RANDOM DISTANCE EXPANDING MAPPINGS, THERMODYNAMIC FORMALISM, GIBBS MEASURES, AND FRACTAL GEOMETRY

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Contents

1. Introduction

In this paper we define random distance expanding dynamical systems. We develop the appropriate thermodynamic formalism of such systems. We obtain in particular the existence and uniqueness of invariant Gibbs states, the appropriate pressure function and exponentially fast convergence of iterates of Perron-Frobenius operators resulting, in particular, in an exponential decay of correlations. We also obtain the formula for the derivative of the expected value of the pressure function. Next, we define and investigate in detail conformal random expanding repellers. Applying the developed machinery of thermodynamic formalism we prove a version of Bowen's formula (obtained in a somewhat different context in [4]) which identifies the Hausdorff dimension of almost all fibers with the only zero of expected value of the pressure function. We then turn to more refined fractal properties. We move into two directions.

Firstly, we investigate the h-dimensional Hausdorff and packing measures of fibers, h being their Hausdorff dimension. We introduce the (rather generic) class of essential random conformal Gibbs-systems (G-systems) and the (rather exceptional) class of quasi-deterministic G-systems. We show that unlike in the classical deterministic case (see for ex. [13]) for essential random conformal G-systems the h-dimensional Hausdorff measure vanishes while the packing measure is infinite. This in particular refutes the conjecture stated in [4] that the h-dimensional Hausdorff measure of fibers is always positive and finite. This fact has further striking geometrical consequences. Namely, all fibers of an essential random conformal G-system are not bi-Lipschitz equivalent to any fiber of any quasi-deterministic system, in particular any deterministic system. In particular, almost every fiber of an essential random system is not a geometric circle not even a piecewise analytic curve. For deterministic systems circles are possible. The quasi-deterministic systems behave, on the other hand, more like deterministic ones. The Hausdorff and packing measures are positive and finite and the Hausdorff dimension of all fibers is the same.

The second direction concerns the multifractal spectrum of Gibbs measures on fibers. We show that the multifractal formalism is valid, i.e. the multifractal spectrum is given by the Legendre transform of a temperature function and it is continuous.

Unlike [3], [8] and other our considerations are restricted neither to symbol systems nor to expanding maps on full smooth manifolds. Our concept of distance expanding random dynamical systems comprises both of them and includes an abundance of systems that are neither of them. We do not need any Markov partitions or (even auxiliary) symbol dynamics to carry on our investigations.

Unlike recent trends to employ the method of Hilbert metric and Birkhoff's cones (like for example in [7, 9, 14]), our approach stems from a more classical method presented in [5] and undertaken in [8]. Developing it in the context of random dynamical systems we demonstrate that it works well and does not lead to too complicated (at least to our taste) technicalities.

Throughout the paper we were paying special attention to the measurability issues. This concerns particularly the conformal measures which are one of the main objects of the thermodynamical formalism. In the RDS setting they are given by a family of conditional measures ν_x, $x \in X$ the base space of the RDS. These measures are produced in a pointwise manner and measurability of $x \mapsto \nu_x$ is not immediately obvious. As a matter of fact measurability was not clear to us even in the symbol of full manifold expanding case until very recently Kifer's paper [9] has arrived to take care of these cases.

Last but not least, throughout the entire paper we were avoiding, in hypothesis, absolute constants as much as possible. Our feeling is that in the context of random systems all (or at least as many as possible) absolute constants appearing in deterministic systems should become measurable functions, possibly with some integrability conditions. With this respect the thermodynamic formalism developed in our paper represents also, up to our knowledge, new achievements in the theory of random symbol dynamics or random expanding maps on full manifolds.

2. Generalities and Settings on RDS

Suppose $(X, \mathcal{F}, m, \theta)$ is a measure preserving dynamical system with invertible and ergodic map $\theta : X \to X$ which is refered to as the base map. Suppose also (Y, ϱ) is a Polish space which we assumed to be normalized so that $diam(Y) = 1$. To the space Y there is always associated its (most natural) Borel σ–algebra $\mathcal{B}_Y$ of subsets of Y. Let $\mathcal{K}_Y$ be the space space of all compact subsets of Y topologized by the Hausdorff metric. Assume further that that a measurable mapping $X \ni x \mapsto \mathcal{J}_x \in \mathcal{K}_Y$ is given. Let

$$\mathcal{J} = \bigcup_{x \in X} \{x\} \times \mathcal{J}_x \ . \tag{2.1}$$

It follows then from [6] that $\mathcal{J}$ is measurable in the sense that $\mathcal{J} \in \mathcal{F} \otimes \mathcal{B}_Y$, the product σ-algebra of $\mathcal{F}$ and $\mathcal{B}_Y$. Set $\mathcal{A} = \mathcal{F} \otimes \mathcal{B}_Y|_{\mathcal{J}}$. Suppose finially that for every $x \in X$ a continuous mapping

$$T_x : \mathcal{J}_x \to \mathcal{J}_{S(x)}$$

is defined. We consider the skew-product map $T : \mathcal{J} \to \mathcal{J}$ defined by the formula

$$T(x, z) = (\theta(x), T_x(z)). \tag{2.2}$$

If T is measurable, it is called a bundle random dynamical system over the base $\theta : X \to X$. For every $n \geq 0$ we define the map

$$T_x^n := T_{\theta^{n-1}(x)} \circ ... \circ T_x : \mathcal{J}_x \to \mathcal{J}_{S^n(x)}.$$

Before proceeding to measurably expanding RDS, which are our main objects of interest in this paper, and in order to fix ideas, let us look at some examples. Keep $\theta : X \to X$, a measure preserving ergodic dynamical system and for every $x \in X$ consider a rational function $F_x : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. As in the deterministic case, the dynamics of $(F_x^n)_n$ splits the sphere into two parts, the *Fatou set* which is the part one which this family behaves normal

$$\mathcal{F}_x = \left\{ z \in \hat{\mathbb{C}} : (F_x^n)_n \text{ is normal on some neighborhood of } z \right\}$$

and its complement the *Julia set* $\mathcal{J}_x = \hat{\mathbb{C}} \setminus \mathcal{F}_x$. This set carries on the chaotic part of the dynamics, the part we are interested in. From these definitions we have that $F_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$ for every $x \in X$. Setting then $T_x = F_x|_{\mathcal{J}_x}$, the global skew-product map $T : \mathcal{J} \to \mathcal{J}$ is a bundle RDS provided it is measurable.

Occasionally, but by no means always, $\theta : X \to X$ can be just a two-sided shift map and the invariant measure m a Bernoulli measure. If the maps T_x depend then only on the 0-th coordinate then the iterates T_x^n, $n \in \mathbb{Z}$, form a sequence of independent random variables. More examples, all already conformal, can be found in Section 8. General references on RDS are the book by L. Arnold [1] and the survey articles [10, 11].

Throughout the paper we will consider two types of bundle random dynamical systems, namely measurably expanding and conformal measurably expanding ones. We define the former now.

Definition 2.1. *Let $T : \mathcal{J} \to \mathcal{J}$ be a measurable skew-product (bundle RDS) defined as in (2.2), i.e. by the formula*

$$T(x, z) = (\theta(x), T_x(z))$$

and suppose that the bundle mappings $T_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$ are continuous open and surjective. This RDS is measurably expanding if for some $\xi, \eta > 0$, the following conditions hold:

- *Uniform Openess,*
- *Measurably Expanding,*
- *Log-integrability of the Degree,*
- *Topological exactness,*
- *Measurability of the Transfer Operator.*

These conditions are precisely defined below. If in addition Y is a smooth Riemannian manifold and the mappings T_x are restrictions of $\mathcal{C}^{1+\alpha}$–conformal mappings, then the system is called conformal measurably expanding.

The first part of the paper and, in particular, the whole thermodynamical formalism is developed for general measurable expanding RDS. The geometric applications start with Section 6 and they are done for conformal RDS. Let us now precisely define the first four conditions; the measurability of the transfer operator will be given in Section 3.3 and details on conformal systems are at the beginning of Section 6.

As usual, we will denote by $B(z,r)$ a ball in (Y,ϱ) centered at z and of radius r and by $B_x(z,r) = B(z,r) \cap J_x$ the corresponding ball in the subspace J_x.

Uniform Openess 2.2. $T_x(B_x(z,\eta)) \supset B_{\theta(x)}\big(T_x(z), \xi\big)$ *for every* $(x,z) \in \mathcal{J}$.

Measurably Expanding 2.3. *There exists a measurable function* $\gamma : X \to (1,+\infty)$, $x \mapsto \gamma_x$, *with* $\log \gamma \in L^1(m)$ *and such that for a.e.* $x \in X$

$$\varrho(T_x(z_1), T_x(z_2)) \geq \gamma_x \varrho(z_1, z_2) \quad \textit{whenever} \quad \varrho(z_1,z_2) < \eta,\ z_1, z_2 \in \mathcal{J}_x.$$

Put

$$\chi = \int_X \log \gamma dm > 0. \tag{2.3}$$

Note that condition (2.3) implies that $T_x|_{B(z,\eta)}$ is one–to–one for every $(x,z) \in \mathcal{J}$. Together with the compacity of the spaces $\mathcal{J}_x$ it follows that the numbers

$$deg(T_x^j) := \max_{y \in \mathcal{J}_{x_j}} \# T_x^{-j}(\{y\}) \quad , \quad j \geq 1 \ ,$$

are finite. Here and in the following $x_j = \theta^j(x)$, $j \in \mathbb{Z}$. Concerning the degree, we only assume the following.

Log–integrability of the Degree 2.4. *For every $j \geq 1$, the function $x \mapsto deg(T_x^j)$ is measurable and $\log deg(T_x^j) \in L^1(m)$.*

Topological Exactness 2.5. *There exists a measurable function $x \mapsto n_r(x)$ such that for every $r > 0$ and a.e. $x \in X$*

$$T_x^{n_r(x)}(B_x(z,r)) = \mathcal{J}_{S^{n_r(x)}(x)} \quad \textit{for every } z \in \mathcal{J}_x \tag{2.4}$$

and there exists a function $j \in L^1(m)$ such that for almost every $x \in X$

$$T^{j(x)}_{x_{-j(x)}}(\mathcal{B}(\xi)) = \mathcal{J}_x \quad , \quad \mathcal{B}(\xi) \ \textit{being any ball of radius}\ \xi \ \textit{of}\ \mathcal{J}_{x_{-j(x)}} \ . \tag{2.5}$$

For example in the deterministic case and in the context of rational dynamics the two preceeding conditons immediately follow from Montel's theorem.

Here are some general remarks. If it does not lead to misunderstanding we will frequently identify $\mathcal{J}_x$ and $\{x\} \times \mathcal{J}_x$. By T^n we denote the n–th composition of T

and by T_x^n its composition with the projection to the second coordinate. We also define

$$B(y,\xi)=B_x(z,\xi) \quad for \;\; y=(x,z)\in\mathcal{J}\ .$$

Conditions (2.2) and (2.3) imply that, for every $y=(x,z)\in\mathcal{J}$ there exists a unique continuous inverse branch $T_y^{-1}:B(T(y),\xi)\to B(y,\eta)$ of T_x sending $T_x(z)$ to z. By (2.3) we have

$$\varrho(T_y^{-1}(z_1),T_y^{-1}(z_2))\le\gamma_x^{-1}\varrho(z_1,z_2) \tag{2.6}$$

for $z_1,z_2\in B(T(y),\xi)$, and

$$T_y^{-1}(B(T(y),\xi))\subset B(y,\gamma_x^{-1}\xi)\subset B(y,\xi). \tag{2.7}$$

Hence, for every $n\ge 0$, the composition

$$T_y^{-n}=T_y^{-1}\circ T_{T(y)}^{-1}\circ\ldots\circ T_{T^{n-1}(y)}^{-1}:B(T^n(y),\xi)\to B(y,\xi) \tag{2.8}$$

is well-defined and has the following properties. The function $T_y^{-n}:B(T^n(y),\xi)\to B(y,\xi)$ is continuous, $T^n\circ T_y^{-n}=\mathrm{Id}_{B(T^n(y),\xi)}$, $T_y^{-n}(T_x^n(z))=z$ and

$$\varrho(T_y^{-n}(z_1),T_y^{-n}(z_2))\le(\gamma_x^n)^{-1}\varrho(z_1,z_2) \tag{2.9}$$

for $z_1,z_2\in B(T^n(y),\xi)$, where $\gamma_x^n=\gamma_x\gamma_{\theta(x)}\cdots\gamma_{\theta^{n-1}(x)}$. Moreover,

$$T_x^{-n}(B(T^n(y),\xi))\subset B(y,(\gamma_x^n)^{-1}\xi)\subset B(y,\xi). \tag{2.10}$$

3. Potentials and transfer operator

3.1. **Spaces of continuous functions and measures adapted to bundle RDS.** Generalities on the following function spaces can be found in Bogenschütz [2] and in Crauel's book [6].

We denote $\mathcal{C}_m^0(\mathcal{J})$ the space of measurable mappings $x\mapsto g_x$ with $g_{x|\mathcal{J}_x}$ continuous. In the literature this space is also denoted by $L^0(X,\mathcal{C}(Y))$. It contains the subspace $\mathcal{C}_m^1(\mathcal{J})$ (often denoted $L^1(X,\mathcal{C}(Y))$) which is the set of functions $g\in\mathcal{C}_m^0(\mathcal{J})$ for which the integral

$$\|g\|_1:=\int_X\|g_x\|_\infty\,dm(x)<\infty.$$

Equipped with the norm $\|.\|_1$, $\mathcal{C}_m^1(\mathcal{J})$ is a Banach space.

Let $\mathcal{M}(Y)$ be the space of finite (Radon) measures on Y and set

$$L^\infty(X,\mathcal{M}(Y))=\{\nu_*:X\to\mathcal{M}(Y)\,,x\mapsto\nu_x\ with\ ess\,sup\,|\nu_*|<\infty\}.$$

The later space can be identified with the dual of $\mathcal{C}_m^1(\mathcal{J})$ with duality

$$<\nu,g>=\int_X v_x(g_x)\,dm(x)=\int_X\int_{J_x}g_x\,d\nu_x\,dm(x)$$

and $\mathcal{M}_m^1(\mathcal{J})$, the space of probabilities on $\mathcal{J}$ with marginal m (or, in other words, such that the projection of ν on X equals m), can be identified with the weak unit sphere of $L^\infty(X,\mathcal{M}(Y))$. It follows then from Alaoglu's theorem that $\mathcal{M}_m^1(\mathcal{J})$ is compact with respect to the weak topology coming from the duality described above. This topology is also called the narrow topology.

It is important to have in mind that, by disintegration, a measure $\nu\in\mathcal{M}_m^1(\mathcal{J})$ if and only if there is a family of probabilities $\nu_x\in\mathcal{M}^1(\mathcal{J}_x)$ depending measurably on x, i.e. the map $x\mapsto\int_{\mathcal{J}_x}g_x\,d\nu_x$ is measurable from X to $\mathbb{R}$ for every $g\in L^1(X,\mathcal{C}(Y))$, and such that $d\nu(y)=d\nu_x(z)dm(x)$. The family $(\nu_x)_{x\in X}$ is the canonical system of conditional measures of ν with respect to the measurable partition $\{\mathcal{J}_x\}_{x\in X}$ of $\mathcal{J}$.

Let us also mention that convergence with respect to the narrow topology does not imply convergence of the conditional measures. A counter–example can be found in [6].

3.2. **Hölder spaces and distortion properties.** Fix $\alpha \in (0,1]$. By $\mathcal{H}^\alpha(\mathcal{J}_x)$ we denote denote the space of Hölder continuous functions on $\mathcal{J}_x$ with an exponent α. This means that $\varphi_x \in \mathcal{H}^\alpha(\mathcal{J}_x)$ if and only if $\varphi_x \in C(\mathcal{J}_x)$ and $v(\varphi_x) < \infty$ where

$$v_\alpha(\varphi_x) := \inf\{H_x : |\varphi(z_1) - \varphi(z_2)| \le H_x \varrho_x^\alpha(z_1, z_2)\}.$$

where the infinimum is taken over all $z_1, z_2 \in \mathcal{J}_x$ with $\varrho(z_1, z_2) \le \eta$.

Definition 3.1. *A function $\varphi \in C_m^1(\mathcal{J})$ is called Hölder continuous with an exponent α provided that there exists a measurable function $H : X \to [1, +\infty)$, $x \mapsto H_x$, such that $\log H \in L^1(m)$ and such that $v_\alpha(\varphi_x) \le H_x$ for a.e. $x \in X$. We denote the space of all Hölder functions with fixed α and H by $\mathcal{H}_m^\alpha(\mathcal{J}, H)$, the vector space of all α–Hölder functions by $\mathcal{H}_m^\alpha(\mathcal{J}) = \bigcup_H \mathcal{H}_m^\alpha(\mathcal{J}, H)$ and, finally, the space of all Hölder functions by $\mathcal{H}_m(J) = \bigcup_{\alpha>0} \mathcal{H}_m^\alpha(J)$.*

For every function $g : \mathcal{J} \to \mathbb{C}$ and a.e $x \in X$ let

$$S_n g_x = \sum_{j=0}^{n-1} g_x \circ T_x^j, \tag{3.1}$$

and, if $g : X \to \mathbb{C}$, then $S_n g = \sum_{j=0}^{n-1} g \circ \theta^j$.

Lemma 3.2. *Let $\varphi \in \mathcal{H}_m^\alpha(\mathcal{J}, H)$. If $n \ge 1$, $y = (x, z) \in \mathcal{J}$ and $w_1, w_2 \in B(T_x^n(z), \xi)$, then*

$$|S_n\varphi_x(T_y^{-n}(w_1)) - S_n\varphi_x(T_y^{-n}(w_2))| \le \varrho^\alpha(w_1, w_2) \sum_{j=0}^{n-1} H_{\theta^j(x)} (\gamma_{\theta^j(x)}^{n-j})^{-\alpha}.$$

Proof. We have by (2.9) and Hölder continuity of φ that

$$\begin{aligned}
|S_n\varphi_x(T_y^{-n}(w_1)) - S_n\varphi_x(T_y^{-n}(w_2))| &\le \sum_{j=0}^{n-1} |\varphi_x(T_x^j(T_y^{-n}(w_1))) - \varphi_x(T_x^j(T_y^{-n}(w_2)))| \\
&= \sum_{j=0}^{n-1} \left| \varphi_x(T_{T_x^j(y)}^{-(n-j)}(w_1)) - \varphi_x(T_{T_x^j(y)}^{-(n-j)}(w_2)) \right| \\
&\le \sum_{j=0}^{n-1} \varrho^\alpha\big(T_{T_x^j(x)}^{-(n-j)}(w_1), T_{T_x^j(x)}^{-(n-j)}(w_2)\big) H_{\theta^j(x)} \\
&\le \varrho^\alpha(w_1, w_2) \sum_{j=0}^{n-1} H_{\theta^j(x)} (\gamma_{\theta^j(x)}^{n-j})^{-\alpha}.
\end{aligned}$$

□

For almost every $x \in X$ set

$$Q_x = Q_{x,H} = \sum_{j=1}^{\infty} H_{\theta^{-j}(x)} (\gamma_{\theta^{-j}(x)}^j)^{-\alpha}. \tag{3.2}$$

Lemma 3.3. *The function $x \mapsto Q_x$ is measurable and m-a.e. finite. Moreover, for every $\varphi \in \mathcal{H}_m^\alpha(\mathcal{J}, H)$,*

$$|S_n\varphi_x(T_y^{-n}(w_1)) - S_n\varphi_x(T_y^{-n}(w_2))| \leq Q_{\theta^n(x)}\varrho^\alpha(w_1, w_2)$$

for all $n \geq 1$, a.e. $x \in X$, every $z \in \mathcal{J}_x$ and every $w_1, w_2 \in B(T^n(z), \xi)$ and where again $y = (x, z)$.

