On numerical invariants of locally

Cohen–Macaulay schemes in $\mathbb{P}^n$

by

Le Tuan Hoa, Rosa M. Miró–Roig

and Wolfgang Vogel*

*Max–Planck–Institut
für Mathematik
Gottfried–Claren–Straße 26
D–5300 Bonn 3

Federal Republic of Germany

Sektion Mathematik
Martin–Luther–Universität
Postfach
DDR–4010 Halle

German Democratic Republic

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Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

Mathematisches Institut
Wegelerstr. 10
5300 Bonn 1
Federal Republic of Germany

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§ 0. Introduction

Theories of modules with finite local cohomology have been developed during the last decade under the influence of the advance of the theory of Buchsbaum modules (see, e.g., [33], [34], [28]). One of the main purpose of this paper is to give a new perspective of such a theory for equidimensional and locally Cohen–Macaulay schemes in $\mathbb{P}^n$. The main objective of this work is bounding Castelnuovo's regularity and the genus of such schemes. For getting these bounds we describe two approaches: We reduce our problem to the case of points with the Uniform Position property by cutting with sufficiently general linear subspaces, and we apply our results of § 2. On the other hand, we study annihilators of local cohomology. Using this approach we will improve Castelnuovo bounds proved, e.g., in [30], [21], [1]. These sharper bounds yield some applications. For example, we solve the problem posed in [31] on p. 367 (see our corollary 3.5). Moreover, both approaches provide a motivation for a new structure which is obtained by introducing (k,ℓ)–Buchsbaum (modules) schemes (see also [14]). Therefore an aim of this paper is also to extend to (k,ℓ)–Buchsbaum schemes some of the results known for arithmetically Buchsbaum subschemes in $\mathbb{P}^n$. For example, this yields new upper bounds even for the genus of curves in $\mathbb{P}^3$ not contained in any surface of degree $< s$.

§ 1. Notations and preliminary results

Throughout this paper we work over an algebraically closed field of characteristic zero. We set $S = K[X_0,\ldots,X_n]$, $m = (X_0,\ldots,X_n) \subset S$ and $\mathbb{P}^n = \mathrm{Proj}(S)$. We restrict our attention to subschemes of $\mathbb{P}^n$ which are equidimensional and locally Cohen–Macaulay (this will be understood from now on). Given a subscheme $X \subset \mathbb{P}^n$, we denote by I_X (resp. $I(X)$) the ideal sheaf (resp. the defining ideal) of X . For a coherent sheaf F on X , $F(n)$ as usual will be $F \otimes \mathcal{O}_X(n)$ and we let $h^iF(n) = \dim_k H^i(X,F(n))$. Finally, given a sheaf E on $\mathbb{P}^n$ we denote by H^i_*E the graded S–module $\oplus_t H^i(\mathbb{P}^n,E(t))$ and we will say that E is m–regular if $H^iE(m-i) = 0$ for $i = 1,\ldots,n$ (see [27], definition 1 and theorem 2).

Definition 1.1 ([11], [23]). A subscheme $X \subset \mathbb{P}^n$ of codimension r is called k–Buchsbaum if and only if $m^kH^i_*I_X = 0$ for all $i = 1,\ldots,n-r$. We will say that X is strictly k–Buchsbaum if it is k–Buchsbaum but not $(k-1)$–Buchsbaum.

Remark 1.1.1: Let $X \subset \mathbb{P}^n$ be a 2–codimensional subscheme. X is arithmetically Buchsbaum in the sense of [28] if and only if X is 1–Buchsbaum (cf. [10]).

Remark 1.1.2: It is well knwon that for 2–codimensional $X \subset \mathbb{P}^n$ the property of being 1–Buchsbaum is inherited by general intersection with linear subspaces; and it follows from [11] that if $X \subset \mathbb{P}^n$ is a k–Buchsbaum subscheme and $H \subset \mathbb{P}^n$ is a general hyperplane, then $X \cap H$ is 2k–Buchsbaum. However, in general, the property of being k–Buchsbaum is not preserved when we cut with a general hyperplane.

Hence, it makes sense the following definition:

Definition 1.2. Let $X \subset \mathbb{P}^n$ be a subscheme of codimension r . X is (k,ℓ)–Buchsbaum,

$0 \leq \ell \leq n-r$, if the following conditions hold:

(a) X is a k–Buchsbaum subscheme of $\mathbb{P}^n$,

(b) $X \cap H$ is a k–Buchsbaum subscheme for all general $H \subset \mathbb{P}^n$ linear subspace of dimension $\geq n-\ell$.

