Partitions of projective planes and construction of polyhedra.

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Abstract

In this paper we prove some properties of finite projective planes. In particular, we find the partitions of Desarguesian projective planes of order q^2 by planes of order q. These properties help us to construct polyhedra whose links are incidence graphs of finite projective planes of order q. The polyhedra of this type are interesting because of their universal coverings, which are buildings.

Introduction

We will call a *polyhedron* a two-dimensional complex, which is obtained from several oriented p-gons by identification of corresponding sides. Let's consider a point of the polyhedron. Let's take a sphere of a small radius at this point. The intersection of the sphere with the polyhedron is a graph, which is called the *link* at this point.

We consider the polyhedra with the following properties: all links of all vertices are incidence graphs of finite projective planes of order q. The theory of finite projective planes is well developed (see, for example [4]), but we are interested in their properties, which can be useful for the construction of polyhedra.

Let C_p be a polyhedron whose faces are p-gons and whose links are incidence graphs of finite projective planes of order q. We equip every face of C_p with the hyperbolic metric, such that all sides of the polygons are geodesics and all angles are $\pi/3$, sides which are identified have the same length. Then the universal covering of such polyhedron is a hyperbolic building, see [1],[6].

In the case $p = 3$, i.e. C_p is a simplex, we can also give an euclidean metric to every face. In this metric sides which are identified also have the same length and all sides of the triangles are geodesics. The universal coverings of these polyhedra with euclidean metric are euclidean buildings, see [2], [3].

The main results of our paper are the following theorems. The first one is about partitions of Desarguesian projective planes of order q^2 by planes of order q.

Theorem 1. Let $\mathcal{P}^2(\mathbb{F}_{q^2})$ be a finite projective plane of order q^2 over the finite field $\mathbb{F}_{q^2}$. Then $\mathcal{P}^2(\mathbb{F}_{q^2})$ contains q^2-q+1 disjoint subplanes of order q $\mathcal{P}_1,\dots,\mathcal{P}_n$, where $n=q^2-q+1$. If P is the set of points of $\mathcal{P}^2(\mathbb{F}_{q^2})$ and P_i is the set of points of $\mathcal{P}_i$, then $P=\cup P_i, i=1,\dots,n$; $P_i\cap P_j=\emptyset$ for $i\neq j$.

Also, we prove the existence of a polyhedron with a given number of vertices, whose links are incidence graphs of finite projective planes of order q in every vertex and with a given number of sides of every face.

Theorem 2. For every n,k,q there exists a polyhedron with n vertices, k-gonal faces and incidence graphs of projective planes of order q as links in all vertices.

Partitions of Desarguesian projective planes of order q^2 by planes of order q.

We want to construct an automorphism χ of $\mathcal{P}^2(\mathbb{F}_{q^2})$ which generates a cyclic subgroup $\langle\chi\rangle$ of the automorphism group of $\mathcal{P}^2(\mathbb{F}_{q^2})$ such that $\langle\chi\rangle$ acts simply transitively on $\mathcal{P}^2(\mathbb{F}_{q^2})$. We use the description of $\mathcal{P}^2(\mathbb{F}_{q^2})$ by means of homogeneous coordinates and identify $(\mathbb{F}_{q^2})^3$ with $\mathbb{F}_{q^6}$. The multiplicative group of $\mathbb{F}_{q^6}$ is cyclic with generator $x\in\mathbb{F}_{q^6}\setminus\{0\}$. Let m_x denote the multiplication map $\mathbb{F}_{q^6}\setminus\{0\}\in\mathbb{F}_{q^6}\setminus\{0\}: y\mapsto xy$. This map is $\mathbb{F}_{q^2}$-linear and thus induces an automorphism of $\mathcal{P}^2(\mathbb{F}_{q^2})$ which we denote by χ. The natural group homomorphism $\phi:\langle m_x\rangle\to\langle\chi\rangle$ is surjective. The cyclic group $\langle\chi\rangle$ is called a *Singer cycle.* It acts simply transitively on $\mathcal{P}^2(\mathbb{F}_{q^2})$.

