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SYMMETRIC MONOIDAL NONCOMMUTATIVE SPECTRA, STRONGLY SELF-ABSORBING C^*-ALGEBRAS, AND BIVARIANT HOMOLOGY

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Abstract. Continuing our earlier work we construct symmetric monoidal ∞-categorical models for separable C^*-algebras $\mathtt{SC}^*_\infty$ and noncommutative spectra $\mathtt{NSp}$ using the framework of Higher Algebra due to Lurie. We study localizations of $\mathtt{SC}^*_\infty$ and colocalizations of $\mathtt{NSp}$ with respect to any strongly self-absorbing C^*-algebra. We analyse the homotopy categories of the localizations of $\mathtt{SC}^*_\infty$ and characterize them by a universal property. We also describe the colocalized subcategories of $\mathtt{hNSp}$ spanned by the stabilizations of C^*-algebras in the purely infinite case. As a consequence we compute the noncommutative stable cohomotopy of the $ax+b$-semigroup C^*-algebra arising from any number ring. We also introduce and study the nonconnective version of Quillen's nonunital K'-theory in the framework of stable ∞-categories. We perform computations in the case of stable and $\mathcal{O}_\infty$-stable C^*-algebras.

Introduction

In [20] we constructed a stable presentable ∞-category of noncommutative spectra $\mathtt{NSp}$. It is an ideal framework to carry out stable homotopy theory of noncommutative spaces. Thom constructed a triangulated category $\mathtt{NSH}$ and referred to it as the noncommtative stable homotopy category [30] (see Remark 1.8). The author used the ∞-category $\mathtt{NSp}$ to prove that $\mathtt{NSH}$ is a topological triangulated category as defined by Schwede [28]. Nevertheless, a very important part of the homotopy theory package, viz., the symmetric monoidal structure was left out of the discussion in [20]. In the present article we use Lurie's Higher Algebra [17] to construct a symmetric monoidal stable presentable ∞-category of noncommutative spectra $\mathtt{NSp}$ (see Theorem 1.5).

Toms–Winter introduced a class of simple C^*-algebras called *strongly self-absorbing C^*-algebras* [31], which play a pivotal role in Elliott's Classification Program. Prominent examples of such C^*-algebras, which are also purely infinite, are Cuntz algebras $\mathcal{O}_2$, $\mathcal{O}_\infty$, and tensor products of UHF algebras of infinite type with $\mathcal{O}_\infty$. In the sequel we construct *smashing localizations* of the ∞-category of separable C^*-algebras $\mathtt{SC}^*_\infty$ with respect to arbitrary strongly self-absorbing C^*-algebras. We describe the homotopy categories of the localized ∞-categories (see Proposition 2.8) and derive several useful results. At the level of homotopy categories we also obtain a univeral characterization in this setting (see Theorem 2.13).

It was noticed in [20] that the homotopy category of noncommutative spectra $\mathtt{hNSp}$ is not an *algebraic* triangulated category and the question was raised whether it contains algebraic triangulated subcategories, which would facilitate computations enormously. With an eye towards such algebraization problems we colocalize the stable ∞-category $\mathtt{NSp}$ with respect

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to the stabilization of any strongly self-absorbing C^*-algebra. In the purely infinite case we describe the homotopy category of the colocalized ∞-category spanned by the stabilizations of C^*-algebras (cf. Theorems 3.4, 3.8, and 3.10). Although these results do not settle the algebraization problem, they demonstrate that certain colocalized subcategories are indeed amenable to computation as they reduce to familiar bivariant homology theories. As a consequence we prove that the canonical map from noncommutative stable cohomotopy to topological K-theory is an isomorphism for $\mathcal{O}_\infty$-stable C^*-algebras. Using the results of [6, 16] this isomorphism enables us to complete the computation of noncommutative stable cohomotopy (see Disambiguation 3.11) of $ax+b$-semigroup C^*-algebras arising from number rings (see Theorem 3.12). The $\mathcal{Z}$-stable situation, where $\mathcal{Z}$ is the Jiang–Su algebra, is the most interesting case from the viewpoint of classification. This case is not considered in detail in this article and it will be done elsewhere.

Algebraic K-theory does not (directly) make sense for topological spaces. The appropriate theory in this context is Waldhausen's A-theory [32], which is homotopy invariant but not excisive. One needs a calculus of functors to analyse it. However, algebraic K-theory does make sense for a noncommutative space. Indeed, one can view a noncommutative space or a C^*-algebra (with unit for the time being) simply as a unital complex algebra and study its algebraic K-theory. Now algebraic K-theory satisfies excision on the category of C^*-algebras [29] but it is not homotopy invariant. Roughly speaking, a spectrum valued functor F on k-algebras satisfies excision, where k is a field, if for every short exact sequence $0 \to A \to B \to C \to 0$ the induced diagram $F(A) \to F(B) \to F(C)$ is a homotopy (co)fiber sequence. Thus one needs algebraic K-theory to treat unital and nonunital algebras on an equal footing (note that A is strictly nonunital unless the extension is trivial). Quillen introduced a K'_0-theory for nonunital algebras in [23], whose higher (connective) version was developed by the author in [22]. The author's motivation in that article was categorification of topological $\mathbb{T}$-duality. The higher version of K'_0-theory was called KQ-theory by the author in ibid. so that a conflict with G-theory (or K'-theory of pseudo-coherent modules) could be avoided. In the final part of this article we define nonconnective KQ-theory and show that for stable and $\mathcal{O}_\infty$-stable C^*-algebras it agrees naturally with their nonconnective algebraic as well as topological K-theory (see Theorem 4.15 for a more general result). From the computational viewpoint the following picture emerges:

Theorem (Remark 4.17)**.** For stable and $\mathcal{O}_\infty$-stable separable C^*-algebras the four possible invariants, viz., noncommutative stable cohomotopy, nonconnective KQ-theory, nonconnective algebraic K-theory, and topological K-theory are all naturally isomorphic.

At least the assertion in the $\mathcal{O}_\infty$-stable case for all four invariants appears to be new (see also [4, 19]). The results in this part rely on various properties of algebraic K-theory in the setting of stable ∞-categories established by Blumberg–Gepner–Tabuada [1]. We also obtain an ∞-categorical version of an earlier result of the author on categorification of topological $\mathbb{T}$-duality [22] (see Theorem 4.12 and Remark 4.13). In this article we have decided to change some terminology used previously (also by the author) in order to align ourselves with the conventions in topology. For the benefit of the reader we record them here:

- $\mathtt{NSH}^{\mathrm{op}}$ = noncommutative stable homotopy category,
- $(\mathtt{NSH}^f)^{\mathrm{op}}$ = homotopy category of noncommutative finite spectra,
- $\mathtt{NSH}(\mathbb{C}, A)$ = noncommutative stable cohomotopy of A,
- $\mathtt{NSH}(A, \mathbb{C})$ = noncommutative stable homotopy of A.

Notations and conventions: Throughout this article $\hat{\otimes}$ will denote the maximal C^*-tensor product. All C^*-algebras are assumed to be separable unless otherwise stated. For any ∞-category $\mathcal{C}$ we denote by $\mathtt{h}\mathcal{C}$ its homotopy category. A functor between ∞-categories will implicitly mean an ∞-functor, i.e., a map of underlying simplicial sets.

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1. The symmetric monoidal structure and localizations of $\mathtt{SC}^*_\infty$

Recall from [20] that there is an ∞-category of noncommutative pointed spaces $\mathtt{N}\mathcal{S}_*$ as well as a stable ∞-category of noncommutative spectra $\mathtt{NSp}$, which is obtained after a localization of the stabilization of the ∞-category $\mathtt{N}\mathcal{S}_*$. In this section we construct a symmetric monoidal structure on $\mathtt{N}\mathcal{S}_*$ (resp. $\mathtt{NSp}$) generalizing the smash product of pointed finite CW complexes (resp. finite spectra).

Let $\mathtt{Fin}_*$ denote the category, whose objects are pointed sets $\langle n\rangle = \{*, 1, \cdots, n\}$ with $*$ being the basepoint and whose morphisms are pointed maps. Let $\mathrm{N}(\mathtt{Fin}_*)$ denote its nerve. A *symmetric monoidal* ∞*-category* $\mathcal{C}^\otimes$ is a coCartesian fibration of simplicial sets $p : \mathcal{C}^\otimes \to \mathrm{N}(\mathtt{Fin}_*)$ with the property: for each $n \geqslant 0$ there is an equivalence $\mathcal{C}^\otimes_{\langle n\rangle} \simeq (\mathcal{C}^\otimes_{\langle 1\rangle})^n$ induced by the maps $\{\rho^i : \langle n\rangle \to \langle 1\rangle\}_{1\leqslant i\leqslant n}$. One should regard $\mathcal{C} := \mathcal{C}^\otimes_{\langle 1\rangle}$ as the ∞-category, which is symmetric monoidal. It is customary to work with the underlying symmetric monoidal category $\mathcal{C}$, leaving out the rest of the structure as implicitly understood. A symmetric monoidal ∞-category can also be regarded as a commutative monoid object in $\mathtt{Cat}_\infty$, which is the ∞-category of ∞-categories. For further details the readers may consult [17].

Proposition 1.1. The categories $\mathtt{SC}^*_\infty$ and $\mathtt{N}\mathcal{S}_* := \mathrm{Ind}({\mathtt{SC}^*_\infty}^{\mathrm{op}})$ are symmetric monoidal ∞-categories. Moreover, the tensor product functor $\otimes : \mathtt{N}\mathcal{S}_* \times \mathtt{N}\mathcal{S}_* \to \mathtt{N}\mathcal{S}_*$ preserves small colimits in each variable separately and $j : {\mathtt{SC}^*_\infty}^{\mathrm{op}} \to \mathtt{N}\mathcal{S}_*$ is symmetric monoidal.

Proof. It is well-known that the topological category $\mathtt{SC}^*$ is symmetric monoidal under the maximal C^*-tensor product $\hat{\otimes}$. As a consequence its topological nerve $\mathtt{SC}^*_\infty$ is a symmetric monoidal ∞-category. The symmetric monoidal structure on $\mathtt{SC}^*_\infty$ endows ${\mathtt{SC}^*_\infty}^{\mathrm{op}}$ with a symmetric monoidal structure $\otimes$ that is uniquely defined up to a contractible space of choices (see Remark 2.4.2.7 of [17]). Since $\otimes$ commutes with finite colimits in ${\mathtt{SC}^*_\infty}^{\mathrm{op}}$ and the symmetric monoidal structure extends to the Ind-completion $\mathtt{N}\mathcal{S}_* := \mathrm{Ind}({\mathtt{SC}^*_\infty}^{\mathrm{op}})$, all other assertions follow from Corollary 6.3.1.13 of ibid.. $\square$

Note that the ∞-category $\mathtt{N}\mathcal{S}_*$ is pointed and it follows from Proposition 6.3.2.11 of [17] that there is an equivalence $\mathtt{N}\mathcal{S}_* \otimes \mathcal{S}_* \simeq \mathtt{N}\mathcal{S}_*$.

Lemma 1.2. The stabilization $\mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ is a symmetric monoidal stable ∞-category and the ∞-functor $\Sigma^\infty : \mathtt{N}\mathcal{S}_* \to \mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ is symmetric monoidal.

