ON LINEAR FORMS WITH COEFFICIENTS IN $\mathbb{N}\zeta(1+\mathbb{N})$ The detailed version, Part 7

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In memory of Professor Alexandr Buchstab

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§0. Foreword.

Here I complete the proof of the following theorem.

Theorem *Let* $\gamma = 43.464412$,

$$\phi_i(x_1, x_2) = \sum_{k=1}^{2}(2ik - i - k + 2)\,\zeta(i + k + 1)\,x_k, \qquad i = 1, 2.$$

For any $d \in \mathbb{R}$, *let* $\|d\|$ *denote the distance between* d *and* $\mathbb{Z}$. *There exists a positive constant* c *such that*

$$\|\phi_1(x_1, x_2)\| + \|\phi_2(x_1, x_2)\| \geq c(|x_1| + |x_2|)^{-\gamma},$$

as soon as $x_1 \in \mathbb{Z},\ x_2 \in \mathbb{Z}$, *and* $|x_1| + |x_2| > 0$.

Corollary. *If* $p \in \mathbb{Q},\ q \in \mathbb{Q}, p^2 + q^2 > 0$, *then*

$$\{2p\zeta(3) + 3q\zeta(4), 3p\zeta(4) + 6q\zeta(5)\} \not\subset \mathbb{Q}$$

and therefore

$$\{\zeta(3 + 2k)/\zeta(4), (12\zeta(3)\zeta(5) - 9(\zeta(4))^2)/\zeta(4)\} \not\subset \mathbb{Q},$$

where $k = 0, 1$.

Acknowledgments. I express my deepest thanks to Professors
B.Z.Moroz,
I.I. Piatetski-Shapiro,
A.G.Aleksandrov,
P.Bundshuh
and S.G.Gindikin
for help and support.

§1. Proof of the Theorem.

Let

(1) $$\alpha^*_{\nu,k}=(-1)^{\nu+\nu\Delta+k}\binom{\nu\Delta}{k}^3\binom{\nu\Delta+k}{\nu\Delta-\nu}^3,$$

(2) $$\beta^*_{\nu,k}=\alpha^*_{\nu,k}\left(-\left(\sum_{\kappa=\nu+k+1}^{\nu\Delta+k}1/\kappa\right)-\left(\sum_{\kappa=1}^{\nu\Delta-k}1/\kappa\right)+\left(\sum_{\kappa=1}^{k}1/\kappa\right)\right),$$

(3)
$$2\gamma^*_{\nu,k}=9\alpha^*_{\nu,k}\left(-\left(\sum_{\kappa=\nu+k+1}^{\nu\Delta+k}1/\kappa^2\right)-\left(\sum_{\kappa=1}^{\nu\Delta-k}1/\kappa^2\right)+\left(\sum_{\kappa=1}^{k}1/\kappa\right)\right)^2-$$
$$3\alpha^*_{\nu,k}\left(\left(\sum_{\kappa=\nu+k+1}^{\nu\Delta+k}1/\kappa^2\right)-\left(\sum_{\kappa=1}^{\nu\Delta-k}1/\kappa^2\right)-\left(\sum_{\kappa=1}^{k}1/\kappa^2\right)\right),$$

where $k=0,\dots,\nu\Delta$.

(4) $$\alpha^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k=f_1^*(z;\nu),$$

(5) $$\beta^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k$$

(6) $$\gamma^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k,$$

(7) $$\phi^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-3}+$$
$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-2}+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-1}=$$

$$(-z)^\nu\sum_{t=1}^{\nu} z^{-t}\big(t^{-3}\alpha^*(z;\nu)+t^{-2}\beta^*(z;\nu)+t^{-1}\gamma^*(z;\nu)\big)+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\alpha^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-3}+(-z)^\nu\sum_{k=0}^{\nu\Delta}\beta^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-2}+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-1};$$

(8) $$\psi^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}3\alpha^*_{\nu,k}\,z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-4}\ +$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}2\beta^*_{\nu,k}\,z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-3}+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}\,z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-2}=$$

$$=(-z)^\nu\sum_{t=1}^{\nu} z^{-t}\big(t^{-4}3\alpha^*(z;\nu)+t^{-3}2\beta^*(z;\nu)+t^{-2}\gamma^*(z;\nu)\big)+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}3\alpha^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-4}+(-z)^\nu\sum_{k=0}^{\nu\Delta}2\beta^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-3}+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-2};$$

(9) $$\xi^*(z;\nu)=(-z)^\nu\sum_{k=0}^{\nu\Delta}6\alpha^*_{\nu,k}z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-5}\ +$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}3\beta^*_{\nu,k}z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-4}+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1}^{k+\nu}z^{-t}t^{-3}=$$

$$=(-z)^\nu\sum_{t=1}^{\nu} z^{-t}\big(6t^{-5}\alpha^*(z;\nu)+3t^{-4}\beta^*(z;\nu)+t^{-3}\gamma^*(z;\nu)\big)+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}6\alpha^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-5}+(-z)^\nu\sum_{k=0}^{\nu\Delta}3\beta^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-4}+$$

$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}\gamma^*_{\nu,k}z^k\sum_{t=1+\nu}^{k+\nu}z^{-t}t^{-3}.$$

For a prime $p\in\mathbb{N}$, let v_p stand for the p-adic valuation on $\mathbb{Q}$, and $\pi(x)$ denotes the number of prime $p\in\mathbb{N}\cap(-\infty,\,x]$, where $x\in\mathbb{R}$. So, $\pi(x)=0$ for $x<2$. Let $\{\alpha,\,\beta,\,\gamma\}\subset\mathbb{R}$. If $\pi(\alpha)<\pi(\beta)$ and $1\le\gamma$, then let $d_{\alpha,\beta,\gamma}$ denotes the smallest positive integer d satisfying the condition

(10) $$v_p\left(\frac{d}{\kappa}\right)\geq 0$$

for all the primes $p \in \mathbb{N} \cap (\alpha, \beta]$ and all the $\kappa \in \mathbb{N} \cap [1, \gamma]$. We let, by definition, $d_{\alpha,\beta,\gamma} = 1$ if either $\gamma < 1$, or $\pi(\beta) \le \pi(\alpha)$. Clearly,

$$d_{\alpha,\beta,\gamma} = \prod_{\alpha<p\le\beta} p^{[ln(\gamma)/ln(p)]} \le \gamma^{max(\pi(\beta)-\pi(\alpha),0)}. \tag{11}$$

Lemma 7.1. *If* $1 \le \alpha \le \beta$, $1 \le \gamma \le \gamma_0\alpha$ *for some* $\gamma_0 > 0$, *then for any* $\varepsilon > 0$ *there exists* $C_0(\gamma_0, \varepsilon)$ *satisfying the following inequalities:*

$$d_{\alpha,\beta,\gamma} \le C_0(\gamma_0, \varepsilon)\exp(\alpha(\beta/\alpha - 1 + \varepsilon)), \tag{12}$$

and

$$d_{1,\alpha,\gamma} \le C_0(\gamma_1, \varepsilon)\exp(\alpha(1 + \varepsilon)). \tag{13}$$

Proof (see [64], Lemma 4). It may be assumed that $\beta \le \gamma$; then it follows from the inequalities (11) – (13) that

$$\begin{gathered} d_{\alpha,\beta,\gamma} \le \\ exp((\ln(\gamma_0) + \ln(\alpha))\times \\ ((\beta-\alpha)/\ln(e\alpha) + O(\beta-\alpha)/\ln^2(e\alpha)) \le \\ C_0(\gamma_0, \varepsilon)\exp(\alpha(\beta/\alpha - 1 + \varepsilon)), \end{gathered} \tag{14}$$

$$\begin{gathered} d_{1,\alpha,\gamma} \le \\ \exp((\ln(\gamma_1) + \ln(\alpha))(\alpha/\ln(e\alpha) + O(\alpha/\ln^2(e\alpha))) \le \\ C_0(\gamma_0, \varepsilon)\exp((\alpha(1 + \varepsilon)). \blacksquare \end{gathered} \tag{15}$$

