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Abstract

A strongly reflective modular form with respect to an orthogonal group of signature $(2, n)$ determines a Lorentzian Kac–Moody algebra. We find a new geometric application of such modular forms: we prove that if the weight is larger than n then the corresponding modular variety is uniruled. We also construct new reflective modular forms and thus provide new examples of uniruled moduli spaces of lattice polarised K3 surfaces. Finally we prove that the moduli space of Kummer surfaces associated to $(1, 21)$-polarised abelian surfaces is uniruled.

1 Reflective modular forms

Let L be an even integral lattice with a quadratic form of signature $(2, n)$ and let

$$\mathcal{D}(L) = \{[Z] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (Z, Z) = 0,\ (Z, \overline{Z}) > 0\}^{+}$$

be the associated n-dimensional bounded symmetric Hermitian domain of type IV (here $+$ denotes one of its two connected components). We denote by $\mathrm{O}^{+}(L)$ the index 2 subgroup of the integral orthogonal group $\mathrm{O}(L)$ preserving $\mathcal{D}(L)$. For any $v \in L \otimes \mathbb{Q}$ such that $v^2 = (v, v) < 0$ we define the *rational quadratic divisor*

$$\mathcal{D}_v = \mathcal{D}_v(L) = \{[Z] \in \mathcal{D}(L) \mid (Z, v) = 0\} \cong \mathcal{D}(v_L^{\perp})$$

where $v_L^{\perp}$ is an even integral lattice of signature $(2, n-1)$. If $\Gamma < \mathrm{O}^{+}(L)$ is of finite index we define the corresponding *modular variety*

$$\mathcal{F}_L(\Gamma) = \Gamma \backslash \mathcal{D}(L),$$

which is a quasi-projective variety of dimension n. The most important subgroups of $\mathrm{O}^{+}(L)$ are the stable orthogonal groups

$$\widetilde{\mathrm{O}}^{+}(L) = \{g \in \mathrm{O}^{+}(L) \mid g|_{L^{\vee}/L} = \mathrm{id}\}, \quad \widetilde{\mathrm{SO}}^{+}(L) = \mathrm{SO}(L) \cap \widetilde{\mathrm{O}}^{+}(L)$$

where $L^{\vee}$ is the dual lattice of L. Modular varieties of orthogonal type appear in algebraic geometry. A prime example are moduli spaces of polarised abelian surfaces and K3 surfaces or, more generally, the moduli spaces of polarised holomorphic symplectic varieties (see [GHS2]–[GHS3]).

Let $k > 0$ and $\chi : \Gamma \to \mathbb{C}^*$ be a character or multiplier system (of finite order) of Γ. By $M_k(\Gamma, \chi)$ we denote the space of modular forms of weight k and character χ with respect to Γ.

Definition 1.1 *A modular form $F \in M_k(\Gamma, \chi)$ is called* **reflective** *if*

$$\operatorname{Supp}(\operatorname{div} F) \subset \bigcup_{\substack{r \in L/\pm 1 \\ r \text{ is primitive} \\ \sigma_r \in \Gamma \text{ or } -\sigma_r \in \Gamma}} \mathcal{D}_r(L) \tag{1}$$

where $\sigma_r : l \to l - \frac{2(r,l)}{(r,r)} r$ is the reflection with respect to r. We call F **strongly reflective** *if the multiplicity of any irreducible component of* $\operatorname{div} F$ *is equal to one.*

This definition is motivated by the following result proved in [GHS1, Corollary 2.13].

Proposition 1.2 *Let $\operatorname{sign}(L) = (2, n)$ and $n \geq 3$. The union of the rational quadratic divisors in (1) is equal to the ramification divisor $\operatorname{Bdiv}(\pi_\Gamma)$ of the modular projection*

$$\pi_\Gamma : \mathcal{D}(L) \to \Gamma \backslash \mathcal{D}(L).$$

The most famous example of a strongly reflective modular form is the Borcherds form $\Phi_{12} \in M_{12}(\mathrm{O}^+(II_{2,26}), \det)$ defined in [B]. It is known that

$$\operatorname{div} \Phi_{12} = \bigcup_{r \in R_{-2}(II_{2,26})/\pm 1} \mathcal{D}_r(II_{2,26})$$

where $R_{-2}(II_{2,26})$ denotes the set of -2-vectors in the even unimodular lattice $II_{2,26}$.

Strongly reflective modular forms are very rare. They determine Lorentzian Kac–Moody algebras (see [B], [GN1]). The following theorem proved in 2010 shows that the existence of a strongly reflective modular form of large weight $k \geq n$ implies that the corresponding modular variety has special geometric properties.

Theorem 1.3 *(see [G4]) Let $\operatorname{sign}(L) = (2, n)$ and $n \geq 3$. Let $F_k \in M_k(\Gamma, \chi)$ be a strongly reflective modular form of weight k and character (of finite order) χ where $\Gamma < \mathrm{O}^+(L)$ is of finite index. By $\kappa(X)$ we denote the Kodaira dimension of X. Then*

$$\kappa(\Gamma \backslash \mathcal{D}(L)) = -\infty$$

if $k > n$, or $k = n$ and F_k is not a cusp form. If $k = n$ and F_k is a cusp form then

$$\kappa(\Gamma_\chi \backslash \mathcal{D}(L)) = 0,$$

where $\Gamma_\chi = \ker(\chi \cdot \det)$ is a subgroup of Γ.

Below we prove a stronger theorem which allows us to conclude that the variety $\Gamma\backslash\mathcal{D}(L)$ is *uniruled*. As an application, using reflective modular forms, we give new examples of uniruled moduli spaces in §3.