Proof. Thanks to the previous Lemma, one way to argue is via the identification of stable ∞-categories $\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \simeq \mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} := \mathrm{Fun}^{\mathrm{R}}({\mathtt{N}\mathcal{S}_*}^{\mathrm{op}}, \mathtt{Sp})$ (see Example 6.3.1.22 of [17]; here the tensor product is taken in the category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$). Using the stabilization $\Sigma^\infty : \mathcal{S}_* \to \mathtt{Sp}$ of pointed spaces, the stabilization of noncommutative pointed spaces can be regarded as

$$\mathtt{N}\mathcal{S}_* \simeq \mathtt{N}\mathcal{S}_* \otimes \mathcal{S}_* \to \mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} \simeq \mathtt{Sp}(\mathtt{N}\mathcal{S}_*).$$

$\square$

Recall from [20] that there is an ∞-functor $\Pi^{\mathrm{op}} : \mathtt{SC}^{*\,\mathrm{op}}_{\infty} \to \mathtt{NSp}$. This arises as a composition of the following ∞-functors

$$\mathtt{SC}^{*\,\mathrm{op}}_{\infty} \xrightarrow{j} \mathtt{N}\mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \xrightarrow{L_S} S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) =: \mathtt{NSp}.$$

Here S is a strongly saturated collection generated by the image of the set of morphisms $S_0 = \{\mathrm{C}(f) \to \ker(f) \,|\, f : A \to B \text{ surjective in } \mathtt{SC}^*\}$ in $\mathtt{SC}^{*\,\mathrm{op}}_{\infty}$ under $\Sigma^\infty \circ j$. In ibid. the localization functor $L_S : \mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \to S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ was simply denoted by L. Let T be the strongly saturated collection generated by $j(S_0)$ inside $\mathtt{N}\mathcal{S}_*$. Thus we obtain an accessible localization $L_T : \mathtt{N}\mathcal{S}_* \to T^{-1}\mathtt{N}\mathcal{S}_*$ with respect to T.

Proposition 1.3. The localization functor $L_T : \mathtt{N}\mathcal{S}_* \to T^{-1}\mathtt{N}\mathcal{S}_*$ is a symmetric monoidal ∞-functor between symmetric monoidal ∞-categories.

Proof. By Proposition 2.2.1.9 and Example 2.2.1.7 of [17] (see also Lemma 3.4 of [12]) we need to verify that for any L_T-equivalence $g : X \to Y$ and any $Z \in \mathtt{N}\mathcal{S}_*$ the induced map $g \otimes \mathrm{id}_Z : X \otimes Z \to Y \otimes Z$ is also an L_T-equivalence. Since T is by construction a strongly saturated collection, the L_T-equivalences precisely coincide with T (see Proposition 5.5.4.15 of [18]). Using the exactness of the maximal C^*-tensor product one can check the following: if $\theta(f) : \ker(f) \to \mathrm{C}(f)$ is the canonical map in $\mathtt{SC}^*_\infty$ for any surjection $f : A \to B$ in $\mathtt{SC}^*$, then for any $C \in \mathtt{SC}^*$ the map $\theta(f) \otimes \mathrm{id}_C : \ker(f)\hat{\otimes}C \to \mathrm{C}(f)\hat{\otimes}C$ is the same as $\theta(f \otimes \mathrm{id}_C) : \ker(f \otimes \mathrm{id}_C) \to \mathrm{C}(f \otimes \mathrm{id}_C)$. Thus we have shown that for any $\theta(f)^{\mathrm{op}} \in j(S_0)$ and any $C \in \mathtt{SC}^{*\,\mathrm{op}}_{\infty}$ the map $\theta(f)^{\mathrm{op}} \otimes \mathrm{id}_C \in j(S_0) \subset T$. Since $\otimes$ commutes with small colimits in $\mathtt{N}\mathcal{S}_*$ the same holds for all $Z \in \mathtt{N}\mathcal{S}_*$ from Definition 5.5.4.5 part (2) of [18], i.e., for any $g \in j(S_0)$ and any $Z \in \mathtt{N}\mathcal{S}_*$ the map $g \otimes \mathrm{id}_Z \in T$. The rest follows from the explicit construction of the strongly saturated collection T from $j(S_0)$. $\square$

Corollary 1.4. The stable ∞-category $T^{-1}\mathtt{N}\mathcal{S}_*\otimes\mathtt{Sp}$ is symmetric monoidal and the canonical ∞-functor

$$\mathtt{N}\mathcal{S}_* \simeq \mathtt{N}\mathcal{S}_* \otimes \mathcal{S}_* \xrightarrow{L_T\otimes\mathrm{id}} T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathcal{S}_* \xrightarrow{\mathrm{id}\otimes\Sigma^\infty} T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp}$$

is symmetric monoidal.

Theorem 1.5. There is an equivalence of stable ∞-categories $\mathtt{NSp} \simeq T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp}$.

Proof. The ∞-category $T^{-1}\mathtt{N}\mathcal{S}_*$ is presentable and one has an equivalence $\mathtt{Sp}(T^{-1}\mathtt{N}\mathcal{S}_*) \simeq T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp}$ (see Example 6.3.1.22 of [17]). Thus it suffices to show that $\mathtt{Sp}(T^{-1}\mathtt{N}\mathcal{S}_*) \simeq S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) =: \mathtt{NSp}$. Using Corollary 1.4.2.23 of ibid. one obtains the dotted ∞-functor F (unique up to equivalence) making the following diagram commute

$$\begin{tikzcd}
\mathtt{N}\mathcal{S}_* \arrow[d, "L_T"] & & \mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \arrow[ll, "\Omega^\infty_*"'] \arrow[d, dashed, "F"] \\
T^{-1}\mathtt{N}\mathcal{S}_* & & \mathtt{Sp}(T^{-1}\mathtt{N}\mathcal{S}_*). \arrow[ll, "\Omega^\infty_*"']
\end{tikzcd}$$

Using the characterization of localization (see Proposition 5.2.7.12 of [18]) one concludes that the there is a unique factorization $\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \xrightarrow{L_S} S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \xrightarrow{\overline{F}} \mathtt{Sp}(T^{-1}\mathtt{N}\mathcal{S}_*)$ making the following diagram commute up to equivalence:

$$\begin{tikzcd}
\mathtt{Sp}(\mathtt{N}\mathbb{S}_*) \arrow[r, "L_S"] \arrow[d, "F"'] & S^{-1}\mathtt{Sp}(\mathtt{N}\mathbb{S}_*) \arrow[dl, dashed, "\overline{F}"] \\
\mathtt{Sp}(T^{-1}\mathtt{N}\mathbb{S}_*) &
\end{tikzcd}$$

with $\overline{F}$ exact. Using the same characterization of localization one obtains below the dotted ∞-functor G (unique up to equivalence) making the following diagram commute:

$$\begin{tikzcd}
\mathtt{N}\mathbb{S}_* \arrow[r, "\Sigma^\infty"] \arrow[d, "L_T"] & \mathtt{Sp}(\mathtt{N}\mathbb{S}_*) \arrow[d, "L_S"] \\
T^{-1}\mathtt{N}\mathbb{S}_* \arrow[r, dashed, "G"] & S^{-1}\mathtt{Sp}(\mathtt{N}\mathbb{S}_*).
\end{tikzcd}$$

It follows from Corollary 1.4.4.5 of [17] that up to equivalence there is again a unique exact functor $\overline{G} : \mathtt{Sp}(T^{-1}\mathtt{N}\mathbb{S}_*) \to S^{-1}\mathtt{Sp}(\mathtt{N}\mathbb{S}_*)$ such that the following diagram commutes:

$$\begin{tikzcd}
T^{-1}\mathtt{N}\mathbb{S}_* \arrow[r, "\Sigma^\infty"] \arrow[d, "G"'] & \mathtt{Sp}(T^{-1}\mathtt{N}\mathbb{S}_*) \arrow[dl, dashed, "\overline{G}"] \\
S^{-1}\mathtt{Sp}(\mathtt{N}\mathbb{S}_*). &
\end{tikzcd}$$

The ∞-functors $\overline{F}$ and $\overline{G}$ are inverse equivalences of stable ∞-categories. □

Definition 1.6. The stable presentable symmetric monoidal ∞-category of noncommutative spectra is by definition

$$\mathtt{NSp} := T^{-1}\mathtt{N}\mathbb{S}_* \otimes \mathtt{Sp},$$

equipped with the symmetric monoidal stabilization functor $\Sigma_T^\infty : \mathtt{N}\mathbb{S}_* \stackrel{L_T}{\to} T^{-1}\mathtt{N}\mathbb{S}_* \stackrel{\Sigma^\infty}{\to} \mathtt{NSp}$.

Remark 1.7. Thanks to the identification $S^{-1}\mathtt{Sp}(\mathtt{N}\mathbb{S}_*) \simeq T^{-1}\mathtt{N}\mathbb{S}_* \otimes \mathtt{Sp}$ in the above Theorem 1.5, our new definition of $\mathtt{NSp}$ is backward compatible with the one in Definition 4.18 of [20].

Remark 1.8. Since ${\mathtt{hSC}^*_\infty}^{\mathrm{op}}$ is the homotopy category of noncommutative pointed (compact metrizable) spaces, it seems very natural to consider $\mathtt{NSH}^{\mathrm{op}}$ as its suspension stabilization. Thus we propose to (re)define

$$\mathtt{NSH}^{\mathrm{op}} = \text{ noncommutative stable homotopy category},$$

deviating from the terminology in [30, 20, 21]. Naturally we refer to its triangulated subcategory $(\mathtt{NSH}^f)^{\mathrm{op}}$ as the homotopy category of noncommutative finite spectra; see Definition 2.1 of [21], where $\mathtt{NSH}^f$ was called the homotopy category of noncommutative finite spectra.

Corollary 1.9. The homotopy category of noncommutative spectra $\mathtt{hNSp}$ is a tensor triangulated category, containing $\mathtt{NSH}^{\mathrm{op}}$ as a full tensor triangulated subcategory. It also contains $(\mathtt{NSH}^f)^{\mathrm{op}}$ as a full tensor triangulated subcategory.

2. Localizations of $\mathtt{SC}^*_\infty$

A separable unital C^*-algebra $\mathcal{D}$ ($\mathcal{D} \neq \mathbb{C}$) is called *strongly self-absorbing* if the there is an isomorphism $\phi : \mathcal{D} \to \mathcal{D}\hat{\otimes}\mathcal{D}$ that is approximately unitarily equivalent to $\mathrm{id}_\mathcal{D} \otimes \mathbf{1}_\mathcal{D}$ [31]. In ibid. the authors introduced and conducted an elaborate study of strongly self-absorbing C^*-algebras mainly with applications to the Elliott's Classification Program in mind. We are going to use these C^*-algebras to construct interesting (co)localizations of noncommutative spaces and spectra.

Remark 2.1. In [9] the authors showed the for any strongly self-absorbing C^*-algebra $\mathcal{D}$ the map $\mathrm{id}_\mathcal{D} \otimes \mathbf{1}_\mathcal{D}$ is homotopic to an isomorphism $\phi : \mathcal{D} \to \mathcal{D}\hat{\otimes}\mathcal{D}$. In ibid. the result was asserted under the K_1-injectivity condition, which later turned out to be redundant (see Remark 3.3. of [33]).

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. Then a map $e : \mathbf{1} \to E$ exhibits E as an *idempotent object* if $\mathrm{id}_E \otimes e : E \simeq E \otimes \mathbf{1} \to E \otimes E$ is an equivalence in $\mathcal{C}$ (see, for instance, Definition 6.3.2.1 of [17]). We immediately observe

Lemma 2.2. Any strongly self-absorbing C^*-algebra $\mathcal{D}$ is an idempotent object in $\mathtt{SC}^*_\infty$. The same assertion holds for $\mathbb{K}$.

