AN INVOLUTION PERMUTING FLOER HOMOLOGY

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Abstract. We give an example of an irreducible homology sphere with an involution acting non-trivially on Floer homology.

Let k be the right-handed granny knot shown in Figure 1, and consider the involution on its exterior given by the 180° rotation with respect to the axis indicated in Figure 1.

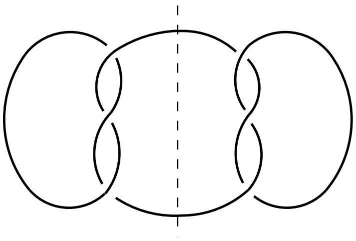

Figure 1

Let Σ be the integral homology 3–sphere obtained by (-1)–surgery on the knot k. The involution on the exterior of k extends to an orientation preserving involution $\sigma : \Sigma \to \Sigma$.

Theorem 1. *The involution $\sigma_* : I_*(\Sigma) \to I_*(\Sigma)$ induced on the Floer homology by σ is non-trivial, $\sigma_* \neq \mathrm{id}$.*

Note that the manifold Σ is irreducible. Many examples of reducible homology 3-spheres with involutions acting non-trivially on Floer homology have been known, see [9] including the discussion on [1].

The proof goes as follows. We first describe the representation variety of $\pi_1\Sigma$ and compute the indices of the generators in the Floer chain complex. After that, it is not difficult to see that σ induces a non-trivial action on the Floer chain complex. However, since the generators are present in both parities, it is not clear if the boundary operators vanish. To show that the action is actually non-trivial on the Floer *homology* we use a cobordism argument and the Floer exact triangle. In the last section we describe the representations of $\pi_1\Sigma$ fixed by σ in terms of the branch set of $\Sigma \to S^3$.

I wish to thank Olivier Collin, Jose Montesinos and Daniel Ruberman for useful discussions.

Research was supported in part by NSF Grant 9704204 and MPI Bonn.

1. THE MANIFOLD Σ

The integral homology sphere Σ admits many equivalent descriptions. We list several of them in this section and prove that they are actually equivalent.

Description 1. The result of (-1)–surgery on the right-handed granny knot k shown in Figure 1. Note that k is a connected sum of two right-handed trefoils.

Description 2. The double branched cover of S^3 with the branch set shown in Figure 2.

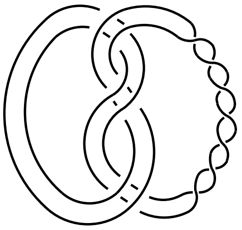

FIGURE 2

Description 3. The graph manifold obtained by plumbing according to the diagram shown in Figure 3. In fact, this description shows that Σ is the link of a singularity. For basics on graph manifolds and singularities see [4].

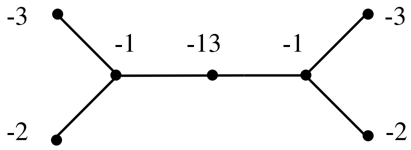

FIGURE 3

Description 4. The splice of two copies of the Brieskorn homology sphere $\Sigma(2,3,7)$ along the singular fibers of degree 7.

Description 5. The manifold obtained by gluing together two copies of the right-handed trefoil exterior along the boundary tori according to the map given in the standard meridian-longitude bases by the matrix

$$\varphi = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Description 6. Let k_1 and k_2 be the singular fibers of degree 7 in two copies of $\Sigma(2,3,7)$, and k_1^* and k_2^* the images of their cores in the surgered manifolds $\Sigma(2,3,7) - k_1 = \Sigma(2,3,7) - k_2 = \Sigma(2,3,13)$. Then Σ is the result of $(+1)$–surgery of the connected sum $\Sigma(2,3,13)\#\Sigma(2,3,13)$ along the knot $k_1^* \# k_2^*$.

The knot k is strongly invertible with respect to the given involution, so the equivalence of descriptions 1 and 2 follows from [8]. The equivalence of descriptions 1 and 3 is an exercise in Kirby calculus, see Figure 4.

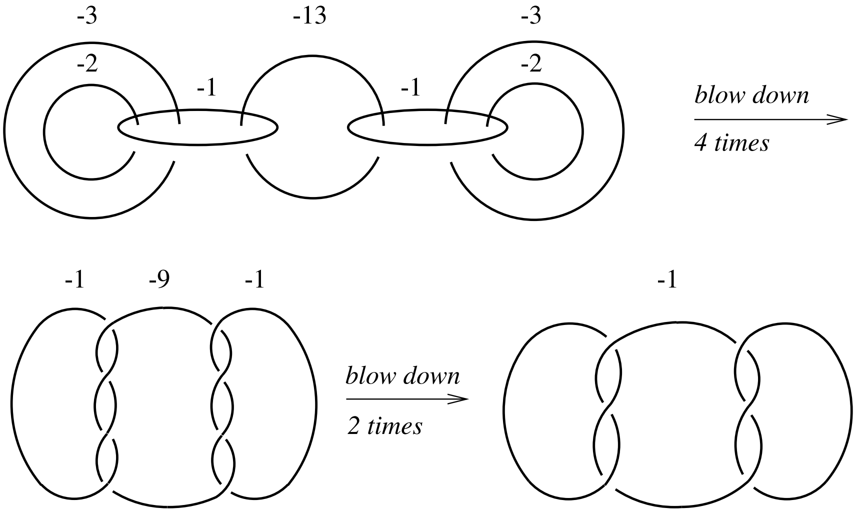

Figure 4

The manifold $\Sigma(2,3,13)$ is obtained from $\Sigma(2,3,7)$ by (-1)–surgery on k_1, see Figure 5, where the core k_1^* in $\Sigma(2,3,13)$ is also shown.