Remarks 1.2.1. (a) Any k–Buchsbaum curve $X \subset \mathbb{P}^n$ is a $(k,1)$–Buchsbaum curve. However, there exist examples of k–Buchsbaum subschemes $X \subset \mathbb{P}^n$ of dimension > 1 which are not $(k,1)$–Buchsbaum subschemes.

(b) It follows from remark 1.1.2 that any 1–Buchsbaum subscheme $X \subset \mathbb{P}^n$ of codimension 2 is a $(1,n-2)$–Buchsbaum subscheme. Moreover, if $k=1$ then the $(1,\ell)$–Buchsbaum property does yield the theory of ℓ–Buchsbaum schemes as developed in [25].

Finally, we recall some known results on finite set of points needed later on.

Given a closed subscheme $X \subset \mathbb{P}^n$ we will denote by H_X the Hilbert function of X , i.e. $H_X(n) = \dim_k(S/I(X))_n$, where $(S(I(X))_n$ is the n–th graded component of the homogeneous coordinate ring of X . The first differences of the Hilbert function will be denoted by $\Delta H_X(n) = H_X(n) - H_X(n-1)$.

Definition 1.3 ([20]). A set $X \subset \mathbb{P}^n$ of points has the Uniform Position Property (UPP for short) if for any subset X' of X of d' points and any $n \geq 0$, it is $H_{X'}(n) = \min(d',H_X(n))$.

Lemma 1.4. Let $X \subset \mathbb{P}^n$ be a reduced, irreducible, nondegenerate subscheme of codimension r and degree d . Then, $X \cap L_r$ is a set of d points with the UPP, for all general linear subspace of dimension r .

Proof: It follows from Bertini's theorem and the uniform position lemma for a general hyperplane section of a curve ([20]).

For any $0 \leq a,\ b \in \mathbb{Z}$, we set $[a/b] := \max\{t \in \mathbb{Z} \mid a \geq tb\}$. Finally, given a set $X \subset \mathbb{P}^n$ of d points, we denote $\delta(X) := \min\{t \mid H_X(t) = d\}$.

Lemma 1.5 (See [26], Lemma 4). Let $X \subset \mathbb{P}^n$, $n \geq 2$, be a set of d points with the UPP spanning $\mathbb{P}^n$. Assume that X is not lying on a hypersurface of degree $< s$ and set

$$p := \min\{t \in \mathbb{N} \mid \binom{n+t}{t} > d-1-\left[\binom{n+s-1}{n}-1\right]\left[(d-1)/\left[\binom{n+s-1}{n}-1\right]\right]\}$$

Then, we have $0 \leq p \leq s-1$ and $\delta(X) \leq (s-1)\left[(d-1)/\left[\binom{n+s-1}{n}-1\right)\right] + p$.

Lemma 1.6 ([20]). Let $X \subset \mathbb{P}^n$ be a set of d points with the UPP. Then, for any integers $t_1,\dots,t_m$ one has $H_X(t_1+\dots+t_m) \geq \min(d, \sum_{i=1}^m H_X(t_i)-m+1)$.

§ 2. Numerical invariants

In a recent years there has been a great deal of work on the subject of arithmetically Buchsbaum (briefly a.B.) subschemes and, in particular, on 2–codimensional a.B. subschemes in $\mathbb{P}^n$ as a natural extension of arithmetically Cohen–Macaulay 2–codimensional subschemes in $\mathbb{P}^n$. A sample of the kind of problems considered may be found in [2–10], [12], [16–18], [22–24], [29], [32]. The purpose of the following sections is to extend and to improve these considerations by studying the following numerical invariants.

Given a reduced, irreducible, non–degenerate subscheme $X \subset \mathbb{P}^n$ of codimension r, we will use the following notations:

$d = d(X) =$ degree of X

$g = g(X) =$ geometric genus of X

$s = s(X) = \min\{t \mid H^0(\mathbb{P}^n, I_X(t)) \neq 0\}$

$e = e(X) = \max\{t \mid H^{n-r+1}(\mathbb{P}^n, I_X(t)) \neq 0\}$

$c_i = c_i(X) = \max\{t \mid H^i(\mathbb{P}^n, I_X(t)) \neq 0\}$ for $i = 1,\dots,n-r$

$b_i = b_i(X) = \min\{t \mid H^i(\mathbb{P}^n, I_X(t)) \neq 0\}$ for $i = 1,\dots,n-r$ ($c_i = b_i = -\infty$, if $H^i_* I_X = 0$)

and

$r = r(X) = \min\{t \mid H^i(\mathbb{P}^n, I_X(t-i)) = 0$ for $i = 1,\dots,n\}$ index of regularity (in the sense of Castelnuovo–Mumford) or Castelnuovo's regularity.

