Normal Subgroups of Symplectic Groups Over Rings

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Abstract. We consider a module with an alternating form over a commutative ring. Under certain conditions, which hold, for example, when the form is non-singular and the module is projective of rank ≥ 6 and contains a unimodular vector, we describe all subgroups of the symplectic group which are normalized by symplectic transvections. This generalizes many previous results of Dickson, Abe, Klingenberg, Bak, et el.

Key words: mormal subgroups, symplectic groups, alternating forms.

1. Introduction.

Let R be a commutative associative ring with 1. For any integer $n \geq 1$, let $Sp_{2n}R$ be the standard symplectic group and $Ep_{2n}R$ its subgroup generated by elementary symplectic matrices [11], [37], [54], [62].

When R is a field, Dickson [20] proved that $Sp_{2n}R = Ep_{2n}R$ (by the way, the term "symplectic" was coined later, so Dickson wrote about "abelian linear groups $SA(2n, R)$"). Moreover, he showed that this group modulo its center (which consists of $\pm 1_{2n}$) is simple with the following three exeptions: R consists of 2 elements and $n = 1$ (in this case $Sp_{2n}R = SL_2R$ is isomorphic to the symmetric group S_3); R consists of 3 elements and $n = 2$ (in this case $Sp_{2n}R$ is isomorphic to the alternating group A_4) R consists of 2 elements and $n = 2$ (in this case, $Sp_{2n}R = Sp_4R$ is isomorphic to the symmetric group S_6). In all these 3 cases, the commutator subgroup of $Sp_{2n}R = Ep_{2n}R$ is a proper non-central normal subgroup. See also [5], [21], [42] [46] about symplectic groups over fields.

Klingenberg [23] described all normal subgroups of $Sp_{2n}R$ for a local ring R such that the characteristic of the residue field $R/\mathrm{rad}(R)$ is not 2 and its cardinality is not 3. Abe [1] reduced the conditions on the local ring R to the following condition: the residue field has more than 3 elements when $n = 1$ and more than two elements when $n = 2$. When $2R \neq R$, his answer involves some additive subgroups of R which are more general than ideals (he called them special submodules associated with ideals; later [3] the result were extended to other rings R). See also [13]- [17], [19], [25] [26] [31], [33]-[35], [43], [49]-[53] about Sp_{2n} over local, semilocal, and other "zero-dimensional" rings R.

Mennicke [37] and Bass-Milnor-Serre [11] described all normal subgroups of $Sp_{2n}R$ when R is the ring of integers $\mathbf{Z}$ or, more generally, a Dedikind ring of arithmetic type and $n \geq 2$. Note that the normal subgroup structure of $Sp_2R = SL_2R$ is very different and essentially intractable even when $R = \mathbf{Z}$ [27] - [30], [39], [40], [38] or another Dedikind ring of arithmetic type with finite GL_1R [18], [22], [41], [45].

The normal subgroup structure of $Sp_{2n}R$ for any R with "infinite" n was studied in [4], [9] , [32], [44], [61]. Bak [6] announced a description of all subgroups of $Sp_{2n}R$ when $n \geq 3$ and is greater than a certain dimension of R; see [7] for proofs.

Kopeiko [24] showed that $Ep_{2n}R$ is normal in $Sp_{2n}R$ for any R when $n \geq 2$. Later this was redescovered in part by Taddei [47].

Using localization and patching, a complete description of all subgroups H of $Sp_{2n}R$ which are normalized by $Ep_{2n}R$, was obtained in [58] in general context of Chavallwey groups, provided that $n \geq 2$, R has no residue fields of 2 elements in the caee $n = 2$, and

(1) for every element z of R there are r, s in R such that $z = 2rz + sz^2$.

The condition (1) is necessary for the standard description of those H"s in terms of ideals of R, as can be seen from the case of local ring R (see [1], [3]). It was claimed in [58] that without the condition (1), a complete description of H 's is possible in more general terms. This was proved by Abe [2].

Here we improve on Abo's result extending it to symplectic groups of alternating forms F on R-modules V. Our proofs here use localization and patching. The approach to description of normal subgroups was introduced in [57] for general linear groups $GL_n R$, $n \geq 3$. Later it was used for orthogonal [60] and Chevalley [2] , [46], [58] groups.

As a departure from the setting of [6], [7], [9], our R-module V need not be finitely generated or projective, and our alternating form F need not be non-singular. Instead of non-singularity, we impose another condition which is equivalent to non-singularity in the case of a finitely generated projective V.

Singular F on a finitely generated free V over local and semilocal rings R was studied in [13]-[16], [43]. The answer inviolves tableaux of ideals.

2. Statement of results

A *alternating form* F on an R-module V is a bilinear form F on V such that $F(v, v) = 0$ for all v in V. We do not require that $F = Q - Q^{T}$, i.e. $F(u, v) = Q(u, v) - Q(v, u)$ for all u, v in V, where Q is a bilinear form on V, although such a form Q exists when V is projective. Note that any alternating form F is skew-symmetric, i.e. $F(u, v) = -F(v, u)$ for all u, v in V.

The symplectic group $\mathrm{Sp}_F R$ is the group of all automorpisms of the R-module V which preserve an alternating form F. Let $\mathrm{Gp}_F R$ denote the group of all automorphisms which multiply the form by a unit of R.

For every e, u in V such that $F(e, u) = 0$ and any x in R we define (following [56]) $\tau(e, u, x)$ in $\mathrm{Sp}_F R$ by

$\tau(e, u, x)v = v + uF(e, v) + eF(u, v) + exF(e, v).$

An element v of V is called *F-unimodular* if $F(V, v) = R$, i.e. $F(u, v) = 1$ for some u in V. The elements $\tau(e, u, x)$ as above with unimodular e are called *symplectic transvections*. We denote by $\mathrm{Ep}_F R$ the subgroup of $\mathrm{Sp}_F R$ generated by all symplectic transvection. Clearly (see (14) below) $\mathrm{Ep}_F R$ is normal in $\mathrm{Gp}_F R$. Here we give another description of $\mathrm{Ep}_F R$, where a *hyperbolic pair* means a pair u, v of vectors with $F(u, v) = 1$.

PROPOSITION 2. The group $\mathrm{Ep}_F R$ coincides with the subgroup of $\mathrm{Sp}_F R$ generated by all elements $\tau(e, 0, r)$, where $r \in R$ and $e \in V$ is either F-unimodular or orthogonal to a hyperbolic pair in V.

The main goal of this paper is to describe all subgroups H of $\mathrm{Gp}_F R$ normalized by $\mathrm{Ep}_F R$. It is much easier to describe the centralizer of $\mathrm{Ep}_F R$. If $\mathrm{Ep}_F R$ is trivial, its centralizer in $\mathrm{Gp}_F R$ is $\mathrm{Gp}_F R$. Otherwise, i.e. when an F-unimodular vector in V exists, i.e. the Witt index of F is at least 1, we will show in Section 3 below that the centralizer consists of all scalar authomorphisms of V:

PROPOSITION 3. If V contains an F-unimodular vector, then the centralizer of $\mathrm{Ep}_F R$ in $\mathrm{Gp}_F R$ consists of all scalar authomorphisms of V, and hence coincides with the center of $\mathrm{Gp}_F R$.

We define a *symplectic ideal of* R as a pair (A, B), where A is an ideal of R and B is an additive subgroup of A such that $r^2 b, 2a, a^2 r \in B$ for all r in R, b in B, and a in A.

Note that the condition (1) above is equivalent to the following: $B = A$ for every symplectic ideal (A, B) of R. Under different names, our symplectic ideals appeared first in [1], and then in [2]) [3], [6], [7], [9], [10], [12], [31], [54], [56].

Given any symplectic ideal (A, B) of R and any vector e in V, we define $T(e;A, B)$ as the subgroup of $\mathrm{Ep}_F R$ generated by all $\tau(e, 0, b)$ with b in B and by all $\tau(e, ua, 0)$ with a in A and u in V such that $F(e, u) = 0$. It is easy to check (see the identity (12) below) that $T(e;A, B)$ consists of all $\tau(e, u, r)$ with $u \in e^{\perp}A$, $r \in |u|$, where $e^{\perp} = \{v \in V : F(e, v) = 0\}$ is the orthogonal complement of e in V and where the map $|\,|: VA \to A/B$ is defined by

$$\Big|\sum_{1 \le i \le n} v_i a_i\Big| = B + \sum_{1 \le i < j \le n} F(v_i a_i, v_j a_j), \text{ where } v_i \in V, a_i \in A.$$

It is easy to check that this is well-defined, i.e. $|v| \in A/B$ does not depend on choice of presentation $v = \sum v_i a_i$.

Let $\mathrm{Ep}_F(A, B)$ denote the subgroup of $\mathrm{Ep}_F R$ generated by all $T(e;A, B)$, where e ranges over all F-unimodular vectors in V. Clearly, $\mathrm{Ep}_F(A, B)$ is a normal subgroup of $\mathrm{Sp}_F R$, and $\mathrm{Ep}_F(R,R) = \mathrm{Ep}_F R$.

THEOREM 4. Assume that $\dim(F \bmod P) \ge 4$ for every maximal ideals P of R. Let e_1, e_2 be vectors in V with $F(e_1, e_2) = 1$. Then the group $\mathrm{Ep}_F R$. is generated by its subgroups $T(e_1, R, R)$ and $T(e_2, R, R)$. Moreover, for any symplectic ideal (A, B) of R, the group $\mathrm{Ep}_F(A, B)$ coincides with the normal subgroup of $\mathrm{Ep}_F R$ generated by $T(e_1, R, R)$.

The condition $\dim(F \bmod P) \ge 2m$ (used in Theorem 4 with $m = 2$) means that there are vectors v_i in V such that the matrix $(F(v_i, v_j)_{1 \le i,j \le 2m})$ over R is invertible modulo P. Since F is alternating, this number $2m$ must be even. In the case of a non-singular F, the condition is equivalent to $\dim_{R/P} V/VP \ge 2m$.

The dimension condition in the Theorem 3 is necessary. Without this condition, the first conclusion would give that $E_2R = Ep_2R$ is normal in $GL_2R = Gp_2R$, which is not true in general [18]. However E_2R is normal in GL_2R when $E_2R = SL_2R$ (which is the case under the first Bass stable range condition [8] and for some other rings [55]) or R is a topological ring with GL_1R open in R [59].

We define $Gp_F(A, B)$ to be the set of all g in Gp_FR such that there is $\alpha \in GL_1R$ and $c \in R$ such that $(c^2 - \alpha)R \subset B$, $F(gu, gv) = \alpha F(u, v)$, $gv - vc \in VA$ and $F(vc, gv) + B = |gv - vc|$ for all $u, v \in V$. It is easy to check that $Gp_F(A, B)$ is a normal subgroup of $Gp_F(R, R) = Gp_FR$. The group $Gp_F(0, 0)$ is the group of scalar automorphisms of V.

For any two subgroups H_1 and H_2 of a group G we denote by $[H_1, H_2]$ the subgroup of G generated by all commutators $[h_1, h_2] = h_1h_2h_1^{-1}h_2^{-1}$ with h_1 in H_1 and h_2 in H_2. It is easy to check that $[H_1, H_2]$ is normalized by both H_1 and H_2. THEOREM 5. Assume that V contains an F-unimodular vector, that $\dim(F \bmod P) \geq 4$ for every maximal ideals P of R, and that $\dim(F \bmod P) \geq 6$ for every ideal P of index 2 in R. Then Ep_FR is generated by its subgroups $\tau(e, 0, R)$, where e ranges over all F-unimodular vectors e in V. Moreover, for any symplectic ideal (A, B) of R, $Gp_F(A, B)$ is the centralizer of Ep_FR in Gp_FR modulo $Ep_F(A, B)$, i.e. it consists of all g in Gp_FR such that $[g, Ep_FR] \subset Ep_F(A, B)$. COROLLARY 6. Under the conditions of Theorem 5, for any symplectic ideal (A, B) of R, every subgroup H of $Gp_F(A, B)$ containing $Ep_F(A, B)$ is normalized by Ep_FR. Moreover, for any symplectic tranvection g in Gp_FR and any h in H the commutator $[g, h]$ is product of symplectic transvections in H.

