Hecke orbits

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This is a report on work, mostly *joint with Ching-Li Chai.*

1) Moduli spaces and Hecke orbits. We write $\mathcal{A}_g \to \mathrm{Spec}(\mathbb{Z})$ for the moduli space of polarized abelian varieties. However from §3 we will write $\mathcal{A}_g$ instead of $\mathcal{A}_g \otimes \mathbb{F}_p$, the moduli space of polarized abelian varieties in characteristic p.

Let $[(A,\mu)] = x \in \mathcal{A}_g$. We say that $[(B,\nu)] = y$ is in the Hecke orbit of x, notation $y \in \mathcal{H}(x)$, if there exists a diagram

$$(A,\mu)_\Omega \xleftarrow{\varphi} (C,\zeta) \xrightarrow{\psi} (B,\nu)_\Omega;$$

here A and B are in the same characteristic, Ω is some algebraically closed field, and $\varphi : C \to A$ and $\psi : C \to B$ are isogenies such that

$$\varphi^*(\mu) = \zeta = \psi^*(\nu).$$

If moreover the degrees of φ and ψ are both some power of a prime number ℓ, different from a given p, we write $y \in \mathcal{H}_\ell(x)$.

If A and B are both in characteristic p and φ and ψ are both of α_p-coverings, then we write $y \in \mathcal{H}_\alpha(x)$.

If A and B are both in characteristic p and φ and ψ have degrees not divisible by p we write $y \in \mathcal{H}^{(p)}(x)$.

Question. $\boxed{\text{Given } (A,\mu)\text{; what is the Zariski closure of the Hecke orbit } \mathcal{H}(x)?}$

2) Over $\mathbb{C}$. In case (A,μ) is defined over $\mathbb{C}$, it is easy to see that $\mathcal{H}(x)$ is classically everywhere dense in $\mathcal{A}_g(\mathbb{C})$; hence

$$\overline{\mathcal{H}(x)} = \mathcal{A}_g \otimes \mathbb{C}.$$

3) A theorem by Ching-Li Chai in 1995. From now on we work in characteristic p. We say an abelian variety A of dimension g is *ordinary* if $A(k)[p] \cong (\mathbb{Z}/p)^g$. We say an elliptic curve is *supersingular* if it is not ordinary. The following facts are not difficult to prove / well known.

(3a) For an *ordinary* elliptic curve E its moduli point x has a Hecke orbit which is everywhere dense in $\mathcal{A}_1$. In this case even $\mathcal{H}_\ell(x)$ is everywhere dense in $\mathcal{A}_1$ for every prime number $\ell \neq p$.

(3b) For a *supersingular* elliptic curve its moduli point $x \in \mathcal{A}_{1,1}$ has a Hecke orbit which is nowhere dense in $\mathcal{A}_1$. In fact, $\mathcal{H}(x) \cap \mathcal{A}_{1,1}$ is finite.

We see that in general, and in contrast with characteristic zero, a Hecke orbit need not be dense in the moduli space. What can we expect? What is the Zariski closure of a Hecke orbit?

(3b) **Theorem,** Chai, 1995, see [1]. *For an ordinary abelian variety A the Hecke orbit of (A, μ) is everywhere Zariski dense in the moduli space.*

This is a deep result. The proof uses various methods, the most crucial being showing that the closure of the Hecke orbit in $\mathcal{A}_{g,1}$ contains, the "cusp at infinity". A tricky computation then shows that around this point the Hecke orbit is dense.

(3c) In this paper by Chai we find the following remark by M. Larsen. *Let (E, λ) be an ordinary elliptic curve with its principal polarization. It is not difficult to show that the Hecke orbit of $(A, \mu) := (E, \lambda)^g$ is everywhere dense in the moduli space.*

4) Methods and ideas. We like to determine the Zariski closure of every Hecke orbit in positive characteristic. Perhaps the question is not so interesting, but we will see that methods developed in order to answer this question give insight into structure of $\mathcal{A}_g \otimes \mathbb{F}_p$.

- Structure of $A[p^\infty]$ carries information about A.
- This is used to define *two stratifications* and *two foliations* of $\mathcal{A}_g$. E.g. see [8], [12] and [14]. Interplay between these will provide useful information.
- Note that this information is typical for characteristic p geometry. We do not have "continuous" paths, nor complex uniformization, but we do have quite a lot of other structure, which enables us to study properties in characteristic p.
- We use "interior boundaries": instead of degenerating the abelian varieties, we can "make the p-structure more special".
- At ordinary points we have Serre-Tate canonical coordinates. These can be generalized to "central leaves" of $\mathcal{A}_g$.
- Every abelian variety over a finite field admits sufficiently many Complex Multiplications (as Tate showed). However a new notion "hypersymmetric abelian varieties" is more restrictive, see [4]. Such cases can be considered as analogous to abelian varieties of CM-type in characteristic zero.
- As in [1] the method of Hilbert Modular Varieties will be of technical importance.

