ALGEBRAIC COBORDISM

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0. Introduction

The classical construction of the Thom spectrum MU in the stable homotopy category $\mathcal{SH}$ is easy to lift to the $\mathbb{P}^1$-spectrum MGL in the Morel-Voevodsky $\mathbb{A}^1$ stable homotopy over a field k, $\mathcal{SH}(k)$. In this talk, we describe our joint work with Fabien Morel, constructing an algebro-geometric theory Ω^* on smooth quasi-projective schemes over a field k over characteristic zero, which should give a description of the "geometric part" $\oplus_n \mathrm{MGL}^{2n,n}$ of the cohomology theory $\mathrm{MGL}^{*,*}$ represented by MGL.

Throughout this note, k will be a field, $\mathbf{Sm}_k$ denotes the category of smooth quasi-projective k-schemes (as a full subcategory of k-schemes) and **GrRing** is the category of graded, commutative rings with unit. For details of the results described here, we refer to [1] and [2].

1. Oriented cohomology

Abstracting the basic properties of complex cobordism leads to the following definition:

Definition 1. An *oriented cohomology theory A on* $\mathbf{Sm}_k$ consists of

D1. An additive functor $A^* : \mathbf{Sm}_k^{\mathrm{op}} \to \mathbf{GrRing}$.

D2. For each projective morphism $g : Y \to X$ in $\mathbf{Sm}_k$ of relative dimension $d := \dim_k Y - \dim_k X$, an $A^*(X)$-linear map

$$g_* : A^*(Y) \to A^{*-d}(X)$$

which satisfy

The author gratefully acknowledges the support of the Humboldt Foundation through the Wolfgang Paul Program, and support of the NSF via grant DMS 0140445.

A1. $\mathrm{id}_* = \mathrm{id}$, $(f \circ g)_* = f_* \circ g_*$.

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ Z \arrow[r, "f"'] & X \end{tikzcd}$$

is a transverse cartesian square in $\mathbf{Sm}_k$ with g projective of pure relative dimension, then $f^* \circ g_* = g'_* \circ f'^*$.

A2. For $L \to X$ a line bundle over $X \in \mathbf{Sm}_k$ with 0-section $s : X \to L$, set $c_1(L) := s^*(s_*(1_X))$, where $1_X \in A^0(X)$ is the unit. Let $E \to X$ be a rank $n+1$ vector bundle, $\mathbb{P}(E) \to X$ the associated projective space bundle, $O(1) \to \mathbb{P}(E)$ the tautological quotient bundle and $\xi := c_1(O(1)) \in A^1(\mathbb{P}(E))$. Then $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module with basis $1, \xi, \ldots, \xi^n$.

A3. Let $p : V \to X$ be an affine space bundle. Then $p^* : A^*(X) \to A^*(V)$ is an isomorphism.

Examples of oriented cohomology theories:

$$\begin{aligned} &X \mapsto \mathrm{CH}^*(X);\ \text{the Chow ring},\\ &X \mapsto K_0^{alg}(X)[\beta, \beta^{-1}],\\ &X \mapsto \mathrm{MGL}^{2*,*}(X) := \oplus_n \mathrm{MGL}^{2n,n}(X). \end{aligned}$$

In the second example, the β is a variable with degree -1, put in to keep track of the relative dimension under push-forward:

$$g_*([E]\beta^i) := [Rg_*E]\beta^{i+d}.$$

1.1. **The formal group law.** Let $\mathbb{L}$ denote the Lazard ring, i.e., the coefficient ring of the universal formal group law $F_{\mathbb{L}} \in \mathbb{L}[[u, v]]$. Let A be an oriented cohomology theory. Applying A2 twice gives the isomorphism

$$\lim_{\substack{\leftarrow \\ m,n}} A^*(\mathbb{P}^n \times \mathbb{P}^m) \cong A^*(k)[[u, v]],$$

with u going to $c_1(O(1, 0))$ and v going to $c_1(O(0, 1))$. Thus, there is a unique $F_A \in A^*(k)[[u, v]]$ with $F(u, v)$ going over to $c_1(O(1, 1))$. It follows that

$$F(c_1(L), c_1(M)) = c_1(L \otimes M)$$

for all pairs of line bundles L, M on some $X \in \mathbf{Sm}_k$. It easily follows that F_A defines a formal group law over $A^*(k)$, hence there is a canonical homomorphism $\phi_A : \mathbb{L} \to A^*(k)$ classifying F_A.

Example 2. $F_{\mathrm{CH}}(u, v) = u + v$ and $F_{K_0}(u, v) = u + v - \beta uv$.

Theorem 3. *Let k be a field of characteristic zero. Then there is a universal oriented cohomology theory Ω^* on $\mathbf{Sm}_k$. The map $\phi_\Omega : \mathbb{L} \to \Omega^*(k)$ is an isomorphism. If $i : Z \to X$ is a codimension d closed embedding in $\mathbf{Sm}_k$, the sequence*

$$\Omega^{*-d}(Z) \xrightarrow{i_*} \Omega^*(X) \xrightarrow{j^*} \Omega^*(X \setminus Z) \to 0$$

is exact.

We now assume k has characteristic zero. Fix an irreducible $X \in \mathbf{Sm}_k$. Let $\deg : \Omega^*(X) \to \Omega^*(k)$ be the composition

$$\Omega^*(X) \xrightarrow{\text{res}} \Omega^*(k^*(X)) \xrightarrow{\phi^{-1}_{\Omega_{k(X)}}} \mathbb{L} \xrightarrow{\phi_{\Omega_k}} \Omega^*(k).$$

For each proper integral subscheme $Z \subset X$, fix a resolution of singularities $\tilde{Z} \to X$, giving the map $f_Z : \tilde{Z} \to X$. We have the *generalized degree formula*

Theorem 4. *Let α be in $\Omega^*(X)$. There are proper integral subschemes $Z_i \subset X$ and elements $\alpha_i \in \Omega^*(k)$, $i = 1, \dots, n$, such that*

$$\alpha - (\deg \alpha) \cdot 1_X = \sum_{i=1}^{n} \alpha_i \cdot f_{Z*}(1_{\tilde{Z}}).$$

This result implies the so-called "degree formulas" of Rost *et al.* for characteristic classes.

1.2. **Comparison theorems.** For a formal group law (F, R), we have the classifying homomorphism $\mathbb{L} \to R$, giving the oriented theory $\Omega^*_{F,R} := \Omega^* \otimes_{\mathbb{L}} R$. We write

$$\Omega^*_+ := \Omega^*_{u+v,\mathbb{Z}};\ \Omega^*_\times := \Omega^*_{u+v-\beta uv,\mathbb{Z}[\beta,\beta^{-1}]}.$$

Theorem 5. *The maps $\Omega^*_+ \to \mathrm{CH}^*$, $\Omega^*_\times \to K_0^{alg}[\beta, \beta^{-1}]$ induced by the classifying maps $\Omega^* \to \mathrm{CH}^*$, $\Omega^* \to K_0^{alg}[\beta, \beta^{-1}]$ are isomorphisms. The classifying map $\Omega^* \to \mathrm{MGL}^{2*,*}$ is surjective with torsion kernel.*

References

[1] Levine, M. and Morel, F., *Algebraic cobordism I*, preprint, Feb. 2002, http://www.math.uiuc.edu/K-theory/0547/index.html .
[2] Levine, M., *Algebraic cobordism II*, preprint, June 2002, http://www.math.uiuc.edu/K-theory/0577/index.html .

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