Low-dimensional Representations of $\mathrm{Aut}(F_2)$

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1 Introduction

Let $F_2 = \langle x, y \rangle$ be the free group of rank 2 with generators x and y. We will denote the automorphism group $\mathrm{Aut}(F_2)$ by Φ_2. There is a well known open problem concerning the linearity of this group : Is it true that Φ_2 has a faithful linear representation? Magnus and Tretkoff [9] have conjectured that there is no such representation over any field. In the case of free groups of rank ≥ 3, the automorphism group is not linear [6].

The above conjecture is closely connected with the old problem of linearity of the braid groups (see [1, 4]). It was proved in [4] that if B_4, the braid group on four strings, has a faithful representation of degree m, then Φ_2 has a faithful representation of degree $2m$. For a very recent account of representations of braid groups see [2].

We consider a more general problem of describing all representations of Φ_2 of degree n for small n. Very little is known about this problem : we know only the paper [3] where it is proved that Φ_2 has no faithful 3-dimensional representations over any field of characteristic 0.

We shall now recall some facts about the structure of Φ_2. For $a \in F_2$ let f_a be the inner automorphism of F_2 defined by a, i.e., $(z)f_a = a^{-1}za$ for all $z \in F_2$. (In order to conform with the usage in [8], we write f_a on the right hand side of the element to which it is applied.) Since F_2 has trivial center, the homomorphism $a \mapsto f_a$ is injective, and we use it to identify F_2 with its image in Φ_2.

It is well known [8, p. 169] that Φ_2 is generated by the following three elements :

$$\begin{aligned} P &: x \mapsto y, \quad y \mapsto x; \\ U &: x \mapsto xy, \quad y \mapsto y; \\ \sigma &: x \mapsto x^{-1}, \quad y \mapsto y; \end{aligned}$$

and has a presentation consisting of the following relations :

$$P^2 = \sigma^2 = (\sigma P)^4 = (P\sigma PU)^2 = (UP\sigma)^3 = 1, \quad (U\sigma)^2 = (\sigma U)^2. \tag{1}$$

Let $\rho : \Phi_2 \to \mathrm{GL}(V)$ be a linear representation, where V is an n-dimensional vector space over K. We can construct new representations :

$$P \to \epsilon_1 \rho(P), \quad U \to \epsilon_2 \rho(U), \quad \sigma \to \epsilon_3 \rho(\sigma), \tag{2}$$

where $\epsilon_i = \pm 1$ and $\epsilon_1 \epsilon_2 \epsilon_3 = 1$.

We say that a representation ρ' of Φ_2 is *weakly equivalent* to the representation ρ if ρ' is equivalent to one of the representations (2) or their dual representations.

Our main result can be stated as follows.

Theorem. *Consider indecomposable representations ρ of Φ_2 of degree $n \leq 4$, over an algebraically closed field K, such that $\rho(F_2) \neq 1$. There are no such representations if*

$n \leq 2$. If $\rho(\Phi_2)$ is infinite then, up to weak equivalence, there exist for $n = 3$ only one such representation, and for $n = 4$ two if char $K \neq 2, 3$, one if char $K = 3$, and none if char $K = 2$. All the representations mentioned above are reducible, and are listed in the last section. If $\rho(\Phi_2)$ is finite, ρ factorizes through the natural homomorphism $\Phi_2 \to \Gamma_i$, where Γ_i are some finite groups of small orders defined in Lemma 2.

Corollary. *Φ_2 has no faithful representation of degree $n \leq 4$ over any field.*

If $\rho(F_2) = 1$, then ρ factorizes through the natural homomorphism $\Phi_2 \to \Phi_2/F_2 \simeq \mathrm{GL}(2, \mathbf{Z})$. It is easy to show that there exist infinitely many nonequivalent idecomposable 4-dimensional representations of $\mathrm{GL}(2, \mathbf{Z})$.

From our theorem it follows that for $n \leq 4$ there are only finitely many nonequivalent n-dimensional representations of Φ_2 such that $\rho(F_2) \neq 1$, and in all these cases $\rho(F_2)$ is a solvable group. On the other hand, already for $n = 6$ there exists a one-parameter family of irreducible nonequivalent representations of Φ_2 such that $\rho(F_2)$ contains a free non-Abelian subgroup. Hence it is impossible to extend our theorem to dimensions $n \geq 6$. This also explains why the proof of our theorem involves a lot of computations.

We indicate briefly how to construct the family mentioned above. For that purpose we make use of the braid group B_4 and the well known 3-dimensional Bürau representation β_t depending on a parameter t. This can be modified to obtain a one-parameter family of 3-dimensional representations β_t^* of B_4/Z_4, where Z_4 is the center of B_4. We recall that there is an embedding $B_4/Z_4 \to \Phi_2$ (see [4]) such that the image of B_4/Z_4 in Φ_2 has index 2. The representations β_t^* induce 6-dimensional representations of Φ_2 having the properties stated above. The claim about the existence of free non-Abelian subgroups follows from [10].

For $n \geq 6$ it would be interesting to describe the character variety of n-dimensional representations of Φ_2. For the case of braid group B_4, the character variety of 3-dimensional representations was recently described by Formanek [5].

In the last section of our paper we describe also some new 4-dimensional representations of B_4. Two of them are at the same time indecomposable and reducible. It would be interesting to find some applications of these representations.

By using our identification of F_2 with a subgroup of Φ_2, we have $y = (\sigma U)^2$ and $x = PyP$. Furthermore we have :

$$U^{-1}xU = xy, \quad Uy = yU, \quad \sigma y = y\sigma, \quad \sigma x\sigma = x^{-1}. \tag{3}$$

The elements U and y generate a free Abelian group of rank 2. We introduce the element $\omega = P\sigma P$, which satisfies :

$$\omega^2 = 1, \;\; \sigma\omega = \omega\sigma, \;\; \omega U\omega = U^{-1}, \;\; \omega y\omega = y^{-1}. \tag{4}$$

The subgroup $D_4 = \langle P, \sigma\rangle$ of Φ_2 is a dihedral group of order 8. We shall use some elementary facts about the representations of D_4 over fields of characteristic $\neq 2$.

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2 Some general facts and lemmas

In this section we recall some general facts about Φ_2 and its representations. We prove two lemmas concerning some particular factor groups of Φ_2. The proof of the theorem proper will begin in the next section.

In our proof we shall use the following simple fact : Any two primitive elements of F_2 are conjugate in Φ_2. Recall that $a \in F_2$ is called *primitive* if there exists $b \in F_2$ such that $\{a, b\}$ is a free basis of F_2. In order to prove the above fact, let a and b be primitive elements of F_2. Then it is clear that there exists $\phi \in \Phi_2$ such that $(a)\phi = b$. This implies that $\phi^{-1} \circ f_a \circ \phi = f_b$, and, by using our identification, we obtain $\phi^{-1} \cdot a \cdot \phi = b$. Thus our claim is proved.

In particular, the elements x and xy are conjugate in Φ_2. So $xy = z^{-1}xz$ for some $z \in \Phi_2$. This shows that y is a commutator in Φ_2, and consequently F_2 is contained in the commutator subgroup of Φ_2.

Given a linear representation $\rho : \Phi_2 \to \mathrm{GL}(V)$, for the sake of simplicity, we shall refer to the eigenvalues, trace, determinant, ... of $\rho(y)$ as the eigenvalues, trace, determinant, ... of y, and similarly for other elements of Φ_2. Since F_2 is contained in the commutator subgroup of Φ_2, we have

$$\det(y) = 1. \tag{5}$$

Now assume that $\rho(F_2) \neq 1$, or equivalently, that $\rho(y) \neq 1$. Under this hypothesis we claim that $\rho(y)$ is not a scalar operator. Indeed, if $\rho(y)$ were a scalar, then we would have $\rho(x) = \rho(y)$ and $\rho(xy^{-1}) = 1$. This is impossible since y and xy^{-1} are conjugate in Φ_2 and $\rho(y) \neq 1$.