Lemma 7.2. *If*

$$\nu > 2/\Delta, \tag{16}$$

$$p > \nu\Delta, \tag{17}$$

then

$$v_p(\beta^*_{\nu,k}) \ge 0 \tag{18}$$

$$v_p(\gamma^*_{\nu,k}) \ge 0 \tag{19}$$

Proof (see [58], p.23). If $p > \nu\Delta + k$, then (18) – (19) directly follow from the equalities (1) – (3). If $\nu\Delta < p \le \nu\Delta + k$, then we must consider in the expressions (2) – (3) only the case $\nu + k < \kappa \le \nu\Delta + k$; if $\nu\Delta < p \le \nu + k$, then we have

$$p < \kappa \le \nu\Delta + k \le 2\nu\Delta < 2p$$

and, consequently, $(\kappa, p) = 1$; therefore in this case we must consider only

$$p > max(\nu + k, \nu\Delta).$$

But then $v_p(\kappa) \le 1$,

$$v_p\left(\binom{\nu\Delta+k}{\nu\Delta-\nu}^3\right) \ge 3,$$

$$v_p\left(\kappa^{-2}\binom{\nu\Delta+k}{\nu\Delta-\nu}^3\right) \ge 1.$$

■

According to the Lemma 7.2,

$$d_{1,\nu\Delta,2\nu\Delta}\beta^*_{\nu,k} \in \mathbb{Z}, \tag{20}$$

$$(d_{1,\nu\Delta,2\nu\Delta})^2\gamma^*_{\nu,k} \in \mathbb{Z}. \tag{21}$$

Let $\nu \in \mathbb{N}$, $\Delta \in \mathbb{N}$, $\Delta \ge 2$. Let us consider the numbers

$$\binom{\nu\Delta}{k}\binom{\nu\Delta+k}{\nu\Delta-\nu}, \tag{22}$$

where $k = 0, \ldots, \nu\Delta$. How small may be chosen $r(\nu) \in (0, +\infty) \cap \mathbb{Q}$ such that

$$r(\nu)\binom{\nu\Delta}{k}\binom{\nu\Delta+k}{\nu\Delta-\nu} \in \mathbb{Z}$$

for all the $k = 0, \ldots, \nu\Delta$? Probably G.V. Chudnovsky was the first man, who discovered, that $r(\nu)$ may be chosen sufficiently small; Hata ([17]) in details studied this effect. Therefore I name such $r(\nu)$ by Chudnovsky-Hata's multiplier.

Lemma 7.3. *Let $p \in \mathbb{N}$ is a prime number such that*

$$2\{\nu\Delta/p\} < \{\nu(\Delta-1)/p\}. \tag{23}$$

Then

$$v_p\left(\binom{\nu\Delta}{k}\binom{\nu\Delta+k}{\nu\Delta-\nu}\right) \ge 1, \tag{24}$$

where $k = 0, \ldots, \nu\Delta$.

Proof. See [77], section 2. ■

Let $d^*_1(\nu)$ denotes the product of all the prime numbers $p \in \mathbb{N}$, which satisfy to the condition (23). Then, according to the Lemma 7.3,

$$(d^*_1(\nu))^{-3}\alpha^*_{\nu,k} \in \mathbb{Z} \tag{25}$$

for any $k = 0, \ldots, \nu\Delta$.

Lemma 7.4. *If (16) holds, then*

$$d_{1,\nu\Delta,2\nu\Delta}(d^*_1(\nu))^{-3}\beta^*_{\nu,k} \in \mathbb{Z}, \tag{26}$$

$$(d_{1,\nu\Delta,2\nu\Delta})^2(d^*_1(\nu))^{-3}\gamma^*_{\nu,k} \in \mathbb{Z}, \tag{27}$$

where $k = 0, \ldots, \nu\Delta$.

Proof. We prove that

$$v_p\left(d_{1,\nu\Delta,2\nu\Delta}(d_1^*(\nu))^{-3}\beta^*_{\nu,k}\in\mathbb{Z}\right)\geq 0,\tag{28}$$

$$v_p\left((d_{1,\nu\Delta,2\nu\Delta})^2(d_1^*(\nu))^{-3}\gamma^*_{\nu,k}\right)\geq 0\tag{29}$$

for any prime number $p\in\mathbb{N}$.

We partially repeat the proof of the Lemma 7.2. If $p>\nu\Delta+k$, then, since $\kappa\leq\nu\Delta+k$, it follows that $v_p(\kappa)=0$, and, according to (25), the relations (28) – (29) hold. If $p>\nu\Delta$, then, since $\kappa\leq\nu\Delta+k\leq 2\nu\Delta<2p$, it follows that $v_p(\kappa)\leq 1$; moreover, if $p\leq\nu+k<\kappa$, then, since $\kappa<2p$, it follows that $v_p(\kappa)=0$, and, in view of (25), the relations (28) – (29) hold. On the other hand, if $p>max(\nu\Delta,\nu+k)$ then the inequality (23) turns into equality with both sides equal to zero. Therefore $v_p(d_1^*(\nu))=0$ and the relations (18) – (19) hold. Finally if $p\leq\nu\Delta$, then, since $\kappa\leq 2\nu\delta$, it follows that $v_p(d_{1,\nu\Delta,2\nu\Delta}/\kappa)\geq 0$ and, according to (25), the relations (18) – (19) hold. ■

Let

$$D^{**}(\nu)=d_{1,\nu(\Delta+1),2\nu\Delta}.\tag{30}$$

Then $d_{1,\nu\Delta,2\nu\Delta}$ is divisor of $D^{**}(\nu)$ and according to the Lemma 7.4,

$$D^{**}(\nu)(d_1^*(\nu))^{-3}\beta^*_{\nu,k}\in\mathbb{Z},\tag{31}$$

$$(D^{**}(\nu))^2(d_1^*(\nu))^{-3}\gamma^*_{\nu,k}\in\mathbb{Z},\tag{32}$$

where $k=0,\ldots,\nu\Delta$.

Lemma 7.5. *If (16) holds, then*

$$(D^{**}(\nu))^5(d_1^*(\nu))^{-3}\alpha^*(z;\nu)\in\mathbb{Z}[z],\tag{33}$$

$$(D^{**}(\nu))^5(d_1^*(\nu))^{-3}\beta^*(z;\nu)\in\mathbb{Z}[z],\tag{34}$$

$$(D^{**}(\nu))^5(d_1^*(\nu))^{-3}\gamma^*(z;\nu)\in\mathbb{Z}[z],\tag{35}$$

$$(D^{**}(\nu))^5(d_1^*(\nu))^{-3}\phi^*(z;\nu)\in\mathbb{Z}[z],\tag{36}$$

$$(D^{**}(\nu))^5(d_1^*(\nu))^{-3}\psi^*(z;\nu)\in\mathbb{Z}[z],\tag{37}$$

$$(D^{**}(\nu))^5(d_1^*(\nu))^{-3}\xi^*(z;\nu)\in\mathbb{Z}[z].\tag{38}$$

Proof. The relations (33) – (34) directly follow from (4) – (6), (31) – (32) and (25). It follows from (7) – (9), (25), (31) – (32) and (30) that

$$(-1)^\nu(D^{**}(\nu))^3(d_1^*(\nu))^{-3}\phi^*(z;\nu)=\tag{39}$$

$$\sum_{k=0}^{\nu\Delta}(d_1^*(\nu))^{-3}\alpha^*_{\nu,k}z^{k+\nu-t}\sum_{t=1}^{k+\nu}(D^{**}(\nu))^3t^{-3}+$$
$$\sum_{k=0}^{\nu\Delta}(D^{**}(\nu))(d_1^*(\nu))^{-3}\beta^*_{\nu,k}z^{k+\nu-t}\sum_{t=1}^{k+\nu}(D^{**}(\nu))^2t^{-2}+$$
$$\sum_{k=0}^{\nu\Delta}(D^{**}(\nu))^2(d_1^*(\nu))^{-3}\gamma^*_{\nu,k}z^{k+\nu-t}\sum_{t=1}^{k+\nu}(D^{**}(\nu))t^{-1}\in\mathbb{Z}[z],$$

(40) $$(-1)^\nu(D^{**}(\nu))^4(d_1^*(\nu))^{-3}\psi^*(z;\nu)=$$
$$(-z)^\nu\sum_{k=0}^{\nu\Delta}3(d_1^*(\nu))^{-3}\alpha^*_{\nu,k}\sum_{t=1}^{k+\nu}z^{k+\nu-t}(D^{**}(\nu))^4t^{-4}+$$
$$\sum_{k=0}^{\nu\Delta}2(D^{**}(\nu))(d_1^*(\nu))^{-3}\beta^*_{\nu,k}\sum_{t=1}^{k+\nu}z^{-t}(D^{**}(\nu))^3t^{-3}+$$
$$\sum_{k=0}^{\nu\Delta}(D^{**}(\nu))^2(d_1^*(\nu))^{-3}\gamma^*_{\nu,k}\sum_{t=1}^{k+\nu}z^{k+\nu-t}(D^{**}(\nu))^2t^{-2}\in\mathbb{Z}[z],$$