Proof. By (2.3), $\log \gamma \in L^1(m)$. Then using Birkhoff's Ergodic Theorem for θ^{-1} we get that

$$\lim_{j\to\infty} \frac{1}{j} \sum_{k=0}^{j-1} \log \gamma_{\theta^{-j}(x)} = \chi.$$

for m-a.e. $x \in X$. Therefore, there exists a measurable function $C_\gamma : X \to [1, +\infty)$ m-a.e. finite such that

$$C_\gamma^{-1}(x)e^{j\chi/2} \leq \gamma^j_{\theta^{-j+1}(x)} \tag{3.3}$$

for all $j \geq 0$ and a.e. $x \in X$. Moreover, since $\log H \in L^1(m)$ it follows again from Birkhoff's Ergodic Theorem that

$$\lim_{j\to\infty} \frac{1}{j} \log H_{\theta^{-j}(x)} = 0$$

for a.e. $x \in X$. There thus exists a measurable function $C_H : X \to [1, +\infty)$ such that

$$H_{\theta^j(x)} \leq C_H(x)e^{j\alpha\chi/4} \text{ and } H_{\theta^{-j}(x)} \leq C_H(x)e^{j\alpha\chi/4} \tag{3.4}$$

for all $j \geq 0$ and a.e. $x \in X$. Then, for a.e. $x \in X$, all $n \geq 0$ and all $a \geq j \geq n-1$, we have

$$H_{\theta^j(x)} = H_{\theta^{-(n-j)}(\theta^n(x))} \leq C_H(\theta^n(x))e^{(n-j)\alpha\chi/4}.$$

Therefore, still with $x_n = \theta^n(x)$,

$$\begin{aligned} Q_{x_n} &= \sum_{j=0}^{n-1} H(x_j)(\gamma_{x_j}^{n-j})^{-\alpha} \leq \sum_{j=0}^{n-1} C_H(x_n)e^{(n-j)\alpha\chi/4}C_\gamma^\alpha(x_{n-1})e^{-\alpha(n-j)\chi/2} \\ &\leq C_\gamma^\alpha(x_{n-1})C_H(x_n)\sum_{j=0}^{n-1} e^{-\alpha(n-j)\chi/4} \leq C_\gamma^\alpha(x_{n-1})C_H(x_n)(1 - e^{-\alpha\chi/4})^{-1}. \end{aligned}$$

Hence

$$Q_x \leq C_\gamma^\alpha(\theta^{-1}(x))C_H(x)(1 - e^{-\alpha\chi/4})^{-1}.$$

The remaining affirmation follows now directly from Lemma 3.2. ∎

3.3. **Transfer operator.** We are let to consider *transfer operators*

$$\mathcal{L} = \mathcal{L}_\varphi : \mathcal{C}_m^0(\mathcal{J}) \to \mathcal{C}_m^0(\mathcal{J})$$

where we always take the potential φ to be a function of the Hölder space $\mathcal{H}_m^\alpha(\mathcal{J})$. The image of a function $g \in \mathcal{C}_m^0(\mathcal{J})$ by $\mathcal{L}$ is given by $(\mathcal{L}g)(\theta(x), w) = \mathcal{L}_x g_x(w)$ where

$$\mathcal{L}_x g_x(w) = \sum_{T_x(z)=w} g_x(z)e^{\varphi_x(z)} \ , \ x \in X \text{ and } w \in \mathcal{J}_{\theta(x)}\,. \tag{3.5}$$

Notice that the bundle operator $\mathcal{L}_x$ maps $\mathcal{C}(\mathcal{J}_x)$ into $\mathcal{C}(\mathcal{J}_{\theta(x)})$. It is indeed a positive bounded linear operator whose norm is bounded above by

$$\|\mathcal{L}_x\|_\infty \leq deg(T_x)exp(\|\varphi\|_\infty). \tag{3.6}$$

For every $n > 1$ and a.e. $x \in X$, we denote

$$\mathcal{L}_x^n := \mathcal{L}_{\theta^{n-1}(x)} \circ ... \circ \mathcal{L}_x : \mathcal{C}(\mathcal{J}_x) \to \mathcal{C}(\mathcal{J}_{\theta^n(x)})\,.$$

Note that

$$\mathcal{L}_x^n g_x(w) = \sum_{z \in T_x^{-n}(w)} g_x(z) e^{S_n\varphi_x(z)}\ ,\ w \in \mathcal{J}_{\theta^n(x)}\,, \tag{3.7}$$

where $S_n\varphi_x(z)$ has been defined in (3.1). The dual operator $\mathcal{L}_x^*$ maps $C^*(\mathcal{J}_{\theta(x)})$ into $C^*(\mathcal{J}_x)$.

In order to ensure that $\mathcal{L}g \in \mathcal{C}_m^0(\mathcal{J})$ for every $g \in \mathcal{C}_m^0(\mathcal{J})$ we make the following assumption.

Measurability of the Transfer Operator 3.4. *The transfer operator is measurable if for every $g \in \mathcal{C}_m^0(\mathcal{J})$, $x \mapsto \mathcal{L}_x g_x$ is measurable.*

4. The RPF–theorem

Let $\mu \in \mathcal{M}^1_m(\mathcal{J})$ and let μ_x be the conditional measures of μ. The measure μ is called a *Gibbs state* for $\varphi \in \mathcal{H}^\alpha_m(\mathcal{J})$ provided that there exists a measurable function $D_\varphi : X \to [1, +\infty)$ and an integrable function $P_\cdot(\varphi) \in L^1(m)$ such that

$$(D_\varphi(\theta^n(x)))^{-1} \le \frac{\mu_x(T_y^{-n}(B(T^n(y), \xi)))}{\exp(S_n\varphi(y) - S_nP_x(\varphi))} \le D_\varphi(\theta^n(x)) \tag{4.1}$$

for every $n \ge 0$, a.e. $x \in X$ and every $z \in \mathcal{J}_x$ and with $y = (x, z)$.

The measure $\mu \in \mathcal{M}^1_m(\mathcal{J})$ is called T–*invariant* if $\mu \circ T^{-1} = \mu$. In terms of the conditional measures this equivalently means that $\mu_x \circ T_x^{-1} = \mu_{\theta(x)}$ for a.e. $x \in X$ (see [1, Theorem 1.4.5]).

In this part we will establish the following version of Ruelle-Perron-Frobenius (RPF) Theorem.

Theorem 4.1. *Let $\varphi \in \mathcal{H}^\alpha_m(\mathcal{J})$ and let $\mathcal{L} = \mathcal{L}_\varphi$ be the associated transfer operator. Then the following holds.*

1. *There exists a unique $\nu \in \mathcal{M}^1_m(\mathcal{J})$ and a measurable positive function λ such that $\mathcal{L}_x^*\nu_{\theta(x)} = \lambda_x\nu_x$ for a.e. $x \in X$ and with* $\operatorname{supp}(\nu_x) = \mathcal{J}_x$.
2. *There exists a positive function $q \in \mathcal{C}^0_m(\mathcal{J})$ with $q_x \in \mathcal{H}_\alpha(\mathcal{J}_x)$ and such that $\mathcal{L}_x q_x = \lambda_x q_{S(x)}$ for a.e. $x \in X$. If normalized by $\int_{Y_x} q_x d\nu_x = 1$, then this function is unique.*
3. *The measures $\mu_x := q_x\nu_x$ are the conditional measures of a measure $\mu \in \mathcal{M}^1_m(\mathcal{J})$ which is the unique T–invariant Gibbs state for the potential φ. Moreover, μ (and hence also ν) is ergodic.*

Notice that the function $P = P(\varphi)$ in the definition of the Gibbs state (4.1) is

$$P = \log \lambda\ .$$

In order to prove Theorem 4.1 we will, in a first step, reduce the base X to an orbit

$$\mathcal{O}_x = \{\theta^n(x);\, n \in \mathbb{Z}\} \quad , \quad x \in X.$$

The reason for this is that only then we can deal with sequentially compact spaces on which the transfer (or related) operators act continuously. Conformal and invariant Gibbs states can then be obtained, for example, by fixed point methods like in the deterministic case. Once this is done for almost every orbit, we have natural candidates of conditional measures for the desired Gibbs measures for the initial RDS over X. However, one must carefully check measurable dependence of these conditional measures. With the notations of Theorem 4.1, $x \mapsto \nu_x$ and also of $x \mapsto q_x$ must be shown to be measurable. This turns out to be a delicate task (which often has been neglected in the literature) and we will get it as a consequence of the following important result.

Theorem 4.2. *Let $\varphi \in \mathcal{H}^\alpha_m(\mathcal{J})$, set $\hat\varphi_x = \varphi_x + \log q_x - \log q_{\theta(x)} \circ T - \log \lambda_x$ where $x \mapsto q_x$ is a (a priori not necessarily measurable) function such that $\tilde{\mathcal{L}} q = q$. Denote $\hat{\mathcal{L}} := \mathcal{L}_{\hat\varphi}$. Then, for a.e. $x \in X$ and all $g_x \in C(\mathcal{J}_x)$,*

$$\hat{\mathcal{L}}_x^n g_x \to \int g_x q_x d\nu_x \quad \textit{as } \ n \to \infty.$$

Notice that in addition we also obtain uniform bounds for this convergence; they are given in the Propositions 4.14 and 4.16.

4.1. **The RPF–theorem for θ–orbits.** Let in the following $\mathcal{O}_{x_0}$ be the θ–orbit of some $x_0 \in X$. We may suppose that this orbit is chosen so that all the involved measurable functions are defined and finite on the points of $\mathcal{O}_{x_0}$. For every $x \in \mathcal{O}_{x_0}$, let $\varphi_x \in \mathcal{C}(\mathcal{J}_x)$ be the continuous potential of the bundle operator $\mathcal{L}_x : C(\mathcal{J}_x) \to C(\mathcal{J}_{\theta(x)})$ which has been defined in (3.5).

4.1.1. *Conformal measures.* Consider the space of probabilities

$$\mathcal{P}(\mathcal{O}_{x_0}) := \prod_{x\in\mathcal{O}_{x_0}} \mathcal{M}^1(\mathcal{J}_x).$$

Proposition 4.3. *There exists* $(\nu_x)_{x\in\mathcal{O}_{x_0}} \in \mathcal{P}(\mathcal{O}_{x_0})$ *such that*

$$\mathcal{L}_x^*\nu_{\theta(x)} = \lambda_x \nu_x \quad \textit{for every } x \in \mathcal{O}_{x_0}$$

where $\lambda_x = \nu_{\theta(x)}(\mathcal{L}_x 1\!\!1)$.

Remark 4.4. *For the iterated dual operator* $(\mathcal{L}_x^n)^* = \mathcal{L}_x^* \circ ... \circ \mathcal{L}_{\theta^{n-1}(x)}^*$ *we get from Proposition 4.3 that*

$$(\mathcal{L}_x^n)^*\nu_{\theta^n(x)} = \lambda_x^n \nu_x$$

if we set

$$\lambda_x^n := \lambda_x \lambda_{\theta(x)} \cdot ... \cdot \lambda_{\theta^{n-1}(x)} .$$

Proof. A simple observation is that the map $\Psi_x : \mathcal{M}^1(\mathcal{J}_{\theta(x)}) \to \mathcal{M}^1(\mathcal{J}_x)$ defined by

$$\Psi_x(\nu_{\theta(x)}) = \frac{\mathcal{L}_x^*\nu_{\theta(x)}}{\mathcal{L}_x^*\nu_{\theta(x)}(1\!\!1)}$$

is weakly continuous. Consider then the global map $\Psi : \mathcal{P}(\mathcal{O}_{x_0}) \to \mathcal{P}(\mathcal{O}_{x_0})$ given by

$$\nu = (\nu_x)_{x\in\mathcal{O}_{x_0}} \longmapsto \Psi(\nu) = \big(\Psi_x \nu_{\theta(x)}\big)_{x\in\mathcal{O}_{x_0}} .$$

Weak continuity of the Ψ_x implies continuity of Ψ with respect to the coordinate convergence. The space $\mathcal{P}(\mathcal{O}_{x_0})$ being convex and compact for this topology, we can apply the Schauder-Tychonov fixed point theorem to get $\nu \in \mathcal{P}(\mathcal{O}_{x_0})$ fixed point of Ψ, i.e.

$$\mathcal{L}_x^*\nu_{\theta(x)} = \lambda_x\nu_x \quad \textit{where } \lambda_x = \mathcal{L}_x^*\nu_{\theta(x)}(1\!\!1) = \nu_{\theta(x)}(\mathcal{L}_x(1\!\!1))$$

for every $x \in \mathcal{O}_{x_0}$. □

Using standard techniques one get's out of Proposition 4.3 the following (indeed equivalent) statement.

Lemma 4.5. *For every* $n \geq 0$ *there exists a finite partition* $\{A_k\}$ *of* $\mathcal{J}_x$ *into measurable sets such that* $T^n|_{A_k}$ *is a measurable isomorphism. In addition, for every measurable set* $A \subset \mathcal{J}_x$,

$$\nu_{\theta^n(x)}(T^n(A)) \leq \sum_k \nu_{\theta^n(x)}(T^n(A\cap A_k)) = \lambda_x^n \int_A e^{-\S_n\varphi} d\nu_x.$$

If $T^n_{x\,|A}$ *is one–to–one then*

$$\nu_{\theta^n(x)}(T_x^n(A)) = \lambda_x^n \int_A e^{-\theta_n\varphi} d\nu_x.$$

Here is one more useful estimation.

Lemma 4.6. *For every $x \in \mathcal{O}_{x_0}$ and $n \geq 1$,*

$$\inf_{z\in\mathcal{J}_x} e^{S_n\varphi_x(z)} \leq \frac{\lambda_x^n}{\deg(T_x^n)} \leq \sup_{z\in\mathcal{J}_x} e^{S_n\varphi_x(z)}.$$

Moreover, for every $z \in \mathcal{J}_x$, every $r > 0$ and with $N = n_r(x)$,

$$\nu_x(B(z,r)) \geq D(x,r), \tag{4.2}$$

where

$$D(x,r) := \left(\deg(T_x^N)\right)^{-1} \exp\left\{\inf_{a\in B(z,r)} S_N\varphi_x(a) - \sup_{b\in B(z,r)} S_N\varphi_x(b))\right\}.$$

It follows that the set $\mathcal{J}_x$ is a topological support of ν_x.

Proof. Since

$$\nu_{\theta^n(x)}(\mathcal{L}_x^n 1\!\!1) = ((\mathcal{L}_x^n)^*\nu_{\theta^n(x)})(1\!\!1) = \lambda_x^n \nu_x(1\!\!1) = \lambda_x^n,$$

we get that

$$\inf_{z\in\mathcal{J}_x} e^{S_n\varphi_x(z)} \leq \frac{\lambda_x^n}{\deg(T_x^n)} \leq \sup_{z\in\mathcal{J}_x} e^{S_n\varphi_x(z)}.$$

Now fix an arbitrary $z \in \mathcal{J}_x$ and $r > 0$. Put $n = n_r(x)$ (see the topological exactness condition (2.5)). Then by Lemma 4.5

$$\nu_x(B(z,r))\lambda_x^n \sup_{a\in B(z,r)} e^{-S_n\varphi_x(a)} \geq \lambda_x^n \int_{B(z,r)} e^{-S_n\varphi_x} d\nu_x \geq 1.$$

Thus

$$\begin{aligned}\nu_x(B(z,r)) &\geq (\lambda_x^n)^{-1} \exp\left\{\inf_{a\in B(z,r)} S_n\varphi_x(a)\right\} \\ &\geq (\deg(T_x^n))^{-1} \exp\left\{\inf_{a\in B(z,r)} S_n\varphi_x(a) - \sup_{b\in B(z,r)} S_n\varphi_x(b))\right\}.\end{aligned}$$

□

4.1.2. *Invariant density.* Consider now the normalized operator $\tilde{\mathcal{L}}$ given by

$$\tilde{\mathcal{L}}_x = \lambda_x^{-1}\mathcal{L}_x \quad , \quad x \in X. \tag{4.3}$$

Proposition 4.7. *For every $x \in \mathcal{O}_{x_0}$, there exists a function $q_x \in \mathcal{H}^\alpha(\mathcal{J}_x)$ such that*

$$\tilde{\mathcal{L}}_x q_x = q_{\theta(x)} \quad \textit{for every } x \in \mathcal{O}_{x_0}.$$

Moreover, $q_x(z_1) \leq \exp\{Q_x\varrho^\alpha(w_1,w_2)\}q_x(z_2)$ for all $z_1, z_2 \in \mathcal{J}_x$ with $\varrho(z_1,z_2) \leq \xi$ and

$$1/C_\varphi(x) \leq q_x \leq C_\varphi(x) \tag{4.4}$$

for some measurable function C_φ depending only on the potential φ.

In order to prove this statement we first need good estimate for the iterates of the normalized operator applied to the constant function $1\!\!1$. Let $j = j(x)$ be the index given by topological exactness (cf. (2.5) and define

$$C_\varphi(x) := \deg(T_{x_{-j}}^j)\exp\{Q_{x_{-j}} + \|S_j\varphi_{x_{-j}}\|_\infty\} \max_{n=0,\dots,j} \frac{\sup_{z\in\mathcal{J}_x} \mathcal{L}_{x_{-n}}^n 1\!\!1(z)}{\inf_{z\in\mathcal{J}_x} \mathcal{L}_{x_{-n}}^n 1\!\!1(z)} \geq 1. \tag{4.5}$$

Lemma 4.8. *For all $w_1, w_2 \in \mathcal{J}_x$ and $n \geq 1$*

$$\frac{\tilde{\mathcal{L}}^n_{x_{-n}} \mathbb{1}(w_1)}{\tilde{\mathcal{L}}^n_{x_{-n}} \mathbb{1}(w_2)} = \frac{\mathcal{L}^n_{x_{-n}} \mathbb{1}(w_1)}{\mathcal{L}^n_{x_{-n}} \mathbb{1}(w_2)} \leq C_\varphi(x). \tag{4.6}$$

If in addition $\varrho(w_1, w_2) \leq \xi$, then

$$\frac{\tilde{\mathcal{L}}^n_{x_{-n}} \mathbb{1}(w_1)}{\tilde{\mathcal{L}}^n_{x_{-n}} \mathbb{1}(w_2)} \leq \exp\{Q_x \varrho^\alpha(w_1, w_2)\}. \tag{4.7}$$

Moreover,

$$1/C_\varphi(x) \leq \tilde{\mathcal{L}}^n_{x_{-n}} \mathbb{1}(w) \leq C_\varphi(x) \quad \textit{for every } w \in \mathcal{J}_x \textit{ and } n \geq 1. \tag{4.8}$$

Proof. First, (4.7) immediately follows from Lemma 3.3. Notice also that

$$\exp\{Q_x \varrho^\alpha(w_1, w_2)\} \leq \exp Q_x \tag{4.9}$$

since $\operatorname{diam}(Y) = 1$. The global version (4.6) can be shown as follows: If $n = 0, \dots, j(x)$, then for every $w_1, w_2 \in \mathcal{J}_x$

$$\mathcal{L}^n_{x_{-n}} \mathbb{1}(w_1) \leq \frac{\sup_{w \in \mathcal{L}_x} \mathcal{L}^n_{x_{-n}} \mathbb{1}(w)}{\inf_{w \in \mathcal{L}_x} \mathcal{L}^n_{x_{-n}} \mathbb{1}(w)} \mathcal{L}^n_{x_{-n}} \mathbb{1}(w_2) \leq C_\varphi(x) \mathcal{L}^n_{x_{-n}} \mathbb{1}(w_2).$$

Next, let $n > j = j(x)$. Take $w_1' \in T^{-j}_{x_{-j}}(w_1)$ such that

$$e^{S_j \varphi(w_1')} \mathcal{L}^{n-j}_{x_{-n}} \mathbb{1}(w_1') = \sup_{y \in T^{-j}_{x_{-j}}(w_1)} e^{S_j \varphi(y)} \mathcal{L}^{n-j}_{x_{-n}} \mathbb{1}(y)$$

and $w_2' \in T^{-j}_{x_{-j}}(w_2)$ such that $\varrho_{x_{-j}}(w_1', w_2') \leq \xi$. Then by (4.7) and (4.9)

$$\begin{aligned} \mathcal{L}^n_{x_{-n}} \mathbb{1}(w_1) &= \mathcal{L}^j_{x_{-j}} (\mathcal{L}^{n-j}_{x_{-n}} \mathbb{1})(w_1) \leq \deg(T^j_{x_{-j}}) e^{S_j \varphi(w_1')} \mathcal{L}^{n-j}_{x_{-n}} \mathbb{1}(w_1') \\ &\leq \deg(T^j_{x_{-j}}) e^{S_j \varphi(w_1')} e^{Q_{x_{-j}}} \mathcal{L}^{n-j}_{x_{-n}} \mathbb{1}(w_2') \leq C_\varphi(x) \mathcal{L}^n_{x_{-n}} \mathbb{1}(w_2). \end{aligned}$$

This shows (4.6). By Proposition 4.3

$$\int_{\mathcal{J}_x} \tilde{\mathcal{L}}^n_{x_{-n}}(\mathbb{1}) d\nu_x = \int_{\mathcal{J}_{x_{-n}}} \mathbb{1} d\nu_{x_{-n}} = 1, \tag{4.10}$$

we can find $w, w' \in \mathcal{J}_x$ such that $\tilde{\mathcal{L}}^n_{x_{-n}} \mathbb{1}(w) \leq 1$ and $\tilde{\mathcal{L}}^n_{x_{-n}} \mathbb{1}(w') \geq 1$. Therefore, by the already proved part of this lemma, we get for every $w \in \mathcal{J}_x$ and every $n \geq 1$ that

$$1/C_\varphi(x) \leq \tilde{\mathcal{L}}^n_{x_{-n}} \mathbb{1}(w) \leq C_\varphi(x).$$

□

Proof of Proposition 4.7. Let $x \in \mathcal{O}_{x_0}$. Then by Lemma 4.8, for every $k \geq 0$ and all $w_1, w_2 \in \mathcal{J}_x$ with $\varrho(w_1, w_2) \leq \xi$

$$|\tilde{\mathcal{L}}^k_{x_{-k}} \mathbb{1}(w_1) - \tilde{\mathcal{L}}^k_{x_{-k}} \mathbb{1}(w_2)| \leq C_\varphi(x) 2 Q_x \varrho^\alpha(w_1, w_2)$$

and $1/C_\varphi(x) \leq \tilde{\mathcal{L}}^k_{x_{-k}} \mathbb{1} \leq C_\varphi(x)$. It follows that the sequence

$$h^n_x := \frac{1}{n} \sum_{k=0}^{n-1} \tilde{\mathcal{L}}^k_{x_{-k}} \mathbb{1} \quad , \ n \geq 1,$$

is equicontinuous for every $x \in \mathcal{O}_{x_0}$. The orbit $\mathcal{O}_{x_0}$ being (at most) countable there exists $n_j \to \infty$ such that $h_x^{n_j} \to h_x$ uniformly for every $x \in \mathcal{O}_{x_0}$. The operators $\tilde{\mathcal{L}}_x$ being bounded we clearly get $\tilde{\mathcal{L}}_x h_x = h_{\theta(x)}$, $x \in \mathcal{O}_{x_0}$. The normalized version

$$q_x = \frac{h_x}{\int_{\mathcal{J}_x} h_x \, dx} \quad , \quad x \in \mathcal{O}_{x_0},$$

has all the required properties. □

4.2. **Proof of Theorem 4.2.** This proof is an adaption of Bowen's strategy [5] to the random setting.

