Quantization of Symplectic Transformations and the Lefschetz Fixed Point Theorem

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Introduction

The present paper is aimed at the generalization of the Atiyah–Bott–Lefschetz fixed point theorem [1] in the situation of symplectic geometry. Let us briefly describe the general statement of the problem.

Let

$$\widehat{A} : \mathcal{H}_1 \to \mathcal{H}_2$$

be a Fredholm operator acting in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and let $\widehat{U}_1$ and $\widehat{U}_2$ be invertible operators such that the diagram

$$\begin{array}{ccc} \mathcal{H}_1 & \xrightarrow{\widehat{A}} & \mathcal{H}_2 \\ {\scriptstyle \widehat{U}_1}\downarrow & & \downarrow{\scriptstyle \widehat{U}_2} \\ \mathcal{H}_1 & \xrightarrow{\widehat{A}} & \mathcal{H}_2 \end{array} \qquad (1)$$

commutes. It is easy to see that in this case the space $\operatorname{Ker} \widehat{A}$ is an invariant subspace of the operator $\widehat{U}_1$ and, hence, this operator acts on the kernel of the operator $\widehat{A}$. Similarly, the

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operator $\widehat{U}_2$ is correctly defined on the cokernel of the operator $\widehat{A}$. Since the spaces $\operatorname{Ker}\widehat{A}$ and $\operatorname{Coker}\widehat{A}$ are finite-dimensional (the operator $\widehat{A}$ is a Fredholm one) the number

$$\mathcal{L}\left(\widehat{A},\widehat{U}_1,\widehat{U}_2\right) = \operatorname{Trace}\left(\widehat{U}_1\Big|_{\operatorname{Ker}\widehat{A}}\right) - \operatorname{Trace}\left(\widehat{U}_2\Big|_{\operatorname{Coker}\widehat{A}}\right)$$

is defined. This number is called *the Lefschetz number* of the diagram (1). We remark that if $\widehat{U}_1 = \mathrm{id}$, $\widehat{U}_2 = \mathrm{id}$ then the Lefschetz number $\mathcal{L}\left(\widehat{A},\mathrm{id},\mathrm{id}\right)$ coincides with the *index* of the operator $\widehat{A}$.

The Lefschetz number can also be defined in more general situation. Namely, let

$$0 \longrightarrow \mathcal{H}_1 \xrightarrow{\widehat{A}_1} \mathcal{H}_2 \xrightarrow{\widehat{A}_2} \dots \xrightarrow{\widehat{A}_{N-1}} \mathcal{H}_N \longrightarrow 0 \tag{2}$$

be a complex with finite-dimensional cohomology (for example, an elliptic complex on a smooth compact manifold without boundary) and let $\widehat{U}_1,\dots,\widehat{U}_N$ be operators defining an endomorphism of this complex:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{H}_1 & \xrightarrow{\widehat{A}_1} & \mathcal{H}_2 & \xrightarrow{\widehat{A}_2} & \dots & \xrightarrow{\widehat{A}_{N-1}} & \mathcal{H}_N & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widehat{U}_1} & & \downarrow{\scriptstyle \widehat{U}_2} & & & & \downarrow{\scriptstyle \widehat{U}_N} & & \\
0 & \longrightarrow & \mathcal{H}_1 & \xrightarrow{\widehat{A}_1} & \mathcal{H}_2 & \xrightarrow{\widehat{A}_2} & \dots & \xrightarrow{\widehat{A}_{N-1}} & \mathcal{H}_N & \longrightarrow & 0
\end{array} \tag{3}$$

Then the Lefschetz number of diagram (3) is defined as an alternative sum

$$\mathcal{L} = \sum_{k=1}^{N}(-1)^k \operatorname{Trace}\left(\widehat{U}_k\Big|_{H^k}\right),$$

where H^k are the cohomology of complex (2).

Let us consider the two important particular cases of the above general construction.

a) Classical Lefschetz theorem. In this case complex (2) is the classical de-Rham complex on a smooth compact manifold M without boundary

$$0 \to \Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Lambda^n(M) \to 0$$

and mappings $\widehat{U}_k$ are determined by some smooth mapping $f : M \to M$ with the graph being transversal to the diagonal. In this case the Lefschetz number equals to

$$\mathcal{L} = \sum_{k=0}^{n}(-1)^k \operatorname{Trace}\left(\widehat{U}_k\Big|_{H^k(M)}\right),$$

where $\widehat{U}_k$ is the action of the induced mapping f^* on forms of the degree k and $H^k(M)$ are cohomology groups of the manifold M.

The following statement is the classical result by S. Lefschetz.

The formula

$$\mathcal{L} = \sum_{\alpha_k} (-1)^{\sigma(\alpha_k)},$$

is valid, where the sum is taken over all fixed points of the mapping f and $\sigma(\alpha_k)$ is the intersection index between the graph of the mapping f and the diagonal at the point α_k.

b) Atiyah–Bott–Lefschetz theorem. In the paper [1] M. F. Atiyah and R. Bott had formulated the following generalization of the Lefschetz theorem to the case of elliptic complexes. Let $E_i \to M$, $i = 1, 2, \dots, N$ be vector bundles over M and let

$$0 \longrightarrow C^\infty(E_1) \xrightarrow{A_1} C^\infty(E_2) \xrightarrow{A_2} \dots \xrightarrow{A_{N-1}} C^\infty(E_N) \longrightarrow 0$$

be an elliptic complex. Consider a mapping

$$f : M \to M$$

with its graph transversal to the diagonal and suppose that f acts on sections of the bundles E_k in accordance to the formula

$$f : u(x) \mapsto T_k(x) u(f(x)) \stackrel{\text{def}}{=} \widehat{U}_k(u),$$

where $T_k(x)$ is an endomorphism of the bundle E_k (such endomorphisms of the space $C^\infty(E_k)$ of sections of the bundle E_k are called *geometrical*).Then the following statement is valid.

The formula takes place

$$\mathcal{L} = \mathcal{L}\left(\widehat{A}, \widehat{U}\right) = \sum_{\alpha_k} \frac{\sum_{j=1}^N (-1)^{j-1} \operatorname{Trace} T_j(\alpha_k)}{|\det\, (1 - f_*(\alpha_k))|},$$

the outer summation is performed over all fixed points of the mapping f.

