log canonical singularity at some point $O \in G$, where

$$\theta = \frac{845m}{1056} + \frac{455c}{264} + \frac{845a}{24288} + \frac{65b}{736} - \frac{8}{33}.$$

Let us show that $0 < \theta < 1$. Obviously, we have

$$0 \leqslant \tilde{M}_y \cdot \tilde{\Omega} = \bar{\Omega} \cdot \bar{M}_y - \frac{b}{13 \cdot 23} = \frac{8 + 38c}{13 \cdot 33} - \frac{a + 23m}{23 \cdot 33} - \frac{b}{13 \cdot 23},$$

which gives $184 + 874c \geqslant 299m + 13a + 33b$. Similarly, we have

$$0 \leqslant \tilde{M}_y \cdot \tilde{E} = \bar{\Omega} \cdot E - \frac{b}{14 \cdot 23} = \frac{a}{13 \cdot 23} - \frac{b}{14 \cdot 23},$$

which implies that $a \geqslant b$. So far, we obtained the system of inequalities

$$\begin{cases} 4 + 43m \geqslant a + 14c, \\ 184 + 874c \geqslant 299m + 13a + 33b, \\ 184 + 874c \geqslant 299m + 13a, \\ 304/4485 > m > 88/2795, \\ 2/23 \geqslant c > 1/20, \\ a + 14c > 448/65, \\ 31012/4485 > a \geqslant b, \end{cases}$$

which is still consistent, but it implies that $\theta < 1$. If $\theta \leqslant 0$, then the log pair

$$\left(\tilde{X}, \frac{65}{32}\tilde{\Omega} + \frac{65m}{32}\tilde{L}_{xz} + \frac{65c}{32}\tilde{M}_y + \left(\frac{1495m}{1056} + \frac{455c}{528} + \frac{65a}{1056} - \frac{4}{33}\right)\tilde{E}\right),$$

is not log canonical at the point O as well. Thus, if $\theta \leqslant 0$, then

$$\frac{4}{13 \cdot 14} \geqslant \frac{4}{13 \cdot 14} + \theta\frac{23}{13 \cdot 14} = \left(\frac{65}{32}\tilde{\Omega} + \frac{65m}{32}\tilde{L}_{xz} + \frac{65c}{32}\tilde{M}_y + \left(\frac{1495m}{1056} + \frac{455c}{528} + \frac{65a}{1056} - \frac{4}{33}\right)\tilde{E}\right) \cdot G > \frac{1}{14},$$

which is absurd. Hence, we see that $1 > \theta > 0$.

Suppose that $O \neq O_{13}$ and $O \neq O_{14}$. Then $O \notin \tilde{E} \cup \tilde{M}_x$, and it follows from Lemma 1.4.6 that

$$\frac{b}{13 \cdot 14} = -\frac{b}{23} G^2 = \tilde{\Omega} \cdot G > \frac{32}{65},$$

which implies that $b > 448/5$. But $31012/4485 > a \geqslant b$, which is a contradiction.

Therefore, we see that either $O = O_{13}$ or $O = O_{14}$.

Suppose that $O = O_{13}$. Then $O \notin \tilde{E}$, and it follows from Lemma 1.4.6 that

$$\frac{8 + 38c - 13m}{13 \cdot 33} - \frac{a}{33 \cdot 23} - \frac{b}{13 \cdot 23} = \bar{\Omega} \cdot \bar{M}_y - \frac{b}{13 \cdot 23} = \tilde{\Omega} \cdot \tilde{M}_y > \frac{32\big(1 - \theta\big)}{13 \cdot 65},$$

which implies that $c > 12/65$. But $c < 2/23$, which is a contradiction.

Thus, we see that $O = O_{14}$. Then $O \notin \tilde{M}_y$. Hence, it follows from Lemma 1.4.6 that

$$\frac{a - b}{14 \cdot 23} = \tilde{\Omega} \cdot \tilde{E} > \frac{32\big(1 - \theta\big)}{14 \cdot 65},$$

which implies that $130a + 845m + 1820c > 1312$. Applying Lemma 1.4.6 again, we see that

$$\frac{65}{32} \frac{b}{13 \cdot 14} = \frac{65}{32} \tilde{\Omega} \cdot G > \frac{37}{462} - \frac{1495m}{14784} - \frac{65c}{1056} - \frac{65a}{14784},$$

which implies that $1495m + 910c + 65a + 165b \geqslant 1184$. Thus, we obtain the system of inequalities

$$\begin{cases} 130a + 845m + 1820c > 1312, \\ 1495m + 910c + 65a + 165b \geqslant 1184, \\ 4 + 43m \geqslant a + 14c, \\ 184 + 874c \geqslant 299m + 13a + 33b, \\ 184 + 874c \geqslant 299m + 13a, \\ 304/4485 > m > 88/2795, \\ 2/23 \geqslant c > 1/20, \\ a + 14c > 448/65, \\ 31012/4485 > a \geqslant b, \end{cases}$$

which is, unfortunately, consistent. So, we must blow up the point O_{14}.

Let $\phi\colon \hat{X} \to \tilde{X}$ be a weighted blow up of O_{14} with weights $(1, 9)$, let F be the exceptional curve of ϕ, let $\hat{\Omega}$, $\hat{L}_{xz}$, $\hat{M}_y$, $\hat{E}$ and $\hat{G}$ be the proper transforms of Ω, L_{xz}, M_y and E, Grespectively. Then

$$K_{\hat{X}} \equiv \phi^*(K_{\tilde{X}}) - \frac{8}{14}F,\ \hat{G} \equiv \phi^*(G) - \frac{9}{14}F,\ \hat{E} \equiv \phi^*(\tilde{E}) - \frac{1}{14}F,\ \hat{\Omega} \equiv \phi^*(\tilde{\Omega}) - \frac{d}{14}F,$$

where d is a positive rational number.

The curve F contains one singular point A_9 of the surface $\hat{X}$ such that A_9 is a singular point of type $\frac{1}{9}(1, 4)$. Then $\hat{G} \not\ni A_9 \in \hat{E}$ and $\hat{E} \cap \hat{G} = \varnothing$. The log pull back of $(X, \frac{65}{32}D)$ is the log pair

$$\left(\hat{X}, \frac{65}{32}\hat{\Omega} + \frac{65m}{32}\hat{L}_{xz} + \frac{65c}{32}\hat{M}_y + \left(\frac{1495m}{1056} + \frac{455c}{528} + \frac{65a}{1056} - \frac{4}{33}\right)\hat{E} + \theta\hat{G} + \nu F\right),$$

which must have a non-log canonical singularity at some point $A \in F$, where

$$\nu = \frac{65m}{168} + \frac{65c}{96} + \frac{65a}{3864} + \frac{325b}{10304} + \frac{d}{14} + \frac{4}{21}.$$

Obviously, the inequality $\nu > 0$ holds. Let us show that $\nu < 1$. Indeed, we have

$$\frac{a - b}{14 \cdot 23} - \frac{d}{9 \cdot 14} = \hat{E} \cdot \hat{\Omega} \geqslant 0 \leqslant \hat{G} \cdot \hat{\Omega} = \frac{b}{13 \cdot 14} - \frac{d}{14},$$

which implies that $b \geqslant 13d$ and $9(a-b) \geqslant 23d$. Thus, we obtain the system of inequalities

$$\begin{cases} 130a + 845m + 1820c > 1312, \\ 1495m + 910c + 65a + 165b \geqslant 1184, \\ 4 + 43m \geqslant a + 14c, \\ 184 + 874c \geqslant 299m + 13a + 33b, \\ 184 + 874c \geqslant 299m + 13a, \\ 304/4485 > m > 88/2795, \\ 2/23 \geqslant c > 1/20, \\ a + 14c > 448/65, \\ 31012/4485 > a \geqslant b \geqslant 13d, \\ 9(a-b) \geqslant 23d, \end{cases}$$

which is consistent, but it implies that $\nu < 1$.

Suppose that $A \neq A_9$ and $A \notin \hat{G}$. Then $A \notin \hat{E} \cup \hat{G}$, and it follows from Lemma 1.4.6 that

$$\frac{d}{9} = \hat{\Omega} \cdot F > \frac{32}{65},$$

which is impossible, because $31012/4485 > a \geqslant b \geqslant 13d$. We see that either $A = A_9$ or $A \in \hat{G}$.

Suppose that $A \in \hat{G}$. Then it follows from Lemma 1.4.6 that

$$\frac{65d}{32 \cdot 9} + \theta = \left(\frac{65}{32}\hat{\Omega} + \theta\hat{G}\right) \cdot F > 1,$$

because $A \notin \hat{E}$. Applying Lemma 1.4.6 again, we see that the inequality

$$\frac{65}{32}\left(\frac{b}{13 \cdot 14} - \frac{d}{14}\right) + \nu = \left(\frac{65}{32}\hat{\Omega} + \nu F\right) \cdot \hat{G} > 1,$$

holds. Therefore, we obtain the system of inequalities

$$\begin{cases} 1320b + 11960m + 20930c + 520a > 16192 + 2277d, \\ 16445d + 58305m + 125580c + 2535a + 6435b > 90528, \\ 130a + 845m + 1820c > 1312, \\ 1495m + 910c + 65a + 165b \geqslant 1184, \\ 4 + 43m \geqslant a + 14c, \\ 184 + 874c \geqslant 299m + 13a + 33b, \\ 184 + 874c \geqslant 299m + 13a, \\ 304/4485 > m > 88/2795, \\ 2/23 \geqslant c > 1/20, \\ a + 14c > 448/65, \\ 31012/4485 > a \geqslant b \geqslant 13d, \\ 9(a-b) \geqslant 23d, \end{cases}$$

which is inconsistent. Hence, we see that $A = A_9$. By Lemma 1.4.6, we have

$$\frac{65}{32}\left(\frac{a-b}{14 \cdot 23} - \frac{d}{9 \cdot 14}\right) + \frac{\nu}{9} = \left(\frac{65}{32}\hat{\Omega} + \nu F\right) \cdot \hat{E} > \frac{1}{9},$$

because A is not contained in $\hat{G}$. Applying Lemma 1.4.6 once again, we see that the inequality

$$\frac{65d}{32 \cdot 9} + \frac{1}{9}\left(\frac{1495m}{1056} + \frac{455c}{528} + \frac{65a}{1056} - \frac{4}{33}\right) = \left(\frac{65}{32}\hat{\Omega} + \left(\frac{1495m}{1056} + \frac{455c}{528} + \frac{65a}{1056} - \frac{4}{33}\right)\hat{E}\right) \cdot F > \frac{1}{9}$$

holds. Therefore, we obtain the system of inequalities

$$\begin{cases}
2145d+1495m+910c+65a>1184,\\
2275a+11960m+20930c>25024+2277d+780b\\
130a+845m+1820c>1312,\\
1495m+910c+65a+165b\geqslant 1184,\\
4+43m\geqslant a+14c,\\
184+874c\geqslant 299m+13a+33b,\\
184+874c\geqslant 299m+13a,\\
304/4485>m>88/2795,\\
2/23\geqslant c>1/20,\\
a+14c>448/65,\\
31012/4485>a\geqslant b\geqslant 13d,\\
9(a-b)\geqslant 23d,
\end{cases}$$

which is inconsistent. The obtained contradiction completes the proof. □

Lemma 3.4.10. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 23, 51, 83, 166)$. Then $\mathrm{lct}(X) = 91/40$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2+y^5z+xz^3+x^{11}y=0,$$

the surface X is singular at the point O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{91}{40}=\mathrm{lct}\left(X,\frac{4}{13}C_x\right)<\mathrm{lct}\left(X,\frac{4}{23}C_y\right)=\frac{115}{24},$$

which implies, in particular, that $\mathrm{lct}(X)\leqslant 91/40$.

Suppose that $\mathrm{lct}(X) < 91/40$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{91}{40}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(663))$ contains x^{51}, $y^{13}x^{28}$, $y^{26}x^5$ and z^{13}, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P \in C_x$. Then

$$\frac{8}{27\cdot 51}=D\cdot C_x\geqslant\begin{cases}
\dfrac{\mathrm{mult}_P(D)}{23} & \text{if } P=O_y,\\
\dfrac{\mathrm{mult}_P(D)}{51} & \text{if } P=O_z,\\
\mathrm{mult}_P(D) & \text{if } P\neq O_y \text{ and } P\neq O_z,
\end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 40/91$. Thus, we see that $P = O_x$. Then

$$\frac{8}{13\cdot 51}=D\cdot C_y\geqslant\frac{\mathrm{mult}_P(D)}{13}>\frac{40}{91\cdot 13}>\frac{8}{13\cdot 51},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 91/40$. □

3.5. Sporadic cases with $I = 5$

Lemma 3.5.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 13, 19, 25, 63)$. Then $\mathrm{lct}(X) = 13/8$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^2t+yt^2+xy^4+x^4z=0,$$

and X is singular at O_x, O_y, O_z and O_t. We have

$$\mathrm{lct}\left(X,\frac{5}{13}C_y\right)=\frac{13}{18}<\mathrm{lct}\left(X,\frac{5}{11}C_x\right)=\frac{33}{20}<\mathrm{lct}\left(X,\frac{5}{19}C_z\right)=\frac{57}{25}<\mathrm{lct}\left(X,\frac{5}{25}C_t\right)=\frac{25}{11},$$

which implies, in particular, that $\mathrm{lct}(X)\leqslant 13/8$.

The curve C_x is reducible. We have $C_x = L_{xt} + M_x$, where L_{xt} and M_x are irreducible curves such that L_{xt} is given by $x = t = 0$, and M_x is given by $x = z^2 + yt = 0$. Then

$$L_{xt} \cdot L_{xt} = \frac{-27}{13 \cdot 19},\ M_x \cdot M_x = \frac{-28}{13 \cdot 25},\ L_{xt} \cdot M_x = \frac{2}{13},\ D \cdot L_{xt} = \frac{5}{13 \cdot 19},\ D \cdot M_x = \frac{10}{13 \cdot 25},$$

and $O_y \in C_x$. Note that C_x is smooth outside of the point O_y.

The curve C_y is reducible. We have $C_y = L_{yz} + M_y$, where L_{yz} and M_y are irreducible curves such that L_{yz} is given by $y = z = 0$, and M_y is given by $y = x^4 + zt = 0$.

$$L_{yz} \cdot L_{yz} = \frac{-31}{11 \cdot 25},\ M_y \cdot M_y = \frac{-24}{19 \cdot 25},\ L_{yz} \cdot M_y = \frac{4}{25},\ D \cdot L_{yz} = \frac{5}{11 \cdot 25},\ D \cdot M_y = \frac{20}{19 \cdot 25},$$

and the only singular point of the curve C_y is O_t. We have $M_y \cdot M_x = 31/475$ and $L_{xt} \cdot L_{yz} = 0$.