Proof. For a strongly self-absorbing C^*-algebra $\mathcal{D}$ the canonical unital $*$-homomorphism $\mathbb{C} \to \mathcal{D}$ exhibits it as an idempotent object in $\mathtt{SC}^*_\infty$ (see Remark 2.1). For $\mathbb{K}$ the map $\mathbb{C} \to \mathbb{K}$ sending $1 \mapsto e_{11}$ exhibits $\mathbb{K}$ as an idempotent object in $\mathtt{SC}^*_\infty$. $\square$

Remark 2.3. If $E \in \mathcal{C}$ is an idempotent object, then $L_E : \mathcal{C} \to \mathcal{C}$ of the form $L_E(X) = - \otimes E$ is a localization. In [12] the authors called localizations $L_E : \mathcal{C} \to \mathcal{C}$ of the form $L_E(X) = - \otimes E$ for some $E \in \mathcal{C}$ *smashing localizations* in keeping with the terminology prevalent in stable homotopy theory. Any smashing localization $L_E : \mathcal{C} \to \mathcal{C}$ is compatible with the symmetric monoidal structure on $\mathcal{C}$ and, in fact, $L_E\mathcal{C}$ inherits a symmetric monoidal structure from $\mathcal{C}$, such that $L_E : \mathcal{C} \to L_E\mathcal{C}$ becomes symmetric monoidal (see Proposition 2.2.1.9 and Proposition 6.3.2.7 of [17]). By abuse of notation we are sometimes going to drop the object E from the smashing localization L_E and denote it simply by L.

Example 2.4. Smashing localizations of the ∞-category of separable C^*-algebras $\mathtt{SC}^*_\infty$ produces interesting results. By definition $\mathtt{SC}^*_\infty$ is opposite to the ∞-category of noncommutative pointed compact Hausdorff spaces. We present a few pertinent examples here.

(1) If $L(A) = A \otimes \mathbb{K}$, then we denote the smashing localization $L\mathtt{SC}^*_\infty$ by $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$. It is the ∞-category of *C^*-stable C^*-algebras.* For finite pointed CW complexes (X, x) and (Y, y) the homotopy set $\mathtt{hSC}^*_\infty[\mathbb{K}^{-1}](L(\mathrm{C}(X, x)), L(\mathrm{C}(Y, y)))$ is the connective E-theory group denoted by $\mathrm{kk}((Y, y),(X, x))$ in [8] (see Remark 2.11 below).

(2) If $L(A) = A \otimes \mathcal{D}$, where $\mathcal{D}$ is a strongly self-absorbing C^*-algebra, then we denote the smashing localization $L\mathtt{SC}^*_\infty$ by $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$. We refer to it as the ∞-category of *$\mathcal{D}$-stable C^*-algebras.* From the perspective of Elliott's Classification Program the ∞-category $\mathtt{SC}^*_\infty[\mathcal{Z}^{-1}]$ would be the most interesting localization, where $\mathcal{Z}$ is the Jiang–Su algebra. We call it the ∞-category of *$\mathcal{Z}$-stable C^*-algebras.*

(3) If $\mathcal{D} = \mathcal{O}_\infty$ we call $\mathtt{SC}^*_\infty[\mathcal{O}_\infty^{-1}]$ the ∞-category of *strongly purely infinite C^*-algebras.* The suspension stable version of this category will be analysed in the next section.

Proposition 2.5. Let us suppose that there is a unital embedding $\iota_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}'$ of strongly self-absorbing C^*-algebras. Then $\mathcal{D}'$ is an idempotent object in $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$.

Proof. Consider the following commutative diagram in $\mathtt{SC}^*$

$$\begin{tikzcd} \mathcal{D}' \arrow[rr, "\mathrm{id}_{\mathcal{D}'}\otimes \mathbf{1}_{\mathcal{D}'}"] \arrow[dr, "\mathrm{id}_{\mathcal{D}'}\otimes \mathbf{1}_{\mathcal{D}}"'] & & \mathcal{D}'\hat{\otimes}\mathcal{D}' \\ & \mathcal{D}'\hat{\otimes}\mathcal{D}. \arrow[ur, "\mathrm{id}_{\mathcal{D}'}\otimes \iota_{\mathcal{D}}"'] & \end{tikzcd}$$

Since $\mathcal{D}'$ is strongly self-absorbing $\mathrm{id}_{\mathcal{D}'} \otimes \mathbf{1}_{\mathcal{D}'}$ is homotopic to an isomorphism $\mathcal{D}' \to \mathcal{D}'\hat{\otimes}\mathcal{D}'$. It follows from Proposition 5.12 of [31] that $\mathrm{id}_{\mathcal{D}'} \otimes \mathbf{1}_{\mathcal{D}}$ is homotopic to an isomorphism $\mathcal{D}' \to \mathcal{D}'\hat{\otimes}\mathcal{D}$ demonstrating that $\mathcal{D}'$ is $\mathcal{D}$-stable. It follows that $\mathrm{id}_{\mathcal{D}'} \otimes \iota_{\mathcal{D}}$ is a homotopy equivalence. Observe that the unit object in $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$ is $\mathcal{D}$. Thus the unital embedding $\iota_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}'$ exhibits $\mathcal{D}'$ as an idempotent object in $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$. ☐

Corollary 2.6. In the localized ∞-category $\mathtt{SC}^*_\infty[\mathcal{Z}^{-1}]$ every strongly self-absorbing C^*-algebra is an idempotent object.

Proof. The assertion follows from the characterization of $\mathcal{Z}$ as the initial object in the homotopy category of strongly self-absorbing C^*-algebras with unital $*$-homomorphisms (see Corollary 3.2 of [33]). ☐

Remark 2.7. In view of the above Corollary one may construct $\mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$ for any strongly self-absorbing C^*-algebra $\mathcal{D}$ as a localization of $\mathtt{SC}^*_\infty[\mathcal{Z}^{-1}]$. Thus isomorphisms in $\mathtt{SC}^*_\infty[\mathcal{Z}^{-1}]$ contain the most refined information amongst all smashing localizations with respect to strongly self-absorbing C^*-algebras.

For any $A, B \in \mathtt{SC}^*$ we denote by $[A, B]$ the homotopy classes of $*$-homomorphisms $A \to B$.

Proposition 2.8. For any $A, B \in \mathtt{SC}^*$ and any strongly self absorbing C^*-algebra $\mathcal{D}$ there is a natural isomorphism

$$\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L(A), L(B)) \cong [A, B\hat{\otimes}\mathcal{D}].$$

Proof. Let us first observe that there is an identification

$$\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L(A), L(B)) \cong \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D}).$$

There is an element $\theta_A = \mathrm{id}_A \otimes \mathbf{1}_{\mathcal{D}} \in \mathtt{SC}^*(A, A\hat{\otimes}\mathcal{D})$ sending $a \mapsto a \otimes \mathbf{1}_{\mathcal{D}}$. This induces a map

$$K : \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D}) \to \mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D})$$

by precomposing with $[\theta_A]$ (here $[-]$ denotes the homotopy class). Using the fact that $\mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}\hat{\otimes}\mathcal{D}$ is homotopic to an isomorphism $\gamma : \mathtt{SC}^*(\mathcal{D}, \mathcal{D}\hat{\otimes}\mathcal{D})$, we deduce that the map $\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}$ is homotopic to an isomorphism $\gamma_B \in \mathtt{SC}^*(B\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D}\hat{\otimes}\mathcal{D})$. Now we define a map

$$L : \mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D}) \to \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D})$$

as follows: $L([\phi]) = [\gamma_B^{-1} \circ (\phi \otimes \mathrm{id}_{\mathcal{D}})]$. Observe that $K \circ L([\phi]) = [\gamma_B^{-1} \circ (\phi \otimes \mathrm{id}_{\mathcal{D}})] \circ [\theta_A] = [\gamma_B^{-1} \circ (\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}) \circ \phi]$. Since $[\mathrm{id}_B \otimes \mathrm{id}_{\mathcal{D}} \otimes \mathbf{1}_{\mathcal{D}}] = [\gamma_B]$ the composition $K \circ L = \mathrm{id} : \mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D}) \to \mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D})$.

Now $L \circ K([\psi]) = L([\psi \circ \theta_A]) = [\gamma_B^{-1} \circ ((\psi \circ \theta_A) \otimes \mathrm{id}_{\mathcal{D}})]$. Let $\tau_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ denote the tensor flip map, which is also homotopic to the identity. A verification on the simple tensors demonstrates that $[(\mathrm{id}_B \otimes \tau_{\mathcal{D}}) \circ ((\psi \circ \theta_A) \otimes \mathrm{id}_{\mathcal{D}})] = [\gamma_B \circ \psi]$. It follows that $L \circ K = \mathrm{id} : \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D}) \to \mathtt{hSC}^*_\infty(A\hat{\otimes}\mathcal{D}, B\hat{\otimes}\mathcal{D})$. It remains to observe that $\mathtt{hSC}^*_\infty(A, B\hat{\otimes}\mathcal{D}) \cong [A, B\hat{\otimes}\mathcal{D}]$ (see [20]). ☐

Observe that the subset $\{s_i s_j^* \,|\, i, j \in \mathbb{N}\} \subset \mathcal{O}_\infty$ generates a copy of the compact operators $\mathbb{K}$ inside $\mathcal{O}_\infty$. Let $\iota : \mathbb{K} \to \mathcal{O}_\infty$ denote the canonical inclusion.

Proposition 2.9. In the C^*-stable ∞-category $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$ the map $\iota : \mathbb{K} \to \mathcal{O}_\infty$ exhibits $\mathcal{O}_\infty$ as an idempotent object.

Proof. Consider the diagram $\mathcal{O}_\infty \xrightarrow{\theta} \mathcal{O}_\infty\hat{\otimes}\mathbb{K} \xrightarrow{\phi} \mathcal{O}_\infty\hat{\otimes}\mathcal{O}_\infty$ in $\mathtt{SC}^*$. The map θ sends $a \mapsto a \otimes e_{11}$ and the map $\phi = \mathrm{id}_{\mathcal{O}_\infty} \otimes \iota$. The composite $\phi \circ \theta$ is homotopic to $\mathrm{id}_{\mathcal{O}_\infty} \otimes \mathbf{1}_{\mathcal{O}_\infty} : \mathcal{O}_\infty \to \mathcal{O}_\infty\hat{\otimes}\mathcal{O}_\infty$, whence it is an equivalence in $\mathtt{SC}^*_\infty$. The map θ is an equivalence in $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$. It follows that $\phi = \mathrm{id}_{\mathcal{O}_\infty} \otimes \iota$ is an equivalence in $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$. ☐

Corollary 2.10. The ∞-category $\mathtt{SC}^*_\infty[\mathcal{O}_\infty^{-1}]$ can be obtained as a localization of $\mathtt{SC}^*_\infty[\mathbb{K}^{-1}]$.

Remark 2.11. It is well-known that $\mathtt{hSC}^*_\infty[\mathbb{K}^{-1}](A, B) \cong [A, B\hat{\otimes}\mathbb{K}]$. Isomorphisms in $\mathtt{hSC}^*_\infty[\mathbb{K}^{-1}]$ between C^*-algebras of the form $\mathrm{C}(X, x)\hat{\otimes}\mathbb{K}$, where (X, x) is a finite pointed CW complex, can be detected in terms of connective kk-theory (see Theorem 2.4 of [8]). The connective kk-theory should not be confused with Cuntz kk-theory for m-algebras (or locally convex algebras).

Corollary 2.12. Consider the following problem: Given two finite pointed CW complexes (X, x) and (Y, y) are the C^*-algebras $\mathrm{C}(X, x)\hat{\otimes}\mathcal{O}_\infty$ and $\mathrm{C}(Y, y)\hat{\otimes}\mathcal{O}_\infty$ homotopy equivalent? In view of the above Remark 2.11 a sufficient criterion can be obtained in terms of connective kk-theory. Homotopy equivalences of matrix bundles can also be detected by connective E-theory [30].

