# RENORMALIZATION FOR DUMMIES

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Abstract. This talk is based on joint work with Alain Connes "Renormalization and motivic Galois theory", where we investigate the nature of divergences in quantum field theory, showing that they are organized in the structure of a certain "motivic Galois group", which is uniquely determined and universal with respect to the set of physical theories. The renormalization group can be identified canonically with a one parameter subgroup. The group is obtained through a Riemann-Hilbert correspondence. Its representations classify equisingular flat vector bundles, where the equisingularity condition is a geometric formulation of the fact that in quantum field theory the counterterms are independent of the choice of a unit of mass. As an algebraic group scheme, it is a semi-direct product by the multiplicative group of a pro-unipotent group scheme whose Lie algebra is freely generated by one generator in each positive integer degree. There is a universal singular frame in which all divergences disappear. When computed as iterated integrals, its coefficients are certain rational numbers that appear in the local index formula of Connes-Moscovici. When working with formal Laurent series over the field of rational numbers, the data of equisingular flat vector bundles define a Tannakian category whose properties are reminiscent of a category of mixed Tate motives.

## 1. The Connes–Kreimer theory of renormalization

The main idea of renormalization is to correct the original Lagrangian of a quantum field theory by an infinite series of counterterms, labelled by the Feynman graphs that encode the combinatorics of the perturbative expansion of the theory. These counterterms have the effect of cancelling the ultraviolet divergences. Thus, in the procedure of perturbative renormalization, one introduces a counterterm $C(\Gamma)$ in the initial Lagrangian for every divergent one particle irreducible (1PI) Feynman diagram $\Gamma$. In the case of a *renormalizable* theory, all the necessary counterterms $C(\Gamma)$ can be obtained by modifying the numerical parameters that appear in the original Lagrangian.

One of the most effective renormalization techniques in quantum field theory, developed by 't Hooft and Veltman [9], is dimensional regularization (DimReg). It is widely used in perturbative calculations. It is based on an analytic continuation of Feynman diagrams to complex dimension $d \in \mathbb{C}$, in a neighborhood of the integral dimension $D$ at which UV divergences occur. For the complex dimension $d \to D$, the analytically continued integrals become singular and the expression admits a Laurent series expansion. Thus, within the framework

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of dimensional regularization, one can implement a renormalization by minimal subtraction, where the singular part of the Laurent series in $z = d - D$ is subtracted at each order in the loop expansion.
The Bogoliubov–Parasiuk preparation, or BPHZ method (for Bogoliubov–Parasiuk–Hepp–Zimmermann) accounts for the presence of subdivergences in the renormalization process and the possible appearance of non-local terms via the following inductive procedure.
The BP preparation of a graph $\Gamma$, whose divergent integral we denote by $U(\Gamma)$, is given by the formal expression

$$\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma\subset\Gamma} C(\gamma)U(\Gamma/\gamma), \tag{1.1}$$

where the sum is over divergent subgraphs. The $C(\gamma)$ are inductively defined counterterms, obtained (in the minimal subtraction scheme) by taking the pole part (here denoted by $T$) of the Laurent expansion in $z = d - D$ of a divergent expression,

$$C(\Gamma) = -T(\overline{R}(\Gamma)) = -T\left(U(\Gamma) + \sum_{\gamma\subset\Gamma} C(\gamma)U(\Gamma/\gamma)\right). \tag{1.2}$$

The renormalized value of $\Gamma$ is then given by the formula

$$R(\Gamma) = \overline{R}(\Gamma) + C(\Gamma) = U(\Gamma) + C(\Gamma) + \sum_{\gamma\subset\Gamma} C(\gamma)U(\Gamma/\gamma). \tag{1.3}$$

This extraction of a renormalized value from divergent Feynman integrals was related in the work of Connes and Kreimer to a Hopf algebra constructed out of the Feynman diagrams of the theory. The main conceptual breakthrough in the Connes–Kreimer theory of perturbative renormalization [2], [3] consists of showing that the BPHZ recursive formulae (1.1), (1.2), (1.3) can be understood mathematically in terms of a well known procedure of extraction of finite values known as the Birkhoff factorization of loops.
In fact, one can consider the affine group scheme $G$ dual to the commutative Hopf algebra of Feynman graphs and describe the unrenormalize integrals $U(\Gamma)$ by assigning a loop $\gamma(z)$ defined on an infinitesimal circle $C$ around the critical dimension $D$, with values in the pro-unipotent Lie group $G(\mathbb{C})$. Connes and Kreimer gave an explicit recursive formula for the Birkhoff factorization