The curve C_z is reducible. We have $C_z = L_{yz} + M_z$, where M_z is an irreducible curve that is given by the equations $z = t^2 + xy^4 = 0$. The only singular point of C_z is O_x. We have

$$L_{yz} \cdot L_{xt} = 0,\ M_z \cdot M_z = \frac{12}{11 \cdot 13},\ L_{yz} \cdot M_z = \frac{2}{11},\ D \cdot M_z = \frac{10}{11 \cdot 13}.$$

The curve C_t is reducible. We have $C_t = L_{xt} + M_t$, where M_t is an irreducible curve that is given by the equations $t = y^4 + x^3 z = 0$. The only singular point of C_t is O_z. We have

$$M_t \cdot M_t = \frac{56}{11 \cdot 19},\ L_{xt} \cdot M_t = \frac{4}{19},\ D \cdot M_t = \frac{20}{11 \cdot 19}.$$

We suppose that $\mathrm{lct}(X) < 13/8$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{13}{8} D)$ is not log canonical at some point $P \in X$. Let us derive a contradiction.

Suppose that $P \notin C_x \cup C_y \cup C_z \cup C_t$. Then there is a unique curve $Z \subset X$ that is cut out by

$$\alpha y t^2 = x^4 z$$

such that $P \in Z$, where $0 \neq \alpha \in \mathbb{C}$. The curve Z is reducible. Indeed, we have

$$L_{xt} \subset \mathrm{Supp}(Z) \supset L_{yz},$$

and we can write $Z = C + pL_{xt} + qL_{yz}$, where $p \in \mathbb{Z}_{>0} \ni q$, and C is a curve on X whose support does not contains the curves L_{xt} and L_{yz}. Let us prove that C is irreducible and find p and q.

The open subset $Z \setminus (Z \cap C_x)$ of the curve Z is a $\mathbb{Z}_{11}$-quotient of the affine curve

$$\alpha y t^2 - z = z^2 t + y t^2 + y^4 + z = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

which is isomorphic to an affine septic curve $R_x \subset \mathbb{C}^2$ that is given by the equation

$$\alpha^2 y \Big(t^5 + y^3 + \big(1 + \alpha\big) t^2\Big) = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big),$$

which implies that the curve C is irreducible, the inequality $\mathrm{mult}_P(C) \leqslant 6$ and the equality

$$q = \begin{cases} 1 \text{ if } \alpha \neq -1, \\ 2 \text{ if } \alpha = -1, \end{cases}$$

hold. But $p = 2$, because the subset $Z \setminus (Z \cap C_y)$ is a $\mathbb{Z}_{13}$-quotient of the curve

$$t^2 - \frac{z x^4}{\alpha} = z^2 t + x + \frac{\alpha + 1}{\alpha} x^4 z = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, z, t\big]\Big).$$

Therefore, we see that $P \in C$ and we have the following possibilities:

- the inequality $\alpha \neq -1$ holds, $p = 2 \neq q = 1$ and

$$C \cdot L_{xt} = \frac{117}{247},\ C \cdot L_{yz} = \frac{94}{275},\ C \cdot C = \frac{8636}{5225},\ D \cdot C = \frac{244}{1045};$$

- the equality $\alpha = -1$ holds, $p = q = 2$ and

$$C \cdot L_{xt} = \frac{117}{247},\ C \cdot L_{yz} = \frac{5}{11},\ C \cdot C = \frac{179}{209},\ D \cdot C = \frac{45}{209}.$$

We see that C is irreducible and $\mathrm{mult}_P(C) \leqslant 6$. Then the log pair

$$\Big(X,\ \frac{13}{8\cdot 63}\Big(C+pL_{xt}+qL_{yz}\Big)\Big)$$

must be log canonical at the point P. By Remark 1.4.7, we may assume that $\mathrm{Supp}(D)$ does not contain at least one curve among the curves C, L_{xt} and L_{yz}. Put

$$D=\epsilon C+\Xi,$$

where Ξ is an effective $\mathbb{Q}$-divisor such that $C\not\subset\mathrm{Supp}(\Xi)$. Now we obtain the inequality $\epsilon\leqslant 5/94$, because either $\epsilon=0$, or $L_{xt}\cdot\Xi\geqslant 0$, or $L_{zy}\cdot\Xi\geqslant 0$. On the other hand, we see that

$$D\cdot C=\epsilon C^2+\Xi\cdot C\geqslant \epsilon C^2+\mathrm{mult}_P\big(\Xi\big)=\epsilon C^2+\mathrm{mult}_P\big(D\big)-\epsilon\mathrm{mult}_P\big(C\big)>\epsilon C^2+\frac{8}{13}-6\epsilon,$$

which implies that $\epsilon>2594/40755$. But $\epsilon\leqslant 5/94$. Thus, we see that $P\in C_x\cup C_y\cup C_z\cup C_t$.

It follows from Remark 1.4.7 that we may assume that $\mathrm{Supp}(D)$ does not contains are least one irreducible component of the curves C_x, C_y, C_z, C_t.

Suppose that $P\in L_{xt}$. Put $D=\delta L_{xt}+\Theta$, where Θ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{xt}. If $\delta\neq 0$, then

$$\frac{10}{13\cdot 25}=D\cdot M_x=\big(\delta L_{xt}+\Theta\big)\cdot M_x\geqslant \delta L_{xt}\cdot M_x=\frac{2\delta}{13},$$

which implies that $\delta\leqslant 1/5$. Then it follows from Lemma 1.4.6 that

$$\frac{5+27\delta}{13\cdot 19}=\big(-K_X-\delta L_{xt}\big)\cdot L_{xt}=\Theta\cdot L_{xt}>\begin{cases}\dfrac{8}{13}\ \text{if}\ P\not\in\mathrm{Sing}(X),\\ \dfrac{8}{13\cdot 19}\ \text{if}\ P=O_z,\\ \dfrac{8}{13\cdot 13}\ \text{if}\ P=O_y,\end{cases}$$

which implies that $\delta>3/27$. But $\delta\leqslant 1/5$. Thus, we see that $P\not\in L_{xt}$.

Suppose tat $P\in L_{yz}$ and $P\neq O_t$. Arguing as in the previous case, we obtain a contradiction.

Suppose that $P\in M_x$ and $P\neq O_t$. Then P is a smooth point of X, because $P\not\in L_{xt}$. Put

$$D=eM_x+\Upsilon,$$

where Υ is an effective $\mathbb{Q}$-divisor such that $M_x\not\subset\mathrm{Supp}(\Upsilon)$. If $e\neq 0$, then

$$\frac{5}{13\cdot 19}=D\cdot L_{xt}=\big(eM_x+\Upsilon\big)\cdot L_{xt}\geqslant eL_{xt}\cdot M_x=\frac{2e}{13},$$

which implies that $e\leqslant 5/38$. Then it follows from Lemma 1.4.6 that

$$\frac{10+28e}{13\cdot 25}=\big(-K_X-eM_x\big)\cdot M_x=\Upsilon\cdot M_x>\frac{8}{13},$$

which implies that $e>95/14$. But $e\leqslant 5/38$. Thus, we see that $P\not\in M_x$ or $P=O_t$.

Arguing as above, we see that either $P\not\in M_y$ or $P=O_t$. Then $P\in M_z\cup M_t$ or $P=O_t$.

Suppose that $P\in M_z$. Put $D=sM_z+\Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve M_x. If $s\neq 0$, then

$$\frac{5}{11\cdot 25}=D\cdot M_z=\big(sM_z+\Delta\big)\cdot L_{yz}\geqslant sM_x\cdot L_{xt}=\frac{2s}{11},$$

which implies that $s\leqslant 1/10$. Then it follows from Lemma 1.4.6 that

$$\frac{10}{11\cdot 13}=D\cdot M_x=sM_x^2+\Delta\cdot M_X>sM_x^2+\frac{8}{13}\geqslant\frac{8}{13}>\frac{10}{11\cdot 13},$$

which is a contradiction. Thus, we see that $P\not\in M_z$. Similarly, we see that $P\not\in M_t$.

The obtained contradiction shows that $P=O_t$. Then

$$\frac{5}{11\cdot 25}=D\cdot L_{yz}>\frac{8}{13\cdot 25}>\frac{5}{11\cdot 25}$$

whenever $L_{yz}\not\subset\mathrm{Supp}(D)$. Thus, we see that $L_{yz}\subset\mathrm{Supp}(D)$. Then $M_y\not\subset\mathrm{Supp}(D)$. Put

$$D=mL_{yz}+cM_x+\Omega,$$

where $m>0$ and $c\geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor such that $L_{yz}\not\subset\mathrm{Supp}(\Omega)\not\supset M_x$. Then

$$\frac{20}{19\cdot 25}=D\cdot M_y=\big(mL_{yz}+cM_x+\Omega\big)\cdot M_y\geqslant\frac{4m}{25}+\frac{\mathrm{mult}_{O_t}(D)-m}{25}>\frac{3m+\frac{8}{13}}{25},$$

which implies that $m<36/247$. Then it follows from Lemma 1.4.6 that

$$\frac{5+31m}{11\cdot 25}=\big(-K_X-mL_{yz}\big)\cdot L_{yz}=(\Omega+cM_x)\cdot L_{yz}>\frac{8}{13\cdot 25},$$

which implies that $m>23/403$. We will see later that $c>0$ as well.

Arguing as above, we see obtain an inconsistent system of inequalities

$$\begin{cases}\dfrac{20}{19\cdot 25}>\dfrac{3m+\frac{1216}{905}}{25},\\ \dfrac{5+31m}{11\cdot 25}>\dfrac{1216}{905\cdot 25},\end{cases}$$

in the case when $(X,\frac{1216}{905}D)$ is not log canonical at O_t. We see that $\mathrm{lct}(X)\geqslant 1216/905$.

Let $\pi\colon\bar{X}\to X$ be a weighted blow up of O_t with weights $(11,19)$, let E be the exceptional curve of π, let $\bar{\Omega}$, $\bar{L}_{yz}$, $\bar{M}_x$, $\bar{M}_y$ be the proper transforms of Ω, L_{yz}, M_x, M_y, respectively. Then

$$K_{\bar{X}}\equiv\pi^*(K_X)+\frac{5}{25}E,\ \bar{L}_{yz}\equiv\pi^*(L_{yz})-\frac{19}{25}E,\ \bar{M}_y\equiv\pi^*(M_y)-\frac{19}{25}E,\ \bar{M}_x\equiv\pi^*(M_x)-\frac{11}{25}E,$$

and there is a positive rational number a such that

$$\bar{\Omega}\equiv\pi^*(\Omega)-\frac{a}{25}E.$$

The curve E contains two singular points Q_{11} and Q_{19} of $\bar{X}$ such that Q_{11} is a singular point of type $\frac{1}{11}(2,3)$, and Q_{19} is a singular point of type $\frac{1}{19}(11,13)$. Then

$$\bar{L}_{yz}\cup\bar{M}_y\not\ni Q_{19}\in\bar{M}_x\not\ni Q_{11}=\bar{L}_{yz}\cap\bar{M}_y,$$

which implies that $\bar{L}_{yz}\cap\bar{M}_x=\varnothing$. The log pull back of the log pair $(X,\frac{13}{8}D)$ is the log pair

$$\left(\bar{X},\frac{13}{8}\bar{\Omega}+\frac{13m}{8}\bar{L}_{yz}+\frac{13c}{8}\bar{M}_x+\left(\frac{247m}{200}+\frac{143c}{200}+\frac{13a}{200}-\frac{1}{5}\right)E\right),$$

which must have non-log canonical singularity at some point $Q\in E$. We have

$$0\leqslant\bar{L}_{yz}\cdot\bar{\Omega}=\frac{5}{11\cdot 25}+\frac{31m-a-11c}{11\cdot 25},$$

which implies that $a+11c\leqslant 5+31m$. But $m<36/247$. Hence, we see that $a<2351/247$ and

$$\left(\frac{247m}{200}+\frac{143c}{200}+\frac{13a}{200}-\frac{1}{5}\right)<1.$$

The log pull back of the the log pair $(X,\frac{13}{8}D)$ is effective if and only if the inequality

$$19m+11c+a\geqslant 40/13$$

holds. On the other hand, if $19m+11c+a\leqslant 40/13$, then the log pair

$$\left(\bar{X},\frac{13}{8}\bar{\Omega}+\frac{13m}{8}\bar{L}_{yz}+\frac{13c}{8}\bar{M}_x\right)$$

is not log canonical at the point Q as well. Thus, if $19m+11c+a\leqslant 40/13$, then

$$\frac{40}{13\cdot 11\cdot 19}\geqslant\frac{a+19m+11c}{11\cdot 19}=\left(\bar{\Omega}+m\bar{L}_{yz}+c\bar{M}_x\right)\cdot E>\begin{cases}\dfrac{8}{13}\ \text{if}\ Q_{19}\ne Q\ne Q_{11},\\ \dfrac{8}{13}\dfrac{1}{11}\ \text{if}\ Q=Q_{11},\\ \dfrac{8}{13}\dfrac{1}{19}\ \text{if}\ Q=Q_{19},\end{cases}$$

which is absurd. Thus, the log pull back of $(X,\frac{13}{8}D)$ is effective.

Suppose that $Q \neq Q_{11}$ and $Q \neq Q_{19}$. Then $Q \notin \bar{L}_{yz} \cup \bar{M}_x$. By Lemma 1.4.6, we have

$$\frac{a}{19 \cdot 11} = \frac{a}{25}E^2 = \bar{\Omega} \cdot E > \frac{8}{13},$$

because $E^2 = -25/209$. Then $a > 1672/13$, which is impossible, because $a < 2351/247$.

Therefore, we see that either $Q = Q_{11}$ or $Q = Q_{19}$.

Suppose that $Q = Q_{11}$. Then $Q \notin \bar{M}_x$. Hence, it follows from Lemma 1.4.6 that

$$\left(\frac{13}{8}\bar{\Omega} + \left(\frac{247m}{200} + \frac{143c}{200} + \frac{13a}{200} - \frac{1}{5}\right)E\right) \cdot \bar{L}_{yz} > \frac{1}{11},$$

but $\bar{L}_{yz} \cdot E = 1/11$ and $\bar{L}_{yz} \cdot \bar{M}_x = 0$. Moreover, we have

$$\bar{\Omega} \cdot \bar{L}_{yz} = \left(\bar{\Omega} + c\bar{M}_x\right) \cdot \bar{L}_{yz} = \left(D - mL_{yz}\right) \cdot L_{yz} - \frac{a + 11c}{25 \cdot 11} = \frac{5 + 31m - a - 11c}{11 \cdot 25},$$

which immediately implies that $m > 19/130$. But $m < 36/247$, which is a contradiction.

Thus, we see that $Q = Q_{19}$. Then $Q \notin \bar{L}_{yz}$, and it follows from Lemma 1.4.6 that

$$\left(\frac{13}{8}\bar{\Omega} + \left(\frac{247m}{200} + \frac{143c}{200} + \frac{13a}{200} - \frac{1}{5}\right)E\right) \cdot \bar{M}_x > \frac{1}{19},$$

but we have $\bar{M}_x \cdot E = 1/19$. Therefore, it follows from the equality

$$\bar{\Omega} \cdot \bar{M}_x = \Omega \cdot M_x - \frac{a}{25 \cdot 19} = D \cdot M_x - mL_{yz} \cdot M_x - cM_x \cdot M_x - \frac{a}{25 \cdot 19} = \frac{10 - 13m + 28c}{13 \cdot 25} - \frac{a}{25 \cdot 19},$$

which implies that $c > 2/27$. But $c < 5/28$. However, we have no contradiction here.