Now we demonstrate that the homotopy category of the smashing localization $\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}]$ admits a universal characterization much like KK-theory. The localization ∞-functor $L_{\mathcal{D}} : \mathtt{SC}^*_\infty \to \mathtt{SC}^*_\infty[\mathcal{D}^{-1}]$ induces a canonical (ordinary) functor $L_{\mathcal{D}} : \mathtt{SC}^* \to \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}]$. Recall that a functor $F : \mathtt{SC}^* \to \mathcal{C}$ ($\mathcal{C}$ an ordinary category) is called $\mathcal{D}$-*stable* if F sends the morphism $A \to A\hat{\otimes}\mathcal{D}$ mapping $a \mapsto a \otimes \mathbf{1}_{\mathcal{D}}$ to an isomorphism in $\mathcal{C}$ for all $A \in \mathtt{SC}^*$.

Theorem 2.13. The functor $L_{\mathcal{D}} : \mathtt{SC}^* \to \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}]$ is the universal homotopy invariant and $\mathcal{D}$-stable functor on $\mathtt{SC}^*$.

Proof. Let us first show that functor $L_{\mathcal{D}}$ is homotopy invariant and $\mathcal{D}$-stable. It is easy to verify that it is homotopy invariant. It follows from the arguments in the proof of Proposition 2.8 that the map $\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L_{\mathcal{D}}(A\hat{\otimes}\mathcal{D}), L_{\mathcal{D}}(B)) \to \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L_{\mathcal{D}}(A), L_{\mathcal{D}}(B))$ induced by $A \to A\hat{\otimes}\mathcal{D}$ is an isomorphism for all $B \in \mathtt{SC}^*$. For any $B \in \mathtt{SC}^*$ the map

$$\mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L_{\mathcal{D}}(B), L_{\mathcal{D}}(A)) \to \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}](L_{\mathcal{D}}(B), L_{\mathcal{D}}(A\hat{\otimes}\mathcal{D}))$$

is equivalent to that map $[B, A\hat{\otimes}\mathcal{D}] \to [B, A\hat{\otimes}\mathcal{D}\hat{\otimes}\mathcal{D}]$ once again by Proposition 2.8. This map is induced by $A\hat{\otimes}\mathcal{D} \to A\hat{\otimes}\mathcal{D}\hat{\otimes}\mathcal{D}$ sending $a \otimes d \mapsto a \otimes \mathbf{1}_{\mathcal{D}} \otimes d$. Since $\mathcal{D}$ is strongly self-absorbing one easily sees $[B, A\hat{\otimes}\mathcal{D}] \to [B, A\hat{\otimes}\mathcal{D}\hat{\otimes}\mathcal{D}]$ is an isomorphism. Since $L_{\mathcal{D}}$ is surjective on objects we conclude that $L_{\mathcal{D}}$ is $\mathcal{D}$-stable.

Let $F_i : \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}] \to \mathcal{C}$ with $i = 1, 2$ be two functors making the following diagram commute

(1)
$$\begin{tikzcd} \mathtt{SC}^* \arrow[rr, "L_{\mathcal{D}}"] \arrow[dr, "F"'] & & \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}] \arrow[dl, dashed, "F_i"] \\ & \mathcal{C}. & \end{tikzcd}$$

On objects they are both determined by $\mathcal{D}$-stability $F_i(A\hat{\otimes}\mathcal{D}) \cong F(A\hat{\otimes}\mathcal{D}) \cong F(A)$. Similarly, on each morphism $\phi : A\hat{\otimes}\mathcal{D} \to B\hat{\otimes}\mathcal{D}$ the value of $F_i(\phi)$ is uniquely determined by the following diagram:

$$\begin{tikzcd} F_i(A\hat{\otimes}\mathcal{D}) \arrow[r, "F_i(\phi)"] & F_i(B\hat{\otimes}\mathcal{D}) \\ F(A) \arrow[u, "\cong"] \arrow[r, "F(\phi)"] & F(B). \arrow[u, "\cong"'] \end{tikzcd}$$

For the existence note that for any homotopy invariant and $\mathcal{D}$-stable functor $F : \mathtt{SC}^* \to \mathcal{C}$ there is a functor $\overline{F} : \mathtt{hSC}^*_\infty[\mathcal{D}^{-1}] \to \mathcal{C}$ sending $A\hat{\otimes}\mathcal{D}$ to $F(A\hat{\otimes}\mathcal{D}) \cong F(A)$ that makes the above diagram (1) commute (up to a natural isomorphism). $\square$

3. Coidempotent objects and colocalizations of $\mathtt{NSp}$

Let us remind the readers that the ∞-functor $\Pi^{\mathrm{op}} : \mathtt{SC}^{*\,\mathrm{op}}_\infty \to \mathtt{NSp}$ arises as a composition of the following ∞-functors

$$\mathtt{SC}^{*\,\mathrm{op}}_\infty \xrightarrow{j} \mathtt{N}\mathcal{S}_* \xrightarrow{L_T} T^{-1}\mathtt{N}\mathcal{S}_* \xrightarrow{\Sigma^\infty} \mathtt{Sp}(T^{-1}\mathtt{N}\mathcal{S}_*) \simeq T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} =: \mathtt{NSp}.$$

For any separable C^*-algebra A one ought to regard $\Pi^{\mathrm{op}}(A)$ as its suspension spectrum after localization with respect to T. Hence we are going to reset $\Sigma^\infty_T A := \Pi^{\mathrm{op}}(A)$. Owing to the symmetric monoidal structure on $\mathtt{NSp}$ that we established earlier, one may consider the endofunctor $-\otimes \Sigma^\infty_T A : \mathtt{NSp} \to \mathtt{NSp}$ for any $A \in \mathtt{SC}^{*\,\mathrm{op}}_\infty$.

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. We say that a map $e : E \to \mathbf{1}$ exhibits E as a *coidempotent object* in $\mathcal{C}$ if the dual map $e^{\mathrm{op}} : \mathbf{1} \to E$ exhibits E as an idempotent object in $\mathcal{C}^{\mathrm{op}}$. Recall that the symmetric monoidal structure on $\mathcal{C}$ endows $\mathcal{C}^{\mathrm{op}}$ with a symmetric monoidal structure that is uniquely defined up to a contractible space of choices.

Lemma 3.1. If $\mathcal{D}$ is a strongly self-absorbing C^*-algebra, then $j(\mathcal{D})$ is a coidempotent object in $\mathtt{N}\mathcal{S}_*$. The same assertion holds for $\mathbb{K}$, i.e., $j(\mathbb{K})$ is a coidempotent object in $\mathtt{N}\mathcal{S}_*$.

Proof. Let X stand for $\mathcal{D}$ or $\mathbb{K}$. Since X is an idempotent object in $\mathtt{SC}^*_\infty$, it becomes a coidempotent object in $\mathtt{SC}^{*\,\mathrm{op}}_\infty$. Consequently, $j(X)$ becomes a coidempotent object in $\mathtt{N}\mathcal{S}_*$ (since $j : \mathtt{SC}^{*\,\mathrm{op}}_\infty \to \mathtt{N}\mathcal{S}_*$ is a fully faithful symmetric monoidal ∞-functor). $\square$

Lemma 3.2. For any strongly self-absorbing C^*-algebra $\mathcal{D}$, the stabilization $\Sigma^\infty_T\mathcal{D}$ is a coidempotent object in $\mathtt{NSp}$. The same assertion holds for $\mathbb{K}$, i.e., $\Sigma^\infty_T\mathbb{K}$ is a coidempotent object in $\mathtt{NSp}$.

Proof. Since $\Sigma^\infty : \mathtt{N}\mathcal{S}_* \to \mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ and $L_T : \mathtt{Sp}(\mathtt{N}\mathcal{S}_*) \simeq \mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} \to T^{-1}\mathtt{N}\mathcal{S}_* \otimes \mathtt{Sp} \simeq S^{-1}\mathtt{Sp}(\mathtt{N}\mathcal{S}_*)$ are both symmetric monoidal ∞-functors, the assertion follows from the previous Lemma. $\square$

Recall that an ∞-functor $R : \mathcal{C} \to \mathcal{C}$ is called a *colocalization* if $R : \mathcal{C} \to R\mathcal{C}$ is the right adjoint to the inclusion $R\mathcal{C} \subset \mathcal{C}$; in particular, the inclusion is the left adjoint to R and hence preserves all small colimits.

Proposition 3.3. Let A be a strongly self-absorbing C^*-algebra or $\mathbb{K}$. The ∞-functors $R_1 : \mathtt{N}\mathcal{S}_* \to \mathtt{N}\mathcal{S}_*$ and $R_2 : \mathtt{NSp} \to \mathtt{NSp}$ given by $R_1(X) = X \otimes j(A)$ and $R_2(X) = X \otimes \Sigma^\infty_T A$ are colocalization functors.

Proof. The assertions follow from the dual of Proposition 6.3.2.4 of [17]. $\square$

3.1. **Colocalizations and purely infinite strongly self absorbing C^*-algebras.** The list of known examples of strongly self-absorbing C^*-algebras is rather limited. The list includes Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the Jiang–Su algebra $\mathcal{Z}$, UHF algebras of infinite type, and tensor products of $\mathcal{O}_\infty$ with UHF algebras of infinite type. It follows from the results of Kirchberg that strongly self-absorbing C^*-algebras are either stably finite or purely infinite. In the purely infinite case Toms–Winter completely classified all strongly self-absorbing C^*-algebras satisfying UCT (Corollary page 4022 [31]), viz., they are $\mathcal{O}_2$, $\mathcal{O}_\infty$ and tensor products of $\mathcal{O}_\infty$ with UHF algebras of infinite type. We are particularly interested in the purely infinite ones since $ax+b$-semigroup C^*-algebras of number rings are all purely infinite (Corollary 8.2.11 of [6]). Among the strongly self-absorbing purely infinite C^*-algebras $\mathcal{O}_\infty$ plays a distinguished role in the classification program. The C^*-algebra $A\hat{\otimes}\mathcal{O}_\infty$ is purely infinite for any $A \in \mathtt{SC}^*$ [15]. Deviating slightly from the predictable pattern the colocalization of $\mathtt{NSp}$ by the ∞-functor $R_{\Sigma^\infty_T \mathcal{D}}(-) = - \otimes \Sigma^\infty_T \mathcal{D}$ is denoted by $\mathtt{NSp}[\mathcal{D}^{-1}]$ (and not by $\mathtt{NSp}[(\Sigma^\infty_T \mathcal{D})^{-1}]$). In what follows we are going to drop the object $\Sigma^\infty_T \mathcal{D}$ from the colocalization functor $R_{\Sigma^\infty_T \mathcal{D}}$ and denote it simply by R.

Thanks to Proposition 3.3 above one can study colocalizations of both $\mathtt{N}\mathcal{S}_*$ and $\mathtt{NSp}$ with respect to a strongly self-absorbing C^*-algebra $\mathcal{D}$ or $\mathbb{K}$. We are mostly interested in the (suspension) stable situation. Let us call the full ∞-subcategory of $\mathtt{NSp}$ (or of its colocalization) spanned by $\Sigma^\infty_T A$ for all $A \in \mathtt{SC}^*_\infty$ (or $\Sigma^\infty_T A$ followed by the the colocalization functor) the *C^*-core*. In the sequel we describe the homotopy category of the C^*-core of the colocalization of $\mathtt{NSp}$ when $\mathcal{D}$ is a purely infinite strongly self-absorbing C^*-algebra. We leave out the cases involving the stably finite ones for a future project.