5) Newton polygons. A Newton polygon for an abelian variety is a polygon

- starting at $(0, 0)$, and ending at $(h = 2g, d = g)$,
- lower convex,
- with breakpoints in $\mathbb{Z} \times \mathbb{Z}$,

- and slopes β wit $0 \leq \beta \leq 1$.
- A NP is called symmetric if the slopes β and $1-\beta$ appear with the same multiplicity.

Every p-divisible group in characteristic p determines a Newton Polygon; basically its slopes are given as "the p-adic values of the eigenvalues of the Frobenius morphism". This statement is correct over $\mathbb{F}_p$. In general more theory is necessary in order to give the definition of the NP of a p-divisible group. See [9]. For an abelian variety one defines the Newton Polygon $\mathcal{N}(A)$ to be the NP of $A[p^\infty]$; the NP of an abelian variety is symmetric (Manin, FO).

A theorem by Diedonné en Manin says that over an *algebraically closed field k isogeny classes* of p-divisible groups are classified by Newton Polygons. See [9].

An example: we write σ for the NP where all slopes are equal to $1/2$. This is called the supersingular Newton polygon. A non-trivial fact (Tate, FO, Shioda, Deligne): $\mathcal{N}(A) = \sigma$ if and only if $A \otimes k \sim E^g$, where E is a supersingular elliptic curve.

6) Stratifications and foliations.

6a) NP: $A[p^\infty]$ up to $\sim_k$.
We write:

$$\mathcal{W}^0_\xi(\mathcal{A}_g) = \{[(A,\mu)] \mid \mathcal{N}(A) = \xi\}.$$

Here ξ is a symmetric NP. These are called te open Newton Polygon strata.

Theorem (Grothendieck, Katz), see [8].

$$\mathcal{W}^0_\xi(\mathcal{A}_g) \subset \mathcal{A}_g$$

is localy closed.

The "interior boundary" of $\mathcal{W}^0_\xi(\mathcal{A}_{g,1})$ was predicted by a conjecture, the "principally polarized version" of a conjecture by Grothendieck. For proofs see [11], and [13].

6b) Fol $A[p^\infty]$ up to $\cong_k$.
For $x = [(A,\mu)]$ we write

$$\mathcal{C}(x) = \{[(B,\nu)] \mid \exists\Omega \ \ (A,\mu)[p^\infty]_\Omega \cong (B,\nu)[p^\infty]_\Omega, \quad T_\ell(A,\mu)_\Omega \cong T_\ell(B,\nu)_\Omega \ \ \forall \ell \neq p\}.$$

Here Ω is some algebraically closed field. This is called "the central leaf through x".

Theorem. *For $x \in \mathcal{W}^0_{\mathcal{A}_g}$:*

$$\mathcal{C}(x) \subset \mathcal{W}^0_{\mathcal{A}_g}$$

is closed.
See [14]. This uses the notion of "slope filtrations" as developed by T. Zink, and a theorem in [16].

An obvious remark, which will be of use later:
if $y \in \mathcal{C}(x)$, say y, x both defined over the same perfect field, then $\mathcal{C}(y) = \mathcal{C}(x)$.

Remark. The "interior boundaries" of central leaves are mysterious, although S. Harashita and I have a conjecture how they should look like.

6c) EO $A[p]$ up to $\cong_k$.
For (A, μ), where μ is a principal polarization, we write φ for the isomorphism class of $(A, \mu)[p] \otimes k$.

$$S_\varphi = \{[(B, \nu)] \mid \exists \Omega \ \ (A, \mu)[p]_\Omega \cong (B, \nu)[p]_\Omega\}.$$

Theorem.

$$S_\varphi \subset \mathcal{A}_{g,1}$$

is a localy closed subset. Every stratum S_φ is quasi-affine.
See [12]. These strata are called EO-strata, where the E refers to T. Ekedahl. The "interior boundaries" of these strata are determined in [12]. Note that if the *dimension of $S_{[}\varphi]$ is positive* then its closure has extra points inside $\mathcal{A}_{g,1}$, i.e. the "interior boundary"

$$\partial(S_\varphi) := \overline{S_\varphi} \quad - \quad S_\varphi \quad \text{is not empty.}$$

7) The Hecke Orbit conjecture.
$\boxed{\textbf{HO}}$ **Conjecture** (FO, 1995), **theorem** (Chai & FO, manuscript in preparation).

$$\forall x \in \mathcal{W}^0_\xi(\mathcal{A}_g) \quad \mathcal{H}(x) \textit{ is dense in } \mathcal{W}^0_\xi(\mathcal{A}_g).$$

See [10], [14]. A detailed proof will be given in [7]. For a preliminary survey of a proof see [3].