Let $\psi\colon \tilde{X} \to \bar{X}$ be a weighted blow up of Q_{19} with weights $(11, 13)$, let G be the exceptional curve of ψ, let $\tilde{\Omega}$, $\tilde{L}_{yz}$, $\tilde{M}_x$, $\tilde{E}$ be the proper transforms of Ω, L_{yz}, M_x, E, respectively. Then

$$K_{\tilde{X}} \equiv \psi^*(K_{\bar{X}}) + \frac{5}{19}G,\ \tilde{M}_x \equiv \psi^*(\bar{M}_x) - \frac{11}{19}G,\ \tilde{E} \equiv \psi^*(E) - \frac{13}{19}G,\ \tilde{\Omega} \equiv \psi^*(\bar{\Omega}) - \frac{b}{19}G,$$

where b is a positive rational number.

The curve G contains two singular points O_{11} and O_{13} of $\tilde{X}$ such that O_{11} is a singular point of type $\frac{1}{11}(2,3)$, and O_{13} is a singular point of type $\frac{1}{13}(1,2)$. Then

$$\tilde{E} \not\ni O_{13} \in \tilde{M}_x \not\ni O_{11} \in \tilde{E},$$

where $\tilde{E} \cap \tilde{M}_x = \varnothing$. The log pull back of the log pair $(X, \frac{13}{8}D)$ is the log pair

$$\left(\tilde{X}, \frac{13}{8}\tilde{\Omega} + \frac{13m}{8}\tilde{L}_{yz} + \frac{13c}{8}\tilde{M}_x + \left(\frac{247m}{200} + \frac{143c}{200} + \frac{13a}{200} - \frac{1}{5}\right)\tilde{E} + \theta G\right),$$

which must have non-log canonical singularity at some point $O \in G$, where

$$\theta = \frac{143c}{100} + \frac{13b}{152} + \frac{169a}{3800} + \frac{169m}{200} - \frac{2}{5}.$$

Let us show that $\theta < 1$. Indeed, we have

$$0 \leqslant \tilde{M}_x \cdot \tilde{\Omega} = \frac{10}{13 \cdot 25} + \frac{28}{13 \cdot 25} - \frac{a + 19m}{19 \cdot 25} - \frac{b}{19 \cdot 13},$$

which implies that $25b \leqslant 190 + 532c - 13(a + 19m)$. Then $\theta < 1$, because $c > 2/27$ and $c < 5/38$.

Let us show that $\theta > 0$. If $\theta \leqslant 0$, then the log pair

$$\left(\tilde{X}, \frac{13}{8}\tilde{\Omega} + \frac{13m}{8}\tilde{L}_{yz} + \frac{13c}{8}\tilde{M}_x + \left(\frac{247m}{200} + \frac{143c}{200} + \frac{13a}{200} - \frac{1}{5}\right)\tilde{E}\right),$$

is not log canonical at the point O as well. Thus, if $\theta \leqslant 0$, then

$$\frac{5}{11 \cdot 13} + \theta\frac{19}{11 \cdot 13} = \left(\frac{13}{8}\tilde{\Omega} + \frac{13m}{8}\tilde{L}_{yz} + \frac{13c}{8}\tilde{M}_x + \left(\frac{247m}{200} + \frac{143c}{200} + \frac{13a}{200} - \frac{1}{5}\right)\tilde{E}\right) \cdot G > \frac{1}{13},$$

which implies that $\theta > 6/19$, which is absurd. Hence, we see that $1 > \theta > 0$.

Suppose that $O \neq O_{11}$ and $O \neq O_{13}$. Then $O \notin \tilde{E} \cup \tilde{M}_x$, and it follows from Lemma 1.4.6 that

$$\frac{b}{11 \cdot 13} = -\frac{b}{19}G^2 = \tilde{\Omega} \cdot G > \frac{8}{13},$$

because $G^2=-19/143$. Thus, we see that $b>88$. On the other hand, the inequalities

$$0\leqslant\left(\tilde{\Omega}+m\tilde{L}_{yz}\right)\cdot\tilde{E}=\frac{a+19m-b}{11\cdot 19}$$

hold. Then $a+19m\geqslant b>88$. Thus, we obtain the system of inequalities

$$\begin{cases} a+19m\geqslant b>88,\\ 25b\leqslant 190+532c-13(a+19m),\\ 5/38>c>2/27,\end{cases}$$

which is inconsistent. Therefore, we see that either $O=O_{11}$ or $O=O_{13}$.

Suppose that $O=O_{13}$. Then $O\not\in\tilde{E}$, and it follows from Lemma 1.4.6 that

$$\frac{190+532c-25b-13(a+19m)}{19\cdot 13\cdot 25}=\tilde{\Omega}\cdot\tilde{M}_x>\frac{56}{845}-\frac{22c}{325}-\frac{b}{247}-\frac{a}{475}-\frac{m}{25},$$

which implies that $c>3/13$. But $c<5/28$, which is a contradiction.

Thus, we see that $O=O_{11}$. Then $O\not\in\tilde{M}_x$. Hence, it follows from Lemma 1.4.6 that

$$\frac{a+19m-b}{19\cdot 11}=\left(\tilde{\Omega}+m\tilde{L}_{yz}\right)\cdot\tilde{E}>\frac{56}{715}-\frac{2c}{25}-\frac{b}{209}-\frac{13a}{5225}-\frac{13m}{275},$$

which implies that $22c>280/13-2(a+19m)$. Applying Lemma 1.4.6 again, we see that

$$\frac{b}{11\cdot 13}=\tilde{\Omega}\cdot G>\frac{48}{65}-\frac{19m}{25}-\frac{11c}{25}-\frac{a}{25},$$

which implies that $13(a+19m)+143c+25b>240$. Note that $\bar{M}_y\not\subset\mathrm{Supp}\,(\bar{\Omega})$. Thus we have

$$0\leqslant\bar{\Omega}\cdot\bar{M}_y=\Omega\cdot M_y-\frac{a+19m}{25}E\cdot\bar{M}_x=\frac{20-31c}{19\cdot 25}-\frac{a+19m}{25\cdot 11},$$

which implies that $19(a+19m)\leqslant 220-341c$. Similarly, we see that

$$\frac{20}{19\cdot 25}-\frac{31c}{19\cdot 25}-\frac{4m}{25}=(D-cMx-mL_{yz})\cdot M_y=\Omega\cdot M_y\geqslant\frac{\mathrm{mult}_{O_t}(\Omega)}{25}>\frac{8/13-m-c}{25},$$

which implies that $108/13>12c+57m$. Thus, we obtain the system of inequalities

$$\begin{cases} 19(a+19m)\leqslant 220-341c,\\ 25b\leqslant 190+532c-13(a+19m),\\ 13(a+19m)+143c+25b>240,\\ 22c>280/13-2(a+19m),\\ 108/13>12c+57m,\\ a+11c\leqslant 5+31m,\\ 5/38>c>2/27,\end{cases}$$

which is, unfortunately, consistent. So, we must blow up the point O_{11}.

Let $\phi\colon\hat{X}\to\tilde{X}$ be a weighted blow up of O_{11} with weights $(2,3)$, let F be the exceptional curve of ϕ, let $\hat{\Omega}$, $\hat{L}_{yz}$, $\hat{M}_x$, $\hat{E}$ be the proper transforms of Ω, L_{yz}, M_x, E, respectively. Then

$$K_{\hat{X}}\equiv\phi^*(K_{\tilde{X}})-\frac{6}{11}F,\ \hat{G}\equiv\phi^*(G)-\frac{3}{11}F,\ \hat{E}\equiv\phi^*(\tilde{E})-\frac{2}{11}F,\ \hat{\Omega}\equiv\phi^*(\tilde{\Omega})-\frac{d}{11}F,$$

where d is a positive rational number. Then $F^2=-11/6$ and

$$\hat{\Omega}\cdot F=\frac{d}{6},\ \left(\hat{\Omega}+m\hat{L}_{yz}\right)\cdot\hat{E}=\frac{a+19m-b}{11\cdot 19}-\frac{d}{33},\ \hat{\Omega}\cdot\hat{G}=\frac{b}{11\cdot 13}-\frac{d}{22},\ F\cdot\hat{G}=\frac{1}{2},\ F\cdot\hat{E}=\frac{1}{3}.$$

The curve F contains two singular points A_2 and A_3 of the surface $\hat{X}$ such that A_2 is a singular point of type $\frac{1}{2}(1,1)$, and A_3 is a singular point of type $\frac{1}{3}(1,2)$. Then

$$\hat{E}\not\ni A_2\in\hat{G}\not\ni A_3\in\hat{E},$$

where $\hat{E}\cap\hat{G}=\varnothing$. The log pull back of the log pair $(X,\frac{13}{8}D)$ is the log pair

$$\left(\hat{X},\frac{13}{8}\hat{\Omega}+\frac{13m}{8}\hat{L}_{yz}+\frac{13c}{8}\hat{M}_x+\left(\frac{247m}{200}+\frac{143c}{200}+\frac{13a}{200}-\frac{1}{5}\right)\hat{E}+\theta\hat{G}+\nu F\right),$$

which must have non-log canonical singularity at some point $A \in F$, where

$$\nu = \frac{91m}{200} + \frac{13c}{25} + \frac{91a}{3800} + \frac{39b}{1672} + \frac{13d}{88} + \frac{2}{5}.$$

Obviously, the inequality $\nu > 0$ holds. Let us show that $\nu < 1$. Indeed, we have

$$\frac{a+19m-b}{11\cdot 19} - \frac{d}{33} = \hat{E}\cdot\left(\hat{\Omega} + m\hat{L}_{yz}\right) \geqslant 0 \leqslant \hat{G}\cdot\hat{\Omega} = \frac{b}{11\cdot 13} - \frac{d}{22},$$

which implies that $2b \geqslant 13d$ and $3(a+19m-b) \geqslant 19d$. But the system of inequalities

$$\begin{cases} 2b \geqslant 13d, \\ 3(a+19m-b) \geqslant 19d, \\ 1001(a+19m) + 21736c + 975b + 6175d \geqslant 25080, \\ 19(a+19m) \leqslant 220 - 341c, \\ 25b \leqslant 190 + 532c - 13(a+19m), \\ 13(a+19m) + 143c + 25b > 240, \\ 22c > 280/13 - 2(a+19m), \\ 108/13 > 12c + 57m, \\ a + 11c \leqslant 5 + 31m, \\ 5/38 > c > 2/27, \end{cases}$$

is inconsistent. Hence, we see that $1 > \nu > 0$.

Suppose that $A \neq A_2$ and $A \neq A_3$. Then $A \notin \hat{E} \cup \hat{G}$, and it follows from Lemma 1.4.6 that

$$\frac{d}{6} = \hat{\Omega}\cdot F > \frac{8}{13},$$

which implies that $d > 48/13$. But the system of inequalities

$$\begin{cases} d > 48/13, \\ 2b \geqslant 13d, \\ 3(a+19m-b) \geqslant 19d, \\ 19(a+19m) \leqslant 220 - 341c, \\ 25b \leqslant 190 + 532c - 13(a+19m), \\ 13(a+19m) + 143c + 25b > 240, \\ 22c > 280/13 - 2(a+19m), \\ 108/13 > 12c + 57m, \\ a + 11c \leqslant 5 + 31m, \\ 5/38 > c > 2/27, \end{cases}$$

is inconsistent. Therefore, we see that either $A = A_2$ or $A = A_3$.

Suppose that $A = A_2$. Then it follows from Lemma 1.4.6 that

$$\frac{13d}{48} + \frac{1}{2}\left(\frac{143c}{100} + \frac{13b}{152} + \frac{169a}{3800} + \frac{169m}{200} - \frac{2}{5}\right) = \left(\frac{13}{8}\hat{\Omega} + \theta\hat{G}\right)\cdot F > \frac{1}{2},$$

because $A \notin \hat{E}$. Applying Lemma 1.4.6 again, we see that the inequality

$$\frac{13}{48}\left(\frac{b}{11\cdot 13} - \frac{d}{22}\right) + \frac{1}{2}\left(\frac{91m}{200} + \frac{13c}{25} + \frac{91a}{3800} + \frac{39b}{1672} + \frac{13d}{88} + \frac{2}{5}\right) = \left(\frac{13}{8}\hat{\Omega} + \nu F\right)\cdot\hat{G} > \frac{1}{2},$$

holds. Therefore, we obtain the system of inequalities

$$\begin{cases} 2b \geqslant 13d, \\ 3(a+19m-b) \geqslant 19d, \\ 16302c + 975b + 507(a+19m) + 6175d > 15960, \\ 1976c + 91(a+19m) + 175b > 2280, \\ 19(a+19m) \leqslant 220 - 341c, \\ 25b \leqslant 190 + 532c - 13(a+19m), \\ 13(a+19m) + 143c + 25b > 240, \\ 22c > 280/13 - 2(a+19m), \\ 108/13 > 12c + 57m, \\ a + 11c \leqslant 5 + 31m, \\ 5/38 > c > 2/27, \end{cases}$$

which is inconsistent. Hence, we see that $A = A_3$. By Lemma 1.4.6, we have

$$\frac{13d}{48} + \frac{1}{3}\left(\frac{247m}{200} + \frac{143c}{200} + \frac{13a}{200} - \frac{1}{5}\right) = \left(\frac{13}{8}\hat{\Omega} + \left(\frac{247m}{200} + \frac{143c}{200} + \frac{13a}{200} - \frac{1}{5}\right)\hat{E}\right) \cdot F > \frac{1}{3},$$

because A is not contained in $\hat{G}$. Applying Lemma 1.4.6 again, we see that the inequality

$$\frac{13}{4}\left(\frac{a+19m-b}{11\cdot 19} - \frac{d}{33}\right) + \frac{1}{3}\left(\frac{91m}{200} + \frac{13c}{25} + \frac{91a}{3800} + \frac{39b}{1672} + \frac{13d}{88} + \frac{2}{5}\right) = \left(\frac{13}{8}\hat{\Omega} + \nu F\right) \cdot \hat{E} > \frac{1}{3},$$

holds. Therefore, we obtain the system of inequalities

$$\begin{cases} 2b \geqslant 13d, \\ 3(a+19m-b) \geqslant 19d, \\ 286c + 26(a+19m) + 325d > 480, \\ 143c + 13(a+19m) > 165, \\ 19(a+19m) \leqslant 220 - 341c, \\ 25b \leqslant 190 + 532c - 13(a+19m), \\ 13(a+19m) + 143c + 25b > 240, \\ 22c > 280/13 - 2(a+19m), \\ 108/13 > 12c + 57m, \\ a + 11c \leqslant 5 + 31m, \\ 5/38 > c > 2/27, \end{cases}$$

which is inconsistent. The obtained contradiction completes the proof. □

Lemma 3.5.2. Suppose that and $(a_0, a_1, a_2, a_3, d) = (11, 25, 37, 68, 136)$. Then $\mathrm{lct}(X) = 11/6$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$xy^5 + x^9 z + yz^3 + t^2 = 0,$$

and X is singular at the points O_x, O_y and O_z.