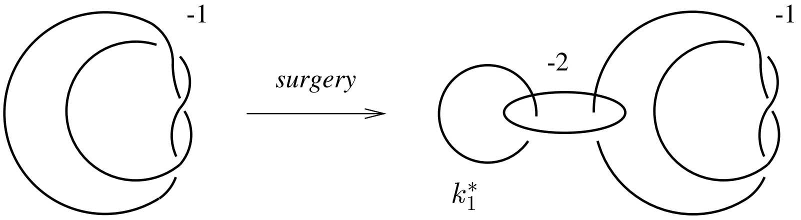

Figure 5

Then the manifold $\Sigma(2,3,13)\#\Sigma(2,3,13) + k_1^*\#k_2^*$ looks as in Figure 6. This proves the equivalence of descriptions 1 and 6. The algorithm of [4] for splicing plumbing diagrams produces the diagram in Figure 3. Descriptions 4 and 5 are obtained from each other by a change of basis.

2. The representation variety

This section is devoted to describing the representation space of $\pi_1\Sigma$ and the induced action of σ.

Lemma 2. *The representation space of* $\pi_1\Sigma$ *consists of four isolated points and four circles. The representations are non-degenerate in the Morse–Bott sense.*

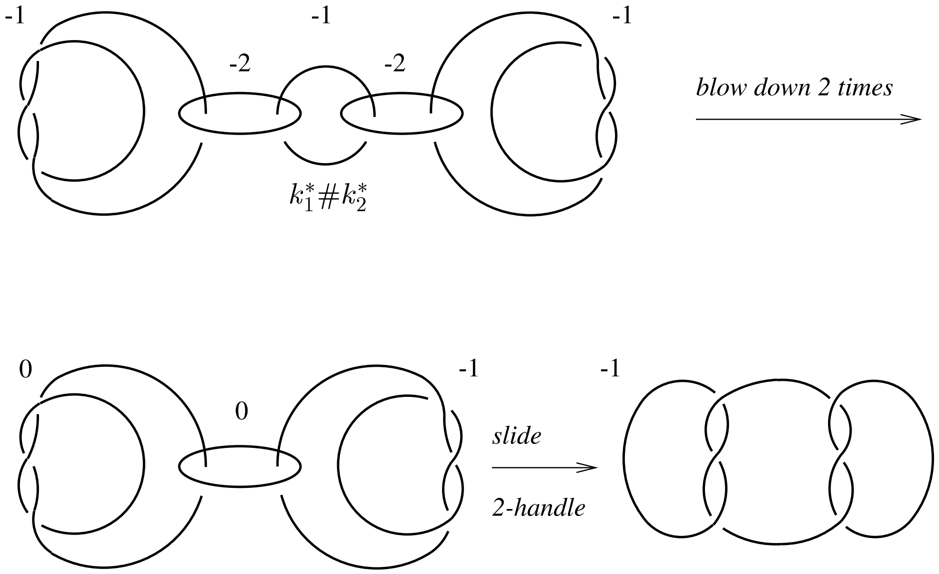

FIGURE 6

After we compute the Floer indices in the next section, the Floer chain complex of Σ will be obtained from this by perturbing the circles.

Proof. We begin with finding a presentation for the fundamental group of the homology sphere Σ. It can be easily read off any of the descriptions of Σ given above. For example, in description 4 we start with the exterior in $\Sigma(2,3,7)$ of the singular fiber of order 7, whose fundamental group has presentation

$$\langle\, x_1, x_2, h \mid x_1^2 = h^{-1},\ x_2^3 = h\,\rangle$$

with the canonical meridian $(x_1x_2)^7h$ and longitude $(x_1x_2)^{-6}h^{-1}$, see [4], Lemma 7.5. Then

$$\begin{aligned}\pi_1\Sigma = \langle\, & x_1, x_2, \xi_1, \xi_2, h, \gamma \mid x_1^2 = h^{-1},\ x_2^3 = h,\ \xi_1^2 = \gamma^{-1},\ \xi_2^3 = \gamma, \\ & (x_1x_2)^7h = (\xi_1\xi_2)^{-6}\gamma^{-1},\ (x_1x_2)^{-6}h^{-1} = (\xi_1\xi_2)^7\gamma\,\rangle, \\ = \langle\, & x_1, x_2, \xi_1, \xi_2, h, \gamma \mid x_1^2 = h^{-1},\ x_2^3 = h,\ \xi_1^2 = \gamma^{-1},\ \xi_2^3 = \gamma, \\ & x_1x_2 = \xi_1\xi_2,\ (x_1x_2)^{-13} = (\xi_1\xi_2)^{-13} = \gamma h = h\gamma\,\rangle.\end{aligned}$$

In what follows, we will identify the Lie group SU(2) with the group S^3 of unit quaternions, and the subgroup U(1) of diagonal matrices with the group S^1 of unit complex numbers.