4.2.1. *Positive cones of Hölder functions.* For $s \geq 1$, set

$$\Lambda_x^s = \Big\{ g \in \mathcal{C}(\mathcal{J}_x);\ g \geq 0,\ \nu_x(g) = 1 \text{ and } g(w_1) \leq e^{sQ_x \varrho^\alpha(w_1,w_2)} g(w_2) \\ \text{for all } w_1, w_2 \in \mathcal{J}_x \text{ with } \varrho(w_1,w_2) \leq \xi \Big\}. \tag{4.11}$$

The connection to Hölder functions is the following.

Lemma 4.9. *If $g \in \mathcal{H}^\alpha(\mathcal{J}_x)$ and $g \geq 0$, then*

$$\frac{g + v(g)/Q_x}{\nu_x(g) + v(g)/Q_x} \in \Lambda_x^1.$$

Proof. Consider $h = g + v(g)/Q_x$. The get the inequality from the definition of the cone we take $z_1, z_2 \in \mathcal{J}_x$ such that $h(z_1) > h(z_2)$. Then

$$\frac{h(z_1)}{h(z_2)} - 1 = \frac{|h(z_1) - h(z_2)|}{|h(z_2)|} \leq \frac{v(g)\varrho^\alpha(z_1,z_2)}{v(g)/Q_x} = Q_x \varrho^\alpha(z_1,z_2).$$

□

Lemma 4.10. *There exists a measurable function $C_{\max} : X \to (0,\infty)$ such that $\|g\|_\infty \leq C_{\max}(x)$ for a.e. $x \in X$ and every $g \in \Lambda_x^s$.*

Proof. Let $g \in \Lambda_x^s$ and let $z \in \mathcal{J}_x$. Since $g \geq 0$ we get

$$\int_{B(z,\xi)} g \, d\nu_x \leq \int_{\mathcal{J}_x} g \, d\nu_x = 1,$$

and therefore there exists $b \in \overline{B}(z,\xi)$ such that

$$g(b) \leq \frac{1}{\nu_x(B(z,\xi))} \leq D(x,\xi)$$

where $N = n_\xi(x)$ and where the latter inequality is due to Lemma 4.6. Hence

$$g(z) \leq e^{sQ_x \varrho^\alpha(b,z)} g(b) \leq C_{\max}(x) := e^{sQ_x} D(x,\xi). \tag{4.12}$$

□

The important property of the cones Λ^s are their invariance with respect to the normalized operator $\tilde{\mathcal{L}}_x = \lambda_x^{-1} \mathcal{L}_x$.

Lemma 4.11. *Let $g \in \Lambda_x^s$. Then, for every $n \geq 1$,*

$$\frac{\tilde{\mathcal{L}}_x^n g(w_1)}{\tilde{\mathcal{L}}_x^n g(w_2)} \leq \exp\left\{ sQ_{x_n} \varrho^\alpha(w_1,w_2) \right\} , \quad w_1, w_2 \in \mathcal{J}_{\theta^n(x)} \text{ with } \varrho(w_1,w_2) \leq \xi.$$

Consequently $\tilde{\mathcal{L}}_x^n(\Lambda_x^s) \subset \Lambda_{\theta^n(x)}^s$ for a.e. $x \in X$ and all $n \geq 1$.

Notice that the constant function $1\!\!1 \in \Lambda_x^s$ for every $s \geq 1$. For this particular function we already have proven this distortion estimation in Lemma 4.8.

Proof. Let $\varrho_{\theta^n(x)}(w_1, w_2) \leq \xi$ and let $z_1 \in T_x^{-n}(w_1)$ If $y = (x, z_1)$ we denote in the following $z_2 = T_y^{-n}(w_2)$. With these notations we obtain from Lemma 3.2

$$\frac{\tilde{\mathcal{L}}_x^n g(w_1)}{\tilde{\mathcal{L}}_x^n g(w_2)} \leq \sup_{z_1 \in T_x^{-n}(w_1)} \frac{e^{S_n\varphi_x(z_1)}g(z_1)}{e^{S_n\varphi_x(z_2)}g(z_2)} \tag{4.13}$$

$$\leq \sup_{z_1 \in T_x^{-n}(w_1)} \frac{\exp\left(\sum_{j=0}^{n-1} H_{\theta^j(x)}(\gamma_{\theta^j(x)}^{n-j})^{-\alpha}\varrho^\alpha(w_1, w_2)\right)e^{S_n\varphi_x(z_2)}e^{sQ_x\varrho^\alpha(z_1,z_2)}g(z_2)}{e^{S_n\varphi_x(z_2)}g(z_2)}$$

$$\leq \exp\left\{\varrho^\alpha(w_1, w_2)\Big(sQ_x(\gamma_x^n)^{-\alpha} + \sum_{j=0}^{n-1} H_{\theta^j(x)}(\gamma_{\theta^j(x)}^{n-j})^{-\alpha}\Big)\right\}.$$

Since

$$Q_x(\gamma_x^n)^{-\alpha} + \sum_{j=0}^{n-1} H_{\theta^j(x)}(\gamma_{\theta^j(x)}^{n-j})^{-\alpha} = Q_{\theta^n(x)}, \tag{4.14}$$

the Lemma follows. □

Here is one more estimation.

Lemma 4.12. *There is a measurable function* $C_{\min} : X \to (0, \infty)$ *such that*

$$\tilde{\mathcal{L}}_{x_{-i}}^i g \geq C_{\min}(x) \quad \text{for every } i \geq j(x) \text{ and } g \in \Lambda_{x_{-i}}^s.$$

Proof. First, let $i = j(x)$. Since

$$\int_{\mathcal{J}_{x_{-i}}} g d\nu_{x_{-i}} = 1,$$

there exists $a \in \mathcal{J}_{x_{-i}}$ such that $g(a) \geq 1$. By definition of $j(x)$, for any point $w \in \mathcal{J}_x$ there exists $z \in T_{x_{-i}}^{-i}(x) \cap B(a, \xi)$. Therefore

$$\tilde{\mathcal{L}}_{x_{-i}}^i g(w) \geq e^{S_i\varphi_{x_{-i}}(z)}g(z) \geq e^{S_i\varphi_{x_{-i}}(z)}e^{-sQ_x}g(a) \geq C_{\min}(x)$$

where

$$C_{\min}(x) := \exp\Big(-sQ_x - \|S_{j(x)}\varphi_{x_{-j(x)}}\|_\infty\Big) \leq 1. \tag{4.15}$$

The case $i > j(x)$ follows from the previous one, since $\tilde{\mathcal{L}}_{x_{-i}}^{i-j(x)} g_{x_{-i}} \in \Lambda_{x_{-j(x)}}$. □

We now start the heart of the proof of Theorem 4.2.

Lemma 4.13. *There exists* $\beta_x \in (0, 1)$ *such that for a.e.* $x \in X$*, every* $i > j(x)$ *and* $g_{x_{-i}} \in \Lambda_{x_{-i}}^s$*, there exists* $h_x \in \Lambda_x^s$ *such that*

$$(\tilde{\mathcal{L}}^i g)_x = \tilde{\mathcal{L}}_{x_{-i}}^i g_{x_{-i}} = \beta_x q_x + (1 - \beta_x)h_x\ .$$

Proof. First, set

$$\beta_{x,r} := \frac{1 - \exp\left(H_{x_{-1}}\gamma_{x_{-1}}^{-\alpha} r^\alpha\right)}{1 - \exp(-2sQ_x r^\alpha)}.$$

and

$$\beta_x := \frac{C_{\min}(x)}{C_\varphi(x)} \cdot \inf_{r \in (0,\xi]} \beta_{x,r}. \tag{4.16}$$

Since by (4.14)

$$sQ_x = sQ_{x_{-1}}\gamma_{x_{-1}}^{-\alpha} + sH_{x_{-1}}\gamma_{x_{-1}}^{-\alpha}, \tag{4.17}$$

$$H_{x_{-1}}\gamma_{x_{-1}}^{-\alpha} = sQ_x - (sQ_{x_{-1}} + (s-1)H_{x_{-1}})\gamma_{x_{-1}}^{-\alpha} < 2sQ_x.$$

Hence the limit $\lim_{r\to 0^+}\beta_{x,r}$ is not only positive but also strictly smaller than one. It follows, in particular, from (4.5)and (4.15) that

$$0 < \beta_x < \frac{C_{\min}(x)}{C_\varphi(x)} \le 1.$$

Now let $i \ge j(x)$ and $g_{x_{-i}} \in \Lambda^s_{x_{-i}}$. By Lemma 4.12

$$\tilde{\mathcal{L}}^i_{x_{-i}} g_{x_{-i}} \ge C_{\min}(x).$$

Then by (4.4) for all $w, z \in Y_x$ with $\varrho_x(z,w) < \xi$,

$$\begin{aligned}
&\beta_x\Big(\exp\big(sQ_x\varrho_x^\alpha(z,w)\big)q_x(z) - q_x(w)\Big)\\
&\quad\le \beta_x\Big(\exp\big(sQ_x\varrho_x^\alpha(z,w)\big) - \exp\big(-sQ_x\varrho_x^\alpha(z,w)\big)\Big)q_x(z)\\
&\quad\le \beta_x\Big(\exp\big(sQ_x\varrho_x^\alpha(z,w)\big) - \exp\big(-sQ_x\varrho_x^\alpha(z,w)\big)\Big)C_\varphi(x)\\
&\quad\le \beta_x C_\varphi(x)\big(1-\exp(-2sQ_x\varrho_x^\alpha(z,w))\big)\exp\big(sQ_x\varrho_x^\alpha(z,w)\big)\\
&\quad\le \Big(\exp\big(sQ_x\varrho_x^\alpha(z,w)\big) - \exp\big((sQ_{x_{-1}}+H_{x_{-1}})\gamma_{x_{-1}}^{-\alpha}\varrho_x^\alpha(z,w)\big)\Big)\tilde{\mathcal{L}}^i_{x_{-i}}g_{x_{-i}}(z).
\end{aligned}$$

Since by (4.13), for $h \in \Lambda^s_{x_{-1}}$,

$$\tilde{\mathcal{L}}_{x_{-1}}h(z) \le \exp\big((sQ_{x_{-1}}+H_{x_{-1}})\gamma_{x_{-1}}^{-\alpha}\varrho_x^\alpha(z,w)\big)\tilde{\mathcal{L}}_{x_{-1}}h(w),$$

$$\tilde{\mathcal{L}}^i_{x_{-i}}g_{x_{-i}}(z) \le \exp\big((sQ_{x_{-1}}+H_{x_{-1}})\gamma_{x_{-1}}^{-\alpha}\varrho_x^\alpha(z,w)\big)\tilde{\mathcal{L}}^i_{x_{-i}}g_{x_{-i}}(w).$$

Then we have that

$$\begin{aligned}
&\beta_x\Big(\exp\big(sQ_x\varrho_x^\alpha(z,w)\big)q_x(z) - q_x(w)\Big)\\
&\qquad\qquad\le \exp\big(sQ_x\varrho_x^\alpha(z,w)\big)\tilde{\mathcal{L}}^i_{x_{-i}}g_{x_{-i}}(z) - \tilde{\mathcal{L}}^i_{x_{-i}}g_{x_{-i}}(w)
\end{aligned}$$

and then

$$\tilde{\mathcal{L}}^i_{x_{-i}}g_{x_{-i}}(w) - \beta_x q_x(w) \le \exp\big(sQ_x\varrho_x^\alpha(z,w)\big)\big(\tilde{\mathcal{L}}^i_{x_{-i}}g_{x_{-i}}(z) - \beta_x q_x(z)\big).$$

Moreover, $\beta_x q_x \le C_{\min}(x) \le \tilde{\mathcal{L}}^i_{x_{-i}}g_{x_{-i}}$. Hence the function

$$h_x := \frac{\tilde{\mathcal{L}}^i_{x_{-i}}g_{x_{-i}} - \beta_x q_x}{1-\beta_x} \in \Lambda^s_x.$$

□

We are now ready to establish a first result on the exponential convergence.

Proposition 4.14. *Let $s > 1$. There exists $B = B(s) < 1$ and a measurable function $A : X \to (0,\infty)$ such that for every (not necessarily measurable) function $g : \mathcal{J} \to \mathbb{R}$ with $g_x \in \Lambda^s_x$ for a.e. $x \in X$ the following holds:*

$$\|(\tilde{\mathcal{L}}^n g)_x - q_x\|_\infty = \|\tilde{\mathcal{L}}^n_{x_{-n}}g_{x_{-n}} - q_x\|_\infty \le A(x)B^n$$

for a.e. $x \in X$ and every $n \ge 1$.

Proof. Let i_n be a sequence of integers such that $i_{n+1} \geq j(x_{-s_n})$ where $s_n = \sum_{k=1}^{n} i_n$, $n \geq 1$, and where $s_0 = 0$. Lemma 4.13 yields the existence of a function $h_x^{(1)} \in \Lambda_x^s$ such that

$$\left(\tilde{\mathcal{L}}^{i_1} g\right)_x = \beta_x q_x + (1-\beta_x) h_x^{(1)} = \Big(1-(1-\beta_x)\Big) q_x + (1-\beta_x) h_x^{(1)}.$$

Since

$$\begin{aligned}\left(\tilde{\mathcal{L}}^{i_2+i_1} g\right)_x = \tilde{\mathcal{L}}^{i_2}_{x_{i_2}} \left(\tilde{\mathcal{L}}^{i_1} g\right)_{x_{i_2}} &= \tilde{\mathcal{L}}^{i_2}_{x_{i_2}} \left(\beta_{x_{i_2}} q_{x_{i_2}} + (1-\beta_{x_{i_2}}) h^{(1)}_{x_{i_2}}\right) \\ &= \beta_{x_{i_2}} q_x + (1-\beta_{x_{i_2}}) \tilde{\mathcal{L}}^{i_2}_{x_{i_2}} \left(h^{(1)}_{x_{i_2}}\right)\end{aligned}$$

it follows again from Lemma 4.13 that there is $h_x^{(2)} \in \Lambda_x^s$ such that

$$\begin{aligned}\left(\tilde{\mathcal{L}}^{i_2+i_1} g\right)_x &= \beta_{x_{i_2}} q_x + (1-\beta_{x_{i_2}}) \Big\{\beta_x q_x + (1-\beta_x) h_x^{(2)}\Big\} \\ &= \Big(1-(1-\beta_{x_{i_2}})(1-\beta_x)\Big) q_x + (1-\beta_{x_{i_2}})(1-\beta_x) h_x^{(2)}.\end{aligned}$$

It follows now by induction that, for every $n \geq 2$ and a.e. $x \in X$, there exists $h_x^{(n)} \in \Lambda_x^s$ such that

$$\left(\tilde{\mathcal{L}}^{s_n} g\right)_x = \left(\tilde{\mathcal{L}}^{i_n+\ldots+i_1} g\right)_x = (1-\Pi_x^{(n)}) q_x + \Pi_x^{(n)} h_x^{(n)}$$

where we set

$$\Pi_x^{(n)} = (1-\beta_x) \prod_{k=2}^{n} (1-\beta_{x_{i_k}}).$$

Since $h_x^{(n)} \in \Lambda_x^s$, we have $|h_x^{(n)}| \leq C_{\max}(x)$. Therefore,

$$\left|\left(\tilde{\mathcal{L}}^{s_n} g\right)_x - \left(1-\Pi_x^{(n)}\right) q_x\right| \leq C_{\max}(x) \left|\Pi_x^{(n)}\right| \tag{4.18}$$

for every $n \geq 2$ and a.e. $x \in X$.