The curves C_x and C_y are reduced and irreducible. We have

$$\frac{11}{6} = \mathrm{lct}\left(X, \frac{5}{11}C_x\right) < \mathrm{lct}\left(X, \frac{5}{25}C_y\right) = \frac{55}{18},$$

which implies thatlct$(X) \leqslant 11/6$.

Suppose that $\mathrm{lct}(X) < 11/6$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $(X, \frac{11}{6}D)$ is not log canonical at some point P. by Remark 1.4.7 we may assume that the support of D does not contain C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(407))$ contains x^{37}, z^{11} and $x^{12}y^{11}$, we see that $P \in \mathrm{Sing}(X) \cup C_x$ by Lemma 1.4.10.

Suppose that $P \in C_x$. Then

$$\frac{10}{25 \cdot 37} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{25} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{37} \text{ if } P = O_z, \\ \mathrm{mult}_P(D) \text{ if } P \neq O_y \text{ and } P \neq O_z, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 6/11$. Thus, we see that $P = O_x$. Then

$$\frac{10}{11 \cdot 37} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{11} > \frac{6}{121} > \frac{10}{11 \cdot 37},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 11/6$. □

Lemma 3.5.3. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 19, 41, 68, 136)$. Then $\mathrm{lct}(X) = 91/50$.

Proof. The surface X is defined by the quasihomogeneous equation

$$x^9 y + xz^3 + y^5 z + t^2 = 0,$$

and X is singular at the points O_x, O_y and O_z.

The curves C_x and C_y are reduced and irreducible. Then

$$\frac{91}{50} = \mathrm{lct}\left(X, \frac{5}{13} C_x\right) < \mathrm{lct}(X, \frac{5}{19} C_y) = \frac{19}{6},$$

which implies that $\mathrm{lct}(X) \leqslant \frac{50}{91}$.

Suppose that $\mathrm{lct}(X) < 91/50$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $(X, \frac{91}{50} D)$ is not log canonical at some point P. By Remark 1.4.7 we may assume that the support of D does not contain C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(533))$ contains x^{41}, z^{13} and $x^3 y^{26}$, we see that $P \in \mathrm{Sing}(X) \cup C_x$ by Lemma 1.4.10.

Suppose that $P \in C_x$. Then

$$\frac{10}{19 \cdot 41} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{19} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{41} \text{ if } P = O_z, \\ \mathrm{mult}_P(D) \text{ if } P \neq O_y \text{ and } P \neq O_z, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 50/91$. We see that $P = O_x$. Then

$$\frac{10}{13 \cdot 41} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{13} > \frac{50}{91 \cdot 13} > \frac{10}{13 \cdot 41},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 91/50$. □

3.6. Sporadic cases with $I = 6$

Lemma 3.6.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (5, 7, 8, 9, 23)$. Then $\mathrm{lct}(X) = 5/8$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$yz^2 + y^2 t + xt^2 + x^3 z = 0,$$

and X is singular at O_x, O_y, O_z and O_t. We have

$$\mathrm{lct}\left(X, \frac{6}{5} C_x\right) = \frac{5}{8} < \mathrm{lct}\left(X, \frac{6}{7} C_y\right) = \frac{7}{9} < \mathrm{lct}\left(X, \frac{6}{8} C_z\right) = \frac{6}{7} < \mathrm{lct}\left(X, \frac{6}{9} C_t\right) = 1,$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 5/8$.

The curve C_x is reducible. We have $C_x = L_{xy} + M_x$, where L_{xy} and M_x are irreducible curves such that L_{xy} is given by $x = y = 0$, and M_x is given by $x = z^2 + yt = 0$. Then

$$L_{xy} \cdot L_{xy} = \frac{-11}{8 \cdot 9},\ M_x \cdot M_x = \frac{-4}{7 \cdot 9},\ L_{xy} \cdot M_x = \frac{2}{9},\ D \cdot L_{xy} = \frac{6}{8 \cdot 9},\ D \cdot M_x = \frac{12}{7 \cdot 9},$$

and $L_{xy} \cap M_x = O_t$. Note that C_x is smooth outside of the point O_t.

The curve C_y is reducible. We have $C_y = L_{xy} + M_y$, where M_y is an irreducible curve such that M_y is given by $y = t^2 + x^2 z = 0$. Then

$$M_y \cdot M_y = \frac{1}{5}, \quad L_{xy} \cdot M_y = \frac{1}{4}, \ D \cdot M_y = \frac{3}{10},$$

and $L_{xy} \cap M_y = O_z$. The only singular point of the curve C_y is O_z.

The curve C_z is reducible. We have $C_z = L_{zt} + M_z$, where L_{zt} and M_z are irreducible curves such that L_{zt} is given by $x = y = 0$, and M_z is given by $z = tx + y^2 = 0$. Then

$$L_{zt} \cdot L_{zt} = \frac{-6}{35}, \ M_z \cdot M_z = \frac{-2}{45}, \ L_{zt} \cdot M_z = \frac{2}{5}, \ D \cdot L_{zt} = \frac{6}{35}, \ D \cdot M_z = \frac{4}{15},$$

and $L_{zt} \cap M_z = O_x$. The only singular point of C_z is O_x. We have $L_{xy} \cdot L_{zt} = 0$ and $L_{xy} \cdot M_z = 1/9$.

The curve C_t is reducible. We have $C_t = L_{zt} + M_t$, where M_t is an irreducible curve that is given by the equations $t = x^3 + z^2 y = 0$. Then

$$M_t \cdot M_t = \frac{3}{7 \cdot 8}, \ L_{zt} \cdot M_t = \frac{3}{7}, \ D \cdot M_t = \frac{9}{28},$$

and $L_{zt} \cap M_t = O_y$. The only singular point of C_t is O_y.

We suppose that $\mathrm{lct}(X) < 5/8$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{5}{8}D)$ is not log canonical at some point $P \in X$. Let us derive a contradiction.

Suppose that $P \notin C_x \cup C_y \cup C_z \cup C_t$. Then there is a unique curve $Z_\alpha \subset X$ that is cut out by

$$xt + \alpha y^2 = 0$$

such that $P \in Z$, where $0 \neq \alpha \in \mathbb{C}$. The curve Z_α is reduced. But it is always reducible. Indeed, one can easily check that

$$Z_\alpha = C_\alpha + L_{xy}$$

where C_α is a reduced curve whose support contains no L_{xy}. Let us prove that C_α is irreducible if $\alpha \neq 1$.

The open subset $Z_\alpha \setminus (Z_\alpha \cap C_x)$ of the curve Z_α is a $\mathbb{Z}_5$-quotient of the affine curve

$$t + \alpha y^2 = 0 = yz^2 + y^2 t + t^2 + z = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

which is isomorphic to a plane affine curve that is given by the equation

$$y\Big(\alpha(\alpha-1)y^4 + z + z^2 y\Big) = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big),$$

which implies that the curve C_α is irreducible and $\mathrm{mult}_P(C_\alpha) \leqslant 3$ if $\alpha \neq 1$.

The case $\alpha = 1$ is special. Namely, if $\alpha = 1$, then

$$C_1 = R_1 + M_z,$$

where R_1 is a reduced curve whose support contains no C_1. Arguing as in the case $\alpha \neq 1$, we see that R_1 is irreducible and R_1 is smooth at the point P.

By Remark 1.4.7, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible components of the curve Z_α.

Suppose that $\alpha \neq 1$. Then elementary calculations imply that

$$C_\alpha \cdot L_{xy} = \frac{25}{8 \cdot 9}, \ C_\alpha \cdot C_\alpha = \frac{449}{360}, \ D \cdot C_\alpha = \frac{41 \cdot 6}{360},$$

and we can put $D = \epsilon C_\alpha + \Xi$, where Ξ is an effective $\mathbb{Q}$-divisor such that $C_\alpha \not\subset \mathrm{Supp}(\Xi)$. Now we obtain the inequality $\epsilon \leqslant 6/25$, because either $\epsilon = 0$, or $L_{xy} \cdot \Xi \geqslant 0$. On the other hand, we see that

$$\frac{41 \cdot 6}{360} = D \cdot C_\alpha = \epsilon C_\alpha^2 + \Xi \cdot C_\alpha \geqslant \epsilon C_\alpha^2 + \mathrm{mult}_P\big(\Xi\big) = \epsilon C_\alpha^2 + \mathrm{mult}_P\big(D\big) - \epsilon \mathrm{mult}_P\big(C_\alpha\big) > \epsilon C_\alpha^2 + \frac{5}{8} - 3\epsilon,$$

which is impossible, because $\epsilon \leqslant 6/25$.

Thus, we see that $\alpha = 1$. Then elementary calculations imply that

$$R_1 \cdot L_{xy} = \frac{17}{8 \cdot 9}, \ R_1 \cdot R_1 = \frac{13}{8 \cdot 9}, \ M_z \cdot R_1 = \frac{28}{45}, \ D \cdot R_1 = \frac{30}{8 \cdot 9},$$

and we can put $D = \epsilon_1 R_1 + \Xi_1$, where Ξ_1 is an effective $\mathbb{Q}$-divisor such that $R_1 \not\subset \mathrm{Supp}(\Xi_1)$. Now we obtain the inequality $\epsilon_1 \leqslant 12/25$, because either $\epsilon_1 = 0$, or $L_{xy} \cdot \Xi_1 \geqslant 0$ or $M_z \cdot \Xi_1 \geqslant 0$. By Lemma 1.4.6, we see that

$$\frac{30 - 13\epsilon_1}{72} = \Xi_1 \cdot R_1 > \frac{5}{8},$$

which is impossible, because $\epsilon_1 \leqslant 12/25$. Thus, we see that $P \in C_x \cup C_y \cup C_z \cup C_t$.

It follows from Remark 1.4.7 that we may assume that $\mathrm{Supp}(D)$ does not contains are least one irreducible component of the curves C_x, C_y, C_z, C_t.

Suppose that $P = O_z$. If $L_{yz} \not\subset \mathrm{Supp}(D)$, then

$$\frac{1}{12} = D \cdot L_{yz} \geqslant \frac{\mathrm{mult}_P(D)}{8} > \frac{1}{5},$$

which is a contradiction. If $M_y \not\subset \mathrm{Supp}(D)$, then

$$\frac{3}{10} = D \cdot M_y \geqslant \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(M_y)}{8} = \frac{2\mathrm{mult}_P(D)}{8} > \frac{2}{5},$$

which is a contradiction. Thus, we see that $P \neq O_z$. Similarly, we see that $P \neq O_x$ and $P \neq O_y$.

Suppose that $P \in L_{xy}$. Put $D = \delta L_{xy} + \Theta$, where Θ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{xy}. If $\delta \neq 0$, then

$$\frac{4}{21} = D \cdot M_x = \big(\delta L_{xy} + \Theta\big) \cdot M_x \geqslant \delta L_{xy} \cdot M_x = \frac{2\delta}{9},$$

which implies that $\delta \leqslant 6/7$. Then it follows from Lemma 1.4.6 that

$$\frac{6 + 11\delta}{72} = \big(-K_X - \delta L_{xy}\big) \cdot L_{xy} = \Theta \cdot L_{xy} > \begin{cases} \frac{8}{5} \text{ if } P \neq O_t, \\ \frac{8}{45} \text{ if } P = O_t, \end{cases}$$

which implies that $\delta > 34/55$ and $P = O_t$, because $\delta \leqslant 6/7$. Then

$$\frac{4}{21} = D \cdot M_x = \big(\delta L_{xy} + \Theta\big) \cdot M_x \geqslant \delta L_{xy} \cdot M_x + \frac{\mathrm{mult}_P(D) - \delta}{9} > \frac{2\delta}{9} + \frac{8/5 - \delta}{9},$$

which implies that $\delta < 4/35$. But $\delta > 34/35$. Thus, we see that $P \neq L_{xy}$. Then $P \notin \mathrm{Sing}(X)$.

Suppose that $P \in M_x$. Put $D = eM_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $M_x \not\subset \mathrm{Supp}(\Upsilon)$. If $e \neq 0$, then

$$\frac{6}{72} = D \cdot L_{xy} = \big(eM_x + \Upsilon\big) \cdot L_{xy} \geqslant eL_{xy} \cdot M_x = \frac{2e}{9},$$

which implies that $e \leqslant 3/8$. Then it follows from Lemma 1.4.6 that

$$\frac{4 + 4e}{21} = \big(-K_X - eM_x\big) \cdot M_x = \Upsilon \cdot M_x > \frac{8}{5},$$

which is impossible, because $e \leqslant 3/8$. Thus, we see that $P \notin M_x$. Similarly, we see that $P \notin L_{zt} \cup M_y \cup M_z \cup M_t$, which is a contradiction. □

Lemma 3.6.2. Suppose that $(a_0, a_1, a_2, a_3, d) = (7, 10, 15, 19, 45)$. Then $\mathrm{lct}(X) = 35/54$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + y^3 z + xt^2 + x^5 y = 0,$$

the surface X is singular at the point O_x, O_y, O_t. The surface X is also singular at a point Q such that $Q \neq O_y$ and O_y and Q are cut out on X by the equations $x = t = 0$.

The curve C_x is reducible. We have $C_x = L_{xz} + Z_x$, where L_{xz} and Z_x are irreducible and reduced curves such that L_{xz} is given by the equations $x = z = 0$, and Z_x is given by the equations $x = z^2 + y^3 = 0$. Then

$$L_{xz} \cdot L_{xz} = \frac{-23}{10 \cdot 19},\ Z_x \cdot Z_x = \frac{-16}{10 \cdot 19},\ L_{xz} \cdot Z_x = \frac{3}{19},$$

and $L_{xz} \cap Z_x = O_t$. The curve C_y is irreducible and

$$\frac{35}{54} = \mathrm{lct}\Big(X, \frac{6}{7} C_x\Big) < \mathrm{lct}\Big(X, \frac{6}{10} C_y\Big) = \frac{25}{18},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 35/54$.

Suppose that $\mathrm{lct}(X) < 35/54$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{35}{54}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, we may assume that either $L_{xz} \not\subseteq \mathrm{Supp}(D)$, or $Z_x \not\subseteq \mathrm{Supp}(D)$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(105))$ contains x^{15}, y^7x^5 and z^7, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = O_t$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{6}{10 \cdot 19} = D \cdot L_{xz} \geqslant \frac{\mathrm{mult}_P\big(D\big)}{19} > \frac{54}{35 \cdot 19} > \frac{6}{10 \cdot 19},$$

which is a contradiction. If $Z_x \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{12}{10 \cdot 19} = D \cdot Z_x \geqslant \frac{\mathrm{mult}_P\big(D\big)\mathrm{mult}_P(Z_x)}{15} > \frac{54 \cdot 2}{35 \cdot 19} > \frac{12}{10 \cdot 19},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P \in L_{xz}$. Put $D = mL_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{12}{10 \cdot 19} = -K_X \cdot Z_x = D \cdot Z_x = \big(mL_{xz} + \Omega\big) \cdot Z_x \geqslant mL_{xz} \cdot Z_x = \frac{3m}{19},$$

which implies that $m \leqslant 2/5$. Then it follows from Lemma 1.4.6 that

$$\frac{6 + 23m}{10 \cdot 19} = \big(-K_X - mL_{xz}\big) \cdot L_{xz} = \Omega \cdot L_{xz} > \begin{cases} \dfrac{54}{35} \text{ if } P \neq O_y, \\ \dfrac{54}{35}\dfrac{1}{10} \text{ if } P = O_y, \end{cases}$$

which is impossible, because $m \leqslant 2/5$. Thus, we see that $P \notin L_{xz}$.