Thus, generalizing the Lefschetz theorem Atiyah and Bott use an arbitrary elliptic complex instead of the de–Rham complex preserving the 'geometrical' nature of an endomorphism U of the considered complex. This means that the mentioned endomorphism is, as before, determined by some mapping $f : M \to M$ of the manifold M.

It seems, however, that more natural is to consider endomorphisms connected with some symplectic (canonical) transformation

$$g : T^*M \to T^*M$$

rather than with a mapping of the manifold M itself.[1]

Such endomorphisms can be realized on sections of the corresponding bundles as Fourier integral operators (obtained by quantization of canonical transformations) associated with these endomorphisms g (see, for example, [2]).

The aim of this paper is to obtain the corresponding generalization of the Atiyah–Bott–Lefschetz theorem in the framework of the classical quantization, that is, in the situation of $1/h$-pseudodifferential operators [2]. In such a situation we obtain an *asymptotic* formula of the Lefschetz type, which expresses the leading term of the asymptotic expansion of the Lefschetz number as $h \to 0$ in terms of fixed points of the symplectic transformation g (see Theorem 4 below).

The interesting particular case [3] of such a construction is a case when the considered symplectic transformation is determined by a Hamilton flow g_t along trajectories of Hamilton vector field

$$V(H) = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p}$$

determined by some real Hamilton function $H(x,p)$. In this case the corresponding Fourier integral operators can be represented in the form

$$U_k = e^{-\frac{i}{h}\widehat{H}_k t},$$

where $\widehat{H}_k$, $k = 1,2$ are pseudodifferential operators with principal symbol H. Here the Lefschetz number is strongly connected with *spectral properties* of the operators $\widehat{H}_k$. Actually, let us consider the simplest case of an elliptic operator:

$$\begin{array}{ccccccc}
0 & \longrightarrow & C^\infty(E_1) & \xrightarrow{\widehat{A}} & C^\infty(E_2) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle e^{-\frac{i}{h}\widehat{H}_1 t}} & & \Big\downarrow{\scriptstyle e^{-\frac{i}{h}\widehat{H}_2 t}} & & \\
0 & \longrightarrow & C^\infty(E_1) & \xrightarrow{\widehat{A}} & C^\infty(E_2) & \longrightarrow & 0
\end{array}$$

Suppose that the operators $\widehat{H}_1$ and $\widehat{H}_2$ has a discrete spectrum and denote by λ_j^1 and λ_j^2 the eigenvalues of the operators $\widehat{H}_1$ and $\widehat{H}_2$ correspondingly. Then it is evident that the Lefschetz number can be written down in the form

$$\mathcal{L}\left(\widehat{A}, \widehat{H}_1, \widehat{H}_2\right) = \sum e^{-\frac{i}{h}\lambda_j^1 t} - \sum e^{-\frac{i}{h}\lambda_j^2 t},$$

[1] We recall that for any smooth manifold the total space of the cotangent bundle has the standard symplectic structure (see, for example, [2]).

where the first sum is taken over eigenvalues corresponding to the kernel of the operator $\widehat{A}$ and the second over eigenvalues corresponding to the cokernel of this operator. Such a situation arises, for example, in the simplest scattering problem. Actually, let us consider Hamiltonians

$$\widehat{H}_2 = -h^2\Delta, \quad \widehat{H}_1 = -h^2\Delta + \widehat{T}$$

where $\widehat{T}$ is an $1/h$–differential operator with finite in (x,p) symbol. Then there exist the operator

$$\widehat{W} = \lim_{t\to+\infty} e^{-\frac{i}{h}\widehat{H}_1 t} e^{\frac{i}{h}\widehat{H}_2 t} = 1 + \widehat{B}$$

where $\widehat{B}$ is an operator of trace class. One can show that the operator $\widehat{H}_1$ has continuous spectrum and a finite number of eigenvalues. The operator $\widehat{W}$ maps the space corresponding to the continuous spectrum of the operator $\widehat{H}_1$ on the whole space L_2 and the kernel of this operator coincides with the subspace corresponding to the discrete spectrum of this operator. In this situation the Lefschetz number

$$\mathcal{L}\left(\widehat{W}, e^{-\frac{i}{h}\widehat{H}_1 t}, e^{\frac{i}{h}\widehat{H}_2 t}\right) = \sum_{k=1}^{N} e^{-\frac{i}{h}\lambda_k t}$$

is simply the Fourier transform of the part of the spectral measure of the operator $\widehat{H}_1$ corresponding to the discrete spectrum of this operator.

Let us desrcibe briefly the contents of the paper.

Section 1 is aimed at the computation of *the asymptotics of the trace* of a Fourier integral operator associated with some symplectic transformation. The result of this section is formulated in the theorem expressing the leading term of the asymptotical expansion of the trace of Fourier integral operator in terms of fixed points of the corresponding symplectic transformation.

In Section 2 we consider the case when the symplectic transformation is determined by *a flow* along trajectories of some Hamilton vector field. In this case fixed points of the symplectic transformation g_t correspond to closed trajectories of the Hamilton vector field and the coefficients of the asymptotic expansion obtained in the previous section can be expressed in terms of integrals along closed trajectories of the Hamilton vector field.

In Section 3 we derive *the generalized Lefschetz formula* in situation of symplectic transformations. Here we obtain the expression of the leading term of the asymptotic expansion of the Lefschetz number as $h \to 0$. To simplify the notation we restrict ourselves to the case of elliptic operator though all considerations can be carried out in the case of elliptic complexes as well.

In conclusion we remark that all the constructions of the present paper can be carried out also in the case when the principal symbol $H_0(x,p)$ of the operator $\widehat{H}$ which determines the Hamilton flow g_t is a *complex-valued* function. In this case one should use the *s-analytic theory* and, in particular, the formula of asymptotic expansion of rapidly oscillating integral with complex phase function (see [2]).

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1 The Trace of Fourier Integral Operator

Let M be a smooth compact manifold without boundary and let

$$g : T^*M \to T^*M$$

be a symplectic transformation which satisfies the following condition on non-degeneracy.

Condition 1 The transformation g has only *finite number of fixed points* $\alpha_1, \ldots, \alpha_k$ such that the determinant $\det(1 - g_*)$ does not vanish at each of these points.Here

$$g_* : TT^*M \to TT^*M$$

is the induced mapping of the tangent spaces.