Proof of Theorem 1. We want to construct a partition of $\mathcal{P}^2(\mathbb{F}_{q^2})$ into q^2-q+1 subplanes of order q by means of the Singer cycle. The field $\mathbb{F}_q$ is embedded in $\mathbb{F}_{q^2}$ in exactly one way, and so we have a "standard subplane" of $\mathcal{P}^2(\mathbb{F}_{q^2})$ of order q, namely $\mathcal{P}^2(\mathbb{F}_q)$. Again we identify $\mathbb{F}_{q^2}^3$ with $\mathbb{F}_{q^6}$ and $\mathbb{F}_q^3$ with $\mathbb{F}_{q^3}\subseteq\mathbb{F}_{q^6}$.

We want to show that the orbits of the map χ^{q^2-q+1} are projective subplanes of order q. For this purpose we consider first the orbit of the point p of $\mathcal{P}^2(\mathbb{F}_{q^2})$ corresponding to the one-dimensional subspace of $\mathbb{F}_{q^2}^3$ generated by $\gamma(1)=(1,0,0)\in\mathbb{F}_{q^2}^3$, where γ is an $\mathbb{F}_{q^2}$-linear bijection $\gamma:\mathbb{F}_{q^6}\to\mathbb{F}_{q^2}^3$ which maps $\mathbb{F}_{q^3}\subseteq\mathbb{F}_{q^6}$ onto $\mathbb{F}_q^3$. Under the map γ^{-1}, this space corresponds simply to the subspace generated by $1\in\mathbb{F}_{q^6}$ in the $\mathbb{F}_{q^2}$-vector space $\mathbb{F}_{q^6}$, which is simply $\mathbb{F}_{q^2}1=\mathbb{F}_{q^2}\subseteq\mathbb{F}_{q^6}$. We first show that $\langle\chi^{q^2-q+1}\rangle p\subseteq\mathcal{P}^2(\mathbb{F}_q)$. For this purpose we have to prove that for any $z\in\mathbb{Z}$, the element $\gamma^{-1}(m_x^{z(q^2-q+1)}1)=\gamma^{-1}(x^{z(q^2-q+1)})$ of the $\mathbb{F}_{q^2}$-vector space $\mathbb{F}_{q^2}^3$ has coordinates which, up to a factor from $\mathbb{F}_{q^2}\setminus\{0\}$, lie in $\mathbb{F}_q$. By means of our identifications using the map γ this simply means that $x^{z(q^2-q+1)}/\lambda$ lies in $\mathbb{F}_{q^3}\setminus\{0\}$ for some suitable element λ of $\mathbb{F}_{q^2}\setminus\{0\}$. This is again equivalent to say that $x^{z(q^2-q+1)} \in (\mathbb{F}_{q^2}\setminus\{0\})(\mathbb{F}_{q^3}\setminus\{0\}) \subseteq \mathbb{F}_{q^6}\setminus\{0\}$. So, it suffices to understand why this is true.

It is easy to see that $\mathbb{F}_q = \mathbb{F}_{q^2} \cap \mathbb{F}_{q^3}$. Hence, we have

$$|(\mathbb{F}_{q^2}\setminus\{0\})(\mathbb{F}_{q^3}\setminus\{0\})| = |\mathbb{F}_{q^2}\setminus\{0\}||\mathbb{F}_{q^3}\setminus\{0\}| / |\mathbb{F}_q\setminus\{0\}| = (q^2-1)(q^3-1)/(q-1)$$

So, $(\mathbb{F}_{q^2}\setminus\{0\})(\mathbb{F}_{q^3}\setminus\{0\})$ is the subgroup of the cyclic group $\mathbb{F}_{q^6}\setminus\{0\}$ of order $(q^2-1)(q^3-1)/(q-1)$, whose elements y are characterized by the property that $y^{(q^2-1)(q^3-1)/(q-1)} = 1$. But for $x^{z(q^2-q+1)}$ we have $(x^{z(q^2-q+1)})^{(q^2-1)(q^3-1)/(q-1)} = 1$ because of $(q^2-q+1)(q^2-1)(q^3-1)/(q-1) = q^6-1$. So, we have proved that $x^{z(q^2-q+1)} \in (\mathbb{F}_{q^2}\setminus\{0\})(\mathbb{F}_{q^3}\setminus\{0\})$ and hence that $\langle\chi^{q^2-q+1}\rangle p \subseteq \mathcal{P}^2(\mathbb{F}_q)$.