Suppose that $P \in Z_x$. Put $D = \epsilon Z_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_x \not\subset \mathrm{Supp}(\Delta)$. If $\epsilon \neq 0$, then

$$\frac{6}{10 \cdot 19} = -K_X \cdot L_{xz} = D \cdot L_{xz} = \big(\epsilon Z_x + \Delta\big) \cdot L_{xz} \geqslant \epsilon L_{xz} \cdot Z_x = \frac{3\epsilon}{19},$$

which implies that $m \leqslant 1/5$. Then it follows from Lemma 1.4.6 that

$$\frac{6 + 16\epsilon}{10 \cdot 19} = \big(-K_X - \epsilon Z_x\big) \cdot Z_x = \Delta \cdot Z_x > \begin{cases} \dfrac{54}{35} \text{ if } P \neq Q, \\ \dfrac{54}{35}\dfrac{1}{5} \text{ if } P = Q, \end{cases}$$

which is impossible, because $\epsilon \leqslant 1/5$. Thus, we see that $P \notin Z_x$.

We see that $P \notin C_x$ and $P \in \mathrm{Sing}(X)$. Then $P = O_x$. We have

$$\frac{18}{7 \cdot 19} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P\big(D\big)}{7} > \frac{54}{35 \cdot 7} > \frac{18}{7 \cdot 19},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 35/54$. □

Lemma 3.6.3. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 19, 29, 53, 106)$. Then $\mathrm{lct}(X) = 55/36$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$x^7z + xy^5 + yz^3 + t^2 = 0.$$

Note that X is singular at O_x, O_y and O_z. The curves C_x and C_y are irreducible. It is easy to see

$$\mathrm{lct}(X, \frac{6}{11}C_x) = \frac{55}{36} < \mathrm{lct}(X, \frac{6}{19}C_y) = \frac{57}{28}.$$

Suppose that $\mathrm{lct}(X) < \frac{55}{36}$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $(X, \frac{55}{36}D)$ is not log canonical.

For a smooth point $P \in X \setminus C_x$ and an effective $\mathbb{Q}$-divisor $D \equiv -K_X$, we have

$$\mathrm{mult}_P D \leqslant \frac{6 \cdot 319 \cdot 106}{11 \cdot 19 \cdot 29 \cdot 53} < \frac{36}{55}$$

since $H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(319))$ contains x^{29}, z^{11}, $x^{10}y^{11}$. Therefore, either there is a point $P \in C_x$ such that $\mathrm{mult}_P D > \frac{36}{55}$ or we have $\mathrm{mult}_{O_x} D > \frac{36}{55}$. Since the pairs $(X, \frac{6\cdot 55}{11\cdot 36}C_x)$ and $(X, \frac{6\cdot 55}{19\cdot 36}C_y)$ are log canonical and the curves C_x and C_y are irreducible, we may assume that the support of D does not contain the curves C_x and C_y. Then we can obtain

$$\mathrm{mult}_{O_x} D \leqslant 11 C_y \cdot D \leqslant \frac{11\cdot 19\cdot 106\cdot 6}{11\cdot 19\cdot 29\cdot 53} < \frac{36}{55}$$

and for any point $P \in C_x$

$$\mathrm{mult}_P D \leqslant 29 C_x \cdot D \leqslant \frac{29\cdot 11\cdot 106\cdot 6}{11\cdot 19\cdot 29\cdot 53} < \frac{36}{55}.$$

Therefore, $\mathrm{lct}(X) = \frac{55}{36}$. □

Lemma 3.6.4. Suppose that $(a_0, a_1, a_2, a_3, d) = (13, 15, 31, 53, 106)$. Then $\mathrm{lct}(X) = 45/28$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$x^7 z + xy^5 + yz^3 + t^2 = 0,$$

and X is singular at the points O_x, O_y and O_z.

The curves C_x and C_y are reduced and irreducible. We have

$$\frac{45}{28} = \mathrm{lct}\left(X, \frac{6}{15}C_y\right) < \mathrm{lct}\left(X, \frac{6}{13}C_x\right) = \frac{65}{36},$$

which implies thatlct$(X) \leqslant 45/28$.

Suppose that $\mathrm{lct}(X) < 45/28$. Then there is an effective $\mathbb{Q}$-divisor $D \equiv -K_X$ such that the pair $(X, \frac{45}{28}D)$ is not log canonical at some point P. by Remark 1.4.7 we may assume that the support of D does not contain C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(403))$ contains x^{31}, z^{13}, xy^{26}, we see that $P \in \mathrm{Sing}(X) \cup C_x$ by Lemma 1.4.10.

Suppose that $P \in C_x$. Then

$$\frac{12}{14\cdot 31} = D\cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{15} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{31} \text{ if } P = O_z, \\ \mathrm{mult}_P(D) \text{ if } P \neq O_y \text{ and } P \neq O_z, \end{cases}$$

which implies that $P = O_z$, because $\mathrm{mult}_P(D) > 28/45$. Then

$$\frac{12}{13\cdot 31} = D\cdot C_y \geqslant \frac{\mathrm{mult}_P(D)\mathrm{mult}_P(C_y)}{31} > \frac{56}{45\cdot 30} > \frac{12}{13\cdot 31},$$

because $\mathrm{mult}_P(C_y) = 2$. Thus, we see that $P = O_x$. Then

$$\frac{12}{13\cdot 31} = D\cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{13} > \frac{28}{45\cdot 13} > \frac{12}{13\cdot 31},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 45/28$. □

3.7. Sporadic cases with $I = 7$

Lemma 3.7.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (11, 13, 21, 38, 76)$. Then $\mathrm{lct}(X) = 13/10$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^5 + x^5 z = 0.$$

Note that X is singular at O_x, O_y and O_z.

The curves C_x and C_y are irreducible. We have

$$\frac{55}{42} = \mathrm{lct}(X, \frac{7}{11}C_x) > \mathrm{lct}(X, \frac{7}{13}C_y) = \frac{13}{10},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 13/10$.

Suppose that $\mathrm{lct}(X) < 13/10$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{13}{10}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of D does not contain the curves C_x and C_y.

Suppose that $P \in C_x$ and $P \notin \mathrm{Sing}(X)$. Then

$$\frac{10}{13} < \mathrm{mult}_P(D) \leqslant D \cdot C_x = \frac{2}{39} < \frac{10}{13},$$

which is a contradiction. Suppose that $P \in C_y$ and $P \notin \mathrm{Sing}(X)$. Then

$$\frac{10}{13} < \mathrm{mult}_P(D) \leqslant D \cdot C_y = \frac{2}{33} < \frac{10}{13},$$

which is a contradiction. Suppose that $P = O_x$. Then

$$\frac{10}{13}\frac{1}{11} < \frac{\mathrm{mult}_{O_x}(D)}{11} \leqslant D \cdot C_y = \frac{2}{33} < \frac{10}{13}\frac{1}{11},$$

which is a contradiction. Suppose that $P = O_z$. Then

$$\frac{10}{13}\frac{2}{21} < \frac{2\mathrm{mult}_{O_z}(D)}{21} = \frac{\mathrm{mult}_{O_z}(D)\mathrm{mult}_{O_z}(C_y)}{21} \leqslant D \cdot C_y = \frac{2}{33} < \frac{10}{13}\frac{2}{21},$$

which is a contradiction. Suppose that $P = O_y$. Then

$$\frac{10}{13}\frac{1}{13} < \frac{\mathrm{mult}_{O_y}(D)}{13} \leqslant D \cdot C_x = \frac{2}{39} < \frac{10}{13}\frac{1}{13},$$

which is a contradiction. Thus, we see that $P \in X \setminus \mathrm{Sing}(X)$ and $P \notin C_x \cup C_y$.

Let $\mathcal{L}$ be the pencil on X that is cut out by the pencil

$$\lambda x^{13} + \mu y^{11} = 0,$$

where $[\lambda : \mu] \in \mathbb{P}^1$. Then the base locus of the pencil $\mathcal{L}$ consists of the point O_z.

Let C be the unique curve in $\mathcal{L}$ that passes through the point P. Arguing as in the proof of Lemma 3.3.1, we see that the curve C is irreducible. On the other hand, the curve C is a double cover of the curve

$$\lambda x^{13} + \mu y^{11} = 0 \subset \mathbb{P}(11, 13, 21) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z\big]\Big)$$

such that $\lambda \neq 0$ and $\mu \neq 0$. Thus, the inequality $\mathrm{mult}_P(C) \leqslant 2$ holds. In particular, the log pair $(X, \frac{7}{110}C)$ is log canonical. Thus, we may assume that the support of D does not contain the curve C and hence we obtain

$$\frac{10}{13} < \mathrm{mult}_P(D) \leqslant D \cdot C = \frac{2}{3} < \frac{10}{13},$$

which is a contradiction. □

3.8. Sporadic cases with $I = 8$

Lemma 3.8.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (7, 11, 13, 23, 46)$. Then $\mathrm{lct}(X) = 35/48$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^3z + xz^3 + x^5y = 0,$$

the surface X is singular at the point O_x, O_y and O_z.

The curves C_x, C_y and C_z are irreducible. We have

$$\frac{35}{48} = \mathrm{lct}\left(X, \frac{8}{7}C_x\right) < \mathrm{lct}\left(X, \frac{8}{13}C_z\right) = \frac{91}{80} < \mathrm{lct}\left(X, \frac{8}{11}C_y\right) = \frac{55}{48},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 35/48$.

Suppose that $\mathrm{lct}(X) < 35/48$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{35}{48}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x, C_y and C_z.

Suppose that $P \in C_x$. Then

$$\frac{16}{11\cdot 13} = D\cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{11} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)\mathrm{mult}_{O_z}(C_x)}{13} \text{ if } P = O_z, \\ \mathrm{mult}_P(D) \text{ if } P \neq O_x \text{ and } P \neq O_z, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 48/35$ and $\mathrm{mult}_{O_z}(C_x) = 2$.

We see that $P \neq O_z$. Suppose that $P \in C_y$. Then

$$\frac{16}{7\cdot 13} = D\cdot C_y \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{7} \text{ if } P = O_x, \\ \mathrm{mult}_P(D) \text{ if } P \neq O_x, \end{cases}$$

which is impossible, because $\mathrm{mult}_P(D) > 48/35$. Thus, we see that $P \in C_y$. Then $P \notin \mathrm{Sing}(X)$.

Let us show that $P \notin C_z$. Suppose that $P \in C_z$. Then

$$\frac{16}{7\cdot 11} = D\cdot C_z \geqslant \mathrm{mult}_P(D) > \frac{48}{35},$$

which is a contradiction. Thus, we see that $P \notin C_z$.

We see that $P \notin C_x \cup C_y \cup C_z$. Then there is a unique curve $Z \subset X$ that is cut out by

$$x^4y = \alpha z^3$$

such that $P \in Z$, where $0 \neq \alpha \in \mathbb{C}$. We see that $C_x \not\subset \mathrm{Supp}(Z)$. But the open subset $Z\setminus(Z\cap C_x)$ of the curve Z is a $\mathbb{Z}_7$-quotient of the affine curve

$$y - \alpha z^3 = t^2 + y^3z + z^3 + y = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

which is isomorphic to a plane affine curve $R_x \subset \mathbb{C}^2$ that is given by the equation

$$t^2 + \alpha^3z^{10} + (1+\alpha)z^3 = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big),$$

which is irreducible if $\alpha \neq -1$. We see that Z is irreducible if $\alpha \neq -1$.

It follows from the equation of the curve R_x that the log pair $(X, \frac{35}{48}Z)$ is log canonical at the point P. By Remark 1.4.7, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve Z.

Suppose that $\alpha \neq -1$. Then $Z \not\subseteq \mathrm{Supp}(D)$ and

$$\frac{48}{77} = D\cdot Z \geqslant \mathrm{mult}_P(D) > \frac{48}{35},$$

which is a contradiction. Thus, we see that $\alpha = -1$.

We have $Z = Z_1 + Z_2$, where Z_1 and Z_2 are irreducible reduced curves such that

$$Z_1\cdot Z_1 = Z_1\cdot Z_1 = \frac{742}{77},\ Z_1\cdot Z_2 = \frac{10}{7} + \frac{12}{11} = \frac{194}{77},$$

and $Z_1 \cap Z_2 = O_x \cup O_y$. We may assume that $P \in Z_1$.

Put $D = mZ_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $Z_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{24}{77} = -K_X\cdot Z_2 = D\cdot Z_2 = \big(mZ_1 + \Omega\big)\cdot Z_2 \geqslant mZ_1\cdot Z_2 = \frac{194m}{77},$$

which implies that $m \leqslant 12/97$. Then it follows from Lemma 1.4.6 that

$$\frac{24 - 742m}{77} = \big(-K_X - mZ_1\big)\cdot Z_1 = \Omega\cdot Z_1 > \frac{48}{35},$$

which is a contradiction. The obtained contradiction completes the proof. ◻

Lemma 3.8.2. Suppose that $(a_0, a_1, a_2, a_3, d) = (7, 18, 27, 37, 81)$. Then $\mathrm{lct}(X) = 35/72$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + y^3 z + x t^2 + x^9 y = 0,$$

the surface X is singular at the point O_x, O_y, O_t. The surface X is also singular at a point Q such that $Q \neq O_y$ and O_y and Q are cut out on X by the equations $x = t = 0$.

The curve C_x is reducible. We have $C_x = L_{xz} + Z_x$, where L_{xz} and Z_x are irreducible and reduced curves such that L_{xz} is given by the equations $x = z = 0$, and Z_x is given by the equations $x = z^2 + y^3 = 0$. Then

$$L_{xz} \cdot L_{xz} = \frac{-47}{18 \cdot 37},\ Z_x \cdot Z_x = \frac{-40}{18 \cdot 37},\ L_{xz} \cdot Z_x = \frac{3}{37},$$

and $L_{xz} \cap Z_x = O_t$. The curve C_y is irreducible and

$$\frac{35}{72} = \mathrm{lct}\left(X, \frac{8}{7} C_x\right) < \mathrm{lct}\left(X, \frac{8}{18} C_y\right) = \frac{15}{8},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 35/72$.