Note that $b_i \leq c_i$ and $r \geq c_i + 1 + i$ for $i = 1,\dots,n-r$.

Before proving the main results of this section, we will give the following technical but key lemma which we needed in the sequel.

Lemma 2.1. Let $X \subset \mathbb{P}^n$ be a k–Buchsbaum subscheme of codimension r and $k \geq 1$. Assume that $H^i(\mathbb{P}^n, I_X(t)) \neq 0$ for some $t \in \mathbb{Z}$ and $i \in \{1,\dots,r\}$. Then:

(a) For some $t \leq p \leq t + k-1$ there exists $0 \neq x \in H^i I_X(p)$ such that x is annihilated by all $L \in m$.

(b) For some $t-k+1 \leq p \leq t$ there exists $x \in H^iI_X(p)$ such that $x \notin mH^iI_X(p-1)$.

Proof: (a) We argue by contradiction. If (a) does not hold, then for any $0 \neq x \in H^iI_X(t)$ there exist linear forms $L_1,\dots,L_k$ such that $(L_1 \dots L_k)x \neq 0$, contradicting k–Buchsbaum.

(b) We also argue by contradiction. Let $0 \neq x \in H^iI_X(t)$, then there exist $\alpha_0^{t-1},\dots,\alpha_n^{t-1}$ in $H^iI_X(t-1)$ such that $x = \sum_{i=0}^{n} x_i\alpha_i^{t-1}$. But each $\alpha_j^{t-1} \in H^iI_X(t-1)$ can be written as a linear combination of elements in $H^iI_X(t-2)$ with coefficients $x_0,\dots,x_n$. Continuing in this way we get $0 \neq x = x_0^k\alpha_0^{t-k} + \dots + x_n^k\alpha_\gamma^{t-k}$ with $\alpha_j^{t-k} \in H^iI_X(t-k)$ ($\gamma = \binom{k+n}{n}$), again contradicting k–Buchsbaum.

Proposition 2.2. Let $X \subset \mathbb{P}^n$ be a k–Buchsbaum subscheme of codimension r and $k \geq 1$. Then: $s(X)-k \leq s(X \cap H) \leq s(X)$ and $b_i(X \cap H) \leq b_i(X)$ for $i = 1,\dots,n-r-1$ and for all general hyperplane $H \subset \mathbb{P}^n$.

Proof: Obviously $s(X \cap H) \leq s(X)$ and $b_i(X \cap H) \leq b_i(X)$ for $i = 1,\dots,n-r-1$. Let L be a linear form defining H, let $\sigma = s(X \cap H)$, and let F be a general form of degree k. We have the commutative diagram:

$$\begin{array}{ccccccc}
H^0I_X(\sigma) & \longrightarrow & H^0I_{X\cap H}(\sigma) & \xrightarrow[g]{} & H^1I_X(\sigma-1) & \xrightarrow[L]{} & H^1I_X(\sigma) \\
\downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \downarrow \\
H^0I_X(\sigma+k) & \xrightarrow[h]{} & H^0I_{X\cap H}(\sigma+k) & \xrightarrow[g]{} & H^1I_X(\sigma+k-1) & \xrightarrow[L]{} & H^1I_X(\sigma+k)
\end{array}$$

Let $0 \neq x \in H^0I_{X\cap H}(\sigma)$. By commutativity, $0 = Fg(x) = gF(x)$. Hence $0 \neq F(x) \in \mathrm{Ker}(g)$, i.e. there exists $0 \neq z \in H^0I_X(\sigma+k)$ such that $h(z) = F(x)$. So $s(X) \leq \sigma+k$.

As a consequence we have the following lower bound for the degree:

Corollary 2.3. Let $X \subset \mathbb{P}^n$ be an irreducible, reduced, non–degenerate subscheme of codimension r. Assume that X is (k,ℓ)–Buchsbaum, $0 \leq \ell \leq n-r$. Then, $d = d(X) \geq \binom{r+s'-1}{s'-1}$ where $s' := \max(2, s-(\ell-1)k-2^{n-\ell-r}k)$.

Proof: Consider a generic section $X' := X \cap L_r$ of X with a sufficiently general linear space L_r of dimension r where we set $L_r = H_1 \cap \ldots \cap H_{n-r}$. By Bertini's theorem, X' is not lying on a linear space of dimension $r-1$. On the other hand, it follows from remark 1.1.2, definition 1.2 and proposition 2.2 that X' is not lying on a hypersurface of degree $< s-(\ell-1)k-2^{n-\ell-r}k$.

Thus, X' is not lying on a hypersurface of degree $< s'$. Hence, $d = d(X) = d(X') \geq \binom{r+s'-1}{s'-1}$.

