TRIANGULATED CATEGORIES OF SINGULARITIES AND D-BRANES IN LANDAU-GINSBURG MODELS

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Dedicated to the blessed memory of Andrei Nikolaevich Tyurin – adviser and friend

Introduction

In spite of physics terms in the title, this paper is pure mathematical and its purpose is to fill some gaps in our knowledges about singularities of algebraic varieties and about categories related to them. It seems that two different types of categories can be associated with singularities (or singularities of maps). The first type categories are connected with vanishing cycles and closely related to the categories which were introduced in [18] for symplectic Picard-Lefschetz pencil. The second type categories are pure algebraic and are coming from the derived categories of coherent sheaves. This type categories will be the center of this work. An important notion here is a concept of a perfect complex, which was introduced in [1]. A perfect complex is a complex of sheaves which locally quasi-isomorphic to a complex of locally free sheaves of finite type (for good reference see also [19]).

We can attach to any algebraic variety X the bounded derived category of coherent sheaves $\mathbf{D}^b(\mathrm{coh}(X))$. This category admits a triangulated structure. The derived category of coherent sheaves has a triangulated subcategory $\mathfrak{Perf}(X)$ formed by the perfect complexes. If the variety X is smooth then any coherent sheaf has a finite resolution of locally frees of finite type and the subcategory of perfect complexes coincides with whole $\mathbf{D}^b(\mathrm{coh}(X))$. But for singular varieties this property does not fulfill. We introduce a notion of triangulated category of singularities $\mathbf{D}_{Sg}(X)$ as the quotient of the triangulated category $\mathbf{D}^b(\mathrm{coh}(X))$ by the full triangulated subcategory of perfect complexes $\mathfrak{Perf}(X)$. The category $\mathbf{D}_{Sg}(X)$ reflects the properties of the singularities of X and does not depend on whole X . For example we prove that it is invariant with respect to a localization in the Zarisky topology (Proposition 1.14). The category $\mathbf{D}_{Sg}(X)$ has good properties when the variety X is Gorenstein. In this case if the locus of singularities is proper then all Hom's between any objects are finite-dimensional vector spaces (Corollary 1.24).

The investigation of such type categories is inspired by the Homological Mirror Symmetry Conjecture ([15]).

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Mainly works on topological string theory consider the case of topological N=2 superconformal sigma-models with a Calabi-Yau target space. In this case the field theory has two topologically twisted versions: A-models and B-models. The corresponding D-branes are called A-branes and B-branes. The mirror symmetry should exchange these two classe of D-branes. From the mathematical point of view the category of B-branes on Calabi-Yau is the derived category of coherent sheaves ([15],[3]). For A-branes on Calabi-Yau manifolds it has been proposed that the relevant category is the so-called Fukaya category, whose objects are, roughly speaking, Lagrangian submanifolds equipped with flat vector bundles ([15]). The Homological Mirror Symmetry Conjecture asserts that if two Calabi-Yaus X and Y are mirror to each other, then the derived category of coherent sheaves on X is equivalent to the Fukaya category of Y, and vice versa.

On the other hand, phisicists also consider more general topological twisted N=2 field theories and corresponding D-branes. One class of such theories is given by the sigma-model with a Fano variety as a target space. Another set of examples is provided by N=2 Landau-Ginsburg models. In many cases these two classes of N=2 theories are related by mirror symmetry ([10]). For example, the sigma model with target $\mathbb{P}^n$ is mirror to a Landau-Ginsburg model which is given by superpotential

$$W = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n}$$

on $\mathbb{C}^{*n}$. A general definition of Landau-Ginsburg model involves, besides a choice of a target space, a choice of a holomorphic function W on this manifold. In particular, non-trivial Landau-Ginsburg models require non-compact target space.

For Fano varieties we have the derived categories of coherent sheaves (B-branes) and also if we have a symplectic form we can propose the relevant Fukaya category (A-branes). Thus if one wants to extend the Homological Mirror Symmetry Conjecture to the non-Calabi-Yau case, one should understand D-branes in Landau-Ginsburg models.

Categories of A-branes in Landau-Ginsburg models are studied in [9] and in [18] from the mathematical point of view. Mirror symmetry relates B-branes on a Fano variety (coherent sheaves) with A-branes in a LG model. In the case when the Fano variety is $\mathbb{P}^n$, this prediction has been tested in [9] (and for $\mathbb{P}^2$ in [18] from the mathematical viewpoint).

One can also consider the Fukaya category (A-branes) on a Fano variety. One can expect that in this case the Fukaya category is equivalent to the category of B-branes in the mirror Landau-Ginsburg model. If we assume the mirror symmetry, we can predict a description of the Fukaya category on a Fano variety by studying B-branes in the mirror LG model. As a rule it is easy to understand B-branes, rather that A-branes.

The mathematical definition for the category of B-branes in Landau-Ginsburg models was proposed by M.Kontsevich. Roughly speaking, he suggests that the superpotential W deforms complexes of coherent sheaves to "twisted" complexes, i.e the composition of differentials now is not zero, but it is equal to the multiplication with W. This "twisting" also breaks $\mathbb{Z}$-grading down to $\mathbb{Z}/2$-grading. The equivalence of this definition with the physics notion of B-branes in LG models was tested in paper [11] in the case of usual quadratic superpotential $W = x_1^2 + \cdots + x_n^2$.

We establish a connection between categories of B-branes in Landau-Ginsburg models and triangulated categories of singularities. We consider the singular fibres of the map W and show that the triangulated categories of singularities of these fibres are equivalent to the categories of B-branes (Theorem 3.8). In particular this gives us that, in spite of the fact that the category of B-branes is defined using the total space X, it depends only on the singular fibres of the superpotential. It seems that this result can be useful for precise calculations of the categories of B-branes in Landau-Ginsburg models.

Another result which is also useful for calculations connects the categories of B-branes in different dimensions (Theorem 2.1). It says the following: Let $W : X \to \mathbb{C}$ be a superpotential. Consider the manifold $Y = X \times \mathbb{C}^2$ and another superpotential $W' = W + xy$ on Y, where x, y are coordinates on $\mathbb{C}^2$. Then the category of B-branes in the Landau-Ginsburg model on Y is equivalent to the category of B-branes in the Landau-Ginsburg model on X. Actually, we prove that the triangulated category of singularities of the fiber X_0 over point 0 is equivalent to the triangulated category of singularities of the fiber Y_0 (Theorem 2.1). But taking in mind that the categories of singularities are equivalent to the categories of B-branes (Theorem 3.8) we obtain the connection between B-branes mentioned above.

In the end we give a calculation of the category of B-branes in the Landau-Ginsburg model with the superpotential $W = z_0^n + z_1^2 + \cdots + z_{2k}^2$ on $\mathbb{C}^{2k+1}$. The singularity of this superpotential corresponds to the Dynkin diagram A_{n-1}. This category has $n-1$ indecomposible objects. We describe morphisms between them, the translation functor and the triangulated structure on this category.

1. Singularities and triangulated categories

1.1. **Triangulated categories and localization.** In this section we remind definitions of a triangulated category and its localization which were introduced in [20] (see also [5],[12],[13]). Let $\mathcal{D}$ be an additive category. The structure of a triangulated category on $\mathcal{D}$ is defined by giving of the following data:

a) an additive autoequivalence $[1] : \mathcal{D} \longrightarrow \mathcal{D}$ (it is called a translation functor),

b) a class of exact (or distinguished) triangles:

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1],$$

which must satisfy the set of axioms T1–T4.

T1. a) For each object X the triangle $X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow X[1]$ is exact.

b) Each triangle isomorphic to an exact triangle is exact.

c) Any morphism $X \xrightarrow{u} Y$ can be included in an exact triangle
$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$.

T2. A triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is exact if and only if the triangle $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ is exact.

T3. For any two exact triangles and two morphisms f, g the diagram below

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, dotted, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1].
\end{tikzcd}$$

can be completed to a morphism of triangles by a morphism $h : Z \to Z'$.

T4. For each pair of morphisms $X \xrightarrow{u} Y \xrightarrow{v} Z$ there is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "x"] \arrow[d, "v"'] & Z' \arrow[r] \arrow[d, "w"] & X[1] \arrow[d, equal] \\
X \arrow[r] & Z \arrow[r, "y"] \arrow[d] & Y' \arrow[r, "w'"] \arrow[d, "t"] & X[1] \arrow[d, "u[1]"] \\
 & X' \arrow[r, equal] \arrow[d, "r"] & X' \arrow[r, "r"] \arrow[d] & Y[1] \\
 & Y[1] \arrow[r, "x[1]"] & Z'[1] &
\end{tikzcd}$$

where the first two rows and the two central columns are exact triangles.

A functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ between two triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ is called exact if it commutes with the translation functors, i.e there fixed a natural isomorphism $t : F \circ [1]_{\mathcal{D}} \xrightarrow{\sim} [1]_{\mathcal{D}'} \circ F$ and it transforms all exact triangles into exact triangles, i.e for each exact triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ in $\mathcal{D}$ the triangle

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{t_X Fw} FX[1]$$

is exact.

A full additive subcategory $\mathcal{N} \subset \mathcal{D}$ is called full triangulated subcategory, if the following conditions hold: it is closed with respect to the translation functor in $\mathcal{D}$ and if it contains any two objects of an exact triangle in $\mathcal{D}$ then it contains the third object of this triangle as well.

Now we remind the definition of a localization of categories. Let $\mathcal{C}$ be a category and let Σ be a class of morphisms in $\mathcal{C}$. It is well-known [4] that there is a category $\mathcal{C}[\Sigma^{-1}]$ and a functor

$Q : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ which is universal among the functors making the elements of Σ invertible. The category $\mathcal{C}[\Sigma^{-1}]$ has a good description if Σ is a multiplicative system.

A family of morphisms Σ in a category $\mathcal{C}$ is called *a multiplicative system* if it satisfies the following conditions:

M1 all identical morphisms id_X belongs to Σ ;

M2 the composition of two elements of Σ belong to Σ ;

M3 any diagram $X' \xleftarrow{s} X \xrightarrow{u} Y$, with $s \in \Sigma$ can be completed to a commutative square

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "s"'] & Y \arrow[d, dashed, "t"] \\ X' \arrow[r, dashed, "u'"] & Y' \end{tikzcd}$$

with $t \in \Sigma$ (the same when all arrows reversed);

M4 for any two morphisms f, g the existence of $s \in \Sigma$ with $fs = gs$ is equivalent to the existence of $t \in \Sigma$ with $tf = tg$.

If Σ is a multiplicative system then $\mathcal{C}[\Sigma^{-1}]$ has the following description. The objects of $\mathcal{C}[\Sigma^{-1}]$ are the objects of $\mathcal{C}$. The morphisms from X to Y in $\mathcal{C}[\Sigma^{-1}]$ are pairs (s, f) in $\mathcal{C}$ of the form

$$X \xrightarrow{f} Y' \xleftarrow{s} Y, \qquad s \in \Sigma$$

modulo the following equivalence relation: $(f, s) \sim (g, t)$ iff there is a commutative diagram

$$\begin{tikzcd} & Y' \arrow[d] & \\ X \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & Y''' & Y \arrow[ul, "s"'] \arrow[l, "r"'] \arrow[dl, "t"] \\ & Y'' \arrow[u] & \end{tikzcd}$$

with $r \in \Sigma$.