Indeed, by Theorem 5, $[Ep_FR, H] \subset [Ep_FR, Gp_F(A, B)] \subset Ep_F(A, B) \subset H$.

THEOREM 7. Under the conditions of Theorem 5,

$$Ep_F(A, B) = [Ep_F(A, B), Ep_FR] = [Ep_F(A, B), Sp_FR] = [Gp_F(A, B), Ep_FR]$$

for every symplectic ideal (A, B) of R.

Since the group $\mathrm{Sp}_4\mathbb{Z}/2\mathbb{Z} = \mathrm{Ep}_4\mathbb{Z}/2\mathbb{Z}$ is not perfect, we have to require that the dimension of F modulo P is not 4 for any ideal P of index 2 in R. Note that the group $\mathrm{Ep}_2R = \mathrm{E}_2R$ is not perfect for small fields and for many other rings R.

By Corollary 6, every subgroup H of $\mathrm{Gp}_F(A, B)$ containing $\mathrm{Ep}_F(A, B)$ is normalized by Ep_FR. We want to prove the converse: for every subgroup H of Gp_FR which is normalized by Ep_FR there is a symplectic ideal (A, B) of R such that $\mathrm{Ep}_F(A, B) \subset H \subset \mathrm{Gp}_F(A, B)$. For this to be true, we will need some conditions on F, besides the existence of an F-unimodular vector in V.

First of all, as we did in Theorem 6, we want to exclude the case when $V = R^2$. In the case, there are non-standard normal subgroups of $\mathrm{Sp}_FR = \mathrm{SL}_2R$ (even for $R = \mathbb{Z}$ [27], [28], [30], [36], [39], [40], [41] and other small dimensional rings [18], [22], [29], [38]) unless we impose rather severe restrictions on R [17], [45], [59]. Since the group $\mathrm{Sp}_4\mathbb{Z}/2\mathbb{Z}$ has a non-standard normal subgroup (its commutator subgroup which is proper subgroup), we have to require that the dimension of F modulo P is not 4 for any ideal P of index 2 in R

Finally, we have to impose a condition on F which is weaker than its non-singularity. Namely, we will assume that $v \in VF(v, V)$ for every vector v in V. That is, for every vector v there is a finite set of vectors u_i, w_i in V such that $v = \sum w_iF(v, u_i)$. When V is finitely generated projective, this condition is equivalent to the condition that F is non-singular, i.e. the assignement $u \mapsto F(u, ?)$ gives an bijection $V \to \mathrm{Hom}_R(V, R)$. In general, the condition means that the map $V/VA \to \mathrm{Hom}_{R/A}(V/A, R/A)$ is injective for every ideal A of R.

Here is the main result of this paper.

THEOREM 8. Under the conditions of Theorem 5, assume that $v \in VF(v, V)$ for every vector v in V. Then a subgroup H of Gp_FR is normalized by Ep_FR if and only if $\mathrm{Ep}_F(A, B) \subset H \subset \mathrm{Gp}_F(A, B)$ for a symplectic ideal (A, B) of R, and if and only if the commutator $[g, h]$ is a product of symplectic transvections in H for every symplectic tranvection g in Gp_FR and every h in H.

3. Proof of Proposition 2

First we list some easy to check relations for $\tau(e, u, x)$.. Let e, u, v be in V, x, y in R, and g in $Gp(q, R)$. Assume that $F(e, u) = F(e, v) = 0$. Then:

(9) $\tau(e, u, x)v = v$ when $F(u\ v) = 0$; in particular, $\tau(e, u, x)e = e$;

(10) $\tau(ey, u, x) = \tau(e, uy, xy^2)$;

(11) $\tau(e, u + ey, x) = \tau(e, u, x + 2y)$;

(12) $\tau(e, u, 0) = \tau(u, e, 0)$;

(13) $\tau(e, u, x)\tau(e, v, y) = \tau(e, u + v, x + y + F(u, v))$;

in particular, $\tau(e, u, x)^{-1} = \tau(e, -u, -x)$;

(14) $g\tau(e, u, x)g^{-1} = \tau(ge, gu/\alpha(g), x/\alpha(g)))$ for every g in $\mathrm{Gp}_F R$, where $\alpha(g) \in \mathrm{GL}_1 R$ is such that $F(gw, gw') = \alpha(g)F(w, w')$ for all w, w' in V,

in particular,

(15) when $ge = e$ and $g \in \mathrm{Sp}_F R$ (i.e. $\alpha(g) = 1$), we have $g\tau(e, u, x)g^{-1} = \tau(e, gu, x)$ and $[g, \tau(e, u, x)] = \tau(e, gu, x)\tau(e, -u, -x) = \tau(e, gu - u, F(u, gu))$.

Now we are ready to prove Proposition 2. Let H be the subgroup of $\mathrm{Ep}_F R$ generated by the subgroups $\tau(e, 0, R)$, where e ranges over all vectors e in V which are either F-unimodular or orthogonal to a hyperbolic pair in V. Clearly, H is a normal subgroup of $\mathrm{Gp}_F R$. We want to prove that $H = \mathrm{Ep}_F R$.

By the definition of $\mathrm{Ep}_F R$, it contains $\tau(e, 0, R)$ for every F-unomodular vector e in V. Let us show that $\mathrm{Ep}_F R \ni \tau(e, 0, r)$ when $r \in R$ and e is orthogonal to a hyperbolic pair e_1, e_2 in V. Indeed,

$\tau(e, 0, r) = \tau(e, e_1, 0)\ \tau(e, e_2 r, 0)\ \tau(e, -e_1 - e_2 r, 0)$

$= \tau(e_1, e, 0)\ \tau(e_2, er, 0)\ \tau(e_1 + e_2 r, -e, 0) \in \mathrm{Ep}_F R$ by (10), (12), (13), because the vectors e_1, e_2, and $e_1 + e_2 r$ are F-unimodular.

Thus, $H \subset \mathrm{Ep}_F R$. Let us show now that $\mathrm{Ep}_F R \subset H$.

By the definition of $\mathrm{Ep}_F R$, it suffices to show that $H \supset \mathrm{T}(e, R, R)$ for any F-unimodular vector e in V, i.e. $H \ni \tau(e, u, r)$ for an arbitrary symplectic transvection $\tau(e, u, r)$, where $u \in e^{\perp}$ and $r \in R$.

We pick a vector e' in V with $F(e, e') = 1$, and set $r' = F(u, e')$, $v = u - er'$. Then $u = er' + v$ with v orthogonal to both e and e'. By (11),(13),

$$\tau(e, u, r) = \tau(e, v, 0)\, \tau(e, 0, r + 2r').$$

So it remains to show that $\tau(e, v, 0) \in H$.

By (15),

$H \ni [\tau(e, 0, 1), \tau(v, e', 0)] = \tau(v, e, -1)$, hence

$H \ni \tau(v, e, -1)\, \tau(v, 0, 1) = \tau(v, e, 0) = \tau(e, v, 0)$.

4. Proof of Proposition 3

In this section we assume that V contains an F-unimodular vector. We fix a hyperbolic pair e_1, e_2 in V. So $F(e_1, e_2) = 1$ and $e_1R + e_2R$ is a hyperbolic plane in V. Let $U = (e_1R + e_2R)^{\perp}$ denote the orthogonal complement of $e_1R + e_2R$ in V. So $V = (e_1R + e_2R) \perp U$.

LEMMA 16. Under the conditions of Theorem 2, the centralizer of $T(e_1, R,R)$ *in* $\mathrm{Gp}_F R$, *is* Z_1 $\mathrm{Gp}_F(0,0)$ where $\mathrm{Gp}_F(0,0) \subset \mathrm{Gp}_F R$, is the subgroup of all scalar authomorphisms of V and Z_1 *is* the center of $T(e_1, R,R)$, which consists of $\tau(e_1, u, x))$ in $T(e_1, R,R)$, with $2F(u, V) = 0$.

Proof. Let g be in $\mathrm{Gp}_F R$ and commute with each element of $T(e_1, R,R)$. In particular, $g\,\tau(e_1, 0, 1) = \tau(e_1, 0, 1)g$, hence $g\,\tau(e_1, 0, 1)e_2 = \tau(e_1, 0, 1)ge_2$, i.e. $ge_2 + ge_1 = ge_2 + e_1F(e_1, ge_2)$, i.e. $ge_1 = e_1F(e_1, ge_2)$. Since the vector ge_1 is F-unimodular, it follows that $F(e_1, ge_2)R = R$. Replacing g by its scalar multiple $gF(e_1, ge_2)^{-1}$, we can assume that $ge_1 = e_1$. Since $F(ge_1, ge_2) = 1$, the vector ge_2 has the form $ge_2 = e_2 + e_1c + w$ with $c \in R$ and $w \in U$. So $ge_2 = \tau(e_1, w, c)e_2$. Set now $h = \tau(e_1, w, c)^{-1}g$. Then $he_1 = e_1$ and $he_2 = e_2$, hence $hU = U$. The equality $g\tau(e_1, u, x)g^{-1} = \tau(e_1,u, x)$ for an arbitrary $\tau(e_1,u, x)$ in $T(e_1, R,R)$, with u in U takes the form

$\tau(e_1, hu, x + 2F(w, hu)) = \tau(e_1, u,x)$, hence $h = 1$, $g = \tau(e_1, w, c)$, and $2F(w,U) = 0$. Thus, g (after it was multiplied by a scalar) belongs to the center of $T(e_1,R,R)$. Lemma 13 is proved.

Remark. The intersection of $\mathrm{Gp}_F(0, 0)$ and Z_1 is trivial.

Notation. For any vectors e, e' in V, let $E(e, e'; R)$ denote the subgroup of $\mathrm{Sp}_F R$ generated by $T(e, R, R)$ and $T(e', R, R)$.

COROLLARY 17. The centralizer of $E(e_1, e_2; R)$ in $Gp_F R$. coincides with the group $\mathrm{Gp}_F(0,0)$ of scalar authomorphisms of V. In particular, $\mathrm{Gp}_F(0,0)$ is exactly the center of $Gp_F R$.

Proof. Let $g \in Gp_F R$ commute with every element of $T(e_1, R, R)$ and $T(e_2, R, R)$. By Lemma 13, $g \in T(e_1, R, R)\,\mathrm{Gp}_F(0,0) \cap T(e_2, R, R)\,\mathrm{Gp}_F(0,0) = \mathrm{Gp}_F(0,0)$. (Since $ge_2 \in e_2 R$, the $T(e_1, A, A)$ -component of g is 1, so $g \in \mathrm{Gp}_F(0,0)$, i.e. g is multiplication by an invertible scalar on V.)

Remark. Corollary 17 contains Proposition 2, because $E(e_1, e_2; R) \subset \mathrm{Ep}_F R$.

THEOREM 18. Assume that V contains an F-unimodular vector. Let (A, B) be a symplectic ideal of R and $g \in \mathrm{Gp}_F R$. If $[g, \mathrm{Ep}_F R] \in \mathrm{Gp}_F(A, B)$, then $g \in \mathrm{Gp}_F(A, B)$.