By measurability of β and j one can find $M < 1$ and $J \geq 1$ such that the set

$$G := \{x : \beta_x \geq 1-M \text{ and } j(x) \leq J\}$$

has a positive measure greater or equal to 3/4. Let n_k be a sequence such that $n_0 = 0$ and, for $k > 0$, $x_{-Jn_k} \in G$ and

$$\#\{n : 0 \leq n < n_k \text{ and } x_{-Jn} \in G\} = k-1.$$

Applying Birkhoff's Ergodic Theorem to the function θ^{-J} we have that for almost every $x \in X$

$$\lim_{n\to\infty} \frac{\#\{0 \leq m \leq n-1 : \theta^{-Jm}(x) \in G\}}{n} = E(\mathbb{1}_G|\mathcal{I}_J)(x)$$

where $E(\mathbb{1}_G|\mathcal{I}_J)$ is the conditional expectation of $\mathbb{1}_G$ with the respect to the σ-algebra $\mathcal{I}_J$ of θ^{-J}-invariant sets. Note that if a measurable set A is θ^{-J}-invariant, then set $\cup_{j=0}^{J-1}\theta^j(A)$ is θ^{-1}-invariant. If $m(A) > 0$, then from the ergodicity of θ^{-1} we get that $m(\cup_{j=0}^{J-1}\theta^j(A)) = 1$ and then from invariantness $m(A) \geq 1/J$. Hence we get that for almost every x the sequence n_k is infinite and

$$\lim_{k\to\infty} \frac{k}{n_k} \geq \frac{3}{J4}. \tag{4.19}$$

Fix $N \geq 0$ and take $l \geq 0$ so that $Jn_l \leq N < Jn_{l+1}$. Define a finite sequence s_k by $s_k = Jn_k$ for $k < l$ and $s_l = N$ and observe that by (4.19) $N \leq Jn_l \leq 2J^2 l$. Then (4.4) and (4.18) give

$$\begin{aligned} ||\tilde{\mathcal{L}}^N_{x_{-N}} g_{x_{-N}} - q_x||_\infty &\leq \Big|\Big|\tilde{\mathcal{L}}^N_{x_{-N}} g_{x_{-N}} - \Big(1 - \prod_{k=0}^{l-1}(1-\beta_k)\Big) q_x \Big|\Big|_\infty + ||q_x||_\infty \prod_{k=0}^{l-1}(1-\beta_k) \\ &\leq M^l \big(C_\varphi(x) + C_{\max}(x)\big) \leq (\sqrt[2J^2]{M})^N \big(C_\varphi(x) + C_{\max}(x)\big). \end{aligned}$$

This shows the Proposition. □

Consider now the potential $\hat{\varphi} = \varphi + \log q_x - \log q_{\theta(x)} \circ T - \log \lambda_x$ and the associated transfer operator $\hat{\mathcal{L}}_x := \mathcal{L}_{\hat{\varphi},x}$. Then

$$\hat{\mathcal{L}}_x g = \frac{1}{q_{\theta(x)}} \tilde{\mathcal{L}}_x (g q_x) \quad \textit{for every } g \in \mathcal{C}^0_m(\mathcal{J}).$$

Consequently

$$\hat{\mathcal{L}}_x \mathbb{1}_x = \mathbb{1}_{\theta(x)}. \tag{4.20}$$

Now, let

$$\mu_x := q_x \nu_x.$$

From conformality of ν_x (see Proposition 4.3) it follows that

$$\begin{aligned} \hat{\mathcal{L}}^*_x(\mu_{\theta(x)})(g) &= \int_{\mathcal{J}_{\theta(x)}} \hat{\mathcal{L}}_x(g) d\mu_{\theta(x)} = \lambda_x^{-1} \int_{\mathcal{J}_{\theta(x)}} (\mathcal{L}_x g q_x) d\nu_{\theta(x)} \\ &= \lambda_x^{-1} \mathcal{L}^*_x(\nu_{\theta(x)})(g q_x) = \nu_x(g q_x) = \mu_x(g). \end{aligned} \tag{4.21}$$

It follows that, if $f_x, g_{\theta(x)} \circ T_x \in L^2(\mu_x)$, then
(4.22)

$$\begin{aligned} \mu_x\big((g_{\theta(x)} \circ T_x) f_x\big) &= \hat{\mathcal{L}}^*_x(\mu_{\theta(x)})\big((g_{\theta(x)} \circ T_x) f_x\big) = \mu_{\theta(x)}\Big(\hat{\mathcal{L}}^*_x\big((g_{\theta(x)} \circ T_x) f_x\big)\Big) \\ &= \mu_{\theta(x)}\big(g_{\theta(x)} \hat{\mathcal{L}}_x(f_x)\big), \end{aligned}$$

since

$$\hat{\mathcal{L}}_x\big((g_{\theta(x)} \circ T_x) f_x\big) = g_{\theta(x)} \hat{\mathcal{L}}_x(f_x).$$

We also have then

$$\mu_x(g_{\theta(x)} \circ T_x) = \mu_{\theta(x)}(g_{\theta(x)}) \tag{4.23}$$

for $g_{\theta(x)} \in L^1(\mu_{\theta(x)})$. In other words, if $x \mapsto \mu_x$ is measurable, then it defines an invariant measure. In order to establish this measurability we first need the following version of exponential convergence along positive θ–orbits.

Lemma 4.15. *Let $s > 1$ and let $g : \mathcal{J} \to \mathbb{R}$ be any function such that $g_x \in \mathcal{H}^\alpha(\mathcal{J}_x)$ for a.e. $x \in X$, then, with the notations of Proposition 4.14,*

$$\Big|\Big|\hat{\mathcal{L}}^n_x g_x - \Big(\int g_x d\mu_x\Big)\mathbb{1}\Big|\Big|_\infty \leq C_\varphi(\theta^n(x))\Big(\Big(\int |g_x|\, d\mu_x\Big) + 4\frac{v(g_x q_x)}{Q_x}\Big) A(\theta^n(x)) B^n,$$

where $B = B(s)$.

Proof. Fix $s > 1$. First suppose that $g_x \geq 0$. Consider the function

$$h_x = \frac{g_x + v(g_x)/Q_x}{\Delta_x} \quad \text{where} \quad \Delta_x := \nu_x(g_x) + v(g_x)/Q_x.$$

It follows from Lemma 4.9 that h_x belongs to the cone Λ_x^s and from Proposition 4.14 we have

$$
\begin{aligned}
(4.24)\quad & \Big|\Big|\tilde{\mathcal{L}}_x^n g_x - \Big(\int g_x\, d\nu_x\Big) q_{\theta^n(x)}\Big|\Big|_\infty \\
&\le \Big|\Big|\Delta_x \tilde{\mathcal{L}}_x^n h_x - \frac{v(g_x)}{Q_x}\tilde{\mathcal{L}}_x^n \mathbb{1}_x - \Big(\int g_x\, d\nu_x\Big) q_{\theta^n(x)}\Big|\Big|_\infty \\
&= \Big|\Big|\Delta_x \tilde{\mathcal{L}}_x^n h_x - \Delta_x q_{\theta^n(x)} + \frac{v(g_x)}{Q_x}\Big(q_{\theta^n(x)} - \tilde{\mathcal{L}}_x^n \mathbb{1}_x\Big)\Big|\Big|_\infty \\
&\le \Big(\Delta_x + \frac{v(g_x)}{Q_x}\Big) A(\theta^n(x))B^n.
\end{aligned}
$$

Then applying this inequality for $g_x q_x$ and using (4.4) we get

$$
\begin{aligned}
& \Big|\Big|\hat{\mathcal{L}}_x^n g_x - \Big(\int g_x\, d\mu_x\Big) \mathbb{1}_{\theta^n(x)}\Big|\Big|_\infty \\
&\le \Big|\Big|\frac{1}{q_{\theta^n(x)}}\Big|\Big| \cdot \Big|\Big|\tilde{\mathcal{L}}_x^n (g_x q_x) - \Big(\int g_x q_x\, d\nu_x\Big) q_{\theta^n(x)}\Big|\Big|_\infty \\
&\le C_\varphi(\theta^n(x))\Big(\Big(\int g_x\, d\mu_x\Big) + 2\frac{v(g_x q_x)}{Q_x}\Big) A(\theta^n(x))B^n.
\end{aligned}
$$

So we have the desired estimation for non-negative g_x. In general case we can use the standard trick and we write $g_x = g_x^+ - g_x^-$, where $g_x^+, g_x^- \ge 0$. With this the lemma follows. □

The estimation obtained in Lemma 4.15 has the inconvenience to depend on the evaluation of a measurable function, namely $C_\varphi A$, along the positive θ–orbit. In particular, it is not clear at all from this statement that $||\hat{\mathcal{L}}_x^n g_x|| \to \Big(\int g_x d\mu_x\Big)$ as $n \to \infty$. Therefore we provide the following improvement of Lemma 4.15 which, in particular, shows Theorem 4.2.

Proposition 4.16. *We have*

$$\Big|\Big|\hat{\mathcal{L}}_x^n g_x - \Big(\int g_x d\mu_x\Big) \mathbb{1}_{\theta^n(x)}\Big|\Big|_\infty \longrightarrow 0 \quad , \quad n \to \infty \, ,$$

uniformly m–a.e. for every (not necessarily measurable) function $g : \mathcal{J} \to \mathbb{R}$ for which $g_x \in \mathcal{C}(\mathcal{J}_x)$ for a.e. $x \in X$.

Proof. First of all, we may suppose that the functions $g_x \in \mathcal{H}^\alpha(\mathcal{J}_x)$ since otherwise it suffices to approximate it by a uniformly convergent sequence of Hölder functions. Now, let $\mathcal{A} > 0$ be sufficiently big such that the set

$$X_\mathcal{A} = \{x \in X;\, A(x) \le \mathcal{A}\}$$

has positive measure. Notice that, by ergodicity of m, some iteration of a.e. $x \in X$ is in the set $X_\mathcal{A}$. Then by Poincaré recurrence theorem, for a.e. $x \in X$, there exists a sequence $n_j \to \infty$ such that $\theta^{n_j}(x) \in X_\mathcal{A}$, $j \ge 1$. Therefore we get, for such an $x \in X_\mathcal{A}$, from Lemma 4.15 that

$$\Big|\Big|\hat{\mathcal{L}}_x^{n_j} g_x - \Big(\int g_x d\mu_x\Big) \mathbb{1}_{\theta^{n_j}(x)}\Big|\Big|_\infty \Big(\int |g_x|\, d\mu_x + 4\frac{v(g_x q_x)}{Q_x}\Big)^{-1} \le \mathcal{A} B^{n_j}$$

for every $j \geq 1$. Finally, one can get rid of the subsequence n_j by employing the monotonicity argument that already appears in Walters paper [15]: since $\hat{\mathcal{L}}_x 1\!\!1_x = 1\!\!1_{\theta(x)}$, we have for every $w \in \mathcal{J}_{\theta(x)}$

$$\inf_{z \in \mathcal{J}_x} g_x(z) \leq \sum_{z \in T_x^{-1}(w)} g_x(z) e^{\psi(z)} \leq \sup_{z \in \mathcal{J}_x} g_x(z).$$

Consequently the sequence $(M_{n,x})_{n\in\mathbb{N}} = (\sup_{w \in \mathcal{J}_{\theta^n(x)}} \hat{\mathcal{L}}_x^n g_x(w))_{n\in\mathbb{N}}$ is decreasing. Similarly we have an increasing sequence $(m_{n,x})_{n\in\mathbb{N}} = (\inf_{w \in \mathcal{J}_{\theta^n(x)}} \hat{\mathcal{L}}_x^n g_x(w))_{n\in\mathbb{N}}$. The proposition follows since both sequences converge on some subsequence. □

We can now make one of the main steps in the proof of Theorem 4.1 and establish unicity and, most importantly, measurability of the Gibbs measures and of the function λ.

Proof of Theorem 4.1 excepted ergodicity and the Gibbs property. We recall that

$$\hat{\mathcal{L}}_x^n g_x(w) = \sum_{z \in T_x^{-n}(w)} g_x(z) e^{S_n \hat{\varphi}(z)} = (\lambda_x^n q_{\theta^n(x)}(w))^{-1} \big(\mathcal{L}_x^n g_x q_x\big)(w).$$

Then

$$\lim_{n\to\infty} \frac{\mathcal{L}_x^n g_x(w_n)}{\mathcal{L}_x^n 1\!\!1(z_n)} = \lim_{n\to\infty} \frac{\hat{\mathcal{L}}_x^n (g_x/q_x)(w_n)}{\hat{\mathcal{L}}_x^n (1/q_x)(z_n)} = \frac{\nu_x(g_x)}{\nu_x(1\!\!1)} = \nu_x(g_x) \tag{4.25}$$

for every sequences $w_n, z_n \in \mathcal{J}_{\theta^n(x)}$. Hence the measures ν_x are determined uniquely. Moreover, since the transfer operator is supposed to be measurable, and since for some fixed $w \in \mathcal{J}_x$, the functions

$$x \mapsto \frac{\mathcal{L}_x^n g_x(T_x^n(w))}{\mathcal{L}_x^n 1\!\!1(T_x^n(w))}, \quad n \geq 1,$$

are measurable the limit $x \mapsto \nu_x(g_x)$ is also measurable.

Uniqueness and measurability of the sequences of λ's follows now from the formula

$$\mathcal{L}_x^*(\nu_{\theta(x)})(1\!\!1) = \lambda_x. \tag{4.26}$$

Concerning the eigenfunction q of $\tilde{\mathcal{L}}$, measurability and uniqueness follows from Proposition 4.14 since it shows that

$$\tilde{\mathcal{L}}_{x_{-n}}^n 1\!\!1 = \frac{\mathcal{L}_{x_{-n}}^n 1\!\!1}{\lambda_{x_{-n}}^n} \longrightarrow q_x.$$

□

4.3. **Exponential Decay of Correlations and Ergodicity of μ.** For a function $f_x \in L^1(\mu_x)$ we denote the L^1–norm with respect to μ_x by

$$\|f_x\|_1 := \int |f_x| d\mu_x.$$

Proposition 4.17. *There exists a θ–invariant set $X' \subset X$ of full m–measure such that, for every $x \in X'$, every $f_{\theta^n(x)} \in L^1(\mu_{\theta^n(x)})$ and every $g_x \in \mathcal{H}^\alpha(\mathcal{J}_x)$,*

$$\big|\mu_x\big((f_{\theta^n(x)} \circ T_x^n) g_x\big) - \mu_{\theta^n(x)}(f_{\theta^n(x)})\mu_x(g_x)\big| \leq A_*(g_x, \theta^n(x)) B^n ||f_{\theta^n(x)}||_1$$

where

$$A_*(g_x,\theta^n(x)) := C_\varphi(\theta^n(x))\Big(\Big(\int |g_x|\, d\mu_x\Big) + 4\frac{\upsilon(g_x q_x)}{Q_x}\Big)A(\theta^n(x)).$$

Proof. Set $h_x = g_x - \int g_x d\mu_x$ and note that by (4.22) and (4.20) we have that

$$\begin{aligned}(4.27)\quad &\mu_x\big((f_{\theta^n(x)}\circ T_x^n)g_x\big) - \mu_{\theta^n(x)}(f_{\theta^n(x)})\mu_x(g_x)\\ &= \mu_{\theta^n(x)}\big(f_{\theta^n(x)}\hat{\mathcal{L}}_x^n(g_x)\big) - \mu_{\theta^n(x)}(f_{\theta^n(x)})\mu_x(g_x) = \mu_{\theta^n(x)}\big(f_{\theta^n(x)}\hat{\mathcal{L}}_x^n(h_x)\big).\end{aligned}$$

Since Lemma 4.15 yields

$$||\hat{\mathcal{L}}_x^n h_x||_\infty \le A_*(g_x,\theta^n(x))B^n,$$

it follows from (4.27) that

$$\begin{aligned}&\big|\mu_x\big((f_{\theta^n(x)}\circ T_x^n)g_x\big) - \mu_{\theta^n(x)}(f_{\theta^n(x)})\mu_x(g_x)\big|\\ &\qquad\le \int \big|f_{\theta^n(x)}\hat{\mathcal{L}}_x^n(h_x)\big| d\mu_{\theta^n(x)} \le A_*(g_x,\theta^n(x))B^n\int \big|f_{\theta^n(x)}\big| d\mu_{\theta^n(x)}.\end{aligned}$$

□

Using similar arguments like in Proposition 4.16 we obtain the following.

Corollary 4.18. *Let $f_{\theta^n(x)} \in L^1(\mu_{\theta^n(x)})$ and $g_x \in L^1(\mathcal{J}_x)$ where $x \in X'$, X' still the set given by Lemma 4.17. If $||f_{\theta^n(x)}||_1 \neq 0$ for all n, then*

$$\frac{\big|\mu_x\big((f_{\theta^n(x)}\circ T_x^n)g_x\big) - \mu_{\theta^n(x)}(f_{\theta^n(x)})\mu_x(g_x)\big|}{||f_{\theta^n(x)}||_1} \longrightarrow 0 \quad \text{as } n\to\infty.$$

Remark 4.19. *Note that if $||f_{\theta^n(x)}||_1$ does not grow exponentially fast then*

$$(4.28)\qquad \big|\mu_x\big((f_{\theta^n(x)}\circ T_x^n)g_x\big) - \mu_{\theta^n(x)}(f_{\theta^n(x)})\mu_x(g_x)\big| \longrightarrow 0 \quad \text{still as } n\to\infty.$$

This is for example the case if $x \mapsto \log ||f_x||_1$ is m-integrable since Birkhoff's Ergodic Theorem implies $(1/n)\log||f_{\theta^n(x)}||_1 \to 0$ a.e. So then (4.28) holds, in particular, for $f \in C^1_m(\mathcal{J})$.

This remark permits us to do the next step in the proof of Theorem 4.1.

Proposition 4.20. *The measure μ is ergodic.*

Proof. Let B be a measurable set such that $T^{-1}(B) = B$ and, for $x \in X$, denote by B_x the set $\{y \in \mathcal{J}_x : (x,y) \in B\}$. Then we have that $T_x^{-1}(B_{\theta(x)}) = B_x$. Now let

$$X_0 := \{x \in X : \mu_x(B_x) > 0\}.$$

This is clearly a θ-invariant subset of X. We will show that, if $m(X_0) > 0$, then $\mu_x(B_x) = 1$ for a.e. $x \in X_0$. Since θ is ergodic with respect to m this implies ergodicity of T with respect to μ.

Define a function f by $f_x := 1\!\!1_{B_x}$. Clearly $f_x \in L^1(\mu_x)$ and $f_{\theta^n(x)} \circ T_x^n = f_x$ m–a.e. We can suppose that this holds for every $x \in X'$ where X' is given by Lemma 4.17. Let $x \in X' \cap X_0$ and let $g_x \in C(\mathcal{J}_x)$ be any continuous function with $\int g_x d\mu_x = 0$. Then from (4.28) we obtain that $\mu_x\big((g_{\theta^n(x)}\circ T_x^n)f_x\big) \to 0$ as $n\to\infty$. Consequently

$$(4.29)\qquad \int_{B_x} g_x\, d\mu_x = 0.$$

Since this holds for every mean zero $g_x \in C(\mathcal{J}_x)$ we have $\mu_x(B_x) = 1$ for every $x \in X' \cap X_0$. This finishes the proof of the ergodicity of T with respect to μ. □

4.4. **The pressure and Gibbs property.** From (4.25) and (4.26) we have the following.

Corollary 4.21. $\lambda_x = \nu_{\theta(x)}(\mathcal{L}_x 1\!\!1) = \lim_{n\to\infty} \frac{\mathcal{L}^{n+1}_{x,\varphi} 1\!\!1(w_{n+1})}{\mathcal{L}^{n}_{\theta(x),\varphi} 1\!\!1(w_{n+1})}$.

Now we can define a measurable pressure function by the formula

$$x \mapsto P_x(\varphi) = \log \lambda_x.$$

Note that by Corollary 4.21 we can obtain an alternative definition of $P_x(\varphi)$, namely

$$P_x(\varphi) = \log(\nu_{\theta(x)}(\mathcal{L}_x 1\!\!1)) = \lim_{n\to\infty} \log \frac{\mathcal{L}^{n+1}_x 1\!\!1(w_{n+1})}{\mathcal{L}^{n}_{\theta(x)} 1\!\!1(w_{n+1})} \tag{4.30}$$

where w_n is any sequence of points from $\mathcal{J}_{\theta^n(x)}$.

Definition-Proposition 4.22. *The function $P_x(\varphi)$ is bounded above by an integrable function. Therefore, the global (integrated or expected) pressure φ given by*

$$E\big(P(\varphi)\big) = \int_X P_x(\varphi) dm(x)$$

is well defined.

Proof. Remember that $\mathcal{L}_x 1\!\!1(w) = \sum_{T_x(z)=w} e^{\varphi_x(z)} \le \deg(T_x) e^{\|\varphi\|_\infty}$. Hence

$$P_x(\varphi) = \log(\nu_{\theta(x)}(\mathcal{L}_x 1\!\!1)) \le \log(\deg(T_x)) + \|\varphi\|_\infty.$$

Since the function $\varphi \in \mathcal{C}^1_m(\mathcal{J})$ and since $\log(\deg(T_x)) \in L^1(m)$ by assumption, we are done. □

The equality (4.25) yields alternative formulas for the global pressure. To get them, observe that by Birkhoff's Ergodic Theorem

$$E\big(P(\varphi)\big) = \lim_{n\to\infty} \frac{1}{n} \log \lambda^n_x \quad for\ a.e.\ \ x \in X.$$

In addition, by (4.26),

$$\lambda^n_x = \lambda^n_x \nu_x(1\!\!1) = \nu_{\theta^n(x)}(\mathcal{L}^n_x(1\!\!1)).$$

Thus, it follows that

$$\frac{1}{n} \log \lambda^n_x = \lim_{k\to\infty} \frac{1}{n} \log \frac{\mathcal{L}^{k+n}_x 1\!\!1_x(w_{k+n})}{\mathcal{L}^{k}_{\theta^n(x)} 1\!\!1_{\theta^n(x)}(w_{k+n})}.$$

However, we can get a more interesting formula.

Lemma 4.23. *For almost every $x \in X$ there exists a sequence $n_j \to \infty$ such that for every $\varphi \in \mathcal{H}^\alpha_m(\mathcal{J})$ we have*

$$E\big(P(\varphi)\big) = \lim_{j\to\infty} \frac{1}{n_j} \log \mathcal{L}^{n_j}_x 1\!\!1(w_{n_j})$$

where $w_{n_j} \in \mathcal{J}_{\theta^{n_j}(x)}$ are any arbitrary chosen points.