Proposition 2.4. Let $X \subset \mathbb{P}^n$ be a k–Buchsbaum subscheme of codimension r and $k \geq 1$. Then:

(a) $e(X) \geq c_{n-r}(X)-k-1$.

(b) $c_{n-r}(X)-k+1 \leq e(X \cap H)$ and $c_i(X)-k+1 \leq c_i(X \cap H)$ for $i = 1,\ldots,n-r-1$ and for all general hyperplane $H \subset \mathbb{P}^n$.

(c) $e(X) + 1 \leq e(X \cap H) \leq e(X)+k+1$ for all general hyperplane $H \subset \mathbb{P}^n$.

Proof: (a) Suppose that $H^{n-r+1}I_X(n) = 0$ for all $n \geq c_{n-r}-k-1$. Therefore, for all general hyperplane $H \subset \mathbb{P}^n$ we have the exact cohomology sequence:

$$H^{n-r}I_X(n) \longrightarrow H^{n-r}I_X(n+1) \longrightarrow H^{n-r}I_{X\cap H}(n+1) \longrightarrow 0$$

for all $n \geq c_{n-r}(X)-k-1$. By lemma 2.1, for some $t \geq c_{n-r}(X)-k+1$ there exists $x \in H^{n-r}I_X(t)$ such that $x \notin mH^{n-r}I_X(t-1)$. Now, take two linear forms L_1 and L_2 , we have the commutative diagram:

$$\begin{tikzcd}
H^{n-r}I_X(t-2) \arrow[r, "L_1"'] \arrow[d, "L_2"] & H^{n-r}I_X(t-1) \arrow[r, "h"'] \arrow[d, "L_2"] & H^{n-r}I_{X\cap H}(t-1) \arrow[r] \arrow[d, "g"] & 0 \\
H^{n-r}I_X(t-1) \arrow[r, "L_1"'] & H^{n-r}I_X(t) \arrow[r, "f"'] & H^{n-r}I_{X\cap H}(t) \arrow[r] \arrow[d] & 0 \\
 & & 0 &
\end{tikzcd}$$

Let $x \in H^{n-r}I_X(t)$ be as indicated. In particular, $f(x) \neq 0$. Since g and h are surjective, there exists $0 \neq y \in H^{n-r}I_X(t-1)$ such that $g(h(y))) = f(x)$. Moreover, $f(L_2y) = f(x)$ by commutativity. Hence, $(L_2y-x) \in \mathrm{Ker}(f) = \mathrm{Im}(L_1)$, i.e. there exists $z \in H^{n-r}I_X(t-1)$ such that $x = L_2y - L_1z$, contradicting the choice of x .

(b) For all general hyperplane $H \subset \mathbb{P}^n$ and for all $n > c_i(X \cap H)$ we have

$$H^iI_X(n-1) \longrightarrow H^iI_X(n) \longrightarrow H^iI_{X\cap H}(n) = 0 \ .$$

Taking k–times H , we get

$$H^iI_X(n-1) \longrightarrow H^iI_X(n+k-1) \longrightarrow 0 \ .$$

By hypothesis, X is k–Buchsbaum. Thus, $H^iI_X(n+k-1) = 0$ for $n > c_i(X \cap H)$. Equivalently, $c_i(X)-k+1 \leq c_i(X \cap H)$. Similarly, we prove $c_{n-r}(X)-k+1 \leq e(X\cap H)$.

(c) From the surjective map

$$H^{n-r}I_{X\cap H}(e(X)+1) \longrightarrow H^{n-r+1}I_X(e(X)) \longrightarrow 0 \ ,$$

we get $e(X) + 1 \le e(X \cap H)$. To proving that $e(X \cap H) \le e(X)+k+1$ we argue by contradiction. If not, $H^{n-r}I_{X\cap H}(e(X)+k+2) \neq 0$ and hence we have $H^{n-r}I_X(e(X)+k+2) \neq 0$ that is to say $c_{n-r} \geq e(X)+k+2$ which contradicts (a).

Remark. Let $X \subset \mathbb{P}^n$ be a k–Buchsbaum subscheme of codimension r with $k \geq 1$. Is then $e(X) \geq s(X) - (n-r+1)(k+1)$?

Proposition 2.4 has an interesting consequence which will be used in next section.

Corollary 2.5. Let $X \subset \mathbb{P}^n$ be a k–Buchsbaum subscheme of codimension r with $k \geq 1$. If $X \cap H$ is m–regular; then X is $(m+k-1)$–regular.

Proof: It follows from Proposition 2.4.

