ON THE FOUR-VERTEX THEOREM

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1) Introduction

In 1912 Adolf Kneser proved that on any simple closed plane curves there are at least four vertices, i.e. points where the curvature is stationary [11]. For convex curves this had been established already in 1909 by Mukhopadhyaya [13]. Since then vertices of closed plane curves have been studied from various viewpoints. Otto Haupt, to whom the present paper is dedicated, made contributions in connection with his theory of geometric orders [6,7]. A survey of the history of the four-vertex theorem and an extensive bibiliography can be found in [2] (see also the more recent papers [14,15]).

In this paper we want to prove the four-vertex theorem for a more general class of closed curves, that may also have self-intersections. For example, since every simple closed curve in $\mathbb{R}^2$ has its tangent winding number n equal to one, it seems possible at first sight that the four-vertex theorem might hold for all closed plane curves with $n = 1$. This however is not true. Figure 1 indicates for every $n \geq 0$ a closed plane curve with tangent winding number n and only two vertices.

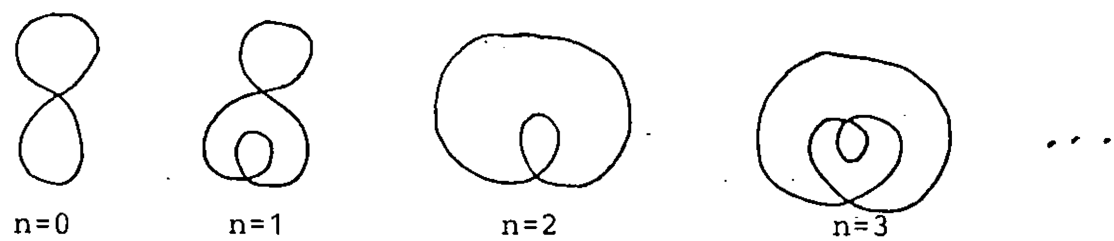
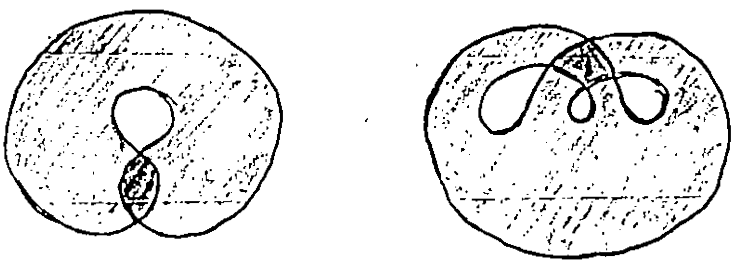


Figure 1

By the Schoenflies theorem [12] any embedding $f: S^1 \longrightarrow \mathbb{R}^2$ extends to an embedding $\hat{f}: D^2 \longrightarrow \mathbb{R}^2$ (here $D^2 = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$, $S^1 = \partial D^2$). In other words, any simple closed curve in $\mathbb{R}^2$ bounds an embedded disc. We say that an immersion $f: S^1 \longrightarrow \mathbb{R}^2$ bounds an immersed surface of genus g, if there is a compact surface M^2 of genus g with connected boundary $\partial M = S^1$ and an immersion $\hat{f}: M^2 \longrightarrow \mathbb{R}$ which extends f. For example, the curve in Figure 2a bounds an immersed disk (genus zero) and the one in Figure 2b an immersed surface of genus one. Immersed surfaces in $\mathbb{R}^2$ were studied in some detail by Banchoff and Kauffman [9,10].



a) b)

Figure 2

In section 4 we will prove the following theorem (even a slightly improved version):

<u>Theorem:</u> Any closed curve in $\mathbb{R}^2$ that bounds an immersed surface has at least four vertices.

Since a curve bounding a surface of high genus is necessarily quite complicated it is very plausible that it should be possible to obtain a better estimate (than the one given by the theorem) in terms of the genus.

<u>Conjecture:</u> If a closed curve in $\mathbb{R}^2$ bounds an immersed surface of genus $g \geq 1$, then it has at least $4g+2$ vertices.

Figure 3 indicates for every genus $g \geq 1$ an immersed surface in $\mathbb{R}^2$ having only $4g+2$ vertices on its boundary.