Proof. Applying Proposition 2 to R/A, V/VA, and $F \pmod A$ instead of R, V, and F and using that the map $\mathrm{Ep}_F(R) \to \mathrm{Ep}_F(R/A)$ is onto, we conclude that g is a scalar modulo A, i.e. there is $c \in R$ such that $gv - cv \in VA$ for all $v \in V$. In prticular $c^2 - \alpha(g) \in A$, where $\alpha(g) = F(ge_1, ge_2) \in \mathrm{GL}_1 R$ is such that $F(gu, gv) = \alpha(g) F(u, v)$ for all $u,v \in V$.

We claim now that $(c^2 - \alpha(g))R \subset B$ and that $F(e_1 c, ge_1) + B = |ge_1 - e_1 c|$.

To prove this, we write $ge_1 = e_1 x + e_2 y' + w$ with $x = F(ge_1, e_2)$, $y' = F(e_1, ge_2)$, and $w \in U$. We have $x - c \in A$, $y' \in A$, $w \in UA$. Now we pick $x' \in R$ such that $xx' - 1 \in A$ and $z \in |wx'|$. We set $g' = \tau(e_2, wc', z)$ with $\tau(e_2, wc', z) \in \mathrm{Ep}_F(A, B)$. We have $g'e_1 = \tau(e_2, wc', z)ge_1 = \tau(e_2, wc', z)(e_1 x + e_2 y' + w) = e_1 x + e_2 y + wa$ with $a = 1 - xx' \in A$ and $y = y' - z \in A$.

Our claim takes the following form: $(x^2 - \alpha(g))R \subset B$ and that $xy \in B$.

For an arbitrary r in R we set $h = [g', \tau(e_1, 0, r)] \in \mathrm{Gp}_F(A, B)$. Then
$he_2 = \tau(g'e_1, 0, r/\alpha(g))(e_2 - e_1 r) = e_2 - e_1 r + g'e_1 F(g'e_1, e_2 - e_1 r)r/\alpha(g))$
$= e_2(1 + rxy/\alpha(g) + r^2y^2/\alpha(g)) + e_1(rx^2/\alpha(g)) - r + r^2xy/\alpha(g)) + war(x + ry)/\alpha(g)$.

Since $Ry^2 \subset B$, the equality $|he_2 - e_2| = F(he_2, e_2) + B$ takes the form $rx^2/\alpha(g) - r \in B$, i.e. $r(x^2 - \alpha(g)) \in B$.

We have proved that $(x^2 - \alpha(g))R \subset B$ which is equivalent to $(c^2 - \alpha(g))R \subset B$ because $x - c \in A$.

Now we consider $h^{-1}e_2 = [\tau(e_1, 0, r), g']e_2 = \tau(e_1, 0, r)\,\tau(g'e_1, 0, -r/\alpha(g))e_2$
$= \tau(e_1, 0, r)\,(e_2 - g'e_1 F(g'e_1, e_2)r/\alpha(g) = e_2 - g'e_1 rx/\alpha(g) + e_1 F(e_1, e_2 - g'e_1 rx/\alpha(g))r$
$= e_2(1 - rxy/\alpha(g)) + e_1(r - rx^2/\alpha(g) - xyr^2/\alpha(g)) - warx/\alpha(g)$.

Since $Ry^2 \subset B$ and $(1 - x^2/\alpha(g))R \subset B$, the equality $|h^{-1}e_2 - e_2| = F(h^{-1}e_2, e_2) + B$ takes the form $xyr^2/\alpha(g)) \in B$. Setting $r = x$, we obtain that $xy \in B$.

Thus, our claim is proved. Similarly, $F(ec, ge) + B = |ge - ec|$ for every F-unimodular vector e in V. Note that V is spanned by F-unimodular vectors. Namely, $v = e_1 s + e_2 t + w = e_1 + e_2 t + w + e_1(s - 1)$ for an arbitrary vector v in V, where $s, t \in R$, $w \in U$, and vectors $e_1 + e_2 t + w$ and e_1 are F-unimodular. So $F(ec, ge) + B = |ge - ec|$ for every vector e in V. Thus, we have proved that $g \in \mathrm{Gp}_F(A, B)$.

Remark. Theorem 18 with $A = 0$ implies Proposition 2.

5. Proof of Theorem 4

Let e_1, e_2 and $U = (e_1R + e_2R)^{\perp}$ be as defined before Lemma 16. For any symplectic ideal (A, B) of R and any two vectors e, e' in V, let $E(e,e'; R, A, B)$ denote the normal subgroup of $E(e,e'; R)$ (see the notation before Corollary 17) generated by $T(e; A, B)$ and $T(e', A, B)$. In particular, $E(e,e'; R, R, R) = E(e,e'; R)$

We want to prove that $E(e_1, e_2; R, A, B) = \mathrm{Ep}_F(A, B)$, i.e. that $E(e_1, e_2; R, A, B)$ does not depend on choice of a hyperbolic pair e_1, e_2 under the conditions of Theorem 4. LEMMA 19. *For any symplectic ideal (A, B) of R, any two vectors $e, e' \in V$, and any vector $e'' \in V$ orthogonal to e, e' we have $E(e,e'; R, A, B) \supset T(e'', As^2, Bs^2)$, where $s = F(e,e')$.*

Proof. Let $\tau(e'', uas^2, bs^2) \in T(e'', As^2, Bs^2)$, where $u \in V, F(e'', u) = 0, a \in A, b \in B$. We have to prove that $\tau(e'', uas^2, bs^2) \in E(e,e'; R, A, B)$.

Case 1: $u = 0$. Then $\tau(e'', uas^2, bs^2) = \tau(e'', uas^2, bs^2) = \tau(e'', 0, bs^2) = \tau(e'', -ebs, bs^2)$ $\tau(e'', ebs, 0) \in E(e,e'; R, A,B)$, because $\tau(e'', -ebs, bs^2) = [\tau(e, 0, -b), \tau(e'', e', 0)] \in E(e,e'; R, A, B)$, where $\tau(e'', e', 0) = \tau(e', e'', 0) \in T(e', R, R)$ by (12), and $\tau(e'', ebs, 0) = \tau(e, e''bsz, 0) \in T(e; A, B)$ also by (12).

General case. Set $r = F(e, u) \in R$, $r' = F(e', u) \in R$ and $w = us - e'r + er'$. Then w is orthogonal to e, e', and e''.

By (13), $\tau(e''t, uas^2, bs^2) = \tau(e'', uas^2, bs^2)$

$= \tau(e'', was, 0)\, \tau(e'', e'ars, 0)\, \tau(e'', ear's, 0)\, \tau(e'', 0, b's^2)$, where $b' = b + rr'sa^2 \in B$.

By (12), $\tau(e'', e'ars, 0) \in T(e'; Ax, Bx) \subset E(e,e'; R, A, B)$ and

$\tau(e'', ear's, 0) \in T(e; Ax, Bx) \subset E(e,e'; R, A, B)$.

By Case 1, $\tau(e, 0, b's^2) \in E(e,e'; R, A, B)$.

Moreover $\tau(e''\ was, 0) = [\tau(e, wa, 0), \tau(e'', e', 0)] \in E(e,e'; R, A, B)$, because $\tau(e'', e', 0) = \tau(e', -e'', 0) \in T(e', R,R)$ by (12).

Thus, $\tau(e''t, uas^2, bs^2) \in E(e,e'; R, A, B)$.

COROLLARY 20. *For any symplectic ideal (A, B) of R, any two vectors $e, e' \in V$, and any two vectors $w, w' \in V$ orthogonal to e, e' we have $E(e,e'; R, A, B) \supset E(ws^2, w's^2; R, A, B)$, where $s = F(e,e')$.*

Proof. We have to prove that $ghg^{-1} \in E(e,e'; R, A, B)$ whenever $g \in E(ws^2, w's^2, R)$ and $h \in T(ws^2, A, B) \cup T(ws^2, A, B)$. By Lemma 19, $h \in E(e,e'; R, A, B)$ and $g \in E(e,e'; R, R,R) = E(e,e'; R)$. So, $ghg^{-1} \in E(e,e'; R, A, B)$.

LEMMA 21. Let P be a maximal ideal of R. Supose that $\dim(F \bmod P) \geq 4$. Let $e, e' \in V$ and $F(e, e') \in S = R \setminus P$. Then there is $s \in S$ such that $E(e_1, e_2; R, A, B) \supset T(e; As^2, Bs^2)$ for all symplectic ideals (A, B) of R.

Proof. We write $e = v + u$ with $v \in e_1R + e_2R$ and $u \in U$.

If $F(U, u)$ intersects S, then we find v in U with $F(u, v) = s_0 \in S$. By Corollary 20, $E(e_1, e_2; R, A, B) \supset E(u, v; R, A, B)$ and $E(u, v; R, A, B) \supset T(e; As_0{}^2, Bs_0{}^2)$. So

$E(e_1, e_2; R, A, B) \supset T(e; As^2, Bs^2)$ with $s = s_0$.

If $F(U, u)$ does not intersect S, i.e. $F(U, u) = F(V, u) \subset P$ then $F(V, v)$ intersects S. We find a vecot v' in $e_1R + e_2R$ with $F(v, v') = s_1 \in S$, and a pair $w, w' \in U$ with $F(w, w') = s_2 \in S$. By Corollary 20,

$E(e_1, e_2; R, A, B) \supset E(w, w'; R, A, B) \supset E(vs_2{}^2, v's_2{}^2; R, A, B)$. By Lemma 19,

$E(vs_2{}^2, v's_2{}^2; R, A, B) \supset T(e; As_2{}^2s_1{}^8, Bs_2{}^2s_1{}^8)$.

So $E(e_1, e_2; R, A, B) \supset T(e; As^2, Bs^2)$ with $s = s_2s_1{}^4 \in S$.

Now we can complete our proof of Theorem 4. We have to prove that $\tau(e, ua, b) \in E(e_1, e_2; R, A, B)$ for any F-unimodular vector $e \in V$, any vector $u \in V$ orthogonal to e, any $a \in A$, and any $b \in B$. By Lemma 21, for every maximal ideal P of R there is $s \in R$ outside P such that $E(e_1, e_2; R, A, B) \supset \tau(e, uaRs^2, 0)$. Writing 1 as a linear combination of those s^2, we obtain an element of $E(e_1, e_2; R, A, B)$ of the form $\tau(e, ua, ra^2)$ with $r \in R$.

It remains to show that $\tau(e, 0, b') \in E(e_1, e_2; R, A, B)$ with $b' = b - ra^2 \in B$. By Lemma 21, for every maximal ideal P of R there is $s \in R$ outside P such that $\tau(e, 0, b'r^2s^2) \in E(e_1, e_2; R, A, B)$ for all $r \in R$. Writing 1 as the square of a linear combination of those s, and using that $E(e_1, e_2; R, A, B) \supset \tau(e, eb'R, 0) = \tau(e, 0, 2b'R)$, we obtain that $\tau(e, 0, b') \in E(e_1, e_2; R, A, B)$.

6. Proof of Theorem 5

To prove the first conclusion of the theorem we need only the following condition: $\dim(F \bmod P) \geq 6$ for every maximal ideal P of R of index 2. We denote by H the subgroup of $\mathrm{Ep}_F R$ generated by its subgroups $\tau(e, 0, R)$, where e ranges over all F-unimodular vectors e in V. Clearly, H is a normal subgroup of $\mathrm{Gp}_F R$. We want to prove that $H = \mathrm{Ep}_F R$. By the definition of $\mathrm{Ep}_F R$, it suffices to show that H contains an arbitrary symplectic transvection $\tau(e, u, r)$.