8) The almost-product-structure. Let W be an irreducible component of $\mathcal{W}^0_\xi(\mathcal{A}_g \otimes k)$ and let $x \in W$. For the notion of an "isogeny leaf" $I(x)$, the smallest connected subset of $\mathcal{H}_\alpha(x)$ containing x, see [14]. This is also constructed as part of the mod p reduction of a Rapoport-Zink space.

There exist reduced, irreducible schemes T and J and a finite surjective morphism

$$\Phi : T \times J \to W$$

such that for every $t \in T$, we have that

$$\Phi(\{t\} \times J) \textit{ is a irreducible component of an isogeny leaf inside } W$$

and for every $j \in J$, we have that

$$\Phi(T \times \{j\}) \textit{ is an irreducible component of a central leaf.}$$

I.e. "Central leaves and isogeny leaves give, up to a finite map, a product structure on every component of a Newton Polygon stratum".

9) Reductions.
9a) We write $\mathbf{HO}_\ell$ for the conjecture that for every x the Hecke-ℓ-orbit $\mathcal{H}_\ell$ is dense in the central leaf $\mathcal{C}(x)$. Analogous definition for $\mathbf{HO}^{(p)}$.

In fact, what can be proved:

$$(\mathbf{HO}_\ell \text{ for at least one } \ell \neq p) \quad \Longleftrightarrow \quad \mathbf{HO}^{(p)}$$

By the almost-product-structure we see that

$$\mathbf{HO}_\ell \text{ for at least one } \ell \neq p \Longleftrightarrow \mathbf{HO}^{(p)} \Longrightarrow \mathbf{HO}.$$

9b) In order to show **HO** for every x it suffices to show **HO** for every $x \in \mathcal{A}_g(\mathbb{F})$, where $\mathbb{F} = \overline{\mathbb{F}_p}$.

9c) We write $\mathbf{HO}_{\ell,\mathrm{discrete}}$ for:
For every non-supersingular $x \in \mathcal{A}_g$ the central leaf $\mathcal{C}(x)$ is absolutely irreducible.

9d) We write $\mathbf{HO}_{\ell,\mathrm{contin}}$ for:
For every non-supersingular $x \in \mathcal{A}_g$ the Zariski closure of the Hecke orbit $\mathcal{H}_\ell(x)$ contains an irreducible component of the same dimension as $\mathcal{C}(x)$; i.e. $\mathcal{H}_\ell(x)$ is dense in at least one irreducible component of $\mathcal{C}(x)$.

9e) We conclude:

$$\mathbf{HO}_{\ell,\mathrm{discrete}} \quad + \quad \mathbf{HO}_{\ell,\mathrm{contin}} \Longrightarrow \mathbf{HO}.$$

9f) For any $y \in \mathcal{H}(x)$ there is a finite-to-finite (Hecke) correspondence

$$\mathcal{C}(x)_k \quad \longleftarrow \quad T \quad \longrightarrow \quad \mathcal{C}(y)_k.$$

9f) We conclude that we need only show **HO** for moduli points over $\mathbb{F}$ and their central leaves inside $\mathcal{A}_{g,1} \otimes \mathbb{F}$.

10) Hypersymmetric abelian varieties.
Note that Tate showed that for any abelian variety A over a finite field the natural maps

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \xrightarrow{\sim} \mathrm{End}(T_\ell(A)),$$

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}(A[p^\infty])$$

are isomorphisms.

Definition. *An abelian variety over $\mathbb{F} := \overline{\mathbb{F}_p}$ is said to be hypersymmetric if the natural map*

$$\mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_p \xrightarrow{\sim} \mathrm{End}(A[p^\infty])$$

is an isomorphism.

It is not difficult to prove that for any p and for any symmetric Newton polygon there exists a hypersymmetric abelian variety having that Newton Polygon. For details see [4].

Here is a fact which will be used.

For every $x \in \mathcal{A}_g$ the central leaf $\mathcal{C}(x)$ contains a hypersymmetric point.

Sketch of a proof. One shows that any central leaf admits a Hecke correspondence with a central leaf inside $\mathcal{A}_{g,1}$, use (9f). Hence we assume that $x \in \mathcal{A}_{g,1}$. As every supersingular abelian variety is hypersymmetric we are done in that case. Assume that $x \in \mathcal{W}^0_\xi$, with $\xi \neq \sigma$. In that case $W_\xi := \mathcal{W}^0_\xi(\mathcal{A}_{g,1})$ is a geometrically irreducible. Hence W_ξ contains a hypersymmetric point. By the almost-product-structure, see (8), there is a H_α-action moving that point into a given central leaf. $\square$

We give some ideas leading to a proof of the Hecke Orbit conjecture (apologies, many details are missing in this description).