Suppose that $\mathrm{lct}(X) < 35/78$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{35}{72} D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, we may assume that either $L_{xz} \not\subseteq \mathrm{Supp}(D)$, or $Z_x \not\subseteq \mathrm{Supp}(D)$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(189))$ contains x^{27}, $y^7 x^9$ and z^7, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P = O_t$. If $L_{xz} \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{8}{18 \cdot 37} = D \cdot L_{xz} \geqslant \frac{\mathrm{mult}_P(D)}{37} > \frac{72}{35 \cdot 37} > \frac{8}{18 \cdot 37},$$

which is a contradiction. If $Z_x \not\subseteq \mathrm{Supp}(D)$, then

$$\frac{16}{18 \cdot 37} = D \cdot Z_x \geqslant \frac{\mathrm{mult}_P(D)}{37} > \frac{72}{35 \cdot 37} > \frac{16}{18 \cdot 37},$$

which is a contradiction. Thus, we see that $P \neq O_t$.

Suppose that $P \in L_{xz}$. Put $D = m L_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{16}{18 \cdot 37} = -K_X \cdot Z_x = D \cdot Z_x = \big(m L_{xz} + \Omega\big) \cdot Z_x \geqslant m L_{xz} \cdot Z_x = \frac{3m}{37},$$

which implies that $m \leqslant 8/27$. Then it follows from Lemma 1.4.6 that

$$\frac{8 + 47m}{18 \cdot 37} = \big(-K_X - m L_{xz}\big) \cdot L_{xz} = \Omega \cdot L_{xz} > \begin{cases} \dfrac{72}{35} & \text{if } P \neq O_y, \\ \dfrac{72}{35} \dfrac{1}{18} & \text{if } P = O_y, \end{cases}$$

which is impossible, because $m \leqslant 8/27$. Thus, we see that $P \notin L_{xz}$.

Suppose that $P \in Z_x$. Put $D = \epsilon Z_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_x \not\subset \mathrm{Supp}(\Delta)$. If $\epsilon \neq 0$, then

$$\frac{8}{18 \cdot 37} = -K_X \cdot L_{xz} = D \cdot L_{xz} = \big(\epsilon Z_x + \Delta\big) \cdot L_{xz} \geqslant \epsilon L_{xz} \cdot Z_x = \frac{3\epsilon}{37},$$

which implies that $m \leqslant 4/27$. Then it follows from Lemma 1.4.6 that

$$\frac{16 + 40\epsilon}{18 \cdot 37} = \big(-K_X - \epsilon Z_x\big) \cdot Z_x = \Delta \cdot Z_x > \begin{cases} \dfrac{72}{35} & \text{if } P \neq Q, \\ \dfrac{72}{35} \dfrac{1}{9} & \text{if } P = Q, \end{cases}$$

which is impossible, because $\epsilon \leqslant 5/27$. Thus, we see that $P \notin Z_x$.

We see that $P \notin C_x$ and $P \in \mathrm{Sing}(X)$. Then $P = O_x$. We have

$$\frac{24}{7 \cdot 37} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{7} > \frac{72}{35 \cdot 7} > \frac{24}{7 \cdot 37},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 35/72$. ∎

3.9. Sporadic cases with $I = 9$

Lemma 3.9.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (7, 15, 19, 32, 64)$. Then $\mathrm{lct}(X) = 35/54$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^3 z + xz^3 + x^7 y = 0,$$

the surface X is singular at the point O_x, O_y and O_z, the curves C_x and C_y are irreducible, and

$$\frac{35}{54} = \mathrm{lct}\left(X, \frac{9}{7}C_x\right) < \mathrm{lct}\left(X, \frac{9}{15}C_y\right) = \frac{25}{18},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 35/54$.

Suppose that $\mathrm{lct}(X) < 35/2$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{35}{18}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(133))$ contains x^{10}, y^7x^4 and z^7, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P \in C_x$. Then

$$\frac{6}{95} = D \cdot C_x \geqslant \begin{cases} \dfrac{\mathrm{mult}_P(D)}{15} \text{ if } P = O_y, \\ \dfrac{\mathrm{mult}_P(D)}{19} \text{ if } P = O_z, \\ \mathrm{mult}_P(D) \text{ if } P \neq O_y \text{ and } P \neq O_z, \end{cases} > \begin{cases} \dfrac{54}{35 \cdot 15} \text{ if } P = O_y, \\ \dfrac{54}{35 \cdot 19} \text{ if } P = O_z, \\ \dfrac{54}{35} \text{ if } P \neq O_y \text{ and } P \neq O_z, \end{cases} > \frac{6}{95}$$

which is a contradiction. Thus, we see that $P = O_x$. Then

$$\frac{18}{133} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P(D)}{7} > \frac{54}{35 \cdot 7} > \frac{18}{133},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 35/54$. □

3.10. Sporadic cases with $I = 10$

Lemma 3.10.1. Suppose that $(a_0, a_1, a_2, a_3, d) = (7, 19, 25, 41, 82)$. Then $\mathrm{lct}(X) = 7/12$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^3 z + xz^3 + x^9 y = 0,$$

and X is singular at the points O_x, O_y and O_z.

The curves C_x and C_y are reducible. We have

$$\frac{7}{12} = \mathrm{lct}\left(X, \frac{10}{7}C_x\right) < \mathrm{lct}\left(X, \frac{10}{19}C_y\right) = \frac{19}{12},$$

which implies, in particular, that $\mathrm{lct}(X) \leqslant 7/12$.

Suppose that $\mathrm{lct}(X) < 7/12$. Then there is a $\mathbb{Q}$-effective divisor $D \equiv -K_X$ such that the pair $(X, \frac{7}{12}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curves C_x and C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(175))$ contains x^{25}, x^6y^7 and z^7, it follows from Lemma 1.4.10 that $P \in \mathrm{Sing}(X) \cup C_x$.

Suppose that $P \in C_x$. Then

$$\frac{4}{95} = D \cdot C_x \geqslant \begin{cases} \dfrac{12}{7} \text{ if } P \neq O_y \text{ and } P \neq O_z, \\ \dfrac{12}{7}\dfrac{1}{19} \text{ if } P = O_y, \\ \dfrac{12}{7}\dfrac{1}{25} \text{ if } P = O_z, \end{cases}$$

which is a contradiction. Thus, we see that $P \notin C_x$. Then $P = O_x$. We have

$$\frac{4}{35} = D \cdot C_x \geqslant \frac{\mathrm{mult}_P(D)}{7} > \frac{12}{49}$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X)=7/12$. □

Lemma 3.10.2. Suppose that $(a_0,a_1,a_2,a_3,d)=(7,26,39,55,117)$. Then $\mathrm{lct}(X)=7/18$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3+y^3z+xt^2+x^{13}y=0,$$

the surface X is singular at the point O_x, O_y, O_t. The surface X is also singular at a point Q such that $Q\neq O_y$ and O_y and Q are cut out on X by the equations $x=t=0$.

The curve C_x is reducible. We have $C_x=L_{xz}+Z_x$, where L_{xz} and Z_x are irreducible and reduced curves such that L_{xz} is given by the equations $x=z=0$, and Z_x is given by the equations $x=z^2+y^3=0$. Then

$$L_{xz}\cdot L_{xz}=\frac{-71}{26\cdot 55},\ Z_x\cdot Z_x=\frac{-32}{13\cdot 55},\ L_{xz}\cdot Z_x=\frac{3}{55},$$

and $L_{xz}\cap Z_x=O_t$. The curve C_y is irreducible and

$$\frac{7}{18}=\mathrm{lct}\Big(X,\frac{10}{7}C_x\Big)<\mathrm{lct}\Big(X,\frac{10}{26}C_y\Big)=\frac{13}{6},$$

which implies, in particular, that $\mathrm{lct}(X)\leqslant 7/18$.

Suppose that $\mathrm{lct}(X)<7/18$. Then there is a $\mathbb{Q}$-effective divisor $D\equiv -K_X$ such that the pair $(X,\frac{7}{18}D)$ is not log canonical at some point P. By Remark 1.4.7, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, we may assume that either $L_{xz}\not\subseteq\mathrm{Supp}(D)$, or $Z_x\not\subseteq\mathrm{Supp}(D)$.

Since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(273))$ contains x^{39}, y^7x^{13} and z^7, it follows from Lemma 1.4.10 that $P\in\mathrm{Sing}(X)\cup C_x$.

Suppose that $P=O_t$. If $L_{xz}\not\subseteq\mathrm{Supp}(D)$, then

$$\frac{2}{11\cdot 26}=D\cdot L_{xz}\geqslant\frac{\mathrm{mult}_P(D)}{55}>\frac{18}{7\cdot 55}>\frac{2}{11\cdot 26},$$

which is a contradiction. If $Z_x\not\subseteq\mathrm{Supp}(D)$, then

$$\frac{20}{26\cdot 55}=D\cdot Z_x\geqslant\frac{\mathrm{mult}_P(D)}{55}>\frac{18}{7\cdot 55}>\frac{20}{26\cdot 55},$$

which is a contradiction. Thus, we see that $P\neq O_t$.

Suppose that $P\in L_{xz}$. Put $D=mL_{xz}+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz}\not\subseteq\mathrm{Supp}(\Omega)$. If $m\neq 0$, then

$$\frac{20}{26\cdot 55}=-K_X\cdot Z_x=D\cdot Z_x=\big(mL_{xz}+\Omega\big)\cdot Z_x\geqslant mL_{xz}\cdot Z_x=\frac{3m}{55},$$

which implies that $m\leqslant 10/39$. Then it follows from Lemma 1.4.6 that

$$\frac{10+71m}{26\cdot 55}=\big(-K_X-mL_{xz}\big)\cdot L_{xz}=\Omega\cdot L_{xz}>\begin{cases}\dfrac{18}{7}\ \text{if}\ P\neq O_y,\\ \dfrac{18}{7}\dfrac{1}{26}\ \text{if}\ P=O_y,\end{cases}$$

which implies that $m>920/497$. But we already proved that $m\leqslant 10/39$. Thus, we see that $P\notin L_{xz}$.

Suppose that $P\in Z_x$. Put $D=\epsilon Z_x+\Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_x\not\subseteq\mathrm{Supp}(\Delta)$. If $\epsilon\neq 0$, then

$$\frac{10}{26\cdot 55}=-K_X\cdot L_{xz}=D\cdot L_{xz}=\big(\epsilon Z_x+\Delta\big)\cdot L_{xz}\geqslant\epsilon L_{xz}\cdot Z_x=\frac{3\epsilon}{55},$$

which implies that $m\leqslant 5/39$. Then it follows from Lemma 1.4.6 that

$$\frac{20+32\epsilon}{13\cdot 55}=\big(-K_X-\epsilon Z_x\big)\cdot Z_x=\Delta\cdot Z_x>\begin{cases}\dfrac{18}{7}\ \text{if}\ P\neq Q,\\ \dfrac{18}{7}\dfrac{1}{13}\ \text{if}\ P=Q,\end{cases}$$

which is impossible, because $\epsilon\leqslant 5/39$. Thus, we see that $P\notin Z_x$.

We see that $P \notin C_x$ and $P \in \mathrm{Sing}(X)$. Then $P = O_x$. We have

$$\frac{6}{77} = D \cdot C_y \geqslant \frac{\mathrm{mult}_P\big(D\big)}{7} > \frac{18}{49} > \frac{6}{77},$$

which is a contradiction. Thus, we see that $\mathrm{lct}(X) = 7/18$. □

Part 4. The Big Table

Log del Pezzo surface with $I = 1$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(2, 2n+1, 2n+1, 4n+1)$	$8n+4$	$\frac{2}{(2n+1)(4n+1)}$	8	1	y^4, y^3z, y^2z^2, yz^3, z^4, xt^2, x^nyt, x^nzt, $x^{2n+1}y^2$, $x^{2n+1}yz$, $x^{2n+1}z^2$	$O_t = \frac{1}{4n+1}(1,1)$ $O_yO_z = 4 \times \frac{1}{2n+1}(1,2n)$
$(1,2,3,5)$	10	$\frac{1}{3}$	9	1^{a} $\frac{7}{10}^{b}$	t^2, yzt, y^2z^2, y^5, xz^3, xy^2t, xy^3z, x^2zt, x^2yz^2, x^2y^4, x^3yt, x^3y^2z, x^4z^2, x^4y^3, x^5t, x^5yz, x^6y^2, x^7z, x^8y, x^{10}	$O_z = \frac{1}{3}(1,1)$
$(1,3,5,7)$	15	$\frac{1}{7}$	9	1^{c} $\frac{8}{15}^{d}$	z^3, yzt, y^5, xt^2, xy^3z, x^2yz^2, x^2y^2t, x^3zt, x^3y^4, x^4y^2z, x^5z^2, x^5yt, x^6y^3, x^7yz, x^8t, x^9y^2, $x^{10}z$, $x^{12}y$, x^{15}	$O_t = \frac{1}{7}(3,5)$
$(1,3,5,8)$	16	$\frac{2}{15}$	10	1	t^2, yzt, y^2z^2, xz^3, xy^5, x^2y^2t, x^2y^3z, x^3zt, x^3yz^2, x^4y^4, x^5yt, x^5y^2z, x^6z^2, x^7y^3, x^8t, x^8yz, $x^{10}y^2$, $x^{11}z$, $x^{13}y$, x^{16}	$O_y = \frac{1}{3}(1,1)$ $O_z = \frac{1}{5}(1,1)$
$(2,3,5,9)$	18	$\frac{1}{15}$	7	2^{e} $\frac{11}{6}^{f}$	t^2, yz^3, y^3t, y^6, xy^2z^2, x^2zt, x^2y^3z, x^3yt, x^3y^4, x^4z^2, x^5yz, x^6y^2, x^9	$O_z = \frac{1}{5}(1,2)$ $O_yO_t = 2 \times \frac{1}{3}(1,1)$
$(3,3,5,5)$	15	$\frac{1}{15}$	5	2	t^3, zt^2, z^2t, z^3, y^5, xy^4, x^2y^3, x^3y^2, x^4y, x^5	$O_xO_y = 5 \times \frac{1}{3}(1,1)$ $O_zO_t = 3 \times \frac{1}{5}(1,1)$
$(3,5,7,11)$	25	$\frac{5}{231}$	5	$\frac{21}{10}$	z^2t, y^5, xt^2, xy^3z, x^2yz^2, x^3yt, x^5y^2, x^6z	$O_x = \frac{1}{3}(1,1)$ $O_z = \frac{1}{7}(3,5)$ $O_t = \frac{1}{11}(5,7)$
$(3,5,7,14)$	28	$\frac{2}{105}$	6	$\frac{9}{4}$	t^2, z^2t, z^4, xy^5, x^2y^3z, x^3yt, x^3yz^2, x^6y^2, x^7z	$O_x = \frac{1}{3}(1,1)$ $O_y = \frac{1}{5}(1,2)$ $O_zO_t = 2 \times \frac{1}{7}(3,5)$
$(3,5,11,18)$	36	$\frac{2}{165}$	6	$\frac{21}{10}$	t^2, y^5z, xz^3, xy^3t, x^2y^6, x^3yz^2, x^5y^2z, x^6t, x^7y^3, x^{12}	$O_y = \frac{1}{5}(1,1)$ $O_z = \frac{1}{11}(5,7)$ $O_zO_t = 2 \times \frac{1}{3}(1,1)$

a: if C_x has an ordinary double point, b: if C_x has a non-ordinary double point, c: if the defining equation of X contains yzt, d: if the defining equation of X does not contain yzt, e: if C_y has a tacknodal double point, f: if C_y has no tacknodal points.