We pick a vector e' in V with $F(e, e') = 1$, and set $U' = (eR + e'R)^{\perp}$, $r' = F(u, e')$, $v = u - er'$. Then $u = er' + v$ with v orthogonal to both e and e'. By (11),(13),

$\tau(e, u, r) = \tau(e, v, 0)\, \tau(e, 0, r + 2r')$.

So it remains to show that $\tau(e, v, 0) \in H$. It suffices to show that for every maximal ideal P of R there is $s \in S = R \setminus P$ such that $\tau(e, U's, 0) \subset H$.

If $\mathrm{card}(R/P) \neq 2$, then we pick $t_0 \in R$ such that $t_0^2 - t_0 = s \in S$. By (15), $H \ni [\tau(e, 0, t'), \tau(v, e't, 0)] = \tau(v, ett', -t't^2) = f(t, t')$ for all $t, t' \in R$ and all $v \in U'$, hence

$H \ni f(t_0, 1)^{-1} f(1, t_0^2) = \tau(v, e(t_0^2 - t_0), 0)$

$= \tau(v, es, 0) = \tau(e, vs, 0)$.

If $\mathrm{card}(R/P) = 2$, then we use the condition of the theorem and pick two orthogonal pairs (v, v'), (w, w') in U' with $s_1 = F(v, v') \in S$ and $s_2 = F(w, w') \in S$.

We have $H \ni [\tau(e, 0, 1), \tau(v, e', 0)] = \tau(v, e, -1)$, hence

$H \ni [\tau(e', -w', 0), \tau(v, e, -1)] = \tau(v, w', 0)$, and $H \ni [\tau(w, et, 0), \tau(v, w', 0)] = \tau(v, ets_2, 0) = \tau(e, vts_2, 0)$ for all t in R.

Thus, $\tau(e, vs_2 R, 0) \subset H$. For an arbitrary $u' \in U'$ we have $u's_1 = vx + u''$ with $x = F(u', v')$ and $F(u'', v) = 0$. We have

$\tau(e, u''s_2 s_1, 0) = [\tau(u'', -v', 0), \tau(e, vs_2, 0)] \in H$, hence

$\tau(e, u's, 0) = \tau(e, u' s_2 s_1^2, 0) = \tau(e, vxs_2 s_1, 0)\, \tau(e, u''s_2 s_1, 0) \in H$ with $s = s_2 s_1^2 \in S = R \setminus P$.

The first half of Theorem 5 is proved. Now we have the second half to prove.

By Theorem 3, we have only the inclusion $[\mathrm{Gp}_F(A, B), \mathrm{Ep}_F R\,] \subset \mathrm{Ep}_F(A, B)$ to prove. Note that both $\mathrm{Gp}_F(A, B)$ and $\mathrm{Ep}_F R$ normalize $\mathrm{Ep}_F(A, B)$.

By the first conclusion of the theorem, it suffices to show that $[\mathrm{Gp}_F(A, B), \tau(e, 0, R)] \subset \mathrm{Ep}_F(A, B)$ for any F-unimodular vector e in V. In other words, we want to prove that the subgroups $\mathrm{Gp}_F(A, B)$ and $\tau(e, 0, R)$ commute modulo $\mathrm{Ep}_F(A, B)$.

It suffices to show that for every maximal ideal P of R and any g in $\mathrm{Gp}_F(A, B)$ there is $s \in S = R \setminus P$ such that $[g, \tau(e, 0, Rs)] \subset \mathrm{Ep}_F(A, B)$.

We will prove this using only the following condition: $\dim(F \bmod P) \geq 4$.

Case 1: there is w, w' in V orthogonal to both e and ge and such that $F(w, w') = s \in S = R \setminus P$. Let $\alpha \in \mathrm{GL}_1 R$ and $c \in R$ be such that $(c^2 - \alpha)R \subset B$, $F(gu, gv) = \alpha F(u, v)$, $gv - vc \in VA$ and $F(v, gv) + B = |\, gv - vc\,|$ for all $u, v \in V$. For any r in R we write

$\tau(ec, 0, rs) = \tau(ec, w, 0)\, \tau(ec, w'r, 0)\, \tau(ec, -w - w'r, 0)$

$= \tau(w, ec, 0)\, \tau(w', ecr, 0)\, \tau(w + w'r, -ec, 0)$

and $\tau(ge, 0, rs) = \tau(ge, w, 0)\, \tau(ge, w'r, 0)\, \tau(ge, -w - w'r, 0)$

$= \tau(w, ge, 0)\, \tau(w', ger, 0)\, \tau(w + w'r, -ge, 0)$, hence

$\tau(ge, 0, rs)\, \tau(ec, 0, rs)^{-1}$

$= \tau(w, ge, 0)\, \tau(w', ger, 0)\, \tau(w + w'r, -ge, 0)\, (\, \tau(w, ec, 0)\, \tau(w', ecr, 0)\, \tau(w + w'r, -ec, 0)\,)^{-1}$

$= h_1 (g_2 h_2 g_2^{-1})\, (g_3 h_3 g_3^{-1})$, where

$h_3 = \tau(w + w'r, -ge, 0)\, \tau(w + w'r, -ec, 0)^{-1} = \tau(w + w'r, ec - ge, -F(ge, ec)) \in \mathrm{Ep}_F(A, B)$,

$g_3 = \tau(w, ec, 0)\, \tau(w', ecr, 0) \in \mathrm{Ep}_F R$,

$h_2 = \tau(w', ger, 0)\, \tau(w', ecr, 0)^{-1} = \tau(w', ger - ecr, -F(ger, ecr)) \in \mathrm{Ep}_F(A, B)$,

$g_2 = \tau(w, ec, 0) \in \mathrm{Ep}_F R$,

and $h_1 = \tau(w, ge, 0)\, \tau(w, ec, 0)^{-1} = \tau(w, ge - ec, -F(ge, ec)) \in \mathrm{Ep}_F(A, B)$.

So $\tau(ge, 0, rs)\,\tau(ec, 0, rs)^{-1} \in \mathrm{Ep}_F(A, B)$, hence $[g,\ \tau(e, 0, \alpha rs)]$
$= g\,\tau(e, 0, \alpha rs)g^{-1}\,\tau(e, 0, \alpha rs)^{-1} = \tau(ge, 0, rs)\ \tau(e, 0, \alpha rs)^{-1}$
$= \tau(ge, 0, rs)\,\tau(ec, 0, rs)^{-1}\,(\tau(e, 0, rs(c^2-\alpha)) \in \mathrm{Ep}_F(A, B)$ for all r in R.
Thus, $[g,\ \tau(e, 0, Rs)] \subset \mathrm{Ep}_F(A, B)$.

General case. We pick a vector $e' \in V$ such that $F(e, e') = 1$ and write $ge = ex + e'y + u$ with $x = F(ge, e'),\ y = F(e, ge) \in R,\ u \in U = (Re + Re')^{\perp}$. Since $g \in \mathrm{Gp}_F(A, B)$, we have $(x^2 - \alpha(g))R \subset B$, $y \in A$, $u \in UA$, and $xy + B = |u|$.

Set $h = \tau(e',\ ux/\alpha(g),\ xy/\alpha(g))$. Then $hge = ex + e'ya + ua$, where $a = 1 - x^2/\alpha(g))$, $aR = (x^2 - \alpha(g))R \subset B$. Note that $xy/\alpha(g) - xy(x/\alpha(g))^2 = axy/\alpha(g) \in B$, hence $h \in \mathrm{Ep}_F(A,B)$. Since $ge = ex + e'y + u$ is F-unimodular and $a - 1 \in xR$, we can find $u' \in U$ and $r \in R$ such that $y' = y + F(u', u)a + rx \in S$. Set $h' = \tau(e', u'a, ra)h \in \mathrm{Ep}_F(A,B)$. Then $hge = ex + e'ay' + ua - u'a$.

Now we pick v, v' in U with $F(v, v') \in S$ and set $w = vy' + eF(u - u', v), w' = v'y' + eF(u - u', v')$. Then $F(w, w') = F(v, v')y'^2 \in S$ and $F(e, w) = F(e, w') = F(hge,w) = F(hge, w') = 0$. By Case 1. $[hg, \tau(e, 0, Rs)] \subset \mathrm{Ep}_F(A, B)$ for some $s \in S$, hence
$[g, \tau(e, 0, Rs)] \subset \mathrm{Ep}_F(A,B)$.

7. Proof of Theorem 7

By Theorem 5, it suffices to prove that $\mathrm{Ep}_F(A, B) \subset [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$, i.e. $T(e_1, A, B) \subset [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$, i.e. $\tau(e_1, uax, b) \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$ for all $u \in U = (e_1 R + e_2 R)^{\perp}$, $a \in A$, and $b \in B$, where e_1, e_2 is a hyperbolic pair in V.

LEMMA 22. Under the condition of Theorem 4, for any maximal ideal P of R there is $s \in S = R \setminus P$ such that $\tau(e_1, uas, 2sa' + bs^2) \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$ for all a, a' in A and b in B.

Proof. We pick vectors $e_3, e_4 \in U$ such that $s_0 = F(e_3, e_4) \in S$.

Case 1: $a = b = 0$. Then

$\tau(e_1, uas, 2sa' + bs^2) = \tau(e_1, 0\, 2sa')$

$= [\tau(e_1, e_3 a', 0), \tau(e_1, e_4, 0)] \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$ for $s = s_0 = F(e_3, e_4) \in S$.

Case 2: $a' = b = 0$ and the image $\pi(u)$ of u in U_P is F_P-unimodular. We pick $v \in U$ such that $s' = F(u, v) \in S$.

If $\mathrm{card}(R/P) \neq 2$, then we pick r in R with $r - r^2 \in S$ and set $f(y,t)$ $= \tau(e_1, uasty, -y(as't)^2) = [\tau(u, 0, y), \tau(e_1, vat, 0)] \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$ for any r,t in R, where $\tau(u, 0, r) \in \mathrm{Ep}_F R$ by Lemma 19 with $x = 1$. Now

$f(1,r) f(r^2, 1)^{-1} = \tau(e_1, uas'(r - r^2), 0) \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$.

So we are done with $s = s'(r - r^2) \in S$.

If $\mathrm{card}(R/P) = 2$, then $\dim(F \bmod P) \geq 6$ by the condition of Theorem 5. So we can find e, e' in U orthogonal to u, v so that $F(e, e') \in S$ Although e need not be F-unimodular, $\tau(e, u, 0) \in \mathrm{Ep}_F R$ by Lemma 19 with $x = 1$. So

$\tau(e_1, uas, 0) = [\tau(e, u, 0), \tau(e_1, e'a, 0)] \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$ for any $a \in A$, where $s = F(e, e') \in S$.

Case 3: $u = 0$ and $a' = 0$. Then $[\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R] \ni$
$[\tau(e_3, 0\, b), \tau(e_1, e_4, 0)] = \tau(e_1, e_3 b s_0, -b s_0{}^2)$ for all $b \in B$.

On the other hand, by Case 2 there is $s_1 \in S$ such that $[\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R] \ni \tau(e_1, e_3 b s_1, 0)$ for all $b \in B$. So for $s = s_0 s_1$ we obtain that $[\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R] \ni$
$\tau(e_1, e_3 bs, 0)\tau(e_1, e_3 bs, -bs^2)^{-1} = \tau(e_1, 0, bs^2)$ for all $b \in B$.

General case. We write $us_0 = e_3 t + e_4 t' + w = e_3 + e_4 t' + w + e_3 (t - 1)$ with $t = F(u, e_4), t' = F(e_3, u) \in R$ and $w \in U$ orthogonal to both e_3 and e_4. Then:

$\tau(e_1, 0, 2s_0 a') \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$ for all a' in A by Case 1;

$\tau(e_1, (e_3 + e_4 t' + w) a s_1, 0) \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$ for all $a \in A$ for a suitable $s_1 \in S$ by Case 2;

$\tau(e_1, e_3(t - 1) a s_2, 0) \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$ for all $a \in A$ with a suitable $s_2 \in S$ by Case 2;

$[\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R] \ni \tau(e_1, 0, b s_3{}^2)$ for all $b \in$ B with a suitable s_3 in S.