Let $\alpha : \pi_1\Sigma \to \mathrm{SU}(2)$ be an irreducible representation. At least one of the quaternions $\alpha(h)$, $\alpha(\gamma)$ has to be equal to ± 1; otherwise the representation α would be reducible as $h\gamma = \gamma h$ and h commutes with x_1, x_2 and γ commutes with ξ_1, ξ_2. In fact, the cases $\alpha(h) = 1$ and $\alpha(\gamma) = 1$ should also be excluded:

if say $\alpha(h)$ were equal to 1 we would have $\alpha(x_1) = \pm 1$ which would imply that α is reducible.

Suppose that $\alpha(\gamma) \neq \pm 1$ and, if necessary, conjugate α so that $\alpha(\gamma) \in \mathbb{C}$. Then $\alpha(\xi_1)$ and $\alpha(\xi_2)$ are complex numbers and $\alpha(h) = -1$. The unit complex numbers $\alpha(\gamma)$, $\alpha(\xi_1)$, and $\alpha(\xi_2)$ satisfy the following equations,

$$\alpha(\xi_1)^2 = \alpha(\gamma)^{-1}, \quad \alpha(\xi_2)^3 = \alpha(\gamma), \quad (\alpha(\xi_1)\alpha(\xi_2))^{13} = -\alpha(\gamma)^{-1},$$

which are equivalent to the equations

$$\alpha(\xi_1)^2 = -\alpha(\gamma)^3, \quad \alpha(\xi_2)^3 = \alpha(\gamma)^{-2}, \quad \alpha(\gamma)^7 = 1.$$

Therefore, $\alpha(\xi_1)$ and $\alpha(\xi_2)$ are uniquely determined by $\alpha(\gamma)$. The remaining relations give

$$\alpha(x_1)^2 = -1, \quad \alpha(x_2)^3 = -1, \quad (\alpha(x_1)\alpha(x_2))^7 = -1.$$

Here we used the fact that $\alpha(\gamma) = -\alpha(x_1)\alpha(x_2)$. The pairs $\alpha(x_1)$, $\alpha(x_2)$ satisfying these relations are, up to conjugacy, in a bijective correspondence with irreducible representations of $\pi_1\Sigma(2,3,7)$. According to [3], there are two such representations, with the traces $\operatorname{tr}\alpha(x_1) = 0$, $\operatorname{tr}\alpha(x_2) = -1$, and $\operatorname{tr}(\alpha(x_1)\alpha(x_2)) = 2\cos(\pi m/7)$, $m = 3, 5$. Their Floer indices are 3 and 7, and we will call them α_3 and α_7, respectively. We denote the corresponding representations of $\pi_1\Sigma$ by $\alpha_3 * \theta$ and $\alpha_7 * \theta$ (where θ is to signify the fact that the representation is reducible on the "right" component). By symmetry, we get representations $\theta * \alpha_3$ and $\theta * \alpha_7$.

Now suppose that $\alpha(h) = \alpha(\gamma) = -1$ and conjugate α, if necessary, to assure that $\alpha(x_1)\alpha(x_2)$ is a complex number. The relations on x_1 and x_2 imply

$$\alpha(x_1)^2 = -1, \quad \alpha(x_2)^3 = -1, \quad (\alpha(x_1)\alpha(x_2))^{13} = 1,$$

so that the pairs $\alpha(x_1)$, $\alpha(x_2)$ are, up to conjugation, in a bijective correspondence with the irreducible representations of $\pi_1\Sigma(2,3,13)$. There are four such representations with the traces $\operatorname{tr}\alpha(x_1) = 0$, $\operatorname{tr}\alpha(x_2) = -1$, and $\operatorname{tr}(\alpha(x_1)\alpha(x_2)) = 2\cos(\pi p_\varphi/13)$ with $p_\varphi = 4, 6, 8, 10$, see [3]. The situation with $\alpha(\xi_1)$ and $\alpha(\xi_2)$ is completely analogous. Given a representation α, we can conjugate the quaternions $\alpha(\xi_1)$ and $\alpha(\xi_2)$ by a unit complex number without changing $\alpha(x_1)$ and $\alpha(x_2)$ to get a new representation not conjugate to α. Thus we have a full circle of irreducible representations for every p_φ. We end up with four circles of representations in addition to the four isolated representations that we already had. The fact that the representations are non-degenerate follows from Lemma 5.1 of [7]. ☐

Note that the representation space of $\pi_1\Sigma$ can also be read off a pillowcase picture, see Figure 7. For an explanation to such pictures see [7]. The four dots in the interior represent the images of the circles, and the four isolated representations show up at the bottom and at the left edge.