The aim of this section is the computation of the asymptotics as $h \to 0$ of the trace of Fourier integral operator $T(g,\varphi)$ with a symbol φ corresponding to the symplectic transformation g. Let us recall briefly the definition of the operator $T(g,\varphi)$ (see, for example, [2]).

Denote by $L \subset T^*M \times T^*M$ a submanifold of the symplectic space $T^*(M \times M) \cong T^*M \times T^*M$ which is the graph of the transformation g. We remark that $L = \operatorname{graph} g$ is a *Lagrangian manifold* in $T^*M \times T^*M$ with respect to the Hamiltonian structure

$$\pi_1^*\omega - \pi_2^*\omega = \omega_1 - \omega_2,$$

where

$$\pi_i : T^*M \times T^*M \to T^*M$$

are the canonical projections and the form ω determines the symplectic structure on the space T^*M. It is evident that L is diffeomorphically projected on the second factor, that is, that

$$\pi_2|_L : L \to T^*M$$

is a diffeomorphism. Hence, there exists the canonical measure

$$\mu = \pi_2^*(\omega^n)$$

on the manifold L (here $\omega^n = \omega \wedge \ldots \wedge \omega$ (n times)).

Let now φ be a smooth function on the symplectic space T^*M. Then the operator $T(g, \varphi)$ is defined as an integral operator with the kernel

$$K(x, y) = \left(-\frac{i}{2\pi h}\right)^{n/2} k_{(L,\mu)}\left(\pi_2^*(\varphi)\right), \tag{4}$$

where $k_{(L,\mu)}$ is Maslov's canonical operator on the quantized Lagrangian manifold L with the non-degenerate measure μ (see [2]). The function φ is called a *symbol* of the Fourier integral operator $T(g, \varphi)$.

Let us describe the symbol classes which will be used below. Denote by $\Sigma^{m,\mu}$ the set of C^∞-functions $\varphi(x, p, h)$ on $T^*M \times [0, 1]$ which satisfy the estimates

$$|D_x^\alpha D_p^\beta \varphi(x, p, h)| \leq C_{\alpha\beta} h^\mu (1 + |p|^2)^{(m-|\beta|)/2} \tag{5}$$

(where (x, p) are canonical coordinates on T^*M) and may be represented in the form

$$\varphi(x, p, h) = \sum_{k=0}^{N} h^{\mu+k} \varphi_k(x, p) + R_N(x, p, h)$$

for any integer N. Here the function $R_N(x, p, h)$ must satisfy estimates (5) with m replaced by $m + N + 1$.

These symbol classes are modification to the case of compact manifold without boundary of the classes $\Sigma_{1,0}^{m,\mu}$ introduced by M. Shubin [4]. We remark that all the theory developed below remaines valid also in the space $\mathbf{R}^n$ if we modify the estimate (5) in the following way

$$|D_x^\alpha D_p^\beta \varphi(x, p, h)| \leq C_{\alpha\beta} h^\mu (1 + |x|^2 + |p|^2)^{(m-|\alpha|-|\beta|)/2}.$$

Suppose now that the transformation g has the operator $T(g, \varphi)$ belongs to the class $\Sigma^{m,\mu}$ with sufficiently large negative m. Then the following affirmation takes place.

Theorem 1 *The formula*

$$\operatorname{Trace} T(g, \varphi) \equiv \sum_k \exp\left\{\frac{i}{h} S\left(\alpha_k\right)\right\} \frac{\varphi\left(\alpha_k\right)}{\sqrt{\det\left(1 - g_*\left(\alpha_k\right)\right)}} \pmod{O(h)}, \tag{6}$$

is valid. Here the sum is taken over all fixed points α_k of the transformation g, S is a nonsingular action on the manifold L and the choice of the branch of the square root is described below.

Remark 1 Certainly, one can show that there exists an asymptotic expansion of the trace of the operator $T(g, \varphi)$ up to an *arbitrary* power of h. However, the computation of explicit formulas for the corresponding terms of this asymptotic expansion is rather complicated and we present here only the computation of the leading term.

Remark 2 In the case when fixed points of the mapping g are degenerate, that is, they do not satisfy Condition 1, one can also obtain the asymptotic expansion of the trace using normal forms of the phase function at stationary points (see [5]).

All the rest of this section is the proof of Theorem 1, that is, the computation of the asymptotical expansion of the integral

$$\operatorname{Trace} T(g, \varphi) = \int_M K(x, x) dx \tag{7}$$

as $h \to 0$, here dx is some fixed positive measure on the manifold M.

To compute integral (7) we shall use the following special coordinates systems.

First, we denote by (x, p) (correspondingly, (y, q)) the canonical coordinate systems in the first (correspondingly, second) factor of the product $T^*M \times T^*M$. Thus, the structure forms in the symplectic spaces T^*M and $T^*M \times T^*M$ have the form

$$dp \wedge dx \quad \text{and} \quad dp \wedge dx - dq \wedge dy,$$

correspondingly. We suppose also, that coordinate systems are chosen in such a way that the measure dx mentioned above has the unit density.

It is easy to show that the Lagrangian manifold L can be covered by canonical charts of the type U_I with the coordinates $(x^I, p_{\overline{I}}, q)$ (here, as it is usual in the canonical operator theory [2], by $I \subset \{1, \dots, n\}$ we denote some subset of indices, by $\overline{I}$ we denote its complement in $\{1, \dots, n\}$, x^I is the tuple $(x^{i_1}, \dots, x^{i_m})$ where $I = \{i_1, \dots, i_m\}$, and $p_{\overline{I}}$ has similar sense).

Let us express the objects included in definition (4) of the canonical operator in the described local coordinates.