(41) $$(-1)^\nu(D^{**}(\nu))^5(d_1^*(\nu))^{-3}\xi^*(z;\nu)=$$
$$(-z)^\nu\sum_{k=0}^{\nu\Delta}6(d_1^*(\nu))^{-3}\alpha^*_{\nu,k}\sum_{t=1}^{k+\nu}z^{k+\nu-t}(D^{**}(\nu))^5t^{-5}+$$
$$+(-z)^\nu\sum_{k=0}^{\nu\Delta}3(D^{**}(\nu))(d_1^*(\nu))^{-3}\beta^*_{\nu,k}\sum_{t=1}^{k+\nu}z^{k+\nu-t}(D^{**}(\nu))^4t^{-4}+$$
$$(-z)^\nu\sum_{k=0}^{\nu\Delta}(D^{**}(\nu))^2(d_1^*(\nu))^{-3}\gamma^*_{\nu,k}\sum_{t=1}^{k+\nu}z^{k+\nu-t}(D^{**}(\nu))^3t^{-3}\in\mathbb{Z}[z].$$

■

Though in the initial work I used computer for calculation, I shall make here all calculations 'by hands' using only calcuator of the firm 'Casio.' Therefore everyone can check them up. I take below $\Delta=13$. According to (13) and (30),

(42) $$(\Delta+1-\varepsilon)\nu-O(1)\le\ln(D^{**}(\nu))\le(\Delta+1+\varepsilon)\nu+O(1),$$

with any $\varepsilon>0$ and $O(1)$ depending only from ε. For $\Delta=13$ we have

$$(14-\varepsilon)\nu-O(1)\le\ln(D^{**}(\nu))\le(14+\varepsilon)\nu+O(1).$$

In view of (32) and (45) in [77],

$$\nu(\Delta I-\varepsilon)+O(1)\le\log(d_1^*(\nu))\le\nu(\Delta I+\varepsilon)+O(1),$$

where

$$I=I(\Delta)=\ln(\Delta)-\frac{\Delta-1}{2\Delta}\ln(\Delta-1)-\frac{\Delta+1}{2\Delta}\ln(\Delta+1)-$$

$$\frac{\pi}{2\Delta}\sum_{1\le\kappa\le(\Delta-1)/2}\cot(\pi\kappa/(\Delta-1))+$$
$$\frac{\pi}{2\Delta}\sum_{1\le\kappa\le(\Delta+1)/2}\cot(\pi\kappa/(\Delta+1)).$$

For $\Delta=13$ we have

$$I=\ln(13)-\frac{6}{13}\ln(12)-\frac{7}{13}\ln(14)-$$
$$\frac{\pi}{26}\sum_{1\le\kappa\le6}\cot(\pi\kappa/12))+$$
$$\frac{\pi}{26}\sum_{1\le\kappa\le7}\cot(\pi\kappa/14).$$

Further we have

$$\ln(12)=2,48490665...;\ln(13)=2,564949357...;\ln(14)=2,63905733...;$$
$$\frac{6}{13}\ln(12)=1,146879992...;\frac{7}{13}\ln(14)=1,42103087...;$$
$$\cot(\pi/12)+\cot(2\pi/12)+\cot(3\pi/12)+\cot(4\pi/12)+\cot(5\pi/12)+$$
$$cot(6\pi/12)=2/\sin(pi/6)+2/\sin(pi/3)+1=5+4/\sqrt{3}=7.309401077...;$$
$$2/\sin(\pi/7)=4,609529742;\ \cot(\pi/14)+\cot(2\pi/14)+\cot(3\pi/14)+$$
$$\cot(4\pi/14)+\cot(5\pi/14)+\cot(6\pi/14)+\cot(7\pi/14)=$$
$$2/\sin(\pi/7)+2/\sin(2\pi/7)+2/\sin(3\pi/7)=$$
$$2/\sin(\pi/7)+4\sin(5\pi/14)\cos(\pi/14)/(\sin(2\pi/7)\cos(\pi/14))=$$
$$2/\sin(\pi/7)+4cos(\pi/7)/\sin(2\pi/7)=4/\sin(\pi/7)=9,219059484...;$$
$$\frac{\pi}{26}\sum_{1\le\kappa\le7}\cot(\pi\kappa/4)-$$
$$\frac{\pi}{26}\sum_{1\le\kappa\le6}\cot(\pi\kappa/12)=$$
$$\frac{\pi}{26}(9,219059484...-7,309401077...)=$$
$$(9,219059484...-7,309401077...)=(1,909658407...)\pi/26=0,230744954...$$
$$I=\ln(13)-\frac{6}{13}\ln(12)-\frac{7}{13}\ln(14)-$$
$$\frac{\pi}{26}\sum_{1\le\kappa\le6}\cot(\pi\kappa/12))+$$
$$\frac{\pi}{26}\sum_{1\le\kappa\le7}\cot(\pi\kappa/14)=$$
$$2,564949357...-1,146879992...-1,42103087...+0,230744954...=$$
$$=0.227783449....$$
$$3\Delta I=39\times0,227783449...=...8,883554511...,$$

Let

$$\omega_2(\Delta) = 5(\Delta+1) - 3\Delta I(\Delta) = \tag{43}$$

$$5(\Delta+1) - \frac{3}{2}(\Delta-1)\ln\left(\frac{\Delta}{\Delta-1}\right) +$$

$$\frac{3}{2}(\Delta+1)\ln\left(\frac{\Delta+1}{\Delta}\right) +$$

$$\frac{3\pi}{2}\sum_{1\le\kappa\le(\Delta-1)/2}\cot(\pi\kappa/(\Delta-1)) -$$

$$\frac{3\pi}{2}\sum_{1\le\kappa<(\Delta+1)/2}\cot(\pi\kappa/(\Delta+1)).$$

Then

$$\omega_2(13) = 70 - 8,883554511... = 61.11644549..., \tag{44}$$

and

$$\nu(\omega_2(13)-\varepsilon) - O(1) \le \ln((D^{**}(\nu))^5/(d_1^*(\nu)^3) \le \tag{45}$$

$$\nu(\omega_2(13)+\varepsilon) + O(1)$$

with any $\varepsilon > 0$ and $O(1)$ depending only from ε.

In view of (5.27) in [71], we must calculate the value $|h^{\sim}(\eta_1^{\wedge}(1,\pi/3,\delta_0)|$; we compute also the value $|h^{\sim}(\eta_0^{\wedge}(1,\pi/3,\delta_0)|$. According to (3.1.43) and (3.1.71) in [62], for our case $\Delta = 13$ we have

$$q_1 = 36/49 = 0,734693877..., \; q_2 = 676/49 = 13,79591837... \tag{46}$$

$$q_0 = 12\times 169/49 = 41,3877551... . \tag{47}$$

In view of (3.1.10) in [62],

$$|D_0(1,\pi/3,1/13)|^2 = 2 + (1/676)^2 - 1/676 + 1 + 1/676 = 3 + (1/676)^2 =$$

$$1370929/(676)^2 = 3,000002188...,$$

$$|D_0(1,\pi/3,1/13)| = 1170,866773.../676 = 1,732051439... .$$

In view of (3.1.41) in [62],

$$p_1 = 8(|R_0^*(r,\psi,\delta_0)|^2 + |R_0(r,\psi,\delta_0)|^2)/(1+\delta_0)^2 =$$

$$8\left(r^2 + rt + \frac{1}{4} + |D_0(r,\psi,\delta_0)|\right)/(1+\delta_0)^2,$$

and in our case

$$p_1 = 8\left(7/4 + |D_0(1,\pi/3,1/3)|\right)/(1+1/13)^2 = \tag{48}$$

$$(169/196)8\left(1,75 + 1,732051439...\right) = 24.0190487... .$$

In view of (3.1.42) in [62],

$$p_2 = 8\left(1 - \frac{1}{26} + \frac{1}{676} + |D_0(1, \pi/3, 1/3)|\right)/(1 + 1/13)^2 = \tag{49}$$