$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z), \quad \forall z \in C, \tag{1.4}$$

of $\gamma(z)$ into a part $\gamma_+(z)$ regular at $z = 0$ (*i.e.* at the cricital dimension $D$) and a part $\gamma_-(z)$ holomorphic in $\mathbb{P}^1(\mathbb{C}) \smallsetminus \{0\}$.
The affine group scheme $G$ dual to the Connes–Kreimer Hopf algebra is called the "group of diffeographisms" of the physical theory, as it acts on the coupling constants of the theory thourgh a representation in the group of formal diffeomorphisms tangent to the identity (*cf.* [2], [3]).
When applied to the group of diffeographisms, the explicit form of the Birkhoff factorization (1.4) yield the counterterms (1.2) and the renormalized values (1.3) in the BPHZ renormalization procedure.

Another important point in the Connes–Kreimer theory is that the loop $\gamma(z)$ in fact depends on a mass scale $\mu$, while, due to physical considerations, the negative piece of the Birkhoff factorization, which produces the counterterms (1.2), is in fact independent of $\mu$.

## 2. Renormalization and the Riemann–Hilbert correspondence

In our work [4] [5] (see also the survey [6]), we identified the loops $\gamma_\mu(z) = \gamma_{\mu,-}(z)^{-1}\gamma_{\mu,+}(z)$ with solutions of suitable differential equations. This uses an unpublished result of 't Hooft, which shows that the counterterms only depend on the beta-function of the theory (the infinitesimal generator of the renormalization group flow). Connes and Kreimer proved this result in their context in the form of a scattering formula for $\gamma_-(z)$.

We used a description through iterated integrals, writing the time-ordered exponential of physicists in the form

$$\mathbf{T}e^{\int_a^b \alpha(t)\,dt} = 1 + \sum_1^\infty \int_{a\le s_1\le\cdots\le s_n\le b} \alpha(s_1)\cdots\,\alpha(s_n)\prod ds_j\,, \tag{2.1}$$

for a $\mathfrak{g}(\mathbb{C})$-valued smooth function $\alpha(t)$, where $t \in [a,b] \subset \mathbb{R}$ is a real parameter, and the product taken in the dual algebra $\mathcal{H}^\vee$. When paired with any element $x \in \mathcal{H}$, (2.1) reduces to a finite sum, which defines an element in $G(\mathbb{C})$.

We are particularly interested in the following property of the expansional: (2.1) is the value $g(b)$ at $b$ of the unique solution $g(t) \in G(\mathbb{C})$ with value $g(a) = 1$ of the differential equation

$$dg(t) = g(t)\,\alpha(t)\,dt\,. \tag{2.2}$$

The idea of reformulating a Birkhoff factorization problem in terms of a class of differential equations is familiar to the analytic approach to the Riemann–Hilbert problem. In our setting, this reformulation allows us to associate to the BPHZ procedure of perturbative renormalization a class of differential systems up to a natural equivalence, which we then study via the Riemann–Hilbert correspondence.

We start by considering the data of loops $\gamma_\mu(z)$ satisfying the conditions

$$\frac{\partial}{\partial\mu}\,\gamma_{\mu^-}(z) = 0\,. \tag{2.3}$$

and

$$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)), \qquad \forall t \in \mathbb{R}. \tag{2.4}$$

These two conditions admit a geometric description in terms of connections as follows.

Let $\pi : B \to \Delta$ be a principal $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$-bundle, identified with $\Delta \times \mathbb{C}^*$ by the non-canonical choice of a section $\sigma : \Delta \to B$, $\sigma(0) = y_0$. Physically, the latter corresponds to a choice of the Planck constant. Let $P = B \times G(\mathbb{C})$ be the trival principal $G(\mathbb{C})$-bundle, and $B^*$ and $P^*$ the restrictions to the punctured disk $\Delta^*$. Here $\Delta$ represents the complexified dimensions $D - z$ of DimReg and $\mathbb{C}^*$ the possible choices of normalization of the integral "in dimension $D - z$".