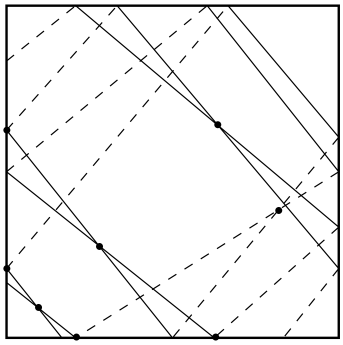

FIGURE 7

Lemma 3. *The involution σ^* induced on the representation space of Σ interchanges $\alpha_3 * \theta$ with $\theta * \alpha_3$ and $\alpha_7 * \theta$ with $\theta * \alpha_7$. It acts identically on the circles.*

Proof. The induced involution σ_* on the fundamental group can be given by the following explicit formula,

$$\begin{array}{rcl} h & \mapsto & \gamma^{-1}, \\ x_1 & \mapsto & (\xi_1\xi_2)^2\xi_1^{-1}(\xi_1\xi_2)^{-2}, \\ x_2 & \mapsto & (\xi_1\xi_2)^3\xi_2^{-1}(\xi_1\xi_2)^{-3}, \end{array}$$

and similarly on the other triple of generators. The first claim is now clear since σ_* interchanges, up to conjugacy, the two triples of generators.

Each representation α with $\alpha(h) = \alpha(\gamma) = -1$ and $\alpha(x_1x_2) = \alpha(\xi_1\xi_2) \in \mathbb{C}$ can be written in the form $(X_1, X_2; uX_1u^{-1}, uX_2u^{-1})$ for some $u \in \mathbb{C}$, where $X_1 = \alpha(x_1)$ and $X_2 = \alpha(x_2)$. The action of σ^* is then given by the formula

$$\begin{aligned} \sigma^*\alpha = (u(X_1X_2)^2X_1^{-1}(X_1X_2)^{-2}u^{-1}, u(X_1X_2)^3X_2^{-1}(X_1X_2)^{-3}u^{-1}; \\ (X_1X_2)^2X_1^{-1}(X_1X_2)^{-2}, (X_1X_2)^3X_2^{-1}(X_1X_2)^{-3}), \end{aligned}$$

which is conjugate to

$$(X_1^{-1}, X_1X_2^{-1}X_1^{-1}; u^{-1}X_1^{-1}u, u^{-1}X_1X_2^{-1}X_1^{-1}u).$$

According to Lemma 6 of [10], there exists a unit quaternion ρ such that $\rho X_1\rho^{-1} = X_1^{-1}$, $\rho X_2\rho^{-1} = X_1X_2^{-1}X_1^{-1}$, $\rho u\rho^{-1} = u^{-1}$, and $\rho^2 = -1$. Therefore, $\sigma^*\alpha$ is conjugate to

$$\begin{aligned} (X_1, X_2; \rho u^{-1}X_1^{-1}u\rho^{-1}, \rho u^{-1}X_1X_2^{-1}X_1^{-1}u\rho^{-1}) = \\ (X_1, X_2; uX_1u^{-1}, uX_2u^{-1}) = \alpha. \end{aligned}$$

□

Remark. We point out that we define Floer index by using anti-self-dual equations rather than self-dual equations used in [3] and [7]. This is the reason why the Floer indices of α_k, $k = 3, 7$, are odd. In general, our Floer indices μ_1 are related to the Floer indices μ_2 of [3] and [7] by the formula $\mu_1 + \mu_2 = 5 \mod 8$.

3. THE FLOER CHAIN COMPLEX

The paper [7] shows how to compute the Floer indices of representations of certain graph homology spheres by reducing the computation to previously known cases. It provides explicit formulas in the case when the homology sphere is of the type $X \cup Y$ where X and Y are the exteriors of a regular fiber in a Seifert fibered homology sphere. Our manifold Σ is of this type, see description 5, since the 3-sphere S^3 can be Seifert fibered so that the right-handed trefoil is a regular fiber.

Lemma 4. *The Floer indices of* $\alpha_k * \theta$ *and* $\theta * \alpha_k$ *are equal to* $k \mod 8$.

Proof. The (-1)–filling of the right-hand trefoil exterior is $\Sigma(2,3,7)$, and the Floer indices of α_k over $\Sigma(2,3,7)$ are equal to k mod 8, $k = 3, 7$. According to Theorem 4.4 of [7] the Floer index of $\alpha_k * \theta$ and that of α_k differ by

$$\operatorname{sign} B(e^{2i\psi_k}) + \dim\ker B(e^{2i\psi_k}).$$

Here $\operatorname{tr}\alpha_k(\xi_1\xi_2) = 2\cos(\psi_k)$ with ψ_k equal either $3\pi/7$ or $5\pi/7$ depending on k, and $B(t) = (1-t)V + (1-t^{-1})V^t$ where

$$V = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

is a Seifert matrix for the right-handed trefoil. Since the roots of

$$\det B(t) = (t + t^{-1} - 2)(t + t^{-1} - 1)$$

are 1 and $e^{\pm\pi i/3}$, it has zero kernel at $e^{6\pi i/7}$ and $e^{10\pi i/7}$. In fact, $\det B(t)$ is negative at these points so the signatures of both $B(e^{6\pi i/7})$ and $B(e^{10\pi i/7})$ are equal to 0. ☐