Since $\langle\chi\rangle$ acts simply transitively on $\mathcal{P}^2(\mathbb{F}_{q^2})$, the stabilizer of the point p under the action of $\langle\chi^{q^2-q+1}\rangle$ is trivial and hence the length of the orbit of p under this group action is equal to $|\langle\chi^{q^2-q+1}\rangle| = (q^4+q^2+1)/(q^2-q+1) = q^2+q+1$. Since $\mathcal{P}^2(\mathbb{F}_q)$ has q^2+q+1 elements, too, we conclude that $\langle\chi^{q^2-q+1}\rangle p = \mathcal{P}^2(\mathbb{F}_q)$. So we have shown that one of the orbits, namely $\langle\chi^{q^2-q+1}\rangle p$, is indeed equal to a projective subplane of order q, namely $\mathcal{P}^2(\mathbb{F}_q)$.

We want to show now that all other orbits of the map χ^{q^2-q+1} are projective subplanes of order q. Let's consider the orbit $\langle\chi^{q^2-q+1}\rangle r$ for some arbitrary point $r \in \mathcal{P}^2(\mathbb{F}_{q^2})$. Since $\langle\chi\rangle$ acts transitively on $\mathcal{P}^2(\mathbb{F}_{q^2})$ there is some $n \in \mathbb{N}$ such that $r = \chi^n p$, where $p \in \mathcal{P}^2(\mathbb{F}_{q^2})$ is the same as above. But $\langle\chi^{q^2-q+1}\rangle(\chi^n p) = \chi^n\langle\chi^{q^2-q+1}\rangle p$, which shows that the orbit of r under the action of $\langle\chi^{q^2-q+1}\rangle p$ is the image of the orbit $\langle\chi^{q^2-q+1}\rangle p = \mathcal{P}^2(\mathbb{F}_q)$ under the automorphism χ^n of $\mathcal{P}^2(\mathbb{F}_{q^2})$. Of course, an automorphism of $\mathcal{P}^2(\mathbb{F}_{q^2})$ maps subplanes of order q onto subplanes of order q. So, we have shown that $\langle\chi^{q^2-q+1}\rangle r$ is a subplane of $\mathcal{P}^2(\mathbb{F}_{q^2})$ of order q for arbitrary $r \in \mathcal{P}^2(\mathbb{F}_{q^2})$, and we are finished.

Example 1. We give the partition of the projective plane of order four by projective planes of order two. The incidence matrix of the plane of order four is of size 5x21. In this matrix we can see three disjoint submatrices 3x7, which are incidence matrices of projective planes of order two. We denote points of the subplane $\mathcal{P}_i, i = 1, 2, 3$ by numbers from $7i-6$ to $7i$. Every row in the matrix 5x21 represents a line in the projective plane of order four and every row in the matrices 3x7 represents a line in the corresponding plane of order two.

5	6	7	12	18
1	2	5	8	21
3	4	5	14	16
1	3	6	10	19
2	4	6	9	15
1	4	7	11	17
2	3	7	13	20

12	13	14	1	15
8	9	12	3	17
10	11	12	2	16
8	10	13	4	18
9	11	13	5	19
8	11	14	6	20
9	10	14	7	21

19	20	21	4	12
15	16	19	7	8
17	18	19	2	14
15	17	20	5	10
16	18	20	1	9
15	18	21	3	11
16	17	21	6	13

Hamiltonian cycles in the incidence graphs of finite projective planes

Definition. A closed circuit in a connected graph, which crosses every vertex of the graph exactly once, is called a *Hamiltonian cycle*.

Corollary. The incidence graph of a finite projective plane $\mathcal{P}^2(\mathbb{F}_q)$ (of order q) which admits a Singer cycle has a Hamiltonian cycle.