1. Let

$$S_I = S_I\left(x^I, p_{\overline{I}}, q\right)$$

be an action in the chart U_I of the Lagrangian manifold L. This means that the equations of L are

$$
\begin{aligned}
x^{\overline{I}} &= -\frac{\partial S_I}{\partial p_{\overline{I}}}\left(x^I, p_{\overline{I}}, q\right) \stackrel{\text{def}}{=} x^{\overline{I}}\left(x^I, p_{\overline{I}}, q\right), \\
p_I &= \frac{\partial S_I}{\partial x^I}\left(x^I, p_{\overline{I}}, q\right) \stackrel{\text{def}}{=} p_I\left(x^I, p_{\overline{I}}, q\right), \\
y &= \frac{\partial S_I}{\partial q}\left(x^I, p_{\overline{I}}, q\right) \stackrel{\text{def}}{=} y\left(x^I, p_{\overline{I}}, q\right).
\end{aligned}
\tag{8}
$$

The action S_I can be constructed in the explicit form in the following way. First, we denote by S (the non–singular action) a solution of the Pfaffian equation

$$
dS = p\,dx\big|_L \tag{9}
$$

on the Lagrangian manifold L. We recall that the manifold L is supposed to be quantized. In particular, this means that equation (9) has a global solution on L defined up to an additive constant. To fix this constant we choose a point α^* on the manifold L (which will be referred below as *base* point of the manifold L) and suppose that $S(\alpha^*) = 0$. Then the function $S(\alpha)$ at the point $\alpha \in L$ on the Lagrangian manifold is given by the formula

$$
S(\alpha) = \int\limits_{\alpha^*}^{\alpha} p\,dx \tag{10}
$$

and the function $S_I\left(x^I, p_{\overline{I}}, q\right)$ is equal to[2]

$$
S_I(\alpha) = S(\alpha) - \left(x^{\overline{I}} p_{\overline{I}} - yq\right)\Big|_L
$$

expressed in the local coordinates of the chart U_I (the Legendre transform of the function $S(\alpha)$).

2. The expression for the density of the measure μ in the local coordinates of the chart U_I is

$$
\mu_I\left(x^I, p_{\overline{I}}, q\right) = \frac{dy \wedge dq}{dx^I \wedge dp_{\overline{I}} \wedge dq} = \det \frac{\partial y}{\partial (x^I, p_{\overline{I}})}. \tag{11}
$$

The argument of the function (11) (which is used below for calculation of the square root of this function) is defined in the following way [2].

Let U_I and U_J be two canonical charts on the Lagrangian manifold L with non–empty intersection. Denote

$$
I_1 = I \cap J, \quad I_2 = I \setminus J, \quad I_3 = J \setminus I, \quad I_4 = \overline{I} \cap \overline{J}.
$$

[2]Here and below we accept the usual in the tensor analysis summation convention.

Then in the intersection $U_I \cap U_J$ we obtain

$$\arg\mu_I = \arg\mu_J + \pi\alpha_{IJ}, \tag{12}$$

where

$$\begin{aligned} \alpha_{IJ} &= \frac{1}{\pi}\sum \arg\lambda_{k,IJ} + |I_2|, \\ -\frac{3\pi}{2} &< \arg\lambda_{k,IJ} \leq \frac{\pi}{2} \end{aligned} \tag{13}$$

and $\lambda_{k,IJ}$ are the eigenvalues of the matrix

$$\frac{\partial\left(-p_{I_2}, x^{I_3}\right)}{\partial\left(x^{I_2}, p_{I_3}\right)}. \tag{14}$$

(The fact that the manifold L is a Lagrangian one leads to the symmetricity of matrix (14). Hence, the inequalities (13) are correct.)

Now let l be a path joining the base point α^* with a point $\alpha \in U_I$ of the chart U_I. Let us fix some type U_{I_0} of the chart in a neighbourhood of the point α^* and some (arbitrarily chosen) value of the argument of μ_{I_0} at the point α^*. Let

$$\{U_{I_0}, \ldots, U_{I_N} = U_I\}$$

be a chain of charts covering the path l. Define the index of the path l (or, more exactly, the index of the chain of charts along l) by the formula

$$\operatorname{ind} l = \sum_{k=0}^{N-1} \alpha_{I_k I_{k+1}}.$$

Then, due to relation (12) it is evident that the argument of μ_I in the chart U_I is equal to

$$\arg\mu_I = \arg\mu_{I_0} + \pi \operatorname{ind} l. \tag{15}$$

Due to the fact that the manifold L is quantized, the argument $\arg\mu_I$ given by (15) does not depend on the choice of the path l (as well as on the choice of a chain of charts).

Now the local expression of canonical operator (4) can be represented in the form

$$k_{(L,\mu)}\left(\pi_2^*\varphi\right) = F^{1/h}_{p_{\bar{I}}\to x^{\bar{I}}} \overline{F}^{1/h}_{q\to y} \left\{ e^{\frac{i}{h}S_I\left(x^I, p_{\bar{I}}, q\right)} \sqrt{\mu_I\left(x^I, p_{\bar{I}}, q\right)}\varphi\left(x^I, p_{\bar{I}}, q\right) \right\},$$

where $F^{1/h}$ is $1/h$–Fourier transformation and $\overline{F}^{1/h}$ is its conjugate [2].

Evidently, with the help of a partition of unity the computation of the trace (7) of the operator $T(g,\varphi)$ can be reduced to the case when the support of the function φ is contained in one of the canonical charts U_I. In this case we have

$$\begin{aligned} \operatorname{Trace} T(g,\varphi) &= (-1)^{|\bar{I}|/2}\left(\frac{i}{2\pi h}\right)^{n+|\bar{I}|/2} \int \exp\left\{\frac{i}{h}\left[x^{\bar{I}}p_{\bar{I}} - qx + S_I\left(x^I, p_{\bar{I}}, q\right)\right]\right\} \\ &\times \sqrt{\mu_I\left(x^I, p_{\bar{I}}, q\right)}\varphi\left(x^I, p_{\bar{I}}, q\right)\, dp_{\bar{I}}\, dq\, dx, \end{aligned} \tag{16}$$

where the argument of -1 equals $-\pi$ in accordance to (13).