Lemma 1. *Denote by Γ the quotient group of Φ_2 obtained by adding the new defining relation $[U, (P\sigma)^2] = 1$ to the presentation* (1). *Then the image of F_2 in Γ is trivial.*

Proof. Since $(P\sigma)^2 = \sigma\omega = \omega\sigma$ and $\omega U \omega = U^{-1}$, we have $\sigma\omega U\omega\sigma U^{-1} = y^{-1}$. Hence, in Γ we have $y = 1$, and consequently also $x = 1$. ∎

In the next lemma and its proof we denote by C_k a cyclic group of order k, by Q the quaternion group of order 8, by S_k the symmetric group of degree k, and by $E(2^k)$ an elementary Abelian group of order 2^k.

Lemma 2. *By adding new relations to the presentation* (1), *we obtain some finite quotient groups as follows :*

(i) *relation $U^2 = 1$, quotient group $\Gamma_1 \simeq C_2 \times S_4$;*

(ii) *relation $[U, \sigma] = 1$, quotient group $\Gamma_2 \simeq C_2 \times S_4$;*

(iii) *relations $U^4 = (\sigma U)^4 = 1$, quotient group $\Gamma_3 \simeq E(64) \rtimes S_3$;*

(iv) *relations $U^4 = [P, (\sigma U)^4] = 1$, quotient group $\Gamma_4 \simeq (Q\#Q) \rtimes S_4$;*

where # denotes the central product. In particular Γ_1 and Γ_2 have order 48, *Γ_3 order* 384, *and Γ_4 order* 768.

Proof. It is straightforward to check that there exist surjective homomorphisms $f : \Gamma_1 \to \{\pm 1\} \times S_4$ and $g : \Gamma_2 \to \{\pm 1\} \times S_4$ given by :

$$f(U) = (-1, (13)), \quad f(P) = (1, (23)), \quad f(\sigma) = (-1, (12)(34));$$

and

$$g(U) = (-1, (1234)), \quad g(P) = (1, (23)), \quad g(\sigma) = (-1, (13)(24)).$$

To prove (i) and (ii) it suffices to show that $|\Gamma_1| \leq 48$ and $|\Gamma_2| \leq 48$, respectively. Let Γ be the common factor group of Γ_1 and Γ_2 obtained from the presentation of Φ_2 by adding the relations $U^2 = 1$ and $\sigma U = U\sigma$. These relations are equivalent to $U^2 = 1$, $(\sigma U)^2 = 1$, and so we have $\Gamma_1/\langle x, y\rangle \simeq \Gamma \simeq \Gamma_2/\langle x, y\rangle$.

In Γ we have $1 = (UP\sigma)^3 = UPU\sigma P\sigma UP\sigma = UPUPU\omega\sigma P\sigma = (UP)^3\omega$. Thus $(UP)^6 = 1$, and since $\omega = P\sigma P$, we have $\sigma \in \langle U, P\rangle$. It follows that $|\Gamma| \leq 12$.

In Γ_1 we have $x = U^{-2}xU^2 = U^{-1}xyU = U^{-1}xUy = xy^2$, and so $y^2 = 1$. It follows that $|\langle x, y\rangle| \leq 4$, and so $|\Gamma_1| \leq 48$. Thus (i) is proved.

In Γ_2 we have $y = (\sigma U)^2 = U^2$ and $y^{-1}xy = U^{-2}xU^2 = U^{-1}xUy = xy^2$. Hence $yxy = x$, and by conjugating by P we obtain $xyx = y$. So $x^2 = y^{-2}$. As $xyx^{-1} = y^{-1}$, by conjugating the equality $x^2 = y^{-2}$ by x, we obtain $x^2 = y^2$, and so $x^4 = 1$. If $x^2 \neq 1$ in Γ_2, then $\langle x, y\rangle = Q$ is the quaternion group. If $(P\sigma)^2 \neq 1$, as Γ has no elements of order 4, we have $(P\sigma)^2 = x^2$. It follows that $(P\sigma)^2$ is central in Γ_2, and Lemma 1 gives a contradiction. We conclude that $x^2 = 1$ in Γ_2, and so $|\Gamma_2| \leq 48$. Hence (ii) holds.

We now prove (iv). Let $G = (Q\#Q') \rtimes S_4$ where Q' is another copy of Q. We have $Q = \{\pm 1, \pm i, \pm j, \pm k\}$, where $1, i, j, k$ are the quaternionic units, and analogously $Q' = \{\pm 1, \pm i', \pm j', \pm k'\}$. We now describe the action of S_4 on $Q\#Q'$. First of all, both Q and Q' are normal in G. The normal 4-group, say V, of S_4 acts trivially on Q, while the subgroup S_3 acts as follows :

$$(12): \quad i \to j, \quad j \to i\,;$$

$$(123): \quad i \to -j, \quad j \to k.$$

The alternating subgroup A_4 acts trivially on Q' and the odd permutations interchange i' and j'. It is now straightforward to verify that there is a surjective homomorphism $h : \Gamma_4 \to G$ such that :

$$h(U) = (kj', (1432)), \quad h(P) = (1, (12)), \quad h(\sigma) = (jj', (13)(24)).$$

In order to prove (iv), it suffices to show that $|\Gamma_4| \leq 768$. In Γ_4 we have $x = U^{-4}xU^4 = xy^4$, and so $x^4 = y^4 = 1$. As $y = (\sigma U)^2$, P and y^2 commute in Γ_4, and so $x^2 = y^2$ and $|\langle x, y\rangle| \leq 8$. Let Δ be the factor group $\Gamma_4/\langle x, y\rangle$. Clearly $\Delta \simeq \mathrm{GL}_2(\mathbf{Z})/N$, where N is the normal closure in $\mathrm{GL}_2(\mathbf{Z})$ of $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$. The image of N in the modular group $\mathrm{SL}_2(\mathbf{Z})/\{\pm 1\}$ is the unique normal subgroup of level 4, and so it has index 24. For these facts we refer the reader to [11, Chapter VIII]. Hence the index of N in $SL_2(\mathbf{Z})$ is at most 48, and in $GL_2(\mathbf{Z})$ at most 96. It follows that $|\Gamma_4| \leq 96 \cdot 8 = 768$ and (iv) is proved.

We have shown above that h is an isomorphism. Since $\Gamma_3 = \Gamma_4/P$ where P is the normal closure of $y^2 = (\sigma U)^4$ in Γ_4, and $h(y)^2 = (-1, 1)$, (iii) follows from (iv). ■

This lemma was proved first by using GAP, the symbolic computation package [7]. Subsequently we have constructed the homomorphisms f, g, h and succeded to eliminate the reliance on GAP in our proof.

3 Representations of degree 2 and 3

For $n = 1$ the assertion of the theorem is obvious. In this section we prove the assertion of the theorem when $n = 2$ or 3 and char $K \neq 2$.

Let $n = 2$. Since $\rho(F_2) \neq 1$, Lemma 1 implies that $\rho(P\sigma)^2 \neq 1$, and so the restriction of ρ to D_4 is faithful. Hence we may assume that

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $\sigma y = y\sigma$ and $\det(y) = 1$, we have

$$\rho(y) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

As $x = PyP$, we have $\rho(xy) = 1$. Since y and xy are conjugate, we obtain that $\lambda = 1$, a contradiction.

Now let $n = 3$. By Lemma 1, V is a sum of two irreducible D_4-modules : a 2-dimensional and a 1-dimensional. Up to weak equivalence, we may assume that

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

As $\sigma y = y\sigma$, we have

$$\rho(y) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{pmatrix}.$$

From $(\omega y)^2 = 1$, we obtain that $c(b-e) = d(b-e) = 0$ and $b^2 = e^2 = cd + 1$.