$$(169/196)8\,(651/676 + 1,732051439...) = 18,59047727... .$$

In view of (3.1.70) and (3.1.61) in [62],

$$p_0 = 8(|R^*_{-1}(r, \psi, \delta_0)|^2 + |R_{-1}(r, \psi, \delta_0)|^2)/(1 + \delta_0)^2 =$$

$$8\left(r^2 + (2 + \delta_0)^2/4 + r(2 + \delta_0)\cos(\psi) + |D_0(r, \psi, \delta_0)|\right)/(1 + \delta_0)^2,$$

and in our case

$$p_0 = 8\,(2107/676 + |D_0(1, \pi/3, 1/3)|)\,/(1 + 1/13)^2 = \tag{50}$$

$$(169/196)8\,(3,116863905... + 1,732051439...) = 33,44762013... .$$

Further we have

$$p_1/2 = 24,0190487.../2 = 12,00952435...\,,$$

$$p_2/2 = 18,59047727.../2 = 9,295238635...\,,$$

$$p_0/2 = 33,44762013.../2 = 16,72381007...\,,$$

$$\sqrt{(p_1/2)^2 - q_1} = \sqrt{(12,00952435...)^2 - 0,734693877...} = 11,97889733...\,,$$

$$\sqrt{(p_2/2)^2 - q_2} = \sqrt{(9,295238635...)^2 - 13,79591837...} = 8,520888622...\,,$$

$$\sqrt{(p_0/2)^2 - q_0} = \sqrt{(16,72381007...)^2 - 41,3877551...} = 15,43690604... .$$

In view of (3.1.37) in [62],

$$|\eta_1(1, \pi/3, 1/13)|^2 = p_1/2 - \sqrt{(p_1/2)^2 - q_1} = \tag{51}$$

$$12,00952435... - 11,97889733... = 0,03062702...$$

$$|\eta_0(1, \pi/3, 1/13)|^2 = p_1/2 + \sqrt{(p_1/2)^2 - q_1} = \tag{52}$$

$$12,00952435... + 11,97889733... = 23,98842168...$$

In view of (3.1.37) in [62],

$$|\eta_1(1, \pi/3, 1/13) + 1|^2 = p_2/2 - \sqrt{(p_2/2)^2 - q_2} = \tag{53}$$

$$9,295238635... - 8,520888622... = 0,774350013...$$

$$|\eta_0(1, \pi/3, 1/13) + 1|^2 = p_2/2 + \sqrt{(p_2/2)^2 - q_2} = \tag{54}$$

$$9,295238635... + 8,520888622... = 17,816127257...$$

In view of (3.1.61) in [62], in our case

$$s = \cos(\psi/2) = \cos(\pi/6) = \frac{\sqrt{3}}{2} > \delta_0/4 = \frac{1}{52}.$$

Therefore, in view of (3.1.65) in [62],

$$|\eta_1(1,\pi/3,1/13)-1|^2 = p_0/2-\sqrt{(p_0/2)^2-q_0} = \tag{55}$$

$$16,72381007...-15,43690604... = 1,28690403...\,,$$

$$|\eta_0(1,\pi/3,1/13)-1|^2 = p_0/2+\sqrt{(p_0/2)^2-q_0} = \tag{56}$$

$$16,72381007...+15,43690604... = 32,16071611...$$

Before to go further, let us check the previous calculations.

$$|\eta_0(1,\pi/3,1/13)|^2|\eta_1(1,\pi/3,1/13)|^2 =$$

$$23,98842168...\times 0,03062702.. = 0.73469387...,$$

what is almost equal to above $q_1 = 36/49 = 0,734693877...\,,$

$$|\eta_0(1,\pi/3,1/13)+1|^2|\eta_1(1,\pi/3,1/13)+1|^2 =$$

$$17,816127257...\times 0,774350013... = 13.79591837...,$$

what is (almost) equal to above $q_2 = 13,79591837...\,,$

$$|\eta_0(1,\pi/3,1/13)-1|^2|\eta_1(1,\pi/3,1/13)-1|^2 =$$

$$32,16071611...\times 1,28690403... = 41,38775517...,$$

what is almost equal to above $q_0 = 41,3877551...\,,$

$$|\eta_1(1,\pi/3,1/13)+1|^2+|\eta_1(1,\pi/3,1/13)-1|^2 =$$

$$0,774350013...+1.28690403... = 2,061254043...\,,$$

what "almost" is equal to

$$2(1+|\eta_1(1,\pi/3,1/13)|^2) = 2(1+0,03062702...) = 2,06125404...\,;$$

$$|\eta_0(1,\pi/3,1/13)+1|^2+|\eta_0(1,\pi/3,1/13)-1|^2 =$$
$$17,816127257...+32,16071611... = 49,97684336...\,,$$

what "almost" is equal to

$$2(1+|\eta_0(1,\pi/3,1/13)|^2) = 2(1+23,98842168...) = 49,97684336...\,;$$

In view of (3.1.28) in [62],

$$\ln(|h^{\sim}(\eta_1(1,\pi/3,1/13)) =$$

$$1,5\ln(|(\eta_1(1,\pi/3,1/13))^2-1|^2)+36\ln(13/12)-6\ln 2+$$
$$18\ln(|\eta_1(1,\pi/3,1/13)|^2).$$

Further we have

$$1,5\ln(|(\eta_1(1,\pi/3,1/13))-1|^2 = 1,5\ln(0,774350013...) = -0,383596941...,$$

$$1,5\ln(|(\eta_1(1,\pi/3,1/13))+1|^2 = 1,5\ln(1.28690403...) = 0,378359035...,$$

$$36\ln(13/12) = 2,881537476...$$
$$18\ln(|\eta_1(1,\pi/3,1/13)|^2) = 18\ln(0,03062702...) = -62.74570776...,$$
$$\ln(64) = 4.158883083...$$

(57) $$-\ln(|h^{\sim}(\eta_1(1,\pi/3,1/13)|) =$$
$$4,158883083... + 62,74570776... + 0,383596941...-$$
$$0,378359035... - 2,881537476 = 64.02829127... .$$

Let

(58) $$\gamma_1^*(\Delta) = -\ln(|h^{\sim}(\eta_1(1,\pi/3,1/Delta)|) - \omega_2(\Delta)$$

I view of (44) and (57),

(59) $$\gamma_1^*(13) = 64.02829127... - 61.11644549... =$$
$$2,911845783... > 0,$$

and

(60) $$-\nu(\gamma_1^*(13)+\varepsilon) + O(1) \leq$$
$$\ln(|h^{\sim}(\eta_1(1,\pi/3,1/13))|^{\nu}(D^{**}(\nu))^5/(d_1^*(\nu)^3)) \leq$$
$$\nu(61.11644549... - 64.02829127... + \varepsilon) + O(1) =$$
$$-\nu(\gamma_1^*(13)-\varepsilon) + O(1) \to -\infty,$$

if $0 < \varepsilon < \gamma_1^*(13)/2$ and $\nu \to +\infty$.

Now we compute also the value

$$\ln(|h^{\sim}(\eta_0(1,\pi/3,1/13))|) =$$
$$1,5\ln(|(\eta_0(1,\pi/3,1/13))^2-1|^2) + 36\ln(13/12) - 6\ln 2 +$$
$$18\ln(|\eta_0(1,\pi/3,1/13)|^2) =$$
$$1,5\ln(|(\eta_0(1,\pi/3,1/13))^2-1|^2) - 6\ln(2) + 18\ln((169/144)|\eta_0(1,\pi/3,1/13)|^2).$$

Further we have

$$\ln(|(\eta_0(1,\pi/3,1/13))^2-1|^2) =$$
$$\ln(17,816127257...) + \ln(32,16071611...) = 6,350849784...$$
$$1,5\ln(|(\eta_0(1,\pi/3,1/13))^2-1|^2) = 9,526274675...$$
$$\ln((169/144)|\eta_0(1,\pi/3,1/13)|^2) =$$
$$\ln(169/144) + \ln(23,98842168...) = 3,337656699...,$$
$$18\ln((169/144)|\eta_1(1,\pi/3,1/13)|^2) = 60,07782059...,$$
$$\ln(64) = 4.158883083...$$

(61) $$\ln(|h^{\sim}(\eta_0(1,\pi/3,1/13))|) = 9,526274675... -$$
$$4,158883083... + 60,07782059... = 65,44521218... .$$

Let

$$\gamma_0^*(\Delta)=\ln(|h^{\sim}(\eta_0(1,\pi/3,1/\Delta)|)+\omega_2(\Delta) \tag{62}$$