So for $s' = s_1 s_2 s_3 \in S$ and $s = s_0 s_1 s_2 s_3 \in S$ we obtain that $\tau(e_1, uas, 2sa' + bs^2)$
$= \tau(e_1, 0, 2sa')\, \tau(e_1, (e_3 + e_4 t' + w) as', 0) \cdot \tau(e_1, e_3(t - 1) ass', 0)\ \tau(e_1, 0, bs^2 + t'(t - 1) a^2 ss')$
$\in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$ foir all a, a' in A and b in B.

Lemma 22 is proved. Now, for fixed u, a, b, we set

$Y_1 = \{r \in R : \tau(e_1, uar, 0) \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]\}$,

$Y_2 = \{r \in R : \tau(e_1, 0, 2ra') \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]\}$,

$Y_3 = \{r \in R : \tau(e_1, 0, b3^2) \in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]\}$.

By Lemma 22, each Y_i contains Rs for an element s outside an arbiitrary maximal ideal P of R. Clearly, Y_1 and Y_2 are additive subgroups of R. So $Y_1 = Y_2 = R$. Now it is clear that Y_3 is an additive subgroups of R, hence $Y_3 = R$.

Therefore, $\tau(e_1, uas, 2sa' + bs^2) = \tau(e_1, uar, 0)\, \tau(e_1, 0, 2ra')\, \tau(e_1, 0, b3^2)$
$\in [\mathrm{Ep}_F(A, B), \mathrm{Ep}_F R]$.

8. Proof of Theorem 8

In this section we assume that there are vectors e_1, e_2 in V with $F(e_1, e_2) = 1$. As above, we set $U = (e_1R + e_2R)^{\perp}$.

Let H be a subgroup of $\mathrm{Gp}_F R$ normalized by $\mathrm{Ep}_F R$. Denote by A the ideal of R generated by all $F(U, u)$, where $u \in U$ and $\tau(e_1, u, r) \in H$ for some r in R (depending on u). Let B be the set of all $b \in R$ such that $\tau(e_1, 0, b) \in H$. Clearly, B is an additive subgroup of R.

LEMMA 23. $2A \subset B$.

Proof. It suffices to show that $2F(u,v) \in B$ whenever $u,v \in U, r \in R$, and $\tau(e_1, u, r) \in H$. We have $H \supset [H, \mathrm{Ep}_F R] \ni [\tau(e_1, u, r), \tau(e_1, v, 0)] = \tau(e_1, 0, 2F(u,v))$, hence $2F(u,v) \in B$ by the definition of B.

LEMMA 24. Suppose that $\dim(U \bmod P) \geq 2$ for every maximal ideal P of R. Then $B \subset A$.

Proof. The dimension condition means that 1 can be written as a sum of elements $F(u, v)$ with u, v in U. So it suffices to produce $\tau(e_1, vbF(u,v), *)$ in H for arbitrary u,v in U and b in B. We have $H \supset [H, \mathrm{Ep}_F R] \ni$

$[\tau(e_2, v, 0), \tau(e_1, 0, b)] = [\tau(e_1, 0, -b), \tau(v, e_2, 0)]$

$= \tau(v, e_2 - e_1b, 0)\, \tau(v, -e_2, 0) = \tau(v, -e_1b, -b)$, hence

$H \ni [\tau(e_1, u, 0), \tau(v, -e_1b, -b)] = \tau(e_1, u, 0)\, \tau(e_1, -\tau(v, -e_1b, -b)u, 0)$

$= \tau(e_1, u, 0)\, \tau(e_1, -u + e_1F(v, u) + vbF(v, u), 0) = \tau(e_1, vbF(v, u), -bF(v, u)^2)$.

LEMMA 25. Under the condition of Lemma 24, for any $w \in U$ and any $a \in A$ there is $t \in R$ such that $\tau(e_1, wa, t) \in H$.

Proof. It suffices to consider the case $a = F(u, v)$, where $u, v \in U, r \in R$, $\tau(e_1, u, r) \in H$. Set

$Y = \{s \in R : \tau(e_1, was, t) \in H$ for some $t \in R\}$.

We want to prove that $Y \ni 1$. Since Y is an additive subgroup of R, it suffices to show that $Y \supset Rs$ for an element s of R outside an arbitrary maximal ideal P of R.

We pick e, e' in V with $F(e, e') = s_0$ in $S = R \setminus P$. We write $ws_0 = ez + e'z' + w'$ with $z = F(w, e')$, $z' = F(e, w)$, w' orthogonal to e, e'. Similarly, we write $us_0 = ex + e'x' + u'$ and $vs_0 = ey + e'y' + v'$ with u' and v' orthogonal to e, e'. Note that $F(us_0, vs_0) = as_0^2 = yz' - zy' + F(u', v')$.

By Lemma 19, $\tau(e, v', y), \tau(e', 0, cs_0) \in \mathrm{Ep}_F R$ for any c in R, so

$H \supset [\mathrm{Ep}_F R, H] \ni [\tau(e, v', y), \tau(e_1, u, r)] = \tau(e_1, \tau(e, v', y)u, r)\, \tau(e_1, -u, -r)$

$= \tau(e_1, -eF(u', v') + eyx' + v'x's_0, ?)$, hence

$H \supset [\mathrm{Ep}_F R, H] \ni [\tau(e', 0, cs_0), \tau(e_1, -eF(u', v') + eyx' + v'x's_0, ?)]$

$= \tau(e_1, e'cs_0^2(F(u', v') - yx'), ?)$.

Moreover, $H \supset [\mathrm{Ep}_F R, H] \ni [\tau(e', 0, 1), \tau(e_1, u, r)] = \tau(e_1, -e'x, ?)$, hence $H \supset [\mathrm{Ep}_F R, H] \ni [\tau(e, 0, 1), \tau(e_1, -e'x, ?)] = \tau(e_1, -exs_0, ?)$, hence $H \supset [\mathrm{Ep}_F R, H] \ni$

$[\tau(e', 0, cy'), \tau(e_1, -exs_0, ?)] = \tau(e_1, e'cxy's_0^2, ?)$. So $H \ni$

$\tau(e_1, e'cs_0^2(F(u', v') - yx'), ?)\, \tau(e_1, e'cxy's_0^2, ?) = \tau(e_1, e'cs_0^2(F(u', v') - yx' + xy'), ?) = \tau(e_1, e'cas_0^4, ?)$.

Recall that c here is an arbitrary element of R. So $H \ni \tau(e_1, e'c(z's_0 - 1)as_0^4, ?)$.

By Lemma 19, $f = \tau(e, w, z) \in \mathrm{Ep}_F R$. So

$H \ni f\tau(e_1, e'cas_0^4, ?)f^{-1} = \tau(e_1, fe'cas_0^4, ?)$.

Therefore $H \ni \tau(e_1, e'c(z's_0 - 1)as_0^4, ?)\, \tau(e_1, fe'cas_0^4, ?)$

$= \tau(e_1, (e'(z's_0 - 1) + \tau(e, w, z)e')\, cas_0^4, ?) = \tau(e_1, wcas_0^6, ?)$.

Thus, $Y \supset Rs$ with $s = s_0^6$ in $S = R \setminus P$.

COROLLARY 26. Under the coditions of Theorem 5, (A, B) is a symplectic ideal of R.

Proof. Let $r \in R, a \in A, b \in B$. By Lemmas 23 and 24, $2a, \in B$ and $b \in A$. It remains to prove that br^2, $ra^2 \in B$.

To prove that $ra^2 \in B$, it suffices to show that for any maximal ideal P of R there is $s \in S = R \setminus P$ such that $a^2 sR \subset B$.

We pick vectors $e_3, e_4 \in U$ such that $s_0 = F(e_3, e_4) \in S$.

By Lemma 25, for any c in R we have $\tau(e_1, e_4ca, ?) \in H$. So for any d in R we have
$H \supset [\mathrm{Ep}_F R, H] \ni [\tau(e_3, 0, d), \tau(e_1, e_4ca, ?)] = \tau(e_1, e_3acds_0, -a^2c^2ds_0^2) = f(c, d)$.
So $H \ni f(c, d) f(1, dc^2)^{-1} = \tau(e_1, e_3a(c-c^2)ds_0, 0)$ and
$H \ni \tau(e_1, e_3a(c-c^2)ds_0, 0) f(c-c^2, d)^{-1} = \tau(e_1, 0, a^2(c-c^2)^2ds_0^2)$,
i.e. $a^2(c-c^2)^2ds_0^2 \in B$.

If $\mathrm{card}(R/P) \neq 2$, we can choose c such that $c^2 - c$ is in S, hence $a^2sR \subset B$ for $s = (c-c^2)^2s_0^2 \in S$.

If $\mathrm{card}(R/P) = 2$, we pick vectors e, e' in U orthogonal to e_3, e_4 and such that $F(e, e') \in S$. By Lemma 19, $\tau(e, e_3d, 0) \in \mathrm{Ep}_F R$. So $H \supset [\mathrm{Ep}_F R, H] \ni$
$[\tau(e, e_3d, 0), \tau(e_1, e'a, ?)] = \tau(e_1, e_3adF(e, e'), 0)$, hence
$H \ni f(1, -dF(e, e'))\tau(e_1, e_3adF(e, e'), 0) = \tau(e_1, 0, a^2dF(e, e')s_0^2)$,
i.e. $sa^2R \subset B$ for $s = F(e, e')s_0^2 \in S$.

We have proved that $ra^2 \in B$.

Now we have to prove that $br^2 \in B$. Since $2A \subset B$, it suffices to show that for any maximal ideal P of R there is $s \in S = R \setminus P$ such that $br^2s^2 \in B$.

Let e_3 and e_4 be as above. We have seen that for any $a \in A$ there is $s \in S$ such that
(27) $\tau(e_1, e_3ads, 0) \in H$ for all $d \in R$.
We will use this with a, d replaced by b, r. We have
$H \supset [H, \mathrm{Ep}_F R] \ni [\tau(e_1, 0, b), \tau(e_3, e_2r, 0)]\,\tau(e_1, e_3brs, 0)$
$= \tau(e_3, e_1brs, -br^2s^2)\tau(e_1, e_3brs, 0) = \tau(e_3, 0, -br^2s^2)$, hence $H \supset [H, \mathrm{Ep}_F R] \ni$
$[\tau(e_3, 0, -br^2s^2), \tau(e_1, e_4, 0)]\,\tau(e_1, e_3br^2s^2, 0)$
$= \tau(e_1, -e_3br^2s^2, br^2s^2)\,\tau(e_1, e_3br^2s^2, 0)$
$= \tau(e_1, 0, br^2s^2)$.
Thus, $br^2s^2 \in B$.

COROLLARY 28. Under the coditions of Theorem 5, $H \supset \mathrm{Ep}_F(A, B)$,

Proof. By Theorem 4, it suffices to show that $H \supset T(e_1, A, B)$. By the definition of B, $H \supset \tau(e_1, 0, B)$. So it remains to show that $\tau(e_1, wa, 0) \in H$ for any $u \in U$ and any $a \in A$.

Set $Y = \{t \in R: \tau(e_1, wat, 0) \in H\}$. We want to prove that $1 \in Y$. Since Y is closed under addition, it suffices to show that for any maximal ideal P of R there is an element $s' \in S = R \setminus P$ such that $Rs' \subset Y$. i.e. $\tau(e_1, was'r, 0) \in H$ for all r in R.