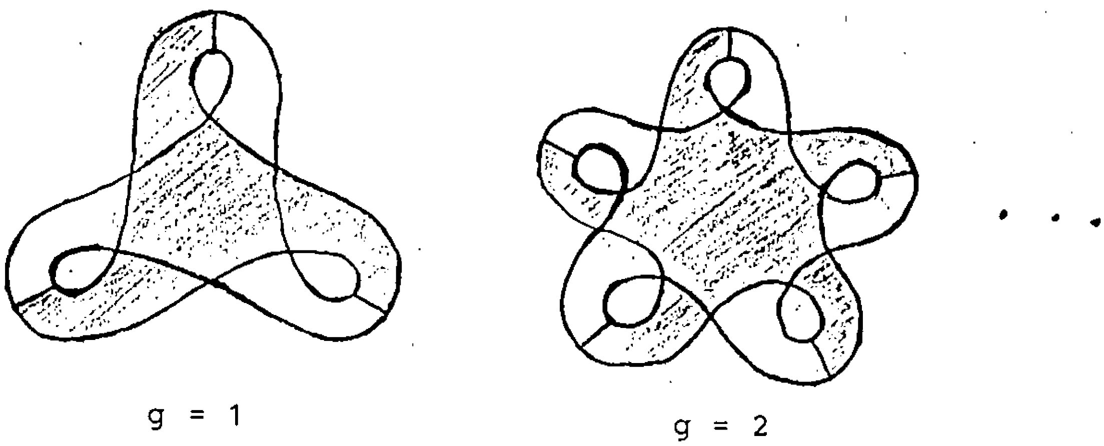


Figure 3

We have indicated in Figure 3 for each surface M^2 a number of $2g+1$ cuts which divide M^2 into two topological discs (one of which is emphasized by shading). The vertices are just on the endpoints of these cuts. The first surface in Figure 3 was invented in another context by G. Francis [4].

The proof of our main theorem is based on a certain proposition (Proposition 3 below) due to G. Valette [16]. Since Valettes paper is somewhat inaccessible, we give a proof of this proposition.

Different approaches to the study of vertices on plane curves with self-intersections can be found in [5,8].

2) Honest vertices of plane curves

Let $x:\mathbb{R} \longrightarrow \mathbb{R}^2$ be a closed (i.e. L-periodic) curve of class C^∞, parametrized by arclength. Let $t = x'$ and u be the Frenet frame of x, $k:\mathbb{R} \longrightarrow \mathbb{R}$ the curvature of x. Then we have $n' = nt$, $t' = -kn$.

x is said to have a vertex at $s \in \mathbb{R}$ if $k'(s) = 0$. A vertex at s is called an honest vertex if k fails to be a strictly monotone function in every neighborhood of s.

Proposition 1: Every closed curve in $\mathbb{R}^2$ has at least two honest vertices. If there are only finitely many honest vertices then their number is even and each honest vertex corresponds to a local maximum or a local minimum of the curvature.

Proof: Any point where the maximum or the minimum of the curvature is attained in an honest vertex. Thus there are at least two honest vertices. Suppose there are only finitely many honest vertices $\{s_0,\dots,s_n\} \in S^1 = \mathbb{R}/\mathbb{Z}\cdot L$. Then k is strictly monotone on each component of $S^1 - \{s_0,\dots,s_n\}$. This clearly implies the remaining assertions.

□

We will often find it useful to identify $\mathbb{R}^2$ with $\mathbb{C}$ and to compactify $\mathbb{R}^2 = \mathbb{C}$ by a "point at infinity" ∞. Orientation preserving Möbius transformations of the Riemann sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$ are then represented as fractional linear mappings

$$(1) \qquad z \longmapsto \frac{az+b}{cz+d}\,, \qquad ad - bc = 1\,.$$

The point is that Möbius transformations take (honest) vertices of a plane curve to (honest) vertices. This follows easily from

<u>Lemma 1:</u> The sign of k' is unchanged under orientation preserving Möbius transformations of $\mathbb{R}^2 \cup \{\infty\}$.

