Max-Planck-Institut für Mathematik
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by

Alexander Fel'shtyn
Nikita Luchnikov
Evgenij Troitsky

Max-Planck-Institut für Mathematik
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Alexander Fel'shtyn
Nikita Luchnikov
Evgenij Troitsky

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Instytut Matematyki
Uniwersytet Szczecinski
ul. Wielkopolska 15
70-451 Szczecin
Poland

Department of Mechanics and Mathematics
Moscow State University
119991 GSP-1 Moscow
Russia

MPIM 14-27

REIDEMEISTER CLASSES AND TWISTED INNER REPRESENTATIONS

ALEXANDER FEL'SHTYN, NIKITA LUCHNIKOV, AND EVGENIJ TROITSKY

Abstract. As it is known from the previous research, the study of the structure and counting of Reidemeister classes (twisted conjugacy classes) of an automorphism $\phi : G \to G$, i.e. classes $x \sim gx\phi(g^{-1})$, is closely related to the study the twisted inner representation of a discrete group G, i.e. a representation on $\ell^2(G)$ corresponding to the action $g \mapsto xg\phi(x^{-1})$ $(x, g \in G)$ of G on itself. In the present paper we study twisted inner representations from a more general point of view, but the questions under consideration are still close to the important relations to Reidemeister classes.

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1. Introduction

Let $\varphi : G \to G$ be an automorphism of of a (discrete) group G. The *Reidemeister number* $R(\phi)$ is the number of *Reidemeister classes*, i.e. the equivalence classes of the following relation

$$g \sim xg\phi(x^{-1}), \qquad g, x \in G.$$

This field was extensively developed recently [7, 10, 11, 9, 30, 13, 23, 8, 12, 16, 28, 17] and obtained numerous interesting applications not only in Topology, Dynamics and Group Theory, but also in the Non-commutative geometry [4] and even in the public key cryptography [25, 26].

The *inner representation* γ_G of a group G (we consider in the present paper only discrete groups and only unitary representations) was a subject of an intensive study in many papers. It is defined as

$$[\gamma_G(x)](f)(g) = f(xgx^{-1}), \quad f \in \ell^2(G),\ x, g \in G.$$

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The most developed directions are related to spectral comparison with regular representation (see [20, 21, 22, 19, 15]) and to the study of inner amenability, i.e. the property $1_G \prec \gamma_G$ (see [1, 5, 18, 29]).

The present paper is aimed to start the study and advertise the field related to a modification of the notion of the inner representation, namely, the *twisted inner representation* γ_G^ϕ defined by

$$\gamma_G^\phi(x)(f)(g) = f(xg\phi(x^{-1})), \quad x, g \in G, \quad f \in \ell^2(G).$$

This notion and its elementary properties were used in [14] for the counting of Reidemeister classes of an automorphism ϕ (considered as orbits of the corresponding twisted action of G on itself).

In the present paper we study twisted inner representations from a more general point of view, but the questions under consideration are still close to the important relations to Reidemeister numbers discussed in the above cited papers.

After presenting preliminary results and giving definitions and known results in Sect. 2, we discuss in Sect. 3 some weak containments of representations under consideration. In particular, we prove under supposition of finiteness of stabilizers of ϕ-twisted action, that γ_G^ϕ is weakly contained in the regular representation λ_G. In Sect. 4 we obtain a more strong version of this statement: $\gamma_G^\phi \prec \lambda_G$ if and only if the mentioned stabilizer $C_\phi(a)$ is amenable for all $a \in G$. In Sect. 5 it is proved that $\lambda_G \prec \gamma_G^\phi$ for any ICC group G. In Sect. 6 we consider an automorphism ϕ of a finitely generated residually finite group G with finite Reidemeister number. Then G is ϕ-inner amenable in an appropriate sense if and only if it is amenable. This differs from the case of inner amenability (i.e. Id-inner amenability).

The results of Sect. 4 and 5 are obtained by N.L., of Sect. 6 — by E.T. and of Sect. 3 — by A.F. and E.T. jointly.