§ 3. Bounds for Castelnuovo's regularity

A. Castelnuovo bounds for (k,ℓ)–Buchsbaum schemes in $\mathbb{P}^n$.

Using the results of § 2 we will give a bound for Castelnuovo's regularity of the ideal sheaf of (k,ℓ)–Buchsbaum subschemes of $\mathbb{P}^n$. Examples show that the bounds of the following theorem are optimal.

Theorem 3.1. Let $k \geq 1$ be an integer. Let $X \subset \mathbb{P}^n$ be an irreducible, reduced, non–degenerate subscheme of codimension r . Assume that X is (k,ℓ)–Buchsbaum, $0 \leq \ell \leq n-r$. Then, we have:

$$r(X) \leq (s'-1)\left[(d(X)-1)/\left[\binom{r+s'-1}{s'-1}-1\right]\right]+p+1+\alpha(X) \ ,$$

where $s' := \max(2,s-(\ell-1)k-2^{n-\ell-r}k)$,
$p := \min\{t \in \mathbb{Z} \mid \binom{r+\ell}{t} > d(X)-1-\left[\binom{r+s'-1}{s'-1}-1\right]\left[(d(X)-1)/\left[\binom{r+s'-1}{s'-1}-1\right]\right]\}$, and
$\alpha(X) := k(\ell+1)-n+r+2k(2^{n-\ell-r-1}-1)$.

Proof: As in Corollary 2.3 we have that a generic section $X' := X \cap L_r$ of X with a sufficiently general linear space $L_r = H_1 \cap \ldots \cap H_{n-r}$ of dimension r is a set of points not lying on a hypersurface of degree $< s'$. By lemma 1.4, X' is a set of points with the UPP. Hence lemma 1.5, definition 1.2, remark 1.1.2 and corollary 2.5 yield our bound.

In the following examples we will see that there exist curves in $\mathbb{P}^3$, surfaces in $\mathbb{P}^4$, and three–folds in $\mathbb{P}^5$ which reach the bounds given in Theorem 3.1. Unfortunately, we do not known in general how good the bounds of Theorem 3.1 are.

Example 3.1.1. (a) Let $C \subset \mathbb{P}^3$ be a general smooth connected rational quartic. It is well known that C is (1,1)–Buchsbaum, $s(C) = s'(C) = 2$, $p = 1$, $\alpha = 0$ and $r = 3$. Hence, this example shows that the bounds of Theorem 3.1 are sharp.

(b) Let $C \subset \mathbb{P}^3$ be a general smooth connected rational quintic. By [23] Corollary 2.10 C is (2,1)–Buchsbaum. Moreover, an easy calculation shows that $s(C) = 3$, $s'(C) = 2$, $p = 0$, $\alpha = 1$ and $r = 4$. Hence, this example also shows that the bounds of Theorem 3.1 are sharp.

Example 3.1.2. Let $X \subset \mathbb{P}^4$ be the Veronese surface in $\mathbb{P}^4$. Using the following locally free resolution of its ideal sheaf I_X:

$$0 \longrightarrow 2\,\mathcal{O}_{P^4} \longrightarrow \Omega^1_{P^4}(2) \longrightarrow I_X(3) \longrightarrow 0$$

we can prove that X is (1,2)–Buchsbaum and $r(X) = 3$.

On the other hand, $s(X) = 3$, $s'(X) = 2$, $p = 1$, and $\alpha = 0$. Thus, the Veronese in $\mathbb{P}^4$ gives an example proving that the bounds of Theorem 3.1 are optimal.

Example 3.1.3. Let $X \subset \mathbb{P}^5$ be a smooth connected three–fold in $\mathbb{P}^5$ such that its ideal sheaf I_X has a locally free resolution of the following kind:

$$0 \longrightarrow 2\,\mathcal{O}_{P^5} \longrightarrow \Omega^1_{P^5}(2) \longrightarrow I_X(4) \longrightarrow 0$$

One can easily compute that X is (1,3)–Buchsbaum, $r(X) = 4$, $s(X) = 4$, $s'(X) = 2$, $p = 0$, and $\alpha = 0$. Thus, X gives us an example proving that the bounds of Theorem 3.1 are optimal.

Remark 3.1.4. Let $X \subset \mathbb{P}^n$ be an irreducible, reduced, non–degenerate subscheme. The

Castelnuovo's bounds of Theorem 3.1 are given in terms of $d(X)$ and $s(X)$. For codimension 2 subschemes, using the well known correspondance between codimension 2 subschemes in $\mathbb{P}^n$ and rank 2 reflexive sheaves and applying the vanishing results on cohomology given in [23] Proposition 1.4 and Proposition 1.5, we can find stronger Castelnuovo's bounds in terms of $d(X)$, $s(X)$, and $e(X)$. Seee [23], Theorem 3.1 and Theorem 3.3, for the case of curves in $\mathbb{P}^3$.