Theorem 3.4. For any $A, B \in \mathtt{SC}^*$ there is a natural isomorphism

$$\mathtt{hNSp}[\mathcal{O}_\infty^{-1}](R(\Sigma^\infty_T A), R(\Sigma^\infty_T B)) \cong \mathrm{E}_0(B, A).$$

Proof. By construction there is a natural identification

$$\mathtt{hNSp}[\mathcal{O}_\infty^{-1}](R(\Sigma^\infty_T A), R(\Sigma^\infty_T B)) \cong \mathtt{hNSp}(\Sigma^\infty_T (A\hat{\otimes}\mathcal{O}_\infty), \Sigma^\infty_T (B\hat{\otimes}\mathcal{O}_\infty)),$$

where we used the fact that $\Sigma^\infty_T : \mathtt{hSC}^{*\,\mathrm{op}}_\infty \to \mathtt{hNSp}$ is symmetric monoidal (see Corollary 1.4). Now consider the canonical composition of $*$-homomorphisms $\mathbb{K} \hookrightarrow \mathcal{O}_\infty \xrightarrow{\theta} \mathcal{O}_\infty\hat{\otimes}\mathbb{K}$. Here $\theta : \mathcal{O}_\infty \to \mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ is the corner embedding $a \mapsto a \otimes e_{11}$. We ought to view this as a diagram $\mathbb{K} \leftarrow \mathcal{O}_\infty \leftarrow \mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ in $\mathtt{SC}^{*\,\mathrm{op}}_\infty$. Tensoring the diagram with A and applying $\Sigma^\infty_T(-)$ leads to the following diagram in $\mathtt{hNSp}$

$$\Sigma^\infty_T(A\hat{\otimes}\mathbb{K}) \leftarrow \Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty) \xleftarrow{\Theta} \Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}).$$

Now we apply the functor $\mathtt{hNSp}(-,\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty))$ to this diagram and use Theorem 4.25 of [20] to obtain

$$\begin{tikzcd}
\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathbb{K}),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[r, "\cong"] \arrow[d] & \mathtt{NSH}^{\mathrm{op}}(A\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty) \arrow[d] \\
\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[r, "\cong"] \arrow[d, "\Theta"] & \mathtt{NSH}^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty, B\hat{\otimes}\mathcal{O}_\infty) \arrow[d] \\
\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[r, "\cong"] & \mathtt{NSH}^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty).
\end{tikzcd}$$

Observe that for any $E,F\in\mathtt{SC}^*$ there is a natural map $\mathtt{NSH}(E,F)\to\mathrm{E}_0(E,F)$, which becomes an isomorphism as soon as F is stable (see Theorem 4.1.1. of [30]). Thus we may modify the above diagram as follows:

$$\begin{tikzcd}
\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathbb{K}),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[r, "\cong"] \arrow[d] & \mathrm{E}_0^{\mathrm{op}}(A\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty) \arrow[d] \\
\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[r] \arrow[d, "\Theta"] & \mathrm{E}_0^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty, B\hat{\otimes}\mathcal{O}_\infty) \arrow[d] \\
\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[r, "\cong"] & \mathrm{E}_0^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty).
\end{tikzcd}$$

Since the diagram $\mathbb{K}\hookrightarrow\mathcal{O}_\infty\xrightarrow{\theta}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ produces a E-equivalence, the right vertical composition is an isomorphism. It follows that the left vertical composition is also an isomorphism, i.e., the natural map

$$\Theta:\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty))\to\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty))$$

induced by $\theta:\mathcal{O}_\infty\to\mathcal{O}_\infty\hat{\otimes}\mathbb{K}$ is split surjective.

Now consider the composition of $*$-homomorphisms $\mathcal{O}_\infty\xrightarrow{\theta}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}\xrightarrow{\kappa}\mathcal{O}_\infty$ with $\kappa(a\otimes e_{ij})=s_ias_j^*$. Since $\kappa\circ\theta$ is homotopic to an isomorphism in $\mathtt{SC}^*$, the composition in the induced diagram in $\mathtt{hNSp}$ (after tensoring with A and applying $\Sigma^\infty_T(-)$)

$$\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty)\leftarrow\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K})\leftarrow\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty)$$

is an isomorphism in $\mathtt{hNSp}$. Applying the functor $\mathtt{hNSp}(-,\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty))$ we see that the dotted composite

$$\begin{tikzcd}
\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[r, "\Theta"] \arrow[dr, dashed] & \mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[d] \\
& \mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty))
\end{tikzcd}$$

must be an isomorphism. It follows that Θ is split injective and consequently an isomorphism. Now in the commutative diagram

$$\begin{tikzcd}
\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[r] \arrow[d, "\Theta"] & \mathrm{E}_0^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty, B\hat{\otimes}\mathcal{O}_\infty) \arrow[d] \\
\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}),\Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty)) \arrow[r, "\cong"] & \mathrm{E}_0^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathbb{K}, B\hat{\otimes}\mathcal{O}_\infty)
\end{tikzcd}$$

the right vertical arrow is an isomorphism due to the C^*-stability of E-theory, whence the top horizontal arrow must also be an isomorphism. Finally, we observe that

$$\mathrm{E}_0^{\mathrm{op}}(A\hat{\otimes}\mathcal{O}_\infty, B\hat{\otimes}\mathcal{O}_\infty) \cong \mathrm{E}_0(B, A)$$

due to the $\mathcal{O}_\infty$-stability of E-theory in both variables and all the identifications made thus far were natural. □

Remark 3.5. The above Theorem demonstrates that the colocalized ∞-category $\mathtt{NSp}[\mathcal{O}_\infty^{-1}]$ produces an ∞-categorical model for an enlarged version of the opposite of bivariant E-theory category. Of course, if the separable C^*-algebras in sight are nuclear, then one can replace E-theory by KK-theory.

Remark 3.6. An inspection of the proof of Theorem 3.4 demonstrates that actually a stronger result holds, viz.,

$$\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty),\Sigma^\infty_T B) \cong \mathtt{NSH}(B, A\hat{\otimes}\mathcal{O}_\infty) \cong \mathrm{E}_0(B, A)$$

for any $A, B \in \mathtt{SC}^*$.

Corollary 3.7. The nonconnective algebraic K-theory of $\mathcal{O}_\infty$-stable separable C^*-algebras factors through the essential image of $\Sigma^\infty_T : \mathtt{hSC}^{*\ \mathrm{op}}_\infty \to \mathtt{hNSp}[\mathcal{O}_\infty^{-1}]$.

Proof. It was shown in [19] that the nonconnective algebraic K-theory of $\mathcal{O}_\infty$-stable C^*-algebras agrees naturally with their topological K-theory. The assertion now follows since topological K-theory, which is naturally isomorphic to E-theory, has the desired property. □

Now let $\mathcal{Q}$ denote any UHF algebra of infinite type, so that $\mathcal{O}_\infty\hat{\otimes}\mathcal{Q}$ is a purely infinite strongly self-absorbing C^*-algebra.

Theorem 3.8. For any $A, B \in \mathtt{SC}^*$ there is a natural isomorphism

$$\mathtt{hNSp}[(\mathcal{O}_\infty\hat{\otimes}\mathcal{Q})^{-1}](R(\Sigma^\infty_T A), R(\Sigma^\infty_T B)) \cong \mathrm{E}_0(B\hat{\otimes}\mathcal{Q}, A\hat{\otimes}\mathcal{Q}).$$

Proof. As before we first observe that

$$\mathtt{hNSp}[(\mathcal{O}_\infty\hat{\otimes}\mathcal{Q})^{-1}](R(\Sigma^\infty_T A), R(\Sigma^\infty_T B)) \cong \mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathcal{Q}), \Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathcal{Q})).$$

Arguing as in the previous Theorem one then proves that

$$\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathcal{Q}), \Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_\infty\hat{\otimes}\mathcal{Q})) \cong \mathrm{E}_0(B\hat{\otimes}\mathcal{Q}, A\hat{\otimes}\mathcal{Q}).$$

□

Example 3.9. If $\mathcal{Q}$ is the universal UHF algebra, then the C^*-core of the colocalization of $\mathtt{NSp}$ by the ∞-functor $-\otimes\Sigma^\infty_T(\mathcal{O}_\infty\hat{\otimes}\mathcal{Q})$ produces an ∞-categorical model for the opposite of rationalized bivariant E-theory category. Indeed, it is well known that tensoring with the universal UHF algebra rationalizes E-theory, e.g., it follows from the Theorem in Section 3 of [9] that

$$\mathrm{E}_i(\mathcal{O}_\infty\hat{\otimes}\mathcal{Q}, A\hat{\otimes}\mathcal{Q}) \cong \mathrm{E}_i(A)\otimes_{\mathbb{Z}}\mathbb{Q} \quad \text{for } i = 0, 1.$$

Now we show that the colocalization of $\mathtt{NSp}$ by $-\hat{\otimes}\Sigma^\infty_T\mathcal{O}_2$ annihilates its C^*-core.

Theorem 3.10. For any $A, B \in \mathtt{SC}^*$ there is a natural isomorphism

$$\mathtt{hNSp}[\mathcal{O}_2^{-1}](R(\Sigma^\infty_T A), R(\Sigma^\infty_T B)) \cong 0.$$

Proof. Once again we first observe that

$$\mathtt{hNSp}[\mathcal{O}_2^{-1}](R(\Sigma^\infty_T A), R(\Sigma^\infty_T B)) \cong \mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_2), \Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_2)).$$

We also know from Theorem 4.25 of [20] that

$$\mathtt{hNSp}(\Sigma^\infty_T(A\hat{\otimes}\mathcal{O}_2), \Sigma^\infty_T(B\hat{\otimes}\mathcal{O}_2)) \cong \mathtt{NSH}(B\hat{\otimes}\mathcal{O}_2, A\hat{\otimes}\mathcal{O}_2).$$

Since $\mathcal{O}_2$ is properly infinite one can again find a diagram in $\mathtt{SC}^*$

$$\mathcal{O}_2 \to \mathcal{O}_2\hat{\otimes}\mathbb{K} \to \mathcal{O}_2,$$

such that the composition is homotopic to an isomorphism (see Proposition 1.1.2 of [24]). Tensoring the diagram with A we get another one

$$A\hat{\otimes}\mathcal{O}_2 \to A\hat{\otimes}\mathcal{O}_2\hat{\otimes}\mathbb{K} \to A\hat{\otimes}\mathcal{O}_2,$$

such that the composition is again homotopic to an isomorphism. Applying the homotopy functor $\mathtt{NSH}(B\hat{\otimes}\mathcal{O}_2, -)$ to the above diagram we find that $\mathtt{NSH}(B\hat{\otimes}\mathcal{O}_2, A\hat{\otimes}\mathcal{O}_2)$ is a summand of $\mathtt{NSH}(B\hat{\otimes}\mathcal{O}_2, A\hat{\otimes}\mathcal{O}_2\hat{\otimes}\mathbb{K}) \cong \mathrm{E}_0(B\hat{\otimes}\mathcal{O}_2, A\hat{\otimes}\mathcal{O}_2\hat{\otimes}\mathbb{K}) \cong \mathrm{E}_0(B\hat{\otimes}\mathcal{O}_2, A\hat{\otimes}\mathcal{O}_2)$. It suffices to show that $\mathrm{E}_0(B\hat{\otimes}\mathcal{O}_2, A\hat{\otimes}\mathcal{O}_2)$ vanishes. Since $\mathcal{O}_2$ is KK-contractible, so is $B\hat{\otimes}\mathcal{O}_2$ and hence it satisfies UCT. Thus one may identify $\mathrm{E}_0(B\hat{\otimes}\mathcal{O}_2, A\hat{\otimes}\mathcal{O}_2) \cong \mathrm{KK}_0(B\hat{\otimes}\mathcal{O}_2, A\hat{\otimes}\mathcal{O}_2)$ and the group $\mathrm{KK}_0(B\hat{\otimes}\mathcal{O}_2, A\hat{\otimes}\mathcal{O}_2)$ evidently vanishes. □

3.2. **Noncommutative stable cohomotopy of $ax+b$-semigroup C^*-algebras of number rings.** Number rings are central objects of study in number theory. To any number ring one can associate an $ax+b$-semigroup C^*-algebra that possesses very intriguing structure [5]. It is an important task to ascertain (co)homological invariants of these C^*-algebras. We begin with a disambiguation.