I view of (44) and (61),

$$\gamma_0^*(13)=61.11644549...+65,44521218...=126,5616577..., \tag{63}$$

$$\begin{gathered}\nu(\gamma_0^*(13)-\varepsilon)+O(1)\le\\ \ln(|h^{\sim}(\eta_0(1,\pi/3,1/13)|^{\nu}(D^{**}(\nu))^5/(d_1^*(\nu)^3)\le\\ \nu(\gamma_0^*(13)+\varepsilon)+O(1).\end{gathered} \tag{64}$$

Let

$$\gamma^*(\Delta)=\gamma_0^*(\Delta)/\gamma_1^*(\Delta). \tag{65}$$

In view of (60) and (64), there exist

$$-\lim_{\nu\to+\infty}\frac{\ln(|h^{\sim}(\eta_0(1,\pi/3,1/13)|^{\nu}(D^{**}(\nu))^5/(d_1^*(\nu)^3)}{\ln(|h^{\sim}(\eta_1(1,\pi/3,1/13)|^{\nu}(D^{**}(\nu))^5/(d_1^*(\nu)^3)}= \tag{66}$$

$$\gamma^*(13)\,(=126,5616577.../2,911845783...=43,46440956...),$$

and

$$\gamma^*(13)<\gamma=43,464412 \tag{67}$$

(of course, if our Theorem is true for some γ, then all the bigger numbers can play the role of γ).

Let

$$h_0(\varepsilon)=\frac{\gamma_0^*(13)+6\varepsilon}{\gamma_1^*(13)-2\varepsilon}, \tag{68}$$

where $\varepsilon\in(0,\gamma_1^*(13)/2)$. Clearly, $h_0(\varepsilon)$ increases together with increasing ε in $(0,\gamma_1^*(13)/2)$, and, in view of (59), (65) and (67),

$$h_0(0)<\gamma.$$

Therefore there exists $\varepsilon_{10}\in(0,\gamma_1^*(13)/2)$ such that

$$h_0(\varepsilon)<\gamma \tag{69}$$

for all the $\varepsilon\in(0,\varepsilon_{10})$. Let

$$R_1=R_1(\varepsilon)=\exp(\gamma_1^*(13)+2\varepsilon), \tag{70}$$

$$R_2=R_2(\varepsilon)=\exp(\gamma_1^*(13)-2\varepsilon), \tag{71}$$

$$r_1^{\wedge}=r_2^{\wedge}=r^{\wedge}=r^{\wedge}(\varepsilon)=\exp(\gamma_0^*(13)+2\varepsilon), \tag{72}$$

where $\varepsilon \in (0, \varepsilon_{10})$. Then

$$R_1 > R_2 > 1, \ln(R_1/R_2) = 4\varepsilon. \tag{73}$$

In view of (69),

$$(\ln(r^\wedge(\varepsilon)R_1(\varepsilon)/R_2(\varepsilon)))/\ln(R_2(\varepsilon)) = h_0(\varepsilon) < \gamma \tag{74}$$

for all the $\varepsilon \in (0, \varepsilon_{10})$. Let, as in (4.1.33) of the paper [65],

$$\alpha_0^*(z;\nu) = \alpha^*(z;\nu), \alpha_1^*(z;\nu) = \beta^*(z;\nu), \alpha_2^*(z;\nu) = \gamma^*(z;\nu), \tag{75}$$

$$\alpha_3^*(z;\nu) = \phi^*(z;\nu), \alpha_4^*(z;\nu) = \psi^*(z;\nu), \alpha_5^*(z;\nu) = \xi^*(z;\nu), \tag{76}$$

where $\nu \in \mathbb{N}$. Let us consider the difference equation (29) in [70] now, i.e. the following difference equation:

$$\sum_{j=0}^{4} q_j^{\wedge *}(\nu^{-1})y(\nu + j) + y(\nu + 5) = 0, \tag{77}$$

where $q_j^{\wedge *}(\nu^{-1}) \in \mathbb{Q}(\nu^{-1})$, if $j = 0, \ldots, 4$, are well defined for $\nu \in [1/\varepsilon_5] + \mathbb{N}$ with some $\varepsilon_5 > 0$ and $q_0^{\wedge *}(\nu^{-1}) \neq 0$, when $\nu \in [1/\varepsilon_5] + \mathbb{N}$. It is shown in [70] that the equation (77) has a solutions

$$y(\nu) = \alpha_k^*(1;\nu), \tag{78}$$

where $k = 0, 1, 3, 4, 5$ and $\nu \in \mathbb{N} + [1/\varepsilon_5]$; according to the Lemma 4.1.3 in [65], solutions (78) compose a basis of the space of the solutions of the equation (77). In view of (2.6) in (60),

$$\alpha_2^*(1,\nu) = \gamma^*(1,\nu) = -(-1)^\nu Res(R_0(t;\nu))^3; t = \infty) = 0,$$

where $\nu \in \mathbb{N}$. Therefore, in view of (2.21), (2.22), (2.19), (2.25) and (2.27), the five solutions

$$y(\nu) = \alpha^*(1;\nu), y(\nu) = \beta^*(1;\nu), \tag{79}$$

$$y(\nu) = f_2^*(1;\nu), y(\nu) = f_4^*(1;\nu), y(\nu) = f_6^*(1;\nu) \tag{80}$$

(like to the five solutions (78)) compose a basis of the space of the solutions of the equation (77). It is proved in [71], that equation (77) is the difference equation of Poincaré type and, if $P(\eta)$ denotes the characterisic polynomial of this equation, then

$$P(\eta) = F_0(\eta)(\eta - \lambda_0), \tag{81}$$

where $\lambda_0 \in \mathbb{Q}$,

$$F_0(\eta) = \prod_{l=0}^{1}\prod_{k=0}^{1} h^\sim(\eta_{k,l}^*) \tag{82}$$

is the polynomial considered in the Lemma 3.1.6 of the of the paper [67], and

$$\eta^*_{k,l} = \eta^\wedge_k(1, (2l-1)\pi/3, 1/13) \tag{83}$$

According to the Lemma 3.1.6 in [67], the polynomial $F_0(\eta)$ is irreducible over $\mathbb{Q}$; therefore the multiplicity of the root $\lambda_0 \in \mathbb{Q}$ of the polynomial $P(\eta)$ is equal to 1.

I apply further the following result [76]:

Lemma 7.6. *Let $s \in \mathbb{N} - 1, n \in \mathbb{N}$,*

$$a^\sim_i \in \mathbb{C},\ a_i(\nu) \in \mathbb{C},$$

$$a_n(\nu) = 1,\ a_i(\nu) - a^\sim_i = O(1/\nu) \tag{84}$$

for $\nu \in \mathbb{N}$ and $i = 0, \dots, n$. Let us consider the following difference equation

$$\sum_{k=0}^{n} a_k(\nu) y(\nu + k) = 0, \tag{85}$$

where $\nu \in \mathbb{N} - 1$. For $m \in \mathbb{N}$ let V_m denotes the linear over $\mathbb{C}$ space of solutions $y = y(\nu)$ of the equation

$$\sum_{k=0}^{n} a_k(\nu) y(\nu + k) = 0, \tag{86}$$

where $\nu \in m + \mathbb{N} - 1$. Let the absolute values of all the roots of the characteristical polynomial

$$T(z) = \sum_{k=0}^{n} a^\sim_k z^k \tag{87}$$

are among the numbers $\{\rho_i\colon 1 \le i \le 1 + s\}$, such that $\rho_{s+1} = 0$ and $\rho_j < \rho_i$ for $1 \le i < j \le s + 1$. Let e_i and k_i denote respectively the sum and the maximum of the multiplicities of those roots, whose absolute value is equal to the number ρ_i, where $i = 1, \dots, s + 1$, and let $k^* = k_{s+1}$. We suppose that, if $s > 0$, then

$$e_i > 0 \tag{88}$$

for $i = 1 \dots, s$. For given $y = y(\nu)$ in $\mathbb{C}^{m-1+\mathbb{N}}$, let

$$\omega_{n,y}(\nu) = \max(|y(\nu)|, \dots, |y(\nu + n - 1)|).$$

Then there exist $A > 0$, $m \in \mathbb{N}$, $\alpha^\wedge(\nu) > 0$ with $\nu \in m + \mathbb{N} - 1$ and the subspaces $V^\vee_{m,1}, \dots, V^\vee_{m,s+1}$ such that