Finally, we want to state two special cases of theorem 3.1.

Corollary 3.1.5: Let $X \subset \mathbb{P}^n$ be an irreducible, reduced, non–degenerate subscheme of codimension r. Assume that X is $(k,n-r)$–Buchsbaum with $k \geq 1$. Then we have

$$r(X) \leq \{\text{degree } X-1/\text{codim } X\} + (n-r)(k-1) + 1 \ .$$

Applying this bound to curves in $\mathbb{P}^n$ we get an upper and lower bound for k. For this we introduce the following notation. If $I(X)$ is the defining ideal of a subscheme X of $\mathbb{P}^n$, we denote by $m(X)$ the highest degree of the homogeneous elements in a minimal basis of $I(X)$. Using corollary 3.1.5, remark 1.2.1, (a) and some well–known results we get

Corollary 3.1.6. Let $X \subset \mathbb{P}^n$ be an irreducible, reduced and non–degenerate curve. Let $k \geq 1$ be an integer such that X is strictly k–Buchsbaum. Then we obtain the following bounds for k:

$$m(X)-\{\text{degree } X-1/\text{codim } X\} \leq r(X)-\{\text{degree } X-1/\text{codim } X\}$$

$$\leq k \leq r(X)-2 \leq \text{degree } X - \text{codim } X-1 \ .$$

B. Castelnuovo bounds for equidimensional and locally Cohen–Macaulay schemes in $\mathbb{P}^n$.

The aim of this section is to improve bounds for Castelnuovo's regularity stated in [30], [21], [1]. Moreover, we will describe some applications of these new bounds. For example, we give an affirmative answer to the problem posed in [31] on p. 367. Before stating our main results, we give some general observations.

Let $A = \bigoplus_{n\geq 0} A_n$ be a graded k–algebra, i.e., $A_0 = K$ and A is generated as a K–algebra by A_1. Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded A–module. The i–th local cohomology module of M with support in the irrelevant ideal $P := \bigoplus_{n>0} A_n$, denoted by $H_P^i(M)$, is also a graded A–module. Let $[M]_i$ denote the i–th graded part of M, i.e., $[M]_i = M_i$. Let p be an integer then let $M(p)$ denote the graded A–module whose underlying module is the same as that of M and whose grading is given by $[M(p)]_i = [M]_{p+i}$ for all integers i. Moreover, we will study the following numerical invariants

$$e(M) := \sup\{n \in \mathbb{Z} \text{ such that } [M]_n \neq 0\} .$$

This invariant is connected with $e(X)$ and $c_i(X)$ for schemes X.

$$\lambda(H_P^i(M)) := \inf\{r \in \mathbb{N} \text{ such that } P^r \cdot H_P^i(M) = 0\} ,$$

$$\Lambda(M) := \sum_{i=0}^{d-1} \begin{bmatrix} d-1 \\ i \end{bmatrix} \lambda(H_P^i(M)) ,$$

where $d > 0$ is the Krull–dimension of M,

$$\operatorname{sgn} H_P^i(M) := \begin{Bmatrix} 1 \text{ if } H_P^i(M) \neq 0 \\ 0 \text{ if } H_P^i(M) = 0 \end{Bmatrix} .$$

Finally, we introduce some more notations. Let X be a subscheme of $\mathbb{P}^n$ with defining ideal $I(X) \subset S := K[x_0,\dots,x_n]$. Consider the K–algebra $A = S/I(X)$. Then Castelnuovo's regularity $\operatorname{reg}(A)$ of A is defined by $r(X)-1$ where $r(X)$ is the index of regularity of X (see section 2). If $\mathfrak{a}$ is any homogeneous ideal of S, we put

$$\mathfrak{a} :_S \langle P \rangle = \{s \in S / \exists t \in \mathbb{Z} \text{ such that } P^t \cdot s \subseteq \mathfrak{a}\} .$$

Moreover, we set for integers $a,b \geq 0$

$$\{a/b\} := \inf\{t \in \mathbb{N} \text{ with } a \leq t \cdot b\} .$$

We will prove the following two statements.