Proof. By Lemma 4.16 we have that, for any sequence of $w_n \in \mathcal{J}_{\theta^n(x)}$

$$\lim_{n\to\infty} \frac{\mathcal{L}_x^n(1\!\!1)(w_n)}{\lambda_x^n q_{\theta^n(x)}} = \lim_{n\to\infty} \hat{\mathcal{L}}_x^n(1\!\!1/q_x)(w_n) = 1.$$

Then for every $K > 1$

$$K^{-1}\lambda_x^n q_{\theta^n(x)}(w_n) \le \mathcal{L}_x^n(1\!\!1)(w_n) \le K\lambda_x^n q_{\theta^n(x)}(w_n)$$

and all $n \ge 1$ sufficiently large (depending on K). Hence

$$\begin{aligned}(4.31)\quad \frac{1}{n}\log(K^{-1}) + \frac{1}{n}\log\lambda_x^n + \frac{1}{n}\log q_{\theta^n(x)}(w_n) &\le \frac{1}{n}\log \mathcal{L}_x^n(1\!\!1)(w_n)\\ &\le \frac{1}{n}\log K + \frac{1}{n}\log\lambda_x^n + \frac{1}{n}\log q_{\theta^n(x)}(w_n).\end{aligned}$$

Since $x \mapsto ||q_x||_\infty$ is measurable, using Poincaré Recurrent Theorem, we get that for almost every x, $||q_{S^n(x)}||_\infty$ is uniformly bounded for infinitely many n's. It suffices now to take $n_j \to \infty$ such that $||q_{S^{n_j}(x)}||_\infty$ stays bounded and to take then the limit $j \to \infty$ in (4.31). □

Lemma 4.24 (Gibbs property of ν_x)**.** *Let $w \in \mathcal{J}_x$, set $y = (x, w)$ and let $n \ge 0$. Then*

$$e^{-Q_{\theta^n(x)}}(D_1(\theta^n(x)))^{-1} \le \frac{\nu_x(T_y^{-n}(B(T^n(y),\xi)))}{\exp(S_n\varphi(y) - S_nP_x(\varphi))} \le e^{Q_{\theta^n(x)}}.$$

where

$$D_1(x) := D(x,\xi).$$

Proof. Fix an arbitrary $z \in \mathcal{J}_x$ and set $y = (x, z)$. Then by Lemma 3.3 and Lemma 4.5

$$\begin{aligned}\frac{\nu_x(T_y^{-n}(B(T^n(y),\xi)))}{\exp(S_n\varphi(y) - S_nP_x(\varphi))} &\le \frac{(\lambda_x^n)^{-1}\nu_{\theta^n(x)}(B(T^n(y),\xi))\sup_{z'\in T_y^{-n}(B(T^n(y),\xi))} e^{S_n\varphi(z')}}{(\lambda_x^n)^{-1}e^{S_n\varphi(y)}}\\ &\le e^{Q_{S^n(x)}}.\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{\nu_x(T_y^{-n}(B(T^n(y),\xi)))}{\exp(S_n\varphi(y) - S_nP_x(\varphi))} &\ge \frac{(\lambda_x^n)^{-1}\nu_{\theta^n(x)}(B(T^n(y),\xi))\inf_{z'\in T_y^{-n}(B(T^n(y),\xi))} e^{S_n\varphi(z')}}{(\lambda_x^n)^{-1}e^{S_n\varphi(y)}}\\ &\ge \nu_{\theta^n(x)}(B(T^n(y),\xi))e^{-Q_{S^n(x)}}.\end{aligned}$$

The lemma follows by (4.2). □

Now we will show the uniqueness of the invariant Gibbs measure. For that we assume that there exists a measurable function L_x such that for every $x \in X$ there exist a finite sequence $y_x^1, \dots, y_x^l \in \mathcal{J}_x$ such that

$$\bigcup_{j=1}^{L_x} B(y_x^j, \xi) = Y_x.$$

Now let ν^i, where $i = 1, 2$, be two probabilities such that $d\nu^i = d\nu_x^i dm$ and suppose that they satisfy the Gibbs condition (4.1):

$$(D_i(\theta^n(x)))^{-1} \le \frac{\nu_x(T_y^{-n}(B(T^n(y),\xi)))}{\exp(S_n\varphi(y) - S_nP_x^i)} \le D_i(\theta^n(x))$$

where $P_x^i \in L^1(m)$.

Lemma 4.25. *The measures ν_x^1 and ν_x^2 are equivalent and*

$$\int P_x^1 dm = \int P_x^2 dm = E(P(\varphi)).$$

Proof. Let A be compact and let $\delta > 0$. By regularity of ν_x^2 we can find $\varepsilon > 0$ such that

$$\nu_x^2(B_x(A,\varepsilon)) \le \nu_x^2(A) + \delta. \tag{4.32}$$

Now, let N_x be a measurable function such that $\xi(\gamma_x^{N_x})^{-1} \le \varepsilon/2$ (see the proof of Lemma 3.3 for the existence of the N_x). Set

$$A_n^j := \{y \in T_x^{-n}(y_{x_n}^j) : A \cap T_y^{-n}(B(y_{x_n}^j, \xi)) \neq \emptyset\}.$$

By measurability of $x \mapsto L_x$, $x \mapsto N_x$ and $x \mapsto D_1(x)$ there exists a set $Z \subset X$ with a positive measure m and constants L, N and D such that for $x \in Z$, $L_x \le L$, $N_x \le N$ and $D_1(x) \le D$. Then, for $n \ge N$, we have

$$A \subset \bigcup_{j=1}^{L} \bigcup_{y \in A_n^j} T_y^{-n} B(y_{x_n}^j, \xi) \subset B_x(A, \varepsilon).$$

Then if $\theta^n(x) \in Z$ we get

$$\nu_x^1(A) \le \sum_{j=1}^{L} \sum_{y \in A_n^j} \nu_x^1(T_y^{-n} B(y_{x_n}^j, \xi)) \le D \sum_{j=1}^{L} \sum_{y \in A_n^j} \exp(S_n\varphi(y) - S_n P_x^1(\varphi)) \tag{4.33}$$

Then by (4.32)

$$\begin{aligned} \nu_x^1(A) &\le D \exp(S_n P_x^2 - S_n P_x^1) \sum_{j=1}^{L} \sum_{y \in A_n^j} \exp(S_n\varphi(y) - S_n P_x^2(\varphi)) \\ &\le D^2 \exp(S_n P_x^2 - S_n P_x^1) \sum_{j=1}^{L} \sum_{y \in A_n^j} \nu_x^2(T_y^{-n} B(y_{x_n}^j, \xi)) \\ &\le D^2 L \exp(S_n P_x^2 - S_n P_x^1) \nu_x^2(B(A,\varepsilon)) \\ &\le D^2 L \exp(S_n P_x^2 - S_n P_x^1)(\nu_x^2(A) + \delta) \end{aligned} \tag{4.34}$$

since for $y \neq y'$ such that $y, y' \in T_x^{-n}(y_{x_n}^j)$, we have that

$$T_y^{-n} B(y_{x_n}^j, \xi) \cap T_{y'}^{-n} B(y_{x_n}^j, \xi) = \emptyset.$$

Hence the difference $S_n P_x^2 - S_n P_x^1$ is bounded from below by some constant, since otherwise taking $A = \mathcal{J}_x$ we would obtain that $\nu_x^1(X) = 0$ on passing to a subsequence in 4.34. Similarly, exchanging ν_x^1 with ν_x^2 we obtain that $S_n P_x^1 - S_n P_x^2$ is bounded from above on the same subsequence. Then, letting δ go to zero, we have that ν_x^1 and ν_x^2 are equivalent.

Note that on the subsequence considered above

$$\begin{aligned} \exp(-S_n P_x^1) \mathcal{L}_x^n 1\!\!1_x(y_n) &= \sum_{y \in T_x^{-n}(y_n)} e^{S_n\varphi_x(y) - S_n P_x^1} \\ &\le D \sum_{y \in T_x^{-n}(y_n)} \nu_x^1(T_y^{-n} B(y_n, \xi)) \le D\nu_x^1(Y_x) = D. \end{aligned}$$

Then

$$\frac{1}{n}\log \mathcal{L}_x 1\!\!1_x(y_n) - \frac{1}{n}\log D \leq \frac{1}{n} S_n P_x^1.$$

On the other hand, by (4.33), on the same subsequence

$$1 = \nu_x^1(Y_x) \leq DL \sum_{y \in T_x^{-n}(y_n)} e^{S_n\varphi_x(y) - S_n P_x^1}$$

for some $y_n \in \{y_{x_n}^1, \ldots, y_{x_n}^L\}$. Therefore, if Z is a set of points on which $||\log q_x||$ is bounded, using (4.31) and the Sandwich Theorem, there exists n_k such that

$$\lim_{k\to\infty} \frac{1}{n_k} S_{n_k} P_x^1 = E(P(\varphi))$$

for m-a.e. $x \in X$. Since P_x^1 is integrable it follows from Birkhoff's Ergodic Theorem that actually the limit $\lim_{n\to\infty} \frac{1}{n} S_n P_x^1$ exists m-almost surely and is equal to $\int P_x^1 dm$. Hence $\int P_x^1 dm = \int P_x^2 dm = E(P(\varphi))$ for m-a.e. $x \in X$. □

Therefore the following is a direct consequence of the theorem about the existence of product measure (or one can use Fubini Theorem).

Proposition 4.26. *The measure μ is a unique T-invariant measure satisfying (4.1).*

5. The pressure as a function of a parameter

5.1. **Pressure as a function.** From now on throughout the section, $\varphi \in \mathcal{H}_m(\mathcal{J})$ be a potential such that there exist measurable functions $L : X \ni x \mapsto L_x \in \mathbb{R}$ and $c : X \ni x \mapsto c_x > 0$ such that

$$S_n\varphi_x(z) \le -nc_x + L_x \tag{5.1}$$

for every $z \in \mathcal{J}_x$ and n. Let $\psi \in \mathcal{H}_m(\mathcal{J})$ be any other Hölder function. Define

$$\varphi^t := t\varphi + \psi,$$

denote $P_x(t)$ the pressure $P_x(\varphi^t)$ and let $E\big(P(t)\big)$ be the global pressure of φ^t.

Lemma 5.1. *The function $P_x : \mathbb{R} \to \mathbb{R}$ is monotone, strictly decreasing, convex (so continuous) and*

$$\lim_{t\to+\infty} P_x(t) = -\infty \quad \textit{and} \quad \lim_{t\to-\infty} P_x(t) = +\infty \qquad m-a.e. \tag{5.2}$$

In addition, the global pressure EP has the same properties.

Proof. Let $t_1 < t_2$, fix for every $x \in X$ a point $w_0 \in \mathcal{J}_x$ and put $w_n = T_x^n(w_0)$. Then by (5.1)

$$\sum_{z\in T_x^{-n}(w_n)} \exp(t_1 S_n\varphi(z)) \exp(S_n\psi(z)) \ge \sum_{z\in T_x^{-n}(w_n)} \exp\Big(t_2 S_n\varphi(z)\Big) \exp\Big((t_2-t_1)(nc_x - L_x)\Big)$$

Therefore,

$$\frac{1}{n}\log\Big(\sum_{z\in T_x^{-n}(w_n)} e^{t_1 S_n\varphi(z)} e^{S_n\psi(z)}\Big) \ge \frac{1}{n}\log\Big(\sum_{z\in T_x^{-n}(w_n)} e^{t_2 S_n\varphi(z)} e^{S_n\psi(z)}\Big) + (t_2-t_1)(c_x - L_x/n)\,.$$

Hence, letting $n \to \infty$, we get

$$P_x(t_2) \ge P_x(t_1) + (t_2 - t_1)c_x\,. \tag{5.3}$$

It directly follows from this inequality that the function $P_x(t)$ is strictly decreasing and the formula (5.2) holds.

Now, let $\lambda \in (0,1)$. By Hölder inequality we have

$$\sum_{z\in T_x^{-n}(w_n)} e^{\lambda(t_1 S_n\varphi(z)+S_n\psi(z))} e^{(1-\lambda)(t_2 S_n\varphi(z)+S_n\psi(z))} \le \Big(\sum_{z\in T_x^{-n}(w_n)} e^{t_1 S_n\varphi(z)+S_n\psi(z)}\Big)^{\lambda} \Big(\sum_{z\in T_x^{-n}(w_n)} e^{t_2 S_n\varphi(z)+S_n\psi(z)}\Big)^{(1-\lambda)}. \tag{5.4}$$

Therefore, $P_x(\lambda t_1 + (1-\lambda)t_2) \le \lambda P_x(t_1) + (1-\lambda)P_x(t_2)$, which shows that P_x is convex and thus continuous.

Concerning the global pressure it suffices to integrate (5.3) and to use that $\int c_x dm > 0$ in order to get (5.2) with P_x replaced by EP. Finally, the convexity of EP follows from (5.4) and Lemma 4.23. □

5.2. **Uniform convergence.** By $\mathcal{L}_{x,t}$ we denote $\mathcal{L}_{x,\varphi^t}$ and by q_x^t we denote the function q_x for φ^t from Proposition 4.7.

Proposition 5.2. *For every t_0 there exist $\varepsilon > 0$, $B_\varepsilon < 1$ and a measurable funcion $x \mapsto A_\varepsilon(x)$ such that, for a.e. $x \in X$ and for every t with $|t_0 - t| < \varepsilon$,*

$$\left|\left|\tilde{\mathcal{L}}_{x,t}^n g_x - \left(\int g_x d\nu_{x,t}\right) q_{\theta^n(x)}\right|\right|_\infty \le \left(\|g_x\|_\infty + 2\frac{v(g_x)}{Q_x}\right) A_\varepsilon(\theta^n(x)) B_\varepsilon^n.$$

for every $g_x \in \mathcal{H}^\alpha(\mathcal{J}_x)$.

Proof. By $C_{\varphi,t}(x)$, $C_{\max,t}(x)$ and $C_{\min,t}(x)$ we denote the constants defined by (4.5), (4.12) and (4.15). Observe that by their definition, there exists a measurable function $C_\varepsilon(x)$, such that for all $t \in I_\varepsilon := (t_0 - \varepsilon, t_0 + \varepsilon)$

$$\begin{aligned} C_{\varphi,t}(x) &\le C_\varepsilon(x) C_{\varphi,t_0}(x), \\ C_{\max,t}(x) &\le (C_\varepsilon(x)) C_{\max,t_0}(x), \\ C_{\min,t}(x) &\ge (C_\varepsilon(x))^{-1} C_{\min,t_0}(x). \end{aligned}$$

Then considering $\beta_x' := (C_\varepsilon(x))^{-2}\beta_x$, where β_x is the constant related to φ^{t_0} we obtain Lemma 4.13 with the same constant β_x' for all φ^t with $t \in I_\varepsilon$. Then we get Lemma 4.14 with $B_\varepsilon < 1$ and $A_\varepsilon(x)$ (the same for all potentials with $t \in I_\varepsilon$). Hence the lemma follows exactly like (4.24). □

5.3. **Derivative of the pressure.** Here we assume that the essential infimum of γ_x is greater than some $\gamma > 1$ and the essential suprema of H_x, $n_\xi(x)$, $j(x)$ are finite. Hence all the functions Q_x, $C_\varphi(x)$, $C_{\max}(x)$, $C_{\min}(x)$ also have the essential suprema finite and then it follows that the same is true about $C_\varepsilon(x)$ and $A_\varepsilon(x)$ from Proposition 5.2 (however note that here these upper bounds depend on t_0 and ε). We also assume that essential supremum of $\|\varphi_x\|$ is bounded. Therefore, by Proposition 5.2 we have that there exists $\tilde{A}_\varepsilon$ such that for all $t \in I_\varepsilon$

$$\left|\left|\frac{\tilde{\mathcal{L}}_{x,t}^n g_x}{q_{\theta^n(x)}} - \left(\int g_x d\nu_{x,t}\right)\right|\right|_\infty \le \left(\|g_x\|_\infty + 2\frac{v(g_x)}{Q_x}\right) \tilde{A}_\varepsilon B_\varepsilon^n. \tag{5.5}$$

Proposition 5.3.

$$\frac{dE(P(t))}{dt} = \int \varphi_x d\mu_x^t dm(x) = \int \varphi d\mu^t.$$

Proof. Let $x \mapsto y(x) \in Y_x$ be a measurable function and let

$$EP(t,n) := \int \frac{1}{n} \log \mathcal{L}_{x,t}^n 1\!\!1_x(y(x_n)) dm(x).$$

Then $\lim_{n\to\infty} EP(t,n) = E(P(t))$ by Lemma 4.23. Fix $x \in X$ and put $y_n := y(x_n)$. Observe that

$$\begin{aligned} \frac{d\mathcal{L}_{x,t}^n 1\!\!1_x(y_n)}{dt} &= \sum_{y \in T_x^{-n}(y_n)} e^{S_n(\varphi_x^t)(y)} S_n\varphi_x(y) \\ &= \sum_{j=0}^{n-1} \sum_{y \in T_x^{-n}(y_n)} e^{S_n(\varphi_x^t)(y)} \varphi_{x_j}(T_x^j y) = \sum_{j=0}^{n-1} \mathcal{L}_{x,t}^n(\varphi_{x_j} \circ T_x^j)(y_n). \end{aligned}$$

Since

$$S_n(\varphi_x^t)(y) = S_j(\varphi_x^t)(y) + S_{n-j}(\varphi_{x_j}^t)(T_x^j y)$$

we have that

$$\mathcal{L}_{x,t}^{n}(\varphi_{x_j}\circ T_x^j)(y(x_n))=\mathcal{L}_{x_j,t}^{n-j}(\varphi_{x_j}\mathcal{L}_{x,t}^{j}1\!\!1_x)(y(x_n)).$$

Then by a version of Leibniz integral rule (see for example [12], Proposition 7.8.4 p. 40)

$$\frac{dEP(t,n)}{dt}=\int\frac{\frac{1}{n}\sum_{j=0}^{n-1}\mathcal{L}_{x_j,t}^{n-j}(\varphi_{x_j}\mathcal{L}_{x,t}^{j}1\!\!1_x)(y(x_n))}{\mathcal{L}_{x,t}^{n}1\!\!1_x(y_n)}dm(x).$$

Observe that

$$\mathcal{L}_{x_j,t}^{n-j}(\varphi_{x_j}\mathcal{L}_{x,t}^{j}1\!\!1)(y_n)=\lambda_x^n\tilde{\mathcal{L}}_{x_j,t}^{n-j}\left(\varphi_{x_j,t}\tilde{\mathcal{L}}_{x,t}^{j}1\!\!1_x\right)(y_n)$$

and

$$\mathcal{L}_{x,t}^{n}1\!\!1_x(y_n)=\lambda_x^n\tilde{\mathcal{L}}_{x,t}^{n}1\!\!1_x(y_n).$$

Then

$$\frac{\mathcal{L}_{x,t}^{n}(\varphi_{x_j}\circ T_x^j)(y_n)}{\mathcal{L}_{x,t}^{n}1\!\!1_x(y_n)}=\frac{\tilde{\mathcal{L}}_{x_j,t}^{n-j}\left(\varphi_{x_j}\tilde{\mathcal{L}}_{x,t}^{j}1\!\!1_x\right)(y_n)}{\tilde{\mathcal{L}}_{x,t}^{n}1\!\!1_x(y_n)}. \tag{5.6}$$

The function $\varphi_{x_j}\tilde{\mathcal{L}}_{x,t}^{j}1\!\!1_x$ is uniformly bounded. So does its Hölder variation. Therefore it follows from (5.5), that there exists a constant A_* such that

$$\left\|\tilde{\mathcal{L}}_{x_j,t}^{n-j}\left(\varphi_{x_j}\tilde{\mathcal{L}}_{x,t}^{j}1\!\!1_x\right)(y_n)/q_{x_n}-\left(\int\varphi_{x_j}\tilde{\mathcal{L}}_{x,t}^{j}1\!\!1_x d\nu_{x_j}^t\right)\right\|_\infty\le A_*B_\varepsilon^{n-j}$$

and

$$\left\|\tilde{\mathcal{L}}_{x,t}^{n}(1\!\!1_x)(y_n)/q_{x_n}-1\!\!1_{x_n}\right\|_\infty\le A_*B_\varepsilon^n,$$

From this by (5.6) it follows that

$$\frac{\int\varphi_{x_j}\tilde{\mathcal{L}}_{x,t}^{j}1\!\!1_x d\nu_{x_j}^t-A_*B_\varepsilon^{n-j}}{1+A_*B_\varepsilon^n}\le\frac{\mathcal{L}_{x,t}^{n}(\varphi_{x_j}\circ T_x^j)(y_n)}{\mathcal{L}_x^n 1\!\!1_{Y_x}(y_n)}\le\frac{\int\varphi_{x_j}\tilde{\mathcal{L}}_{x,t}^{j}1\!\!1_x d\nu_{x_j}^t+A_*B_\varepsilon^{n-j}}{1-A_*B_\varepsilon^n},$$

Since m is S-invariant, we have that

$$\int\int\varphi_{x_j}\tilde{\mathcal{L}}_{x,t}^{j}1\!\!1_x d\nu_{x_j}^t dm(x)=\int\int\varphi_x\tilde{\mathcal{L}}_{x_{-j},t}^{j}1\!\!1_{x_{-j}}d\nu_x^t dm(x).$$

Hence, for large n,

$$\frac{\int\int\varphi_x\left(\frac{1}{n}\sum_{j=0}^{n-1}\tilde{\mathcal{L}}_{x_{-j},t}^{j}1\!\!1_{x_{-j}}\right)d\nu_x^t dm(x)-\frac{1}{n}\sum_{j=0}^{n-1}(A_*B_*^{n-j})}{1+A_*B_*^n}\le\frac{dP(\varphi^t,n)}{dt}$$
$$\le\frac{\int\int\varphi_x\left(\frac{1}{n}\sum_{j=0}^{n-1}\tilde{\mathcal{L}}_{x_{-j},t}^{j}1\!\!1_{x_{-j}}\right)d\nu_x^t dm(x)-\frac{1}{n}\sum_{j=0}^{n-1}(A_*B_*^{n-j})}{1-A_*B_*^n}.$$

Therefore

$$\lim_{n\to\infty}\frac{dEP(t,n)}{dt}=\int\varphi_x d\mu_x^t dm(x)$$

uniformly for $t\in I_\varepsilon$. □

Here are some notations for later use. Let

$\mathcal{H}_m^\alpha(\mathcal{J},H,\varphi,V):=\left\{\psi\in\mathcal{H}_m^\alpha(\mathcal{J},H)\,;\,V_x^{-1}\|\varphi_x\|_\infty\le\|\psi_x\|_\infty\le V_x\|\varphi_x\|_\infty\ m-a.e.\right\}$

where $V:X\to[1,\infty)$ is a measurable function.