Let $e_3, e_4 \in U$ and $s_0 = F(e_3, e_4) \in S$ be as in the proof of Corollary 24 above. We are going to use (26) again. We write $ws_0 = e_3x + e_4y + w'$ with $x, y \in R$ and $w' \in U$ orthogonal to e_3, e_4. Then $ws_0^2 = e_3(xs_0 - 1) + e_3 + e_4ys_0 + w's_0 = e_3(xs_0 - 1) + fe_3$, where $f = \tau(e_4, -w', -y) \in \mathrm{Ep}_F R$ by Lemma 15.

By (27), $h_1 = \tau(e_1, e_3(xs_0 - 1)ars, 0) \in H$ and $h_2 = \tau(e_1, fe_3ars, 0) = f\tau(e_1, e_3ars, 0)f^{-1} \in H$ for all r in R. Since $(xs_0 - 1)ys_0a^2r^2s^2 \in Ra^2 \subset B$ by Corollary 24, $h_3 = \tau(e_1, 0, (xs_0 - 1)ys_0a^2r^2s^2) \in H$. So $\tau(e_1, warss_0^2, 0) = h_3h_2h_1 \in H$, hence $rss_0^2 = rs' \in Y$ for all $r \in R$, where $s' = ss_0^2 \in H$. Corollary 28 is proved.

Originally, our definition of A, B depended on choice of an F-unimodular vector e_1. However Corollary 28 shows that in fact it does not depend. We can also state it as follows;
COROLLARY 29. Under the conditiuons of Theorem 5, $\mathrm{Ep}_F(A, B)$ contains all symplectic transvections in H.

LEMMA 30. Under the conditions of Theorem 5, let $e \in U, v \in V, r, r' \in R, F(e, v) = 0$, and $\tau(e, v, r'), \tau(e, 0, r) \in H$. Then $F(u, V)r_0 \subset A$ and $rr_0^4 \in B$ for every $r_0 \in F(e, V)$.

Proof. We pick a vector $e' \in V$ such that $F(e, e') = r_0$. We have $H \supset [\mathrm{Ep}_F R, H] \ni [\tau(e, 0, r), \tau(e_1, e't, 0)] = \tau(e_1, ertr_0, -rt^2 r_0^2) = f(t)$ for all t in R.

By its definition, $A \supset F(err_0, V) \supset Rrr_0^2$.

By Corollary 28, $H \supset \mathrm{Ep}_F(A, B) \ni \tau(e_1, err_0^2, 0)$. So

$H \ni \tau(e_1, err_0^2, 0) f(r_0)^{-1} = \tau(e_1, 0, rr_0^4)$. By its definition, $B \ni rr_0^4$.

Now we have the inclusiuon $F(u, V)r_0 \subset A$ to prove. It suffices to show that for every maximal ideal P of R there is $s \in S = R \setminus P$ such that $sF(u, V)r_0 \subset A$.

Pick any $v' \in V$ and set $z = F(v, v')$. We have to prove that $r_0 sz \in A$ for some $s \in S$ independent on v'. We write $v' = e_1 x + e_2 y + w$ with $x, y \in R$ and $w \in U$. Note that $F(e, w) = 0$ and $z = F(v, e_1)x + F(v, e_2)y + F(v, w)$.

We have:

$H \ni [\tau(e_1, 0, x), \tau(e, v, r')] = \tau(e, e_1 F(e_1, v)x, ?)$;

$H \ni [\tau(e_2, 0, 1), \tau(e, v, r')] = \tau(e, e_2 F(e_2, v), ?)$, hence

$H \ni [\tau(e_1, 0, y), \tau(e, e_2 F(e_2, v), ?)] = \tau(e, e_1 F(e_2, v)y, ?)$;

$H \ni [\tau(e_2, w, 0), \tau(e, v, r')] = \tau(e, e_2 F(w, v) + wF(e_2, w), ?)$, hence

$H \ni [\tau(e_1, 0, 1), \tau(e, e_2 F(w, v) + wF(e_2, w), ?)] = \tau(e, e_1 F(w, v), ?)$.

So $H \ni \tau(e, e_1 F(e_1, v)x, ?)\, \tau(e, e_1 F(e_2, v)y, ?)\, \tau(e, e_1 F(w, v), ?)$

$= \tau(e, e_1 F(v', v)x, ?) = \tau(e, -e_1 z, ?)$.

If $\mathrm{card}(R/P) \neq 2$, we pick $t_0 \in R$ with $s = t_0^2 - t_0 \in S$. Then for any $t, t' \in R$ we have

$H \ni [\tau(e_2, 0, t), \tau(e, -e_1 z, ?)] = \tau(e, -e_2 tz, -tz^2)$, hence

$H \ni [\tau(e_1, 0, t'), \tau(e, -e_2 tz, -tz^2)] = \tau(e, -e_1 tt'z, -t^2 t' z^2) = f(t, t')$, and

$H \ni f(1, t_0^2) f(t_0, 1)^{-1} = \tau(e, e_1 sz, 0) = \tau(e_1, esz, 0)$.

Thus, $szr_0 \in A$ by the definition of A.

If card$(R/P) = 2$, we invoke the condition of Theorem 5 to find vectors $e_3, e_4 \in U$ orthogonal to e, e' with $s = F(e_3, e_4) \in S$. Then

$H \ni [\tau(e_2, e_3, 0), \tau(e, -e_1 z, ?)] = \tau(e, -e_3 z, 0)$, hence

$H \ni [\tau(e_1, e_4, 0), \tau(e, -e_3 z, 0)] = \tau(e, -e_1 sz, 0) = \tau(e_1, esz, 0)$.

Thus, $szr_0 \in A$ by the definition of A.

LEMMA 31. Under the conditions of Theorem 8, let $h \in H$ and $he = ec$ for some $c \in R$ and an F-unimodular vector $e \in V$. Then $hv - vc \in VA$ and $|hv - vc| = F(hv, vc) + B$ for all $v \in V$.

Proof. Clearly, $c \in \mathrm{GL}_1 R$. For any vector u in V orthogonal to e and any scalar r in R we have

$H \ni [h, \tau(e, u, r)] = \tau(e, huc/\alpha(h) - u, rc^2/\alpha(h) - r - F(hu, uc)/\alpha(h))$.

So (using Lemma 30 and a condition of Theorem 8) $huc/\alpha(h) - u \subset VA$ and

$|huc/\alpha(h) - u| = rc^2/\alpha(h) - r - F(hu, uc)/\alpha(h) + B$ for all $u \in e^{\perp}$, hence (taking $u = 0$) $R(\alpha(h) - c^2) \subset B$. It follows that $hu - uc \subset VA$ and $|hu - uc| = F(hu, uc) + B$ for all $u \in e^{\perp}$.

Pick a vector e' in V with $F(e, e') = 1$. We can write $h = \tau(e, u, r)h'$, where $u \in V' = (eR + e'R)^{\perp}$, $r \in R$, $h' \in \mathrm{Gp}_F(A, B)$, $h'e = ec$, and $h'e' = e'\alpha(h)/c$, $h'v - vc \in VA$ and $|h'v - vc| = F(h'v, vc) + B$ for all v in V.

For any $w \in V'$ we have $H \ni [h, \tau(w, 0, 1)]$, because $\tau(w, 0, 1) \in \mathrm{Ep}_F R$, and $H \ni [h', \tau(w, 0, 1)]$ by Theorem 5. So $H \ni [\tau(e, u, r), \tau(w, 0, 1)] = \tau(e, u, r)\,\tau(e, -u - wF(w,u), -r) = \tau(e, -wF(w,u), ?)$, hence $wF(w,u) \in VA$. It follows that that $u \in VA$.

Incuding $\tau(e, u, r')$ into h', where $r' \in |u|$, we are reduced to the case $u = 0$. In this case, $h = \tau(e, 0, r)h'$, and for any vector $w \in V'$ we have $H \ni [h, \tau(w, e', 0)]$ and $H \ni [h', \tau(w, e', 0)]$, hence $H \ni [\tau(e, 0, r), \tau(w, e', 0)] = \tau(w, er, -r)$. By Lemma 30, $wr \in VA$. So $U'r \subset U'A$, hence $r \in A$. Using Lemma 30 again, we conclude that $r \in B$. Thus, we can include we can include $\tau(e, 0, r)$ into h', i.e. we are reduced to the case when $h = h'$.

LEMMA 32. Under the conditions of Theorem 8, let $h \in H \cap \mathrm{Sp}_F R$. and $hw = w$ for a vector $w \in V$ which is orthogonal to a hyperbolic pair. Then $(hv - v)r_0 \in VA$ and $|(hv - v)r_0|r_0^4 = F(hv, v)r_0^6 + B$ for all $v \in V$ orthogonal to w and all $r_0 \in F(w, V)$.

Proof. We can assume that w is orthogonal to e_1, e_2 i.e. $w \in U$. For any vector v in V orthogonal to w and any scalar r in R we have

$H \ni [h, \tau(w, v, r)] = \tau(w, hv - v, -F(hv, v))$.

By Lemma 30, $(hv - v)r_0 \in VA$. We pick now $z \in |(hv - v)r_0|$. Then

$H \ni \tau(w, (hv - v)r_0, z)$ and

$H \ni \tau(w, -hv\, r_0 + vr_0, -F(hvr_0, vr_0))$, hence $H \ni \tau(w, 0, z - F(hvr_0, vr_0))$.

By Lemma 30, $(z - F(hvr_0, vr_0))r_0^4 \in B$.

Thus, $(hv - v)r_0 \in VA$ and $|(hv - v)r_0|r_0^4 = F(hv, v)r_0^6 + B$ for all $v \in w^{\perp}$.

LEMMA 33. Under the conditions of Theorem 8, assume that $A = 0$. Then $H \subset \mathrm{Gp}_F(A, B) = \mathrm{Gp}_F(0, 0)$.

Proof. Let $h \in H$. We write $he_1 = e_1x + e_2y + u$ with $x = F(he_1, e_2), y = F(e_1, he_1), u \in U$. We set

$h' = [h, \tau(e, 0, 1)] \in H$

Case 1: $y = 0$. Then $h'e_1 = e_1$. So $h' = 1$ by Lemma 31 with $A = 0$. It follows that $u = 0$. So $he_1 = e_1x$. By Lemma 31, $h \in \mathrm{Gp}_F(A, B) = \mathrm{Gp}_F(0, 0)$

Case 2; $y^2 = 0$. Since $h'e_1 = e_1 + he_1y$, we have $h' \in \mathrm{Gp}_F(A, B) = \mathrm{Gp}_F(0, 0)$ by Case 1. It follows that $F(h'e_1, e_2) = xy - 1 - x^2 = 0$ and $ux = 0$, hence $x \in \mathrm{GL}_1R$, and $u = 0$. So $he_1 = e_1x$. By Lemma 31, $h \in \mathrm{Gp}_F(A, B) = \mathrm{Gp}_F(0, 0)$.

Case 3: $y^3 = 0$. Since $h'e_1 = e_1 + he_1y$, we have $h' \in \mathrm{Gp}_F(A, B) = \mathrm{Gp}_F(0, 0)$ by Case 2. It follows that $F(h'e_1, e_2) = xy - 1 - x^2 = 0$ and $ux = 0$, hence $x \in \mathrm{GL}_1R$, and $u = 0$. So $he_1 = e_1x$. By Lemma 31, $h \in \mathrm{Gp}_F(A, B) = \mathrm{Gp}_F(0, 0)$.