$$\lim_{\nu\to\infty} \alpha^\wedge(\nu) = 0,$$

$$V_m = V^\vee_{m,1} \oplus \dots, \oplus V^\vee_{m,s+1},\ \dim_{\mathbb{C}}(V^\vee_{m,i}) = e_i,\ 1 \le i \le s + 1,$$

and, if $y \in V^\vee_{m,\theta}$ for some $\theta \in \{1, ..., s\}$, then

$$\exp(-A(ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \omega_n(y)(m) \le \omega_{n,y}(\nu) \tag{89}$$

for $\nu \in m+\mathbb{N}-1$; moreover, the spaces

$$V^{\wedge}_{m,j} = V^{\vee}_{m,j} \oplus \ldots \oplus V^{\vee}_{m,s+1},$$

where $j = 1 \ldots, s+1$, and, if $s \geq 1$ natural projections π_j,

$$V^{\wedge}_{m,j} \mapsto V^{\vee}_{m,j},$$

where $j = 1 \ldots, s$, have the following properties:
if $y \in V^{\wedge}_{m,\theta}$ for some $\theta \in \{1, ..., s\}$, then

$$\omega_{n,y}(\nu) \leq \exp(A(\ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \omega_{n,y}(m), \tag{90}$$

$$(\omega_{n,\pi_{\theta(y)}}(m) - \alpha(\nu)\omega_{n,y}(m))(\rho_\theta)^\nu \times \tag{91}$$

$$\exp(-A(\ln(\nu) + \nu^{1-1/k_\theta})) \leq \omega_{n,y}(\nu),$$

where $\nu \in m+\mathbb{N}-1$; if

$$k^* > 0, \tag{92}$$

and $y \in V^{\vee}_{m,s+1}\,(= V^{\wedge}_{m,s+1})$, then

$$|y(\nu)| \leq (A/\nu)^{\nu/k^*}\omega_{n,y}(m),$$

where $\nu \in m+\mathbb{N}-1$.

Proof. The proof is given in [76]. ■

Remark 7.1. Clearly, for any $\varepsilon \in (0,1)$ there exists $C_0 > 0$ such that the inequalities (89) and (90) may be respectively replaced by the inequalities

$$(1/C_0)(\rho_\theta \exp(-\varepsilon))^\nu \omega_n(y)(m) \leq \omega_{n,y}(\nu) \tag{93}$$

and

$$\omega_{n,y}(\nu) \leq C_0(\rho_\theta \exp(\varepsilon))^\nu \omega_{n,y}(m). \tag{94}$$

In our case the role of the equation (85) plays the equation (77) and the role of the polynomial (87) plays polynomial (81). In view (5.24) in [71], we have

$$n = 5, s = 3, \rho_1 = |h^{\sim}(\eta^{\wedge}_0(1, \pi/3, 1/13))| = |h^{\sim}(\eta^{\wedge}_0(1, -\pi/3, 1/13))|,$$

$$\rho_2 = |h^{\sim}(\eta^{\wedge}_1(1, \pi, 1/13))| = h^{\sim}(\gamma_1)| = h(6/7),$$

$$\rho_3 = |h^{\sim}(\eta^{\wedge}_1(1, \pi/3, 1/13))| = |h^{\sim}(\eta^{\wedge}_1(1, -\pi, 1/13))|.$$

Since the polynomial $F_0(\eta)$ is irreducible over $\mathbb{Q}$, it follows that

$$e_1 = e_3 = 2, e_1 = k_1 = k_2 = k_3 = 1, k_4 = 0.$$

Since $k_4 = 0$, it follows that $V^{\vee}_{m,4} = 0$, $V^{\wedge}_{m,3} = V^{\vee}_{m,3}$. In view (5.27) in [71], the space $V^{\vee}_{m,3}$ of the Lemma 7.6 contains $f^*_{2+2k}(1;\nu)$ for $k = 1,2$; moreover, since $\dim_{\mathbb{C}}(V^{\vee}_{m,3}) = 2$ and the five solutions (79) –(80), compose a basis of the space V_m, it follows that $f^*_4(1;\nu)$, $f^*_6(1;\nu)$ compose a basis of $V^{\vee}_{m,3} = V^{\wedge}_{m,3}$. If

$$y = Z_1 f^*_4(1;\nu) + Z_2 f^*_6(1;\nu) \tag{95}$$

with $\{Z_1, Z_2\} \subset \mathbb{C}$, we let

(96) $$p^{\vee}(y) = (\max(|Z_1|, |Z_2|).$$

It is well known that function $\omega_{n,y}(m)$ (with variable $y \in V_m$) is a norm on the space V_m. We denote by $p^{\wedge}(y)$ the restriction of the norm $\omega_{n,y}(m)$ on the space $V^{\vee}_{m,3} = V^{\wedge}_{m,3}$. So, on the two-dimensional space $V^{\vee}_{m,3} = V^{\wedge}_{m,3}$ we have two norms $p^{\vee}(y)$ and $p^{\wedge}(y)$. Therefore there exists a constant $C_1 > 0$ such that

(97) $$p^{\vee}(y) \le C_1 p^{\wedge}(y),\ C_1 p^{\vee}(y) \ge p^{\wedge}(y)$$

for any $y \in V^{\vee}_{m,3} = V^{\wedge}_{m,3}$. In view of (93) – (94) and (97),

(98) $$(C_0 C_1))^{-1}(\rho_3 \exp(-\varepsilon))^{\nu} p^{\vee}(y) \le \omega_{n,y}(\nu),$$

(99) $$\omega_{n,y}(\nu) \le C_0 C_1 (\rho_3 \exp(\varepsilon))^{\nu} p^{\vee}(y).$$

In view of (94),

(100) $$|\alpha^*_k(1;\nu)| \le O(1)(\rho_1 \exp(\varepsilon))^{\nu}$$

where $k = 0, 1, 3, 4, 5\,, \nu \in \mathbb{N}$ and $O(1)$ depends only from ε.

Let $\{m,\, n\} \subset \mathbb{N}$,

$$a_{i,k} \in \mathbb{R}$$

for $i = 1\,, \dots\,, m,\ k = 1\,, \dots\,, n$,

$$\alpha^{\wedge}_j(\nu) \in \mathbb{Z}$$

where $j = 1\,, \dots\,, m+n$ and $\nu \in \mathbb{N}$. Let there are $\gamma^{\wedge}_0, r^{\wedge}_1 \ge 1,\ \dots\,,\ r^{\wedge}_m \ge 1$ such that

(101) $$|\alpha_i(\nu)| < \gamma^{\wedge}_0 (r^{\wedge}_i)^{\nu}$$

where $i = 1\,, \dots\,, m$ and $\nu \in \mathbb{N}$. Let

$$y_k(\nu) = -\alpha^{\wedge}_{m+k}(\nu) + \sum_{i=1}^{m} a_{i,k}\alpha^{\wedge}_i(\nu)$$

where $k = 1\,, \dots\,, n$ and $\nu \in \mathbb{N}$. If

(102) $$X = \begin{pmatrix} Z_1 \\ \vdots \\ Z_n \end{pmatrix} \in \mathbb{R}^n,$$

then let

(103) $$q_{\infty}(X) = \max(|Z_1|\,, \dots\,, |Z_n|),$$

$$y^{\wedge}(X) = y^{\wedge}(X, \nu) = \sum_{k=1}^{n} y^{\wedge}_k(\nu) Z_k$$

for $\nu \in \mathbb{N}$, let

$$\phi_i(X) = \sum_{k=1}^{n} a_{i,k} Z_k$$

for $i = 1, \ldots, m$, and let

$$\alpha_0^\wedge(X, \nu) = \sum_{k=1}^{n} \alpha_{m+k}^\wedge(\nu) Z_k$$

for $\nu \in \mathbb{N}$. Clearly,

$$y^\wedge(X, \nu) = -\alpha_0^\wedge(X, \nu) + \sum_{i=1}^{m} \alpha_i^\wedge(\nu)\phi_i(X)$$

for $X \in \mathfrak{R}^n$ and $\nu \in \mathbb{N}$,

$$\alpha_0^\wedge(X, \nu) \in \mathbb{Z}$$

for $X \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}$.