The composition of the morphisms (f, s) and (g, t) is a morphism $(g'f, s't)$ defined from the following diagram, which exists by M3:

$$\begin{tikzcd} & & Z'' & & \\ & Y' \arrow[ur, dashed, "g'"] & & Z' \arrow[ul, dashed, "s'"'] & \\ X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z \arrow[ul, "t"'] \end{tikzcd} .$$

There is a quotient functor

$$Q : \mathcal{C} \longrightarrow \mathcal{C}[\Sigma^{-1}], \quad X \mapsto X, f \mapsto (f, 1)$$

which inverts all elements of Σ and it is universal in this sense (see [4]).

Let $\mathcal{D}$ be a triangulated category and $\mathcal{N} \subset \mathcal{D}$ be a full triangulated subcategory. Denote by $\Sigma(\mathcal{N})$ a class of morphisms s in $\mathcal{D}$ embedding into an exact triangle

$$X \xrightarrow{s} Y \longrightarrow N \longrightarrow X[1]$$

with $N \in \mathcal{N}$. It can be checked that $\Sigma(N)$ is a multiplicative system. We define

$$\mathcal{D}/\mathcal{N} := \mathcal{D}[\Sigma(\mathcal{N})^{-1}].$$

We endow the category $\mathcal{D}/\mathcal{N}$ with a translation functor induced by the translation functor in the category $\mathcal{D}$.

Lemma 1.1. *The category $\mathcal{D}/\mathcal{N}$ becomes a triangulated category by taking for exact triangles such that are isomorphic to the images of exact triangles in $\mathcal{D}$. The quotient functor $Q : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{N}$ anihilates $\mathcal{N}$. Moreover, any exact functor $F : \mathcal{D} \longrightarrow \mathcal{D}'$ of triangulated categories for which $F(X) \simeq 0$ when $X \in \mathcal{N}$ factors uniquely through Q.*

The following lemma, which will be necessary in the future, is evident.

Lemma 1.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be full triangulated subcategories in triangulated categories $\mathcal{C}$ and $\mathcal{D}$ respectively. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint pair of exact functors such that $F(\mathcal{M}) \subset \mathcal{N}$ and $G(\mathcal{N}) \subset \mathcal{M}$. Then they induce functors*

$$\overline{F} : \mathcal{C}/\mathcal{M} \longrightarrow \mathcal{D}/\mathcal{N}, \qquad \overline{G} : \mathcal{D}/\mathcal{N} \longrightarrow \mathcal{C}/\mathcal{M}$$

which are adjoint.

Proposition 1.3. ([20], [12]). *Let $\mathcal{D}$ be a triangulated category and $\mathcal{D}', \mathcal{N}$ be full triangulated subcategories. Let $\mathcal{N}' = \mathcal{D}' \cap \mathcal{N}$. Assume that any morphism $N \to X'$ with $N \in \mathcal{N}$ and $X' \in D'$ admits a factorization $N \to N' \to X'$ with $N' \in \mathcal{N}'$. Then the natural functor*

$$\mathcal{D}'/\mathcal{N}' \longrightarrow \mathcal{D}/N$$

is fully faithful.

1.2. **Triangulated categories of singularities.** Let X be a separated noetherian scheme of finite Krull dimension over a field k such that the category of coherent sheaves $\operatorname{coh}(X)$ has enough locally free sheaves. The last condition means that for any coherent sheaf $\mathcal{F}$ there is a vector bundle $\mathcal{E}$ and an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{F}$. For example any quasi-projective scheme satisfies these conditions. Note that any closed and any open subscheme of X is also noetherian, finite dimension and has enough locally free sheaves. It is clear for a closed subscheme while for an open subscheme U it follows from the fact that any coherent sheaf on U can be obtained as the restriction of a coherent sheaf on X (see [7], ex.5.15).

Denote by $\mathbf{D}^b(\mathrm{coh}(X))$ (resp. $\mathbf{D}^b(\mathrm{Qcoh}(X))$) the bounded derived categories of coherent (resp. quasi-coherent) sheaves on X. These categories have canonical triangulated structures.

Since X is noetherian the natural functor $\mathbf{D}^b(\mathrm{coh}(X)) \longrightarrow \mathbf{D}^b(\mathrm{Qcoh}(X))$ is fully faithful and realizes an equivalence of $\mathbf{D}^b(\mathrm{coh}(X))$ with the full subcategory $\mathbf{D}^b(\mathrm{Qcoh}(X))_{coh} \subset \mathbf{D}^b(\mathrm{Qcoh}(X))$ consisting of all complexes with coherent cohomologies (see [1] II, 2.2.2).

Definition 1.4. *A bounded complex of coherent sheaves will be called* a perfect complex *if it is quasi-isomorphic to a bounded complex of locally free sheaves of finite type.*

Lemma 1.5. *Let X be as above. Then for any bounded above complex of coherent sheaves $C^{\cdot}$ on X there is a bounded above complex of locally frees of finite type $P^{\cdot}$ and an epimorphism of the complexes $P^{\cdot} \twoheadrightarrow C^{\cdot}$ which is a quasi-isomorphism.*

Lemma 1.6. *Any complex $C^{\cdot}$, which is isomorphic to a bounded complex of locally frees in $\mathbf{D}^b(\mathrm{coh}(X))$, is perfect.*

Proof. We can represent the isomorphism in the derived category via calculus of fractions as $P^{\cdot} \stackrel{s}{\leftarrow} E^{\cdot} \stackrel{t}{\rightarrow} C^{\cdot}$, where $P^{\cdot}$ is a bounded complex of locally frees and s, t are quasi-isomorphisms. By Lemma 1.5, there is a bounded above complex $Q^{\cdot}$ of locally frees and quasi-isomorphism $Q^{\cdot} \to E^{\cdot}$. Consider a good truncation $\tau^{\geq -k}Q^{\cdot}$ for sufficiently large k. As $E^{\cdot}$ is bounded there is a morphism $r: \tau^{\geq -k}Q^{\cdot} \to E^{\cdot}$ that is also a quasi-isomorphism. To prove the lemma it is sufficient to show that $\tau^{\geq -k}Q^{\cdot}$ is a complex of locally frees. Consider the composition $sr: \tau^{\geq -k}Q^{\cdot} \to P^{\cdot}$ which is a quasi-isomorphism. The cone of sr is a bounded acyclic complex all terms of which, excepted maybe the leftmost term, are locally free. This implies that the leftmost term is locally free as well, because the kernel of an epimorphism of locally frees is locally free. Thus $\tau^{\geq -k}Q^{\cdot}$ is a bounded complex of locally free sheaves and hence $C^{\cdot}$ is perfect. $\Box$

The perfect complexes form a full triangulated subcategory $\mathfrak{Perf}(X) \subset \mathbf{D}^b(\mathrm{coh}(X))$, which is thick.

Remark 1.7. *Actually, a perfect complex is defined as a complex of $\mathcal{O}_X$-modules locally quasi-isomorphic to a bounded complex of locally frees of finite type. But under our assumption on the scheme any such complex is quasi-isomorphic to a bounded complex of locally frees of finite type(see* [1] *II, or* [19] *§2).*

Definition 1.8. *We define a triangulated category $\mathbf{D}_{Sg}(X)$ as the quotient of the triangulated category $\mathbf{D}^b(\mathrm{coh}(X))$ by the full triangulated subcategory $\mathfrak{Perf}(X)$ and call it as a triangulated category of singularities of X.*

Remark 1.9. It is known that if a scheme X is as above and is regular in addition then the subcategory of perfect complexes coincides with the whole bounded derived category of coherent sheaves. Hence, the triangulated category of singularities is trivial in this case.

We also denote as $\mathfrak{Lfr}(X) \subset \mathbf{D}^b(\mathrm{Qcoh}(X))$ the full triangulated subcategory consisting of objects which are isomorphic to bounded complexes of locally free sheaves in $\mathbf{D}^b(\mathrm{Qcoh}(X))$. By the same argument as in Lemma 1.6 we can show that for any complex $C^{\cdot} \in \mathfrak{Lfr}(X)$ there is a bounded complex of locally frees $P^{\cdot}$ and a quasi-isomorphism $P^{\cdot} \to C^{\cdot}$. It can be easily checked that the subcategory $\mathbf{D}^b(\mathrm{coh}(X)) \cap \mathfrak{Lfr}(X)$ coincides with the subcategory of perfect complexes $\mathfrak{Perf}(X)$.

Define a triangulated category $\mathbf{D}'_{Sg}(X)$ as the quotient $\mathbf{D}^b(\mathrm{Qcoh}(X))/\mathfrak{Lfr}(X)$. The full embedding $\mathbf{D}^b(\mathrm{coh}(X)) \longrightarrow \mathbf{D}^b(\mathrm{Qcoh}(X))$ induces a functor $\mathbf{D}_{Sg}(X) \longrightarrow \mathbf{D}'_{Sg}(X)$. We will show that this functor is also full embedding.

Lemma 1.10. *Let X be as above and let $\mathcal{F}$ be a quasicoherent sheaf on X such that for any point $x \in Sing(X)$ it is locally free in some neighbourhood of x. Then it belongs to the subcategory $\mathfrak{Lfr}(X)$. If $\mathcal{F}$ is coherent in addition then it is perfect as a complex.*

Proof. By Lemma 1.5, there is a bounded above complex $Q^{\cdot}$ of locally frees and quasi-isomorphism $Q^{\cdot} \to \mathcal{F}$.Consider a good truncation $\tau^{\geq -k}Q^{\cdot}$ for sufficiently large k. There is a morphism $r : \tau^{\geq -k}Q^{\cdot} \to \mathcal{F}$ that is also a quasi-isomorphism. To prove the lemma it is sufficient to show that $\tau^{\geq -k}Q^{\cdot}$ is a complex of locally free sheaves. All terms of this complex, except maybe for the leftmost term, are locally free. But for any point $x \in Sing(X)$ the leftmost term is also locally free, because $\mathcal{F}$ is locally free. If now $x \notin Sing(X)$ then there is a neighbourhood U of x which is regular. Hence, the leftmost term is locally free on U under assumption that $k > \dim X$.

If now $\mathcal{F}$ is coherent then we can take the resolution $Q^{\cdot}$ such that all terms are of finite type. Hence, $\mathcal{F}$ is perfect. □

In particular, we get from this lemma that if the support of an object $E^{\cdot} \in \mathbf{D}^b(\mathrm{coh}(X))$ does not meet $Sing(X)$ then $E^{\cdot}$ is perfect.