Case 4: $y^3 \neq 0$. Then there is a maximal ideal P of R such that $y^3 s \neq 0$ for all $s \in S = R \setminus P$. We pick a pair v, v' of vectors in U with $r_0 = F(v, v') \in S$, and set $w = e_1 F(u, v) + v.y$. Then $F(e_1, w) = F(he_1, w) = 0$, $h'w = w$. and $F(w, V) \ni y^2 r_0 \in Sy^2$. By Lemma 32, $(h'e_1 - e_1) y^2 r_0 = 0$, hence $y^3 r_0 = 0$ (because $h'e_1 - e_1 = he_1 ry$).

So Case 4 is impossible.

LEMMA 34. Under the conditions of Theorem 8, $H \subset \mathrm{Gp}_F(A, A)$

Proof. We want to prove that the image of H modulo A consists of scalar automorphisms of R/A-module V/VA. Indeed, otherwise, applying Lemma 33 to this module instead of V, we would obtain a non-trivial symplectic transvection in the image of H modulo A. (We used that the image of $\mathrm{Ep}_F R$ modulo A contains all symplectic transvections of $(V/VA, F \bmod A)$.)

So H would contain an element of the form $\tau(e, u, r)\, g$, where $\tau(e, u, r)$ is a symplectic transvection in $\mathrm{Ep}_F R$ which is non-trivial modulo A and where g is trivial modulo A, hence $g \in \mathrm{Gp}_F(A, A)$. We pick a vector $e' \in V$ with $F(e, e') = 1$ and set $U' = (eR + e'R)^{\perp}$. We can assume that $u \in U'$.

By Lemma 19, $\tau(w, 0, 1) \in \mathrm{Ep}_F R$ for any $w \in U'$, hence $[\tau(w, 0, 1), g] \in \mathrm{Ep}_F(A, A)$ by Theorem 5. It follows that $\tau(e, wF(w, u), ?) = [\tau(w, 0, 1), \tau(e, u, r)] \in H\, \mathrm{Ep}_F(A, A)$. By Corollary 29, applyed to $H\,\mathrm{Ep}_F(A, A)$ instead of H, we obtain that $F(w, u) \in A$. So $F(U', u) \subset A$, hence $u \in UA$.

Including $\tau(e, u, 0)$ into g, we are reduced to the case $u = 0$. In this case we have

$\tau(w, -er, ?) = [\tau(w, e', 0), \tau(e, u, r)] \in H\,\mathrm{Ep}_F(A, A)$, hence $rF(w, U) \subset A$ for all $w \in U'$ by Corollary 29. It follows that $r \in A$. This is a contradiction.

LEMMA 35. Under the conditions of Theorem 8, let $g \in \mathrm{Gp}_F R$ and $ge_1 = e_1 x + e_2 a' + ua$ with $u \in UA$, $a, a' \in A$, $x \in R$, and $xa' \in B$. Then $\tau(ge_1, 0, r)\tau(e_1 x, 0, -r) \in \mathrm{Ep}_F(A, B)$ for all $r \in R$.

Proof. It suffices to show that for each maximal ideal P of R there is $s \in S = R \setminus P$ such that $\tau(ge_1, 0, rs)\tau(e_1 x, 0, -rs) \in \mathrm{Ep}_F(A, B)$ for all $r \in R$.

Case 1: there is w, w' in V orthogonal to both e_1 and ge_1 and such that $F(w, w') = s \in S = R \setminus P$. For any r in R we write

$\tau(e_1x, 0, rs) = \tau(e_1x, w, 0)\ \tau(e_1x, w'r, 0)\ \tau(e_1x, -w - w'r, 0)$

$= \tau(w, e_1x, 0)\ \tau(w', e_1x\, r, 0)\ \tau(w + w'r, -e_1x, 0)$ and $\tau(ge_1, 0, rs)$

$= \tau(ge_1, w, 0)\ \tau(ge_1, w'r, 0)\ \tau(ge_1, -w - w'r, 0) = \tau(w, ge_1, 0)\ \tau(w', ge_1\, r, 0)\ \tau(w + w'r, -ge_1, 0)$,

hence $\tau(ge_1, 0, rs)\ \tau(e_1x, 0, rs)^{-1}$

$= \tau(w, ge_1, 0)\ \tau(w', ge_1r, 0)\ \tau(w + w'r, -ge_1, 0)\ (\ \tau(w, e_1x, 0)\ \tau(w', e_1x\, r, 0)\ \tau(w + w'r, -e_1x, 0)\ )^{-1}$

$= h_1(g_2h_2g_2^{-1})\ (g_3h_3g_3^{-1})$, where

$h_3 = \tau(w + w'r, -ge_1, 0)\ \tau(w + w'r, -e_1x, 0)^{-1} = \tau(w + w'r, e_1x - ge_1, -F(ge_1, e_1x)) \in \mathrm{Ep}_F(A, B)$, $g_3 = \tau(w, e_1x, 0)\ \tau(w', e_1x\, r, 0) \in \mathrm{Ep}_F R$,

$h_2 = \tau(w', ge_1r, 0)\ \tau(w', e_1x\, r, 0)^{-1} = \tau(w', ge_1r - e_1xr, -F(ge_1r, e_1xr)) \in \mathrm{Ep}_F(A, B)$,

$g_2 = \tau(w, e_1x, 0) \in \mathrm{Ep}_F R$, and $h_1 = \tau(w, ge_1, 0)\ \tau(w, e_1x, 0)^{-1}$

$= \tau(w, ge_1 - e_1x, -F(ge_1, e_1x)) \in \mathrm{Ep}_F(A, B)$.

So $\tau(ge_1, 0, rs)\ \tau(e_1x, 0, rs)^{-1} \in \mathrm{Ep}_F(A, B)$.

Case 2: $F(V, u)$ intersects S. Then we can find w' in U such that $F(u, w') = s \in S$ and set $w = u$. The vectors w, w' are orthogonal to both e_1 and ge_1, se we are done by Case 1.

Case 3: $a' \in S$. Then we find vectors v, v' in U such that $F(v, v') \in S$ and set $w = e_1F(u, v) + va'$, $w' = e_1F(u, v') + v'a$. Then $F(w, w') = F(v, v')a'^2 \in S$ and the vectors w, w' are orthogonal to both e_1 and ge_1, se we are done by Case 1.

Case 4: $x \in S$. Then we can find $v \in U$ such that both $F(v, U)$ and $F(u - vx, V)$ intersects S. Set $g' = \tau(e_2, va, 0)g$, so $g'e_1 = e_1x + e_2(a' + F(va, ua)) + (u - vx)a$. By Case 2, there is $s_1 \in S = R \setminus P$ such that $\tau(g'e_1, 0, r\, s_1)\tau(e_1x, 0, -rs_1) \in \mathrm{Ep}_F(A, B)$ for all $r \in R$. Conjugating this by $\tau(e_2, va, 0)$, we obtain that $\tau(ge_1, 0, rs_1)\tau(\ \tau(e_2, -va, 0)\ e_1x,\ 0, -rs_1) \in \mathrm{Ep}_F(A, B)$ for all r in R.

On the other hand, we can apply Case 2 to $g = \tau(e_2, -va, 0)$ and conclude that $\tau(\tau(e_2, -va, 0)e_1, 0, rs_2)\tau(e_1, 0, -rs_2) \in \mathrm{Ep}_F(A, B)$ for some s_2 in S and all r in R.

So $\tau(ge, 0, rs)\tau(ex, 0, -rs) = \tau(ge, 0, rs_1s_2)\tau(ex, 0, -rs_1s_2)$

$=(\tau(ge_1, 0, rs_2s_1)\tau(\tau(e_2, -va, 0)\, e_1x, 0, -rs_2s_1))$

$\cdot(\tau(\tau(e_2, -va, 0)\, e_1, 0, x^2s_1rs_2)\tau(e_1, 0, -x^2s_1rs_2))$

$\in \mathrm{Ep}_F(A, B)$ for all $r \in R$.

General case. Since ge_1 is F-unimodular, Cases 2, 3, 4 cover all possibilities.

LEMMA 36. Under the conditions of Theorem 8, let $e \in V$ be F-unimodular, $h \in H$, $c \in R$ and $hv - vc \in VA$ for all $v \in V$. Then

(36) $(F(he, ec) + t)r^2\alpha(h)^2 + c^2(c^2 - \alpha(h))r \in B$ for all $r \in R$ and all $t \in |he - ec|$.

Proof. Note that in the presence of a hyperbolic pair e, e', the element $\alpha(h) \in \mathrm{GL}_1R$ (such that $F(hu, hv) = \alpha(h)\, F(u, v)$ for all u, v in V) is unique and equal to $F(he, he')$. By Lemma 34, $h \in \mathrm{Gp}_F(A, A)$, i.e. there is $c \in R$ such that $gv - vc \in VA$ for all $v \in V$. Such an element c is not unique, but its coset $c + A$ is unique (under the conditions of Theorem 8), $c + A \in \mathrm{GL}_1R/A$, and $c^2 - \alpha(h) \in A$. Note also the the relation (36) we want to prove does not depend on choice of c in the coset $c + A$ or on choice t in the coset $|he - ec| \in A/B$. It suffices to consider the case $e = e_1$.

We write $he_1 = e_1x + e_2y + u$ with $x = F(he_1. e_2) \in c + A$, $y = F(e_1, he_1) \in A$, $u \in UA$, where $U = (Re_1 + Re_2)^\perp$.

Pick $z \in |u|$. Then $t \equiv (x - c)y + z \pmod B$, hence $F(he, ec) + t \equiv xy + z \pmod B$.

Since $c^2 - \alpha(h) \in A$, $a = 1 - xx' \in A$ for $x' = x/\alpha(h)$.

Set $f = \tau(e_2, ux', zx'^2) \in T(e_2, A, B)$. Then $fhe_1 = e_1x + e_2y' + ua$ with $y' = y - zxx'^2 \in A$. Note that $R(1-(xx')^2) = R(2a - a^2) \subset B$, hence $F(he, ec) + t \equiv xy + z \equiv xy' \pmod B$. (Recall that $2A + a^2R \subset B$.)

Set now $z' = x'y'(1 + a) \in A$ and $f' = \tau(e_2, 0, z') \in T(e_2, A, A)$. Then $ge_1 = f'fhe_1 = e_1x + e_2a' + ua$, where $g = f'fh \in \mathrm{Gp}_F(A, A)$ and $a' = y'a^2$, so $a'R \subset B$.

By Lemma 35, $\tau(ge_1, 0, r)\tau(e_1x, 0, -r) \in \mathrm{Ep}_F(A, B)$ for all $r \in R$. Note that
$[g, \tau(e_1, 0, \alpha(g)r)] = \tau(g e_1, 0, r)\tau(e_1, 0, -\alpha(g)r)$
$= \tau(ge_1, 0, r)\tau(e_1x, 0, -r)\,\tau(e_1, 0, rx^2 - \alpha(g)r)$
$\in \mathrm{Ep}_F(A, B)\,\tau(e_1, 0, r(x^2 - \alpha(g)))$ for all r in R.
Since $h \in H$, $[H, \mathrm{Ep}_F R] \subset H$ and $f \in \mathrm{Ep}_F(A, B) \subset H$, it follows that $k(r) =$
$[f', \tau(e_1, 0, \alpha(g)r)]\,\tau(e_1, 0, rx^2 - \alpha(g)r) \in H$ for all $r \in R$.

Since $k(r)$ fixes every vector in U, we can use Lemma 32 and conclude that $|k(r)e_2 - e_2| = F(k(r)e_2, e_2) + B$, i.e. $dd' \in B$, where
$k(r)e_2 = e_1d + e_2d'$, i.e. $d = F(k(r)e_2, e_2)$ and $d' = F(e_1, k(r)e_2)$.