We consider $G(\mathbb{C})$-connections $\omega$ on $P^*$, which are *equisingular*. This means that $\omega$ is $\mathbb{G}_m$-invariant and its restrictions to sections of the principal bundle $B$ that agree at $0 \in \Delta$ are mutually equivalent, in the sense that they are related by a gauge transformation by a $G(\mathbb{C})$-valued $\mathbb{G}_m$-invariant map $h$ regular in $B$. The property that, when approaching the singular fiber, the type of singularity does not depend on the section along which one restricts the connection but only on the value of the section at $0 \in \Delta$ corresponds to the fact that the counterterms are independent of the mass scale, as in (2.3).
Thus, we have identified a class of differential systems associated to a physical theory, namely the equivalence classes of flat equisingular $G(\mathbb{C})$-valued connections on $P$.
The Riemann–Hilbert correspondence consists of describing a certain category of equivalence classes of differential systems though a representation theoretic datum (*cf. e.g.* [11]). When the category is a neutral Tannakian category, one obtains a classification in terms of finite dimensional linear representations of an affine group scheme.
Rather than working with an assigned renormalizable QFT, it is preferable to formulate the Riemann–Hilbert correspondence in a universal setting. This is achieved by considering, instead of flat equisingular $G(\mathbb{C})$-connections, the category of equivalence classes of all *flat equisingular bundles*. For a specific physical theory, the corresponding category of equivalence classes of flat equisingular $G(\mathbb{C})$-connections can be recovered from this more general setting by considering the subcategory of those flat equisingular bundles that are finite dimensional linear representations of $G^* = G \rtimes \mathbb{G}_m$.
The category of equivalence classes of flat equisingular bundles has as objects $\Theta = (E, \nabla)$ pairs of a finite dimensional $\mathbb{Z}$-graded vector space $E$ and an equisingular flat $W$-connection $\nabla$. To define the latter, we consider the vector bundle $\tilde{E} = B \times E$ with the action of $\mathbb{G}_m$ given by the grading and with the weight filtration defined by $W^{-n}(E) = \oplus_{m \geq n} E_m$. A $W$-connection is a connection on the restriction of $\tilde{E}$ to $B^*$, which is compatible with the weight filtration and induces the trivial connection on the associated graded. The connection $\nabla$ in the data above is a flat $W$-connection that satisfies the equisingular condition, that is, it is $\mathbb{G}_m$-invariant and the restrictions to sections $\sigma$ of $B$ with $\sigma(0) = y_0$ are all $W$-equivalent on $B$, where the equivalence relation is realized by an isomorphism of the vector bundles over $B$, compatible with the filtration and identity on the associated graded, that conjugates the connections. We consider the data $\Theta = (E, \nabla)$ as $W$-equivalence classes.
For a linear map $T : E \to E'$, consider the $W$-connections $\nabla_j$, $j = 1, 2$, on $\tilde{E}' \oplus \tilde{E}$ of the form

$$\nabla_1 = \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix} \quad \text{and} \quad \nabla_2 = \begin{pmatrix} \nabla' & T\nabla - \nabla' T \\ 0 & \nabla \end{pmatrix}, \tag{2.5}$$

where $\nabla_2$ is the conjugate of $\nabla_1$ by the unipotent matrix

$$\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}.$$

Morphisms $T \in \mathrm{Hom}(\Theta, \Theta')$ in the category of equisingular flat bundles are linear maps $T : E \to E'$ compatible with the grading and such that the connections $\nabla_j$ of (2.5) are $W$-equivalent on $B$. The condition is independent of the choice of representatives for the connections $\nabla$ and $\nabla'$.
The category $\mathcal{E}$ of equisingular flat bundles is a tensor category over $k = \mathbb{C}$, with a fiber functor $\omega : \mathcal{E} \to Vect_{\mathbb{C}}$ given by

$$\omega : \Theta = (E, \nabla) \mapsto E. \tag{2.6}$$

In fact, one can refine the construction and work over the field $k = \mathbb{Q}$, since the universal singular frame (see (2.10) below) in which one expresses the connections has rational coefficients. In this case, the fiber functor $\omega : \mathcal{E}_{\mathbb{Q}} \to Vect_{\mathbb{Q}}$ is of the form $\omega = \oplus \omega_n$, with