<u>Proof:</u> We describe the plane curve under consideration by a map $s \longmapsto z(t) \in \mathbb{C}$, $s \in \mathbb{R}$ satisfying $|z'| \equiv 1$. Then we have

$$(2) \qquad k' = \operatorname{Im} \frac{z'''}{z'} = \operatorname{Im} \{z,s\}$$

where s is arclength and

$$(3) \qquad \{z,s\} = \frac{z'''}{z'} - \frac{3}{2}\left(\frac{z''}{z'}\right)^2$$

is the Schwarzian derivative. If t is another parameter then it is known ([1], formula 5.30) that

$$(4) \qquad \{z,t\} = \varphi^{-2}(\{z,s\} + 2A^2 - 2A'),$$

where

$$(5) \qquad \varphi = \frac{dt}{ds}\,, \qquad A = \frac{1}{2}\,\frac{\varphi'}{\varphi}\,.$$

Thus the sign of $\operatorname{Im}\{z,s\}$ is invariant under changes of the parametrization of the curve z. Moreover $\{z,s\}$ itself is known to be invariant under orientation preserving Möbius transformations.

□

3) Spiral arcs on S^2

The following lemma due to A. Kneser ([11], see also [17]) will be crucial for our further discussion:

Lemma 2: Let $x:[a,b] \longrightarrow \mathbb{R}^2$ be a regular curve whose curvature k satisfies $k > 0$ and is strictly increasing in $[a,b]$. Then the circular closed discs D_t bounded by the osculating circles γ_t, $t \in [a,b]$ satisfy

$$(6) \qquad D_t \subset \overset{\circ}{D}_s \quad \text{for} \quad s < t .$$

Proof: The osculating circle γ_t has radius $r(t) = 1/k(t)$ and center $m(t) = x(t) - r(t)n(t)$. We have

$$(7) \qquad m' = r'n,$$

so the length of the evolute $m:[s,t] \to \mathbb{R}^2$ is given by $r(s) - r(t)$. Therefore we have

$$(8) \qquad \|m(s) - m(t)\| \leq r(s) - r(t),$$

which is equivalent to $D_t \subset D_s$. Moreover, since k is strictly monotone in $[s,t]$ there must be some open subinterval of $[s,t]$ where $k' > 0$. Therefore the evolute m contains a strictly convex arc, and hence strict inequality must hold in (7). This implies (5).

□

As an illustration for the lemma Figure 4 shows the osculating circles of a logarithmic spiral (the spiral itself is not drawn).

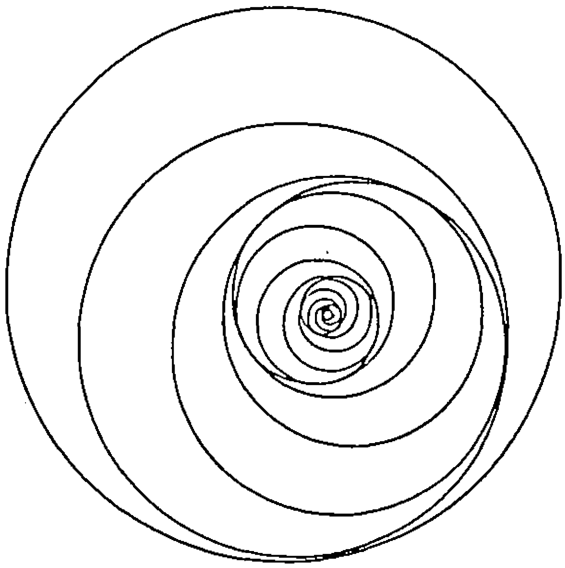


Figure 4

Lemma 2 is not yet quite what we need, because we had to exclude inflection points. However from the viewpoint of Möbius geometry inflection points are just points where the osculating circle passes through ∞. Thus they are not really distinguished points. Considering the whole problem on the Riemann sphere $S^2 = R^2 \cup \{\infty\}$ it is clear that Lemma 2 immediately implies

Lemma 3: Let $x:[a,b] \longrightarrow S^2$ be a regular curve without honest vertices in (a,b). Then one can choose round discs $D_t \subset S^2$ bounded by the osculating circles γ_t such that $D_t \subset \mathring{D}_s$ for $s < t$.

By Lemma 3, a regular curve $x:[a,b] \longrightarrow S^2$ with no honest vertices in (a,b) cannot have self-intersections. It is therefore justified to call such a curve a spiral arc.
We say that a spiral arc $x:[a,b] \longrightarrow S^2 = \mathbb{R}^2 \cup \{\infty\}$ is in normal form if both osculating circles γ_a and γ_b are centered at the origin $0 \in \mathbb{R}^2$.