2. Preliminaries

Denote the stabilizers related to the twisted action of G on itself by

$$St_\phi^{tw}(g,h) := \{k \in G \mid kg\phi(k^{-1}) = h\}, \qquad St_\phi^{tw}(g) := St_\phi^{tw}(g,g).$$

In particular,

$$St_\phi^{tw}(e) = \{k \in G \mid k\phi(k^{-1}) = e\} = C_G(\phi)$$

(fixed elements of ϕ). Evidently, $St_\phi^{tw}(g,h) = \varnothing$ if $h \notin \{g\}_\phi$. Otherwise

$$St_\phi^{tw}(g, sg\phi(s^{-1})) = \{k \in G \mid kg\phi(k^{-1}) = sg\phi(s^{-1})\} = \{k \in G \mid s^{-1}kg\phi(k^{-1}s) = g\},$$

i.e. $St_\phi^{tw}(g, sg\phi(s^{-1})) = s \cdot St_\phi^{tw}(g)$ is a coset of this group. Thus

$$|St_\phi^{tw}(g, sg\phi(s^{-1}))| = |St_\phi^{tw}(g)|. \tag{1}$$

Definition 2.1. A group G is called *residually finite* if for any finite set $K \subset G$, $e \notin K$, there exists a normal group H of finite index such that $H \cap K = \varnothing$. Taking K formed by $g_i^{-1}g_j$ for some finite set $K_0 = \{g_1, \dots, g_s\}$ one obtains an epimorphism $G \to G/H$ onto a finite group, which is injective on K_0.

We will remind now some facts from Representation Theory and Harmonic Analysis (see [3] and [2] for an effective introduction). (Left) *regular representation* λ_G is the unitary representation of G on $\ell^2(G)$ by left translations. The completion $C^*_\lambda(G)$ of $\ell^1(G)$ by the norm of $B(\ell^2(G)$ is called *reduced group C^*-algebra* of G. The completion $C^*(G)$ of $\ell^1(G)$ by the norm of all unitary representations is called *(full) group C^*-algebra* of G. The algebra $C^*_\lambda(G)$ is a quotient of $C^*(G)$.

Non-degenerate representations of $C^*(G)$ are exactly unitary representations of G, in particular, $\widehat{C^*(G)} = \widehat{G}$. For a representation ρ of G we denote by $C^*\rho$ the corresponding representation of $C^*(G)$, and by $C^* \operatorname{Ker} \rho$ the kernel of $C^*\rho$. One introduces on $\widehat{G}$ the *Jacobson-Fell* or *hull-kernel* topology defining the closure of a set X by the following formula

$$\overline{X} = \{[\rho] \,:\, C^*\mathrm{Ker}\rho \supseteq \bigcup_{[\pi]\in X} C^*\mathrm{Ker}\pi\}.$$

This topology can be described in terms of *weak containment*: a representation ρ is weakly contained in representation π (we write $\rho \prec \pi$) if diagonal matrix coefficients of ρ can be approximated by linear combinations of diagonal matrix coefficients of π uniformly on finite sets. Here a *matrix coefficient* of a representation ρ on a Hilbert space H is the function $g \mapsto \langle \rho(g)\xi, \eta\rangle$ on G for some fixed $\xi, \eta \in H$, and a *diagonal* one corresponds to $\xi = \eta$. Then $C^* \operatorname{Ker} \pi \subset C^* \operatorname{Ker} \rho$ if and only if $\rho \prec \pi$. Since

$$C^* \operatorname{Ker}(\rho_1 \oplus \cdots \oplus \rho_m) = \cap_{i=1}^m C^* \operatorname{Ker} \rho_i,$$

$$\rho_1 \oplus \cdots \oplus \rho_m \prec \pi \text{ if } \rho_i \prec \pi, \quad i = 1, \dots, m. \tag{2}$$

Lemma 2.2 (e.g. [2, Lemma F.1.3]). *Let $(\pi, \mathcal{H})$ and $(\rho, \mathcal{K})$ be unitary representations of a topological group G. Let V be a subset of $\mathcal{H}$ such that $\{\pi(x)\chi \,|\, x \in G, \chi \in V\}$ is total in $\mathcal{H}$. The following are equivalent:*

- *$\pi \prec \rho$;*
- *every function of positive type of the form $\langle \pi(.)\chi, \chi\rangle$ with $\chi \in V$ can be approximated, uniformly on compact subsets of G, by finite sums of functions of positive type associated with ρ.*