Lemma 1.11. *Let X be as above. Then any object $A \in \mathbf{D}_{Sg}(X)$ is isomorphic to an object $\mathcal{F}[k]$ where $\mathcal{F}$ is a coherent sheaf.*

Proof. The object A is a bounded complex of coherent sheaves. Let us take locally free bounded above resolution $P^{\cdot} \xrightarrow{\sim} A$ which exists by Lemma 1.5. Consider a brutal truncation $\sigma^{\geq -k}P^{\cdot}$ for sufficiently large $k \gg 0$. Denote by $\mathcal{F}$ the cohomology $H^{-k}(\sigma^{\geq -k}P^{\cdot})$. It is clear that $A \cong \mathcal{F}[k+1]$ in $\mathbf{D}_{Sg}(X)$. □

Lemma 1.12. *Let X be as above. Then for any locally free sheaf $\mathcal{E}$ and for any quasi-coherent sheaf $\mathcal{F}$*

$$\mathrm{Ext}^i(\mathcal{E},\mathcal{F})=0, \qquad \textit{for} \quad i>n.$$

Proof. Let $U_1\cup\ldots\cup U_n$ be an affine cover of X. For all subsequence of indices $I=(i_1,\ldots,i_k)$, let $U_I=U_{i_1}\cap\ldots\cap U_{i_k}$, and let $j_I: U_I\hookrightarrow X$ be the open immersion. As X is separated, each U_I is affine and each j_I is an affine map. Hence, j_{I*} is exact and preserves quasi-coherence.

We consider the Čech hypercover complex of quasi-coherent sheaves

$$0\to\mathcal{F}\to\bigoplus_{i=1}^{n} j_{i*}j_i^*\mathcal{F}\to\bigoplus_{I=(i_1,i_2)} j_{I*}j_I^*\mathcal{F}\to\bigoplus_{I=(i_1,i_2,i_3)} j_{I*}j_I^*\mathcal{F}\to\cdots \tag{1}$$

This is an exact sequence of sheaves.

Since $U_I=\mathit{Spec}(A_I)$ is affine the category of quasi-coherent sheaves on U_I is equivalent to the category of A_I-modules. Therefore, we have

$$\mathrm{Ext}^i(\mathcal{E}, j_{I*}j_I^*\mathcal{F})=\mathrm{Ext}^i(j_I^*\mathcal{E}, j_I^*\mathcal{F})=0 \qquad \text{for all} \qquad i>0,$$

because $j_I^*\mathcal{E}$ corresponds to a projective module over A_I. Thus the nontrivial Ext's from $\mathcal{E}$ to $\mathcal{F}$ are bounded by the length of the complex (1) which is equal to n. □

Proposition 1.13. *For a scheme X as above the natural functor $\mathbf{D}_{Sg}(X)\longrightarrow\mathbf{D}'_{Sg}(X)$ is fully faithful.*

Proof. Consider the full embedding $\mathbf{D}^b(\mathrm{coh}(X))\hookrightarrow\mathbf{D}^b(\mathrm{Qcoh}(X))$. We have $\mathfrak{Perf}(X)=\mathbf{D}^b(\mathrm{coh}(X))\cap\mathfrak{Lfr}(X)$. To prove the proposition we check that the conditions of Proposition 1.3 are fulfilled. Let $Q^\cdot\xrightarrow{t}\mathcal{F}^\cdot$ be a morphism in $\mathbf{D}^b(\mathrm{Qcoh}(X))$ such that $Q^\cdot\in\mathfrak{Lfr}(X)$ and $\mathcal{F}^\cdot\in\mathbf{D}^b(\mathrm{coh}(X))$. Let $P^\cdot\xrightarrow{s}\mathcal{F}^\cdot$ be a quasi-isomorphism where $P^\cdot$ is a bounded above complex of locally frees of finite type. It exists by Lemma 1.5. The brutal truncation $\sigma^{\geq -k}P^\cdot$ is a perfect complex. Consider the map $r:\sigma^{\geq -k}P^\cdot\to\mathcal{F}^\cdot$ induced by s. A cone of the map r is isomorphic to $G[k+1]$ where G is a coherent sheaf. Since $Q^\cdot$ is a bounded complex of locally frees by Lemma 1.12, we obtain that

$$\mathrm{Hom}(Q^\cdot, G[k+1])=0$$

for sufficiently large k. Hence, the map t can be lifted to some map $Q^\cdot\to\sigma^{\geq -k}P^\cdot$. □

Consider a morphism $f:X\to Y$ of finite Tor-dimension (for example flat). It defines a functor $\mathbf{L}f^*:\mathbf{D}^b(\mathrm{coh}(Y))\longrightarrow\mathbf{D}^b(\mathrm{coh}(X))$. It is clear that the functor $\mathbf{L}f^*$ sends a perfect complex on Y to a perfect complex on X. Therefore, we get a functor $\mathbf{L}\bar{f}^*:\mathbf{D}_{Sg}(Y)\longrightarrow\mathbf{D}_{Sg}(X)$. By the same reason there is a functor $\mathbf{L}\bar{f}^*:\mathbf{D}'_{Sg}(Y)\longrightarrow\mathbf{D}'_{Sg}(X)$.

Let $f:X\to Y$ be a morphism of finite Tor-dimension. Suppose that it is proper of locally finite type. Then there is the functor of direct image $\mathbf{R}f_*:\mathbf{D}^b(coh(X))\to\mathbf{D}^b(coh(Y))$ and, moreover, this functor takes perfect complexes to perfect complexes (see [1] III, or [19]). In this case we get a functor $\mathbf{R}\bar{f}_* : \mathbf{D}_{Sg}(X) \to \mathbf{D}_{Sg}(Y)$, which is the right adjoint to $\mathbf{L}\bar{f}^*$.

Now we prove a local property for triangulated categories of singularities.

Proposition 1.14. *Let X be as above and let $j : U \hookrightarrow X$ be an embedding of an open subscheme such that $Sing(X) \subset U$. Then the functor $\bar{j}^* : \mathbf{D}_{Sg}(X) \longrightarrow \mathbf{D}_{Sg}(U)$ is an equivalence of triangulated categories.*

Proof. Since X is separated the map j is affine and, hence, it is exact and it preserves quasi-coherence. This implies that there is the functor $j_* : \mathbf{D}^b(\mathrm{Qcoh}(U)) \to \mathbf{D}^b(\mathrm{Qcoh}(X))$ which is right adjoint to j^*. Take an object $B \in \mathfrak{Lfr}(U)$ and consider $j_*(B)$. It is easy to see that the object $j_*(B)$ belongs to $\mathfrak{Lfr}(X)$. Actually, this condition is local, for U it is fulfilled and for $X \setminus Sing(X)$ as for smooth scheme it is evident. Thus the functor j_* induces the functor

$$\bar{j}_* : \mathbf{D}'_{Sg}(U) \longrightarrow \mathbf{D}'_{Sg}(X).$$

Moreover, the functor $\bar{j}_*$ is right adjoint to $\bar{j}^*$

For any object $A \in \mathbf{D}^b(\mathrm{Qcoh}(X))$ we have a canonical map $\mu_A : A \longrightarrow j_* j^* A$. A cone $C(\mu_A)$ of this map is an object whose support belongs to $X \backslash U$ and does not intersect $Sing(X)$. Hence, by Lemma 1.10 the object $C(\mu_A)$ belongs to the subcategory $\mathfrak{Lfr}(X)$. This gives that μ_A becomes an isomorphism in $\mathbf{D}'_{Sg}(X)$. Therefore, the functor

$$\bar{j}^* : \mathbf{D}'_{Sg}(X) \longrightarrow \mathbf{D}'_{Sg}(U)$$

is fully faithful. On the other hand, we know that $j^* j_*(B) \cong B$ for any $B \in \mathbf{D}^b(\mathrm{Qcoh}(U))$. Hence, $\bar{j}^*$ is an equivalence.

The functor j^* preserves coherence. Thus, using Proposition 1.13, we obtain that the functor

$$\bar{j}^* : \mathbf{D}_{Sg}(X) \longrightarrow \mathbf{D}_{Sg}(U)$$

is fully faithful. Now note that by Lemma 1.11 any object $B \in \mathbf{D}_{Sg}(U)$ is isomorphic to $\mathcal{F}[k]$ where $\mathcal{F}$ is a coherent sheaf on U and any coherent sheaf on U can be obtained as the restriction of a coherent sheaf on X ([7], Ex.5.15). This implies that $\bar{j}^* : \mathbf{D}_{Sg}(X) \longrightarrow \mathbf{D}_{Sg}(U)$ is an equivalence. □

1.3. **Triangulated categories of singularities for Gorenstein schemes.** Remind the definition of a Gorenstein local ring and a Gorenstein scheme.

Definition 1.15. *A local noetherian ring A is called Gorenstein if A as a module over itself has a finite injective resolution.*

It can be shown that if A is Gorenstein than A is a dualizing complex for itself (see [8]). This means that A has finite injective dimension and the natural map

$$M \longrightarrow \mathbf{R}\operatorname{Hom}^{\cdot}(\mathbf{R}\operatorname{Hom}^{\cdot}(M, A), A)$$

is an isomorphism for any coherent A-module M and as consequence for any object from $\mathbf{D}^b(\operatorname{coh}(Spec(A)))$.

Definition 1.16. *A scheme X is Gorenstein if all of its local rings are Gorenstein local rings.*

Remark 1.17. If X is Gorenstein and has finite Krull dimension, then $\mathcal{O}_X$ is a dualizing complex for X , i.e. it has finite injective dimension as quasi-coherent sheaf and the natural map

$$\mathcal{F} \longrightarrow \mathbf{R}\underline{\mathcal{H}om}^{\cdot}(\mathbf{R}\underline{\mathcal{H}om}^{\cdot}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X)$$

is an isomorphism for any coherent sheaf $\mathcal{F}$. In particular, we have that there is an integer n_0 such that $\underline{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{O}_X) = 0$ for each quasi-coherent sheaf $\mathcal{F}$ and all $i > n_0$.

Lemma 1.18. *Let X be as above and Gorenstein. Then for any coherent sheaf $\mathcal{F}$ and $P^{\cdot} \in \mathfrak{Perf}(X)$ there is an integer m depending only on $P^{\cdot}$ such that*

$$\operatorname{Hom}^i(\mathcal{F}, P^{\cdot}) = 0$$

for all $i > m$.

Proof. For any coherent sheaf $\mathcal{F}$ and for any locally free sheaf $\mathcal{P}$ we know that

$$\underline{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{P}) = \underline{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{O}_X) \otimes \mathcal{P} = 0$$

for any $i > n_0$. Using the spectral sequence from local to global Ext's we obtain that

$$\operatorname{Ext}^i(\mathcal{F}, \mathcal{P}) = 0$$

for $i > n_0 + n$ where n is dimension of X . Since $P^{\cdot}$ is a bounded complex there exists m depending only on $P^{\cdot}$ such that

$$\operatorname{Hom}^i(\mathcal{F}, P^{\cdot}) = 0$$

for $i > m$. $\square$

Lemma 1.19. *Let X as above and Gorenstein. The the following conditions on a coherent sheaf $\mathcal{F}$ are equivalent.*

1) *The sheaves $\underline{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{O}_X)$ are trivial for all $i > 0$.*
2) *There is a right locally free resolution*

$$0 \longrightarrow \mathcal{F} \longrightarrow \{Q^0 \longrightarrow Q^1 \longrightarrow Q^3 \cdots\}$$

Proof. 1) $\Rightarrow$ 2). Denote by $\mathcal{F}^{\vee}$ the sheaf $\underline{\mathcal{H}om}(\mathcal{F}, \mathcal{O}_X)$. Consider a left locally free resolution of $\mathcal{F}^{\vee}$. Applying to it the functor $\underline{\mathcal{H}om}(\cdot, \mathcal{O}_X)$ we obtain a right locally free resolution of $\mathcal{F}$, because $\mathcal{O}_X$ is dualizing complex.