Lemma 5. *Two of the circles in the representation space of* $\pi_1\Sigma$ *have Floer index 1 and the other two Floer index 5.*

Proof. This will follows from Theorem 5.2 of [7] after we take care of orientations. Remember that we fibered S^3 so that the right handed trefoil is a regular fiber with the exterior X, and represented Σ as a union $X \cup X$. With this fibration in place, X has a canonical orientation so that

$$\pi_1 X = \langle\, x_1, x_2, h \mid x_1^2 = h^{-1},\ x_2^3 = h \,\rangle$$

where h represents a regular fiber. The canonical meridian and longitude are given as $m = x_1 x_2$ and $\ell = m^{-6}h^{-1}$ with the two copies of X glued according to the map

$$\varphi = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

in the canonical bases in $\pi_1(\partial X)$. Note that h is also a longitude although not a canonical one; it is not null-homologous in X. The pair of curves x_1x_2

and h^{-1} form an oriented basis in $\pi_1(\partial X)$; its orientation is consistent with that of m, ℓ as

$$\det \begin{pmatrix} 1 & -6 \\ 0 & 1 \end{pmatrix} = 1.$$

In the bases $x_1 x_2$, h^{-1} and $\xi_1\xi_2$, γ^{-1} in the two copies of ∂X, the gluing map can be written as

$$\varphi = \begin{pmatrix} u & v \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 13 \\ 0 & -1 \end{pmatrix}.$$

This matrix is the main piece of information needed to apply Theorem 5.2 of [7]. The other ingredients in the notations of [7] are as follows: $\varepsilon_X = \varepsilon_Y = -1$, $a = c = 6$, $b_1 = d_1 = 1$, $b_2 = d_2 = -1$, $e_X = e_Y = 5$, $p_\varphi = 4, 6, 8, 10$, $C = 2$, and $m_X = m_Y = 2$. The Floer index is now given by the formula (remember the remark at the end of the previous section)

$$\frac{2p_\varphi^2}{13} + \frac{2}{13}\sum_{k=1}^{13} \cot^2\left(\frac{\pi k}{13}\right)\sin^2\left(\frac{\pi p_\varphi k}{13}\right) + 2 \mod 8$$

It equals 1 mod 8 if $p_\varphi = 4, 8$, and 5 mod 8 if $p_\varphi = 6, 10$. ☐

The Floer chain complex is obtained from this by perturbing the circles. Given a circle S of Floer index $\mu(S)$, we consider a Morse function $f : S \to \mathbb{R}$ with two critical points β and γ of indices 0 and 1, respectively, and assign to the critical point β the index $\mu(S)$ mod 8 and to the critical point γ the index $\mu(S) - 1$ mod 8. See [3] for the proof that this gives a complex whose homology is Floer homology.

Denote β_n, γ_n the pairs of critical points obtained by perturbing the n-th circle, $n = 1, 2, 3, 4$. Then the Floer chain complex has the following generators

Floer Index	Generators
0	γ_1, γ_3
1	β_1, β_2
2	
3	$\alpha_3 * \theta$, $\theta * \alpha_3$
4	γ_2, γ_4
5	β_2, β_4
6	
7	$\alpha_7 * \theta$, $\theta * \alpha_7$

4. THE BOUNDARY OPERATORS

The Floer chain complex of Σ may *a priori* have non-trivial boundary operators. We will not compute all these operators; however, we will show that the generators $\alpha_k * \theta$ and $\theta * \alpha_k$, $k = 3, 7$, survive in $I_*(\Sigma)$. The idea is as follows. Recall from [3] that $I_3(\Sigma(2,3,7)) = I_7(\Sigma(2,3,7)) = \mathbb{Z}$ and $I_k(\Sigma(2,3,7)) = 0$ otherwise. Since $\alpha_k * \theta$ and $\theta * \alpha_k$ are the only generators in degrees 3 and 7 in the Floer chain complex of Σ, the result follows from theorem below.

Theorem 6. *There exists a degree 0 epimorphism* $I_*(\Sigma) \to I_*(\Sigma(2,3,7)) \oplus I_*(\Sigma(2,3,7))$.

Our proof of Theorem 6 consists of a construction and several lemmas, which will be proved later in this section. Let k_1 be the singular fiber of degree 7 in $\Sigma(2,3,7)$ and perform (-1)-surgery along k_1, see Figure 5. The resulting manifold is $\Sigma(2,3,13)$, and it comes together with a canonical cobordism P_1, the trace of the surgery, which is an oriented 4-manifold with boundary $-\Sigma(2,3,7) \cup \Sigma(2,3,13)$.