$$\omega_n(\Theta) = \mathrm{Hom}(\mathbb{Q}(n), \mathrm{Gr}^W_{-n}(\Theta)),$$

where $\mathbb{Q}(n)$ denotes the object in $\mathcal{E}_{\mathbb{Q}}$ given by the class of the pair of the trivial bundle over $B$ with fiber a one-dimensional $\mathbb{Q}$-vector space placed in degree $n$ and the trivial connection.
Let $\mathcal{F}(1, 2, 3, \cdots)_\bullet$ be the free graded Lie algebra generated by one element $e_{-n}$ in each degree $n \in \mathbb{Z}_{>0}$, and let

$$\mathcal{H}_u = \mathcal{U}(\mathcal{F}(1, 2, 3, \cdots)_\bullet)^\vee \tag{2.7}$$

be the commutative Hopf algebra obtained by considering the graded dual of the enveloping algebra $\mathcal{U}(\mathcal{F})$. We can then identify explicitly the affine group scheme associated to the neutral Tannakian category of flat equisingular bundles as follows (*cf.* [4] [5]).

**Theorem 2.1.** *The category $\mathcal{E}$ of flat equisingular bundles is a neutral Tannakian category, with fiber functor* (2.6)*. It is equivalent to the category* $\mathrm{Rep}_{U^*}$ *of finite dimensional linear representations of the affine group scheme $U^* = U \rtimes \mathbb{G}_m$, where $U$ is the pro-unipotent affine group scheme associated to the Hopf algebra $\mathcal{H}_u$ of* (2.7)*.*

The proof is obtained by expressing the connections in terms of a "universal singular frame"

$$\gamma_U(z, v) = \mathbf{T}\mathbf{e}^{-\frac{1}{z}\int_0^v u^{Y}(e)\frac{du}{u}} \in U\ , \tag{2.8}$$

where we consider the element

$$e = \sum_1^\infty e_{-n}\,, \tag{2.9}$$

in the Lie algebra $\mathrm{Lie}\, U$. The frame (2.8) has coefficients

$$\gamma_U(z, v) = \sum_{n\geq 0} \sum_{k_j>0} \frac{e_{-k_1} e_{-k_2} \cdots e_{-k_n}}{k_1\,(k_1+k_2)\cdots(k_1+k_2+\cdots+k_n)}\, v^{\sum k_j}\, z^{-n}, \tag{2.10}$$

with $e_{-n}$ the generators of $\mathrm{Lie}\, U$. The coefficients are the same as those occurring in in the local index formula of Connes–Moscovici [7].
In [1], Cartier conjectured the existence of a universal group of symmetries, which he called "cosmic Galois group", which acts on the coupling constants of (renormalizable) physical theories and relates the Connes–Kreimer theory of

perturbative renormalization to number theoretic objects such as multiple zeta values. The affine group scheme $U^*$ fulfills these properties. In fact, through the map ([2], [3]) of $G$ to formal diffeomorphisms, and the representation $U^* \to G^*$ specified by the beta function (*cf.* [4])

$$U \xrightarrow{\rho} G \to \mathrm{Diff}(\mathbb{C}^N),$$

the "cosmic Galois group" $U^*$ acts on the coupling constants. Moreover, the affine group scheme $U^*$ is a motivic Galois group. In fact, by results of Goncharov and Deligne (*cf.* [8]), we have the following identification.

**Proposition 2.2.** *There is a (non-canonical) isomorphism*

$$U^* \cong G_{\mathcal{M}_T}(\mathcal{O})\,. \tag{2.11}$$

*of the affine group scheme $U^*$ with the motivic Galois group $G_{\mathcal{M}_T}(\mathcal{O})$ of the scheme $S_N$ of $N$-cyclotomic integers, for $N = 3$ or $N = 4$.*

Notice also that, as $U$ is a pro-unipotent affine group scheme, the element $e$ in (2.9) defines a morphism of affine group schemes

$$\mathbf{rg}\ :\ \mathbb{G}_a\ \to U\,, \tag{2.12}$$

from the additive group $\mathbb{G}_a$ to $U$, which gives a lifting of the renormalization group **rg** to a canonical 1-parameter subgroup of $U^*$. One obtains this way, in particular, a Galois interpretation of the renormalization group.

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