We shall compute the asymptotical expansion of integral (16) by the stationary phase method (see, for example, [2]). To do this we first derive equations for stationary points of the phase function of integral (16)

$$\Phi = x^{\bar{I}}p_{\bar{I}} - qx + S_I\left(x^I, p_{\bar{I}}, q\right) \tag{17}$$

with respect to the variables $(p_{\bar{I}}, q, x)$. Due to relations (8) the equations for the stationary point read

$$\begin{aligned} x^{\bar{I}} &+ \frac{\partial S_I\left(x^I, p_{\bar{I}}, q\right)}{\partial p_{\bar{I}}} = x^{\bar{I}} - x^{\bar{I}}\left(x^I, p_{\bar{I}}, q\right) = 0, \\ p_{\bar{I}} &- q_{\bar{I}} = 0, \\ -q_I &+ \frac{\partial S_I\left(x^I, p_{\bar{I}}, q\right)}{\partial x^I} = -q_I + p_I\left(x^I, p_{\bar{I}}, q\right) = 0, \\ -x &+ \frac{\partial S_I\left(x^I, p_{\bar{I}}, q\right)}{\partial q} = -x + y\left(x^I, p_{\bar{I}}, q\right) = 0. \end{aligned} \tag{18}$$

Equations (18) show that *stationary points of the phase function* (17) *exactly correspond to fixed points of the mapping* g.

Now let us show that if the Condition 1 is satisfied then each stationary point of the phase function (17) are *non–degenerated*. To do this we compute the Hessian of (17) in $(x^I, x^{\bar{I}}, q_I, q_{\bar{I}}, p_{\bar{I}})$ at stationary points. It is equal to

$$\det \operatorname{Hess}(-\Phi) = \begin{vmatrix} -\frac{\partial p_I}{\partial x^I} & 0 & 1 - \frac{\partial p_I}{\partial q_I} & -\frac{\partial p_I}{\partial q_{\bar{I}}} & -\frac{\partial p_I}{\partial p_{\bar{I}}} \\ 0 & 0 & 0 & 1 & -1 \\ 1 - \frac{\partial y^I}{\partial x^I} & 0 & -\frac{\partial y^I}{\partial q_I} & -\frac{\partial y^I}{\partial q_{\bar{I}}} & -\frac{\partial y^I}{\partial p_{\bar{I}}} \\ -\frac{\partial y^{\bar{I}}}{\partial x^I} & 1 & -\frac{\partial y^{\bar{I}}}{\partial q_I} & -\frac{\partial y^{\bar{I}}}{\partial q_{\bar{I}}} & -\frac{\partial y^{\bar{I}}}{\partial p_{\bar{I}}} \\ \frac{\partial x^{\bar{I}}}{\partial x^I} & -1 & \frac{\partial x^{\bar{I}}}{\partial q_I} & \frac{\partial x^{\bar{I}}}{\partial q_{\bar{I}}} & \frac{\partial x^{\bar{I}}}{\partial p_{\bar{I}}} \end{vmatrix} \tag{19}$$

Adding the last column of determinant (19) to the previous one and decomposing this determinant with the hellp of the second row, we get

$$\det \mathrm{Hess}\,(-\Phi) = \begin{vmatrix} A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & A\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + C \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - B\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - D - B\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{vmatrix}, \tag{20}$$

where the matrices A, B, C, and D are given by the relations

$$A = \begin{pmatrix} \frac{\partial y^I}{\partial x^I} & \frac{\partial y^I}{\partial p_{\bar I}} \\ \frac{\partial y^{\bar I}}{\partial x^I} & \frac{\partial y^{\bar I}}{\partial p_{\bar I}} \end{pmatrix}, B = \begin{pmatrix} \frac{\partial p_I}{\partial x^I} & \frac{\partial p_I}{\partial p_{\bar I}} \\ \frac{\partial x^{\bar I}}{\partial x^I} & \frac{\partial x^{\bar I}}{\partial p_{\bar I}} \end{pmatrix}, C = \begin{pmatrix} \frac{\partial y^I}{\partial q_I} & \frac{\partial y^I}{\partial q_{\bar I}} \\ \frac{\partial y^{\bar I}}{\partial q_I} & \frac{\partial y^{\bar I}}{\partial q_{\bar I}} \end{pmatrix}, D = \begin{pmatrix} \frac{\partial p_I}{\partial q_I} & \frac{\partial p_I}{\partial q_{\bar I}} \\ \frac{\partial x^{\bar I}}{\partial q_I} & \frac{\partial x^{\bar I}}{\partial q_{\bar I}} \end{pmatrix}.$$

From the other hand, deriving the variables (x, p) via (y, q) from equations (8) of the Lagrangian manifold L, we obtain equations of the symplectic transformation g. Now, using the implicit function theorem, one can show that the determinant

$$\det(1 - g_*),$$

included in the non–degeneracy condition is equal to

$$(-1)^{|\bar I|}\det A^{-1} \begin{vmatrix} A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & A\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + C \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - B\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - D - B\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{vmatrix}. \tag{21}$$

Comparing formulas (20) and (21) we obtain the relation

$$\det(1 - g_*) = (-1)^{|\bar I|}\mu_I^{-1}\left(x^I, p_{\bar I}, q\right) \det\mathrm{Hess}_{(x,q,p_{\bar I})}(-\Phi) \tag{22}$$

which is valid at fixed points of the mapping g. Relation (22) shows, in particular, that if Condition 1 is valid, then stationary points of the phase function of integral (16) are non–degenerated.

Now the computation of integral (16) by the stationary phase method gives

$$\mathrm{Trace}\, T(g, \varphi) \equiv \sum_k e^{-i\pi\frac{|\bar I|}{2}} \exp\left\{\frac{i}{h}\Phi(\alpha_k)\right\} \frac{\sqrt{\mu_I(\alpha_k)}\varphi(\alpha_k)}{\sqrt{\det \mathrm{Hess}\,(-\Phi)|_{\alpha_k}}} \quad (\mathrm{mod}\, O(h)),$$

where the sum is taken over all fixed points of the mapping g lying in the support of the amplitude function $\varphi\left(x^I, p_{\bar I}, q\right)$.

With the help of formula (22) the latter expression for the trace can be rewritten in the form

$$\operatorname{Trace} T(g,\varphi) \equiv \sum_k \exp\left\{\frac{i}{h}\Phi\left(\alpha_k\right)\right\} \frac{\varphi\left(\alpha_k\right)}{\sqrt{\det(1-g_*\left(\alpha_k\right))}} \quad (\operatorname{mod} O(h)), \tag{23}$$

where the argument of the determinant $\det(1-g_*)$ is chosen in accordance to the following rule:

$$\arg\det(1-g_*) = -i\pi\left|\overline{I}\right| - \arg\mu_I\left(\alpha_k\right) + \arg\det\operatorname{Hess}\left(-\Phi\right)\big|_{\alpha_k}, \tag{24}$$

the argument $\arg\mu_I$ is chosen in accordance with formula (15) and $\arg\det\operatorname{Hess}(-\Phi)$ is equal to

$$\arg\det\operatorname{Hess}(-\Phi) = \sum \arg\lambda_k, \quad -\frac{3\pi}{2} < \arg\lambda_k \leq \frac{\pi}{2}.$$

In the latter relation λ_k are eigenvalues of Hessian matrix.