2) $\Rightarrow$ 1). Consider a brutal truncation $\sigma^{\le k}Q^{\cdot}$ for sufficiently large k . Denote by $\mathcal{G}$ the nontrivial cohomology $H^k(\sigma^{\le k}Q^{\cdot})$. For any $i>0$ we have isomorphisms

$$\underline{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{O}_X) \cong \underline{\mathcal{E}xt}^{i+k+1}(\mathcal{G}, \mathcal{O}_X) = 0$$

The last equality follows from Remark 1.17. □

Lemma 1.20. *Let X be as above and Gorenstein. Let $\mathcal{F}$ be a coherent sheaf which is perfect as a complex. Suppose that $\underline{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i>0$. Then $\mathcal{F}$ is locally free.*

Proof. If $\mathcal{F}$ is perfect then $\mathcal{F}^{\vee} = \underline{\mathcal{H}om}(\mathcal{F}, \mathcal{O}_X)$ is also perfect. Hence, it has bounded locally free resolution $\mathcal{P}^{\cdot} \longrightarrow \mathcal{F}$. Since $\mathcal{O}_X$ is a dualizing complex we obtain a bounded right resolution of $\mathcal{F} \longrightarrow \mathcal{P}^{\cdot}$. Thus $\mathcal{F}$ is locally free. □

Proposition 1.21. *Let X be as above and Gorenstein. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves such that $\underline{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i>0$. Fix N such that $\mathrm{Ext}^i(\mathcal{P}, \mathcal{G}) = 0$ for $i>N$ and for any locally free sheaf $\mathcal{P}$. Then*

$$\mathrm{Hom}_{\mathbf{D}_{Sg}(X)}(\mathcal{F}, \mathcal{G}[N]) \cong \mathrm{Ext}^N(\mathcal{F}, \mathcal{G})/\mathcal{R}$$

where $\mathcal{R}$ is the subspace of elements factoring through locally free, i.e. $e \in \mathcal{R}$ iff $e = \beta\alpha$ with $\alpha : \mathcal{F} \to \mathcal{P}$ and $\beta \in \mathrm{Ext}^N(\mathcal{P}, \mathcal{G})$ where $\mathcal{P}$ is locally free.

Proof. By the definition of a localization any morphism from $\mathcal{F}$ to $\mathcal{G}[N]$ in $\mathbf{D}_{Sg}(X)$ can be represented by a pair of morphisms in $\mathbf{D}^b(\mathrm{coh}(X))$ of the form

$$\mathcal{F} \xleftarrow{s} A \xrightarrow{a} \mathcal{G}[N] \tag{2}$$

such that the cone $C(s)$ is a perfect complex. By Lemma 1.19 there is a right locally free resolution $\mathcal{F} \longrightarrow Q^{\cdot}$. We consider a brutal truncation $\sigma^{\le k}Q^{\cdot}$ for sufficiently large k such that $\mathrm{Hom}(\mathcal{E}[-k-1], C(s)) = 0,$ where $\mathcal{E} = H^k(\sigma^{\le k}Q^{\cdot})$. Such k exists by Lemma 1.18. Using the triangle

$$\mathcal{E}[-k-1] \longrightarrow \mathcal{F} \longrightarrow \sigma^{\le k}Q^{\cdot} \longrightarrow \mathcal{E}[-k]$$

we find that the map $\mathcal{F} \to C(s)$ can be lifted to a map $\sigma^{\le k}Q^{\cdot} \to C(s)$. Therefore, there is a map $\mathcal{E}[-k-1] \to A$ and we get a morphism in $\mathbf{D}_{Sg}(X)$ of the form

$$\mathcal{F} \xleftarrow{s'} \mathcal{E}[-k-1] \xrightarrow{e} \mathcal{G}[N] \tag{3}$$

which is equivalent to the morphism (2). Since $\mathrm{Ext}^i(\mathcal{P}, \mathcal{G}) = 0$ for $i>N$ and for any locally free sheaf $\mathcal{P}$ we get

$$\mathrm{Hom}(\sigma^{\le k}Q^{\cdot}[-1], \mathcal{G}[N]) = 0.$$

Hence, there is a map f which completes the following diagram

$$\begin{tikzcd} & \mathcal{E}[-k-1] \arrow[dl, "s'"'] \arrow[dr, "e"] & \\ \mathcal{F} \arrow[rr, "f"] & & \mathcal{G}[N] \end{tikzcd} .$$

This gives us that the map f is equivalent to the map (3) and as consequence to the map (2). Thus, any morphism from $\mathcal{F}$ to $\mathcal{G}[N]$ in $\mathbf{D}_{Sg}(X)$ is equivalent to a morphism from $\mathcal{F}$ to $\mathcal{G}[N]$ in $\mathbf{D}^b(\mathrm{coh}(X))$. Now if f is equivalent to 0-morphism then repeating the procedure above we find that f factors through a morphism $\sigma^{\le k}Q^{\cdot} \to \mathcal{G}[N]$. By the condition on G any such morphism can be lifted to a morphism $Q^0 \to \mathcal{G}[N]$. Hence, if f is equivalent to the 0-morphism in $\mathbf{D}_{Sg}(X)$ then it factors through Q^0. □

Remark 1.22. If X is affine then N can be taken equal to 0.

Proposition 1.23. *Let X be as above and Gorenstein. Then any object $A \in \mathbf{D}_{Sg}(X)$ is isomorphic to the image of a coherent sheaf $\mathcal{F}$ such that $\underline{\mathcal{E}xt}^i(\mathcal{F},\mathcal{O}_X)=0$ for all $i>0$.*

Proof. An object A is a bounded complex of coherent sheaves. Let us take locally free bounded above resolution $P^{\cdot} \xrightarrow{\sim} A$ which exists by Lemma 1.5. Consider a brutal truncation $\sigma^{\ge -k}P^{\cdot}$ for sufficiently large $k \gg 0$. Denote by $\mathcal{G}$ the cohomology $H^{-k}(\sigma^{\ge -k}P^{\cdot})$. Since A is bounded and X is Gorenstein we know that $\mathbf{R}\underline{\mathcal{H}om}^i(A,\mathcal{O}_X)=0$ for sufficiently large i. This implies that iff $k \gg 0$ then $\underline{\mathcal{E}xt}^i(\mathcal{G},\mathcal{O}_X)=0$ for all $i>0$. Moreover, we get that $A \cong \mathcal{G}[k+1]$ in $\mathbf{D}_{Sg}(X)$.

By Lemma 1.19 there is a right locally free resolution $\mathcal{G} \to Q^0 \to Q^1 \to \cdots$. Consider $\mathrm{Im}\, d_{k-1} = \mathrm{Ker}\, d_k \subset Q^k$ and denote it by $\mathcal{F}$. Applying again Lemma 1.19 we obtain that $\underline{\mathcal{E}xt}^i(\mathcal{F},\mathcal{O}_X)=0$ for all $i \ge 0$. And we have an isomorphism $A \cong \mathcal{G}[k+1] \cong \mathcal{F}$ in $\mathbf{D}_{Sg}(X)$. □

Corollary 1.24. *Let X be as above, proper and Gorenstein and let $U \subset X$ be open subscheme such that $Sing(X) \subset U$. Then $\dim_k \mathrm{Hom}(A,B) < \infty$ for any two objects $A,B \in \mathbf{D}_{Sg}(U)$.*

Proof. We know that for any two objects in $\mathbf{D}^b(\mathrm{coh}(X))$ the space of morphisms is finite dimensional. Now the statement of the corollary immediately follows from Propositions 1.21 and 1.23. □

2. Reduction of dimension

Let X be a separated regular noetherian scheme of finite Krull dimension. Any such scheme has enough locally free sheaves (see [1], II). Let $f : X \to \mathbb{A}^1$ be a flat morphism. Consider the scheme $Y = X \times \mathbb{A}^2$ and a morphism $g = f + xy$ to $\mathbb{A}^1$, where x, y are coordinates on $\mathbb{A}^2$. Denote by $X_0 = X/f$ and $Y_0 = Y/g$ the fibers of f and g respectively over the point 0.

Consider the scheme $Z = Y_0/x$. There are natural maps $i : Z \to Y_0$ and $q : Z \to X_0$ where the former is a closed embedding and the latter is an $\mathbb{A}^1$-bundle. All the schemes, introduced above, are separated noetherian schemes of finite Krull dimension that have enough locally free sheaves.

Consider the composition functor $\mathbf{R}i_*q^* : \mathbf{D}^b(\mathrm{coh}(X_0)) \to \mathbf{D}^b(\mathrm{coh}(Y_0))$ and denote it by Φ_Z . The aim of this paragraph is to prove the following theorem.

Theorem 2.1. *The functor* $\Phi_Z : \mathbf{D}^b(\mathrm{coh}(X_0)) \to \mathbf{D}^b(\mathrm{coh}(Y_0))$ *defined by formula*

$$\Phi_Z(\cdot) = \mathbf{R}i_*q^*(\cdot)$$

induces a functor $\overline{\Phi}_Z : \mathbf{D}_{Sg}(X_0) \longrightarrow \mathbf{D}_{Sg}(Y_0)$ *which is an equivalence.*

At first, we consider partial compactifications of Y and Y_0 . Denote by $\overline{Y}$ the scheme $X \times \mathbb{P}^2$ and let $\overline{Y}_0 \subset \overline{Y}$ be a closed subscheme which is given by equation $fz^2 + xy = 0$, where now x, y, z are projective coordinates on $\mathbb{P}^2$. There is a flat map $\pi : \overline{Y}_0 \to X$ which is a conic bundle over X . The scheme $\overline{Y}_0$ is a partial compactification of Y_0 such that $Sing(\overline{Y}_0) = Sing(Y_0)$. Hence, by Proposition 1.14 we have an equivalence $\mathbf{D}_{Sg}(\overline{Y}_0) \cong \mathbf{D}_{Sg}(Y_0)$ of triangulated categories of singularities.

Consider the cartesian square

$$\begin{array}{ccc}
\widetilde{Z} & \xrightarrow{\ \bar{i}\ } & \overline{Y}_0 \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\
X_0 & \xrightarrow{\ j\ } & X
\end{array}$$

Here $\widetilde{Z} = X_0 \times_X \overline{Y}_0$ is a fiber product. The scheme $\widetilde{Z}$ is a union of two components $\overline{Z}_1 \cup \overline{Z}_2$. Each $\overline{Z}_i$ is isomorphic to $\mathbb{P}^1 \times X_0$ and their intersection is isomorphic to X_0 . The component $\overline{Z}_1$ is a partial compactification of $Z = \mathbb{A}^1 \times X_0$. We denote by i_1, i_2 the closed embeddings of $\overline{Z}_1, \overline{Z}_2$ in $\overline{Y}_0$ and we denote by $W \subset \overline{Y}_0$ the intersection $\overline{Z}_1 \cap \overline{Z}_2$. We know that W is isomorphic to X_0 . Moreover, $Sing(\overline{Y}_0)$ is contained in W and coincides with $Sing(X_0)$ under the isomorphism $W \cong X_0$.