Proof. Let $\langle\chi\rangle$ again denote a Singer cycle of $\mathcal{P}^2(\mathbb{F}_q)$. The candidate for the Hamiltonian cycle in the incidence graph is

$$p, L, \chi(p), \chi_{\mathcal{L}}(L), \chi^2(p), \chi_{\mathcal{L}}^2(L), \dots, \chi^{q^2+q}(p), \chi_{\mathcal{L}}^{q^2+q}(L), \chi^{q^2+q+1}(p) = p,$$

where L is the unique line joining p and $\chi(p)$, and $\chi_{\mathcal{L}}$ is the map induced by χ on the set $\mathcal{L}$ of lines. It is clear by construction that each element in this sequence is incident with the elements before and after this element. Moreover, because of the simply transitive action of $\langle\chi\rangle$ on $\mathcal{P}^2(\mathbb{F}_q)$ each point occurs precisely once in this sequence. This can be seen as follows:

The natural map $\langle\chi\rangle \to \langle\chi_{\mathcal{L}}\rangle$, which assigns to χ the map induced on the set of lines is surjective by construction. Moreover, only the identity in $\langle\chi\rangle$ induces the identity in $\langle\chi_{\mathcal{L}}\rangle$. This is clear since the induced action of an automorphism of a projective plane on the set of lines determines the automorphism uniquely. hence, the natural map $\langle\chi\rangle \to \langle\chi_{\mathcal{L}}\rangle$ is an isomorphism, and we have $|\langle\chi_{\mathcal{L}}\rangle| = q^2 + q + 1$. So, in order to show that the action of $\langle\chi_{\mathcal{L}}\rangle$ is simply transitive on the set of lines $\mathcal{L}$, it suffices to show that the stabilizer $\langle\chi_{\mathcal{L}}\rangle_L$ is trivial for any arbitrary line L. So, let $\chi_{\mathcal{L}}^n \in \langle\chi_{\mathcal{L}}\rangle_L$ for some $L \in \mathcal{L}, n \in \mathbb{N}$. Then, χ^n leaves L invariant. Since $\langle\chi\rangle$ acts simply transitively on $\mathcal{P}^2(\mathbb{F}_q)$, the stabilizers $\langle\chi^n\rangle$ on L have the same length $\mid \langle\chi^n\rangle \mid$, which implies that $\mid \langle\chi^n\rangle \mid$ is a divisor of $\mid L \mid = q+1$. But since $\mid \langle\chi_{\mathcal{L}}\rangle \mid = q^2+q+1$, the order of $\langle\chi^n\rangle$ is also a divisor of q^2+q+1, which implies that $\mid \langle\chi^n\rangle \mid = 1$, since q^2+q+1 and $q+1$ are relatively prime. We conclude that $\chi^n = 1 \in \langle\chi\rangle$. Hence, we also have $\chi_{\mathcal{L}}^n = 1 \in \langle\chi_{\mathcal{L}}\rangle$, which shows that $\langle\chi_{\mathcal{L}}\rangle_L$ is trivial. Hence, $\langle\chi_{\mathcal{L}}\rangle$ acts simply transitively on $\mathcal{L}$, and so each line occurs in the above sequence precisely once. This completes the proof.

Polyhedral presentation.

Definition. Suppose we have n disjoint finite projective planes $\mathcal{P}_1, \ldots, \mathcal{P}_n$ of order q. Let P_i and L_i be the sets of points and lines respectively in $\mathcal{P}_i$; let $P = \cup P_i, L = \cup L_i$, $P_i \cap P_j = \emptyset$ for $i \neq j$ and let λ be a bijection $\lambda : P \to L$.

A set $\mathcal{K}$ of k-tuples $(x_1, x_2, \ldots, x_k)$ will be called a *polyhedral presentation* over P compatible with λ if

(1) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for some $x_3, \ldots, x_k$ if and only if x_2 and $\lambda(x)$ are incident;

(2) $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ implies that $(x_2, x_3, \ldots, x_k, x_1) \in \mathcal{K}$;

(3) given $x_1, x_2 \in P$, then $(x_1, x_2, x_3, \ldots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

We will call λ a *basic bijection.*

In the incidence matrix all points of the projective plane are numbers and rows are lines. To define the basic bijection, we write the number of the point before the corresponding line.

Polyhedral presentations for $n = 1$ and $k = 3$, $q = 2, 3$ were listed in [5]. They were called triangle presentations in that paper, because the case $k = 3$ was considered. Let's take a triangle presentation $\mathcal{T}_{A_1}$ from [5], p.160:

$$(0,1,3), (1,2,4), (2,3,5), (3,4,6), (4,5,0), (5,6,1), (6,0,2).$$

We can express it as an incidence matrix of finite projective plane of order two with a corresponding basic bijection λ_1:

0: 1 4 2
1: 3 4 5
2: 4 3 6
3: 0 4 5
4: 1 5 6
5: 0 2 6
6: 0 1 3

Example 2. Let's take the partition of the projective plane of order four by projective planes of order two from the Example 1. We denote points of the subplane $\mathcal{P}_i, i = 1, 2, 3$ by numbers from $7i - 6$ to $7i$. To each of three disjoint submatrices 3x7, which are incidence matrices of projective planes of order two, we apply a basic bijection induced by λ_1.