If $b \neq e$, then $c = d = 0$, $b = -e = \pm 1$. As $\det(y) = 1$, we have $a = -1$. From $\rho(y) = \mathrm{diag}(-1, b, -b)$ and $\rho(x) = \rho(PyP) = \mathrm{diag}(b, -1, -b)$, we obtain that $\rho(xy) = \mathrm{diag}(-b, -b, 1)$. As $\rho(xy) \neq 1$, we must have $b = 1$. By using the fact that y and U commute, we have

$$\rho(U) = \begin{pmatrix} \alpha & 0 & \beta \\ 0 & \gamma & 0 \\ \delta & 0 & \epsilon \end{pmatrix}.$$

The eqation $Uxy = xU$ implies that $\alpha = \epsilon = 0$. Since $y = (\sigma U)^2$, we must have $\beta\delta = \gamma^2 = 1$. Hence $\rho(U^2) = 1$, and so Lemma 2 applies.

If $b = e$, then $\det(y) = 1$ implies that $a = 1$. Hence

$$\rho(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & b & c \\ 0 & d & b \end{pmatrix}, \quad \rho(x) = \begin{pmatrix} b & 0 & c \\ 0 & 1 & 0 \\ d & 0 & b \end{pmatrix}.$$

Since xy and y are conjugate, we have $\operatorname{tr}(xy) = \operatorname{tr}(y) = 1 + 2b$. This gives $b^2 = 1$, and so $cd = 0$. By replacing ρ by its dual (if necessary) we may assume that $d = 0$.

If $b = -1$, then $Uy = yU$ implies that $\rho(\sigma)$ and $\rho(U)$ commute, and Lemma 2 applies.

If $b = 1$, then $c \neq 0$ and we may assume that $c = 1$. Since $Uy = yU$, we have

$$\rho(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} \alpha & 0 & \beta \\ \gamma & \delta & \epsilon \\ 0 & 0 & \delta \end{pmatrix}.$$

The equation $(\omega U)^2 = 1$ implies that $\beta = 0$, $\delta = \alpha$, and $\alpha^2 = 1$. The equation $Uxy = xU$ implies that $\alpha = 1$ and $\gamma = -1$. Since $y = (\sigma U)^2$, we must have $\epsilon = 1/2$. Thus we obtain

$$\rho(U) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

The equations (6) and (7) define an indecomposable representation of Φ_2. Obviously this representation is reducible.

4 Representations of degree 4

In this section we begin the proof of the theorem when $n = 4$ and char $K \neq 2$. This part of the proof will be completed in the next three sections.

We claim that the eigenvalues of y can be written as

$$\lambda, \lambda^{-1}, \mu, \mu^{-1} \tag{8}$$

for some $\lambda, \mu \in K^*$. If all eigenvalues of y are ± 1, this follows from (5). If y has an eigenvalue $\lambda \neq \pm 1$, then $\omega y \omega = y^{-1}$ implies that λ^{-1} is also an eigenvalue of y. Since $\lambda^{-1} \neq \lambda$, (5) implies that the remaining two eigenvalues of y can be written as μ, μ^{-1}. This proves our claim.

By replacing ρ with a weakly equivalent representation, if necessary, we may assume that

$$\operatorname{tr}(\sigma) = 0, 2. \tag{9}$$

We shall denote by V^+ resp. V^- the eigenspace of σ for eigenvalue $+1$ resp. -1. Since ω and y commute with σ, these subspaces are invariant under ω and y. We shall denote by $\rho(\omega)^+$ and $\rho(y)^+$ the restrictions of $\rho(\omega)$ and $\rho(y)$ to V^+, respectively.

We conclude this section with two lemmas.

Lemma 3. *Let ρ be a 4-dimensional representation of Φ_2 and assume that char $K \neq 2$. If* $\operatorname{tr}(\sigma) = 2$, *then all eigenvalues of y are ± 1.*

Proof. We shall assume that y has an eigenvalue $\lambda \neq \pm 1$ and obtain a contradiction. As $\operatorname{tr}(\sigma) = 2$, $\dim V^+ = 3$ and $\dim V^- = 1$. If $e_4 \in V^-$, $e_4 \neq 0$, then e_4 is an eigenvector of y. Say $y(e_4) = \mu e_4$. Since $\omega y \omega = y^{-1}$ and V^- is ω-invariant, we conclude that $\mu = \pm 1$.

It follows that $\rho(y)^+$ has three distinct eigenvalues λ, λ^{-1}, and μ. Let e_1 and e_3 be eigenvectors of $\rho(y)^+$ belonging to λ and μ, respectively. Set $e_2 = \omega(e_1)$. Then

$$y(e_2) = y\omega(e_1) = \omega y^{-1}(e_1) = \lambda^{-1}\omega(e_1) = \lambda^{-1} e_2,$$

and so $\{e_1, e_2, e_3, e_4\}$ is a basis of V.

Since $\rho(\omega)^+\rho(y)^+\rho(\omega)^+ = \rho(y^{-1})^+$, the subspace Ke_3 is ω-invariant. From $P\sigma P = \omega$ we deduce that $\operatorname{tr}(\omega) = 2$, and so

$$\omega(e_1) = e_2,\ \omega(e_2) = e_1,\ \omega(e_3) = e_3,\ \omega(e_4) = e_4 \ .$$

By identifying linear operators with their matrices with respect to this basis, we have

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \rho(\omega) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}.$$

As U and y commute,

$$\rho(U) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & w & z \end{pmatrix}.$$

The equality $(\omega U)^2 = 1$ implies that $\alpha\beta = 1$ and

$$u^2 = z^2 = 1 - vw, \quad v(u+z) = w(u+z) = 0. \tag{10}$$

The equality $y = (\sigma U)^2$ implies that $\alpha^2 = \lambda$ and

$$u^2 = z^2 = \mu + vw, \quad v(u-z) = w(u-z) = 0. \tag{11}$$

If $\mu = 1$, the above equations imply $v = w = 0$. Hence $\rho(\sigma)$ and $\rho(U)$ commute, and Lemma 2 implies that $\rho(y)^2 = 1$. This contradicts the assumption that $\lambda \neq \pm 1$.

If $\mu = -1$, then (10) and (11) imply that $u = z = 0$ and $vw = 1$. By conjugating by the diagonal matrix $\mathrm{diag}(1,1,1,w)$, we may assume that $v = w = 1$. Thus

$$\rho(U) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since $P\sigma P = \omega$ and $P^2 = 1$, we must have

$$\rho(P) = \begin{pmatrix} a & a & b & c \\ a & a & b & -c \\ d & d & e & 0 \\ f & -f & 0 & 0 \end{pmatrix},$$

where

$$2cf = 1, \quad b(2a+e) = d(2a+e) = 0, \quad e^2 = 4a^2 = 1 - 2bd.$$

By conjugating by $\mathrm{diag}(1,1,f,f)$, we may assume that $c = 1/2$ and $f = 1$.

If $b = d = 0$, then the $(1,4)$ entries in $\rho(UP\sigma))^3 = 1$ give $ae(\alpha^2 - 1) = 0$. As $\alpha^2 \neq 1$, we have $ae = 0$. Since $e^2 = 4a^2$, we have $a = e = 0$. As $\rho(P)$ is nonsingular, we have a contradiction.

If $b \neq 0$ or $d \neq 0$, then $e = -2a$ and by comparing the $(4,3)$ entries in $\rho(UP\sigma))^3 = 1$, we obtain that $a(\alpha^2 - 1) = 0$, and so $a = 0$. By comparing $(4,4)$ entries, we obtain a contradiction. ■

Lemma 4. *Let ρ be a 4-dimensional representation of Φ_2 and assume that char $K \neq 2$. Then the Jordan canonical form of $\rho(y)$ contains no Jordan blocks of size 3.*

Proof. Assume that $\rho(y)$ has a Jordan block of size 3. Then $\mathrm{tr}(\sigma) \neq 0$, and so by (9) we have $\mathrm{tr}(\sigma) = 2$. We can choose a basis of V such that

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \mu \end{pmatrix}.$$

As $\omega y \omega = y^{-1}$, we have $\lambda^2 = 1$. Since $\det(y) = 1$, we have $\lambda = \mu$.