Let $s>1$. There exists $B=B(s)<1$ and a measurable function $A:X\to(0,\infty)$ such that for every (not necessarily measurable) function $g:\mathcal{J}\to\mathbb{R}$ with $g_x\in\Lambda_x^s$ for a.e. $x\in X$ the following holds:

$$\|(\tilde{\mathcal{L}}^n g)_x - q_x\|_\infty = \|\tilde{\mathcal{L}}^n_{x_{-n}} g_{x_{-n}} - q_x\|_\infty \le A(x)B^n$$

for a.e. $x\in X$ and every $n\ge 1$.

6. Fractal structure of Conformal Random Expanding Repellers

In this section we deal with *conformal* random dynamical systems. We prove an appropriate version of Bowen's Formula, show that typically Hausdorff and packing measures on fibers respectively vanish and are infinite, and at the end we perform multifractal analysis of Gibbs states.

Definition 6.1. *Let the ambient space Y be a smooth Riemannian manifold and assume that we deal with mappings $f_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$ that can be extended to a neighborhood of $\mathcal{J}_x$ in Y to conformal $C^{1+\alpha}$ mappings. If in addition*

$$f : (x,z) \mapsto (\theta(x), f_x(z))$$

is a measurably expanding RDS in the sense of Definition 2.1 then we call the system conformal measurably expanding.

By $|f'_x(y)|$ we denote the norm of the derivative of f_x and by $||f'_x||_\infty$ its supremum over $z \in \mathcal{J}_x$. Since the system is expanding we have

$$|f'_x| \geq \gamma_x \quad \text{for a.e.} \quad x \in X, \tag{6.1}$$

where $|f'_x|$ is the similarity factor of the derivative f'_x.

6.1. Bowen's Formula.

For every $t \in \mathbb{R}$ we consider the potential

$$\varphi_t(x,z) = -t \log |f'_x(z)|.$$

The associated topological pressure $P(\varphi_t)$ will be denoted $P(t)$. Let

$$E(P(t)) = \int_X P_x(t)dm(x)$$

be its expected value with respect to the measure m. In view of (6.1), it follows from Lemma 5.1 that the function $t \mapsto E(P(t))$ has a unique zero. Denote it by h. The result of this subsection is the following version of Bowen's formula, identifying the Hausdorff dimension of almost all fibers with the parameter h.

Theorem 6.2 (Bowen's Formula)**.** *The parameter h, i.e. the zero of the function $t \mapsto E(P(t))$, is m-a.e. equal to the Hausdorff dimension $\mathrm{HD}(\mathcal{J}_x)$ of the fiber $\mathcal{J}_x$.*

Proof. Let $(\nu_x^h)_{x\in X}$ be the measures produced in Theorem 4.1 for the potential φ_h. Fix $x \in X$ and $z \in \mathcal{J}_x$ and set again $y = (x,z)$. For every $r \in (0,\xi]$ let $k = k(z,r)$ be the largest number $n \geq 0$ such that

$$B(z,r) \subset f_y^{-n}(B(f_x^n(z),\xi)). \tag{6.2}$$

By the expanding property this inclusion holds for all $0 \leq n \leq k$. Fix such an n. By Lemma 4.24,

$$\nu_x^h(B(z,r)) \leq \nu_x^h(f_y^{-n}(B(f_x^n(z),\xi))) \leq \exp\big(hQ_{\theta^n(x)}\big)|(f_x^n)'(z)|^{-h} \exp(-P_x^n(h)). \tag{6.3}$$

On the other hand,

$$B(z,r) \not\subset f_y^{-(s+1)}(B(f_x^{s+1}(z),\xi))$$

for every $s \geq k$. But, since by Lemma 3.3,

$$B(z, \exp(-Q_{\theta^{s+1}(x)}\xi^\alpha)|(f_x^{s+1})'(z)|^{-1}\xi) \subset f_y^{-(s+1)}(B(f_x^{s+1}(z),\xi)),$$

we get

$$\exp\big(-Q_{\theta^{s+1}(x)}\xi^\alpha\big)|(f_x^{s+1})'(z)|^{-1}\xi \leq r \tag{6.4}$$

and

$$|(f_x^s)'(z)|^{-1} \le \xi^{-1} \exp\big(Q_{\theta^{s+1}(x)}\xi^\alpha\big) r.$$

Inserting this to (6.3) we obtain,

(6.5)

$$\nu_x^h(B(z,r)) \le \xi^{-h} \exp\big(hQ_{\theta^n(x)}\big) \exp\big(hQ_{\theta^{s+1}(x)}\xi^\alpha\big) r^h \exp(-P_x^n(h)) |\big(f_{\theta^n(x)}^{s+1-n}(f_x^n(z))|^h.$$

or equivalently

$$\begin{aligned}(6.6)\quad \frac{\log \nu_x^h(B(z,r))}{\log r} \ge h + \frac{Q_{\theta^n(x)}}{\log r} + \frac{hQ_{\theta^{s+1}(x)}\xi^\alpha}{\log r} + \frac{-h\log\Big(\Big|\big(f_{\theta^n(x)}^{s+1-n}(f_x^n(z))\Big|\Big)}{\log r} \\ + \frac{-h\log\xi}{\log r} + \frac{-P_x^n(h))}{\log r}.\end{aligned}$$

Our goal is to show that

$$\liminf_{r\to 0} \frac{\log \nu_x^h(B(z,r))}{\log r} \ge h \ \text{ for a.e. } \ x \in X \ \text{ and all } \ z \in \mathcal{J}_x.$$

Since the function $x \mapsto Q_x$ is measurable and almost everywhere finite, there exists $M > 0$ such that $m(A) > 0$, where

$$A = \{x \in X : Q_x \le M\}.$$

Fix $n = n_k \ge 0$ to be the largest integer $\le k$ such that $\theta^n(x) \in A$ and $s = s_k$ to be the least integer $\ge k$ such that $\theta^{s+1}(x) \in A$. It follows from Birkhoff's Ergodic Theorem that

$$\lim_{k\to\infty} s_k/n_k = 1.$$

Now, note that by (6.2), the formula

$$f_y^{-n}(B(f_x^n(z),\xi)) \subset B(z, \exp(Q_{\theta^n(x)}\xi^\alpha)|(f_x^n)'(z)|^{-1}\xi)$$

yields

$$r \le \exp(Q_{\theta^n(x)}\xi^\alpha)|(f_x^n)'(z)|^{-1}\xi.$$

Equivalently,

$$-\log r \ge \log|(f_x^n)'(z)| - \xi^\alpha Q_{\theta^n(x)} - \log\xi.$$

Since $\log|(f_x^n)'(z)| \ge \log\gamma_x^n$ and since the function $x \mapsto \log\gamma_x$ is integrable, $\chi = \int \log\gamma dm > 0$, we thus get from Birkhoff's Ergodic Theorem that for a.e. $x \in X$ and all $r > 0$ small enough (so k and n_k and s_k large enough too)

$$(6.7)\qquad -\log r \ge \frac{\chi}{2} n \ge \frac{\chi}{3} s.$$

Remember that $\theta^n(x) \in A$ and $\theta^{s+1}(x) \in A$. We thus obtain from (6.6) that

$$(6.8)\quad \liminf_{r\to 0} \frac{\log \nu_x^h(B(z,r))}{\log r} \ge h - 3h \limsup_{k\to\infty} \frac{1}{s} \log\Big(\Big|\big(f_{\theta^n(x)}^{s+1-n}(f_x^n(z))\Big|\Big) - 2\frac{1}{n} P_x^n(h).$$

for a.e. $x \in X$ and all $z \in \mathcal{J}_x$. But as $\int P_x(h)dm(x) = 0$, we have by Birkhoff's Ergodic Theorem that

$$(6.9)\qquad \lim_{n\to\infty} \frac{1}{n} P_x^n(h) = 0.$$

Also, since the measure μ^h is f-invariant, it follows from Birkhoff's Ergodic Theorem that there exists a measurable set $X_0 \subset X$ such that for every $x \in X_0$ there exists at least one (in fact of full measure μ_x) $z_x \in \mathcal{J}_x$ such that

$$\lim_{j\to\infty} \frac{1}{j} \log \left|\left(f_x^j\right)(z_x)\right| = \hat{\chi} := \int_{\mathcal{J}} \log |f_x'(z)| d\mu^h(x,z) \in (0,+\infty).$$

Hence, remembering that $\theta^n(x)$ and $\theta^{s+1}(x)$ belong to A, we get

$$\begin{aligned}\limsup_{k\to\infty} \frac{1}{s} \log\left(\left|\left(f_{\theta^n(x)}^{s+1-n}(f_x^n(z))\right|\right) &= \limsup_{k\to\infty} \frac{1}{s}\left(\log\left|\left(f_x^{s+1}(z)\right| - \log\left|\left(f_x^n(z)\right|\right)\right.\right.\\ &= \limsup_{k\to\infty} \frac{1}{s}\left(\log\left|\left(f_x^{s+1}(z_x)\right| - \log\left|\left(f_x^n(z_x)\right|\right)\right.\right.\\ &\le \limsup_{k\to\infty} \frac{1}{s} \log\left|\left(f_x^{s+1}(z_x)\right| - \liminf_{k\to\infty} \frac{1}{s}\log\left|\left(f_x^n(z_x)\right|\right.\right.\\ &= \hat{\chi} - \hat{\chi} = 0.\end{aligned}$$

Inserting this and (6.9) to (6.8) we get that

$$\liminf_{r\to 0} \frac{\log \nu_x^h(B(z,r))}{\log r} \ge h. \tag{6.10}$$

Keep $x \in X$, $z \in \mathcal{J}_x$ and $r \in (0,\xi]$. Now, let $l = l(z,r)$ be the least integer ≥ 0 such that

$$f_y^{-l}(B(f_x^l(z),\xi)) \subset B(z,r). \tag{6.11}$$

Then, by Lemma 4.24,

$$\begin{aligned}\nu_x^h(B(z,r)) &\ge \nu_x(f_y^{-l}(B(f_x^l(z),\xi)))\\ &\ge D_1(\theta^l(x)) \exp\left(-Q_{\theta^l(x)}\right) |(f_x^l)'(z)|^{-l} \exp(-P_x^l(h)).\end{aligned} \tag{6.12}$$

On the other hand

$$f_y^{-(l-1)}(B(f_x^{l-1}(z),\xi)) \not\subset B(z,r).$$

But, since

$$f_y^{-(l-1)}(B(f_x^{l-1}(z),\xi)) \subset B(y, \exp(Q_{\theta^{l-1}}(x)\xi^\alpha)|(f_x^{l-1})'(z)|^{-1}\xi),$$

we get

$$r \le \xi \exp(Q_{\theta^{l-1}(x)}\xi^\alpha)|(f_x^{l-1})'(y)|^{-1}. \tag{6.13}$$

Thus

$$|(f_x^{l-1})'(z)|^{-1} \ge \xi^{-1} \exp\left(-Q_{\theta^{l-1}(x)}\xi^\alpha\right) r.$$

Iserting this to (6.12) we obtain,

$$\begin{aligned}\nu_x^h(B(z,r)) \ge \xi^{-h} D_1(\theta^l(x)) e^{-Q_{\theta^l(x)}} |(f_{\theta^{l-1}(x)})'(f_x^{l-1}(z))|^{-h}\cdot\\ \exp\left(-hQ_{\theta^{l-1}(x)}\xi^\alpha\right) r^h \exp(-P_x^l(h)).\end{aligned} \tag{6.14}$$

Now, given any integer $j \ge 1$ large enough, take $R_j > 0$ to be the least radius $r > 0$ such that $f_y^{-j}(B(f_x^j(z),\xi)) \subset B(z,r)$. Then $l(y,R_j) = j$. Since the function Q is measurable and almost everywhere finite, and θ is a measure-preserving transformation, there exist a set $\Gamma \subset X$ with positive measure m and a constant $E > 0$ such that $Q_x \le E$, $D_1(x) \le E$ and $Q_{\theta^{-1}(x)} \le E$ for all $x \in \Gamma$. It follows from Birkhoff's Ergodic Theorem and ergodicity of the map $\theta : X \to X$ that there exists a measurable set $X_1 \subset X$ with $m(X_1) = 1$ such that for every $x \in X_1$ there exists an unbounded increasing sequence $(j_i)_{i=1}^\infty$ such that $\theta^{j_i}(x) \in \Gamma$ for all $i \geq 1$. Formula (6.13) then yields

$$-\log R_{j_i} \geq -E\xi^\alpha + \log \xi + \log |(f_x^{j_i-1}(z)| \geq -E\xi^\alpha + \log \xi + \log \gamma_x^{j_i-1} \geq \frac{\chi}{2} j_i,$$

where the last inequality was written because of the same argument as (6.7) was, intersecting also X_1 with an apropriate measurable set of measure 1. Now we get from (6.14) that

$$\frac{\log \nu_x^h\big(B(z, R_{j_i})\big)}{\log R_{j_i}} \leq h + \frac{2\log E}{\chi j_i} - \frac{2E}{\chi j_i} - \frac{2h}{\chi}\frac{1}{j_i}\log ||(f_{\theta^{j_i-1}(x)})'||_\infty - \frac{2h\xi^\alpha E}{\chi j_i} - \frac{2h\log \xi}{\chi j_i} - \frac{2}{\chi}\frac{1}{j_i}P_x^{j_i}(h).$$

Noting that $\int_X P_x(t)dm(x) = 0$ and applying Birkhoff's Ergodic Theorem, we see that the last term in the above estimate converges to zero. Also $\frac{1}{j_i}\log ||(f_{\theta^{j_i-1}(x)})'||_\infty$ converges to zero because of Birkhoff's Ergodic Theorem and itegrability of the function $x \mapsto \log ||f_x'||_\infty$. Since all the other terms obviously converge to zero, we thus get for a.e. $x \in X$ and all $z \in \mathcal{J}_x$, that

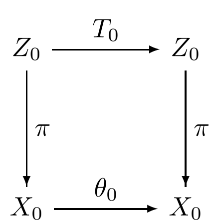
$$\liminf_{r\to 0} \frac{\log \nu_x^h(B(z,r))}{\log r} \leq \liminf_{i\to\infty} \frac{\log \nu_x^h\big(B(z, R_{j_i})\big)}{\log R_{j_i}} \leq h.$$

Combining this with (6.10), we obtain that

$$\liminf_{r\to 0} \frac{\log \nu_x^h(B(z,r))}{\log r} = h$$

for a.e. $x \in X$ and all $z \in \mathcal{J}_x$. This gives that $\mathrm{HD}(\mathcal{J}_x) = h$ for a.e. $x \in X$. We are done. □

6.2. **Random Conformal Expanding Repellers; Hausdorff and Packing measures.** In order to investigate the finer fractal structure of random expanding repellers we need the "measurable constants" defining random distance expanding maps to be absolute and we also need an appropriate version of the law of iterated logarithm. This is the case if the assumptions of Theorem 3.2 in [7] are satisfied. We want to describe this setting now. So, suppose that X_0 and Z_0 are compact metric spaces, $\theta_0 : X_0 \to X_0$ and $T_0 : Z_0 \to Z_0$ are open topologically exact distance expanding maps in the sense as in [13]. We assume that T_0 is a skew-product over Z, i.e. for every $x \in X_0$ there exists a compact metric space $\mathcal{J}_x$ such that $Z_0 = \bigcup_{X\in X_0}\{x\} \times \mathcal{J}_x$ and the following diagram commutes



where $\pi(x, y) = x$ and the projection $\pi : Z_0 \to X_0$ is an open map. We then refer to $T_0 : Z_0 \to Z_0$ and $\theta_0 : X_0 \to X_0$ as a DG system. Note that $T_0(\{x\} \times \mathcal{J}_x) \subset \{\theta_0(x)\} \times \mathcal{J}_{\theta_0(x)}$ and this gives rise to the map $T_x : \mathcal{J}_x \to \mathcal{J}_{\theta_0(x)}$. Since T_0 is distance expanding, conditions (A), (B), (C), (C') and (D) hold with some constants

$\gamma_x \geq \gamma > 1$, $deg(T_x) \leq N_1 < +\infty$ and the numbers $n_r = n_r(x)$ and $j = j(x)$ in fact independent of x. Scrutinizing the proof of Remark 2.9 in [7] one sees that Lipschitz continuity (Denker and Gordin assume differentiability) suffices for it to go through and Lipschitz continuity is incorporated in the definition of expanding maps in [13]. Now assume that $\phi : Z \to \mathbb{R}$ is a Hölder continuous map. Then the hypothesis of Theorems 2.10, 3.1, and 3.2 from [7] are satisfied. Their claims are summarized in the following.

Theorem 6.3. *Suppose that $T_0 : Z_0 \to Z_0$ and $\theta_0 : X_0 \to X_0$ form a DG system and that $\phi : Z \to \mathbb{R}$ is a Hölder continuous potential. Then there exists a Hölder continuous function $P(\phi) : X_0 \to \mathbb{R}$, a measurable collection $\{\nu_x\}_{x \in X_0}$ and a continuous function $q : Z_0 \to [0, +\infty)$ such that*

(a) $\nu_{\theta_0(x)}(A) = \exp\big(P_x(\phi)\big) \int_A e^{-\phi_x} d\nu_x$ *for all $x \in X_0$ and all Borel sets $A \subset \mathcal{J}_x$ such that $T_x|_A$ is one-to-one.*

(b) $\int_{Y_x} q_x d\nu_x = 1$ *for all $x \in X_0$.*

(c) *Denoting for every $x \in X_0$ by μ_x the measure $q_x \nu_x$ we have*

$$\sum_{w \in \theta_0^{-1}(x)} \mu_w(T_w^{-1}(A)) = \mu_x(A)$$

for every Borel set $A \subset \mathcal{J}_x$.