A particular case is:

Lemma 2.3. *Suppose χ is a cyclic vector for π. Then π is weakly contained in ρ if and only if the function $g \mapsto \langle \pi(g)\chi, \chi\rangle$ can be approximated, uniformly on compact subsets of G, by finite sums of functions of positive type associated with ρ.*

An *amenable group* may be characterized in several equivalent ways (see e.g. [2]), in particular:

- There exists an invariant mean on $\ell^\infty(G)$.
- $1_G \prec \lambda_G$, where 1_G is the trivial 1-dimensional representation.
- $C^*(G) = C^*_\lambda(G)$.

We will need the notion of *induced representation* for a representation σ of a subgroup K of a discrete group G on a Hilbert space $\mathcal{H}_\sigma$. Consider the vector space V formed by functions $\chi : G \to \mathcal{H}_\sigma$ such that

(1) the support of χ is contained in a finite union of left cosets of G by K;
(2) $\chi(xk) = \sigma(k^{-1})\,\chi(x)$ for any $x \in G$ and $k \in K$.

Then the induced representation $\operatorname{ind}_K^G \sigma$ of G on V is defined by

$$\left(\operatorname{ind}_K^G \sigma(g)\chi\right)(x) = \chi(g^{-1}x).$$

More detail on the notion can be found e.g. in [2].

Example 2.4. If $K = \{e\}$ and $\sigma = 1_K$ is the trivial 1-dimensional representation, then $\operatorname{ind}_K^G \sigma$ is the left regular representations λ_G.

Lemma 2.5 (see, e.g. [2]). *Let $K \subset H \subset G$ be subgroups.*

(i) *Suppose, σ is a representation of K. Then the representations* $\mathrm{ind}_H^G \left(\mathrm{ind}_K^H \sigma\right)$ *and* $\mathrm{ind}_K^G \sigma$ *are equivalent.*
(ii) *Let $\pi \prec \rho$ be representations of G. Then $\pi|_H \prec \rho|_H$.*
(iii) *Let $\sigma \prec \tau$ be representations of H. Then $\mathrm{ind}_H^G \sigma \prec \mathrm{ind}_H^G \tau$.*
(iv) *Let σ be a representation of H. Then $\sigma \prec \left(\mathrm{ind}_H^G \sigma\right)|_H$.*
(v) $\lambda_G|_H \prec \lambda_H$.

We will need the following class of groups:

Definition 2.6. A discrete group G is called an *ICC-group* (infinite conjugacy classes) if each its conjugacy class (except of $\{e\}$) is infinite.

A group is ICC if and only if its regular representation is factorial [24].
Also we will need the following well known statement by B.Neumann [27]:

Lemma 2.7. *Suppose, G has a finite coset cover*

$$G = \cup_{i=1}^n g_i\, S_i,$$

where $S_1, \dots, S_n$ are some subgroups of G, not necessarily distinct. Then at least one of these subgroups has finite index in G.

3. Twisted inner representation

Our argument here partially follows [20].

Definition 3.1. Denote by γ_G^ϕ the *twisted inner representation* of G on $\ell^2(G)$, i.e.

$$\gamma_G^\phi(x)(f)(g) = f(xg\phi(x^{-1})), \quad x, g \in G, \quad f \in \ell^2(G).$$

Denote $C_\phi(a) := St_\phi^{tw}(a)$, $a \in G$. Evidently, γ_G^ϕ decomposes into a direct sum of representations γ_a^ϕ being restrictions of γ_G^ϕ onto $\{a\}_\phi$ (i.e. on $\ell^2(\{a\}_\phi)$).

Lemma 3.2. *The representation γ_a^ϕ is equivalent to the induced representation* $\mathrm{ind}_{C_\phi(a)}^G 1_{C_\phi(a)}$.