Lemma 2.2. *The closed embedding* $i_1 : \overline{Z}_1 \hookrightarrow \overline{Y}_0$ *is regular and* $\overline{Z}_1$ *is a Cartier divisor in* $\overline{Y}_0$. *The restriction of the line bundle* $\mathcal{O}_{\overline{Y}_0}(\overline{Z}_1)$ *on* $\overline{Z}_1$ *is isomorphic to* $\mathcal{O}_{\overline{Z}_1}(-1) = \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{X_0}$.

Proof. Consider a Cartier divisor in $\overline{Y}_0$ given by the equation $x = 0$. It is the union of $\overline{Z}_1$ and D , where D is a component of $\overline{Y}_0 \setminus Y_0$. Hence, D does not meet the singularities of $\overline{Y}_0$. This implies that D is a Cartier divisor. Therefore, $\overline{Z}_1$ is a Cartier divisor too and i_1 is a regular embedding.

Consider the divisor $\widetilde{Z} = \overline{Z}_1 \cup \overline{Z}_2$. It is equal to $\pi^{-1}(X_0)$ and its restriction on $\overline{Z}_1$ is trivial. The restriction of $\mathcal{O}(\overline{Z}_2)$ on $\overline{Z}_1$ is $\mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{X_0}$ because the intersection $\overline{Z}_2 \cap \overline{Z}_1 = W$ is isomorphic to X_0 . Hence, the restriction of $\mathcal{O}(\overline{Z}_1)$ on $\overline{Z}_1$ is $\mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{X_0}$ $\square$

Denote by p_1 the projection of $\overline{Z}_1$ on X_0 and consider the commutative diagram

$$\begin{array}{ccc} \overline{Z}_1 & \xrightarrow{i_1} & \overline{Y}_0 \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X_0 & \xrightarrow{j} & X \end{array}$$

Proposition 2.3. *The functor* $\mathbf{R}i_{1*}p_1^* : \mathbf{D}^b(\mathrm{coh}(X_0)) \to \mathbf{D}^b(\mathrm{coh}(\overline{Y}_0))$ *is fully faithful.*

Proof. The functor $\Phi_{\overline{Z}_1} = \mathbf{R}i_{1*}p_1^*$ has a right adjoint functor $\Phi_{\overline{Z}_1 *} = \mathbf{R}p_{1*}i_1^{\flat}$ where $i_1^{\flat}(\cdot) \cong \mathbf{L}i_1^*(\cdot \otimes \mathcal{O}(Z_1))[-1]$ (see, for example, [8] Cor.7.3).

At first, note that the functor $p_1^* : \mathbf{D}^b(\mathrm{coh}(X_0)) \longrightarrow \mathbf{D}^b(\mathrm{coh}(\overline{Z_1}))$ is fully faithfull, because by the projection formula we have an isomorphism

$$\mathbf{R}p_{1*}p_1^*(A) \cong A \otimes \mathbf{R}p_{1*}\mathcal{O}_{\overline{Z}_1} \cong A \otimes \mathcal{O}_{X_0} \cong A$$

for any $A \in \mathbf{D}^b(\mathrm{coh}(X_0))$.

Consider the canonical transformation of functors $\mathrm{id} \to \Phi_{\overline{Z}_1 *}\Phi_{\overline{Z}_1}$. We have to show that this transformation is an isomorphism of functors. Take an object $A \in \mathbf{D}^b(\mathrm{coh}(X_0))$ and consider an exact triangle

$$A \longrightarrow \Phi_{\overline{Z}_1 *}\Phi_{\overline{Z}_1}(A) \longrightarrow C \tag{4}$$

To show that $C = 0$ it is sufficient to check that $\mathbf{R}j_*C = 0$, because j is a closed embedding.

Since the functor p_1^* is fully faithful, the triangle (4) is the image of the triangle

$$p_1^*A \longrightarrow i_1^{\flat}\mathbf{R}i_{1*}p_1^*A \longrightarrow B \tag{5}$$

under the functor $\mathbf{R}p_1^*$, where the former morphism is a canonical map induced by the natural transformation $\mathrm{id} \to i_1^{\flat}\mathbf{R}i_{1*}$. Applying the functor $\mathbf{R}i_{1*}$ to the exact triangle (5) we obtain the following triangle

$$\mathbf{R}i_{1*}p_1^*A \longrightarrow \mathbf{R}i_{1*}i_1^{\flat}\mathbf{R}i_{1*}p_1^*A \longrightarrow \mathbf{R}i_{1*}B$$

This triangle is splited by the canonical morphism $\mathbf{R}i_{1*}i_1^{\flat}\mathbf{R}i_{1*}p_1^*A \to \mathbf{R}i_{1*}p_1^*A$ which is obtained by the tensoring product of the object $\mathbf{R}i_{1*}p_1^*A$ with a map α from the following exat triangle

$$i_{1*}\mathcal{O}_{\overline{Z}_1}(\overline{Z}_1)[-1] \xrightarrow{\alpha} \mathcal{O}_{\overline{Y}_0} \longrightarrow \mathcal{O}_{\overline{Y}_0}(Z_1) \longrightarrow i_{1*}\mathcal{O}_{\overline{Z}_1}(\overline{Z}_1)$$

Therefore, the object $\mathbf{R}i_*B$ is isomorphic to $\mathbf{R}i_*p_1^*A(\overline{Z_1}) = \mathbf{R}i_*(p_1^*A \otimes \mathcal{O}_{\overline{Z}_1}(-1))$. Now we have the following sequence of isomorphisms

$$\begin{aligned} \mathbf{R}j_*C &\cong \mathbf{R}j_*\mathbf{R}p_{1*}B \cong \mathbf{R}\pi_*\mathbf{R}i_{1*}B \cong \mathbf{R}\pi_*\mathbf{R}i_{1*}(p_1^*A \otimes \mathcal{O}_{\overline{Z}_1}(-1)) \\ &\cong \mathbf{R}j_*\mathbf{R}p_{1*}(p_1^*A \otimes \mathcal{O}_{\overline{Z}_1}(-1)) \cong \mathbf{R}j_*(A \otimes \mathbf{R}p_{1*}\mathcal{O}_{\overline{Z}_1}(-1)) \cong 0 \end{aligned}$$

Thus the functor $\Phi_{\overline{Z}_1} = \mathbf{R}i_{1*}p_1^*$ is fully faithful. $\square$

Corollary 2.4. *The functor* $\Phi_{\overline{Z}_1} : \mathbf{D}^b(\mathrm{coh}(X_0)) \to \mathbf{D}^b(\mathrm{coh}(\overline{Y}_0))$ *induces a functor* $\overline{\Phi}_Z : \mathbf{D}_{Sg}(X_0) \longrightarrow \mathbf{D}_{Sg}(\overline{Y}_0)$ *which is fully faithful.*

Proof. The functors p_1^* and $i_1^\flat = \mathbf{L}i_1^*(\cdot \otimes \mathcal{O}(\overline{Z}_1))[-1]$ take perfect complexes to perfect complexes as functors of inverse images. The functors $\mathbf{R}i_{1*}$ and $\mathbf{R}p_{1*}$ also preserve perfect complexes, because both morphisms i_1 and p_1 are finite Tor-dimension, proper, and of finite type.

Thus, we get a functor $\overline{\Phi}_{\overline{Z}_1} : \mathbf{D}_{Sg}(X_0) \to \mathbf{D}_{Sg}(Y_0)$ and this functor has the right adjoint $\overline{\Phi}_{\overline{Z}_1 *}$. As the composition $\Phi_{\overline{Z}_1 *}\Phi_{\overline{Z}_1}$ is isomorphic to the identity functor, the composition $\overline{\Phi}_{\overline{Z}_1 *}\overline{\Phi}_{\overline{Z}_1}$ is also isomorphic to the identity functor. □

Thus, we obtained the functor $\overline{\Phi}_{\overline{Z}_1} : \mathbf{D}_{Sg}(X_0) \longrightarrow \mathbf{D}_{Sg}(\overline{Y}_0)$ and showed that this functor is fully faithful. To complete the proof of the theorem we have to check that this functor is an equivalence. We show that any object $\mathcal{E} \in \mathbf{D}_{Sg}(\overline{Y}_0)$ satisfying condition $\overline{\Phi}_{\overline{Z}_1 *}\mathcal{E} = 0$ is zero object. The property to be an equivalence will easy follow from this fact.

Lemma 2.5. *Any object* $A \in \mathbf{D}^b(\mathrm{coh}(\overline{Z}_1))$ *such that* $\mathbf{R}p_{1*}A = 0$ *is isomorphic to an object of the form* $p_1^* B \otimes \mathcal{O}_{\overline{Z}_1}(-1)$ *for some* $B \in \mathbf{D}^b(\mathrm{coh}(X_0))$ *.*

Proof. Denote by B the object $\mathbf{R}p_{1*}(A(1))$. We have a natural map $p_1^*B \otimes \mathcal{O}_{\overline{Z}_1}(-1) \to A$. Denote by C the cone of this map. The object C satisfies the following conditions

$$\mathbf{R}p_{1*}C = 0 \qquad \text{and} \qquad \mathbf{R}p_{1*}(C(1)) = 0,$$

the latter condition is following from the fact that p_1^* is fully faithful. Now any sheaf $\mathcal{O}_{\overline{Z}_1}(n)$ has a resolution of the form

$$\mathcal{O}_{\overline{Z}_1}(-1)^{\oplus n} \longrightarrow \mathcal{O}_{\overline{Z}_1}^{\oplus (n+1)} \longrightarrow \mathcal{O}_{\overline{Z}_1}(n).$$

Tensoring this sequence with the object C and applying the functor $\mathbf{R}p_{1*}$ we get that $\mathbf{R}p_{1*}(C(n)) = 0$ for all n . Hence, the object C is zero, because the sheaf $\mathcal{O}_{\overline{Z}}(1)$ is relatively ample. □

Actually, this Lemma shows us that the category $\mathbf{D}^b(\mathrm{coh}(\overline{Z}_1))$ has a semiorthogonal decomposition of the form

$$\left\langle p_1^*\mathbf{D}^b(\mathrm{coh}(X_0)) \otimes O_{\overline{Z}_1}(-1), \quad p_1^*\mathbf{D}^b(\mathrm{coh}(X_0)) \right\rangle$$

(for definition see [2]). It can be proved for any $\mathbb{P}^1$-bundle and moreover for the projectivization of any bundle. The proof for smooth base can be found in [16] and it works for any base.

The second Lemma is also almost evident.

Lemma 2.6. *Let* $i : Z \hookrightarrow Y$ *be a closed embedding of a Cartier divisor. Let* $\mathcal{E}$ *be a sheaf on* Y *such that its restriction to the complement* $U = X \setminus Z$ *is locally free and* $\mathbf{L}i^*\mathcal{E}$ *is isomorphic to a local free sheaf on* Z *. Then* $\mathcal{E}$ *is local free on* Y *.*

Proof. Let us consider a morphism $\mathcal{E}(-Z)\to\mathcal{E}$. By the assumption it is a monomorphism and the cokernel of this map is a locally free sheaf on Z . To prove that $\mathcal{E}$ is locally free it is sufficient to show that for any closed point $y\in Y$ we have the equalities

$$\mathrm{Ext}^i(\mathcal{E},\mathcal{O}_y)=0$$

for all $i>0$. The sheaf $\mathcal{E}$ is locally free on U . Hence, we only need to consider the case $y\in Z$. We have the following long exact sequence of Ext's

$$\cdots\longrightarrow \mathrm{Ext}^i(\mathcal{E}|_Z,\mathcal{O}_y)\longrightarrow \mathrm{Ext}^i(\mathcal{E},\mathcal{O}_y)\longrightarrow \mathrm{Ext}^i(\mathcal{E}(-Z),\mathcal{O}_y)\longrightarrow \mathrm{Ext}^{i+1}(\mathcal{E}|_Z,\mathcal{O}_y)\longrightarrow\cdots$$

where $\mathcal{E}|_Z$ is a short notation for $i_*i^*\mathcal{E}=\mathbf{R}i_*\mathbf{L}i^*\mathcal{E}$.