Disambiguation 3.11. In [20, 21] the author decided to call the groups $\mathtt{NSH}(\mathbb{C}, -)$ (resp. $\mathtt{NSH}(-, \mathbb{C})$) the noncommutative stable homotopy (resp. noncommutative stable cohomotopy) groups. The terminology was motivated by the fact that $\mathtt{NSH}(\mathbb{C}, -)$ is covariant and $\mathtt{NSH}(-, \mathbb{C})$ is contravariant. However, it was observed in ibid. that $\mathtt{NSH}(\mathbb{C}, -)$ generalizes stable cohomotopy, whereas $\mathtt{NSH}(-, \mathbb{C})$ generalizes stable homotopy of finite pointed CW complexes. In order to align the theory with the terminology familiar to topologists, we rename as follows:

$$\begin{aligned} \mathtt{NSH}(\mathbb{C}, -) &= \text{noncommutative stable cohomotopy} \\ \mathtt{NSH}(-, \mathbb{C}) &= \text{noncommutative stable homotopy} \end{aligned}$$

We also extend the terminology predictably to their graded versions.

Recently Li showed that for a countable integral domain R with vanishing Jacobson radical (which is, in addition, not a field) the left regular $ax+b$-semigroup C^*-algebra $C^*_\lambda(R \rtimes R^\times)$

is $\mathcal{O}_\infty$-stable, i.e., $C^*_\lambda(R \rtimes R^\times)\hat{\otimes}\mathcal{O}_\infty \cong C^*_\lambda(R \rtimes R^\times)$ (see Theorem 1.3 of [16]). Cuntz–Echterhoff–Li computed the topological K-theory of such $ax+b$-semigroup C^*-algebras in [6] as follows:

$$\mathrm{K}_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{[X]\in G\backslash\mathcal{J}} \mathrm{K}_*(C^*(G_X)), \tag{2}$$

where $\mathcal{J}$ is the set of fractional ideal of R, $G = K \rtimes K^\times$, and G_X is the stabilizer of X under the G-action on $\mathcal{J}$. The orbit space $G \setminus \mathcal{J}$ can be identified with the ideal class group of K.

Theorem 3.12. The noncommutative stable cohomotopy of the left regular $ax+b$-semigroup C^*-algebra of the ring of integers R of a number field K is 2-periodic and explicitly given by

$$\mathtt{NSH}(\mathbb{C}, C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{[X]\in G\backslash\mathcal{J}} \mathrm{K}_0(C^*(G_X)).$$

and

$$\mathtt{NSH}(\mathbb{C}, \Sigma C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{[X]\in G\backslash\mathcal{J}} \mathrm{K}_1(C^*(G_X)).$$

Proof. Since $C^*_\lambda(R \rtimes R^\times)$ is $\mathcal{O}_\infty$-stable, there is an identification of noncommutative stable cohomotopy $\mathtt{NSH}(\mathbb{C}, C^*_\lambda(R \rtimes R^\times)) \cong \mathtt{NSH}(\mathbb{C}, C^*_\lambda(R \rtimes R^\times)\hat{\otimes}\mathcal{O}_\infty)$. By Remark 3.6 we conclude that $\mathtt{NSH}(\mathbb{C}, C^*_\lambda(R \rtimes R^\times)\hat{\otimes}\mathcal{O}_\infty) \cong \mathrm{E}_0(\mathbb{C}, C^*_\lambda(R \rtimes R^\times))$. One may identify the E-theory of $C^*_\lambda(R \rtimes R^\times)$ naturally with its topological K-theory (of course, $C^*_\lambda(R \rtimes R^\times)$ is itself nuclear). The results now follow from Equation (2) (the second one after suspension). $\square$

4. Nonconnective KQ-theory: higher and lower nonunital K′-theory

We work exclusively in the category of nonunital (not necessarily unital) k-algebras, denoted by $\mathtt{Alg}_k$, where k is a field of characteristic zero as in [22]. The morphisms in $\mathtt{Alg}_k$ are k-algebra homomorphisms. At certain places in the sequel we are admittedly sloppy regarding size issues; however, as is common in K-theory there will always be a small skeleton that comes to our rescue.

4.1. **Stable ∞-category valued noncommutative motives.** For any k-algebra A let $\tilde{A}$ denote its k-unitization with underlying k-linear space $A \oplus k$ and multiplication $(a,\lambda)(a',\lambda') = (aa' + \lambda a' + \lambda' a, \lambda\lambda')$. The category $\mathtt{Mod}(\tilde{A})$ is an abelian category. In [22] we considered the following differential graded category $\mathtt{HPf}_{\mathtt{dg}}(A)$: its objects are cochain complexes Y of right $\tilde{A}$-modules such that Y is homotopy equivalent to complexes X satisfying

(1) X is homotopy equivalent to a strictly perfect complex,
(2) the canonical map $X \otimes_{\tilde{A}} A \to X$ is a homotopy equivalence.

A k-linear cochain complex worth of morphisms between two such objects is obtained in a standard manner. We are going to consider an ∞-categorical variant of $\mathtt{HPf}_{\mathtt{dg}}$-construction. There is a *differential graded nerve* N_{dg} of a differential graded category (see Construction 1.3.1.6 of [17]). For a differential graded category $\mathcal{C}$ as a simplicial set $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ can be described as follows:

- the 0-simplices are the objects of $\mathcal{C}$,
- the 1-simplices are $\cup_{X,Y\in\mathrm{Ob}(\mathcal{C})}\{f \in \mathcal{C}(X,Y)^0 \mid df = 0\}$.

In order to get an idea about the higher simplices let us note that the differential graded nerve $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ (of a homologically graded differential graded category $\mathcal{C}$) is obtained by applying the homotopy coherent nerve to a Kan complex enriched category constructed out of $\mathcal{C}$. The Kan complex enriched category is obtained by first applying the truncation $\tau_{\geqslant 0}$ to the mapping complexes in $\mathcal{C}$ and then applying the Dold–Kan construction.

Using the above construction we manufacture an ∞-category $\mathrm{C}_\infty(\tilde{A})$ out of the differential graded category of cochain complexes of $\mathtt{Mod}(\tilde{A})$, which turns out to be a stable ∞-category (see Proposition 1.3.2.10 of ibid.). By construction the objects (or 0-simplices) of the differential graded nerve $\mathrm{C}_\infty(\tilde{A})$ are complexes of right $\tilde{A}$-modules. Let us set $\mathtt{HPf}_\infty(A)$ to be the stable ∞-subcategory of $\mathrm{C}_\infty(\tilde{A})$ spanned by the objects of $\mathtt{HPf}_{\mathtt{dg}}(A)$.

Remark 4.1. There is an isomorphism of homotopy categories $\mathtt{hHPf}_\infty(A) \cong \mathtt{h}(\mathtt{HPf}_{\mathtt{dg}}(A)) := \mathrm{H}^0(\mathtt{HPf}_{\mathtt{dg}}(A))$ (see Remark 1.3.1.11 of [17]).

Remark 4.2. In the world of algebra the convention is to consider cohomologically graded complexes and in topology one considers typically homologically graded complexes. In [22] the differential graded complexes were cohomologically graded as it built upon the formalism of [14], whereas in [17] they are homologically graded. The passage between the two is not too difficult (see, for instance, Definitions 3.1.6 and 3.3.1 of [11]).

Let $\mathcal{S}\mathtt{et}_\Delta$ denote the category of simplicial sets with the Joyal model structure, whose fibrant objects are precisely the ∞-categories. We may compose the construction $A \mapsto \mathtt{HPf}_\infty(A)$ with the homotopy coherent nerve construction $\mathrm{N} : \mathcal{S}\mathtt{et}_\Delta \to \mathtt{Cat}_\infty$, where the $\mathtt{Cat}_\infty$ is ∞-category of (small) ∞-categories. Let $\mathtt{Cat}_\infty^{\mathtt{ex}}$ denote the ∞-category of (small) stable ∞-categories with exact functors.

Proposition 4.3. The association $A \mapsto \mathtt{HPf}_\infty(A)$ produces an ∞-functor $\mathrm{N}(\mathtt{Alg}_k) \to \mathtt{Cat}_\infty^{\mathtt{ex}}$.

Proof. Clearly the construction $A \mapsto \tilde{A}$ is functorial. It sends any k-algebra homomorphism to a unital k-algebra homomorphism. Sending $\tilde{A}$ to the differential graded category of cochain complexes of right $\tilde{A}$-modules is also functorial up to a natural isomorphism: $\tilde{A} \to \tilde{B}$ induces the map $- \otimes_{\tilde{A}} \tilde{B}$ between the differential graded categories. Application of the differential graded nerve is again functorial and lands inside $\mathcal{S}\mathtt{et}_\Delta$ (see Proposition 1.3.1.20 of [17]). Now applying the homotopy coherent nerve construction we get an ∞-functor $\mathrm{N}(\mathtt{Alg}_k) \to \mathtt{Cat}_\infty$. Thus we have demonstrated that the association $A \mapsto \mathrm{C}_\infty(\tilde{A})$ produces an ∞-functor $\mathrm{N}(\mathtt{Alg}_k) \to \mathtt{Cat}_\infty$.

Let us now verify that $- \otimes_{\tilde{A}} \tilde{B} : \mathrm{C}_\infty(\tilde{A}) \to \mathrm{C}_\infty(\tilde{B})$ sends an object of $\mathtt{HPf}_\infty(A)$ to an object in $\mathtt{HPf}_\infty(B)$. It is easy to see that $- \otimes_{\tilde{A}} \tilde{B}$ sends strictly perfect complexes of right $\tilde{A}$-modules to strictly perfect complexes of right $\tilde{B}$-modules. The functor also preserves homotopy equivalences whence condition (1) above is preserved. We need to now check that the canonical map $Y \otimes_{\tilde{A}} \tilde{B} \otimes_{\tilde{B}} B \to Y \otimes_{\tilde{A}} \tilde{B}$ is a homotopy equivalence. Since the functor $- \otimes_{\tilde{A}} \tilde{B}$ preserves homotopy equivalences, we may assume that Y is strictly perfect. Since $Y \otimes_{\tilde{A}} \tilde{B} \otimes_{\tilde{B}} B \cong Y \otimes_{\tilde{A}} B$ it suffices to show that $Y \otimes_{\tilde{A}} B \to Y \otimes_{\tilde{A}} \tilde{B}$ is a homotopy equivalence. Tensoring the short exact sequence of $\tilde{A}$-modules $0 \to B \to \tilde{B} \to k \to 0$ with the strictly perfect complex Y, we are reduced to showing $Y \otimes_{\tilde{A}} (\tilde{A}/A) \cong Y \otimes_{\tilde{A}} k$ is acyclic. Since $Y \in \mathtt{HPf}_\infty(A)$ the canonical map $Y \otimes_{\tilde{A}} A \to Y$ is a homotopy equivalence. It follows that $Y \otimes_{\tilde{A}} k$ is acyclic. Consequently, we have an ∞-functor $\mathtt{HPf}_\infty : \mathrm{N}(\mathtt{Alg}_k) \to \mathtt{Cat}_\infty$.