Now it remains only to note that the value of the phase function $\Phi(\alpha_k)$ at a stationary point is equal to

$$\begin{aligned}\Phi\left(\alpha_k\right) &= x^{\overline{I}}p_{\overline{I}} - qx + S_I\left(x^I, p_{\overline{I}}, q\right)\big|_{\alpha_k} \\ &= x^{\overline{I}}p_{\overline{I}} - qx + \left[S - x^{\overline{I}}p_{\overline{I}} + qx\right]\Big|_{\alpha_k} = S\left(\alpha_k\right),\end{aligned}$$

where S is the non–singular action defined above by formula (10). The latter formula completes the proof of Theorem 1.

To conclude this section we present a generalization of the trace theorem to the case when the symplectic transformation g has a *smooth manifold* of fixed points. This generalization will be used below in consideration of the case when the symplectic transformation is generated by a Hamiltonian flow with closed trajectories. Certainly, for computation of the trace of Fourier integral operator one needs to impose some nondegeneracy condition to the manifolld of fixed points. In our case such condition is essentially Condition 1 formulated in the beginning of this section but considered only in transversal directions to the fixed points manifold.

Let us proceed with exact definitions.

As above, we consider a symplectic transformation

$$g: T^*M \to T^*M$$

and we denote by $\mathcal{F}$ the set of its fixed points. We suppose that $\mathcal{F}$ can be decomposed into a finite disjunct union of the components F_k which are smooth compact manifolds without boundary of different dimensions. We remark that, since the fixed points of g correspond to stationary points of the phase function S, the value of this function is constant on each component F_k of the set $\mathcal{F}$. We denote these values by S_k .

Suppose that the following condition is valid.

Condition 2 The kernel of the operator $1 - g_*$ coincides with the tangent space to the fixed points manifold $\mathcal{F}$.

Then the following affirmation takes place.

Theorem 2 *The formula*

$$\operatorname{Trace} T(g, \varphi) \equiv \sum_k \exp\left\{\frac{i}{h} S_k\right\} \int_{F_k} \frac{\varphi dm}{\sqrt{\prod_j \lambda_j}} \pmod{O(h)} \tag{25}$$

is valid. Here dm is a special measure on F_k which will be defined below and λ_j are non-zero eigenvalues of the operator $1 - g_*$.

Since the proof of this Theorem is similar to those of Theorem 1, we present here only the sketch of the proof. To do this we again write down the trace of the operator $T(g, \varphi)$ in the form of integral (15) and then localize it in a neighbourhood of the set $\mathcal{F}$. The integral can be evidently represented as a sum of integrals taken over neighbourhoods of the components F_k of the set $\mathcal{F}$. Then we write down each of these integrals as an iterated integral, the inner integral being taken over (local) variables transversal to F_k and the outer over the variables on the manifold F_k itself.

Let us show that, under Condition 2 the phase function in the inner integral has only non-degenerated stationary points. Denoting, as above, by Φ the phase function of the integral (16), we obtain the following relation

$$\operatorname{Hess}(-\Phi) = M_1 \begin{pmatrix} A & 0 & 0 \\ B & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 - g_* & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} M_2, \tag{26}$$

where M_1 and M_2 are non-degenerated matrices with constant coefficients, A and B are matrices defined above and α and β are given by

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \beta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This formula shows that the kernel of the matrix Hess$(-\Phi)$ exactly coincides with the tangent space of the corresponding component F_k and, hence, this matrix is non-degenerated in transversal directions to this component.

To conclude the proof, it remains to note that, due to formula (26) we have

$$\frac{\sqrt{\mu_I}}{\sqrt{\mathrm{Hess}'(-\Phi)}} = \frac{m}{\sqrt{\prod_j \lambda_j}}.$$

Here $\arg\lambda_j$ are defined as above (see (13)), m is a density of measure dm on F_k, and by Hess$'$ we denote the Hessian along the variables transversal to F_k.

2 Trace Formula for Evolution Operator

In this section we specialize the theorem proved in the previous section for the case when the considered Fourier integral operator is determined by an evolutional Schrödinger equation

$$\begin{cases} ih\frac{\partial U_t}{\partial t} = \widehat{H} U_t, \\ U_t\big|_{t=0} = \mathrm{id}. \end{cases} \tag{27}$$

Here

$$\widehat{H} = H\left(x, -ih\frac{\partial}{\partial x}\right)$$

is a $1/h$-pseudodifferential operator on the manifold M with the full symbol

$$H(x,p) = H_0(x,p) + hH_1(x,p) + h^2 H_2(x,p) + \dots.$$

It is well-known (see, for example [2]) that the operator U_t determined as a solution of problem (27) is a Fourier integral operator (for each fixed value of t). This operator corresponds to the symplectic transformation g_t which is the shift by the time t along trajectories of the Hamilton vector field $V(H_0)$ defined by the principal symbol $H_0(x,p)$ of the operator $\widehat{H}$. It is supposed that the function $H_0(x,p)$ is a real-valued function and that the flow g_t is defined for all values of t.

We shall also use a subprincipal symbol of the operator $\widehat{H}$, that is, the function defined by the formula

$$H_{\mathrm{sub}}(x,p) = iH_1(x,p) - \frac{1}{2}\left[\frac{\partial^2 H_0(x,p)}{\partial x \partial p} - V(H_0)\ln v_x(x)\right],$$

where $v_x(x)$ is a density of the measure dx introduced in the beginning of Section 1.

We shall denote the solution U_t to problem (27) by

$$U_t = e^{-\frac{i}{h}\widehat{H}t}. \tag{28}$$

The aim of this section is obtaining an asymptotical expansion as $h \to 0$ of the trace of operator (28).