Lemma 7. *The homomorphism* $P_{1*} : I_*(\Sigma(2,3,7)) \to I_*(\Sigma(2,3,13))$ *induced by* P_1 *has degree 0 and is injective.*

The manifold $\Sigma(2,3,13)$ bounds a simply-connected oriented 4-manifold Q_1 with $\partial Q_1 = \Sigma(2,3,13)$ obtained from D^4 by gluing two 2-handles as shown on the left in Figure 5. Let X_1 be the boundary connected sum of $\Sigma(2,3,13) \times [0,1]$ and Q_1 along two 3-discs, one in $\Sigma(2,3,13) \times \{1\}$ and the other in ∂Q_1. A schematic picture of the cobordism $P_1 \cup X_1$ is shown in Figure 8.

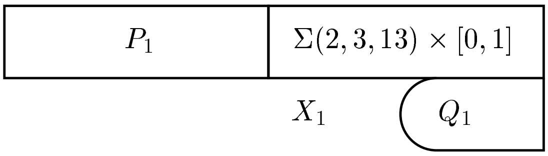

FIGURE 8

Lemma 8. *The cobordism* X_1 *induces an embedding* $X_{1*} : I_*(\Sigma(2,3,13)) \to I_*(\Sigma(2,3,13) \# \Sigma(2,3,13))$ *of degree 0.*

The above construction can be repeated with the two copies of $\Sigma(2,3,13)$ switched to obtain cobordisms P_2, Q_2, and X_2.

Lemma 9. *The images of* X_{1*} *and* X_{2*} *in* $I_*(\Sigma(2,3,13) \# \Sigma(2,3,13))$ *have zero intersection.*

A combination of Lemmas 7, 8, and 9 provides us with a zero degree embedding

$$\begin{aligned}\Phi = (P_1 \cup X_1)_* \,\oplus\, (P_2 \cup X_2)_* :\\ I_*(\Sigma(2,3,7)) \,\oplus\, I_*(\Sigma(2,3,7)) \to I_*(\Sigma(2,3,13)\#\Sigma(2,3,13)).\end{aligned}$$

Now we turn our attention to the description 6 of the manifold Σ. Let $\overline{K}$ be the manifold obtained by 0–surgery of $\Sigma(2,3,13)\#\Sigma(2,3,13)$ along the knot $k_1^*\#k_2^*$, and let Z and Y be the surgery traces from Σ to the connected sum $\Sigma(2,3,13)\#\Sigma(2,3,13)$ and from $\Sigma(2,3,13)\#\Sigma(2,3,13)$ to $\overline{K}$, respectively. Then we have the following diagram with the exact row

$$\begin{tikzcd}
& I_*(\Sigma(2,3,7)) \oplus I_*(\Sigma(2,3,7)) \arrow[d, "\Phi"] & \\
I_*(\Sigma) \arrow[r, "Z_*"] & I_*(\Sigma(2,3,13)\#\Sigma(2,3,13)) \arrow[r, "Y_*"] & I_*(\overline{K})
\end{tikzcd}$$

Lemma 10. *The image of* Φ *belongs to the kernel of* Y_*.

Now, since $\ker Y_* = \operatorname{im} Z_*$, we have that $\operatorname{im}\Phi \subset \operatorname{im} Z_*$, and Z_* maps $I_*(\Sigma)$ onto $\operatorname{im}\Phi \cong I_*(\Sigma(2,3,7)) \oplus I_*(\Sigma(2,3,7))$. This is the epimorphism whose existence was claimed in Theorem 6. Note that it is not canonical.

Proof of Lemma 7. The homomorphism P_{1*} has degree 0 mod 8 because $b_1(P_1) = b_2^+(P_1) = 0$. It can be included in the following Floer exact triangle

$$\begin{tikzcd}
& I_*(\overline{K}_1) \arrow[dl] & & I_{*+2}(\overline{K}_1) \arrow[dl, "W_*"'] & \\
0 = I_*(S^3) \arrow[rr] & & I_*(\Sigma(2,3,7)) \arrow[ul] \arrow[rr, "P_{1*}"] & & I_*(\Sigma(2,3,13)) \arrow[ul]
\end{tikzcd}$$

where $\overline{K}_1$ is obtained by 0–surgery along the right-handed trefoil in S^3. Since $I_{\mathrm{ev}}(\overline{K}_1) = I_{\mathrm{ev}}(\Sigma(2,3,7)) = 0$, the homomorphism W_*, which has degree 1, vanishes by dimension counting, and P_{1*} is a monomorphism. In fact, since $I_{\mathrm{ev}}(\Sigma(2,3,13)) = 0$, the exact triangle splits into the isomorphisms

$$\begin{aligned} I_*(\Sigma(2,3,13)) &= I_*(\Sigma(2,3,7)) \oplus I_{*+2}(\overline{K}_1)\\ &= I_*(\Sigma(2,3,7)) \oplus I_{*+2}(\Sigma(2,3,7)). \end{aligned}$$

□

Proof of Lemma 8. The idea of proof of this lemma is borrowed from [6]. The manifold Q_1 can be embedded in the manifold $\overline{\mathbb{C}P}^2\#\overline{\mathbb{C}P}^2\#\overline{\mathbb{C}P}^2$, see Figure 9, where we first add a (-1)–framed 2–handle to Q_1 and then cap the resulting manifold off with D^4.