Since $\omega\sigma = \sigma\omega$, $\rho(\omega) = A \oplus B$ with A of size 3 and $B = (\pm 1)$. Since $\omega y \omega = y^{-1}$, we have $A \neq 1$ and $\mathrm{tr}(\omega) = \mathrm{tr}(\sigma) = 2$ implies that $B = (1)$. By using $\omega y \omega = y^{-1}$ again, we conclude that $\rho(\omega)$ is upper triangular and that it has the form

$$\rho(\omega) = \begin{pmatrix} 1 & u & u(u-\lambda)/2 & 0 \\ 0 & -1 & \lambda - u & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

By conjugating with a suitable matrix which commutes with $\rho(\sigma)$ and $\rho(y)$, we may assume that $u = 0$.

Since U and y commute, we have

$$\rho(U) = \begin{pmatrix} a & b & c & d \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & e & f \end{pmatrix}, \quad \rho(\omega U) = \begin{pmatrix} a & b & c & d \\ 0 & -a & \lambda a - b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & e & f \end{pmatrix}.$$

From $(\omega U)^2 = 1$ we obtain that $d(a+f) = e(a+f) = 0$, and from $y = (\sigma U)^2$ that $d(a-f) = e(a-f) = 0$. Since $a+f$ or $a-f$ is not zero, it follows that $d = e = 0$. Hence $\rho(U)$ and $\rho(\sigma)$ commute and, by Lemma 2, $\rho(\Phi_2)$ is finite. As $\rho(y)$ has infinite order, we have a contradiction. ■

We now divide the proof into three cases, which will be treated separately in the next three sections.

5 Case 1 : $\lambda \neq \mu, \mu^{-1}$

Up to weak equivalence, we may assume that $\mathrm{tr}(\sigma) = 0, 2$.

Subcase 1 : $\mathrm{tr}(\sigma) = 0$. Both V^+ and V^- have dimension 2. If $\det \rho(y)^+ = 1$, then $\rho(\sigma)$ is a central element of the centralizer of $\rho(y)$ in $\mathrm{GL}(V)$, and in particular it commutes with $\rho(U)$. By Lemma 2, ρ factors through the homomorphism $\Phi_2 \to \Gamma_2$.

Now let $\det \rho(y)^+ \neq 1$. Then the eigenvalues of $\rho(y)^+$ are, say, λ and μ, and those of $\rho(y)^-$ are λ^{-1} and μ^{-1}. Since ω leaves invariant V^+ and V^- and inverts y, it follows that $\lambda = -\mu = \pm 1$ and that $\rho(y)$ and $\rho(\omega)$ commute. By choosing a suitable basis, we may assume that

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s \end{pmatrix}$$

where $r, s = \pm 1$. Then $\rho(\omega)$ and $\rho(y)$ have the form

$$\rho(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & -b \end{pmatrix},$$

where $a, b = \pm 1$. As $\rho(x) \neq \rho(y)$, we have $b = a$. Hence $\rho(\omega y) = \pm 1$. It follows that $\rho(U) = \rho(\omega y U (\omega y)^{-1}) = \rho(U)^{-1}$. Hence ρ factors through the homomorphism $\Phi_2 \to \Gamma_1$ of Lemma 2.

Subcase 2 : $\operatorname{tr}(\sigma) = 2$. Now V^+ has dimension 3 and V^- dimension 1. By Lemma 3, all eigenvalues of y are ± 1, and so $\lambda = -\mu = \pm 1$.

Assume first that $\rho(y)$ is diagonalizable. Then $\rho(y^2) = 1$, and $\rho(\sigma)$, $\rho(\omega)$, and $\rho(y)$ commute. We can diagonalize them simultaneously. By Lemma 1, $\rho(\sigma) \neq \rho(\omega)$. Hence we may assume that

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \epsilon_4 \end{pmatrix},$$

where $\epsilon_i = \pm 1$, $\det(y) = 1$, and $\operatorname{tr}(y) = 0$.

The equations $P^2 = 1$ and $P\sigma P = \omega$ imply that

$$\rho(P) = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 0 & 1/e \\ 0 & 0 & e & 0 \end{pmatrix}.$$

We may assume that $e = 1$. Since $x = PyP$ and $\rho(xy) \neq 1$, we must have $\epsilon_2 = -\epsilon_1$ and $\epsilon_4 = -\epsilon_3$.

If $\epsilon_3 = -\epsilon_1$, then

$$\rho(U) = \begin{pmatrix} u & 0 & 0 & v \\ 0 & f & g & 0 \\ 0 & h & i & 0 \\ w & 0 & 0 & z \end{pmatrix}.$$

The equation $Uxy = xU$ implies that $i = 0$ (and so $gh \neq 0$), $v = w = 0$, and $ac = bc = bd = 0$. Consequently $b = c = 0$. This is impossible since ρ is indecomposable.

If $\epsilon_3 = \epsilon_1$, then

$$\rho(U) = \begin{pmatrix} u & 0 & v & 0 \\ 0 & f & 0 & g \\ w & 0 & z & 0 \\ 0 & h & 0 & i \end{pmatrix}.$$

The equation $Uxy = xU$ now implies that $z = 0$ (and so $vw \neq 0$) and $ad = bc = 0$. This is impossible since $ad - bc = \pm 1$.

Hence $\rho(y)$ is not diagonalizable. By choosing a suitable basis $\{e_1, e_2, e_3, e_4\}$ of V and by replacing λ with $-\lambda$, if necessary, we may assume that

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}.$$

Since $\omega y \omega = y^{-1}$, the subspaces Ke_1, $Ke_1 + Ke_2$, and Ke_3 are ω-invariant. As $\mathrm{tr}(\omega) = \mathrm{tr}(\sigma) = 2$, $\rho(\omega)$ must have the form :

$$\begin{pmatrix} -1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By replacing ρ with its dual representation, we may assume that $\rho(\omega)$ is given by the first of these two matrices. By replacing e_2 with $e_2 + (s/2)e_1$, we may assume that $s = 0$.

As U and y commute, we have

$$\rho(U) = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

From $(\omega U)^2 = 1$, we obtain the equations $\alpha^2 = 1$, $a^2 = d^2 = 1 - bc$, and from $y = (\sigma U)^2$ the equations $\lambda = 1$, $\beta = \alpha/2$, $a^2 = d^2 = bc - 1$. It follows that $a = d = 0$ and $bc = 1$. By conjugating by $\mathrm{diag}(1,1,1,c)$, we may assume that $b = c = 1$. Hence

$$\rho(U) = \begin{pmatrix} \alpha & \alpha/2 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \alpha^2 = 1.$$

Since $P\sigma P = \omega$ and $P^2 = 1$, P must map the eigenspaces of σ to the corresponding eigenspaces of ω. It follows that

$$\rho(P) = \begin{pmatrix} 0 & 0 & 0 & e \\ 0 & f & g & 0 \\ 0 & h & i & 0 \\ 1/e & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{pmatrix} f & g \\ h & i \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The equation $(UP\sigma)^3 = 1$ implies that $f = \alpha$, $i = -\alpha$, $g = 0$, and $h = \alpha/2e$. By conjugating by $\mathrm{diag}(1,1,e,e)$, we may assume that $e = 1$. We compute $\rho(x)$ and find that

$$\rho(x) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & \alpha & 0 & 1 \end{pmatrix}.$$

We obtain indeed an indecomposable representation of Φ_2. The choices $\alpha = 1$ and $\alpha = -1$ give weakly equivalent representations.