This would mean that we got all the objects produced in Section 4 of our paper. However, the map $\theta_0 : X_0 \to X_0$ need not be, and apart from the case when X_0 is finite, is not invertible. But to remedy this situation is easy. We consider the projective limit (Rokhlin's natural extension) $\theta : X \to X$ of $\theta_0 : X_0 \to X_0$. Precisely,

$$X = \{(x_n)_n \leq 0 : \theta_0(x_n) = x_{n+1}\ \forall n \leq -1\}$$

and

$$\theta\big((x_n)_n \leq 0\big) = (\theta_0(x_n))_n \leq 0.$$

Then $\theta : X \to X$ becomes invertible and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ \theta\ } & X \\ \big\downarrow p & & \big\downarrow p \\ X_0 & \xrightarrow{\ \theta_0\ } & X_0 \end{array} \tag{6.15}$$

commutes, where $p\big((x_n)_n \leq 0\big) = x_0$. If in addtion, as we assume from now on, the space X is endowed with a Borel probability θ_0-invariant ergodic measure m_0, then there exists a unique θ-invariant probability measure measure m such that $m \circ \pi^{-1} = m_0$. Let

$$Z := \bigcup_{x \in X} \{x\} \times \mathcal{J}_{x_0}.$$

We define the map $T : Z \to Z$ by the formula

$$T(x, y) = (\theta(x), T_{x_0}(y))$$

and the potential $X \ni x \mapsto \phi(x_0)$ from X to $\mathbb{R}$. We keep for it the same symbol ϕ. Clearly the quadruple (T, θ, m, ϕ) is a Hölder fiber system as defined in Section 2 of our paper. It follows from Theorem 6.3 along with the definition of θ a commutativity of the diagram (6.15) for $x \in X$ all the objects $P_x(\phi) = P_{x_0}(\phi)$,

$\lambda_x = \exp(P_x(\phi))$, $q_x = q_{x_0}$, $\nu_x = \nu_{x_0}$, and $\mu_x = \mu_{x_0}$ enjoy all the properties required in Theorem 4.1 and Theorem 4.2; in particular they are unique. From now on we assume that the measure m is a Gibbs state of a Hölder continuous potential on X (having nothing to do with ϕ or $P(\phi)$; it is only needed for the Law of Iterated Logarithm to hold). The quadruple (T, θ, m, ϕ) is then called a Gibbs-Hölder fiber system (or G-system for short). Since the map $\theta_0 : X_0 \to X_0$ is expanding, since m is a Gibbs state, and since $P(\phi) : X_0 \to \mathbb{R}$ is Hölder continuous, it is well-known (see [13] for example) that the following limit exists

$$\sigma^2(P(\phi)) = \lim_{n\to\infty} \frac{1}{n} \int \Big(S_n(P(\phi)) - nEP(\phi)\Big)^2 dm.$$

This limit is commonly refered to as the asymptotic variance of the function $P(\phi)$. The following theorem of Livsic flavor is (by now) well-known (see [13]).

Theorem 6.4. *Suppose (T, θ, m, ϕ) is a G-Hölder fiber system. Then the following are equivalent.*

(a) $\sigma^2(P(\phi)) = 0$.
(b) *The function $P(\phi)$ is cohomologous to a constant in the class of real-valued continuous functions on X (X_0), meaning that there exists a continuous function $u : X\,(X_0) \to \mathbb{R}$ such that $P(\phi) - (u - u \circ \theta\,(\theta_0)$ is a constant.*
(c) *The function $P(\phi)$ is cohomologous to a constant in the class of real-valued Hölder continuous functions on X (X_0), meaning that there exists a Hölder continuous function $u : X\,(X_0) \to \mathbb{R}$ such that $P(\phi) - (u - u \circ \theta\,(\theta_0)$ is a constant.*
(d) *There exists $R \in \mathbb{R}$ such that $P_x^n(\phi) = nR$ for all $n \geq 1$ and all periodic points $x \in X\,(X_0)$.*

As a matter of fact such theorem is formulated in [13] for non-invertible (θ_0) maps only but it also holds for the Rokhlin's natural extension θ. The following theorem follows directly from [13] and Theorem 6.3 (Hölder continuity of $P(\phi)$).

Theorem 6.5. *(the Law of Iterated Logarithm) If (T, θ, m, ϕ) is a G-Hölder fiber system and $\sigma^2(P(\phi)) > 0$, then*

$$\begin{aligned} -\sqrt{2\sigma^2(P(\phi))} &= \liminf_{n\to\infty} \frac{P_x^n(\phi) - nE(P(\phi))}{\sqrt{n \log\log n}} \\ &\leq \limsup_{n\to\infty} \frac{P_x^n(\phi) - nE(P(\phi))}{\sqrt{n \log\log n}} = \sqrt{2\sigma^2(P(\phi))} \qquad m-a.e. \end{aligned}$$

Now we turn to geometry. Suppose that (f_0, θ_0) is a DG-system endowed with a Gibbs measure m_0 at the base. Suppose also that this system is a random conformal expanding repeller in the sense of Section 6.1 and that the function $\phi : Z \to \mathbb{R}$ given by the formula

$$\phi(x, y) = -\log|f_x'(y)|,$$

is Hölder continuous.

Definition 6.6. *The corresponding system $(f, \theta, m) = (f, \theta, m, \phi)$ (with θ the Rokhlin natural extension of θ_0 as described above) is called conformal random expanding Gibbs-system, or shortly random conformal G-system.*

For every $t \in \mathbb{R}$ the potential $\phi_t = t\phi$, considered in Section 6.2, is also Hölder continuous. As in Section 6.2 denote its topological pressure by $P(t)$. As a direct consequence of Theorem 6.5, we get the following.

Theorem 6.7. *If $(f,\theta,m)=(f,\theta,m,\phi)$ is a random conformal G-system, then*

$$-\sqrt{2\sigma^2(P(t))}=\liminf_{n\to\infty}\frac{P_x^n(t)-nE(P(t))}{\sqrt{n\log\log n}}$$
$$\le\limsup_{n\to\infty}\frac{P_x^n(t)-nE(P(t))}{\sqrt{n\log\log n}}=\sqrt{2\sigma^2(P(t))}$$

for m-a.e. $x\in X$ whenever $\sigma^2(P(t))>0$.

Recall that h is a unique solution to the equation $EP(t)=0$. By Theorem 6.2 (Bowen's Formula) $\mathrm{HD}(\mathcal{J}_x)=h$ for m-a.e. $x\in X$.

Definition 6.8. *We call the random conformal G-system (f,θ,m) quasi-deterministic if and only if $\sigma^2(P(h))=0$; otherwise we call it essential.*

For every $\alpha>0$ let $\mathcal{H}^\alpha$ refer to the α-dimensional Hausdorff measure and let $\mathcal{P}^\alpha$ refer to the α-dimensional packing measure. Recall that a Borel probability measure μ defined on a metric space M is geometric with an exponent α if and only if there exist $A\ge 1$ and $R>0$ such that

$$A^{-1}r^\alpha\le\mu(B(x,r))\le Ar^\alpha$$

for all $x\in M$ and all $0\le r\le R$. The most significant basic properties of geometric measures are the following.

- The measures μ, $\mathcal{H}^\alpha$, and $\mathcal{P}^\alpha$ are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity.
- $0<\mathcal{H}^\alpha(M),\mathcal{P}^\alpha(M)<+\infty$.
- $\mathrm{HD}(M)=h$.

The main result of this section is the following.

Theorem 6.9. *Suppose $(f,\theta,m)=(f,\theta,m,\phi)$ is a random conformal G-system.*

(f) *If (f,θ,m) is essential, then $\mathcal{H}^h(Y_x)=0$ and $\mathcal{P}^h(Y_x)=+\infty$.*
(g) *If, on the other hand, $(f,\theta,m)=(f,\theta,m,\phi)$ is quasi-deterministic, then for every $x\in X$,*
  (g1) *ν_x^h is a geometric measure with exponent h.*
  (g2) *The measures ν_x^h, $\mathcal{H}^h|_{Y_x}$, and $\mathcal{P}^h|_{Y_x}$ are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity independently of $x\in X$ and $y\in Y_x$.*
  (g3) *$0<\mathcal{H}^h(Y_x),\mathcal{P}^h(Y_x)<+\infty$.*
  (g4) *$\mathrm{HD}(Y_x)=h$.*

Proof. Part (f). Remember that by its very definition $EP(h)=\int P_x(h)dm(x)=0$. Fix $\kappa\in(0,\sqrt{2\sigma^2(P(h))})$. It then follows from Theorem 6.7 that there exists a measurable set X_1 with $m(X_1)=1$ such that for every $x\in X_1$ there exists an increasing unbounded sequence $(n_j)_{j=1}^\infty$ (depending on x) of positive integers such that $P_x^{n_j}(h)\le-\kappa\sqrt{n_j\log\log n_j}$ for all $j\ge 1$, or equivalently,

$$\exp\bigl(-P_x^{n_j}(h)\bigr)\ge\exp\bigl(\kappa\sqrt{n_j\log\log n_j}\bigr). \tag{6.16}$$

Since we are in the expanding case, formula (6.12) from the proof of Theorem 6.2 (Bowen's Formula) takes on the following simplyfied form

$$\nu_x(B(z,r))\ge D^{-1}r^h\exp\bigl(-P_x^{l(z,r)}(h)\bigr) \tag{6.17}$$

with some $D \geq 1$ and all $z \in \mathcal{J}_x$. By our uniform assumptions, for all $j \geq 1$ large enough, so disregarding finitely many terms, we may assume without loss of generality, that for all $j \geq 1$, there exists $r_j > 0$ such that $l(z, r_j) = n_j$. Clearly,

$$\lim_{j\to\infty} r_j = 0.$$

It thus follows from (6.17) and (6.16) that

$$\nu_x^h(B(z, r_j)) \geq D^{-1} r_j^h \exp\bigl(-P_x^{n_j}(h)\bigr) \geq D^{-1} r_j^h \exp\bigl(\kappa\sqrt{n_j \log\log n_j}\bigr)$$

for all $x \in X_1$, all $z \in \mathcal{J}_x$ and all $j \geq 1$. Therefore,

$$\limsup_{r\to 0} \frac{\nu_x^h(B(z,r))}{r^h} \geq \limsup_{j\to\infty} \frac{\nu_x^h(B(z,r_j))}{r_j^h} D^{-1} \limsup_{j\to\infty} \exp\bigl(\kappa\sqrt{n_j \log\log n_j}\bigr) = +\infty.$$

Thus $\mathcal{H}^h(\mathcal{J}_x) = 0$. The proof for packing measures is similar. Keep $\kappa \in (0, \sqrt{2\sigma^2(P(h))})$. It then follows from Theorem 6.7 that there exists a measurable set X_2 with $m(X_2) = 1$ such that for every $x \in X_2$ there exists an increasing unbounded sequence $(s_j)_{j=1}^\infty$ (depending on x) of positive integers such that $P_x^{s_j}(h) \geq \kappa\sqrt{s_j \log\log n_s}$ for all $j \geq 1$, or equivalently,

$$\exp\bigl(-P_x^{s_j}(h)\bigr) \leq \exp\bigl(-\kappa\sqrt{s_j \log\log s_j}\bigr). \tag{6.18}$$

Since we are in the expanding case, formula (6.5) from the proof of Theorem 6.2 (Bowen's Formula), applied with $s = k(z, r)$ takes on the following simplified form.

$$\nu_x(B(z, r)) \leq D r^h \exp\bigl(-P_x^{k(z,r)}(h)\bigr) \tag{6.19}$$

with $D \geq 1$ sufficiently large, all $x \in X_2$ and all $z \in \mathcal{J}_x$. By our uniform assumptions, for all $j \geq 1$ large enough, so disregarding finitely many terms, we may assume without loss of generality, that for all $j \geq 1$, there exists $R_j > 0$ such that $k(z, R_j) = s_j$. Clearly,

$$\lim_{j\to\infty} R_j = 0.$$

It thus follows from (6.19) and (6.18) that

$$\nu_x^h(B(z, r_j)) \leq D R_j^h \exp\bigl(-P_x^{s_j}(h)\bigr) \geq D R_j^h \exp\bigl(\kappa\sqrt{s_j \log\log s_j}\bigr)$$

for all $x \in X_2$, all $z \in \mathcal{J}_x$ and all $j \geq 1$. Therefore,

$$\liminf_{r\to 0} \frac{\nu_x^h(B(z,r))}{r^h} \geq \liminf_{j\to\infty} \frac{\nu_x^h(B(z,R_j))}{R_j^h} D \liminf_{j\to\infty} \exp\bigl(-\kappa\sqrt{s_j \log\log s_j}\bigr) = 0.$$

Thus $\mathcal{P}^h(\mathcal{J}_x) = +\infty$. We are done with part (f).

Suppose now that the system $(f, \theta, m) = (f, \theta, m, \phi)$ is quasi-deterministic. It then follows from Theorem 6.4 that there exists a continuous function $u : X \to \mathbb{R}$ such that $P_x(h) = a + u(x) - u(f(x))$, $x \in X$, with some constant $a \in \mathbb{R}$. But $a = 0$ since $\int P_x(h) dm(x) = 0$. Thus $P_x(h) = u(x) - u(f(x))$. Hence, for every $n \geq 1$, $P_x^n(h) = u(x) - u(f^n(x))$. Thus

$$\exp(\inf(u) - \sup(u)) := A^{-1} \leq \exp\bigl(-P_x^n(h)\bigr) \leq A := \exp(\sup(u) - \inf(u))$$

for every $x \in X$ and every $n \geq 1$. So, utilizing (6.17) and (6.19), we get for every $r > 0$ small enough independently of $x \in X$,

$$(AD)^{-1} r^h \leq \nu_x^h(B(y, r)) \leq ADr^h, \quad x \in X, \ z \in \mathcal{J}_x.$$

This means that each ν_x^h, $x \in X$, is a geometric measure with exponent h. Consequently, $\mathcal{H}^h|_{\mathcal{J}_x}$, $\mathcal{P}^h|_{\mathcal{J}_x}$, and ν_x^h are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity independently of $x \in X$ and $z \in \mathcal{J}_x$, $0 < \mathcal{H}^h(\mathcal{J}_x), \mathcal{P}^h(\mathcal{J}_x) < +\infty$, and $\mathrm{HD}(\mathcal{J}_x) = h$ for all $x \in X$. We are done. □

As an immediate consequence of this theorem we get a corollary transparently stating that essential random conformal G-systems are entirely new objects sharing no common ground with determinisitc self-conformal sets.

Corollary 6.10. *Suppose $(f, \theta, m) = (f, \theta, m, \phi)$ is an essential random conformal G-system. Then*

(1) *For m-a.e. $x \in X$ the fiber J_x is not bi-Lipschitz equivalent to any deterministic nor quasi-deterministic self-conformal set.*
(2) *$\mathcal{J}_x$ is not a geometric circle nor even a piecewise smooth curve.*
(3) *If $\mathcal{J}_x$ has a non-degenerate connected component (for example if $\mathcal{J}_x$ is connected, then $h = \mathrm{HD}(\mathcal{J}_x) > 1$.*

7. Multifractal analysis

Let $\varphi \in H_m(Z)$ be such that $E(P(\varphi)) = 0$. Fix $q \in \mathbb{R}$. We will not use the function q_x and therefore this will not cause any confusion. Define auxiliary potentials

$$\varphi_x^{q,t} x(y) := q(\varphi_x(y) - P_x(\varphi)) - t \log |f_x'(y)|.$$

Since $\log |f_x'(y)| \geq \log \gamma_x > 0$, it follows from Lemma 5.1 that for every $q \in \mathbb{R}$ there exists a unique $T(q) \in \mathbb{R}$ such that

$$E(P(\varphi^{q,T(q)})) = 0.$$

Then put

$$\varphi^q := \varphi^{q,T(q)}.$$

Let μ be the invariant Gibbs measure for φ and ν be the φ-conformal measure. For every $\alpha \in \mathbb{R}$ define

$$K_x(\alpha) := \Big\{ y \in Y_x : d_{\mu_x}(y) := \lim_{r \to 0} \frac{\log \mu_x(B(y,r))}{\log r} = \alpha \Big\}.$$

and

$$K_x' := \Big\{ y \in Y_x : \text{the limit } \lim_{r \to 0} \frac{\log \mu_x(B(y,r))}{\log r} \text{ does not exist} \Big\}.$$

This gives us the multifractal decomposition

$$Y_x := \biguplus_{\alpha \geq 0} K_x(\alpha) \uplus K_x'.$$

The multifractal spectrum is the family of functions $\{g_{\mu_x}\}_{x \in X}$ given by the formulas

$$g_{\mu_x}(\alpha) := \mathrm{HD}(K_x(\alpha)).$$

The function $d_{\mu_x}(y)$ is called the local dimension of the measure μ_x at the point y and d_ρ has an analogous meaning for any measure ρ. Since for m almost every $x \in X$ the measures μ_x and ν_x are equivalent with Radon-Nikodym derivatives uniformly separated from 0 and infinity (though the bounds may and usually do depend on x), we conclude that we get the same set $K_x(\alpha)$ if in its definition the measure μ_x is replaced by ν_x. Our goal now is to get a "smooth" formula for g_{μ_x}.

Let μ^q and ν^q be the measures for the potential φ^q given by Theorem 4.1.

Proposition 7.1. *For every $q \in \mathbb{R}$ there exists a measurable set $X_q \subset X$ with $m(X_q) = 1$ and such that, for every $x \in X_q$,*

$$g_{\mu_x}(\alpha(q)) = q\alpha(q) + T(q)$$

where

$$\alpha(q) = -\frac{\int_Z \varphi \, d\mu^q}{\int_Z \log |f'| \, d\mu^q}.$$

Proof. It follows from Lemma 4.24 that for m-a.e. $x \in X$ and all $y \in Y_x$, we have

$$(F_R^q(\theta^n(x)))^{-1} \leq \frac{\nu_x^q(f_y^{-n}(B(f^n(y), R)))}{\exp\big(q(S_n\varphi(y) - P_x^n(\varphi))\big)|(f_x^n)'(y)|^{-T(q)}} \leq F_R^q(\theta^n(x)) \tag{7.1}$$

with some measurable function $F_{R^q} : X \to [1, +\infty)$ depending on R and q. In what follows we keep the notation from the proof of Theorem 6.2. The formulas (6.2) and (6.11) then give for every $j \ge l$ and every $0 \le i \le k$, that

$$(7.2)\quad (F_\xi^q(\theta^j(x)))^{-1} \exp\big(q(S_j\varphi(y) - P_x^j(\varphi))\big)|(f_x^j)'(y)|^{-T(q)} \\ \le \nu_x^q(B(y,r)) \le F_\xi^q(\theta^i(x))) \exp\big(q(S_i\varphi(y) - P_x^i(\varphi))\big)|(f_x^i)'(y)|^{-T(q)}.$$

Now, as in the proof of Theorem 6.2, there exist a measurable set $A = A_q(\xi)$ with positive measure m and a constant $M = M_\xi \ge 1$ such that

$$M^{-1} \le Q_x, F_\xi^q(x), F_\xi^1(x) \le M$$

for all $x \in A$. For every $s \ge 1$ let s_- be the largest integer in $[0, s-1]$ such that $\theta^{s_-}(x) \in A$ and let s_+ be the least integer in $[0, s+1]$ such that $\theta^{s_+}(x) \in A$. It follows from (7.2) applied with $j = l_+$ and $i = k_-$, (6.4) true with $s+1$ replaced by k_+, and (6.13) true with $l-1$ replaced by l_-, that

$$\frac{\log \nu_x^q(B(y,r))}{\log r} \le \frac{-\log M + q\big(S_{l_+}\varphi(y) - P_x^{l_+}(\varphi)\big) - T(q)\log|(f_x^{l_+})'(y)|}{\log \xi + \xi^\alpha M - \log|(f_x^{l_-})'(y)|}$$

and

$$\frac{\log \nu_x^q(B(y,r))}{\log r} \ge \frac{\log M + q\big(S_{k_-}\varphi(y) - P_x^{k_-}(\varphi)\big) - T(q)\log|(f_x^{k_-})'(y)|}{\log \xi - \xi^\alpha M - \log|(f_x^{k_+})'(y)|}.$$

Hence,

$$(7.3)\quad \limsup_{r\to 0} \frac{\log \nu_x^q(B(y,r))}{\log r} \\ \le \limsup_{n\to\infty} \left(q\frac{P_x^{n_+}(\varphi) - S_{n_+}\varphi(y)}{\log|(f_x^{n_-})'(y)|} \right) + T(q) \limsup_{n\to\infty} \frac{\log|(f_x^{n_+})'(y)|}{\log|(f_x^{n_-})'(y)|}$$

and

(7.4)

$$\liminf_{r\to 0} \frac{\log \nu_x^q(B(y,r))}{\log r} \ge \liminf_{n\to\infty} \left(q\frac{P_x^{n_-}(\varphi) - S_{n_-}\varphi(y)}{\log|(f_x^{n_+})'(y)|} \right) + T(q) \liminf_{n\to\infty} \frac{\log|(f_x^{n_-})'(y)|}{\log|(f_x^{n_+})'(y)|}.$$