Let us suppose that the Hamiltonian vector firld $V(H)$ has for given value of t only the finite number of isolated closed trajectories γ_k, $k = 1, 2, \ldots, N$ (including zeroes of the field $V(H)$). Then each fixed point α_k of the transformation g_t for such value of t is determined by some closed trajectory of the Hamiltonian field $V(H)$ (in particular, each zero of the Hamiltonian vector field can be wieved as a closed trajectory for any value of t).

We remark that if a closed trajectory of the vector field $V(H)$ containes more that one point (that is, if it is not generated by a zero of $V(H)$), then this trajectory is a smooth submanifold of fixed points of g_t of dimension 1.

The following statement takes place.

Theorem 3 *Let*

$$\hat{a} = a(x, -ih\frac{\partial}{\partial x})$$

be a pseudodifferential operator with the finite[3] *(in (x,p)) principal symbol $a_0(x,p)$ and suppose that the transformation g_t satisfies Condition 2 above. Then the (mod $O(h)$)-comparizon*

$$\text{Trace}\left\{e^{-\frac{i}{h}\hat{H}t}\hat{a}\right\} \equiv \sum_k \exp\left\{\frac{i}{h}\left[\oint_{\gamma_k} p\,dx - H_0(\alpha_k)t\right] - \oint_{\gamma_k} H_{\text{sub}}(x,p)dt\right\} \oint_{\gamma_k} \frac{a_0 dm}{\sqrt{\prod_j \lambda_j}} \tag{29}$$

is valid. In the case when the fixed point of g_t is determined by a zero of the Hamiltonian vector field, the corresponding term under the summation sign in the latter formula becomes

$$\exp\left\{\frac{i}{h}H_0(\alpha_k)t - H_{\text{sub}}(\alpha_k)t\right\} \frac{a_0(\alpha_k)}{\sqrt{\det(1 - g_{t*}(\alpha_k)}}. \tag{30}$$

Here the argument of the determinant $\det(1 - g_{t*})$ is chosen as it was described in the general case (see formula (24) above).

In the case when all fixed points of the Hamilton flow g_t lying on the support of a_0 are determined by zeroes of the field $V(H_0)$, the corresponding result was obtained by microlocal technique in [3] We shall obtain the result of Theorem 3 as a consequence of Theorem 1 proved in the previous subsection.[4]

[3] One can use a weaker assumption, that is that the symbol $a_0(x,p)$ of the operator $\hat{a}$ belongs to the class $\Sigma^{m,\mu}$ for sufficiently large negative value of m.

[4] In doing so, one *has not use* the microlocalization procedure.

Proof of Theorem 3. To derive formula (29) from formula (25) it suffices to compute the values S and φ of the action and amplitude function included in the right–hand part of formula (25). First of all we remark that the operator

$$\exp\left(-\frac{i}{h}\widehat{H}t\right)\widehat{a}$$

is an integral operator with the kernel

$$K(x,y,t) = \left(-\frac{i}{2\pi h}\right)^{n/2} k_{(L_t,\mu_t)}(\varphi_t),$$

where $L_t = \operatorname{graph} g_t$, μ_t is a measure on L_t equal to $\pi_2^*(\omega^n)$, and φ_t is a function on L_t satisfying the transport equation

$$\widehat{\mathcal{P}}\varphi_t = 0, \quad \varphi_t|_{t=0} = a_0. \tag{31}$$

Here $\widehat{\mathcal{P}}$ is the transport operator

$$\widehat{\mathcal{P}} = \frac{\partial}{\partial t} + V(H_0) + H_{\text{sub}}|_{L_t}.$$

The transport operator can be regarded as an operator on the Lagrangian manifold

$$L = \cup_t(L_t)$$

lying in the symplectic space $T^*M \times T^*M \times T^*\mathbf{R}$ with coordinates $(x,p;y,q;t,E)$ defined with the help of the relation

$$E + H_0(x,p)|_L = 0.$$

Besides, due to the initial data of problem (27) it is evident that the base point of the manifold L must be chosen at $t = 0$.

Let us now compute the values of the action S at fixed points of the mapping g_t. We have

$$S = \int_0^t p\,dx + E\,dt = \int_0^t p\,dx - H_0(x,p)\,dt.$$

Since

$$H_0(x,p) = \text{const}$$

along the trajectories of the field

$$\frac{\partial}{\partial t} + V(H_0)$$

and since the point α_k is a fixed point of the transformation g_t, corresponding to the closed trajectory γ_k of $V(H_0)$, we obtain

$$S(\alpha_k) = \oint_{\gamma_k} p\,dx - H_0(\alpha_k)\,t. \tag{32}$$

Now, solving transport equation (31) we have

$$\varphi_t = \exp\left\{-\int_0^t H_{\text{sub}}(x,p)\,dt\right\} a_0(y,q),$$

the integration is taken along trajectory of the vector field $V(H_0)$ with the origin point (y,q). The latter formula gives the value of the amplitude function φ_t at any fixed point α_k of the transformation g_t in the form

$$\varphi_t(\alpha_k) = \exp\left\{-\oint_{\gamma_k} H_{\text{sub}}(x,p)\,dt\right\} a_0(\alpha_k). \tag{33}$$

Substituting expressions (32) and (33) into formula (25) we obtain formula (29). This completes the proof.

Remark 3 In the case when the fixed point α_k of the transformation g_t is determined by a *zero* of the Hamiltonian vector field $V(H_0)$, formulas (32) and (33) can be rewritten in the form

$$\begin{aligned} S(\alpha_k) &= -H_0(\alpha_k)\,t, \\ \varphi_t(\alpha_k) &= \exp\{-H_{\text{sub}}(\alpha_k)\,t\}\,a_0(\alpha_k) \end{aligned}$$

and we come to the formula derived in [3].