2 A sufficient criterion for uniruledness of orthogonal modular varieties

Recall that a variety X is called *uniruled* if there exists a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$ where Y is a variety with $\dim Y = \dim X - 1$. If Y is uniruled, then $\kappa(Y) = -\infty$. A well known conjecture says that the converse also holds, but this is not known with the exception of dimension 3 (where it follows from [Mi]). Using results of Boucksom, Demailly, Paun and Peternell [BDPP] the conjecture would follow from the abundance conjecture. We shall use the numerical criterion for uniruledness due to Miyaoka and Mori [MM] to formulate a criterion which allows us to prove uniruledness of orthogonal modular varieties in many cases.

Theorem 2.1 *Let $\mathcal{D} = \mathcal{D}(L)$ be a connected component of the type IV domain associated to a lattice L of signature $(2, n)$ with $n \geq 3$ and let $\Gamma \subset \mathrm{O}^+(L)$ be an arithmetic group. Let $\widetilde{B} = \sum_r \mathcal{D}_r$ in $\mathcal{D}$ be the divisorial part of the ramification locus of the quotient map $\mathcal{D} \to \Gamma\backslash\mathcal{D}$ (see (1) and Proposition 1.2). Assume that a modular form F_k with respect to Γ of weight k with a (finite order) character exists, such that*

$$\{F_k = 0\} = \sum_r m_r \mathcal{D}_r$$

where the m_r are non-negative integers. Let $m = \max\{m_r\}$ (which must be > 0 by Koecher's principle). If $k > m \cdot n$, then $\Gamma'\backslash\mathcal{D}$ is uniruled (and thus in particular has Kodaira dimension $-\infty$) for every arithmetic group Γ' contating Γ.

Proof. It is clearly enough to prove the result for Γ since every finite quotient of a uniruled variety is uniruled itself.

We first recall that by [GHS1, Theorem 2.12] every quasi-reflection in $h \in \Gamma$ has the property that $h^2 = \pm\,\mathrm{id}$ and thus h acts as a reflection on $\mathcal{D}$. We choose a toroidal compactification X' of the quotient $X = \Gamma\backslash\mathcal{D}$ for which we can assume that the boundary contains no ramification divisor. Such a compactification exists by [GHS1, Corollary 2.22] and the proof of [GHS1, Corollary 2.29]. Let L be the ($\mathbb{Q}$-)line bundle of modular forms of weight 1. We denote the branch locus in X' by $B = \sum_r \mathcal{D}'_r$ (here we use, by abuse of notation, the same index set for the components as we do for the ramification locus). Then over the regular part X'_{reg} of X' we have

$$K_{X'_{\mathrm{reg}}} = nL - \frac{1}{2}B - D \qquad (2)$$

where $D = \sum_\alpha D_\alpha$ is the boundary.

The assumption about the vanishing locus of the form F_k implies

$$kL = \frac{1}{2} \sum_r m_r \mathcal{D}'_r + \sum_\alpha \delta_\alpha D_\alpha, \quad \delta_\alpha \geq 0.$$

Note that the factor 1/2 in front of the term involving the $\mathcal{D}'_r$ comes from the fact that the map $\mathcal{D}(L) \to X$ is branched of order 2 along B. We rewrite this as

$$kL = \frac{1}{2}\big(mB + \sum_r (m_r - m)\mathcal{D}'_r\big) + \sum_\alpha \delta_\alpha D_\alpha$$

and use this to eliminate $\frac{1}{2}B$ from formula (2). The result is

$$-K_{X'_{\mathrm{reg}}} = (\frac{k}{m} - n)L + \sum_r \frac{m - m_r}{2m}\mathcal{D}'_r + \sum_\alpha \frac{m - \delta_\alpha}{m} D_\alpha.$$

Next we choose a resolution $\widetilde{X} \to X'$. For the canonical bundle on $\widetilde{X}$ we obtain

$$-K_{\widetilde{X}} = (\frac{k}{m} - n)L + \sum_r \frac{m - m_r}{2m}\mathcal{D}'_r + \sum_\alpha \frac{m - \delta_\alpha}{m} D_\alpha + \sum_\beta \varepsilon_\beta E_\beta$$

where the E_β are the exceptional divisors and where we have used the notation $\mathcal{D}'_r$ and D_α also for the strict transform of the corresponding divisors on X'.

We recall the following criterion of Mori and Miyaoka[MM, Theorem 1]: assume that a smooth projective variety Z contains an open subest U such that through every point $x \in U$ there is a curve C with $K_Z.C < 0$. Then Z is uniruled. We want to apply this to $\widetilde{X}$ where we choose $U = X_{\mathrm{reg}}$. Recall that a high multiple $L^{\otimes n_0}$ of L defines a map $\varphi_{L^{\otimes n_0}} : \widetilde{X} \to \mathbb{P}^N$ whose image is the Baily-Borel compactification X^{BB} of X. The restriction of this map to U is an isomorphism onto the image and the codimension of $X^{\mathrm{BB}} \setminus U$ in X^{BB} is at least 2 since the boundary of the Baily-Borel compactification is 1-dimensional and X is normal.