Set $r' = \alpha(g)r = \alpha(h)r \in R$ and $r'' = rx^2 - \alpha(g)r = rx^2 - \alpha(h)r \in A$. Since $f' = \tau(e_2, 0, z')$ $\in T(e_2, A, A)$,
$k(r) = [f', \tau(e_1, 0, r')]\tau(e_1, 0, r'') = \tau(f'e_1, 0, r')\tau(e_1, 0, r''-r')$.
So $k(r)e_2 = \tau(f'e_1, 0, r')(e_1(r''-r') + e_2)$
$= e_1(r''-r') + e_2 + f'e_1r'F(f'e_1, e_1(r''-r') + e_2)$
$= e_1(r''-r') + e_2 + (e_1 - e_2 z')r'(z'(r''-r') + 1) = e_1d + e_2d'$ with $d = r'' + r'z'(r''-r')$ and $d' = 1 - r'z' - r'z'^2(r''-r')$.

So $dd' \in -z'r'^2 + r'' + z'^2R \subset z'r'^2 + r'' + B$, because $z' \in A$. Since $dd' \in B$, we conclude that $z'r'^2 + r'' \in B$. So $z'r'^2x^2 + r''x^2 \in B$, i.e. $x'y'(1+a)r'^2x^2 + r''x^2 \in B$, i.e. $y'r'^2x + x^2r'' \in B$

Recall now that $x - c \in A$, $F(he, ec) + t \equiv xy' \pmod{B}$, $r' = \alpha(h)r$, and $r'' = rx^2 - \alpha(h)r$. Thus, we obtain (36).

Now we can conclude our proof of Theorem 8. Pick $t_1 \in |he_1 - e_1c|$ and $t_2 \in |he_2 - e_2c|$. Then $t_1 + t_2 + F(he_1 - e_1c, he_2 - e_2c) \in |h(e_1 + e_2) - (e_1 + e_2)c|$. We apply Lemma 35 to $e = -e_2$, $e = e_1$, and $e = e_1 + e_2$. Using that $F(he_1 - e_1c, he_2 - e_2c)$
$= \alpha(h) + c^2 - F(he_1, e_2c) - F(e_1c, he_2)$
$= \alpha(h) + c^2 - F(h(e_1 + e_2), (e_1 + e_2)c) + F(he_1, e_1c) + F(he_2, e_2c)$, and that
$2A \subset B$, we obtain that $\alpha(h) + c^2 + c^2(c^2 - \alpha(h))r \in B$ for all $r \in R$, hence $c'^2c^2(c^2 - \alpha(h))R \subset B$ for all $c' \in R$. Picking c' such that $cc' - 1 \in A$, we conclude that $(c^2 - \alpha(h))R \subset B$.. Now Lemma 35 gives that $F(he, ec) + t \in B$ for all F-unimodular vectors $e \in V$. Since V is spanned by its F-unimodular vectors, we conclude that $h \in \mathrm{Gp}_F(A, B)$.

References

1. E.Abe, Chevalley groups over local rings, Tohoku Math. J. 21 (1969), 474-494.
2. E.Abe, Normal subgroups of Chevalley groups over commutative rings, preprint.
3. E.Abe and K.Suzuki, On normal subgroups of Chevalley groups over commutative rings, Tohoku Math. J. 28 (1976), 185-198.
4. D.G. Arrell. The subnormal subgroup structure of the infinite symplectic group, Proc.Edinb.Math.Soc. 25 (1982), 209-216.
5. E. Artin, Geometric Algebra, Willey-Interscience, New York, 1957.
6. A.Bak, On modules with quadratic forms, in Springer Lecture Notes Math. 108 (1969), 55-66.
7. A.Bak, The stable structure of quadratic modules, preprint, 1971
8. H. Bass, K-theory and stable algebra, IHES Publ.Math. 22 (1964), 485-544; 5-60.
9. H. Bass, Unitary algebraic K-theory, in Springer Lecture Notes Math. 343 (1973), 57-205 .
10. H. Bass, L_3 of finite abelian groups, Ann. Math. 99 (1974), 118- 153.
11. H.Bass, J.Milnor and J.-P.Serre, Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$), IHES Publ.Math. 33 (1967), 59-137.
12. H.Bass and W.Parson, Some hybrid symplectic group phenonema,J. Algebra .53:2 (1978), 327-333.
13. C.N. Chang, The structure of the symplectic group over an unramified dyadic field, J. Algebra 30 (1974), 42-50. MR 49#9096.
14. C.N. Chang, The structure of symplectic groups over the integers of dyadic fields, J. Algebra 34 (1975), 172-187. MR 52#592.
15. C.N. Chang, The structure of the symplectic group over semi-local domains, J.Algebra 35 (1975), 457-476. MR 51#5793.
16. C.N. Chang and C.K. Cheng, Symplectic groups over principal ideal domains, J.Algebra 41 (1976), 455-472 . MR 54#12649.

17. D.L.Costa and G.E. Keller, On the normal subgroups of SL(2, A), preprint.
18. P.M.Cohn, On the structure of GL_2 of a ring, IHES Publ. Math. 30 (1966), 5-53.

19. J.Dennin and D.L. MccQuillan, A note on the classical groups over semi-local rings, Proc.Roy.Irish Acad. S.A. 68 (1969), 1-4. RZh 1970.2A220, 3A249.

20. L.E.Dickson, Theory of linear groups in arbitrary field Trans Amer. Math. Soc 2 (1901), 363-394.
21. J.Dieudonne, La géométrie des groupes classiques, 3nd ed. Erg.der Math.und Grenz. Band 5, Springer Verlag 1971.
22. F.Grunewald and S.Schwermer, Free non-abelian quotient of SL_2 over the order of imaginary quadratic number field, J.Algebra 69 (1981), 298-304,
23. W.Klingenber, Symplectic groups over local rings, Am. J. Math. 85 (1963), 232-240.
24. V.I.Kopeiko, Stabilization of symplectic groups over polynomial rings, Mat.Sb 106:1 (1978), 94-107.
25. N.H.J. Lacroix, Two-dimensional linear groups over local rings, Can.J.Math. 21 (1969),106-135. MR 38#5939.
26. N.H.J. Lacroix and C.Levesque, Sur les sous-groupes normaux de SL_2 sur un anneau local, Can. Math. Bull. 26:2 (1983), 209-219.
27. A.W.Mason, Lattice subgroups of normal subgroups of genus zero of the modular group, Proc.London Math.Soc. 24 (1972), 449-469. MR#3635.
28. A.W.Mason, Anomalous normal subgroups of the modular group, Comm. Algebra 11 (1983), 2555-2573.
29. A.W.Mason, Anomalous normal subgroups of $SL_2K[x]$, Quart.J.Math. 36:143 (1985) 345-358.
30. A.W.Mason, Free quations SL_2, Proc.Cambr.Phyl.Soc. 101:3 (1987). 421-429.
31. A.W.Mason, On GL_2 of a local ring in which 2 is not a unit, Can.Math.Bull. 26:2 (1987), 209-219. Z 589.20032.
32. G.Maxwell, Infinite symplectic groups over rings, Comment.Math.Helv. 47 (1972), 254-259.

33. B.R. McDonald, Geometric algebra over local rings, New York-Basel, Marcel Dekker,1976.

34. B.R. McDonald, Linear algebra over commutative rings, New York-Basel, Marcel Dekker,1984.

35. B.R. McDonald and Kirkwood B., The symplectic group over a ring with one in its stable range, Pacific J.Math. 92:1 (1981), 11-125.

36. D.L. McQuillan, Class of normal congruence subgroups of the modular group, Am.J.Math. 87 (1965), 285-296,

37. J.L. Mennicke, Zur Theorie der Siegelschen Modulkgruppe, Math.Ann. 159 (1965), 115-129; MR 31 #5903.

38. B.Fine and M.Newman,The normal subgroup structure of the Picard group Trans. Amer. Math. Soc 302:2 (1987), 769-786.

39. M.Newman, Normal congruence subgroups of the modular group, Am.J.Math. 85 (1963), 419-427.

40. M.Newman, A complete description of normal subgroups of genus one of the modular group, Am.J.Math. 86 (1964), 17-24.

41. M.Newman, Free subgroups and normal subgroups of the modular group, Ill.Math.J. 8 (1964), 262-265.

42. O.T. O'Meara, Lectures on symplectic groups, Amer. Math. Soc. Math.Surveys, No 16, 1978.

43. C.R. Riehm, Structure of the symplectic group over a valuation ring, Am.J.Math. 88 (1966),106-128.

44. E.F. Robertson, Some properties of $Sp_{\Omega}(R)$, J.London Math.Soc. (2)4 (1971), 65-78.

45. J.-P. Serre, Le proble`me des groupes de congruences pour SL_2, Ann. Math. 92 (1970),489-527.MR42#7671=225-3822.

46. R. Steinberg, Lectures on Chevalley groups, Yale U. 1967.

47. G.Taddei, Invariance du sous-groupe symplectique elementarire dans le groupe symplectique sur un anneau, C.R. Acad.Sc. Paris 295 (1982), 1-47.

48. G.Taddei, Normalité des groupes élémentare dans les groupes de Chevalley sur un anneau,Cont.Math. Amer. Math. Soc 55, part II (1986), 693-710.

49. X.P. Tang and J.B. An, The structure of symplectic groups over semilocal rings, Acta Math.Sinica (N.S.) 1:1 (1985), 1-15.

50. S. Tazhetdinov, Subnormal structure of two-dimensional linear groups over local rings, Alg. Logic 22:6 (1983), 707-713.

51. S. Tazhetdinov, Subnormal structure of symplectic groups over local rings, Mat.Zametki 37:2 (1985), 289-298= Math.Notes, p.164-169.

52. S. Tazhetdinov, Subnormal structure of two- dimensional linear groups over rings that are close to fields Alg. Logic 24:4,414-425

53. S. Tazhetdinov, Normal structure of symplectic groups over rings of stable rank 1, Mat.Zametki 39:4 (1986),512-517.

54. L.N. Vaserstein, Stabilization for unitary and orthogonal groups over a ring with involution, Mat.Sb. 81:3 (1970), 328-351 = Math.USSR Sbornik 10, p.307-326.

55. L.N. Vaserstein, On the group SL_2 over Dedekind rings of arithmetic type, Mat.Sbornik 89:2 (1972), 312-322 = Math. USSR Sbornik 18, 321-332.MR 55#8253. Z359.20027. RZh 1973.2A369.

56. L.N. Vaserstein, Stabilization for classical groups over rings, Mat.Sbornik 93:2 (1974), 268-295 = Math. USSR Sbornik 22, 271-303.MR 49#2974. Z287.18015, 305.18007. RZh 1974.9A454.

57. L.N. Vaserstein, On normal subgroups of GL_n over a ring, in Springer Lecture Notes Math. 854 (1980), 456-465.

58. L.N. Vaserstein, On normal subgroups of Chevalley groups over commutative rings, Tohoku Math. J. 38 (1986), 219-230.

59. L.N. Vaserstein, Normal subgroups of the general linear groups over Banach algebras, J. Pure Appl. Algebra 41:1 (1986), 99-112. Z589.20030.

60. L.N. Vaserstein, Normal subgroups of orthogonal groups over commutative rings, Amer. J. Math. to appear.

61. L.N. Vaserstein and A.Mikhalev, On normal subgroups of orthogonal groups over rings with involution, Algebra i Log. 9:6 (1970), 629-632 = Algebra & Logic 9, p.375-377.

62. L.N. Vaserstein and A.A.Suslin, Serre's problem on projective modules over polynomial rings and algebraic K-theory, Izv.Akad.Nauk, ser.mat. 40:5 (1976), 993-1054 = Math.USSR Izv. 10:5, 937-1001. MR 56#5560. Z338.13015, 319.13009. RZh 1972.2A459.