So, we obtain a polyhedral presentation $\mathcal{T}_{35}$ for a projective plane of order four with a basic bijection λ_{35} which is induced by polyhedral presentations of projective planes of order two.

4:	5 6 7	12 18
7:	1 2 5	8 21
2:	3 4 5	14 16
5:	1 3 6	10 19
1:	2 4 6	9 15
3:	1 4 7	11 17
6:	2 3 7	13 20

9:	12 13 14	1 15
11:	8 9 12	3 17
14:	10 11 12	2 16
12:	8 10 13	4 18
10:	9 11 13	5 19
13:	8 11 14	6 20
8:	9 10 14	7 21

18:	19 20 21	4 12
21:	15 16 19	7 8
16:	17 18 19	2 14
19:	15 17 20	5 10
15:	16 18 20	1 9
17:	15 18 21	3 11
20:	16 17 21	6 13

$\mathcal{T}_{35}$ ={(1,9,15), (1,15,9),(2,14,16),(2,16,14),(3,11,17), (3,17,11),(4,12,18)
(4,18,12),(5,10,19),(5,19,10,(6,13,20),(6,20,13),(7,8,21),(7,21,8)
(1,2,3),(1,4,5),(1,6,7),(3,4,6),(3,7,5),(2,5,6),(2,4,7)
(8,9,12),(8,10,13),(8,14,11),(9,14,10),(9,13,11),(12,13,14),(10,11,12)
(15,16,17),(15,18,19),(15,20,21),(17,18,20),(17,21,19),(16,19,20),(16,18,21)}

Suppose we have our partition of the $\mathcal{P}^2(\mathbb{F}_{q^2})$ into subplanes of order q, $\mathcal{P}_1, \dots, \mathcal{P}_n$, where $n = q^2 - q + 1$. Let P_i and L_i be the sets of points and lines respectively in $\mathcal{P}_i$; let $P = \cup P_i, L = \cup L_i$ be the sets of points and lines respectively in $\mathcal{P}^2(\mathbb{F}_{q^2})$. We will form the bijection $\eta : P \to L$ in the following way: for each line l_i^s of $\mathcal{P}_i$ we take the point x_i^s of $\mathcal{P}_{i+1}$ which lies on the same line in $\mathcal{P}^2(\mathbb{F}_{q^2})$ with the points of l_i^s (of course, we take $i+1$ modulo $q^2 - q + 1$, i.e. we take $\mathcal{P}_1$ for $\mathcal{P}_n$). So, we define η as $\eta(x_i^s) = l_i^s$ where $i = 1, \dots, q^2 - q + 1, s = 1, \dots, q^2 + q + 1$.

Conjecture. There is a polyhedral presentation over P compatible with the bijection η , which consists of $(q^2 - q + 1)$-tuples, and their total number is $(q+1)(q^2 + q + 1)$.

We checked this conjecture for small n, for example, for $q = 2$ we have a polyhedral presentation with $k = 3$:

Example 3. As before, we denote points of the subplane $\mathcal{P}_i, i = 1, 2, 3$ by numbers from $7i - 6$ to $7i$. Let's consider the partition of the plane of order by the planes of order two from Example 1. To construct the bijection η in the case $q = 2$ we take points of $\mathcal{P}_{i+1}, i = 1, 2, 3$ (which are numbers from $7i + 1$ to $7i + 7$) for the lines of $\mathcal{P}_i, i = 1, 2, 3$ (which are rows of length three in the incidence matrices). For example, for the line 5 6 7 we take the point 12.