Let $x \in U$. Intersecting with $n - 1$ general hyperplanes through x we obtain a curve C which misses both the boundary and the singular locus of X. Hence we can also consider C as a curve in $\widetilde{X}$. We find that

$$-K_{\widetilde{X}}.C = \left((\frac{k}{m} - n)L + \sum_r \frac{m - m_r}{2m}\mathcal{D}'_r + \sum_\alpha \frac{m - \delta_\alpha}{m} D_\alpha + \sum_\beta \varepsilon_\beta E_\beta \right).C$$

Since C does not meet the boundary and the singular locus of X we have $D_\alpha.C = E_\beta.C = 0$. Moreover, since $m \geq m_r$ and $\mathcal{D}'_r.(n_0 L)^{n-1} > 0$ we have $\frac{m - m_r}{2m}\mathcal{D}'_r.C \geq 0$. Finally, since $k > m \cdot n$ and $L.(n_0 L)^{n-1} > 0$ it follows that $K_{\widetilde{X}}.C < 0$ and thus we can apply the criterion by Miyaoka and Mori. $\square$

3 New examples of uniruled moduli spaces

3.1 The moduli space of Kummer surfaces associated to $(1, 21)$-polarised abelian surfaces.

The moduli space of $(1, t)$-polarised abelian surfaces is a Siegel modular 3-fold $\mathcal{A}_t = \Gamma_t \setminus \mathbb{H}_2$ where $\mathbb{H}_2$ is the Siegel upper-half plane of genus 2 and Γ_t is the corresponding paramodular group which is isomorphic to the integral symplectic group of the symplectic form with elementary divisors $(1, t)$. The paramodular group Γ_t has the maximal extension Γ_t^* in $\mathrm{Sp}_2(\mathbb{R})$ of order $2^{\nu(t)}$ where $\nu(t)$ is the number of prime divisors of t (see [GH1]). We proved in [GH1, Theorem 1.5] that the modular variety $\mathcal{A}_t^* = \Gamma_t^* \setminus \mathbb{H}_2$ can be considered as the moduli spaces of Kummer surfaces associated to $(1, t)$-polarised abelian surfaces. The Kodaira dimension of the moduli space $\mathcal{A}_{21}$ of $(1, 21)$-polarised abelian surfaces is non-negative because the geometric genus $h^{3,0}(\bar{\mathcal{A}}_{21})$ is positive for any smooth compactification of $\mathcal{A}_{21}$ (see [G1]–[G2]). We have a $(4 : 1)$ covering $\mathcal{A}_{21} \to \mathcal{A}_{21}^*$.

Theorem 3.1 *The moduli space $\mathcal{A}_{21}^*$ of Kummer surfaces associated to $(1, 21)$-polarised abelian surfaces is uniruled.*

Proof. The symplectic group of genus 2 can be considered as an orthogonal group of signature $(2, 3)$ (see [G1] and [GH1]). In particular, one has

$$\Gamma_t/\{\pm E_4\} \cong \widetilde{\mathrm{SO}}^+(L_t), \qquad \Gamma_t^*/\{\pm E_4\} \cong \mathrm{O}^+(L_t)/\{\pm E_5\}$$

where $L_t = 2U \oplus \langle -2t \rangle$, $U \cong II_{1,1}$ is the hyperbolic plane (i.e. the even unimodular lattice of signature $(1, 1)$), $2U = U \oplus U$, $\langle -2t \rangle$ is the lattice of rank 1 generated by an element of degree $-2t$, $\mathrm{sign}(L_t) = (2, 3)$ and

$$S_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is the Gram matrix of the quadratic form on L_t in the standard basis used in [GH1]. Therefore the moduli spaces of $(1, t)$-polarised abelian surfaces and associated Kummer surfaces are modular varieties of orthogonal type

$$\mathcal{A}_t \cong \Gamma_t \setminus \mathbb{H}_2 \cong \widetilde{\mathrm{SO}}^+(L_t) \setminus \mathcal{D}(L_t), \qquad \mathcal{A}_t^* \cong \Gamma_t^* \setminus \mathbb{H}_2 \cong \mathrm{O}^+(L_t) \setminus \mathcal{D}(L_t).$$

In what follows we construct a reflective modular form with respect to $\mathrm{O}^+(L_{21})$ and apply Theorem 2.1.

It was proved in [GN2, Main Theorem 2.2.3], that L_t with $t = 21$ belongs to a list of special lattices for which (meromorphic) reflective modular forms exist. More exactly, according to this theorem there are three (meromorphic) reflective forms for $t = 21$. Now we construct a nearly holomorphic (reflective) Jacobi form $\xi_{0,21} \in J_{0,21}$ of weight 0 and index 21 whose Borcherds lifting is a holomorphic reflective modular form. We define this form as a polynomial in standard Jacobi modular forms, namely we put

$$\begin{aligned}\xi_{0,21} = \frac{E_{4,3}}{\Delta_{12}}\Bigg(& E_4\phi_{0,4}\big(-6E_4\phi_{0,3}^2\phi_{0,4}^2 + 10E_{4,1}\phi_{0,3}^3\phi_{0,4} + E_{4,2}\phi_{0,4}^3 - 5E_{4,2}\phi_{0,3}^4\big) \\ & + E_{4,1}E_{4,2}\phi_{0,3}\big(\phi_{0,3}^4 - 4\phi_{0,4}^3\big)\Bigg) - 228\phi_{0,1}^3\phi_{0,3}^2\phi_{0,4}^3 \\ & + \phi_{0,1}^2\phi_{0,3}\phi_{0,4}\big(958\phi_{0,4}^3 + 240\phi_{0,2}^2\phi_{0,4}^2 + 2137\phi_{0,2}\phi_{0,3}^2\phi_{0,4} + 11\phi_{0,3}^4\big) \\ & + \phi_{0,1}\Bigg(24\phi_{0,2}\phi_{0,3}^6 - 27\phi_{0,2}^2\phi_{0,3}^4\phi_{0,4} + (-4080\phi_{0,2}^3\phi_{0,3}^2 - 6273\phi_{0,3}^4)\phi_{0,4}^2 \\ & - 8826\phi_{0,2}\phi_{0,3}^2\phi_{0,4}^3 + 30\phi_{0,4}^5\Bigg) - 75\phi_{0,3}\phi_{0,2}\phi_{0,4}^4 \\ & + \big(7668\phi_{0,3}\phi_{0,2}^3 + 24796\phi_{0,3}^3\big)\phi_{0,4}^3 + \big(1920\phi_{0,3}\phi_{0,2}^5 + 6513\phi_{0,3}^3\phi_{0,2}^2\big)\phi_{0,4}^2 \\ & + (24\phi_{0,3}^3\phi_{0,2}^4 + 96\phi_{0,3}^5\phi_{0,2})\phi_{0,4} - 24\phi_{0,3}^5\phi_{0,2}^3 - 72\phi_{0,3}^7.\end{aligned}$$