Remark 4 In the case when the operators $T(g,\varphi)$ and $\exp\left(-\frac{i}{h}\widehat{H}t\right)$ are acting not on functions but on *sections of some vector bundle* E over the manifold M, formulas (25) and (29) can be rewritten in the form

$$\text{Trace}\,T(g,\varphi) = \sum_k \exp\left\{\frac{i}{h}S_k\right\}\text{Trace}\int_{F_k}\frac{\varphi dm}{\sqrt{\prod_j \lambda_j}} \pmod{O(h)}$$

$$\text{Trace}\left\{e^{-\frac{i}{h}\widehat{H}t}\widehat{a}\right\}$$
$$\equiv \sum_k \exp\left\{\frac{i}{h}\left[\oint_{\gamma_k} p\,dx - H_0(\alpha_k)\,t\right]\right\} \text{Trace}\exp\left\{-\oint_{\gamma_k} H_{\text{sub}}(x,p)\,dt\right\}$$
$$\times \oint_{\gamma_k} \frac{a_0 dm}{\sqrt{\prod_j \lambda_j}} \,(\operatorname{mod} O(h)).$$

Certainly, in the latter formula we suppose that the principal symbol $H_0(x,p)$ is scalar, while all the rest symbols can be matrix ones.

3 Generalization of Atiyah–Bott–Lefschetz Fixed Point Theorem

In this section we shall apply the results of Section 1 to obtaining the theorem of Atyiah–Bott-Lefschetz type for the case when the geometrical endomorphisms of an elliptic complex are given by Fourier integral operators associated with some canonical transformation

$$g : T^*M \to T^*M.$$

Let $E_1 \to M$ and $E_2 \to M$ are vector bundles over M and let

$$\widehat{a} : L^2(M,E_1) \to L^2(M,E_2)$$

be an elliptic pseudodifferential operator acting in sections of these bundles with symbol $a(x,p,h)$ of the class $\Sigma^{m,0}$. The ellipticity of this operator means that there exists a *regulizer* for the operator $\hat{a}$, that is, a $1/h$-pseudodifferential operator

$$\hat{r} = r(x, -ih\,\partial/\partial x)$$

such that symbols of the operators

$$1 - \hat{r} \circ \hat{a}$$

and

$$1 - \hat{a} \circ \hat{r}$$

belong to the class $\Sigma^{-\infty,0} = \bigcap_m \Sigma^{m,0}$. It is also clear that under this condition the operator $\hat{a}$ has finite-dimensional kernel and cokernel, that is, this operator is a Fredholm one.

Let us consider an elliptic complex

$$0 \to L^2(M,E_1) \xrightarrow{\widehat{a}} L^2(M,E_2) \to 0 \tag{34}$$

and let

$$\widehat{U}_1 = T(g, \varphi_1), \quad \widehat{U}_1 = T(g, \varphi_1)$$

be Fourier integral operators of order zero determining an endomorphism of this complex. This means that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L^2(M, E_1) & \xrightarrow{\widehat{a}} & L^2(M, E_2) & \longrightarrow & 0 \\ & & \widehat{U}_1 \Big\downarrow & & \Big\downarrow \widehat{U}_2 & & \\ 0 & \longrightarrow & L^2(M, E_1) & \xrightarrow{\widehat{a}} & L^2(M, E_2) & \longrightarrow & 0 \end{array} \tag{35}$$

commutes.

Remark 5 The nesessity of the requirement that Fourier integral operators $\widehat{U}_1$ and $\widehat{U}_2$ are associated with *one and the same* symplectic transformation g naturally follows from the requirement of commutativity of diagram (35). Actually, due to the fact that this diagram commutes, we have

$$\widehat{a} = \widehat{U}_2^{-1} \circ \widehat{a} \circ \widehat{U}_1.$$

The latter operator is a pseudodifferential one only in the case when the operators $\widehat{U}_1$ and $\widehat{U}_2$ correspond to one and the same symplectic transformation g.

The diagram (35) allows to define *the Lefschetz number*

$$\mathcal{L} = \mathcal{L}\left(\widehat{a}, \widehat{U}_1, \widehat{U}_2\right) = \text{Trace}\left\{\widehat{U}_1\Big|_{\text{Ker}\,\widehat{a}}\right\} - \text{Trace}\left\{\widehat{U}_2\Big|_{\text{Coker}\,\widehat{a}}\right\}.$$

This number is a function of the parameter h and our goal is computation of the leading term of the asymptotical expansion of $\mathcal{L} = \mathcal{L}(h)$ as $h \to 0$.

The following affirmation is valid.

Theorem 4 *Let $\widehat{a}$, $\widehat{U}_1$ and $\widehat{U}_2$ are the above operators and suppose that the transformation $g : T^*M \to T^*M$ satisfies Condition 2 above. Then the formula*

$$\mathcal{L} = \mathcal{L}(h) \equiv \sum_k \exp\left\{\frac{i}{h} S_k\right\} \text{Trace} \int_{F_k} \frac{(\varphi_1 - \varphi_2)\, dm}{\sqrt{\prod_j \lambda_j}} \pmod{O(h)} \tag{36}$$

is valid. In this formula the argument of the determinant $\det(1 - g_*(\alpha_k))$ *is chosen as it is described in Section* 1.

Proof. Let $\widehat{r}$ be a regulizer for the operator $\hat{a}$. As it is shown in [6], the formula

$$\mathcal{L} = \mathrm{Trace}\, \widehat{U}_1 \circ (1 - \widehat{r} \circ \widehat{a}) - \mathrm{Trace}\, \left(\widehat{U}_2 - \widehat{a} \circ \widehat{U}_1 \circ \widehat{r}\right)$$

is valid. The important fact is that this formula does not depend on values of the symbol of the operator $\widehat{r}$ inside a compact set in T^*M. Thus, choosing the operator $\widehat{r}$ in such a way that its symbol is equal to zero in a neighbourhood of the set of fixed points of g and then rewriting the latter formula in the form

$$\mathcal{L} = \mathrm{Trace}\, \widehat{U}_1 \circ (1 - \widehat{r} \circ \widehat{a}) - \mathrm{Trace}\, \widehat{U}_2 \circ (1 - \widehat{a} \circ \widehat{r}), \tag{37}$$

we come to the possibility of using the resuults of Theorem 1 for asymptotical computation of the expression (37) as $h \to 0$. Actually, up to terms of order $O(h^\infty)$ formula (37) can be rewritten in the form

$$\mathcal{L} \equiv \mathrm{Trace}\, U_1' - \mathrm{Trace}\, U_2',$$

where the symbols of the operators U_i' have compact supports and coincide with the symbols of the operators U_j in a vicinity of the set of fixed points of the transformation g. Therefore, we are able to use the result of Theorem 1. It is easy to see that we come to formula (36). This completes the proof of the theorem.

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