6 Case 2 : $\lambda = \mu \neq \pm 1$

By Lemma 3, $\mathrm{tr}\,(\sigma) = 0$, and so both V^+ and V^- have dimension 2. Choose $e_1 \in V^+$, $e_1 \neq 0$, such that $y(e_1) = \lambda e_1$. Then the vector $e_2 = \omega(e_1)$ is in V^+ and $y(e_2) = \lambda^{-1} e_2$. We can choose similarly nonzero vectors e_3, e_4 in V^- such that $y(e_3) = \lambda e_3$, $y(e_4) = \lambda^{-1} e_4$, and $\omega(e_3) = e_4$. With respect to the basis $\{e_1, e_2, e_3, e_4\}$ of V, we have

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \rho(\omega) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ \rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

Since $P\sigma P = \omega$ and $P^2 = 1$, P must map the eigenspaces of σ to the corresponding eigenspaces of ω. It follows that $\rho(P)$ must have the form :

$$\rho(P) = \begin{pmatrix} a & c & \alpha & \gamma \\ a & c & -\alpha & -\gamma \\ b & d & \beta & \delta \\ b & d & -\beta & -\delta \end{pmatrix}.$$

From $P^2 = 1$ it follows that $a = c = \pm 1/2$, $\beta = -\delta = \pm 1/2$, $\alpha = \gamma$, $b = -d$, and $4\alpha b = 1$. By replacing ρ with a weakly equivalent representation, we may assume that $a = 1/2$. By conjugating with the diagonal matrix $\mathrm{diag}(1, 1, 2b, 2b)$, we may assume that $b = \alpha = 1/2$. Hence

$$\rho(P) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \epsilon & -\epsilon \\ 1 & -1 & -\epsilon & \epsilon \end{pmatrix}, \quad \epsilon = \pm 1.$$

Since U and y commute, we have

$$\rho(U) = \begin{pmatrix} u & 0 & v & 0 \\ 0 & u' & 0 & v' \\ w & 0 & z & 0 \\ 0 & w' & 0 & z' \end{pmatrix}.$$

From $y = (\sigma U)^2$ we obtain the equations:

$$v(u - z) = w(u - z) = 0, \quad u^2 = z^2 = vw + \lambda,$$

and from $(\omega U)^2 = 1$ the equality

$$\begin{pmatrix} u' & v' \\ w' & z' \end{pmatrix} = \begin{pmatrix} u & v \\ w & z \end{pmatrix}^{-1}.$$

Assume first that $u \neq z$. Then $v = w = 0$, and consequently $v' = w' = 0$. Furthermore, we have $u' = 1/u$, $z = -u$, and $z' = -1/u$. By using $x = PyP$ and the equation $Uxy = xU$, we obtain $u^2 = 1$. Hence $\lambda = 1$, which is a contradiction.

Hence, we must have $u = z$, and so $u' = z'$. It follows that

$$\rho(U) = \begin{pmatrix} u & 0 & v & 0 \\ 0 & u/\lambda & 0 & -v/\lambda \\ w & 0 & u & 0 \\ 0 & -w/\lambda & 0 & u/\lambda \end{pmatrix}, \quad \lambda = u^2 - vw.$$

If $\epsilon = 1$, by equating the (3,1)-entries of the matrices $\rho(Uxy)$ and $\rho(xU)$, we obtain the equation $\lambda^2(u+w) = u-w$. Similarly, the (4,2)-entries give the equation $\lambda^2(u-w) = u+w$. Hence $\lambda^4 = 1$. As $\lambda \neq \pm 1$, we must have $\lambda^2 = -1$. It follows that $u = 0$ and $w = -\lambda/v$. By equating the (1,1)-entries of the above mentioned matrices, we obtain that $v = 0$, which is impossible.

So we have $\epsilon = -1$. The equation $\rho(Uxy) = \rho(xU)$ now implies that $\lambda^2 = -1$ and $w = -v$. The relation $(UP\sigma)^3 = 1$ implies that

$$4u^2(u-v) = \lambda(3u-v) + \lambda - 1,$$

$$4u^2(u+v) = \lambda(3u+v).$$

By taking into account that $u^2+v^2 = \lambda$, we obtain only one solution : $u = v = -(1+\lambda)/2$. In this case we indeed obtain an indecomposable representation of Φ_2. Since $\rho(U)^4 = 1$ and $\rho(y^2) = -1$, ρ factorizes through the homomorphism $\Phi_2 \to \Gamma_4$ of Lemma 2.

7 Case 3 : $\lambda = \mu = \pm 1$

Recall that D_4 has (up to equivalence) only one 2-dimensional irreducible module and four 1-dimensional ones. Assume that V, as a D_4-module, is a direct sum of two irreducible 2-dimensional modules. On an irreducible 2-dimensional D_4-module the element $(P\sigma)^2$ acts as minus the identity operator and so $\rho(P\sigma)^2$ lies in the center of $\mathrm{GL}(V)$. By Lemma 1, $\rho(F_2) = 1$ and we have a contradiction. The same argument applies when V is a sum of four 1-dimensional D_4-modules. Thus we may assume that V is a direct sum of one 2-dimensional irreducible D_4-module and two 1-dimensional modules.

Subcase 1 : $\mathrm{tr}(\sigma) = 0$. Up to weak equivalence, we may assume that (with respect to a suitable basis of V)

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $r = \pm 1$. As $\omega = P\sigma P$ and $y\sigma = \sigma y$, we have

$$\rho(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \alpha' & \beta' & 0 & 0 \\ \gamma' & \delta' & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \gamma & \delta \end{pmatrix}.$$

Since all eigenvalues of y are equal $\lambda = \pm 1$, we have $\alpha + \delta = 2\lambda$ and $\alpha\delta - \beta\gamma = 1$. Since $\omega y \omega = y^{-1}$, it follows that $\alpha = \delta = \lambda$ and $\beta\gamma = 0$. Similarily $\alpha' = \delta' = \lambda$ and $\beta'\gamma' = 0$.

Up to weak equivalence, we have the following four possibilities :

(i) $\beta' \neq 0$, $\gamma' = \beta = \gamma = 0$;

(ii) $\beta' \neq 0$, $\gamma' = \beta = 0$, $\gamma \neq 0$;

(iii) $\beta' \neq 0$, $\beta \neq 0$, $\gamma' = \gamma = 0$;

(iv) $\beta' = \gamma' = \gamma = 0$, $\beta \neq 0$.

In fact, by using some elementary considerations, one can show that (i) and (iv) are weakly equivalent. Furthermore, by conjugating by a suitable diagonal matrix which commutes with $\rho(P)$, we may assume that the nonzero parameters among β', β, and γ are all equal to 1. We now consider each of the first three possibilities separately.

(i) We have

$$\rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & 0 \\ 0 & e & g & h \\ 0 & f & i & j \end{pmatrix}.$$

The relation $Uxy = xU$ implies that $\lambda = 1$, $h = 0$, $g = a$, and $e = ra$. The relation $y = (\sigma U)^2$ implies that $a^2 = j^2 = 1$, $di = 0$, $(a+j)i = 0$, and $(a-j)f = air$. The relation $(P\sigma PU)^2 = 1$ implies that $c = 0$, $(a-j)i = 0$, $2ab = 1$, and $(a+j)f = air$. It follows that $i = f = 0$. Finally the relation $(UP\sigma)^3 = 1$ implies that $j = -1$, $a = r$, and $d = 0$. Since $d = h = f = i = 0$, ρ is decomposable, contrary to the hypothesis.

(ii) We have

$$\rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & b & e & f \\ 0 & a & f & 0 \\ g & h & c & 0 \\ h & 0 & d & c \end{pmatrix}.$$

From $\rho(Uxy) = \rho(xU)$, by equating $(4,4)$ and $(2,3)$ entries, we find that $c(1-\lambda) = 0$ and $f(1-\lambda) = 0$. As c and f cannot both be 0, we infer that $\lambda = 1$. From $(3,2)$ entries we obtain $g = 0$. The entries $(1,2)$, $(1,3)$, $(4,2)$, and $(4,3)$ provide the equations $a + f = rh$, $c - a = rf$, $a = c + h$, and $f = c + rh$, respectively. These equations imply that $c = -a$, $h = 2a$, $f = -2ar$, and $a(4r - 1) = 0$. As $r = \pm 1$, we obtain $a = 0$, which is impossible since $\rho(U)$ is invertible.