11) "Shaken not stirred".
11a) Theorem. *Every non-supersingular* $W^0_\xi := \mathcal{W}^0\xi(\mathcal{A}_{g,1})$ *is geometrically irreducible.*
Note the amusing fact that W^0_σ has"many components"' for $p \gg 0$, but all other W_ξ are irreducible.

This theorem I conjectured long ago, see [10]. A proof uses "interior boundaries": results in [11], [12], [13], and a description of moduli spaces of supersingular abelian varieties (Tadao Oda-FO, K.-Z.Li-FO); from these results one concludes that Hecke-ℓ operates transitively on the set of geometrically irreducible components of W^0_ξ; then one concludes using [2]. For details see [5].

11b) Theorem. *For every non-supersingular* $x \in \mathcal{A}_g$ *the central leaf* $\mathcal{C}(x)$ *is geometrically irreducible.*
Note that this also works for non-principal polarizations. For details see [5].
Conclusion. $\mathbf{HO}_{\ell,\mathrm{discrete}}$ holds.

11c) We say that a principally abelian variety (B, ν) over k is *split* if there is an isogeny

$$(B, \nu) \sim (B_1, \nu_1) \times \cdots \times (B_r, \nu_r),$$

where the Newton polygon of each of these factors has at most two slopes.

11d) "The Hilbert trick." Note that any abelian variety A over a fintie field has smCM. Hence there exists a commutative, totally real algebra E of rank over $\mathbb{Q}$ equal to the dimension of A such that $E \subset \mathrm{End}^0(A)$. This proves that through any point of $\mathcal{A}_{g,1}(\mathbb{F})$ we can choose the image of a Hilbert Modular variety. For details see [1], and especially see [6], Section 9.

11e) For HMV various strata were studied. Results by Goren-FO, Andreatta-Goren. Finally Chia-Fu Yu showed the discrete **HO** problem for Hilbert Modular Varieties, [17].

11f) Using EO-strata we show that any component of the image of a Hilbert Modular variety contains supersingular points. Here we make essential use of the idea of "interior boundaries".

11g) Write

$$Z(x) = \overline{\mathcal{H}^{(p)}(x)}.$$

Collecting all information obtained up to now one shows:

for every $x \in \mathcal{A}_{g,1}$ *there exists a point* $y \in Z(x) \cap \mathcal{C}(x)$ *which is hypersymmetric and split.*

(This is one of the most difficult and tricky parts of the proof.)

11h) *For a hypersymmetric and split point* $\mathbf{HO}^{(p)}_{\mathrm{contin}}$ *holds.*
Here we see the idea by M. Larsen, already mentioned in [1], see (3c). One ingredient is a generalization of Serre-Tate coordinates to the case of any central leaf, completed at any point.

11i) We see:

$$\mathcal{C}(y) = Z(y) \cap \mathcal{C}(y) \quad \subset \quad Z(x) \cap \mathcal{C}(x) \quad \subset \quad \mathcal{C}(x) = \mathcal{C}(y).$$

Indeed, as $Z(x) \cap \mathcal{C}(x)$ is $\mathcal{H}^{(p)}$-stable, the first inclusion follows. This proves $\mathbf{HO}^{(p)}_{\mathrm{contin}}$ for every $x \in \mathcal{A}_{g,1}(\mathbb{F})$. Hence, using reduction steps, this proves **HO**. □

12) Analogies: three conjectures.
Here are conjectures / theorems, where the basic structure are are quite similar. However methods of proof are very different.

Geometry: a variety V over some field K, of finite type over its prime field.

Arithmetic: a subset Γ of V. Typically the points of Γ are not all defined over some fixed finite extension of K.

Question. What is the closure of Γ inside V? In all three problem we first predict what $\overline{\Gamma}$ should be, and then (try to) prove this to be true.

12a) The Manin-Mumford conjecture. Here $V = A$ is an abelian variety over a field K of characteristic zero. The set Γ is some subset of $\mathrm{Tors}(A)$, a set of torsion points.

The closure of Γ *is a finite union of translates of abelian subvarieties of* A.

This was first proved by M. Raynaud.

12b) The André-Oort conjecture. Here $V = S$ is a Shimura variety over a field K of characteristic zero. The set Γ is some subset of $\mathrm{Spec}(S)$, a set of "special points"; in case S is a moduli scheme of abelian varieties (possibly with some extra structure), a special point is defined by an abelian variety with sufficiently many complex multiplications.

The closure of Γ *should be a finite union of special subvarieties, Hecke translates of Shimura subvarieties.*

It seems that this conjecture has been proved, assuming the generalized RH, by Yafaev - Klingler - Ullmo (using ideas by Edixhoven and Clozel).

12c) The Hecke Orbit conjecture. Here $V = \mathcal{A}_g \otimes \mathbb{F}_p$, and $\Gamma = \mathcal{H}(x)$. See above.

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