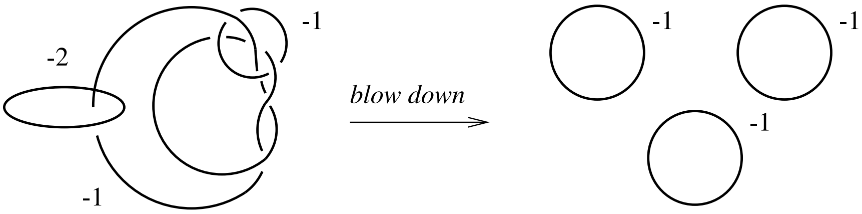

FIGURE 9

Let us denote $3\overline{\mathbb{C}P}^2 \setminus \operatorname{int} Q_1$ by V_1. This is a smooth negative definite manifold with boundary $-\Sigma(2,3,13)$. Let Y_1 be a boundary connected sum of $\Sigma(2,3,13) \times [1,2]$ and V_1 along the two 2–discs, one in $\Sigma(2,3,13) \times \{1\}$ and the other in ∂V_1. Then Y_1 is an oriented cobordism with $\partial Y_1 = -(\Sigma(2,3,13)\#\Sigma(2,3,13)) \cup \Sigma(2,3,13)$. Glue the cobordisms X_1 and Y_1 along $\Sigma(2,3,13)\#\Sigma(2,3,13)$ to get a cobordism from $\Sigma(2,3,13)$ to itself of the form

$$X_1 \cup Y_1 = (\Sigma(2,3,13) \times [0,2]) \,\#\, 3\overline{\mathbb{C}P}^2.$$

Thus $X_1 \cup Y_1$ is a triple blow up of the product cobordism $\Sigma(2,3,13) \times [0,2]$. Therefore, $Y_{1*} \circ X_{1*} = (X_1 \cup Y_1)_* : I_*(\Sigma(2,3,13)) \to I_*(\Sigma(2,3,13))$ is an isomorphism (and hence X_{1*} is injective).

A short proof of the last statement is as follows. The product cobordism $\Sigma(2,3,13) \times [0,2]$ induces the identity map in Floer homology. Blowing it up once corresponds to gluing a 2–handle to $\Sigma(2,3,13) \times [0,2]$ along an unknot in $\Sigma(2,3,13) \times \{2\}$ with framing (-1). The homomorphism induced by this cobordism is represented by the horizontal arrow in the Floer exact triangle, see [5],

$$\begin{tikzcd}
& I_*(\overline{K}_0) \arrow[dl] & \\
I_*(\Sigma(2,3,13)) \arrow[rr] & & I_*(\Sigma(2,3,13)) \arrow[ul]
\end{tikzcd}$$

where $\overline{K}_0 = \Sigma(2,3,13)\#(S^1 \times S^2)$ is obtained by 0–surgery on the unknot. By excision property of Floer homology $I_*(\overline{K}_0)$ vanishes, see [2] or [5]. Therefore, the horizontal arrow is an isomorphism. The second and third blow ups are handled similarly.

To finish the proof we note that the intersection form of X_1 is negative definite, therefore, the degree of X_{1*} is zero. $\square$

Proof of Lemma 9. Let us consider the cobordism $X_1 \cup Y_2$ pictured in Figure 10. Here, Y_2 is a boundary connected sum of the manifold $-V_2$ with $\Sigma(2,3,13) \times [1,2]$, compare with the proof of Lemma 8.

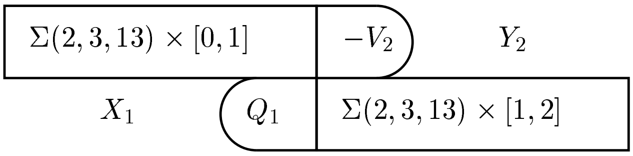

FIGURE 10

It has an obvious splitting along S^3 in the gluing region, therefore, $(X_1\cup Y_2)_*: I_*(\Sigma(2,3,13))\to I_*(\Sigma(2,3,13))$ factors through $I_*(S^3)=0$ and $Y_{2*}\circ X_{1*}=0$. Therefore, $\operatorname{im} X_{1*}\subset\ker Y_{2*}$. On the other hand, $\ker Y_{2*}\cap\operatorname{im} X_{2*}=0$ since $Y_{2*}\circ X_{2*}$ is an isomorphism. This completes the proof. ☐

Proof of Lemma 10. We need to prove that the homomorphism $(P_1\cup X_1\cup Y)_*: I_*(\Sigma(2,3,7))\to I_*(\overline{K})$ is zero. The cobordism $P_1\cup X_1\cup Y$ has a smoothly embedded 2-sphere S obtained by gluing together several 2–discs as follows.