(iii) We have

$$\rho(y) = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ e & f & g & h \\ 0 & e & 0 & g \end{pmatrix}.$$

From $Uxy = xU$ we obtain $a(1-\lambda) = e$ and $e(1-\lambda) = 0$. As a and e are not both zero, we must have $\lambda = 1$. Taking this into account, the same relation implies that $e = 0$, $g = a$, $f - a$, $r = -1$, and $h = a - b - c$. The relation $y = (\sigma U)^2$ implies that $a^2 = 1$ and $a = 2b + c$. From $(P\sigma PU)^2 = 1$ we obtain that $c = 0$, and so $h = a - b$. From $(UP\sigma)^3 = 1$ we find that $a = -1$, $b = -1/2$, and $3d = 1/4$. In particular char $K \neq 3$. Thus $\rho(U)$ is uniquely determined and all the defining relations are satisfied. One can easily check that this representation of Φ_2 is indeed indecomposable.

Subcase 2 : $\mathrm{tr}\,(\sigma) = 2$. By choosing a suitable basis of V, we have

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix},$$

$$\rho(\omega) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & a & b & c \\ 0 & d & e & f \\ 0 & g & h & i \end{pmatrix},$$

where $\alpha, \beta, \lambda = \pm 1$.

By Lemma 4, $\rho(y)$ has no Jordan blocks of size 3, and so $(\rho(y) - \lambda)^2 = 0$. From this equality and $\rho(\omega y)^2 = 1$ we obtain that $\rho(\omega y \omega) = 2\lambda - \rho(y)$. Hence we have $a = e = i = \lambda$ and $f = h = 0$. Now the equation $(\rho(y) - \lambda)^2 = 0$ implies that $bd = cd = bg = cg = 0$. Hence $\rho(y)$ has one of the forms :

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & b & c \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & d & \lambda & 0 \\ 0 & g & 0 & \lambda \end{pmatrix}.$$

By replacing ρ by its dual, we may assume that $\rho(y)$ has the form given by the first of these two matrices. At least one of b and c is not 0. By conjugating by a suitable diagonal matrix, which commutes with $\rho(P)$, we may assume that b and c are either 0 or 1. Hence there are three possibilities to consider :

(i) $b = 1$, $c = 0$;

(ii) $b = 0$, $c = 1$;

(iii) $b = c = 1$.

Furthermore, if $r = 1$ in $\rho(P)$ then, without any loss of generality, it suffices to consider the possibility (i) only. This can be achieved by conjugation by a matrix which commutes with $\rho(\sigma)$ and $\rho(P)$. We analyze each of these possibilities separately.

(i) Since y and U commute, we have

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & 0 & b & c \\ d & e & f & g \\ 0 & 0 & e & 0 \\ h & 0 & i & j \end{pmatrix},$$

where we are now reusing the letters a-j in a different role.

From $Uxy = xU$ we obtain first $e(1-\lambda) = 0$, and so $\lambda = 1$, and then $e = a$, $d = -a$, and $h = 0$. From $y = (\sigma U)^2$ we find that $a^2 = j^2 = 1$, $c(a-j) = 0$, $i(a+j) = 0$, $g(a+j) + ac = 0$, and $ab + 2af + gi = 1$. From $(P\sigma PU)^2 = 1$ we obtain from $(1,4)$ entries that $c(a+j) = 0$. Since $a \neq 0$, this equation when combined with $c(a-j) = 0$ gives $c = 0$. From $(2,4)$ entries we obtain $g(a-j) = 0$. When combined with $g(a+j) = 0$, we conclude that $g = 0$. From $(1,3)$ entries we obtain that $b = 0$. One of the previous equations now gives $f = 1/2a$. Next we exploit the relation $(UP\sigma)^3 = 1$. From $(1,1)$ entries we obtain $a^3 = 1$. Since $a^2 = 1$, it follows that $a = 1$. From $(4,3)$ entries we obtain $i(2r + j) = 0$. As $j^2 = r^2 = 1$, it follows that $i = 0$. Since $c = g = h = i = 0$, ρ is decomposable, and so we have a contradiction.

(ii) We have $r = -1$ and

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & 0 & b & c \\ d & e & f & g \\ h & 0 & i & j \\ 0 & 0 & 0 & e \end{pmatrix},$$

From $Uxy = xU$ we obtain first from $(2,2)$ entries the equation $e(1-\lambda) = 0$, and so $\lambda = 1$. Next from $(3,4)$ entries we obtain $h = 0$, from $(1,4)$ entries $e = a$, and from $(2,4)$ entries $d = a$. From $(\sigma U)^2 = (U\sigma)^2$ by comparing $(1,3)$ entries we obtain $b(a-i) = 0$. Next we use the relation $(P\sigma PU)^2 = 1$. From diagonal entries we find that $a^2 = i^2 = 1$. From $(1,3)$ entries we obtain $b(a+i) = 0$. By combining this equation with $b(a-i) = 0$, we conclude that $b = 0$. From $(1,4)$ entries we find that $c = 0$. Finally we use the relation $(UP\sigma)^3 = 1$. From diagonal entries we find that $a^3 = -1$ and $i^3 = 1$. As $a^2 = i^2 = 1$, we have $a = -1$ and $i = 1$. Now from (1,4) entries we find that $f = 0$, and from (3,4) entries $j = 0$. Since $b = f = h = j = 0$, ρ is decomposable and so we have a contradiction.

(iii) We have $r = -1$ and

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 1 & 1 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} a & 0 & b & c \\ d & e & f & g \\ h & 0 & i & j \\ -h & 0 & e-i & e-j \end{pmatrix},$$

From $Uxy = xU$ we obtain first from (2,2) entries the equation $e(1-\lambda) = 0$, and so $\lambda = 1$. Now the (2,3) entries give $e = -d$, while (2,4) entries give $e = d$. We infer that $e = 0$, which is a contradiction.

8 Characteristic 2 case

Let $n = 2$ and assume only that ρ is nontrivial. Since $(\omega U)^2 = 1$ and $\omega^2 = 1$, it follows that $\det(U) = 1$. Let λ and λ^{-1} be the eigenvalues of U. Since $(P\sigma)^4 = 1$, $\rho(P\sigma)$ is unipotent. As $n = 2$, we have $\rho(P\sigma)^2 = 1$. Hence $\rho(P)$ and $\rho(\sigma)$ commute, and so $\rho(\sigma) = \rho(\omega)$.

Assume first that $\lambda \neq 1$. Since $\omega U \omega = U^{-1}$, we can choose a basis of V such that

$$\rho(U) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \rho(\sigma) = \rho(\omega) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $P^2 = 1$, and $\rho(P)$ commutes with $\rho(\sigma)$, we must have

$$\rho(P) = \begin{pmatrix} a & a+1 \\ a+1 & a \end{pmatrix}$$

for some $a \in K$. By examining the equation $\rho(UP\sigma)^3 = 1$, one can show that $a = 0$ and $\lambda^2 + \lambda + 1 = 0$, i.e., λ is a primitive cube root of 1. Hence we have an indecomposable representations of Φ_2 such that $\rho(\Phi_2) \simeq S_3$.

Assume now that $\lambda = 1$. If $\rho(U) = 1$, then also $\rho(P) = \rho(\sigma)$ and $\rho(\Phi_2) \simeq C_2$. Thus we may assume that

$$\rho(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Now let $\rho(U) \neq 1$. If $\rho(\sigma) \neq 1$, we can choose a basis of V such that

$$\rho(\sigma) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad b \neq 0,$$

because both $\rho(P)$ and $\rho(U)$ commute with $\rho(\sigma)$. From $(UP\sigma)^3 = 1$ we conclude that $a + b = 1$. Hence we obtain a 1-parameter family of non-equivalent indecomposable representation of Φ_2 with $\rho(\Phi_2) \simeq C_2 \times C_2$. If $\rho(\sigma) = 1$, then $\rho(UP)^3 = 1$ implies that either, say,

$$\rho(U) = \rho(P) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

or $\rho(UP)$ has order 3, in which case we may assume that

$$\rho(U) = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where λ is a primitive cube root of 1. Hence we obtain another indecomposable representation of Φ_2 with $\rho(\Phi_2) \simeq S_3$, which is not equivalent to the previous one.