Log del Pezzo surface with $I = 1$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x, y, z, t)$	Singular Points
$(5, 14, 17, 21)$	56	$\frac{4}{1785}$	4	$\frac{25}{8}$	yt^2, y^4, xz^3, x^5yz, x^7t	$O_x = \frac{1}{5}(2, 1)$ $O_z = \frac{1}{17}(7, 2)$ $O_t = \frac{1}{21}(5, 17)$ $O_yO_t = 1 \times \frac{1}{7}(5, 3)$
$(5, 19, 27, 31)$	81	$\frac{3}{2945}$	3	$\frac{25}{6}$	z^3, yt^2, xy^4, x^7yz, $x^{10}t$	$O_x = \frac{1}{5}(2, 1)$ $O_y = \frac{1}{19}(2, 3)$ $O_t = \frac{1}{31}(5, 27)$
$(5, 19, 27, 50)$	100	$\frac{2}{2565}$	4	$\frac{25}{6}$	t^2, yz^3, xy^5, x^7y^2z, $x^{10}t$, x^{20}	$O_y = \frac{1}{19}(2, 3)$ $O_z = \frac{1}{27}(5, 23)$ $O_xO_t = 2 \times \frac{1}{5}(2, 1)$
$(7, 11, 27, 37)$	81	$\frac{3}{2849}$	3	$\frac{49}{12}$	z^3, y^4t, xt^2, x^3y^3z, $x^{10}y$	$O_x = \frac{1}{7}(3, 1)$ $O_y = \frac{1}{11}(7, 5)$ $O_t = \frac{1}{37}(11, 27)$
$(7, 11, 27, 44)$	88	$\frac{2}{2079}$	4	$\frac{35}{8}$	t^2, y^4t, y^8, xz^3, x^4y^3z, $x^{11}y$	$O_x = \frac{1}{7}(3, 1)$ $O_z = \frac{1}{27}(11, 17)$ $O_yO_t = 2 \times \frac{1}{11}(7, 5)$
$(9, 15, 17, 20)$	60	$\frac{1}{765}$	3	$\frac{21}{4}$	t^3, y^4, xz^3, x^5y	$O_x = \frac{1}{9}(4, 1)$ $O_z = \frac{1}{17}(5, 1)$ $O_xO_y = 1 \times \frac{1}{3}(1, 1)$ $O_yO_t = 1 \times \frac{1}{5}(2, 1)$
$(9, 15, 23, 23)$	69	$\frac{1}{1035}$	5	6	t^3, zt^2, z^2t, z^3, xy^4, x^6y	$O_x = \frac{1}{9}(1, 1)$ $O_y = \frac{1}{15}(1, 1)$ $O_xO_y = 1 \times \frac{1}{3}(1, 1)$ $O_zO_t = 3 \times \frac{1}{23}(3, 5)$
$(11, 29, 39, 49)$	127	$\frac{127}{609609}$	3	$\frac{33}{4}$	z^2t, yt^2, xy^4, x^8z	$O_x = \frac{1}{11}(7, 5)$ $O_y = \frac{1}{29}(1, 2)$ $O_z = \frac{1}{39}(11, 29)$ $O_t = \frac{1}{49}(11, 39)$

Log del Pezzo surface with $I = 1$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(11, 49, 69, 128)$	256	$\frac{2}{37191}$	2	$\frac{55}{6}$	t^2, yz^3, xy^5, $x^{17}z$	$O_x = \frac{1}{11}(5,7)$ $O_y = \frac{1}{49}(2,3)$ $O_z = \frac{1}{69}(11,59)$
$(13, 23, 35, 57)$	127	$\frac{127}{596505}$	3	$\frac{65}{8}$	z^2t, y^4z, xt^2, x^8y	$O_x = \frac{1}{13}(9,5)$ $O_y = \frac{1}{23}(13,11)$ $O_z = \frac{1}{35}(13,23)$ $O_t = \frac{1}{57}(23,35)$
$(13, 35, 81, 128)$	256	$\frac{2}{36855}$	2	$\frac{91}{10}$	t^2, y^5z, xz^3, $x^{17}y$	$O_x = \frac{1}{13}(3,11)$ $O_y = \frac{1}{35}(13,23)$ $O_z = \frac{1}{81}(35,47)$

Log del Pezzo surface with $I = 2$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(4, 2n+3, 2n+3, 4n+4)$	$8n+12$	$\frac{1}{(n+1)(2n+3)}$	7	1	y^4, y^3z, y^2z^2, yz^3, z^4, xt^2, $x^{n+2}t$, x^{2n+3}	$O_t = \frac{1}{4n+4}(1,1)$ $O_xO_t = 2 \times \frac{1}{4}(1,1)$ $O_yO_z = 4\times\frac{1}{2n+3}(4, 2n+1)$
$(3, 3n+1, 6n+1, 9n+3)$	$18n+6$	$\frac{8}{3(3n+1)(6n+1)}$	6	1	t^2, y^3t, y^6, xz^3, $x^{n+1}yz^2$, $x^{2n+1}y^2z$, $x^{3n+1}t$, $x^{3n+1}y^3$, x^{6n+2}	$O_z = \frac{1}{6n+1}(3n+1, 3n+2)$ $O_xO_t = 2 \times \frac{1}{3}(1,1)$ $O_yO_t = 2 \times \frac{1}{3n+1}(1,n)$
$(3, 3n+1, 6n+1, 9n)$	$18n+3$	$\frac{4}{9n(3n+1)}$	5	1	z^3, y^3t, xt^2 x^nyz^2, $x^{2n}y^2z$, $x^{3n}y^3$, $x^{3n+1}t$, x^{6n+1}	$O_y = \frac{1}{3n+1}(1,n)$ $O_t = \frac{1}{9n}(3n+1, 6n+1)$ $O_xO_t = 2 \times \frac{1}{3}(1,1)$
$(3, 3n, 3n+1, 3n+1)$	$9n+3$	$\frac{4}{3n(3n+1)}$	7	1	t^3, zt^2, z^2t, z^3, xy^3, $x^{n+1}y^2$, $x^{2n+1}y$, x^{3n+1}	$O_y = \frac{1}{3n}(1,1)$ $O_xO_y = 3 \times \frac{1}{3}(1,1)$ $O_zO_t = 3 \times \frac{1}{3n+1}(1,n)$
$(3, 3n+1, 3n+2, 3n+2)$	$9n+6$	$\frac{4}{(3n+1)(3n+2)}$	5	1	t^3, zt^2, z^2t, z^3, xy^3, $x^{n+1}yt$, $x^{n+1}yz$, x^{3n+2}	$O_y = \frac{1}{3n+1}(1,1)$ $O_zO_t = 3\times\frac{1}{3n+2}(3, 3n+1)$
$(4, 2n+1, 4n+2, 6n+1)$	$12n+6$	$\frac{3}{(2n+1)(6n+1)}$	6	1	z^3, y^2z^2, y^4z, y^6, xt^2, $x^{n+1}yt$, $x^{2n+1}z$, $x^{2n+1}y^2$	$O_x = \frac{1}{4}(1,1)$ $O_t = \frac{1}{6n+1}(1,2)$ $O_xO_z = 1 \times \frac{1}{2}(1,1)$ $O_yO_z = 3 \times \frac{1}{2n+1}(1,n)$
$(2, 3, 4, 5)$	12	$\frac{2}{5}$	5	1[a] $\frac{7}{12}$[b]	z^3, yzt, y^4, xt^2, xy^2z, x^2z^2, x^2yt, x^3y^2, x^4z, x^6	$O_t = \frac{1}{5}(3,4)$ $O_xO_z = 3 \times \frac{1}{2}(1,1)$
$(2, 3, 4, 7)$	14	$\frac{1}{3}$	6	1	t^2, yzt, y^2z^2, xz^3, xy^4, x^2yt, x^2y^2z, x^3z^2, x^4y^2, x^5z, x^7	$O_y = \frac{1}{3}(1,1)$ $O_z = \frac{1}{4}(1,1)$ $O_xO_z = 3 \times \frac{1}{2}(1,1)$

a: if the defining equation of X contains yzt, b: if the defining equation of X contains no yzt.

Log del Pezzo surface with $I = 2$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(3,4,5,10)$	20	$\frac{2}{15}$	5	$\frac{3}{2}$	t^2, z^2t, z^4, y^5, xy^3z, x^2yt, x^2yz^2, x^4y^2, x^5z	$O_x = \frac{1}{3}(1,1)$ $O_yO_t = 1 \times \frac{1}{2}(1,1)$ $O_zO_t = 2 \times \frac{1}{5}(3,4)$
$(3,4,10,15)$	30	$\frac{1}{15}$	7	$\frac{3}{2}$	t^2, z^3, y^5z, xy^3t, x^2yz^2, x^2y^6, x^4y^2z, x^5t, x^6y^3, x^{10}	$O_y = \frac{1}{4}(1,1)$ $O_yO_z = 1 \times \frac{1}{2}(1,1)$ $O_zO_t = 1 \times \frac{1}{5}(3,4)$ $O_xO_t = 2 \times \frac{1}{3}(1,1)$
$(3,7,8,13)$	29	$\frac{29}{546}$	5	1	tz^2, y^3z, t^2x, x^2yz^2, x^3yt, x^5y^2, x^7z	$O_x = \frac{1}{3}(1,1)$ $O_y = \frac{1}{7}(1,2)$ $O_z = \frac{1}{8}(3,7)$ $O_t = \frac{1}{13}(7,8)$
$(3,10,11,19)$	41	$\frac{82}{3135}$	5	1	tz^2, y^3z, xt^2, x^3yz^2, x^4yt, x^7y^2, $x^{10}z$	$O_x = \frac{1}{3}(1,1)$ $O_y = \frac{1}{10}(1,3)$ $O_z = \frac{1}{11}(3,10)$ $O_t = \frac{1}{19}(10,11)$
$(5,13,19,22)$	57	$\frac{6}{715}$	3	$\frac{25}{12}$	t^2y, z^3, xy^4, x^5yz, x^7t	$O_x = \frac{1}{5}(3,4)$ $O_y = \frac{1}{13}(2,3)$ $O_t = \frac{1}{22}(5,19)$
$(5,13,19,35)$	70	$\frac{8}{1235}$	3	$\frac{25}{12}$	t^2, yz^3, xy^5, x^5y^2z, x^7t, x^{14}	$O_y = \frac{1}{13}(2,3)$ $O_z = \frac{1}{19}(5,16)$ $O_xO_t = 2 \times \frac{1}{5}(3,4)$
$(6,9,10,13)$	36	$\frac{4}{195}$	4	$\frac{25}{12}$	t^2z, y^4, xz^3, x^3y^2, x^6	$O_z = \frac{1}{10}(3,1)$ $O_t = \frac{1}{13}(2,3)$ $O_xO_y = 2 \times \frac{1}{3}(1,1)$ $O_xO_z = 2 \times \frac{1}{3}(1,1)$

Log del Pezzo surface with $I = 2$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(7,8,19,25)$	57	$\frac{57}{6650}$	3	$\frac{49}{24}$	ty^4, z^3, xt^2, x^2y^3z, x^7y	$O_x = \frac{1}{7}(5,4)$ $O_y = \frac{1}{8}(7,3)$ $O_t = \frac{1}{25}(8,19)$
$(7,8,19,32)$	64	$\frac{1}{133}$	4	$\frac{35}{16}$	t^2, ty^4, y^8, xz^3, x^3y^3z, x^8y	$O_x = \frac{1}{7}(5,4)$ $O_z = \frac{1}{19}(8,13)$ $O_yO_t = 2 \times \frac{1}{8}(7,3)$
$(9,12,13,16)$	48	$\frac{1}{117}$	3	$\frac{63}{24}$	t^3, y^4, xz^3, x^4y	$O_x = \frac{1}{9}(4,7)$ $O_z = \frac{1}{13}(4,1)$ $O_xO_y = 1 \times \frac{1}{3}(1,1)$ $O_yO_t = 1 \times \frac{1}{4}(1,1)$
$(9,12,19,19)$	57	$\frac{1}{171}$	5	3	t^3, t^2z, tz^2, z^3, xy^4, x^5y	$O_x = \frac{1}{9}(1,1)$ $O_y = \frac{1}{12}(1,1)$ $O_xO_y = 1 \times \frac{1}{3}(1,1)$ $O_zO_t = 3 \times \frac{1}{19}(3,4)$
$(9,19,24,31)$	81	$\frac{3}{1178}$	3	3	t^2y, y^3z, xz^3, x^9	$O_y = \frac{1}{19}(3,4)$ $O_z = \frac{1}{24}(19,7)$ $O_t = \frac{1}{31}(3,8)$ $O_xO_z = 1 \times \frac{1}{3}(1,1)$
$(10,19,35,43)$	105	$\frac{6}{4085}$	3	$\frac{57}{14}$	t^2y, z^3, xy^5, x^7z	$O_x = \frac{1}{10}(3,1)$ $O_y = \frac{1}{19}(16,5)$ $O_t = \frac{1}{43}(2,7)$ $O_xO_z = 1 \times \frac{1}{5}(4,3)$
$(11,21,28,47)$	105	$\frac{5}{3619}$	3	$\frac{77}{30}$	y^5, yz^3, xt^2, x^7z	$O_x = \frac{1}{11}(10,3)$ $O_y = \frac{1}{28}(11,19)$ $O_t = \frac{1}{47}(3,4)$ $O_yO_z = 1 \times \frac{1}{7}(4,5)$

Log del Pezzo surface with $I = 2$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(11,25,32,41)$	107	$\frac{107}{90200}$	3	$\frac{11}{3}$	$t^2y,\ y^3z,\ xz^3,\ x^6t$	$O_x = \frac{1}{11}(3,10)$ $O_y = \frac{1}{25}(11,16)$ $O_z = \frac{1}{32}(25,9)$ $O_t = \frac{1}{41}(11,32)$
$(11,25,34,43)$	111	$\frac{222}{201025}$	3	$\frac{33}{8}$	$t^2y,\ z^2t,\ xy^4,\ x^7z$	$O_x = \frac{1}{11}(3,10)$ $O_y = \frac{1}{25}(1,2)$ $O_z = \frac{1}{34}(11,25)$ $O_t = \frac{1}{43}(11,34)$
$(11,43,61,113)$	226	$\frac{8}{28853}$	2	$\frac{55}{12}$	$t^2,\ yz^3,\ xy^5,\ x^{15}z$	$O_x = \frac{1}{11}(10,3)$ $O_y = \frac{1}{43}(2,3)$ $O_z = \frac{1}{61}(11,52)$
$(13,18,45,61)$	135	$\frac{2}{2379}$	3	$\frac{91}{30}$	$z^3,\ y^5z,\ xt^2,\ x^9y$	$O_x = \frac{1}{13}(2,3)$ $O_y = \frac{1}{18}(13,7)$ $O_t = \frac{1}{61}(2,5)$ $O_yO_z = 1 \times \frac{1}{9}(13,61)$
$(13,20,29,47)$	107	$\frac{107}{88595}$	3	$\frac{65}{18}$	$y^3t,\ yz^3,\ xt^2,\ x^6z$	$O_x = \frac{1}{13}(7,8)$ $O_y = \frac{1}{20}(13,9)$ $O_z = \frac{1}{29}(13,18)$ $O_t = \frac{1}{47}(20,29)$
$(13,20,31,49)$	111	$\frac{111}{98735}$	3	$\frac{65}{16}$	$z^3t,\ y^4z,\ xt^2,\ x^7y$	$O_x = \frac{1}{13}(1,2)$ $O_y = \frac{1}{20}(13,9)$ $O_z = \frac{1}{31}(13,20)$ $O_t = \frac{1}{49}(20,31)$
$(13,31,71,113)$	226	$\frac{8}{28613}$	2	$\frac{91}{20}$	$t^2,\ y^5z,\ xz^3,\ x^{15}y$	$O_x = \frac{1}{13}(6,9)$ $O_y = \frac{1}{31}(13,20)$ $O_z = \frac{1}{71}(31,42)$
$(14,17,29,41)$	99	$\frac{198}{141491}$	3	$\frac{21}{4}$	$t^2y,\ z^2t,\ xy^5,\ x^5z$	$O_x = \frac{1}{14}(3,13)$ $O_y = \frac{1}{17}(12,7)$ $O_z = \frac{1}{20}(14,17)$ $O_t = \frac{1}{41}(14,29)$