Proposition 2: Every spiral arc $x:[a,b] \longrightarrow S^2 = \mathbb{R}^2 \cup \{\infty\}$ can be brought into normal form by applying to x a suitable Möbius transformation $g:S^2 \longrightarrow S^2$.

Proof: By Lemma 3 the osculating circles γ_a and γ_b are disjoint. Now it is known ([3], 10.10.12) that any two disjoint circles in $\mathbb{R}^2 \cup \{\infty\}$ can be made concentric by applying a suitable Möbius transformation.

□

If we ignore the freedom to rotate and to stretch a given spiral arc in normal form, then there are essentially only two different ways to bring a given spiral are into normal form (using orientation preserving Möbius transformations): With any spiral are in normal form also its image under $z \longrightarrow 1/z$ is in normal form.

Let $z:[a,b] \longrightarrow \mathbb{R}^2 \cup \{\infty\}$ be a spiral arc in normal form. By Lemma 3 all osculating circles γ_t (and therefore also the curve z) are contained in the circular region bounded by the concentric circles γ_a and γ_b. Reparametrizing z, if necessary, via $t \longmapsto b+a-t$ we may assume that γ_a has bigger radius than γ_b (Figure 5).

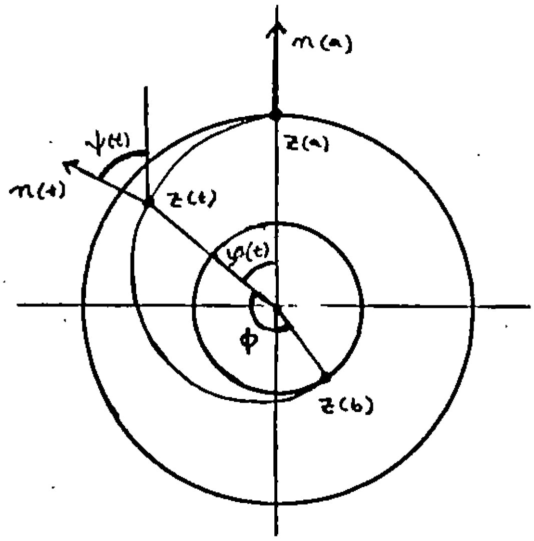


Figure 5

Obviously z and the normal vector n to the curve z can be represented as

(9) $$z(t) = \rho(t)e^{i\varphi(t)} , \qquad n(t) = e^{i\psi(t)}$$

where $\varphi,\psi:[a,b] \longrightarrow \mathbb{R}$ either both are strictly increasing or both or strictly decreasing functions. Furthermore

(10) $$\varphi(a) = \psi(a), \quad \varphi(b) = \psi(b) \quad .$$

The angle $\phi = \varphi(b) - \varphi(a) = \psi(b) - \psi(a)$ is called the <u>winding angle</u> of the spiral arc. ϕ does not depend on the choice of normal form and thus gives as a Möbius geometric invariant for arbitrary spiral arcs on S^2 .

Propositon 3 (Valette [16]: The winding angle Φ of any spiral arc on S^2 satisfies $|\phi| > \pi$.

Proof: Let $z:[a,b] \longrightarrow \mathbb{R}$ be a spiral arc in normal form, $n(t) = e^{i\psi(t)}$ its outward pointing normal vector at $z(t)$, $m = z + rn$ its evolute. Applying to z a euclidean rotation we may assume that ψ is an increasing function and $\psi(a) = 0$. Then, assuming z is parametrized by arclength,

$$(11) \qquad m'(t) = r'e^{i\psi(t)} = r'(\cos\psi(t)+i\sin\psi(t)).$$

Since $m(a) = m(b) = 0$ we conclude

$$(12) \qquad \int_a^b r'(t)\sin\psi(t) = 0.$$

By (10) the assumption $|\phi| \leq \pi$ would imply $\sin\psi(t) > 0$ for $t \in (a,b)$, which contradicts (12) because $r' \leq 0$.

□

4) The four-vertex theorem

In order to establish the four-vertex theorm for a certain class of plane curves it suffices by Proposition 1 to show that it is impossible for a closed curve of the considered kind to have only two honest vertices. Now closed curves on S^2 with only two honest vertices are rather completely described by the following theorem, which is an immediate corollary of Proposition 2:

<u>Theorem 1:</u> Let $z:S^1 \longrightarrow S^2 = \mathbb{R}^2 \cup \{\infty\}$ be a closed curve with only two honest vertices. Then, by applying to z a suitable Möbius transformation, we can assume that the osculating circles at the two vertices are both centered at $0 \in \mathbb{R}^2$. In this case the curve z is the union of two spiral arcs in normal form.