In all of the representaiton mentioned above we have $\rho(y) = \rho(\sigma U)^2 = 1$, and so $\rho(F_2) = 1$. In particular the assertion of the theorem holds if $n = 2$.

Now let $n = 3$ and assume that ρ is indecomposable and $\rho(F_2) \neq 1$. Since $\omega U \omega = U^{-1}$, the eigenvalues of U are λ, λ^{-1}, and 1.

If $\rho(y)$ is diagonalizable, then $\rho(y) \neq 1$ implies that y has three distinct eigenvalues. As $y\sigma = \sigma y$, $\rho(\sigma)$ is diagonalizable. Since $\rho(\sigma)$ is also unipotent, we obtain $\rho(\sigma) = 1$, a contradiction.

Hence $\rho(y)$ is not diagonalizable, and so must be unipotent. Since $yU = Uy$, it follows that $\lambda = 1$, i.e., $\rho(U)$ is unipotent. Consequently $\rho(U)^4 = 1$. Since $y = (\sigma U)^2$ and $\rho(y)$ is unipotent, we conclude that $\rho(y)^2 = 1$. Hence ρ factorizes through the homomorphism $\Phi_2 \to \Gamma_3$.

Finally let $n = 4$. We assume, as in the statement of the theorem, that ρ is indecomposable and that $\rho(F_2) \neq 1$. The eigenvalues of y have the form $\lambda, \lambda^{-1}, \mu, \mu^{-1}$. We divide the proof into three subcases.

Subcase 1 : $\lambda = \mu = 1$. Since $\rho(y)$ is unipotent and $y = (\sigma U)^2$, $\rho(\sigma U)$ is also unipotent. As $n = 4$, we conclude that $\rho(y)^2 = 1$. Since x, y, and xy are conjugate in Φ_2, we have also $\rho(x)^2 = \rho(xy)^2 = 1$. As $\rho(F_2) \neq 1$, we conclude that $\rho(F_2)$ is a four-group. The subspace $W \subset V$ consisting of all vectors v such that $\rho(x)(v) = \rho(y)(v) = v$ has dimension $1, 2$, or 3. Since F_2 is normal in Φ_2, W is Φ_2-invariant.

We choose a basis of W and extend it to a basis of V. With respect to such a basis we have

$$\rho = \begin{pmatrix} \rho' & * \\ 0 & \rho'' \end{pmatrix}$$

where ρ' (resp. ρ'') is the representation of Φ_2 on W (resp. V/W) induced by ρ.

If $\rho(U)$ is unipotent, then $\rho(U^4) = 1$ and so ρ factorizes through the homomorphism $\Phi_2 \to \Gamma_3$. From now, untill the end of this subcase, we shall assume that $\rho(U)$ is not unipotent.

If U has an eigenvalue 1, then we may assume that

$$\rho(U) = \begin{pmatrix} 1 & \alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta^{-1} \end{pmatrix}, \quad \beta \neq 1,$$

with respect to some basis $\{e_1, e_2, e_3, e_4\}$. Since $yU = Uy$, $\rho(y)^2 = 1$, and $\rho(y) \neq 1$, we have

$$\rho(y) = \begin{pmatrix} 1 & \gamma & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma \neq 0.$$

Hence $\beta \cdot \sigma U \sigma(e_3) = (\sigma U)^2(e_3) = y(e_3)$, i.e., $U\sigma(e_3) = \beta^{-1}\sigma(e_3)$. This implies that $\sigma(e_3) = ae_4$ for some $a \in K^*$. As $\sigma^2 = 1$ and $\sigma y = y\sigma$, we infer that

$$\rho(\sigma) = \begin{pmatrix} 1 & \delta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a^{-1} \\ 0 & 0 & a & 0 \end{pmatrix}.$$

An easy computation shows that $\rho(\sigma U)^2 = 1$. As $y = (\sigma U)^2$ and $\rho(y) \neq 1$, we have a contradiction.

Now assume that U has no eigenvalue 1. This implies that $\dim(W) = 2$ and that $\rho'(\Phi_2)$ and $\rho''(\Phi_2)$ are both isomorphic to S_3. For these representations we have $\rho'(P\sigma) = \rho''(P\sigma) = 1$, and consequently $\rho(P\sigma)^2 = 1$. Now Lemma 1 gives a contradiction.

Subcase 2 : $\{\lambda, \lambda^{-1}\} \neq \{\mu, \mu^{-1}\}$. If $\lambda, \mu \neq 1$, then $y\sigma = \sigma y$ and $\sigma^2 = 1$ imply that $\rho(\sigma) = 1$, a contradiction. Now let, say, $\mu = 1$. If $\rho(y)$ is not diagonalizable, its centralizer in $\mathrm{GL}(V)$ is Abelian. Hence $\rho(\sigma)$ and $\rho(U)$ commute. By Lemma 2, ρ factorizes through the homomorphism $\Phi_2 \to \Gamma_2$. We now assume that $\rho(y)$ is diagonalizable. Since σ and y commute, σ leaves invariant the eigenspaces of y. Consequently we can choose a basis of V such that

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $\omega y \omega = y^{-1}$ and $\omega = P\sigma P$, we may also assume that

$$\rho(\omega) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since $Uy = yU$, we have

$$\rho(U) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

From $y = (\sigma U)^2$ we obtain

$$\alpha^2 = \lambda, \quad \beta = \alpha^{-1}, \quad c = a + d, \quad ad + bc = 1,$$

and from $(\omega U)^2 = 1$ we obtain that $a + d = 0$. Consequently $c = 0$, $d = a = 1$. Thus $\rho(\sigma)$ and $\rho(U)$ commute and we can apply Lemma 2.

Subcase 3 : $\lambda = \mu \neq 1$. Both eigenspaces of y have the same dimension. If $\rho(y)$ is not diagonalizable, then the centralizer of $\rho(y)$ in $\mathrm{GL}(V)$ is Abelian and we can use Lemma 2 once again. Now let $\rho(y)$ be diagonalizable. Then both eigenspaces of y have dimension 2, and ω interchanges these eigenspaces. It follows that $1+\omega$ has rank 2. Since $\omega = P\sigma P$, $1+\sigma$ also has rank 2. As y and σ commute, we can choose a basis of V such that

$$\rho(y) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix}, \quad \rho(\sigma) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

Since ω inverts y and commutes with σ, we must have

$$\rho(\omega) = \begin{pmatrix} 0 & 0 & a' & b' \\ 0 & 0 & 0 & a' \\ c' & d' & 0 & 0 \\ 0 & c' & 0 & 0 \end{pmatrix}, \quad a'd' = b'c', \quad a'c' = 1.$$

By conjugating $\rho(\omega)$ by a suitable matrix which commutes with $\rho(y)$ and $\rho(\sigma)$, we may assume that $a' = c' = 1$ and $b' = d' = 0$, i.e.,

$$\rho(\omega) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{13}$$

Since $Uy = yU$, we have

$$\rho(U) = \begin{pmatrix} u' & v' & 0 & 0 \\ z' & w' & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & z & w \end{pmatrix}.$$

From $y = (\sigma U)^2$ we obtain the equations

$$u^2 + z^2 + z(v+w) = w^2 + z(v+w) = \lambda^{-1},$$

and so $z = u + w$ and

$$\lambda^{-1} = uv + vw + wu.$$

The equation $(\omega U)^2 = 1$ gives

$$\begin{pmatrix} u' & v' \\ z' & w' \end{pmatrix} = \begin{pmatrix} u & v \\ z & w \end{pmatrix}^{-1},$$

and so

$$\rho(U) = \begin{pmatrix} \lambda w & \lambda v & 0 & 0 \\ \lambda(u+w) & \lambda u & 0 & 0 \\ 0 & 0 & u & v \\ 0 & 0 & u+w & w \end{pmatrix}. \tag{14}$$