Proof. Indeed, this induced representation T can be realized on $\ell^2(C_\phi(a)\backslash G)$ by the following action $[T(g)(f)](x) = f(xg)$, $x \in C_\phi(a) \backslash G$, $g \in G$, where $C_\phi(a) \backslash G$ is the space of left cosets by $C_\phi(a)$. Let us identify $C_\phi(a) \setminus G$ with $\{a\}_\phi$ by $i(C_\phi(a) \cdot g) = \gamma_G^\phi(g)(a)$. Evidently, this map is well defined and gives a unitary isomorphism

$$I : \ell^2(\{a\}_\phi) \to \ell^2(C_\phi(a) \setminus G), \quad I(f)(x) := f(i(x)).$$

Then

$$[I \circ \gamma_G^\phi(g)(f)](x) = [\gamma_G^\phi(g)(f)](i(x)) = f(gha\phi((gh)^{-1})), \quad x = C_\phi(a) \cdot h,$$
$$[T(g) \circ I(f)](x) = I(f)(xg) = f(i(xg)) = f(\gamma_G^\phi(hg)(a)) = f(gha\phi((gh)^{-1})).$$

Thus, I is an intertwining unitary. $\square$

Theorem 3.3. *Suppose, $|C_\phi(a)| < \infty$, for any $a \in G$. Then γ_G^ϕ is weakly contained in the regular representation λ_G.*

Proof. The characteristic functions $\chi_{C_\phi(a)}$, $a \in G$, are positively definite functions associated to λ_G, because they are finite sums of translations of δ_e. Hence, $\mathrm{ind}_{C_\phi(a)}^G 1_{C_\phi(a)} \prec \lambda_G$ (cf. [2, E.4.4]). By Lemma 3.2 and the decomposition of γ_G^ϕ we obtain $\gamma_G^\phi \prec \lambda_G$. $\square$

This situation naturally arises in the context of the counting of Reidemeister numbers, i.e. the number of Reidemeister classes, i.e. classes $x \sim gx\phi(g^{-1})$. Namely, in [14] the following statement is proved:

Proposition 3.4. *Suppose, G is a finitely generated residually finite group and ϕ is its automorphism with $R(\phi) < \infty$. Then $|C_\phi(a)|$ is uniformly bounded.*

Sketch of the proof. First of all, one can prove (by [17]) the following estimation for an automorphism of a finite group G:

$$|C_G(\phi)| \leqslant R(\phi)^{R(\phi)-1} < 2^{R(\phi)^2}.$$

Hence, for any automorphism ψ of a finite group G

$$\sqrt{\log_2 |C_G(\psi)|} \leqslant R_G(\psi). \tag{3}$$

Let $\{x_1, x_2, \dots\} = C_\Gamma(\phi)$. Then for every n we can find a characteristic subgroup Γ_n of finite index in Γ such that the quotient map $p_n : \Gamma \to \Gamma/\Gamma_n =: G_n$ is injective on $\{x_1, \dots, x_n\}$. Let $\phi_n : G_n \to G_n$ be the induced automorphism. Then $\{p_n(x_1), \dots, p_n(x_n)\} \subset C_{G_n}(\phi_n)$, hence (3) implies

$$R_\Gamma(\phi) \geqslant R_{G_n}(\phi_n) \geqslant \sqrt{\log_2 |C_{G_n}(\phi_n)|} \geqslant \sqrt{\log_2 n}.$$

Since n was arbitrary, we are done with $C_G(\phi)$. As it is explained above, $C_\phi(a) = C_G(\tau_a \circ \phi)$, where τ_a is the inner automorphism defined by a and ϕ and $\tau_a \circ \phi$ have the same Reidemeister numbers. ☐

4. Amenability and weak containment

Now we pass to a generalization of the above result on weak containment.

Theorem 4.1. *For any discrete group G the following properties are equivalent*

(i) $\gamma_G^\phi \prec \lambda_G$;
(ii) $C_\phi(a)$ *is amenable for all* $a \in G$.

Proof. (i)⇒(ii): From Lemma 3.2 we obtain

$$\operatorname{ind}_{C_\phi(a)}^G 1_{C_\phi(a)} \prec \lambda_G.$$

By Lemma 2.5

$$1_{C_\phi(a)} \prec (\operatorname{ind}_{C_\phi(a)}^G 1_{C_\phi(a)})|_{C_\phi(a)} \prec \lambda_G|_{C_\phi(a)} \prec \lambda_{C_\phi(a)}.$$

Thus, $C_\phi(a)$ is amenable.