Lemma 7.7. *Let* $\{l,\, n\} \subset \mathbb{N}$, $\gamma_1^\wedge > 0$, $\gamma_2^\wedge > \frac{1}{2}$, $R_1 \geq R_2 > 1$,

(104) $$\alpha_i = (\log(r_i^\wedge R_1/R_2))/\log(R_2),$$

where $i = 1, \ldots, m$, *let* $X \in \mathbb{Z}^n \backslash \{0\}$,

$$\gamma_3^\wedge = \gamma_1^\wedge (R_1)^{(-\log(2\gamma_2 R_2))/\log(R_2)}, \gamma_4^\wedge = \gamma_3^\wedge \left(\sum_{i=1}^{m} \gamma_0 (r_i^\wedge)^{(log(2\gamma_2^\wedge))/\log(R_2)+l}\right)^{-1}$$

and let for each $\nu \in \mathbb{N} - 1$ *hold the inequalities*

(105) $$\gamma_1^\wedge (R_1)^{-\nu} q_\infty(X) \leq$$
$$\sup\{|y^\wedge(X, \kappa)| \colon \kappa = \nu,\, \ldots,\, \nu + l - 1\},$$

(106) $$|y^\wedge(X, \nu)| \leq \gamma_2^\wedge (R_2)^{-\nu} q_\infty(X)$$

Then

(107) $$\sup\{\|\phi_i(X)\|(q_\infty^{(\mathfrak{K})}(X))^{\alpha_i} \colon i = 1,\, \ldots,\, m\} \geq \gamma_4^\wedge.$$

Proof. Proof may be found in [66], Theorem 2.3.1. ■

In our case

$$m = n = 2\,, a_{1,1} = 2\zeta(3), a_{1,2} = a_{2,1} = 3\zeta(4), a_{2,2} = 6\zeta(5),$$

$$\alpha_1^\wedge(\nu) = (D^{**}(\nu))^5/(d_1^*(\nu)^3))\beta^*(1, \nu),$$
$$\alpha_2^\wedge(\nu) = (D^{**}(\nu))^5/(d_1^*(\nu)^3))\alpha^*(1, \nu),$$
$$\alpha_3^\wedge(\nu) = (D^{**}(\nu))^5/(d_1^*(\nu)^3))\psi^*(1, \nu),$$
$$\alpha_4^\wedge(\nu) = (D^{**}(\nu))^5/(d_1^*(\nu)^3))\xi^*(1, \nu).$$

According to the Lemma 7.5,

$$\alpha_j^\wedge(\nu) \in \mathbb{Z},$$

where $j = 1, \ldots, 4$ and $\nu \in \mathbb{N}$. In view of (100) and (45),

$$|\alpha_j^{\wedge}(\nu)| \le (\rho_1 \exp(\varepsilon))^{\nu} \exp(\nu(\omega_2(13) + \varepsilon) + O(1)) = \tag{108}$$

$$\exp((\gamma_0^*(13) + 2\varepsilon))\nu + O(1),$$

where $j = 1, \ldots, 4, \nu \in \mathbb{N}$ and $O(1)$ depends only from ε.

So, there exists $\gamma_0 > 0$ such that (101) holds with $r_j^{\wedge}$ defined in (72).

The role of $y_k(\nu)$ of the Lemma 7.7 now play

$$y_k(\nu) = (D^{**}(\nu))^5/(d_1^*(\nu)^3))f_{2+2k}^*(1;\nu),$$

where $k = 1, 2, \nu \in m + \mathbb{N}$ with m defined in the Lemma 8.1.

In view of (98), (99) and (45),

$$\exp((\gamma_3^*(13) - 2\varepsilon))\nu + O(1))p^{\vee}(y) = \tag{109}$$

$$(\rho_3 \exp(-\varepsilon))^{\nu} \exp(\nu(\omega_2(13) - \varepsilon) + O(1))p^{\vee}(y) \le \omega_{2,y_k}(\nu),$$

$$\omega_{2,y_k}(\nu) \le \tag{110}$$

$$(\rho_3 \exp(\varepsilon))^{\nu} \exp(\nu(\omega_2(13) + \varepsilon) + O(1))p^{\vee}(y) =$$

$$\exp((\gamma_3^*(13) + 2\varepsilon))\nu + O(1))p^{\vee}(y),$$

where $O(1)$ depends only from ε, $k = 1, 2, , \nu \in m + \mathbb{N}$ with m defined in the Lemma 7.6; moreover in view of (95), (96) and (102),(103) with $n = 2$,

$$p^{\vee}(y) = q_{\infty}(X).$$

So, there exist $\gamma_1^{\wedge} > 0$, $\gamma_2^{\wedge} > 0$ such that the relations (105), (106) hold with R_1, R_2 defined in (70) – (71), with $l = n = 5$ and with $\nu \in m + \mathbb{N}$, where m is defined in the Lemma 7.6 (of course, the condition $\nu \in m + \mathbb{N}$ is not essential, because of the substitution $\nu := \nu - m - 1$ influences only on constants $\gamma_0^{\wedge}, \gamma_1^{\wedge}, \gamma_2^{\wedge}, \gamma_3^{\wedge}$ and $\gamma_4^{\wedge}$). So, all the conditions of the lemma 7.7 are fullfilled. Therefore (107) holds, and, in view of (104) and (74),

$$\alpha_i = h_0(\varepsilon) < \gamma,$$

where $i = 1, 2$.■

§2. Verification of the previous calculations on computer and the values of $\gamma^*(\Delta)$ for other Δ.

Let $Z \in \mathbb{C} \diagdown (-\infty, 0], Z = |Z|(\cos(\alpha) + i\sin(\alpha))$ with $|\alpha| < \pi$. Then

$$w(Z) = ((|Z| + Z)/2)/((|Z| + \Re(Z))/2)^{1/2} =$$

$$(|Z|\cos(\alpha/2) + i\sin(\alpha/2))/(|Z|^{1/2})$$

and $w(Z)^2 = Z$. Let

$$Z = X + iY = 1 - (1 - \delta_0^2)/(2z - 1)^2, \rho = ((|Z| + \Re(Z))/2)^{1/2}.$$

Then

$$|Z| = 2\rho^2 - X\,,\, w = w(Z) = \rho + i\frac{Y}{2\rho},\ \Re(w(Z)) > 0. \tag{111}$$

We put in (3.1.1) of the paper [62] $\eta = \lambda(2z-1)/(1+\delta_0)$, where $z \neq 1/2$. Then, since $z = -r\exp(i\psi)$, it follows that

$$D^{\vee}(z,\eta) = (\eta+1)(\eta+\gamma_1) + 2r\exp(i\psi)(1+\gamma_1)\eta =$$
$$\eta^2 - 2(2z-1)\eta/(1+\delta_0) + \gamma_1 =$$
$$((2z-1)/(1+\delta_0))^2(\lambda^2 - 2\lambda + (1-\delta_0^2)/(2z-1)^2).$$

Clearly,

$$1 - (1-\delta_0^2)/(2z-1)^2) \notin (-\infty, 0]$$

if and only if

$$z \notin \left(\frac{1}{2} - \frac{1}{2\sqrt{1-\delta_0^2}}\,,\frac{1}{2} + \frac{1}{2\sqrt{1-\delta_0^2}}\right),$$

what we suppose further in conection with calculations of

$$|h^{\sim}(\eta_k(1,\pi/3,\delta_0))|$$

for $k = 0,1$. Let $W = w(1-(1-\delta_0^2)/(2z-1)^2)$. Then

$$1/(2z-1)^2 = (1-W^2)/(1-\delta_0^2),$$

and

$$\eta_k^{\wedge}(r,\psi,\delta_0) = \frac{2z-1}{1+\delta_0}(1+(-1)^kW) =$$
$$\frac{1-\delta_0}{2z-1}(1-(-1)^kW)^{-1},$$
$$\eta_k^{\wedge}(r,\psi,\delta_0)^2 = \frac{(2z-1)^2}{(1+\delta_0)^2}(1+(-1)^kW)^2 =$$
$$\frac{(1-\delta_0)^2}{(2z-1)^2}(1-(-1)^kW)^{-2} =$$
$$\frac{1-\delta_0}{1+\delta_0}\frac{1+(-1)^kW}{1-(-1)^kW} =$$
$$\frac{1-(-1)^k\delta_0W + (-1)^kW - \delta_0}{1-(-1)^k\delta_0W - (-1)^kW + \delta_0},$$
$$1 - \eta_k^{\wedge}(r,\psi,\delta_0)^2 = 2\frac{\delta_0 - (-1)^kW}{(1+\delta)(1-(-1)^kW)} =$$
$$2(2z-1)^2(1+(-1)^kW)\frac{\delta_0-(-1)^kW}{(1+\delta)^2(1-\delta_0)}.$$