In this formula we use the following notation: $\Delta_{12}(\tau) = \eta(\tau)^{24} \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$ is the Ramanujan Δ-function, $E_{4,t}$ is the Jacobi-Eisenstein series of weight 4 and index t (see [EZ]) and $\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3}$, $\phi_{0,4}$ are generators of the graded ring of weak Jacobi forms of weight 0 with integral coefficients (see [G3, Theorem 1.9])

$$J_{0,*}^{weak,\mathbb{Z}} = \mathbb{Z}[\phi_{0,1}, \phi_{0,2}, \phi_{0,3}, \phi_{0,4}].$$

The Jacobi forms $\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3}$ are algebraically independent and $4\phi_{0,4} = \phi_{0,1}\phi_{0,3} - 2\phi_{0,2}^2$. The full class of reflective Jacobi forms in [GN2] was obtained using a recursive procedure in terms of Jacobi forms of smaller index. Such formulae are very long to present here. We give in this paper a formula for $\xi_{0,21}$ in terms of the generators. Using explicit formulae for them written in PARI (see [GN1]) we can easily calculate as many terms in the Fourier expansion of $\xi_{0,21}$ as we need. We have

$$\begin{aligned}\xi_{0,21}(\tau, z) = {} & \mathbf{q^{-1}} + 24 + (42r^9 + 168r^8 + \cdots)q \\ & + (\mathbf{3r^{14}} + 322r^{12} + \cdots)q^2 + (420r^{15} + 4152r^{14} + \cdots)q^3 \\ & + (105r^{18} + 2016r^{17} + \cdots)q^4 + (\mathbf{2r^{21}} + 168r^{20} + \cdots)q^5 + O(q^6) \qquad (3)\end{aligned}$$

where $q = e^{2\pi i\tau}$ and $r = e^{2\pi iz}$. The Fourier coefficients in boldface represent all Fourier coefficients $a(n,l)q^nr^l$ in the Fourier expansion of $\xi_{0,21}(\tau, z)$ with indices of negative hyperbolic norm $84n - l^2 < 0$.

The Borcherds lifting $B_{\xi_{0,21}}(Z)$ of the Jacobi form $\xi_{0,21}$ (see [GN1, Theorem 2.1] and [GN2, §2.2]) is a holomorphic modular form of weight 12 and trivial character with respect to $\widetilde{\mathrm{SO}}^+(L_{21})$. We note that a Fourier coefficient $a(n,l)q^nr^l$ of $\xi_{0,t}$ – in our situation we are in the case $t = 21$ – with negative hyperbolic norm $-D = 4tn - l^2 < 0$ determines a divisor $H_D(l)$ with multiplicity $a(n,l)$ of the Borcherds automorphic product $B_{\xi_{0,t}}$ associated to $\xi_{0,t}$, where

$$H_D(l) = \pi_t(\{Z = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathbb{H}_2 \,|\, n\tau + lz + t\omega = 0\}) \subset \Gamma_t \setminus \mathbb{H}_2.$$

This divisor is reflective if and only if $D = l^2 - 4tn$ is a common divisor of $4t$ and $2l$ (see [GN2, Lemma 2.2]). This means that the Borcherds product $B_{\xi_{0,21}}$ has three reflective divisors $H_{84}(0)$, $3H_{28}(14)$ and $2H_{21}(21)$. Here we give an orthogonal reformulation of this fact. For this we represent the index (n,l) of the Fourier coefficient $a(n,l)$ as vector $(k, \frac{l}{2t}, 1)$ in the dual hyperbolic lattice $(L_t^{(1)})^\vee$ of $L_t^{(1)} = U \oplus \langle -2t \rangle$ (see [GN2, §2.2]). In the homogeneous domain $\mathcal{D}(L_{21})$ the Fourier coefficient q^{-1} determines the reflective divisors

$$\mathcal{D}_r, \qquad r \in L_{21}, \quad r^2 = -2, \quad \mathrm{div}(r) = 1, \quad \mathrm{mult}(D_r) = 1$$

where $\mathrm{div}(r)$ is the positive generator of the integral ideal $(r, L_{21}) \subset \mathbb{Z}$. The two other Fourier coefficients determine the divisors

$$\mathcal{D}_u, \qquad u \in L_{21}, \quad u^2 = -2, \quad \mathrm{div}(u) = 2, \quad \mathrm{mult}(\mathcal{D}_u) = 2 \qquad (2q^5r^{21}),$$

$$\mathcal{D}_v, \qquad v \in L_{21}, \quad v^2 = -6, \quad \mathrm{div}(v) = 3, \quad \mathrm{mult}(\mathcal{D}_v) = 3 \qquad (3q^2r^{14}).$$