Theorem 3.2. Let $A = K[x_0,\dots,x_n]/\mathfrak{p}$ be a graded and locally Cohen–Macaulay K–algebra where $\mathfrak{p}$ is a homogeneous prime ideal of the polynomial ring $S := K[x_0,\dots,x_n]$ such that $\mathfrak{p}$ does not contain forms of degree 1 and Krull – dim $A = d \geq 2$. Then we have the following bound for Castelnuovo's regularity

$$\operatorname{reg} A \leq \{\operatorname{degree} \mathfrak{p}-1/\operatorname{codim} \mathfrak{p}\} + \Lambda(A) - \sum_{i=1}^{d-1} \operatorname{sgn} H_P^i(A) .$$

Theorem 3.3: Let K be any infinite field. Let $A = K[x_0,\dots,x_n]/\mathfrak{a}$ be an equidimensional and locally Cohen–Macaulay graded K–algebra of Krull–dimension $d \geq 1$. Then we have the following bound for Castelnuovo's regularity

$$\operatorname{reg} A \leq \operatorname{degree} \mathfrak{a} + \Lambda(A) - \sum_{i=0}^{d-1} \operatorname{sgn} H_P^i(A) .$$

Before embarking on the proofs of these statements we must prove a key lemma which is connected with our Proposition 2.4, (b).

Lemma 3.4. Let A be a graded K–algebra of Krull–dimension $d \geq 1$. Let $x \in [A]_1$ be any parameter for A. Assume that $\operatorname{length}_A([0 :_A x]) < \infty$. Then we have $e(H_P^i(A)) \leq e(H_P^i(A/xA)) + \lambda(H_P^i(A)) - 1$ for all $i \geq 0$.

Proof: There is not loss of generality in assuming that $H_P^i(A) \neq 0$ and $\lambda(H_P^i(A)) < \infty$. We set $p := e(H_P^i(A))$, $q := e(H_P^i(A/xA))$ and $\lambda := \lambda(H_P^i(A)) \geq 1$. We suppose $p \geq q+\lambda$ and look for a contradiction. Consider the following exact sequence (see, e.g., lemma 2.3 of [33]) for $i \geq 0$:

$$[H_P^i(A)]_{j-1} \xrightarrow{x} [H_P^i(A)]_j \longrightarrow [H_P^i(A/xA)]_j .$$

Hence we have for all $j \geq q+1$

$$[H_P^i(A)]_{j-1} \xrightarrow{x} [H_P^i(A)]_j \longrightarrow 0 .$$

Take an element $0 \neq u \in [H_P^i(A)]_p$. Since $\lambda \geq 1$ we have by assumption that $p \geq q+1$. Therefore there is an element, say $u_1 \in [H_P^i(A)]_{p-1}$ such that $u = u_1 \cdot x$. If $\lambda \geq 2$ then there is an element $u_2 \in [H_P^i(A)]_{p-2}$ such that $u_1 = u_2\, x$, that is, $u = u_2\, x^2$. Finally we get $u = u_\lambda\, x^\lambda$ where $u_\lambda \in [H_P^i(A)]_{p-\lambda}$. Since $P^\lambda H_P^i(A) = 0$ we obtain a contradiction $u = 0$, q.e.d.

Proof of theorem 3.2: We induct on d. First we have to prove the result in case $d = 2$. Then, by the General Position Lemma [19], (2.13)–(2.16), there is a generic form, say $x \in [A]_1$ such that $A' = A/x \cdot A : \langle P \rangle$ has the following Castelnuovo bound (see also lemma 1 of [30])

$$\operatorname{reg} A' \leq \{\text{degree } \mathfrak{p}-1/\operatorname{codim} \mathfrak{p}\} \ .$$

Moreover, we have

$$\operatorname{reg} A = \operatorname{reg} A/xA = \max\{\operatorname{reg} A', e(H^0_P(A/xA))\} \ .$$

We consider again the exact sequence

$$0 \longrightarrow H^0_P(A/xA) \longrightarrow H^1_P(A)(-1) \xrightarrow{\ x\ } H^1_P(A) \longrightarrow H^1_P(A/x) \longrightarrow \ldots$$

Hence we obtain

$$\begin{aligned} e(H^0_P(A/xA)) &\leq 1 + e(H^1_P(A)) \\ &\leq 1 + e(H^1_P(A/xA)) + \lambda(H^1_P(A))-1 \quad \text{by lemma 3.4} \\ &\leq \operatorname{reg} A' + \lambda(H^1_P(A))-1 \end{aligned}$$

Therefore we get

$$\begin{aligned} \operatorname{reg} A &\leq \max\{\operatorname{reg} A', \operatorname{reg} A' + \lambda(H^1_P(A))-1\} \\ &= \operatorname{reg} A' + \lambda(H^1_P(A))-1 \\ &\leq \{\text{degree } \mathfrak{p}-1/\operatorname{codim} \mathfrak{p}\}+\lambda(H^1_P(A))-\operatorname{sgn} H^1_P(A) \ . \end{aligned}$$

This completes the proof in case $d = 2$.