Combining Theorem 1 with Proposition 3 we easily obtain as a corollary the classical four-vertex theorem:

<u>Corollary:</u> Every simple closed curve on S^2 has at least four honest vertices.

We now prove a slightly more general version of the theorem announced in the introduction. In contrast to the earlier version we now allow also immersed surfaces in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ that cover the point ∞.

<u>Theorem 2:</u> Any closed curve on S^2 that bounds an immersed surface has at least four honest vertices.

<u>Proof:</u> By the remark at the beginning of this section we have to show that no closed curve with only two honest vertices can bound an immersed surface. Suppose on the contrary that we had a compact surface M^2 with connected boundary $\partial M^2 = S^1$ and an immersion $\hat{f}:M^2 \longrightarrow S^2$ such that the closed curve $f = \hat{f}|_{S^1}$ has only two honest vertices.

We assume that f is the union of two spiral arcs in normal form (cf. Theorem 1).

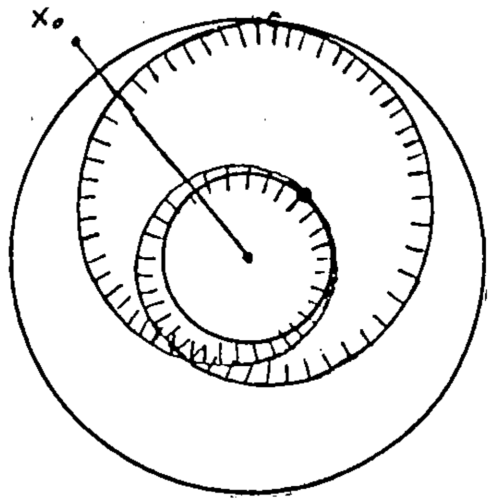


Figure 6

Also we assume (applying, if necessary an inversion in the unit circle) that the surface is on the "inner" (the convex) side of the locally convex curve $\hat{f}|\partial M^2$. See Figure 6 where near ∂M^2 the surface M^2 is indicated by shading.

Let $x_0 \in \mathbb{R}^2$ be some point in the exterior of the bigger one out of the two osculating circles at the vertices. We connect x_0 to the origin in $\mathbb{R}^2$ by the path $t \longmapsto x_t = (1-t)x_0$. This path hits $f(S^1)$ a finite number, say n, times, each time transversally (if it should hit a multiple point of $f(S^1)$ we count of course with multiplicity). n is determined as

$$(13) \qquad n = (|\phi_1| + |\phi_2|)/2\pi \quad ,$$

where ϕ_1 and ϕ_2 are the winding angles of the two mentioned spiral arcs. By Proposition 3 we have $n \geq 2$. For each $t \in [0,1]$ let k_t be the number of preimages of x_t in M^2 under $\hat{f}$. Then, because $\hat{f}$ is an immersion, k_t is locally constant except at the n points where x_t hits $f(S^1)$, where it jumps up by one. We conclude $k_1 \geq 2$, i.e. the origin $0 \in \mathbb{R}^2$ is covered at last twice by $\hat{f}$.

The radial vector field $X(x) = x$ on $\mathbb{R}^2$ extends as a smooth vector field to $S^2 = \mathbb{R}^2 \cup \{\infty\}$. The only critical points of X on S^2 are at 0 and ∞ and both have index 1. Pulling X back to M^2 via the immersion f we obtain on M^2 a smooth vector field Y, all of whose critical points have index 1. By the preceding paragraph Y has at least two critical points.

Moreover Y is transversal (outward pointing) to ∂M^2. We can think about M^2 as $\hat{M}^2 - D$ where $\hat{M}^2$ is a compact surface of genus g without boundary and D is a disc in $\hat{M}^2$. Then Y extends as as a smooth vector field $\hat{Y}$ to $\hat{M}^2$ which has only one further critical point in D, also of index 1. Thus all critical points of $\hat{Y}$ have index 1, and there are at least three such points. This contradicts the Poincaré index formula.

□

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