The divisors $\mathcal{D}_r$ (respectively $\mathcal{D}_u$ and $\mathcal{D}_v$) form one orbit with respect to $\widetilde{\mathrm{SO}}^+(L_{21})$. $\mathrm{O}(L_{21}^\vee/L_{21})$ is the 2-abelian group of order 4 (see [GH1]). The reflection σ_v induces a non-trivial involution in the finite orthogonal group $\mathrm{O}(L_{21}^\vee/L_{21})$ which is different from $-\mathrm{id}$. Therefore $\mathrm{O}^+(L_{21}) = \langle \widetilde{\mathrm{O}}^+(L_{21}), \sigma_v, -E_5 \rangle$. In fact $B_{\xi_{0,21}}(Z)$ is a modular form with respect to the full orthogonal group $\mathrm{O}^+(L_{21})$. This is true because the modular form $B_{\xi_{0,21}}(\sigma_v(Z))$ has the same divisor as $B_{\xi_{0,21}}(Z)$. Therefore they are equal up to a constant according to the Koecher principle.

Now we can apply Theorem 2.1. The modular form $B_{\xi_{0,21}}$ of weight 12 with respect to $\Gamma = \mathrm{O}^+(L_{21})$ has three reflective divisors

$$\mathrm{div}_{\Gamma \setminus \mathcal{D}(L_{21})} B_{\xi_{0,21}} = \pi_\Gamma(D_r) + 2\pi_\Gamma(D_u) + 3\pi_\Gamma(D_v).$$

Since the weight $12 > 3 \cdot 3$ the modular variety $\mathrm{O}^+(L_{21}) \setminus \mathcal{D}(L_{21}) \cong \Gamma_{21}^* \setminus \mathbb{H}_2$ is uniruled according to Theorem 2.1. $\square$

3.2 Uniruled moduli spaces of lattice polarised K3 surfaces.

Let S be a positive definite lattice. We put

$$L(S) = 2U \oplus S(-1), \qquad \mathrm{sign}(L(S)) = (2, 2 + \mathrm{rank}\, S) = (2, 2+n)$$

where $S(-1)$ denotes the corresponding negative definite lattice of rank n. In the applications of this paper S will be A_n $(n \le 7)$, D_n $(n \le 8)$ and E_6 or direct sums of some of them. In what follows we denote by kL the orthogonal sum of k copies of the lattice L and by $L(m)$ the lattice L with quadratic form multiplied by m.

If there exists a primitive embedding of $L(S)$ into the so-called K3 lattice $L_{\mathrm{K3}} = 3U \oplus 2E_8(-1)$ then the modular variety $\widetilde{\mathrm{O}}^+(L(S)) \setminus \mathcal{D}(L(S))$ is the moduli space of lattice polarised K3-surfaces with transcendental lattice $T = L(S)$. The Picard lattice $\mathrm{Pic}(X)$ of a generic member X of this moduli space is the hyperbolic lattice $L(S)^{\perp}_{L_{\mathrm{K3}}}$. See [N], [Do] for more details. If $L(S)$ is 2-elementary, i.e. $L(S)^{\vee}/L(S)$ is a 2-elementary abelian group, then many moduli spaces of lattice polarised K3 surfaces are unirational or rational (see [Ma1], [Ma2]). Here we mainly consider more complicated discriminant groups. For $\mathcal{A}^*_{21}$ the discriminant group is the cyclic group C_{42}. In the examples of this subsection the discriminant group is equal to C_m $(3 \le m \le 8)$ and to C_4^2, C_3^3, C_3^2.

Theorem 3.2 *The modular variety* $\mathcal{M}(S) = \widetilde{\mathrm{O}}^+(L(S)) \setminus \mathcal{D}(L(S))$ *of dimension* $2 + \operatorname{rank} S$ *is uniruled for* S *equal to* A_n $(2 \le n \le 7)$, $2A_3$, $3A_2$, $2A_2$, D_5, D_7 *and* E_6.

We construct strongly reflective modular forms for all $L(S)$ in the theorem using the quasi pullback of the Borcherds form Φ_{12} (see §1). We refer the reader to [BKPS], [GHS1]–[GHS3] for details of the construction of quasi pullback. The proof of the following result can be found in [GHS3, Theorem 8.2 and Corollary 8.12].

Theorem 3.3 *Let* $L \hookrightarrow II_{2,26}$ *be a primitive nondegenerate sublattice of signature* $(2, n)$, $n \ge 3$ *and* $\mathcal{D}_L \hookrightarrow \mathcal{D}_{II_{2,26}}$ *be the corresponding embedding of the homogeneous domains. The set of* -2*-roots*

$$R_{-2}(L^{\perp}) = \{r \in II_{2,26} \mid r^2 = -2,\ (r, L) = 0\}$$

in the orthogonal complement is finite. We put $N(L^{\perp}) = \#R_{-2}(L^{\perp})/2$. *Then the function*

$$\Phi|_L = \left.\frac{\Phi_{12}(Z)}{\prod_{r \in R_{-2}(L^{\perp})/\pm 1} (Z, r)}\right|_{\mathcal{D}_L} \in M_{12+N(L^{\perp})}(\widetilde{\mathrm{O}}^+(L), \det),$$

where in the product over r *we fix a system of representatives in* $R_{-2}(L^{\perp})/\pm 1$. *The modular form* $\Phi|_L$ *vanishes only on rational quadratic divisors of type* $\mathcal{D}_v(L)$ *where* $v \in L^{\vee}$ *is the orthogonal projection to* $L^{\vee}$ *of a* -2*-root* $r \in II_{2,26}$. *Moreover* $\Phi|_L$ *is a cusp form if* $R_{-2}(L^{\perp})$ *is not empty.*