12: 5 6 7
8: 1 2 5
14: 3 4 5
10: 1 3 6
9: 2 4 6
11: 1 4 7
13: 2 3 7

15: 12 13 14
17: 8 9 12
16: 10 11 12
18: 8 10 13
19: 9 11 13
20: 8 11 14
21: 9 10 14

4: 19 20 21
7: 15 16 19
2: 17 18 19
5: 15 17 20
1: 16 18 20
3: 15 18 21
6: 16 17 21

The polyhedral presentation is:
(12,5,15),(14,5,20),(8,5,17),(15,14,3),(15,13,7),(12,7,16),(12,6,17)
(1,20,8),(11,1,16),(16,10,6),(1,18,10),(21,9,6),(10,3,21),(8,2,18)
(13,3,18),(11,4,20),(14,4,21),(17,9,2),(11,7,19),(9,4,19),(2,19,13)

Construction of polyhedra.

We can associate a polyhedron K on n vertices with each polyhedral presentation $\mathcal{K}$ as follows: for every cyclic k-tuple $(x_1, x_2, x_3, \ldots, x_k)$ from the definition we take an oriented k-gon on the boundary of which the word $x_1 x_2 x_3 \ldots x_k$ is written. To obtain the polyhedron we identify the corresponded sides of our polygons respecting orientation. We will say that the polyhedron K *corresponds* to the polyhedral presentation $\mathcal{K}$.

Lemma. A polyhedron K which corresponds to a polyhedral presentation $\mathcal{K}$ has incidence graphs of projective planes of order q as the links.

Proof. Let's look at the incidence matrix of the graph, which is a link in a vertex v of the polyhedron. It follows from the definition of the polyhedral presentation, that a graph with such incidence matrix has to be an incidence graph of a finite projective plane of order q. So, all the links of all vertices are incidence graphs of projective planes of order q.

Proof of Theorem 2. Suppose we have n disjoint finite projective planes of order q $\mathcal{P}_1, \dots, \mathcal{P}_n$. Let P_i and L_i be the sets of points and lines respectively in $\mathcal{P}_i$; let $P = \cup P_i, L = \cup L_i$ and λ be a bijection $\lambda : P \to L$.

The Lemma shows us that to construct the polyhedron it is sufficient to construct a polyhedral presentation $\mathcal{K}$ of k-tuples over P compatible with λ.

To begin, we construct a polyhedral presentation with $k = 3$ with the help of the triangle presentation $\mathcal{T}_0$ given in [5], p.157. Let $\mathcal{T}_0$ be this triangle presentation over P_0 compatible with $\lambda_0 : P_0 \to L_0$, where P_0 and L_0 are the sets of points and lines respectively of a projective plane $\mathcal{P}_0$. Let's consider two projective planes $\mathcal{P}_1$ and $\mathcal{P}_2$, which are isomorphic to $\mathcal{P}_0$. Let $\phi_i : \mathcal{P}_0 \to \mathcal{P}_i, i = 1, 2$ be the corresponding isomorphisms. The 3-tuples in $\mathcal{T}_0$ are of two types: (x, x, x), i.e. all elements are equal, and (x, y, z), where at least two elements are different. To construct our polyhedral presentation $\mathcal{T}'$ we replace in $\mathcal{T}_0$ each 3-tuple of the first type by one 3-tuple $(x, \phi_1(x), \phi_2(x))$ and each 3-tuple of the second type by three 3-tuples $(x, \phi_1(y), \phi_2(z))$, $(\phi_1(x), \phi_2(y), z)$, $(\phi_2(x), y, \phi_1(z))$.

Then, with the help of the previously constructed polyhedral presentation $\mathcal{T}'$ we will construct our polyhedral presentation $\mathcal{K}$. Let's take $k-3$ projective planes $\mathcal{P}_i, i = 3, ..., k-1$ of order q isomorphic to $\mathcal{P}_0$ (and, of course, to $\mathcal{P}_1$ and $\mathcal{P}_2$) such that $\phi_{ij} : \mathcal{P}_j \to \mathcal{P}_i, i = 3, ..., k-1, j = 0, 1, 2$ are the corresponding isomorphisms. The polyhedral presentation $\mathcal{K}$ will be defined as following. There are two cases depending on k, when k is odd and when k is even. If k is odd, i.e. $k = 2l + 1$, we replace in $\mathcal{T}'$ each 3-tuple (x_0, x_1, x_2) , $x_i \in P_i, i = 0, 1, 2$ by

$$(x_0, x_1, x_2, \phi_{13}(x_1), \phi_{24}(x_2), \dots , \phi_{1,2l-1}(x_1), \phi_{2,2l}(x_2))(1)$$