Now, for every $\alpha \in \mathbb{R}$ let

$$\tilde{K}_x(\alpha) = \left\{ y \in Y_x : \lim_{n\to\infty} \frac{P_x^n(\varphi) - S_n\varphi(y)}{\log|(f_x^n)'(y)|} = \alpha \;\text{ and }\; \lim_{n\to\infty} \frac{\log|(f_x^{n_-})'(y)|}{\log|(f_x^{n_+})'(y)|} = 1 \right\}.$$

It then readily follws from (7.3) and (7.4) (recall that $\nu_x^1 = \nu_x$ and $T(1) = 0$) that

$$(7.5)\quad \tilde{K}_x(\alpha) \subset K_x(\alpha)$$

and

$$(7.6)\quad d_{\nu_x^q}(y) = q\alpha + T(q)$$

for all $y \in \tilde{K}_x(\alpha)$. Given $q \in \mathbb{R}$, set

$$(7.7)\quad \alpha(q) = -\frac{\int_Z \varphi\, d\mu^q}{\int_Z \log|f'|\, d\mu^q}$$

In view of Birkhoff's Ergodic Theorem there exists a measurable set $X_q' \subset X$ such that $m(X_q')=1$ and for all $x\in X_q'$ and ν_x^q-a.e. $y\in Y_x$,

$$\lim_{n\to\infty}\frac{1}{n}\log|(f_x^n)'(y)| = \chi := \int_Z \log|f'|d\mu^q > 0,$$

$$\lim_{n\to\infty}\frac{1}{n}P_x^n(\varphi) = \int_{X_q'} P_x(\varphi)dm(x) = E(P(\varphi)) = 0$$

and

$$\lim_{n\to\infty}\frac{1}{n}S_n\varphi(y) = \int_Z \varphi d\mu.$$

Since it also follows from Birkhoff's Ergodic Theorem that for ν_x^q-a.e. $y\in Y_x$ (after intersecting X_q' with another set of measure 1), $\lim_{n\to\infty}\frac{n_-}{n_+}=1$, we thus conclude that for every $x\in X_q'$, $\nu_x^q(\tilde{K}_x(\alpha(q)))=1$. Hence, combining (7.5) and (7.6), we obtain for all $x\in X_q'$ that

$$g_{\mu_x}(\alpha(q)) = \mathrm{HD}(K_x(\alpha(q))) \ge \mathrm{HD}(\tilde{K}_x(\alpha(q))) = q\alpha(q)+T(q). \tag{7.8}$$

Let us now proceed to get the upper bound on $g_{\mu_x}(\alpha(q))$. First observe that there exist a measurable set $\hat{A} := \hat{A}_q(\xi)\subset A_q(\xi)$ with positive measure m and a constant $\hat{M}\ge 1$ such that $\hat{M}^{-1}\le F_{\xi\exp((2\xi^\alpha M)^{-2})}\le \hat{M}$ for all $x\in\hat{A}$. Fix $x\in X$ and suppose that $\theta^n(x)\in\hat{A}$. Let $r_n>0$ be the largest radius such that

$$B(y,r_n)\subset f_y^{-n}(B(f_x^n(y),\xi)). \tag{7.9}$$

It then follows from (7.1) that

$$\nu_x(B(y,r_n)) \le M\exp\big(S_n\varphi(y)-P_x^n(\varphi)\big). \tag{7.10}$$

In view of the definition of r_n, $B(y,2r_n)\not\subset f_y^{-n}(B(f_x^n(y),\xi))$. So, using Lemma 3.3, we therefore get

$$r_n \ge \frac{1}{2}\xi\exp\big((\xi^\alpha Q_{\theta^n(x)})^{-1}\big)|(f_x^n)'(y)|^{-1} \ge \xi\exp\big((2\xi^\alpha M)^{-1}\big)|(f_x^n)'(y)|^{-1}. \tag{7.11}$$

Combining this and (7.10), we get that

$$\frac{\log\nu_x(B(y,r_n))}{\log r_n} \ge \frac{\log M + S_n\varphi(y) - P_x^n(\varphi)}{\log\xi - 2\xi^\alpha M - \log|(f_x^n)'(y)|}. \tag{7.12}$$

Using (7.11) and employing Lemma 3.3 again, we obtain

$$B(y,r_n)\supset B(y,\xi\exp\big((2\xi^\alpha M)^{-1}\big)|(f_x^n)'(y)|^{-1}\big)\supset f_y^{-n}\Big(B\big(f_x^n(y),\xi\exp\big((2\xi^\alpha M)^{-2}\big)\big)\Big).$$

It therefore follws from (7.1) that

$$\nu_x^q(B(y,r_n)) \ge \hat{M}\exp\big(q\big(S_n\varphi(y)-P_x^n(\varphi)\big)\big)|(f_x^n)'(y)|^{-T(q)}.$$

In virtue of (7.9) and Lemma 3.3, we have

$$r_n \le \xi\exp\big(\xi^\alpha Q_{\theta^n(x)}\big)|(f_x^n)'(y)|^{-1} \ge \xi\exp\big(\xi^\alpha M\big)|(f_x^n)'(y)|^{-1}.$$

Hence,

$$\frac{\log\nu_x^q(B(y,r_n))}{\log r_n} \le \frac{-\log\hat{M} + q\big(S_n\varphi(y)-P_x^n(\varphi)\big) - T(q)\log|(f_x^n)'(y)|}{\log\xi + \xi^\alpha M - \log|(f_x^n)'(y)|}. \tag{7.13}$$

Since $m(\hat{A})>0$ and since the map $\theta: X\to X$ is ergodic, there exists a measurable set $X_q''\subset X$ with $m(X_q'')=1$ and such that for all $x\in X_q''$ the set $\{n\ge 1:\theta^n(x)\in$

$\hat{A}\}$ is infinite. Fix $x \in X''_q$ and then $y \in K_x(\alpha)$. It then follows from (7.12) and (7.13) that

$$\liminf_{r\to 0}\frac{\log \nu_x^q(B(y,r))}{\log r} \le \liminf_{n\to\infty}\frac{\log \nu_x^q(B(y,r_n))}{\log r_n} \le q\alpha + T(q).$$

Consequently, $\mathrm{HD}(K_x(\alpha)) \le q\alpha + T(q)$. In particular, $g_{\mu_x}(\alpha(q)) = \mathrm{HD}(K_x(\alpha(q))) \le q\alpha(q) + T(q)$. Combining this with (7.8), we obtain the proposition. □

Now, like in Section 5.3, we assume that the the essential infimum of γ_x is greater than some $\gamma > 1$ and the essential suprema of H_x, $n_\xi(x)$, $j(x)$ are finite. Moreover, about $\varphi \in \mathcal{H}_m(\mathcal{J})$ we assume that there exist constant L and $c > 0$ such that

$$S_n\varphi_x(y) \le -nc + L \tag{7.14}$$

for every $y \in Y_x$ and n and $E(P(\varphi)) = 0$. With these assumptions we can get the following property of the function T.

Corollary 7.2. *The temperature function T is differentiable and*

$$T'(q) = -\alpha(q) = \frac{\int_Z \varphi d\mu^q}{\int_Z \log|f'|d\mu^q} < 0. \tag{7.15}$$

Proof. Since $|f_x'(y)| > 0$, by Proposition 5.3 we obtain that

$$\frac{\partial E(P(q,t))}{dt} = -\int_Z \log|f_x'|d\mu_x^{q,t}dm(x) < 0. \tag{7.16}$$

Then by (7.16) we can use Implicit Function Theorem and find a differentiable function $q \mapsto T(q)$ such that

$$E(P(q,T(q))) = 0.$$

Since $E(P(\varphi^q)) = 0$,

$$0 = \frac{dE(P(\varphi^q))}{dq} = \frac{\partial E(P(q,t))}{\partial q}\Big|_{t=T(q)} + \frac{\partial E(P(q,t))}{\partial t}\Big|_{t=T(q)} T'(q).$$

Then

$$T'(q) = -\frac{\frac{\partial E(P(q,t))}{\partial q}\big|_{t=T(q)}}{\frac{\partial E(P(q,t))}{\partial t}\big|_{t=T(q)}} = -\frac{\int_Z(\varphi_x - P_x)d\mu_x^q dm(x)}{\int_Z -\log|f_x'|d\mu_x^q dm(x)}$$
$$= \frac{\int_Z \varphi_x d\mu_x^q dm(x) - \int_X P_x dm(x)}{\int_Z \log|f_x'|d\mu_x^q dm(x)} = \frac{\int_Z \varphi d\mu^q}{\int_Z \log|f'|d\mu^q}.$$

Hence we obtain 7.15. It follows, in particular, that

$$T'(q) < 0, \tag{7.17}$$

since by (7.14) there exists N such that for every $n \ge N$, m-almost every and for every $y \in \mathcal{J}_x$ and $S_n\varphi_x(y) < 0$. Therefore $\int \varphi d\mu^q < 0$ and we get (7.17) because μ^q is T-invariant. □

8. Examples

In this section, when the all the tools and concepts are introduced and theorems proved, we can illustrate them with a collection of examples. We start the section with presenting a transparent example of an essential random G-system.

Example 8.1. *(Random Cantor Set)* Define

$$f_0(x) = 3x(\mathrm{mod}\,1) \text{ for } x \in [0,1/3] \cup [2/3,1]$$

and

$$f_1(x) = 4x(\mathrm{mod}\,1) \text{ for } x \in [0,1/4] \cup [3/4,1].$$

Let $X = \{0,1\}^{\mathbb{Z}}$, θ be the shift transformation and m be the standard Bernoulli measure. For $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in X$ define $f_x = f_{x_0}$, $f_x^n = f_{\theta^{n-1}(x)} \circ f_{\theta^{n-2}(x)} \circ \ldots \circ f_x$ and

$$\mathcal{J}_x = \bigcap_{n=0}^{\infty} (f_x^n)^{-1}([0,1]).$$

The skew product map defined on $\bigcup_{x\in X} J_x$ by the formula

$$T(x,y) = (\theta(x), f_x(y))$$

generates a . We shall show that this system is essential. To simplify the next calculation, we define recurrently

$$\xi_x(1) = \begin{cases} 3 & \text{if } x_0 = 0 \\ 4 & \text{if } x_0 = 1 \end{cases}, \quad \xi_x(n) = \xi_{\theta^{n-1}(x)}(1)\xi_x(n-1).$$

Consider the potential φ^t defined by the formula

$$\varphi_x^t = -t \log \xi_x(1).$$

Then

$$S_n \varphi_x^t = -t \log \xi_x(n).$$

Let C_n be a cylinder of the order n that is C_n is a subset of $\mathcal{J}_x$ of diameter $(\xi_x(n))^{-1}$ such that $f_x^n|_{C_n}$ is one-to-one and onto $\mathcal{J}_{\theta^n(x)}$. We can project the measure m on $\mathcal{J}_x$ and we call this measure μ_x. In other words, μ_x is such a measure that all cylinders of level n have the measure $1/2^n$. Then by Low of Large Numbers for m-almost every x

$$\lim_{n\to\infty} \frac{\log \mu_x(C_n)}{\log \mathrm{diam}(C_n)} = \frac{\log 2}{(1/n)\log \xi_x(n)} = \frac{\log 4}{\log 12} =: h.$$

Therefore the Hausdorff dimension of $\mathcal{J}_x$ is for m-almost every x constant and equal to h.

Next note that

$$\frac{\mu_x(C_n)}{\mathrm{diam}(C_n)^h} = \exp(-S_n P_x) \tag{8.1}$$

where

$$P_x := \log 2 - h \log \xi_x(1).$$

This will give us the value of the Hausdorff and packing measure. So let $Z_0, Z_1, \ldots$ be independent random variables, each having the same distribution such that the probability of $Z_n = \log 2 - h \log 3$ is equal to the probability of $Z_n = \log 2 - h \log 4$

and is equal to $1/2$. The expected value of Z_n, E, is zero and its standard deviation $\sigma > 0$. Then the Law of the Iterated Logarithm tells us that the probabilities that

$$\liminf_{n\to\infty} \frac{Z_1 + \ldots + Z_n}{\sqrt{n \log\log n}} = -\sqrt{2}\sigma$$

and

$$\limsup_{n\to\infty} \frac{Z_1 + \ldots + Z_n}{\sqrt{n \log\log n}} = \sqrt{2}\sigma$$

are one. Thus the random conformal G-system T is essential. In particular, in view of Theorem 6.9, the Hausdorff measure of almost every fiber J_x vanishes and the packing measure is infinite. Note also that the Hausdorff dimension of fibers is not constant as clearly $\mathrm{HD}(J_{0^\infty}) = \log 2/\log 3$, whereas $\mathrm{HD}(J_{1^\infty}) = \log 2/\log 4 = 1/2$.

We now want to describe some classes of examples coming from complex dynamics. Indeed, having a sequence of rational functions $F = \{f_n\}_{n=0}^\infty$ on the Riemann sphere $\hat{\mathbb{C}}$ we say that a point $z \in \hat{\mathbb{C}}$ is a member of the Fatou set of this sequence if and only if there exists an open set U_z containing z such that the family of maps $\{f_n|_{U_z}\}_{n=0}^\infty$ is normal in the sense of Montel. The Julia set $J(F)$ is defined to be the complement (in $\hat{\mathbb{C}}$) of the Fatou set of F. For every $k \geq 0$ put $F_k = \{f_{k+n}\}_{n=0}^\infty$ and observe that

$$J(F_{k+1}) = f_k(J(F_k)). \tag{8.2}$$

Now, consider the maps $f_c(z) = z^d + c$, $d \geq 2$. Notice that for every $\varepsilon > 0$ there exists $d > 0$ such that if $\{c_n\}_{n=0}^\infty$ is a sequence of points in $\overline{B}(0,\delta)$, then

$$J\big(\{f_{c_n}\}_{n=0}^\infty\big) \subset \{z \in \mathbb{C} : |z| \geq \varepsilon\}.$$

In particular, if $\varepsilon \geq 2/d$ then

$$|f'_{c_k}(z)| \geq 2 \tag{8.3}$$

for all $z \in J\big(\{f_{c_{k+n}}\}_{n=0}^\infty\big)$. Denote δ corresponding to $\varepsilon = 2/d$ by δ_d. Let $\mathcal{F}_d = \{f_c : c \in \overline{B}(0,\delta_d)\}$. Consider an arbitrary ergodig measure-preserving transformation $\theta : X \to X$. Let m be the corresponding invariant probability measure. Let also $H : X \to \mathcal{F}_d$ be an arbitrary measurable function. Abusing slightly notation set $f_x = f_{H(x)}$ for all $x \in X$. For every $x \in X$ let J_x be the Julia set of the sequence $\{f_{\theta^n(x)}\}_{n=0}^\infty$, and then $J = \bigcup_{x\in X} J_x$. Note that, because of (8.2), $f_x(J_x) = J_{\theta(x)}$. Thus, the map

$$T_{\theta,H}(x,y) = (\theta(x), f_x(y)),\ x \in X,\ y \in J_x, \tag{8.4}$$

defines a skew product map in the sense of the first section of our paper. In view of (8.3), when $\theta : X \to X$ is invertible, $T_{\theta,H}$ is a distance expanding random dynamical system, and, since all the maps f_x are conformal, $T_{\theta,H}$ is a conformal measurably expanding system in the sense of Definition 6.1. As an immediate consequence of Theorem 6.2.

Theorem 8.2. *Let $\theta : X \to X$ be an invertible measurable map preserving a probability measure m. Let $H : X \to \mathcal{F}_d$ be an arbitrary measurable function. Finally, let $T_{\theta,H}$ be the distance expanding random dynamical system defined by formula (8.4). Then for almost all $x \in X$ the Hausdorff dimension of the Julia set J_x is equal to the unique zero of the expected value of the pressure function.*

Theorem 8.3. *For the conformal measurably expanding systems $T_{\theta,H}$ defined in Theorem 8.2 the multifractal theorem, Theorem 7.2 holds.*

The construction leading to Theorem 8.2 generates an abundance of examples just because of the freedom in choosing θ and H. We shall now describe only very few of them, selected somewhat arbitrarily according to our taste.

Example 8.4. Set $X = \mathcal{F}_d^{\mathbb{Z}}$. Define $\theta : X \to X$ to be the shift map, i.e. $\theta\big((f_n)_{n=-\infty}^{\infty}\big) = (f_{n+1})_{n=-\infty}^{\infty})$. Take any probability measure λ on $\mathcal{F}_d$ and denote by m the corresponding product measure on $X = \mathcal{F}_d^{\mathbb{Z}}$. Set $H\big((f_n)_{n=-\infty}^{\infty})\big) = f_0$. Note that

$$T_{\theta,H}\big((f_n)_{n=-\infty}^{\infty}), y\big) = \big((f_{n+1})_{n=-\infty}^{\infty}), f_0(y)\big).$$

This example can be generalized as follows.

Example 8.5. Let I be an arbitrary subset of $\mathcal{F}_d$. Set $X = I^{\mathbb{Z}}$. Define $\theta : X \to X$ to be the shift map, i.e. $\theta\big((f_n)_{n=-\infty}^{\infty})\big) = (f_{n+1})_{n=-\infty}^{\infty})$. Take any probability measure λ on I and denote by m the corresponding product measure on $X = I^{\mathbb{Z}}$. Set $H\big((f_n)_{n=-\infty}^{\infty})\big) = f_0$. Note that

$$T_{\theta,H}\big((f_n)_{n=-\infty}^{\infty}), y\big) = \big((f_{n+1})_{n=-\infty}^{\infty}), f_0(y)\big).$$

Note that the above generalization of Example 8.4 is very interesting even if I is finite or countable.

Example 8.6. *Let $X := S^1_{\delta_d} = \{z \in \mathbb{C} : |z| = \delta\}$. Fix an integer $k \geq 2$. Define the map $\theta_0 : X \to X$ by the formula*

$$\theta_0(x) = \delta^{1-k}x^k.$$

Then $\theta_0'(x) = k\delta^{1-k}x^{k-1}$, and therefore $|\theta_0'(x)| = k \geq 2$ for all $x \in X$. The normalized Lebesgue measure λ_0 on X is invariant under θ_0. Define the map $H : X \to \mathcal{F}_d$ by setting $H(x) = f_x$. Then

$$T_{\theta_0,H,0}(x,y) = (k\delta^{1-k}x^{k-1}, g^d + x).$$

Note that $\big(T_{\theta_0,H,0}, \theta_0, \lambda_0\big)$ is a uniformly conformal DG-system and let $(T_{\theta,H}, \theta, \lambda)$ be the corresponding random conformal G-system, both in the sense of Section 6. Theorem 6.9 and Corollary 6.10.

Theorem 8.7. *If $\big(T_{\theta_0,H,0}, \theta_0, \lambda_0\big)$ is the random conformal G-system described above, then*

(f) *$\mathcal{H}^h(J_x) = 0$ and $\mathcal{P}^h(J_x) = +\infty$ for λ-a.e. $x \in X$ if only $\big(T_{\theta_0,H,0}, \theta_0, \lambda_0\big)$ is essential. In consequence, for λ-a.e. $x \in X$ the Julia set J_x is not bi-Lipschitz equivalent to any deterministic self-conformal set. Furthermore, J_x is not a geometric circle nor even a piecewise smooth curve. In fact, if J_x is connected (it suffices to have a non-degenerate connected component) then $h = \mathrm{HD}(J_x) > 1$.*

(g) *If, on the other hand,* $(T_{\theta_0,H,0},\theta_0,\lambda_0)$ *is quasi-deterministic, then for every* $x\in X$*,*

- (g1) ν_x^h *is a geometric measure with exponent* h*.*
- (g2) *The measures* ν_x^h*,* $\mathcal{H}^h|_{J_x}$*, and* $\mathcal{P}^h|_{J_x}$ *are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity independently of* $x\in X$ *and* $y\in J_x$*.*
- (g3) $0<\mathcal{H}^h(J_x),\mathcal{P}^h(J_x)<+\infty$.
- (g4) $\mathrm{HD}(J_x)=h$.

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