Let

$$A_k|1+(-1)^kW|^2 = 1 + |Z| + 2(-1)^k\Re(W) =$$
$$1 + |Z| + 2(-1)^k\rho = (1+(-1)^k\rho)^2 + \rho^2 - X,$$

$$B_k = |\delta_0 - (-1)^k W|^2 = \delta_0^2 + |Z| - 2(-1)^k \delta_0 \Re(W) =$$
$$\delta_0^2 + |Z| - 2(-1)^k \delta_0 \rho = (\delta_0 - (-1)^k \rho)^2 + \rho^2 - X,$$

where $k = 0$, 1. Then

$$|h^{\sim}(\eta_k(1, \pi/3, 1/\Delta))| = 2^{-6}(1-\delta_0)^{3-3\Delta} \times$$
$$\left(\frac{|2z-1|^2}{(1+\delta_0)^2} A_k\right)^{(3\Delta-3)/2} \times$$
$$2^3 |(2z-1)^2|^3 A_k^{3/2} \frac{B_k^{3/2}}{(1+\delta)^6 (1-\delta_0)^3} =$$
$$2^{-3}(1-\delta_0^2)^{-3\Delta}(|2z-1|^2)^{(3\Delta+3)/2}(1+\delta_0)^{-3} A_k^{3\Delta/2} B_k^{3/2}.$$

In our case

$$z = -\exp(\pi i/3),\ 2z - 1 = -2 - i\sqrt{3},\ |2z-1|^2 = 7,$$

$$Z = X + iY = 1 - \frac{\Delta^2 - 1}{49\Delta^2}(1 - 4i\sqrt{3}) =$$
$$\frac{48\Delta^2 + 1}{49\Delta^2} + i\frac{4\Delta^2 - 4}{49\Delta^2}\sqrt{3},$$
$$|Z| = (48\Delta^4 + 1)^{1/2}/(7\Delta^2),$$
$$X = \frac{48\Delta^2 + 1}{49\Delta^2}, \rho = ((|Z| + X)/2)^{1/2} =$$
$$(7(48\Delta^4 + 1)^{1/2} + 48\Delta^2 + 1)/98)^{1/2}/\Delta.$$

Let

$$\omega_k(\Delta) = \ln|h^{\sim}(\eta_k(1, \pi/3, 1/\Delta))|) =$$
$$-3\ln(2) + \frac{3\Delta + 3}{2}\ln(7) + 3\Delta\ln(\Delta^2/(\Delta^2 - 1)) -$$
$$3\Delta\ln((\Delta+1)/\Delta) + \frac{3\Delta}{2}\ln(A_k) + \frac{3}{2}\ln(B_k),$$

where $k = 0, 1$. In view of (58), (62) and (65),

$$\gamma_k^*(\Delta) = (-1)^k \omega_k(\Delta) + (-1)^k \omega_2(\Delta),$$

$$\gamma^*(\Delta) = (\omega_0(\Delta) + \omega_2(\Delta)/(\omega_1(\Delta) - \omega_2(\Delta)).$$

The following program (written in Turbo Pascal) was used in [58] for calculations:

```
Program Fz5hh;
label 1,2;
var k,b:shortint;Delta:real {Delta = Δ};
absZ,rho,A0,A1,B0,B1,om2,omh,omhh,om1,om0,s0,s1,mu:real
```

$\{absZ = |Z|, rho = \rho, A0 = A_0, A1 = A_1,\}$
$\{B0 = B_0, B1 = B_1, om2 = \omega_2, om1 = \omega_2, om0 = \omega_0\}$
$\{s0 = \sum_{1 \le \kappa \le (\Delta-1)/2} \cot(\pi\kappa/(\Delta-1))\}$

$\{s1 = \sum\limits_{1 \leq \kappa \leq (\Delta+1)/2} \cot(\pi\kappa/(\Delta+1))\}$
$\{mu = \gamma^*(\Delta)\}$;
begin;
1: writeln('continue?,yes=1,no=0');
readln (b);
if $(b = 0)$ then goto 2;
writeln('Delta=?');
readln(Delta);
$absZ := sqrt(48 * sqr(sqr(Delta)) + 1)/(7 * sqr(Delta))$;
$rho := sqrt((7 * sqrt(48 * sqr(sqr(Delta)) + 1)+$
$48 * sqr(Delta) + 1)/98)/Delta$;
$A0 := 1 + absZ + 2 * rho$;
$A1 := 1 + absZ - 2 * rho$;
$B0 := 1/sqr(Delta) + absZ - 2 * rho/Delta$;
$B1 := 1/sqr(Delta) + absZ + 2 * rho/Delta$;
$s0 := 0$;
$k := 1$;
while $(k < (Delta - 1)/2)$ do begin
$s0 := s0 + \cos(pi * k/(Delta - 1))/\sin(pi * k/(Delta - 1))$;
$k := k + 1$;
end;
$s1 := 0$;
$k := 1$;
while $(k < (Delta + 1)/2)$ do begin
$s1 := s1 + \cos(pi * k/(Delta + 1))/\sin(pi * k/(Delta + 1))$;
$k := k + 1$;
end;
$om2 := 5 * (Delta + 1) - (3/2) * (Delta - 1) * ln(Delta/(Delta - 1))+$
$(3/2) * (Delta + 1) * ln((Delta + 1)/Delta) + 3 * pi * s0/2 - 3 * pi * s1/2$;
$om1 := -3 * ln(2) + ((3 * Delta + 3)/2) * ln(7)+$
$3 * Delta * ln(sqr(Delta)/(sqr(Delta) - 1))-$
$3 * ln((Delta + 1)/Delta) + (3 * Delta/2) * ln(A1) + (3/2) * ln(B1)$;
$om0 := -3 * ln(2) + ((3 * Delta + 3)/2) * ln(7)+$
$3 * Delta * ln(sqr(Delta)/(sqr(Delta) - 1))-$
$3 * ln((Delta + 1)/Delta) + (3 * Delta/2) * ln(A0) + (3/2) * ln(B0)$;
$mu := -(om0 + om2)/(om1 + om2)$;
writeln('Delta=',Delta,',','mu=',mu);
goto 1;
2:
end.

(It is not necessary to know Turbo Pascal for the use of this program; it is sufficient to rewrite it in MS-DOS format (not in TeX), to place it in the directory of Turbo Pascal, give a command (in command line) turbo fz5hh and then hit Ctrl+F9).

In this program $mu = \gamma^*(\Delta)$.

The following results are obtained:

$$\Delta = 2,\ mu = -5,69729529498727...,$$

$$\Delta = 3,\ mu = -18,4506653923600...,$$
$$\Delta = 4,\ mu = -42,0868798934971...,$$
$$\Delta = 5,\ mu = -3766,31419552490...,$$
$$\Delta = 6,\ mu = 128,630462898873...,$$
$$\Delta = 7,\ mu = 65,78503337027339...,$$
$$\Delta = 8,\ mu = 56,0423180866055...,$$
$$\Delta = 9,\ mu = 48,4350925860344...,$$
$$\Delta = 10,\ mu = 46,5570123894140...,$$
$$\Delta = 11,\ mu = 44,3334143236862...,$$
$$\Delta = 12,\ mu = 44,1044453638024...,$$
$$\Delta = 13,\ mu = 43,46441140066032...,$$
$$\Delta = 14,\ mu = 43,8257111129351...,$$
$$\Delta = 15,\ mu = 43,8522467170842...,$$
$$\Delta = 16,\ mu = 44,4821306835511...,$$
$$\Delta = 17,\ mu = 44,8464590454823...,$$
$$\Delta = 18,\ mu = 45,6162272955407...,$$
$$\Delta = 19,\ mu = 46,61723601535778...,$$
$$\Delta = 20,\ mu = 47,0213561055716....$$

We see that $\Delta = 6$ is the first Δ with $\gamma^*(\Delta) > 0$.

§3. Epilogue.

Much water has flown under the bridges since I announced the above results for the first time. During those eight years, new publications in this subject have appeared. First of all, I wish to refer to the deep research by Rhin and Viola [33]; let me also cite the very interesting papers by Hata [34], by Boll and Rivoal [36], and by Zudilin [37], [38].

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