(ii)⇒(i): One has

$$1_{C_\phi(a)} \prec \lambda_{C_\phi(a)}.$$

By Lemma 2.5 and Example 2.4

$$\operatorname{ind}_{C_\phi(a)}^G 1_{C_\phi(a)} \prec \operatorname{ind}_{C_\phi(a)}^G \left(\operatorname{ind}_{\{e\}}^{C_\phi(a)} 1_{\{e\}} \right).$$

Thus, by Lemmas 2.5 and 3.2 one has

$$\gamma_a^\phi \cong \operatorname{ind}_{C_\phi(a)}^G 1_{C_\phi(a)} \prec \operatorname{ind}_{C_\phi(a)}^G \left(\operatorname{ind}_{\{e\}}^{C_\phi(a)} 1_{\{e\}} \right) \cong \operatorname{ind}_{\{e\}}^G 1_{\{e\}} \cong \lambda_G.$$

Finally, $\sum \gamma_a^\phi = \gamma_G^\phi$ implies $\gamma_G^\phi \prec \lambda_G$. ☐

The place of the following result is more understandable, if we consider the center as the intersection of stabilizers of the inner action.

Theorem 4.2. *Suppose, $\lambda_G \prec \gamma_G^\phi$. Then the center $Z(G)$ has a trivial intersection with the fixed point subgroup $C_G(\phi)$, i.e. $Z(G) \cap C_G(\phi) = \{e\}$.*

Proof. Suppose, there exists $h \in G$, $h \neq e$, $h \in Z(G) \cap C_G(\phi)$. We will show that on the finite set $F := \{e, h\}$ the positive definite function χ_e associated with λ_G can not be approximated by sums of positive definite functions associated with γ_G^ϕ. Indeed, suppose $f \in \ell^2(G)$ and ψ is a positive definite function associated with γ_G^ϕ and f. Since $h \in Z(G) \cap C_G(\phi)$, one has

$$\begin{aligned} \psi(h) &= \langle \gamma_G^\phi(h) f, f \rangle = \sum_{x \in G} f(hx\phi(h^{-1}))\overline{f(x)} \\ &= \sum_{x \in G} f(x)\overline{f(x)} = \langle f, f \rangle = \langle \gamma_G^\phi(e) f, f \rangle = \psi(e). \end{aligned}$$

Thus, all positive definite functions associated with γ_G^ϕ, have the same values at e and h. Thus, χ_e can not be approximated on F. Hence, $\lambda_G \not\prec \gamma_G^\phi$. A contradiction. $\square$

5. The case of ICC groups

ICC groups play an important role both in the theory of inner representations and in the theory of Reidemeister classes. For them we can obtain the "inverse" weak containment.

Theorem 5.1. *Let G be an ICC group. Then $\lambda_G \prec \gamma_G^\phi$.*

Proof. The delta function of unity δ_e is a cyclic vector for λ_G while χ_e is a positive definite function associated with λ_G and δ_e. By Lemma 2.3 it is sufficient to show that for any finite subset $F \subset G$ there exists an element $a \in G$ such that

$$\chi_{C_\phi(a)}|_F = \chi_e|_F.$$

Equivalently, for any finite subset $F \subset G$, such that $e \notin F$, there exists an element $a \in G$ such that

$$C_\phi(a) \cap F = \varnothing.$$

Suppose the opposite: a finite set $F = \{f_1, \dots, f_n\}$ ($f_i \neq e$, $i = 1, \dots, n$) has a non-empty intersection with each $C_\phi(a)$. Thus, for any $a \in G$ there exists $i(a)$ such that

$$f_{i(a)} a \phi(f_{i(a)}^{-1}) = a,$$

or

$$a^{-1} f_{i(a)} a = \phi(f_{i(a)}).$$

Thus, any element of G belongs to one of relative stabilizers of the inner action of G on itself:

$$a \in St(f_i, \phi(f_i)) := \{g \in G \mid g^{-1} f_i g = \phi(f_i)\} = g_i St(f_i, f_i),$$

where g_i is an element of G, and $St(f_i, f_i)$ is a subgroup (see identities before formula (1) in the case $\phi = \mathrm{Id}$). Hence,

$$G = \cup_{i=1}^n g_i St(f_i).$$

By Lemma 2.7 one of $St(f_i, f_i)$ has finite index in G. Thus, the orbit of this f_i under the inner action is finite. I.e. its conjugacy class is finite, while $f_i \neq e$. A contradiction with ICC. $\square$

6. ϕ-inner amenability

We say that G is *ϕ-inner amenable,* if $1_G \prec \gamma_G^\phi$.