Log del Pezzo surface with $I = 3$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(5,7,11,13)$	33	$\frac{27}{455}$	3	$\frac{49}{36}$	$t^2y,\ z^3,\ xy^4,\ x^3yz,\ x^4t$	$O_x = \frac{1}{5}(2,1)$ $O_y = \frac{1}{7}(2,3)$ $O_t = \frac{1}{13}(5,11)$
$(5,7,11,20)$	40	$\frac{18}{385}$	4	$\frac{25}{18}$	$t^2,\ yz^3,\ xy^5,\ x^3y^2z,\ x^4t,\ x^8$	$O_y = \frac{1}{7}(2,3)$ $O_z = \frac{1}{11}(1,4)$ $O_xO_t = 2 \times \frac{1}{5}(2,1)$
$(11,21,29,37)$	95	$\frac{285}{82621}$	3	$\frac{11}{4}$	$t^2y,\ z^2t,\ xy^4,\ x^6z$	$O_x = \frac{1}{11}(5,2)$ $O_y = \frac{1}{21}(1,2)$ $O_z = \frac{1}{29}(11,21)$ $O_t = \frac{1}{37}(11,29)$
$(11,37,53,98)$	196	$\frac{18}{21571}$	2	$\frac{55}{18}$	$t^2,\ yz^3,\ xy^5,\ x^{13}z$	$O_x = \frac{1}{11}(2,5)$ $O_y = \frac{1}{37}(2,3)$ $O_z = \frac{1}{53}(11,45)$
$(13,17,27,41)$	95	$\frac{95}{27183}$	3	$\frac{65}{24}$	$z^2t,\ y^4z,\ xt^2,\ x^6y$	$O_x = \frac{1}{13}(1,2)$ $O_y = \frac{1}{17}(13,7)$ $O_z = \frac{1}{27}(13,17)$ $O_t = \frac{1}{41}(17,27)$
$(13,27,61,98)$	196	$\frac{2}{2379}$	2	$\frac{91}{30}$	$t^2,\ y^5z,\ xz^3,\ x^{13}y$	$O_x = \frac{1}{13}(9,7)$ $O_y = \frac{1}{27}(13,17)$ $O_z = \frac{1}{61}(1,1)$
$(15,19,43,74)$	148	$\frac{18}{12255}$	2	$\frac{57}{14}$	$t^2,\ yz^3,\ xy^7,\ x^7z$	$O_x = \frac{1}{15}(2,7)$ $O_y = \frac{1}{19}(5,17)$ $O_z = \frac{1}{43}(15,31)$

Log del Pezzo surface with $I = 4$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(6, 6n+3, 6n+5, 6n+5)$	$18n+15$	$\frac{8}{(6n+3)(6n+5)}$	5	1	$t^3,\ zt^2,\ z^2t,\ z^3,\ xy^3,\ x^{2n+2}y$	$O_x = \frac{1}{6}(1,1)$ $O_y = \frac{1}{6n+3}(1,1)$ $O_xO_y = 1 \times \frac{1}{3}(1,1)$ $O_zO_t = 3\times \frac{1}{6n+5}(2,2n+1)$
$(6, 6n+5, 12n+8, 18n+9)$	$36n+24$	$\frac{8}{3(6n+3)(6n+5)}$	3	1	$z^3,\ y^3t,\ xt^2,\ x^{2n+1}y^2z,\ x^{6n+4}$	$O_y = \frac{1}{6n+5}(2,2n+1)$ $O_t = \frac{1}{18n+9}(6n+5, 12n+8)$ $O_xO_t = 1 \times \frac{1}{3}(1,1)$
$(6, 6n+5, 12n+8, 18n+15)$	$36n+30$	$\frac{4}{3(3n+2)(6n+5)}$	4	1	$t^2,\ y^3t,\ y^6,\ xz^3,\ x^{2n+2}y^2z,\ x^{6n+5}$	$O_z = \frac{1}{12n+8}(1,3)$ $O_xO_z = 1 \times \frac{1}{2}(1,1)$ $O_xO_t = 1 \times \frac{1}{3}(1,1)$ $O_yO_t = 2\times \frac{1}{6n+5}(2,2n+1)$
$(5,6,8,9)$	24	$\frac{8}{45}$	3	1	$t^2y,\ y^4,\ z^3,\ x^2yz,\ x^3t$	$O_x = \frac{1}{5}(1,3)$ $O_t = \frac{1}{9}(5,8)$ $O_yO_z = 1 \times \frac{1}{2}(1,1)$ $O_yO_t = 1 \times \frac{1}{3}(1,1)$
$(5,6,8,15)$	30	$\frac{2}{15}$	4	1	$t^2,\ y^5,\ yz^3,\ x^2y^2z,\ x^3t,\ x^6$	$O_z = \frac{1}{8}(5,7)$ $O_xO_t = 2 \times \frac{1}{5}(1,3)$ $O_yO_t = 1 \times \frac{1}{3}(1,1)$ $O_yO_z = 1 \times \frac{1}{2}(1,1)$
$(9,11,12,17)$	45	$\frac{20}{561}$	3	$\frac{77}{60}$	$t^2y,\ y^3z,\ xz^3,\ x^5$	$O_y = \frac{1}{11}(3,2)$ $O_z = \frac{1}{12}(11,5)$ $O_t = \frac{1}{17}(3,4)$ $O_xO_z = 1 \times \frac{1}{3}(1,1)$
$(10,13,25,31)$	75	$\frac{24}{2015}$	3	$\frac{91}{60}$	$t^2y,\ z^3,\ xy^5,\ x^5z$	$O_x = \frac{1}{10}(3,1)$ $O_y = \frac{1}{13}(12,5)$ $O_t = \frac{1}{31}(2,5)$ $O_xO_z = 1 \times \frac{1}{5}(3,1)$
$(11,17,20,27)$	71	$\frac{284}{25245}$	3	$\frac{11}{6}$	$t^2y,\ y^3z,\ xz^3,\ x^4t$	$O_x = \frac{1}{11}(2,3)$ $O_y = \frac{1}{17}(11,10)$ $O_z = \frac{1}{20}(17,7)$ $O_t = \frac{1}{27}(11,20)$

Log del Pezzo surface with $I = 4$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(11,17,24,31)$	79	$\frac{158}{17391}$	3	$\frac{33}{16}$	t^2y, tz^2, xy^4, x^5z	$O_x = \frac{1}{11}(2,3)$ $O_y = \frac{1}{17}(1,2)$ $O_z = \frac{1}{24}(11,17)$ $O_t = \frac{1}{31}(11,24)$
$(11,31,45,83)$	166	$\frac{32}{15345}$	2	$\frac{55}{24}$	t^2, yz^3, xy^5, $x^{11}z$	$O_x = \frac{1}{11}(3,2)$ $O_y = \frac{1}{31}(2,3)$ $O_z = \frac{1}{45}(11,38)$
$(13,14,19,29)$	71	$\frac{568}{50141}$	3	$\frac{65}{36}$	ty^3, yz^3, xt^2, x^4z	$O_x = \frac{1}{13}(1,3)$ $O_y = \frac{1}{14}(13,5)$ $O_z = \frac{1}{19}(13,10)$ $O_t = \frac{1}{29}(14,19)$
$(13,14,23,33)$	79	$\frac{632}{69069}$	3	$\frac{65}{32}$	tz^2, y^4z, xt^2, x^5y	$O_x = \frac{1}{13}(2,1)$ $O_y = \frac{1}{14}(13,1)$ $O_z = \frac{1}{23}(13,14)$ $O_t = \frac{1}{33}(14,23)$
$(13,23,51,83)$	166	$\frac{32}{15249}$	2	$\frac{91}{40}$	t^2, y^5z, xz^3, $x^{11}y$	$O_x = \frac{1}{11}(7,6)$ $O_y = \frac{1}{23}(13,14)$ $O_z = \frac{1}{51}(23,32)$

Log del Pezzo surface with $I = 5$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x, y, z, t)$	Singular Points
$(11, 13, 19, 25)$	63	$\frac{63}{2717}$	3	$\frac{13}{8}$	$t^2y,\ tz^2,\ xy^4,\ x^4z$	$O_x = \frac{1}{11}(2, 3)$ $O_y = \frac{1}{13}(1, 2)$ $O_z = \frac{1}{19}(11, 13)$ $O_t = \frac{1}{25}(11, 19)$
$(11, 25, 37, 68)$	136	$\frac{12}{2035}$	2	$\frac{11}{6}$	$t^2,\ yz^3,\ xy^5,\ x^9z$	$O_x = \frac{1}{11}(3, 2)$ $O_y = \frac{1}{25}(2, 3)$ $O_z = \frac{1}{37}(11, 31)$
$(13, 19, 41, 68)$	136	$\frac{60}{10127}$	2	$\frac{91}{50}$	$t^2,\ y^5z,\ xz^3,\ x^9y$	$O_x = \frac{1}{13}(2, 3)$ $O_y = \frac{1}{19}(13, 11)$ $O_z = \frac{1}{41}(19, 27)$

Log del Pezzo surface with $I = 6$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(8, 4n+5, 4n+7, 4n+9)$	$12n+23$	$\frac{9(12n+23)}{2(4n+5)(4n+7)(4n+9)}$	3	1	z^2t, yt^2, xy^3, $x^{n+2}z$	$O_x = \frac{1}{8}(4n+5, 4n+9)$ $O_y = \frac{1}{4n+5}(1,2)$ $O_z = \frac{1}{4n+7}(8, 4n+5)$ $O_t = \frac{1}{4n+9}(8, 4n+7)$
$(9, 3n+8, 3n+11, 6n+13)$	$12n+35$	$\frac{4(12n+35)}{(3n+8)(3n+11)(6n+13)}$	3	1	z^2t, y^3z, xt^2, $x^{n+3}y$	$O_x = \frac{1}{9}(3n+11, 6n+13)$ $O_y = \frac{1}{3n+8}(9, 6n+13)$ $O_z = \frac{1}{3n+11}(9, 3n+8)$ $O_t = \frac{1}{6n+13}(3n+8, 3n+11)$
$(5, 7, 8, 9)$	23	$\frac{23}{70}$	3	$\frac{5}{8}$	y^2t, x^3z, xt^2, yz^2	$O_x = \frac{1}{5}(1,2)$ $O_y = \frac{1}{7}(5,1)$ $O_z = \frac{1}{8}(5,1)$ $O_t = \frac{1}{9}(7,8)$
$(7, 10, 15, 19)$	45	$\frac{36}{665}$	3	$\frac{35}{54}$	z^3, y^3z, xt^2, x^5y	$O_x = \frac{1}{7}(1,5)$ $O_y = \frac{1}{10}(7,9)$ $O_t = \frac{1}{19}(2,3)$ $O_yO_z = 1 \times \frac{1}{5}(1,2)$
$(11, 19, 29, 53)$	106	$\frac{72}{6061}$	2	$\frac{55}{36}$	t^2, yz^3, xy^5, x^7z	$O_x = \frac{1}{11}(8,9)$ $O_y = \frac{1}{19}(2,3)$ $O_z = \frac{1}{29}(11,24)$
$(13, 15, 31, 53)$	106	$\frac{24}{2015}$	2	$\frac{45}{28}$	t^2, y^5z, xz^3, x^7y	$O_x = \frac{1}{13}(5,1)$ $O_y = \frac{1}{15}(13,8)$ $O_z = \frac{1}{31}(15,22)$

Log del Pezzo surface with $I = 7$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(11,13,21,38)$	76	$\frac{14}{429}$	2	$\frac{13}{10}$	$t^2,\ yz^3,\ xy^5,\ x^5z$	$O_x = \frac{1}{11}(2,5)$ $O_y = \frac{1}{13}(2,3)$ $O_z = \frac{1}{21}(11,17)$

Log del Pezzo surface with $I = 8$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(7,11,13,23)$	46	$\frac{128}{1001}$	2	$\frac{35}{48}$	$t^2,\ y^3z,\ xz^3,\ x^5y$	$O_x = \frac{1}{7}(3,1)$ $O_y = \frac{1}{11}(7,1)$ $O_z = \frac{1}{13}(11,10)$
$(7,18,27,37)$	81	$\frac{32}{777}$	3	$\frac{35}{72}$	$y^3z,\ z^3,\ xt^2,\ x^9y$	$O_x = \frac{1}{7}(3,1)$ $O_y = \frac{1}{18}(7,1)$ $O_t = \frac{1}{37}(2,3)$ $O_yO_z = 1 \times \frac{1}{9}(7,1)$

Log del Pezzo surface with $I = 9$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(7,15,19,32)$	64	$\frac{54}{665}$	2	$\frac{35}{54}$	$t^2,\ y^3z,\ xz^3,\ x^7y$	$O_x = \frac{1}{7}(5,4)$ $O_y = \frac{1}{15}(7,2)$ $O_z = \frac{1}{19}(15,13)$

Log del Pezzo surface with $I = 10$

Weight	Degree	K_X^2	Pic	lct(X)	Monomials in $f(x,y,z,t)$	Singular Points
$(7,19,25,41)$	82	$\frac{8}{133}$	2	$\frac{7}{12}$	$t^2,\ y^3z,\ xz^3,\ x^9y$	$O_x = \frac{1}{7}(2,3)$ $O_y = \frac{1}{19}(7,3)$ $O_z = \frac{1}{25}(19,16)$
$(7,26,39,55)$	117	$\frac{1170}{39039}$	3	$\frac{7}{18}$	$y^3z,\ z^3,\ xt^2,\ x^{13}y$	$O_x = \frac{1}{7}(2,3)$ $O_y = \frac{1}{26}(7,3)$ $O_t = \frac{1}{55}(26,39)$ $O_yO_z = 1 \times \frac{1}{13}(7,3)$

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