The core knot k_1^* in $\Sigma(2,3,13)$ bounds an embedded 2–disc D_1 in the 4-manifold P_1 with homological self-intersection -1. Then $D_1\cup_{k_1^*}(k_1^*\times[0,1])$ is an embedded disc in $P_1\cup X_1$ with boundary the knot k_1^* in the first copy of $\Sigma(2,3,13)$ in $\Sigma(2,3,13)\#\Sigma(2,3,13)$. The knot k_2^* in the second copy of $\Sigma(2,3,13)$ bounds an embedded disc D_2' in Q_2. This one comes from the slice 2–disc in D^4 it bounded before the 2–handles were added. It has zero self-intersection. Finally, let D be the axial disc of the 2–handle we attached to $(\Sigma(2,3,13)\#\Sigma(2,3,13))\times[1,2]$ along $k_1^*\#k_2^*$ to get Y. The disc $((k_1^*\#k_2^*)\times[1,2])\cup D$ is embedded in Y, has self-intersection 0, and its boundary is $k_1^*\#k_2^*$. These discs and the sphere S are shown schematically in Figure 11.

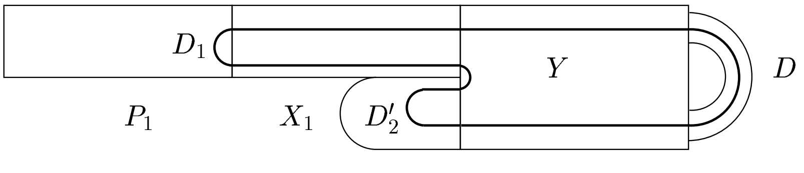

FIGURE 11

The homological self-intersection of the 2–sphere S is (-1); therefore, the cobordism $P_1\cup X_1\cup Y$ splits off a copy of $\overline{\mathbb{C}P}^2$ with a non-trivial SO(3)–bundle over it. Via gluing argument, all L^2–moduli spaces of instantons over such a cobordism are at least 1–dimensional, see [2] or [5]. Therefore, $(P_1\cup X_1\cup Y)_*=0$ and $\operatorname{im}(P_1\cup X_1)_*\subset\ker Y_*$. ☐

5. EQUIVARIANT REPRESENTATION SPACE

The equivariant representation space of $\pi_1\Sigma$ consists of four circles, see Lemma 3. These representations can be pushed down to the quotient

$S^3 = \Sigma/\sigma$ to give SU(2)–representations of $\pi_1(S^3 \setminus k_\sigma)$ where k_σ is the knot pictured in Figure 2, with the meridian represented by a trace-zero matrix. It would be interesting to see the circles directly in the representation space of $\pi_1(S^3 \setminus k_\sigma)$.

The fundamental group of $S^3 \setminus k_\sigma$ has the presentation

$$\pi_1(S^3 \setminus k_\sigma) = \langle\, x_1, x_2, h, u \mid x_1^2 h = 1,\ x_2^3 h^{-1} = 1,\ (x_1 x_2)^{13} h^2 = u^2 \,\rangle,$$

with the canonical meridian uv^{-1} and longitude $uv^{-1}u^{-1}v^{-1}$ where $v = (x_1 x_2)^6 h$. It has irreducible representations of two types. First are representations α with $\alpha(u)^2 \neq -1$. These are in a two-to-one correspondence with the representations of the right-handed trefoil group and, therefore, are two open arcs limiting at reducible representations. Second type are representations α with $\alpha(u)^2 = -1$ so that $\alpha(x_1)$, $\alpha(x_2)$, and $\alpha(h) = -1$ give an irreducible representation of $\pi_1\Sigma(2,3,13)$. There are four such representations, and each of them gives a 2–sphere worth of representations of $\pi_1(S^3 \setminus k_\sigma)$ corresponding to the solutions of the equation $\alpha(u)^2 = -1$ (which is equivalent to $\operatorname{tr} \alpha(u) = 0$). The representation space of $\pi_1(S^3 \setminus k_\sigma)$ is pictured in Figure 12.

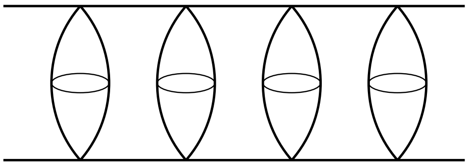

Figure 12

The image in the pillowcase consists of six arcs; the representations of the first type are mapped into two arcs wrapping four times around the pillowcase, and the four arcs corresponding to the 2–spheres lie on the top of each other in the V–pattern, see Figure 13.

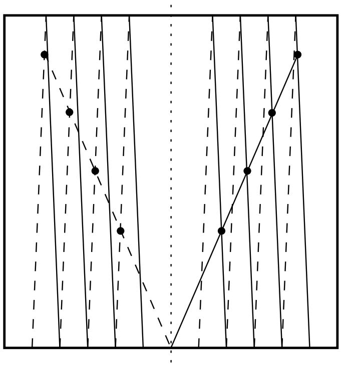

Figure 13

The representations α with $\operatorname{tr} \alpha(m) = 0$ lie on the circle shown in Figure 13 as the vertical dotted line in the middle of the figure. There are four circles of such representations, all mapped to a single point in the pillowcase.

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