The relation of this notion to the usual amenability differs drastically from that of inner amenability, as the following statement shows.

Theorem 6.1. *Let G be a finitely generated residually finite group. Suppose, $R(\phi) < \infty$ for some automorphism $\phi : G \to G$. Then G is amenable if and only if it is ϕ-inner amenable.*

Proof. Suppose, G is ϕ-inner amenable. Thus, by Theorem 3.3 and Proposition 3.4

$$1_G \prec \gamma_G^\phi \prec \lambda_G.$$

For the converse we need a slightly more subtle argument. First of all, by Lemma 3.2

$$\gamma_a^\phi \cong \operatorname{ind}_{C_\phi(a)}^G 1_{C_\phi(a)} \cong \lambda_{G/C_\phi(a)},$$

where the last representation is the quasi-regular one (action on cosets and respectively, on $\ell^2(G/C_\phi(a))$. The last isomorphism is well known (cf. e.g. [2]) and in fact is a part of the proof of Lemma 3.2.

By [6, 1^o, b), p.16], since $C_\phi(a)$ is finite, in particular, amenable, the amenability of G implies amenability of $G/C_\phi(a)$, i.e. $1_G \prec \lambda_{G/C_\phi(a)}$. Since γ_G^ϕ is a direct sum of a finite number of such representations, one has $1_G \prec \gamma_G^\phi$. □

Remark 6.2. In fact, we use only amenability of $C_\phi(a)$, not just the finiteness. Thus, one can obtain from the results of Sections 4 and 5 some generalizations of the above theorem.

References

[1] Erik Bédos and Pierre de la Harpe, *Moyennabilité intérieure des groupes: définitions et exemples*, Enseign. Math. (2) **32** (1986), no. 1-2, 139–157. MR 850556 (87k:43001)

[2] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan's property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008.

[3] J. Dixmier, *C*-algebras*, North-Holland, Amsterdam, 1982.

[4] Siegfried Echterhoff, Heath Emerson, and Hyun Jeong Kim, *A Lefschetz fixed-point formula for certain orbifold C^*-al gebras*, J. Noncommut. Geom. **4** (2010), no. 1, 125–155. MR 2575391 (2011a:19004)

[5] Edward G. Effros, *Property Γ and inner amenability*, Proc. Amer. Math. Soc. **47** (1975), 483–486. MR 0355626 (50 #8100)

[6] Pierre Eymard, *Moyennes invariantes et représentations unitaires*, Lecture Notes in Mathematics, Vol. 300, Springer-Verlag, Berlin, 1972. MR 0447969 (56 #6279)

[7] A. Fel'shtyn, *Dynamical zeta functions, Nielsen theory and Reidemeister torsion*, Mem. Amer. Math. Soc. **147** (2000), no. 699, xii+146. MR 2001a:37031

[8] A. Fel'shtyn and R. Hill, *The Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion*, K-Theory **8** (1994), no. 4, 367–393. MR 95h:57025

[9] A. Fel'shtyn, Yu. Leonov, and E. Troitsky, *Twisted conjugacy classes in saturated weakly branch groups*, Geometriae Dedicata **134** (2008), 61–73.

[10] A. Fel′shtyn and E. Troitsky, *Twisted Burnside-Frobenius theory for discrete groups*, J. Reine Angew. Math. **613** (2007), 193–210. MR MR2377135

[11] A. Fel'shtyn, E. Troitsky, and A. Vershik, *Twisted Burnside theorem for type II_1 groups: an example*, Math. Res. Lett. **13** (2006), no. 5, 719–728.