By construction $\mathtt{HPf}_\infty(A)$ is a stable ∞-category. Thus it suffices to show that the functor $-\otimes_{\tilde{A}} \tilde{B} : \mathtt{HPf}_\infty(A) \to \mathtt{HPf}_\infty(B)$ is exact. The homotopy cofiber sequences in the differential graded category $\mathtt{HPf}_{\mathtt{dg}}(A)$ are equivalent to short exact sequences, that are split exact in each degree. They produce the cofiber sequences in the stable ∞-category $\mathtt{HPf}_\infty(A)$ whence the assertion follows. $\square$

Let us recall from [1] that a diagram $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathtt{Cat}_\infty^{\mathtt{ex}}$ is called *exact* if the sequence of stable presentable ∞-categories $\mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B}) \to \mathrm{Ind}(\mathcal{C})$ is exact, i.e., the composite is trivial, the functor $\mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ is fully faithful, and the canonical map $\mathrm{Ind}(\mathcal{B})/\mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{C})$ is an equivalence. Note that in ibid. the treatment is more general as the notion of exactness is considered in $\mathtt{Cat}_\infty^{\mathtt{ex}(\kappa)}$ for any regular cardinal κ, i.e., the ∞-category of κ-cocomplete small stable ∞-categories and κ-small colimit preserving functors.

Lemma 4.4. For any short exact sequence $A \to B \to C$ in $\mathtt{Alg}_k$ with $A^2 = A$ and B, C unital, the diagram $\mathtt{HPf}_\infty(A) \to \mathtt{HPf}_\infty(B) \to \mathtt{HPf}_\infty(C)$ is exact in $\mathtt{Cat}_\infty^{\mathtt{ex}}$.

Proof. It follows from Lemma 2.14 that the diagram $\mathtt{HPf}_{\mathtt{dg}}(A) \to \mathtt{HPf}_{\mathtt{dg}}(B) \to \mathtt{HPf}_{\mathtt{dg}}(C)$ is an exact sequence of differential graded categories, i.e., $\mathrm{H}^0(\mathtt{HPf}_{\mathtt{dg}}(A)) \to \mathrm{H}^0(\mathtt{HPf}_{\mathtt{dg}}(B)) \to \mathrm{H}^0(\mathtt{HPf}_{\mathtt{dg}}(C))$ is exact, i.e., the composite is trivial, the functor $\mathrm{H}^0(\mathtt{HPf}_{\mathtt{dg}}(A)) \to \mathrm{H}^0(\mathtt{HPf}_{\mathtt{dg}}(B))$ is fully faithful, and the canonical map $\mathrm{H}^0(\mathtt{HPf}_{\mathtt{dg}}(B))/\mathrm{H}^0(\mathtt{HPf}_{\mathtt{dg}}(A)) \to \mathrm{H}^0(\mathtt{HPf}_{\mathtt{dg}}(C))$ is an equivalence after idempotent completion. Proposition 5.15 of [1] says that a diagram $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathtt{Cat}_\infty^{\mathtt{ex}}$ is exact if and only if the sequence of triangulated categories $\mathtt{h}\mathcal{A} \to \mathtt{h}\mathcal{B} \to \mathtt{h}\mathcal{C}$ is exact, i.e., the composite is trivial, the functor $\mathtt{h}\mathcal{A} \to \mathtt{h}\mathcal{B}$ is fully faithful, and the canonical map $\mathtt{h}\mathcal{B}/\mathtt{h}\mathcal{A} \to \mathtt{h}\mathcal{C}$ is an equivalence after idempotent completion. The assertion follows from Remark 4.1 above. $\square$

In [22] the KQ-theory of $A \in \mathtt{Alg}_k$ was defined to be the (connective) algebraic K-theory of the k-linear differential graded category $\mathtt{HPf}_{\mathtt{dg}}(A)$. Any differential graded category $\mathcal{C}$ has an underlying ordinary category $\mathcal{C}'$ with morphisms given by $\mathcal{C}'(X,Y) = \{f \in \mathcal{C}(X,Y)^0 \mid df = 0\}$. The underlying category of the differential graded category $\mathtt{HPf}_{\mathtt{dg}}(A)$ will be denoted by $\mathtt{HPf}(A)$. There is a Waldhausen category structure on $\mathtt{HPf}(A)$. One way to see this is as follows: The category of unbounded cochain complexes of $\tilde{A}$-modules $\mathtt{Ch}(\tilde{A})$ admits a model structure with cohomology isomorphisms as weak equivalences and degreewise epimorphisms as fibrations (see Theorem 2.3.11 of [13]). A map $i : X \to Y$ in $\mathtt{Ch}(\tilde{A})$ is a cofibration if it is a degreewise split monomorphism with cofibrant cokernel (see Proposition 2.3.9 of ibid.). The subcategory of perfect complexes $\mathtt{Perf}(\tilde{A})$, which are the compact objects in $\mathtt{Ch}(\tilde{A})$ [2], is a complete Waldhausen subcategory of the model category $\mathtt{Ch}(\tilde{A})$ in the sense of [10]. The category $\mathtt{HPf}(A)$ is the full subcategory of $\mathtt{Perf}(\tilde{A})$ consisting of cochain complexes X that are homotopy equivalent to strictly perfect complexes and satisfy $X \otimes_{\tilde{A}} A \to X$ is a homotopy equivalence. The strictly perfect complexes of $\tilde{A}$-modules are cofibrant in the model category $\mathtt{Ch}(\tilde{A})$ and the weak equivalences between such complexes are precisely the homotopy equivalences. One can now verify that the Waldhausen category structure on $\mathtt{Perf}(\tilde{A})$ restricts to a Waldhausen category structure on $\mathtt{HPf}(A)$. Summarising, we have

Lemma 4.5. The category $\mathtt{HPf}(A)$ is a Waldhausen subcategory of the model category $\mathtt{Ch}(\tilde{A})$ and the canonical inclusion $\mathtt{HPf}(A) \hookrightarrow \mathtt{Perf}(\tilde{A})$ is Waldhausen exact.

Let $\texttt{Wald}$ denote the category of small Waldhausen categories with Waldhausen exact functors. One may apply the Waldhausen (connective) K-theory functor $\mathbf{K}^w : \texttt{Wald} \to \texttt{Sp}$ to $\texttt{HPf}(A)$ to define its K-theory. In [22] we showed that $\mathrm{KQ}_i(A) \cong \pi_i(\mathbf{K}^w(\texttt{HPf}(A)))$ (see Lemma 2.12 of ibid.). Using the material from Section 7 of [1] we can define the connective K-theory of the stable ∞-category $\texttt{HPf}_\infty(A)$. Let us denote this connective K-theory functor of small stable ∞-categories by $\mathbf{K}^{\mathrm{c}} : \texttt{Cat}^{\texttt{ex}}_\infty \to \texttt{Sp}$.

Lemma 4.6. There is a natural equivalence of spectra $\mathbf{K}^w(\texttt{HPf}(A)) \xrightarrow{\sim} \mathbf{K}^{\mathrm{c}}(\texttt{HPf}_\infty(A))$.

Proof. Since in the previous Lemma we showed that $\texttt{HPf}(A)$ is a Waldhausen subcategory of a model category, the assertion follows from Corollary 7.12 of [1]. □

Remark 4.7. We obtain yet another description of KQ-theory, viz.,

$$\mathrm{KQ}_i(A) \cong \pi_i(\mathbf{K}^{\mathrm{c}}(\texttt{HPf}_\infty(A))).$$

Using the delooping machinery of [26] one can define the nonconnective K-theory spectrum. This task was carried out in Section 9 of [1] in the setting of stable ∞-categories. Let us denote the nonconnective K-theory functor by $\mathbf{K}^{\texttt{nc}} : \texttt{Cat}^{\texttt{ex}}_\infty \to \texttt{Sp}$.

Proposition 4.8. For any short exact sequence in $0 \to A \to B \to C \to 0$ with $A^2 = A$, B, C unital, there is cofiber sequence in $\texttt{Sp}$

$$\mathbf{K}^{\texttt{nc}}(\texttt{HPf}_\infty(A)) \to \mathbf{K}^{\texttt{nc}}(\texttt{HPf}_\infty(B)) \to \mathbf{K}^{\texttt{nc}}(\texttt{HPf}_\infty(C)).$$

Proof. It follows from Lemma 4.4 that $\texttt{HPf}_\infty(A) \to \texttt{HPf}_\infty(B) \to \texttt{HPf}_\infty(C)$ is exact in $\texttt{Cat}^{\texttt{ex}}_\infty$. The assertion now follows since nonconnective algebraic K-theory satisfies localization (see Theorem 9.8 of [1]). □

In [1] the authors constructed the univeral localizing invariant $\mathcal{U}_{\texttt{loc}} : \texttt{Cat}^{\texttt{ex}}_\infty \to \mathcal{M}_{\texttt{loc}}$ and proposed $\mathcal{M}_{\texttt{loc}}$ as a candidate for noncommutative motives in the setting of stable ∞-categories (see Theorem 8.7 of ibid.). We set $\mathcal{M}_\infty(A) := \mathcal{U}_{\texttt{loc}} \circ \texttt{HPf}_\infty(A) : \texttt{Alg}_k \to \mathcal{M}_{\texttt{loc}}$ and call it the *stable ∞-category valued noncommutative motive of* A. In fact, the ∞-category $\mathcal{M}_{\texttt{loc}}$ is itself stable and the exact sequences in $\texttt{Cat}^{\texttt{ex}}_\infty$ produce cofiber sequences in $\mathcal{M}_{\texttt{loc}}$. It follows from Lemma 4.4 that

Lemma 4.9. For any short exact sequence in $0 \to A \to B \to C \to 0$ with $A^2 = A$, B, C unital, there is cofiber sequence in $\mathcal{M}_{\texttt{loc}}$

$$\mathcal{M}_\infty(A) \to \mathcal{M}_\infty(B) \to \mathcal{M}_\infty(C).$$

For any $A \in \texttt{Alg}_k$ let $M_n(A)$ denote the k-algebra of $n \times n$-matrices over A. There is a canonical corner embedding $A \to M_n(A)$ sending $a \mapsto a(e_{11})$.

Proposition 4.10. For any $A \in \texttt{Alg}_k$ there is an equivalence $\mathcal{M}_\infty(A) \simeq \mathcal{M}_\infty(M_n(A))$ induced by the corner embedding $A \to M_n(A)$.