Moreover, all maps $\mathrm{Ext}^i(\mathcal{E},\mathcal{O}_y)\to\mathrm{Ext}^i(\mathcal{E}(-Z),\mathcal{O}_y)$ are zero, because $y\in Z$. Now, since $\mathcal{O}_y=i_*\mathcal{O}_y$, we have isomorphisms

$$\mathrm{Ext}^i(\mathcal{E}|_Z,\mathcal{O}_y)=\mathrm{Hom}^i_Z(\mathbf{L}i^*\mathcal{E}|_Z,\mathcal{O}_y)$$

The complex $\mathbf{L}i^*\mathcal{E}|_Z$ has only 0 and -1 cohomology sheaves which are locally free on Z . Hence,

$$\mathrm{Hom}^i_Z(\mathbf{L}i^*\mathcal{E}|_Z,\mathcal{O}_y)=0$$

for all $i\neq 0,1$.

This gives us that $\mathrm{Ext}^i(\mathcal{E}(-Z),\mathcal{O}_y)=0$ for all $i\neq 0$. Therefore, $\mathcal{E}(-Z)$ is locally free, and $\mathcal{E}$ is locally free as well. $\square$

Using these two lemmas we can prove the following proposition.

Proposition 2.7. *Assume that an object* $\mathcal{E}\in\mathbf{D}_{Sg}(\overline{Y}_0)$ *satisfies the condition* $\overline{\Phi}_{\overline{Z}_1*}\mathcal{E}=0$. *Then* $\mathcal{E}=0$ *in* $\mathbf{D}_{Sg}(\overline{Y}_0)$.

Proof. At first, note that all schemes $X_0,\overline{Z}_1,\overline{Y}_0$ are Gorenstein. By Proposition 1.23 we can assume that $\mathcal{E}$ is a sheaf and $\underline{\mathcal{E}xt}^i(\mathcal{E},\mathcal{O}_X)=0$ for all $i\neq 0$. Note that any such $\mathcal{E}$ is locally free on the complement $X\setminus Sing(X)$. In addition, for such $\mathcal{E}$ we have $\mathbf{L}i_1^*\mathcal{E}\cong i_1^*\mathcal{E}$ is a sheaf.

Denote by $\mathcal{L}$ the relatively ample line bundle on $\overline{Y}_0$ obtained by the restriction of the line bundle $\mathcal{O}_{\overline{Y}}(1)$ on $\overline{Y}=\mathbb{P}^2\times X$. We can see that the object $\mathcal{E}\otimes\mathcal{L}^{\otimes n}$ is isomorphic to $\mathcal{E}$ in the category $\mathbf{D}_{Sg}(\overline{Y}_0)$, because there is an inclusion $\mathcal{E}\to\mathcal{E}\otimes\mathcal{L}^{\otimes n}$ such that the support of the cokernel does not intersect $Sing(\overline{Y}_0)$ and, hence, it is perfect by Lemma 1.10.

Take the sheaf $i_1^*\mathcal{E}$ and denote by $\mathcal{F}$ the sheaf $p_{1*}i_1^*\mathcal{E}$ on X_0 . Tensoring $\mathcal{E}$ with $\mathcal{L}^n$ if it is necessary we can reduce the situation to the case

$$\mathbf{R}^ip_{1*}i_1^*\mathcal{E}=0\quad\text{for all}\quad i>0\quad\text{and}\quad p_1^*\mathcal{F}\xrightarrow{\alpha}i_1^*\mathcal{E}\quad\text{is surjective.}$$

By Lemma 2.5 the kernel of α is a sheaf of the form $p_1^*\mathcal{G}\otimes\mathcal{O}_{\overline{Z}_1}(-1),$ where $\mathcal{G}$ is a sheaf on X_0 . By the assumption, the sheaf $\mathcal{F}$ is perfect as a complex on X_0 . Moreover, the sheaf

$\mathcal{G} \cong p_{1*} i_1^* (\mathcal{E} \otimes \mathcal{L}^{-1})$ is also perfect as a complex on X_0 . Hence, the sheaf $i_1^*\mathcal{E}$ on $\overline{Z}_1$ is a perfect complex on $\overline{Z}_1$. On the other hand, we know that the sheaf $\mathcal{E}$ has a right local free resolution. Hence the sheaf $i_1^*\mathcal{E}$ as the restriction with respect to a regular embedding of a divisor also has a right local free resolution. By Lemma 1.19 this implies that

$$\underline{\mathcal{E}xt}^i(i_1^*\mathcal{E}, \mathcal{O}_{\overline{Z}_1}) = 0 \quad \text{for all} \quad i > 0.$$

By Lemma 1.20 this means that $i_1^*\mathcal{E}$ is locally free on $\overline{Z}_1$. Now apply Lemma 2.6 we get that $\mathcal{E}$ is locally free on whole $\overline{Y}_0$. Hence, it is isomorphic to the zero object in $\mathbf{D}_{Sg}(\overline{Y}_0)$. □

Proof of Theorem 2.1 We already know that $\overline{\Phi}_{\overline{Z}_1}$ is fully faithful. Take an object $A \in \mathbf{D}_{Sg}(\overline{Y}_0)$ and consider the natural map $\overline{\Phi}_{\overline{Z}_1}\overline{\Phi}_{\overline{Z}_1 *}A \to A$. Denote by C its cone. Apply the functor $\overline{\Phi}_{\overline{Z}_1 *}$ to the obtained exact triangle. Since $\overline{\Phi}_{\overline{Z}_1}$ is fully faithful, we get that $\overline{\Phi}_{\overline{Z}_1 *}C = 0$. By Proposition 2.7 the object C is the zero object. Hence, the functor $\overline{\Phi}_{\overline{Z}_1 *}$ is also fully faithful and, consequently, it is an equivalence. It remains only to note that the functor $\overline{\Phi}_Z : \mathbf{D}_{Sg}(X_0) \to \mathbf{D}_{Sg}(Y_0)$ is the composition of the functor $\overline{\Phi}_{\overline{Z}_1} : \mathbf{D}_{Sg}(X_0) \to \mathbf{D}_{Sg}(\overline{Y}_0)$ and the functor $\overline{J}^* : \mathbf{D}_{Sg}(\overline{Y}_0) \to \mathbf{D}_{Sg}(Y_0)$, where $J : Y_0 \hookrightarrow \overline{Y}_0$ is the open embedding. Both of these functors are equivalence. Hence, $\overline{\Phi}_Z$ is an equivalence too. □

3. Triangulated category of B-branes in Landau-Ginsburg model

3.1. **Kontsevich's proposal and categories of pairs.** A mathematical definition of the categories of B-branes in Landau-Ginsburg models is proposed by M.Kontsevich (see also [11]).

By a Landau-Ginsburg model we mean the following data: a smooth variety (or regular scheme) X and a regular function W on X such that the morphism $W : X \longrightarrow \mathbb{A}^1$ is flat (for the definition of B-branes we don't need a symplectic form on X which have to be in a LG model too).

With any point $w_0 \in \mathbb{A}^1$ we can associate a differential $\mathbb{Z}/2\mathbb{Z}$-graded category $DG_{w_0}(W)$, an exact category $Pair_{w_0}(W)$, and a triangulated category $DB_{w_0}(W)$. We give constructions of these categories under the condition that $X = Spec(A)$ is affine (see [11]). The general definition is more sophisticated.

Since the category of coherent sheaves on an affine scheme $X = Spec(A)$ is the same as the category of finitely generated A-modules we will frequently go from sheaves to modules and back. Note that under this equivalence locally free sheaves are the same as projectives modules.

Objects of all these categories are ordered pairs

$$\overline{P} := \left(\; P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \; \right)$$

where P_0, P_1 are finitely generated projective A-modules and the compositions p_0p_1 and p_1p_0 are the multiplications by the element $(W - w_0) \in A$.

Morphisms from $\overline{P}$ to $\overline{Q}$ in the category $DG_{w_0}(W)$ form $\mathbb{Z}/2\mathbb{Z}$-graded complex

$$\mathbb{H}\mathrm{om}(\overline{P},\ \overline{Q}) = \bigoplus_{i,j} \mathrm{Hom}(P_i, Q_j)$$

with a natural grading $(i-j) \mod 2$, and with a differential D acting on homogeneous elements of degree k as

$$Df = q \circ f - (-1)^k f \circ p.$$

The space of morphisms $\mathrm{Hom}(\overline{P},\ \overline{Q})$ in the category $Pair_{w_0}(W)$ is the space of homogeneous of degree 0 morphisms in $DG_{w_0}(W)$ which commutes with the differential. The space of morphisms in the category $DB_{w_0}(W)$ is the space of morphisms in $Pair_{w_0}(W)$ modulo null-homotopic morphisms, i.e.

$$\mathrm{Hom}_{Pair_{w_0}(W)}(\overline{P},\ \overline{Q}) = \mathrm{Z}^0(\mathbb{H}\mathrm{om}(\overline{P},\ \overline{Q})), \qquad \mathrm{Hom}_{DB_{w_0}(W)}(\overline{P},\ \overline{Q}) = \mathrm{H}^0(\mathbb{H}\mathrm{om}(\overline{P},\ \overline{Q})).$$

Thus a morphism $f : \overline{P} \to \overline{Q}$ in the category $Pair_{w_0}(W)$ is a pair of morphisms $f_1 : P_1 \to Q_1$ and $f_0 : P_0 \to Q_0$ such that $f_1p_0 = q_0f_0$ and $q_1f_1 = f_0p_1$. The morphism f is null-homotopic if there are two morphisms $s : P_0 \to Q_1$ and $t : P_1 \to Q_0$ such that $f_1 = q_0t + sp_1$ and $f_0 = tp_0 + q_1s$.

It is clear that the category $Pair_{w_0}(W)$ is an exact category with respect to componentwise monomorphisms and epimorphisms (see definition in [17]).

The category $DB_{w_0}(W)$ can be endowed with a natural structure of a triangulated category. To determine it we have to define a translation functor $[1]$ and a class of exact triangles.

The translation functor can be defined as a functor that takes an object $\overline{P}$ to the object

$$\overline{P}[1] = \Big(P_0 \underset{-p_1}{\overset{-p_0}{\rightleftarrows}} P_1 \Big),$$

i.e. it changes the order of the modules and signs of the morphisms, and takes a morphism $f = (f_0, f_1)$ to the morphism $f[1] = (f_1, f_0)$. We see that the functor $[2]$ is the identity functor.