[12] Alexander Fel'shtyn, *New directions in Nielsen-Reidemeister theory*, Topology Appl. **157** (2010), no. 10-11, 1724–1735. MR 2639839 (2011e:55004)

[13] Alexander Fel′shtyn and Evgenij Troitsky, *Geometry of Reidemeister classes and twisted Burnside theorem*, J. K-Theory **2** (2008), no. 3, 463–506. MR MR2465448

[14] Alexander Fel'shtyn and Evgenij Troitsky, *Twisted conjugacy classes in residually finite groups*, Arxiv e-print 1204.3175, 2012.

[15] Edward Formanek, *The conjugation representation and fusionless extensions*, Proc. Amer. Math. Soc. **30** (1971), 73–74. MR 0291312 (45 #406)

[16] Daciberg Gonçalves and Peter Wong, *Twisted conjugacy classes in wreath products*, Internat. J. Algebra Comput. **16** (2006), no. 5, 875–886. MR MR2274719 (2007m:20051)

[17] Enrico Jabara, *Automorphisms with finite Reidemeister number in residually finite groups*, J. Algebra **320** (2008), no. 10, 3671–3679. MR MR2457715

[18] Paul Jolissaint, *Moyennabilité intérieure du groupe F de Thompson*, C. R. Acad. Sci. Paris Sér. I Math. **325** (1997), no. 1, 61–64. MR 1461398 (98j:20049)

[19] Eberhard Kaniuth, *On the conjugation representation of a locally compact group*, Math. Z. **202** (1989), no. 2, 275–288. MR 1013090 (90k:22003)

[20] Eberhard Kaniuth and Annette Markfort, *The conjugation representation and inner amenability of discrete groups.*, J. Reine Angew. Math. **432** (1992), 23–37.

[21] ———, *Locally compact groups whose conjugation representations satisfy a Kazhdan type property or have countable support*, Math. Proc. Cambridge Philos. Soc. **116** (1994), no. 1, 79–97. MR 1274160 (95d:22006)

[22] ———, *On C^*-algebras associated to the conjugation representation of a locally compact group*, Trans. Amer. Math. Soc. **347** (1995), no. 7, 2595–2606. MR 1277118 (95i:22009)

[23] G. Levitt and M. Lustig, *Most automorphisms of a hyperbolic group have very simple dynamics.*, Ann. Scient. Éc. Norm. Sup. **33** (2000), 507–517.

[24] F. J. Murray and J. von Neumann, *On rings of operators. IV*, Ann. of Math. (2) **44** (1943), 716–808. MR 0009096 (5,101a)

[25] Alexei Myasnikov, Vladimir Shpilrain, and Alexander Ushakov, *Group-based cryptography*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2008. MR 2437984 (2009d:94098)

[26] ———, *Non-commutative cryptography and complexity of group-theoretic problems*, Mathematical Surveys and Monographs, vol. 177, American Mathematical Society, Providence, RI, 2011, With an appendix by Natalia Mosina. MR 2850384 (2012i:68001)

[27] B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. **29** (1954), 236–248. MR MR0062122 (15,931b)

[28] V. Roman'kov, *Twisted conjugacy classes in nilpotent groups*, J. Pure Appl. Algebra **215** (2011), no. 4, 664–671. MR 2738380 (2011k:20065)

[29] Yves Stalder, *Moyennabilité intérieure et extensions HNN*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 2, 309–323. MR 2226017 (2007h:20017)

[30] E. Troitsky, *Noncommutative Riesz theorem and weak Burnside type theorem on twisted conjugacy*, Funct. Anal. Pril. **40** (2006), no. 2, 44–54, In Russian, English translation: *Funct. Anal. Appl.* **40** (2006), No. 2, 117–125.

Instytut Matematyki, Uniwersytet Szczecinski, ul. Wielkopolska 15, 70-451 Szczecin, Poland

E-mail address: felshtyn@mpim-bonn.mpg.de, fels@wmf.univ.szczecin.pl

Dept. of Mech. and Math., Moscow State University, 119991 GSP-1 Moscow, Russia

E-mail address: nikita_luchnikov@mail.ru

Dept. of Mech. and Math., Moscow State University, 119991 GSP-1 Moscow, Russia

E-mail address: troitsky@mech.math.msu.su

URL: http://mech.math.msu.su/~troitsky