Proof. From Proposition 2.4 of [22] we deduce that $\texttt{HPf}_{\texttt{dg}}(A) \cong \texttt{HPf}_{\texttt{dg}}(M_n(A))$ induced by the corner embedding $A \to M_n(A)$. Consequently, $\texttt{HPf}_\infty(A) \simeq \texttt{HPf}_\infty(M_n(A))$ in $\texttt{Cat}^{\texttt{ex}}_\infty$ (see Proposition 4.3). The assertion follows since $\mathcal{M}_\infty(-) = \mathcal{U}_{\texttt{loc}} \circ \texttt{HPf}_\infty(-)$. □

Let $\texttt{C}^*$ denote the category of all C^*-algebras viewed as a subcategory of $\texttt{Alg}_{\mathbb{C}}$. It follows from the Cohen–Hewitt factorization theorem that any $A \in \texttt{C}^*$ satisfies $A^2 = A$. Summarizing, we have the following:

Theorem 4.11. Viewing $\mathtt{C}^*$ as an ordinary category (not a topological category) there is an ∞-functor $\mathrm{N}(\mathtt{C}^*) \to \mathcal{M}_{\mathtt{loc}}$ that satisfies:

1. (matrix stability): $\mathcal{M}_\infty(A) \simeq \mathcal{M}_\infty(M_n(A))$ for all A, and
2. (localization / excision): any short exact sequence $0 \to A \to B \to C \to 0$ produces the following cofiber sequence in $\mathcal{M}_{\mathtt{loc}}$
$$\mathcal{M}_\infty(A) \to \mathcal{M}_\infty(B) \to \mathcal{M}_\infty(C).$$

Proof. Only (2) needs a proof because it has been strengthened (note that B and C are no longer assumed to be unital). For any short exact sequence $0 \to A \to B \xrightarrow{g} C \to 0$ in $\mathtt{C}^*$ we need to show that $\mathcal{M}_\infty(A)$ is the fiber of the induced map $\mathcal{M}_\infty(B) \xrightarrow{g} \mathcal{M}_\infty(C)$. We can form another short exact sequence $0 \to A \to \tilde{B} \xrightarrow{\tilde{g}} \tilde{C} \to 0$ with $\tilde{B}, \tilde{C}$ unital. Now there is a diagram in $\mathcal{M}_{\mathtt{loc}}$

$$\begin{tikzcd}
\mathrm{fib}(g) \arrow[r] \arrow[d] & \mathcal{M}_\infty(B) \arrow[r, "g"] \arrow[d] & \mathcal{M}_\infty(C) \arrow[d] \\
\mathcal{M}_\infty(A) \arrow[r] \arrow[d] & \mathcal{M}_\infty(\tilde{B}) \arrow[r, "\tilde{g}"] \arrow[d] & \mathcal{M}_\infty(\tilde{C}) \arrow[d] \\
\mathcal{M}_\infty(0) \arrow[r] & \mathcal{M}_\infty(\mathbb{C}) \arrow[r, "\cong"] \arrow[u, bend right] & \mathcal{M}_\infty(\mathbb{C}). \arrow[u, bend right]
\end{tikzcd}$$

The bottom two rows and the two columns on the right are cofiber sequences. Since $0 \to B \to \tilde{B} \to \mathbb{C} \to 0$ and $0 \to C \to \tilde{C} \to \mathbb{C} \to 0$ admit splittings $\mathbb{C} \to \tilde{B}$ and $\mathbb{C} \to \tilde{C}$ in $\mathtt{C}^*$ respectively, the indicated splittings exist in the above diagram, i.e., the two columns on the right are split cofiber sequences. The assertion now follows by a diagram chase. $\square$

4.2. C^***-algebras and nonconnective KQ-theory.** Since $\mathcal{M}_{\mathtt{loc}}$ is a stable ∞-category its homotopy category $\mathtt{h}\mathcal{M}_{\mathtt{loc}}$ is triangulated. Consider the triangulated category valued functor $\mathcal{M} = \mathtt{h}\mathcal{M}_\infty : \mathtt{C}^* \to \mathtt{h}\mathcal{M}_{\mathtt{loc}}$ furnished by the above Theorem 4.11. Let $\mathcal{M}^{\mathbb{K}}$ denote the composite functor $\mathcal{M}^{\mathbb{K}} = \mathcal{M}(-\hat{\otimes}\mathbb{K}) : \mathtt{C}^* \to \mathtt{h}\mathcal{M}_{\mathtt{loc}}$. We may restrict this functor to separable C^*-algebas $\mathcal{M}^{\mathbb{K}} : \mathtt{SC}^* \to \mathtt{h}\mathcal{M}_{\mathtt{loc}}$.

Theorem 4.12. The functor $\mathcal{M}^{\mathbb{K}} : \mathtt{SC}^* \to \mathtt{h}\mathcal{M}_{\mathtt{loc}}$ factors uniquely through $\mathtt{KK}$.

Proof. The functor $\mathcal{M}_\infty$ satisfies localization / excision whence the functor $\mathcal{M}$ is a split exact. Since $-\hat{\otimes}\mathbb{K}$ preserves split exact sequences, the functor $\mathcal{M}^{\mathbb{K}}$ is split exact. The functor $\mathcal{M}$ is M_n-stable. Thus $\mathcal{M}^{\mathbb{K}}$ is a C^*-stable (see Proposition 3.31 of [7]). Since the functor $\mathtt{SC}^* \to \mathtt{KK}$ is the universal C^*-stable and split exact functor the assertion follows. $\square$

Remark 4.13. The above Theorem is the key to categorification of topological $\mathbb{T}$-duality. Intuitively, our result asserts that under favourable circumstances topological $\mathbb{T}$-duality induces equivalences of noncommutative motives associated to certain C^*-algebras. For the details we refer the readers to (Example 4.1 of [22] and Section 5 of [19]).

Definition 4.14. We define the nonconnective KQ-theory or $\mathrm{KQ}^{\mathrm{nc}}$-theory groups as
$$\mathrm{KQ}^{\mathrm{nc}}_i(A) := \pi_i(\mathbf{K}^{\mathtt{nc}}(\mathtt{HPf}_\infty(A))) \text{ for all } A \in \mathtt{Alg}_k \text{ and } i \in \mathbb{Z}.$$

Theorem 4.15. Let a C^* algebra B be of the form $A\hat{\otimes}C$, where $C = \mathbb{K}$ or any properly infinite C^*-algebra. Then there is a natural isomorphism $\mathrm{KQ}_i^{\mathrm{nc}}(B) \cong \mathrm{K}_i^{\mathrm{nc}}(B)$ for all $i \in \mathbb{Z}$, where $\mathrm{K}_i^{\mathrm{nc}}(B)$ denotes the i-th nonconnective algebraic K-theory group of B.

Proof. Let us first address the case where $C = \mathbb{K}$ and to this end we set $A_{\mathbb{K}} = A\hat{\otimes}\mathbb{K}$. From Lemma 4.5 we have a canonical Waldhausen exact functor $\mathtt{HPf}(A_{\mathbb{K}}) \to \mathtt{Perf}(\tilde{A_{\mathbb{K}}})$. The composite $\mathtt{HPf}(A_{\mathbb{K}}) \to \mathtt{Perf}(\tilde{A_{\mathbb{K}}}) \xrightarrow{\phi} \mathtt{Perf}(\mathbb{C})$ is trivial, where $\phi = - \otimes^{\mathbb{L}}_{\tilde{A_{\mathbb{K}}}} \mathbb{C}$. It follows that there is a canonical map of stable ∞-categories $\mathtt{HPf}_\infty(A_{\mathbb{K}}) \to F(\phi)$, where $F(\phi)$ is the fiber of the map $\mathrm{N}(\mathcal{M}(\mathtt{Perf}(\tilde{A_{\mathbb{K}}}))^{\mathrm{cf}}) \xrightarrow{\phi} \mathrm{N}(\mathcal{M}(\mathtt{Perf}(\mathbb{C}))^{\mathrm{cf}})$ (see Lemma 7.11 of [1] for the construction of $\mathcal{M}(\mathtt{Perf}(-))$, which is a Waldhausen subcategory of a simplicial model category. Using Lemma 4.6 and Theorem 3.7 of [22] we deduce that the map $\mathtt{HPf}_\infty(A_{\mathbb{K}}) \to F(\phi)$ is a connective K-theory isomorphism, i.e., there is an equivalence $\mathbf{K}^{\mathrm{c}}(\mathtt{HPf}_\infty(A_{\mathbb{K}})) \xrightarrow{\sim} \mathbf{K}^{\mathrm{c}}(F(\phi))$. The connective K-theory spectrum of $F(\phi)$ can be identified with $\mathbf{K}^{\mathrm{c}}(A_{\mathbb{K}})$, i.e., the connective algebraic K-theory spectrum of $A_{\mathbb{K}}$ due to excision [29]. Note that the nonconnective algebraic K-theory spectrum of a stable ∞-category is defined in such a manner (see Section 9 of [1]) so that when applied to a C^*-algebra B it produces the expected result, viz.,

$$\mathbf{K}^{\mathrm{nc}}(\mathtt{HPf}_\infty(B)) \simeq \mathrm{colim}_n\, \Omega^n \mathbf{K}^{\mathrm{c}}(\Sigma^{(n)}_\kappa \mathtt{HPf}_\infty(B)) \simeq \mathrm{colim}_n\, \Omega^n \mathbf{K}^{\mathrm{c}}(\Sigma^n B),$$

where $\Sigma^n B$ denotes the n-th Karoubi delooping of B (see, for instance, [27, 3]). Using localization the nonconnective K-theory spectrum of $F(\phi)$ can be identified with $\mathbf{K}^{\mathrm{nc}}(A_{\mathbb{K}})$ [1], which proves the assertion for stable C^*-algebras.

If C is properly infinite then using Proposition 2.2 of [4] (see also [30]) one obtains a commutative diagram in $\mathtt{C}^*$

(3)
$$\begin{tikzcd} C \arrow[rr, "\iota"] \arrow[dr, "\theta"'] & & M_2(C) \\ & C\hat{\otimes}\mathbb{K}, \arrow[ur, "\kappa"'] & \end{tikzcd}$$

where the top horizontal arrow $\iota : C \to M_2(C)$ is the corner embedding. Tensoring the above diagram with a unital A and applying the functors $\mathrm{KQ}^{\mathrm{nc}}(-)$ and $\mathrm{K}^{\mathrm{nc}}(-)$ along with the natural transformation between them produces a commutative diagram

$$\begin{tikzcd} \mathrm{KQ}^{\mathrm{nc}}_m(A\hat{\otimes}C) \arrow[r] \arrow[d] & \mathrm{KQ}^{\mathrm{nc}}_m(A\hat{\otimes}C\hat{\otimes}\mathbb{K}) \arrow[r] \arrow[d, "\cong"] & \mathrm{KQ}^{\mathrm{nc}}_m(M_2(A\hat{\otimes}C)) \arrow[d] \\ \mathrm{K}^{\mathrm{nc}}_m(A\hat{\otimes}C) \arrow[r] & \mathrm{K}^{\mathrm{nc}}_m(A\hat{\otimes}C\hat{\otimes}\mathbb{K}) \arrow[r] & \mathrm{K}^{\mathrm{nc}}_m(M_2(A\hat{\otimes}C)), \end{tikzcd}$$

where the middle verticle arrow is an isomorphism (since $A\hat{\otimes}C\hat{\otimes}\mathbb{K}$ is stable). Observe that both $\mathrm{KQ}^{\mathrm{nc}}$-theory and K^{nc}-theory are matrix stable whence the top and the bottom horizontal compositions are isomorphisms. The assertion in the unital case now follows by a diagram chase. Finally using excision one can prove the general case. $\square$

Remark 4.16. The argument above actually shows that there is a map of spectra that induces the isomorphism at the level of homotopy groups, which are the $\mathrm{KQ}^{\mathrm{nc}}$-theory and

K^{nc}-theory groups in source and target respectively. The map of connective spectra can also be delooped inductively by a Bass–Heller–Swan splitting argument [25].

Remark 4.17. Observe that $\mathcal{O}_\infty$ is properly infinite whence the above Theorem 4.15 is applicable to $\mathcal{O}_\infty$-stable C^*-algebras. Since we already know that K^{nc}-theory of a stable or an $\mathcal{O}_\infty$-stable C^*-algebra agrees naturally with its topological K-theory (see [29, 4, 19]), we conclude that $\mathrm{KQ}^{\mathrm{nc}}$-theory is naturally isomorphic to topological K-theory for such a C^*-algebra. From the computational viewpoint for such a C^*-algebra it turns out that

$$\text{noncomm. stable cohomotopy} \cong \mathrm{KQ}^{\mathrm{nc}}\text{-theory} \cong \mathrm{K}^{\mathrm{nc}}\text{-theory} \cong \text{top. K-theory}.$$

Let us also remark that topological K-theory is Bott 2-periodic and fairly easy to compute.

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