For any morphism $f : \overline{P} \to \overline{Q}$ from the category $Pair_{w_0}(W)$ we define a mapping cone $C(f)$ as an object

$$C(f) = \Big(Q_1 \oplus P_0 \underset{c_0}{\overset{c_1}{\rightleftarrows}} Q_0 \oplus P_1 \Big)$$

such that

$$c_0 = \begin{pmatrix} q_0 & f_1 \\ 0 & -p_1 \end{pmatrix}, \qquad c_1 = \begin{pmatrix} q_1 & f_0 \\ 0 & -p_0 \end{pmatrix}.$$

There are maps $g : \overline{Q} \to C(f)$, $g = (\mathrm{id}, 0)$ and $h : C(f) \to \overline{P}[1]$, $h = (0, -\mathrm{id})$.

Now we define a standard triangle in the category $DB_{w_0}(W)$ as a triangle of the form

$$\overline{P} \xrightarrow{f} \overline{Q} \xrightarrow{g} C(f) \xrightarrow{h} \overline{P}[1].$$

for some $f \in Pair_{w_0}(W)$.

Definition 3.1. *A triangle* $\overline{P} \to \overline{Q} \to \overline{R} \to \overline{P}[1]$ *in* $DB_{w_0}(W)$ *will be called an exact triangle if it is isomorphic to a standard triangle.*

Proposition 3.2. *The category* $DB_{w_0}(W)$ *endowed with the translation functor* $[1]$ *and the above class of exact triangles becomes a triangulated category.*

Proof. The proof is the same as the proof of the analogous result for a usual homotopic category (see, for example, [5] ot [12]). □

Definition 3.3. *We define a category of B-branes on* X *with the superpotential* W *as a disjoint union* $DB(W) = \coprod_{w \in \mathbb{A}^1} DB_w(W)$.

Note that since X is regular, the set of points on $\mathbb{A}^1$ with singular fibers is finite (see [7], III,Cor.10.7). It will be shown that the category $DB_w(W)$ is trivial if the fiber over point w is smooth. Hence, the category $DB(W)$ is a disjoint union of finitely many numbers of categories.

Actually, the construction of the category $DB_{w_0}(W)$ fits into a general construction of the stable category associated with an exact category. We briefly recall the definition of an exact category, which was introduced by Quillen in [17].

An exact category $\mathcal{E}$ is an additive category together with a choice of a class of sequences $\{F \rightarrowtail E \twoheadrightarrow G\}$ said to be exact. This determines two classes of morphisms: the admissible epimorphisms $E \twoheadrightarrow G$ and the admissible monomorphisms $F \rightarrowtail E$. The exact category is to satisfy the following axioms: Any sequence isomorphic to an exact sequence is exact. In any exact sequence $F \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G$, the map i is a kernel of p and p is a cokernel of i . The class of admisible monomorphisms is closed under composition and is closed under cobase change by pushout along an arbitrary map $F \to F'$. Dually, the class of admissible epimorphisms is closed under composition and under base change by pullback along an arbitrary map $G' \to G$.

This definition is equivalent to the original definition of Quillen (see [14]).

An object $I \in \mathcal{E}$ is injective (resp. P is projective) if the sequence

$$\mathrm{Hom}(E, I) \to \mathrm{Hom}(F, I) \to 0 \qquad (\text{resp. } \mathrm{Hom}(P, E) \to \mathrm{Hom}(P, G) \to 0)$$

is exact for each admissible monomorphism (resp. epimorphism). We say that $\mathcal{E}$ has enough injectives, if for each $F \in \mathcal{E}$ there is an admissible monomorphism in an injective.

If $\mathcal{E}$ has enough injectives then we can define the stable category $\underline{\mathcal{E}}$ as a category which has the same objects as $\mathcal{E}$ and a morphism in $\mathcal{E}$ is the equivalence class $\overline{f}$ of a morphism f of

$\mathcal{E}$ modulo the subgroup of morphisms factoring through an injective in $\mathcal{E}$ (see, for example, [6, 13, 14]).

If $\mathcal{E}$ also has enough projectives (i.e. for each $G \in \mathcal{E}$ there is an admissible epimorphism $P \twoheadrightarrow G$ with projective P), and the classes of projectives and injectives coincide, then $\mathcal{E}$ is called a Frobenius category. The stable category $\underline{\mathcal{E}}$ associated with a Frobenius category has a natural structure of a triangulated category ([6]).

In our case, the category $Pair_{w_0}(W)$ is an exact category with respect to componentwise monomorphisms and epimorphisms. Moreover, it can be shown that $Pair_{w_0}(W)$ is a Frobenius category and the class of injectives consists exactly of homotopic to zero pairs. Hence, the category $DB_{w_0}(W)$ is nothing more than the stable category associated to the exact category $Pair_{w_0}(W)$ and has the natural structure of a triangulated category. It gives another proof of Proposition 3.2.

3.2. **Categories of pairs and categories of singularities.** Further we suppose $w_0 = 0$. Denote by X_0 the fiber of $f : X \to \mathbb{A}^1$ over the point 0. With any pair $\overline{P}$ we can associate a short exact sequence

$$0 \longrightarrow P_1 \xrightarrow{p_1} P_0 \longrightarrow \text{Coker } p_1 \longrightarrow 0 \tag{6}$$

We can attach to an object $\overline{P}$ the sheaf $\text{Coker } p_1$. This is a sheaf on X. But the multiplication by W annihilates it. Hence, we can consider $\text{Coker } p_1$ as a sheaf on X_0. Any morphism $f : \overline{P} \to \overline{Q}$ in $Pair_0(W)$ gives a morphism between cokernels. This way we get a functor $\text{Cok} : Pair_0(W) \longrightarrow coh(X_0)$.

Lemma 3.4. *The functor* Cok *is full.*

Proof. Any map $g : \text{Coker } p_1 \to \text{Coker } q_1$ can be extended to a map of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{p_1} & P_0 & \longrightarrow & \text{Coker } p_1 & \longrightarrow & 0 \\
 & & {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & Q_1 & \xrightarrow{q_1} & Q_0 & \longrightarrow & \text{Coker } q_1 & \longrightarrow & 0,
\end{array}$$

because P_1 and P_0 are projective. To prove the lemma we need to show that $f = (f_1, f_0)$ is a map of pairs, i.e $f_1 p_0 = q_0 f_0$. We have the sequence of equalities

$$q_1(f_1 p_0 - q_0 f_0) = f_0 q_0 p_0 - q_1 q_0 f_0 = f_0 W - W f_0 = 0$$

Since q_1 is an embedding, we obtain that $f_1 p_0 = q_0 f_0$. $\square$

Lemma 3.5. *For any pair* $\overline{P}$ *the coherent sheaf* $\text{Coker } p_1$ *on* X_0 *satisfies the condition*

$$\text{Ext}^i(\text{Coker } p_1, \mathcal{O}_{X_0}) = 0$$

for all $i > 0$.

Proof. First note that X_0 is Gorenstein as a full intersection. Consider the restriction of the sequence (6) on X_0 . We obtain an exact sequence

$$0 \to \text{Coker } p_1 \longrightarrow P_1/W \overset{p_1|_W}{\longrightarrow} P_0/W \longrightarrow \text{Coker } p_1 \to 0.$$

This gives us a periodic resolution of the sheaf Coker p_1

$$0 \to \text{Coker } p_1 \longrightarrow P_1/W \overset{p_1|_W}{\longrightarrow} P_0/W \longrightarrow P_1/W \longrightarrow \cdots$$

Since X_0 is affine and Gorenstein the existance of such resolution implies that

$$\text{Ext}^i(\text{Coker } p_1, \mathcal{O}_{X_0}) = 0$$

for all $i > 0$ (by Lemma 1.19). $\square$

Now we show that the functor Cok induces an exact functor between triangulated categories $DB_0(W)$ and $\mathbf{D}_{Sg}(X_0)$.

Proposition 3.6. *There is a functor F which completes the following commutative diagram*

$$\begin{array}{ccc} Pair_0(W) & \xrightarrow{\text{Cok}} & coh(X_0) \\ \downarrow & & \downarrow \\ DB_0(W) & \xrightarrow{F} & \mathbf{D}_{Sg}(X_0). \end{array}$$

Moreover, the functor F is an exact functor between triangulated categories.

Proof. We have a functor $Pair_0(W) \to \mathbf{D}_{Sg}(X_0)$ which is the composition of Cok and the natural functor from coh(X) to $\mathbf{D}_{Sg}(X_0)$. To prove the existance of a functor F we need to show that any morphism $\overline{f} = (f_1, f_0) : \overline{P} \to \overline{Q}$ which is homotopic to 0 goes to 0-morphism in $\mathbf{D}_{Sg}(X_0)$. Fix a homotopy (t, s) where $t : P_1 \to Q_0$ and $s : P_0 \to Q_1$. Consider the following decomposition of $\overline{f}$:

$$\begin{array}{ccccc} P_1 & \underset{p_0}{\overset{p_1}{\rightleftarrows}} & P_0 & \longrightarrow & \text{Coker } p_1 \\ \downarrow{\scriptstyle (t,f_1)} & & \downarrow{\scriptstyle (s,f_0)} & & \downarrow \\ Q_0 \oplus Q_1 & \underset{c_0}{\overset{c_1}{\rightleftarrows}} & Q_1 \oplus Q_0 & \longrightarrow & Q_0/W \\ \downarrow{\scriptstyle pr} & & \downarrow{\scriptstyle pr} & & \downarrow \\ Q_1 & \underset{q_0}{\overset{q_1}{\rightleftarrows}} & Q_0 & \longrightarrow & \text{Coker } q_1 \end{array} \qquad \text{where} \qquad c_0 = \begin{pmatrix} -q_0 & \text{id} \\ 0 & q_1 \end{pmatrix}, \; c_1 = \begin{pmatrix} -q_1 & \text{id} \\ 0 & q_0 \end{pmatrix}.$$

This gives the decomposition of $F(\overline{f})$ through a locally free object Q_0/W . Hence, $F(\overline{f}) = 0$ in the category $\mathbf{D}_{Sg}(X_0)$. We leave to the reader the proof that the functor F is exact. $\square$

Lemma 3.7. *If $F\overline{P} = 0$ then $\overline{P} = 0$ in $DB_0(W)$.*

Proof. If $F\overline{P}=0$ then $\text{Coker } p_1$ is a perfect complex. This implies that it is locally free by Lemmas 1.20 and 3.5. Hence, there is a map $f:\text{Coker } p_1 \to P_0/W$ which splits the epimorphism $pr: P_0/W \to \text{Coker } p_1$. It can be lifted to a map from $\{P_1 \xrightarrow{p_1} P_0\}$ to $\{P_0 \xrightarrow{W} P_0\}$. Consider a diagram

$$\begin{array}{ccccc}
P_1 & \xrightarrow{p_1} & P_0 & \longrightarrow & \text{Coker } p_1 \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle f} \\
P_0 & \xrightarrow{W} & P_0 & \longrightarrow & P_0/W \\
{\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle pr} \\
P_1 & \xrightarrow{p_1} & P_0 & \longrightarrow & \text{Coker } p_1
\end{array}$$

Since the composition prf is identical, the map (p_0t,u) from the pair $\{P_1 \xrightarrow{p_1} P_0\}$ to itself is homotopic to the identical map of this pair. Hence, there is a map $s: P_0 \to P_1$ such that

$$\text{id}_{P_1} - p_0t = sp_1 \qquad \text{and} \qquad p_1s = \text{id}_{P_0} - u.$$

Moreover, we have the following equalities

$$0 = (up_1 - Wt) = (up_1 - tW) = (u - tp_0)p_1.$$

This gives us that $(u-tp_0)=0$, because no maps from $\text{Coker } p_1$ to P_0. Finally, we obtain two morphisms t and s such that

$$\text{id}_{P_1} = p_0t + sp_1 \qquad \text{and} \qquad \text{id}_{P_0} = p_1s + tp_0.$$

Hence, the pair $\overline{P}$ is isomorphic to the zero object in the category $DB_0(W)$. □

Theorem 3.8. *The functor* $F: DB_0(W) \longrightarrow \mathbf{D}_{Sg}(X_0)$ *is an exact equivalence.*

Proof. By Lemma 3.5 the coherent sheaves $\text{Coker } p_1$ and $\text{Coker } q_1$ satisfy the condition of Proposition 1.21 with $N=0$. This gives an isomorphism

$$\text{Hom}_{\mathbf{D}_{Sg}(X_0)}(\text{Coker } p_1, \text{Coker } q_1) \cong \text{Hom}_{coh(X_0)}(\text{Coker } p_1, \text{Coker } q_1)/\mathcal{R}$$

where $\mathcal{R}$ is the subspace of morphisms factoring through a locally free sheaf. Since Cok is full we get that the functor F is also full.