To apply Theorem 3.3 we need basic properties of the root lattices

$$D_n = \{(x_1, \dots, x_n) \in \mathbb{Z}^n \mid x_1 + \dots + x_n \in 2\mathbb{Z}\},$$
$$A_n = \{(x_1, \dots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid x_1 + \dots + x_{n+1} = 0\}.$$

It is known(see [CS, Ch. 4]) that $A_n^\vee/A_n$ is the cyclic group of order $n+1$ and $D_n^\vee/D_n$ is isomorphic to the cyclic group of order 4 for odd n and to $C_2 \times C_2$ for even n. The discriminant forms are generated by the following elements having *the minimal possible norm* in the corresponding classes modulo A_n or D_n:

$$D_n^\vee/D_n = \{\, 0,\ e_n,\ (e_1 + \dots + e_n)/2,\ (e_1 + \dots + e_{n-1} - e_n)/2 \mod D_n\},$$

$$A_n^\vee/A_n = \{\, \varepsilon_i = \frac{1}{n+1}(\underbrace{i, \dots, i,}_{n+1-i}\ \underbrace{i-n-1, \dots, i-n-1}_{i}\,),\ 1 \le i \le n+1\,\}.$$

If $n \le 7$ then for any i we have $\frac{n}{n+1} \le (\varepsilon_i, \varepsilon_i) = \frac{i(n+1-i)}{n+1} \le 2$. These representations of the discriminant groups of A_n and D_n show that for all A- and D-lattices mentioned in Theorem 3.2 we have

$$\forall\, \bar{a} \in (S^\vee/S)\ \exists\, \alpha \in \bar{a} : (\alpha, \alpha) \le 2. \tag{4}$$

The same property is true for E_6, E_7 and E_8. The discriminant group of E_6 is the cyclic group of order 3. Each of the two non-zero classes of $E_6^\vee/E_6$ contains a vector of square 4/3 (see [CS, Ch. 4, §8.3]).

We first construct a strongly reflective modular form with respect to $\widetilde{\mathrm{O}}^+(2U \oplus A_7(-1))$. The even unimodular lattice $II_{2,26}$ is unique up to isomorphism, but it has 24 different models $II_{2,26} \cong 2U \oplus N(R)(-1)$ where $N(R)(-1)$ is a negative definite even unimodular Niemeier lattice with root system R. If R is empty then $N(\emptyset)$ is the Leech lattice. For example for A_7 we can take $N(2A_7 \oplus 2D_5)$. This gives us an embedding of $L(A_7) = 2U \oplus A_7(-1)$ in $II_{2,26} \cong 2U \oplus N(2A_7 \oplus 2D_5)(-1)$ and a cusp form

$$\Phi|_{L(A_7)} \in S_{60}(\widetilde{\mathrm{O}}^+(L(A_7)), \det), \qquad (|R_2(A_7 \oplus 2D_5)| = 96).$$

$\Phi|_{L(A_7)}$ is strongly reflective. More pecisely we shall prove that the divisor of $\Phi|_{L(A_7)}$ is similar to the divisor of Φ_{12}, namely

$$\operatorname{div}(\Phi|_{L(A_7)}) = \bigcup_{\substack{r \in L(A_7)/\pm 1 \\ (r,r)=-2}} \mathcal{D}_r(L(A_7)). \tag{5}$$

According to Theorem 3.3 the divisors of $\Phi|_{L(A_7)}$ are the rational quadratic divisors $\mathcal{D}_v$ where for $v \in L(A_7)^\vee$ there exists u in the dual lattice of the orthogonal complement of $A_7(-1)$ in the Niemeier lattice $N(2A_7 \oplus 2D_5)(-1)$

such that $v+u \in II_{2,26}$ and $v^2+u^2=-2$. If $v^2=-2$ then $u=0$ and v is a -2-root of both lattices $II_{2,26}$ and $L(A_7)$. Therefore $\Phi|_{L(A_7)}$ vanishes along this divisor. We assume that $-2<v^2<0$. According to (4) there exists $h \in A_7^\vee(-1)$ such that $v \in h+L(A_7)$ and $v^2=h^2$. Moreover v and h are primitive in $L(A_7)^\vee$. If not, then $\frac{h}{m} \in A_7^\vee(-1)$ and $(\frac{h}{m},\frac{h}{m}) \ge -(\frac{2}{m})^2 \ge -\frac{1}{2}$. But $(l,l) \le -\frac{7}{8}$ for any $l \in A_7^\vee(-1)$. According to the Eichler criterion (see [GHS4, Proposition 3.3]) there exists $\gamma \in \widetilde{\mathrm{SO}}^+(L(A_7))$ such that $\gamma(v)=h$. Therefore $\gamma(\mathcal{D}_v)=\mathcal{D}_h$. It means that one can complete $h \in A_7^\vee(-1)$ to a root in the Niemeier lattice $N(2A_7 \oplus 2D_5)$. This is not possible because all roots of any Niemeier lattice $N(R)$ are roots of the root lattice R. Thus property (5) is proved. According to Theorem 2.1 the modular variety $\mathcal{M}(A_7)$ is uniruled. The same proof works for all the other lattices from Theorem 3.1. The reason is that we only used the metric property (4). This finishes the proof of Theorem 3.2.

Remark 1. One can formalize the quasi pullback consideration and to construct more reflective modular forms using Φ_{12} and other reflective modular forms. See the forthcoming paper [GG].