If k is even, i.e. $k = 2l + 2$, we replace in $\mathcal{T}'$ each 3-tuple (x_0, x_1, x_2), $x_i \in P_i, i = 0, 1, 2$ by

$$(x_0, x_1, x_2, \phi_{13}(x_1), \phi_{04}(x_0), \phi_{25}(x_2), \dots , \phi_{1,2l}(x_1), \phi_{2.2l+1}(x_2))(2)$$

It is easy to check, that (1) and (2) satisfy all axioms from the definition of the polyhedral presentation. The proof is finished.

Example 4. We now construct an example of a polyhedral presentation with $k = 5$, $n = 5$, $q = 2$ from the presentation of Example 3. Again we denote the set of points of the plane $\mathcal{P}_i, i = 1, 2, 3, 4, 5$ of order two by numbers from $7i - 6$ to $7i$. The isomorphisms $\phi_{ij} : \mathcal{P}_j \to \mathcal{P}_i, i = 4, 5, j = 1, 2$ can be defined as follows: $\phi_{i,j}(x) = x + 7(i - j)$, where x is a point of $\mathcal{P}_j$ denoted by a number from $7j - 6$ to $7j$.

(5,15,12,22,33),(5,20,14,27,35),(5,17,8,24,29),
(3,15,14,22,35),(7,15,13,22,34),(7,16,12,23,33),
(6,17,12,24,33),(1,20,8,27,29),(1,16,11,23,32),
(6,16,10,23,31),(1,18,10,25,31),(6,21,9,28,30),
(3,21,10,28,31),(2,18,8,25,29),(3,18,13,25,34),
(4,20,11,27,32),(4,21,14,28,35),(2,17,9,24,28),
(7,19,11,26,32),(4,19,9,26,30),(2,19,13,26,34).

Now we will describe a polyhedral presentation associated with $\mathcal{P}^2(\mathbb{F}_q)$ which is different from the polyhedral presentation from the proof of Theorem 2.

Since the Singer cycle acts simply transitively on the set of vertices of $\mathcal{P}^2(\mathbb{F}_q)$, we can present $\mathcal{P}^2(\mathbb{F}_q)$ in the following form: all points are denoted by the numbers from 0 to q^2+q+1, lines are some sets of $q+1$ numbers and there exists some $L_0 = \{i_0, ..., i_q\}$ such that $L_i = \{i+i_0, ..., i+i_q\}$. (This presentation of the finite projective plane was also used in [5] to find all triangle presentations.)

We can write the incidence matrix of a plane of order q as $q+1$ columns, each one is a cyclic permutation of $(0, ..., q^2+q)$.

Let $L_0 = \{i_0, ..., i_q\}$, so a column with number k of the incidence matrix is a cyclic permutation $(i_k, i_k+1, ..., q^2+q, 0, 1, ..., i_k-1)$. If q^2+q+1 is prime, then as a basic bijection we can take some cyclic permutation $(j, j+1, ..., q^2+q, 0, 1, ..., j-1)$ of $(0, ..., q^2+q)$, where j is different from $i_0, ..., i_q$. Then we will have a polyhedral presentation to construct a polyhedron with one vertex and (q^2+q+1)-gonal faces. The number of these faces is $q+1$.

The following example illustrates this construction for $q=3$.

Example 5. For $q=3$ we can take $L_0 = \{0, 1, 4, 6\}$ (see [5]). Also, we will take the permutation $(2, 3, ..., 12, 0, 1)$ as the basic bijection:

2:	0	1	4	6
3:	1	2	5	7
4:	2	3	6	8
5:	3	4	7	9
6:	4	5	6	10
7:	5	6	9	11
8:	6	7	10	12
9:	7	8	11	0
10:	8	9	12	1
11:	9	10	0	2
12:	10	11	1	3
0:	11	12	2	4
1:	12	0	3	5

The polyhedral presentation is:
(2,0,11,9,7,5,3,1,12,10,8,6,4)
(2,1,0,12,11,10,9,8,7,6,5,4,3)
(2,4,6,8,10,12,1,3,5,7,9,11,0)
(2,6,10,1,5,9,0,4,8,12,3,7,11)

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