Now we prove that F is faithful. It is a standard consideration. Let $f:\overline{P}\to\overline{Q}$ be a morphism for which $F(f)=0$. Suppose that f sits in an exact triangle

$$\overline{P} \xrightarrow{f} \overline{Q} \xrightarrow{g} \overline{R}$$

Then the identity map of $F\overline{Q}$ factors through the map $F\overline{Q} \xrightarrow{Fg} F\overline{R}$. Since F is full, there is a map $h:\overline{Q}\to\overline{Q}$ factoring through $g:\overline{Q}\to\overline{R}$ such that $Fh=\text{id}$. Hence, the cone $C(h)$ of map h goes to zero under the functor F. By Lemma 3.7 the object $C(h)$ is the zero object as well, so h is an isomorphism. Thus $g:\overline{Q}\to\overline{R}$ is a split monomorphism and $f=0$.

To complete the proof that F is an equivalence we need to show that every object $A \in \mathbf{D}_{Sg}(X_0)$ is isomorphic to $F\overline{P}$ for some $\overline{P}$. By Proposition 1.23 any object $A \in \mathbf{D}_{Sg}(X_0)$ is isomorphic to the image of a coherent sheaf $\mathcal{F}$ such that $\underline{\mathcal{E}xt}^i(\mathcal{F}, \mathcal{O}_X) = 0$ for all $i > 0$. Consider an epimorphism $P_0 \to \mathcal{F}$ of sheaves on X with locally free P_0. Denote by $p_1 : P_1 \to P_0$ the kernel of this map. Since the multiplication on W gives the zero map on $\mathcal{F}$, there is a map $p_0 : P_0 \to P_1$ such that $p_0 p_1 = W$ and $p_1 p_0 = W$. We get a pair

$$\overline{P} := \left(\begin{array}{c} P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \end{array} \right)$$

And we need only to show that P_1 is locally free. But it is locally free on the complement $X \setminus X_0$ and its restriction to X_0 (in the derived sense) is locally free sheaf because Coker p_1 and P_0/W are locally free on X_0 and there is an exact sequence

$$0 \to \mathrm{Coker}\ p_1 \to P_1/W \to P_0/W \to \mathrm{Coker}\ p_1 \to 0.$$

Hence, by Lemma 2.6 P_1 is locally free as well. $\square$

Corollary 3.9. *The category of B-branes on smooth X with a superpotensial W is equivalent to the disjoint union $\coprod_{w \in \mathbb{A}^1} \mathbf{D}_{Sg}(X_w)$, and this union is finite.*

3.3. **Some simple calculations.** In this paragraph we give a description of the category of B-branes in the Landau-Ginsburg model with a superpotential $W' = z_0^n + z_1^2 + \cdots + z_{2k}^2$ which is given on $\mathbb{C}^{2k+1}$. This superpotential has only one singular point over 0. By Theorem 3.8 the category of B-branes $DB_0(W')$ is equivalent to the triangulated category of singularities $\mathbf{D}_{Sg}(Y_0)$, where Y_0 is the fiber over 0, i.e Y_0 is given by the equation $W' = 0$. By Theorem 2.1 this category is equivalent to the category $\mathbf{D}_{Sg}(X_0)$, where $X_0 = Spec(\mathbb{C}[z]/z^n)$ is the fiber over 0 of the superpotential $W = z^n$ on $\mathbb{C}$. Thus, it is enough to describe the category $\mathbf{D}_{Sg}(X_0)$.

Objects. Denote by A the algebra $\mathbb{C}[z]/z^n$. By Proposition 1.23 any object is coming from a finite-dimensional module M over the algebra A. Any A-module is nothing more as a vector space with an operator L such that $L^n = 0$. Hence any module is a direct sum of the modules V_i for $i = 1, ..., n$, where $V_i = A/z^i$. This is the Jordan decomposition in Jordan blocks. Moreover, the module $V_n = A$ is free, hence it is equal to the zero object in $\mathbf{D}_{Sg}(X_0)$. So indecomposible objects in the category $\mathbf{D}_{Sg}(X_0)$ are

$$V_1, V_2, ..., V_{n-1}.$$

All other objects are finite direct sums of $V_\mu,\ \mu = 1, ...n-1$.

Morphisms. For each pair V_μ, V_ν we fix a morphism

$${}_\nu\alpha_\mu : V_\mu \to V_\nu$$

which is coming from the natural projection if $\mu \geq \nu$ and from the injection that sends $1 \in V_\mu = A/z^\mu$ to $z^{\nu-\mu}$ if $\nu \geq \mu$. All other morphisms are linear combination of compositions of ${}_\nu\alpha_\mu$. There are the following relations:

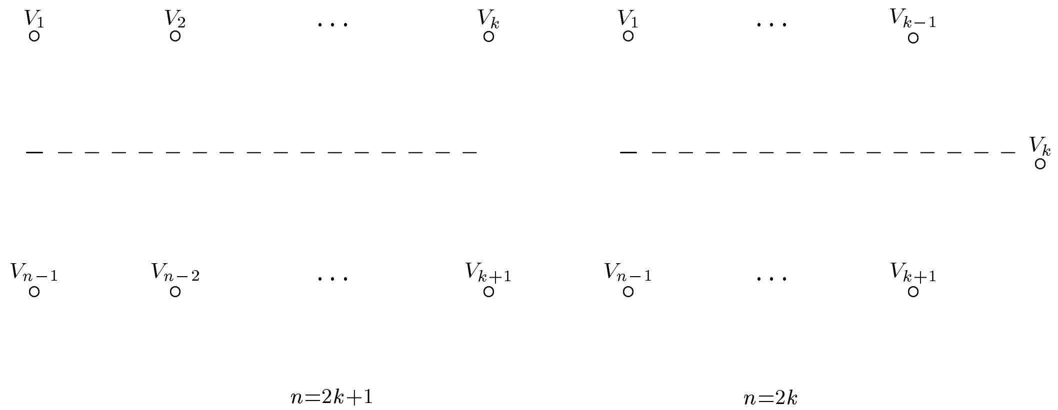
$$
(7) \qquad
\begin{array}{lll}
1) & {}_\mu\alpha_\mu = id_\mu, & \\
2) & {}_\nu\alpha_{\lambda\lambda}\alpha_\mu = {}_\nu\alpha_\mu & \text{if} \quad \nu \geq \lambda \geq \mu \quad \text{or} \quad \nu \leq \lambda \leq \mu, \\
3) & {}_\nu\alpha_{\lambda\lambda}\alpha_\mu = 0 & \text{if} \quad \lambda \geq \mu + \nu \quad \text{or} \quad \lambda + n \leq \mu + \nu, \\
4) & {}_\nu\alpha_{\lambda\lambda}\alpha_\mu = {}_\nu\alpha_{\kappa\kappa}\alpha_\mu & \text{if} \quad \lambda + \kappa = \mu + \nu.
\end{array}
$$

Using this relation we can see that the space $\operatorname{Hom}(V_\mu, V_\nu)$ has a basis formed by the morphisms of the form ${}_\nu\alpha_{\lambda\lambda}\alpha_\mu$, where $\max(\mu,\nu) \leq \lambda < \mu + \nu$. Denote by $\operatorname{depth} V_\mu$ the integer number equals to $\min(\mu, n-\mu)$. We obtain that

$$\dim \operatorname{Hom}(V_\mu, V_\nu) = \min(\operatorname{depth} V_\mu, \operatorname{depth} V_\nu).$$

Moreover, the ring $\operatorname{End}(V_\mu)$ is isomorphic to $\mathbb{C}[x]/x^d$, where $d = \operatorname{depth} V_\mu$.

Translation functor. To see the translation functor it is convinient to draw the following pictures



The translation functor $[1]$ is the reflection with respect to the horisonal line, i.e. it sends V_μ to $V_{n-\mu}$ and takes ${}_\nu\alpha_\mu$ to ${}_{(n-\nu)}\alpha_{(n-\mu)}$. It is easy to see that all relations (7) are preserved.

Exact triangles. For the convenience, we will write $V_{-\mu}$ instead $V_{n-\mu}$ considering all integers modulo n. We also put $V_0 = 0$. At first, any triangle of the form

$$(8) \qquad V_\mu \xrightarrow{{}_\nu\alpha_\mu} V_\nu \xrightarrow{{}_{(\nu-\mu)}\alpha_\nu} V_{(\nu-\mu)} \xrightarrow{h} V_{(-\mu)},$$

where $h = {}_{(-\mu)}\alpha_{(\nu-\mu)}$ if $\nu - \mu \geq 0$ and $h = -{}_{(-\mu)}\alpha_{(\nu-\mu)}$ if $\nu - \mu < 0$, is exact. If now $f : V_\mu \to V_\nu$ is another morphism which is a composition of some α's then using the relations (7) we can represent it as ${}_\nu\alpha_{\lambda\lambda}\alpha_\mu$ where $\lambda > \max(\mu,\nu)$ and $\lambda < \mu + \nu$. In this case the triangle of the form

$$(9) \qquad V_\mu \xrightarrow{f} V_\nu \xrightarrow{g} V_{(\lambda-\mu)} \oplus V_{(\nu-\lambda)} \xrightarrow{h} V_{(-\mu)}$$

is exact, where $g = ({}_{(\lambda-\mu)}\alpha_{\nu}\ , {}_{(\nu-\lambda)}\alpha_{\nu})^t$ and $h = ({}_{(-\mu)}\alpha_{(\lambda-\mu)}, -{}_{(-\mu)}\alpha_{(\nu-\lambda)})$. Note that the triangle (8) is a particular case of the triangle (9) for $\lambda = \max(\mu, \nu)$.

All other exact triangles are isomorphic to the triangles defined above.

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