Remark 2. We expect that one can find a similar construction for the reflective modular form $B_{\xi_{0,21}}$ using a vector of norm 12 in the Leech lattice and the pullback of the Borcherds modular form Φ_{12} (see Remark 4.4 in [GN1]).

3.3 Modular forms with the simplest possible divisor and uniruled modular varieties.

The divisors of the modular forms used in §3.2 are generated by -2-reflections (see (5)). It might happen that a modular group does not contain -2-reflections but it contains -4- or -6-reflections. These divisors are simpler in the sense of [G4] because the Mumford-Hirzebruch volume of such modular divisors is smaller. Three series of strongly reflective modular forms with the simplest divisor were constructed in [G4]. The longest series is the modular tower $D_1 - D_8$. According to [G4, Theorem 3.2] the modular form

$$\Delta_{12-m,D_m} = \mathrm{Lift}\,(\eta^{24-3m}(\tau)\vartheta(\tau,z_1)\cdots\vartheta(\tau,z_m)) \in M_{12-m}(\widetilde{\mathrm{SO}}^+(L(D_m)))$$

with $1 \le m \le 8$ has the following divisor

$$\mathrm{div}\,\Delta_{12-m,D_m} = \bigcup_{\substack{v\in L(D_m)/\pm 1\\ v^2=-4,\ \mathrm{div}(v)=2}} \mathcal{D}_v(L(D_m)) \tag{6}$$

with some modification for $m=4$. In this section we use the modular forms with the simplest divisor for $S=D_2 \cong A_1 \oplus A_1$, A_2 and D_3.

Theorem 3.4 *The following modular varieties are uniruled*

$$\mathcal{SM}(D_3) = \widetilde{\mathrm{SO}}^+(L(D_3)) \setminus \mathcal{D}(L(D_3)),$$

$$\mathcal{M}(2U(3)\oplus A_2(-1)) = \widetilde{\mathrm{O}}^+(2U(3)\oplus A_2(-1))\setminus \mathcal{D}(2U(3)\oplus A_2(-1)),$$
$$\mathcal{SM}^+(L(2A_1)) = \Gamma\setminus\mathcal{D}(L(2A_1)),$$

where $\Gamma = \langle \widetilde{\mathrm{SO}}^+(L(2A_1)), \sigma_{-4}\rangle$ *and* σ_{-4} *is a reflection acting non trivially on the discriminant group of* $L(2A_1)$*.*

Proof. We note that $D_3\cong A_3$. Therefore $\mathcal{SM}(D_3)$ is a double covering of the variety $\mathcal{M}(A_3)$ considered in Theorem 3.2. The second variety is a covering of $\mathcal{M}(A_2)$ because $\widetilde{\mathrm{O}}^+(2U(3)\oplus A_2(-1))$ is a congruence subgroup of $\widetilde{\mathrm{O}}^+(2U\oplus A_2(-1))$. Therefore the claim of the theorem is stronger than similar results of Theorem 3.2 for A_2 and D_3.

In the case of D_3 the ramification divisor of the modular projection

$$\pi^+_{D_3}: \mathcal{D}(L(D_3))\to \widetilde{\mathrm{SO}}^+(L(D_3))\setminus\mathcal{D}(L(D_3))$$

is equal to the divisor of Δ_{9,D_3} (see [G4, Lemma 2.1]). Thus the first modular variety listed in the theorem is uniruled according to Theorem 2.1.

Consider the last case of the theorem. We note that

$$D_2 = \langle e_1+e_2, e_1-e_2\rangle \cong 2A_1,\qquad \mathrm{O}(D_2^\vee/D_2)\cong C_2.$$

The only non trivial element of $\mathrm{O}(D_2^\vee/D_2)$ is realized by the reflection σ_{2e_1}. All -4-vectors with divisor 2 form one $\widetilde{\mathrm{SO}}^+(L(D_2))$-orbit according to the Eichler criterion. Hence by (6) the form Δ_{10,D_2} is strongly reflective with respect to Γ.

To prove uniruledness of the second modular variety we take the modular form

$$\Delta_{9,A_2} = \mathrm{Lift}(\eta^{15}(\tau)\vartheta(\tau,z_1)\vartheta(\tau,z_2)\vartheta(\tau,z_2-z_1)) \in S_9(\widetilde{\mathrm{O}}^+(L(A_2))).$$

According to [G4, Theorem 4.2] we have

$$\operatorname{div}\Delta_{9,A_2} = \bigcup_{\substack{v\in L(A_2)/\pm 1\\ v^2=-6,\ \operatorname{div}(v)=3}} \mathcal{D}_v(L(A_2)).$$

The modular form Δ_{9,A_2} is anti-invariant with respect to reflections σ_v with $v^2=-6$, $\operatorname{div}(v)=3$. These reflections induce the non-trivial element of the finite orthogonal discriminant group $\mathrm{O}(A_2^\vee/A_2)\cong C_2$. Therefore we have $\Delta_{9,A_2}\in S_9(\mathrm{O}^+(L(A_2)),\chi_2)$ where χ_2 is a character of order 2.