The matrix

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

satisfies the equation $\rho(\omega)P_0 = P_0\rho(\sigma)$. Since $\rho(P)$ satisfies the same equation, the matrix $P_0^{-1}\rho(P)$ commutes with σ. Consequently $\rho(P)$ has the form

$$\rho(P) = P_0 \cdot \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ \alpha & \beta & \gamma & \delta \\ 0 & \alpha & 0 & \gamma \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ \alpha & \beta & \gamma & \delta \\ a & a+b & c & c+d \\ \alpha & \alpha+\beta & \gamma & \gamma+\delta \end{pmatrix}.$$

Since $P^2 = 1$, we have the equations:

$$a(a+c) + \alpha(b+d) = 1, \quad \alpha(a+c) = 1, \tag{15}$$

$$\alpha(a+\beta+\delta) = a\gamma, \quad \alpha(\alpha+\gamma) = 0, \tag{16}$$

$$d(\alpha+\gamma) + \gamma(c+\delta) + \delta(\beta+\delta) = 0, \quad \delta(\alpha+\gamma) + \gamma^2 = 1. \tag{17}$$

The second equations of (15),(16), and (17) imply that $\alpha = \gamma = 1$. The second equation of (15) and the first equations of (16) and (17) give $c = \beta = \delta = 1 + a$. From the first equation in (15) we now obtain that $d = 1 + a + b$. Thus

$$\rho(P) = \begin{pmatrix} a & b & 1+a & 1+a+b \\ 1 & 1+a & 1 & 1+a \\ a & a+b & 1+a & b \\ 1 & a & 1 & a \end{pmatrix}$$

By conjugating by the matrix

$$\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

we may assume that

$$\rho(P) = \begin{pmatrix} 0 & t & 1 & 1+t \\ 1 & 1 & 1 & 1 \\ 0 & t & 1 & t \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

where $t = a + b + a^2$. Although $\rho(U)$ will change under this conjugation, it will still have the form (14). By using this expression for $\rho(P)$, we find that

$$\rho(x) = \begin{pmatrix} \lambda^{-1} + rt & rt & rt & rt \\ r & \lambda + rt & 0 & rt \\ rt & rt & \lambda^{-1} + rt & rt \\ 0 & rt & r & \lambda + rt \end{pmatrix}$$

where

$$r = \lambda + \lambda^{-1}.$$

By equating the diagonal entries of the matrices $\rho(xU)$ and $\rho(Uxy)$, we obtain the equations

$$\begin{aligned}
&(v + wt)\lambda^3 + ut\lambda^2 + (v + w + wt)\lambda + w + ut = 0, \\
&(u + wt)\lambda^3 + (u + v + ut)\lambda^2 + wt\lambda + v + ut = 0, \\
&wt\lambda^3 + (v + ut)\lambda^2 + (u + wt)\lambda + u + v + ut = 0, \\
&(v + w + wt)\lambda^3 + (w + ut)\lambda^2 + (v + wt)\lambda + ut = 0.
\end{aligned}$$

By adding the first two equations, we obtain

$$(\lambda + 1) \cdot [v + w + (u + v)\lambda^2] = 0,$$

and by adding the last two, we obtain

$$(\lambda + 1) \cdot [u + v + (v + w)\lambda^2] = 0.$$

Since $\lambda \neq 1$, we have

$$u + v = \lambda^{-2}(v + w) = \lambda^2(v + w),$$

and so $u = v = w$. By (12) and (14), $\rho(\sigma)$ and $\rho(U)$ commute and so, by Lemma 2, $\rho(y)^2 = 1$. This gives $\lambda = 1$, a contradiction.

This completes the proof of the theorem. ■

9 Some indecomposable representations of Φ_2 and B_4

In this section we list all, up to weak equivalence, indecomposable representations ρ of Φ_2 of degree ≤ 4 such that $\rho(F_2) \neq 1$ and $\rho(\Phi_2)$ is infinite. According to the previous section, such reperesentations do not exist if char $K = 2$. We also include an interesting example of an indecomposable representation of degree 4 with $\rho(\Phi_2)$ finite.

One can use the above mentioned representations ρ of Φ_2 in order to construct new representations of B_4. Recall that the braid group B_4 has the following presentation :

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \ : \ [\sigma_1, \sigma_3] = 1, \ \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3\rangle.$$

Furthermore there is a homomorphism $h : B_4 \to \Phi_2$ given by :

$$h(\sigma_1) = PUP, \quad h(\sigma_2) = U\sigma U^{-1}P, \quad h(\sigma_3) = P\sigma U^{-1}\sigma P.$$

For readers convenience, we have also computed the images of σ_i's in each case.

Representation 1. The generators σ, P, and U of Φ_2 are represented by the matrices

$$\rho(\sigma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to verify that these matrices satisfy the defining relations (1) of Φ_2. A simple computation shows that $x = PyP$ and $y = (\sigma U)^2$ are represented by the matrices

$$\rho(x) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Hence $\rho(F_2)$ is a free Abelian group of rank 2.

The corresponding representation of B_4 is determined by :

$$\sigma_1 \to \begin{pmatrix} 1 & -1 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 \to \begin{pmatrix} 1 & -1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Representation 2. The second representation ρ is defined by :

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In this case we find that

$$\rho(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now $\rho(F_2)$ is a solvable group which is not nilpotent.

For B_4 we have :

$$\sigma_1 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & -1 & 0 \\ 1/2 & -1 & 0 & 0 \\ 1/2 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1/2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \sigma_3 \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 0 & 1 & 0 \\ 1/2 & 1 & 0 & 0 \\ -1/2 & 0 & 0 & 1 \end{pmatrix}.$$

Representation 3. If characteristic of K is not 2 or 3, then we have a representation ρ defined by :

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(U) = \begin{pmatrix} 1 & 1/2 & 0 & 1/12 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case we have

$$\rho(x) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case $\rho(F_2)$ is a non-Abelian unipotent group.

The corresponding representation of B_4 is given by :

$$\sigma_1 \to \begin{pmatrix} 1 & 0 & 1/2 & -1/12 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 \to \begin{pmatrix} 1 & 0 & 0 & 1/16 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\sigma_3 \to \begin{pmatrix} 1 & 0 & -1/2 & -1/12 \\ 0 & 1 & 1 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

All three representations above of Φ_2 and B_4 are at the same time indecomposable and reducible.

Representation 4. This representation ρ is defined by :

$$\rho(\sigma) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \rho(P) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix},$$

$$\rho(U) = \frac{1}{2}\begin{pmatrix} -1-i & 0 & -1-i & 0 \\ 0 & -1+i & 0 & 1-i \\ 1+i & 0 & -1-i & 0 \\ 0 & -1+i & 0 & -1+i \end{pmatrix},$$

where $i^2 = -1$. One can show that $\rho(\Phi_2) \simeq (Q\#Q) \rtimes S_3$, a quotient of the group Γ_4 defined in Lemma 2. The images of x and y generate one of the two quaternion groups Q. The basic vectors are common eigenvectors of σ and y and, up to scalar multiples, there are no other common eigenvectors. Since P does not preserve these eigenspaces, ρ has no 1-dimensional invariant subspace. As $\rho(F_2) \neq 1$, ρ cannot be direct sum of two 2-dimensional representations. Hence ρ is irreducible.

In this case the representation of B_4 is given by :

$$\sigma_1 \to \frac{1}{2}\begin{pmatrix} -1 & i & 1 & -i \\ -i & -1 & -i & -1 \\ -1 & -i & -1 & -i \\ -i & 1 & i & -1 \end{pmatrix}, \quad \sigma_2 \to \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix},$$

$$\sigma_3 \to \frac{1}{2}\begin{pmatrix} -1 & -i & 1 & i \\ i & -1 & i & -1 \\ -1 & i & -1 & i \\ i & 1 & -i & -1 \end{pmatrix}.$$

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