If $v^2=-6$ and $\operatorname{div}(v)=3$ then $v/3$ is a primitive vector of $L(A_2)^\vee$ and $(\frac{v}{3},\frac{v}{3}) = -\frac{2}{3}$. Therefore $v/3$ is a -2 vector of the lattice

$$L(A_2)^\vee(3)\cong 2U(3)\oplus A_2^\vee(-3)\cong 2U(3)\oplus A_2(-1).$$

We have $\mathrm{O}^+(L) = \mathrm{O}^+(L^\vee) = \mathrm{O}^+(L^\vee(m))$ and we can thus consider Δ_{9,A_2} as a modular form with respect to the last group. Since $\sigma_v = \sigma_{(v/3)} \in \widetilde{\mathrm{O}}^+(2U(3) \oplus A_2(-1))$ the modular form Δ_{9,A_2} is strongly reflective with respect to $\widetilde{\mathrm{O}}^+(2U(3) \oplus A_2(-1))$. Therefore the second variety of the theorem is also uniruled. $\square$

Remark 1. S. Ma has informed us that he can in fact prove that the variety $\mathcal{SM}^+(L(2A_1))$ is rational.

Remark 2. A modular form of type $\Phi|_{L(S)}$ where S is a root lattice from Theorem 3.2 is the automorphic discriminant of the moduli space of lattice polarised K3 surfaces. It determines a Lorentzian Kac–Moody algebra and gives an arithmetic version of mirror symmetry for K3 surfaces (see [GN3] for more details). The strongly reflective modular forms from §3.3 have similar interpretation for corresponding moduli spaces.

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References

[B] R.E. Borcherds, *Automorphic forms on $O_{s+2,2}(R)$ and infinite products.* Invent. Math. **120** (1995), 161–213.

[BKPS] R.E. Borcherds, L. Katzarkov, T. Pantev, N.I. Shepherd-Barron, *Families of* K3 *surfaces.* J. Algebraic Geom. **7** (1998), 183–193.

[BDPP] S. Boucksom, J.-P. Demailly, M. Paun, Th. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension.* `arXiv:math/0405285`, 39 pp., to appear in J. Algebraic Geom. (2012).

[CS] J.H. Conway, N.J.A. Sloane, *Sphere packings, lattices and groups.* Grundlehren der mathematischen Wissenschaften **290**. Springer-Verlag, New York, 1988.

[Do] I. Dolgachev, *Mirror symmetry for lattice polarized* K3 *surfaces.* J. Math. Sci. **81** (1996), 2599–2630.

[EZ] M. Eichler, D. Zagier, *The theory of Jacobi forms.* Progress in Mathematics **55**. Birkhäuser, Boston, Mass., 1985.

[GG] B. Grandpîerre, V. Gritsenko, *The baby functions of the Borcherds form Φ_{12}*. In preparation.

[G1] V. Gritsenko, *Modular forms and moduli spaces of abelian and* K3 *surfaces.* Algebra i Analiz **6** (1994), 65–102; English translation in St. Petersburg Math. J. **6** (1995), 1179–1208.

[G2] V. Gritsenko, *Irrationality of the moduli spaces of polarized abelian surfaces.* Internat. Math. Res. Notices **6** (1994), 235–243.

[G3] V. Gritsenko, *Elliptic genus of Calabi–Yau manifolds and Jacobi forms.* St. Petersburg Math. Journal **11** (2000), 781–804.

[G4] V. Gritsenko, *Reflective modular forms in algebraic geometry.* `arXiv:1005.3753`, 28 pp.

[GH1] V. Gritsenko, K. Hulek, *Minimal Siegel modular threefolds.* Math. Proc. Cambridge Philos. Soc. **123** (1998), 461–485.

[GH2] V. Gritsenko, K. Hulek, *Commutator coverings of Siegel threefolds.* Duke Mathem. Journal **94** (1998), 509–542.

[GHS1] V. Gritsenko, K. Hulek, G.K. Sankaran, *The Kodaira dimension of the moduli of K3 surfaces.* Invent. Math. **169** (2007), 519–567.

[GHS2] V. Gritsenko, K. Hulek, G.K. Sankaran, *Moduli spaces of irreducible symplectic manifolds.* Compos. Math. **146** (2010), 404–434.

[GHS3] V. Gritsenko, K. Hulek, G.K. Sankaran, *Moduli spaces of K3 surfaces and holomorphic symplectic varieties.* arXiv:1012.4155, 71 pp., "Handbook of Moduli" (ed. G. Farkas and I. Morrison), 2012.

[GHS4] V. Gritsenko, K. Hulek, G.K. Sankaran, *Abelianisation of orthogonal groups and the fundamental group of modular varieties.* J. of Algebra **322** (2009), 463–478.

[GN1] V. Gritsenko, V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. II.* International J. Math. **9** (1998), 201–275.

[GN2] V. Gritsenko, V. Nikulin, *On classification of Lorentzian Kac–Moody algebras.* Russian Math. Survey **57** (2002), 921–981.

[GN3] V. Gritsenko, V. Nikulin, *The arithmetic mirror symmetry and Calabi–Yau manifolds.* Comm. Math. Phys. **200** (2000), 1–11.

[Ma1] S. Ma, *Rationality of the moduli spaces of 2-elementary K3 surfaces.* arXiv:1110.5110, 57 pp.

[Ma2] S. Ma, *The unirationality of the moduli spaces of 2-elementary K3 surfaces* (with an Appendix by K.-I. Yoshikawa). arXiv:1011.1963, 32 pp.

[Mi] Y. Miyaoka, *Deformations of a morphism along a foliation and applications.* Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), 245–268, Proc. Sympos. Pure Math., **46**, Part 1, Amer. Math. Soc., Providence, RI, 1987.

[MM] Y. Miyaoka, S. Mori, *A numerical criterion for uniruledness.* Ann. of Math. (2) **124** (1986), 65–69.

[N] V. Nikulin, *Finite groups of automorphisms of Kählerian* K3 *surfaces.* Trans. Moscow Math. Soc. **2** (1980), 71–135.

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