Let $d \geq 3$. Applying theorem 5.5 of [15] we choose a generic linear form ℓ of S such that $(\mathfrak{p} + \ell \cdot S) : \langle P \rangle$ is a prime ideal, say q . Moreover, we have degree q = degree $\mathfrak{p}$, codim $\mathfrak{p}$ = codim q , and q contains no linear forms (see lemma 3 of [30]). Therefore we get by induction

$$\operatorname{reg} S/q \leq \{\text{degree } \mathfrak{p}-1/\text{codim } \mathfrak{p}\} + \Lambda(S/q) - \sum_{i=1}^{d-2} \operatorname{sgn} H_P^i(S/q) \quad .$$

Hence we obtain by using lemma 3.4 and the above observations in case $d = 2$:

$$\begin{aligned}\operatorname{reg} A = \operatorname{reg} A/\ell \cdot A &= \max\{\operatorname{reg} S/q, e(H_P^0(A/xA))\} \\ &\leq \operatorname{reg} S/q + \lambda(H_P^1(A)) - \operatorname{sgn} H_P^1(A) \\ &\leq \{\text{degree}(\mathfrak{p})-1/\text{codim } \mathfrak{p}\} + \Lambda(S/q) + \lambda(H_P^1(A) \\ &\quad - (\sum_{i=1}^{d-2} \operatorname{sgn} H_P^i(S/q) + \operatorname{sgn} H_P^1(A)) \quad .\end{aligned}$$

Consider again the exact sequence

$$H_P^i(A) \longrightarrow H_P^i(S/q) \longrightarrow H_P^{i+1}(A)(-1) \xrightarrow{\ \ell\ } H_P^{i+1}$$

for $1 \leq i \leq d-2$. Then it follows that

$$\Lambda(S/q) + \lambda(H_P^1(A)) \leq \Lambda(A) \quad .$$

Moreover, if $H_P^i(S/q) = 0$ then we get $H_P^{i+1}(A) = 0$, that is, $\operatorname{sgn} H_P^i(S/q) \geq \operatorname{sgn} H_P^{i+1}(A)$ for $1 \leq i \leq d-2$. Therefore we obtain

$$\operatorname{reg} A \leq \{\text{degree } \mathfrak{p}-1/\text{codim } \mathfrak{p}\} + \Lambda(A) - \sum_{i=1}^{d-1} \operatorname{sgn} H_P^i(A) \, .$$

This concludes the proof of theorem 3.2, q.e.d.

Proof of theorem 3.3: We induct on d. The case $d = 1$ follows from theorem 2 of [30] since $\operatorname{sgn} H^0_P(A) \le 1$. Let $d \ge 2$. Take an element $x \in [A]_1$ such that $0 :_A x \subseteq 0 :_A \langle P \rangle$. Then we get from lemma 2.3 of [33]:

$$\operatorname{reg} A = \max\{\operatorname{reg} A/xA, e(H^0_P(A)\} \quad .$$

Applying lemma 3.4 we obtain

$$\begin{aligned} e(H^0_P(A)) &\le e(H^0_P(A/xA)) + \lambda(H^0_P(A)) - 1 \\ &\le \operatorname{reg} A/xA + \lambda(H^0_P(A)) - 1 \quad . \end{aligned}$$

Therefore we have:

$$\begin{aligned} \operatorname{reg} A &\le \operatorname{reg} A/xA + \lambda(H^0_P(A)) - \operatorname{sgn} H^0_P(A) \\ &\le \operatorname{degree}(\mathfrak{a} + x\cdot S) + \Lambda(A/xA) - \sum_{i=0}^{d-2} \operatorname{sgn} H^i_P(A/xA) \\ &\quad + \lambda(H^0_P(A)) - \operatorname{sgn} H^0_P(A) \text{ , by induction} \\ &\le \operatorname{degree} \mathfrak{a} + \Lambda(A) - \sum_{i=0}^{d-1} \operatorname{sgn} H^i_P(A) \end{aligned}$$

by Bézout's theorem and using the same arguments as in the proof of theorem 3.2. This completes the proof of theorem 3.3, q.e.d.

We will conclude this section by describing some applications and studying some examples. First, we will solve the problem posed in [31] on p. 367. Let V be a reduced, irreducible and nondegenerate subvariety of $\mathbb{P}^n$. Let $\mathfrak{p} \subset K[x_0,\dots,x_n] =: S$ be the defining prime ideal of V. We set $A = S/\mathfrak{p}$. Following [31] V is said to be a variety with minimal syzygies of maximal degree if Castelnuovo's regularity of A is given by

$$\operatorname{reg} A = \{\text{degree } \mathfrak{p}-1/\text{codim } \mathfrak{p}\} + \Lambda(A) \ .$$

The problem of [31] on 367 is to describe the locally Cohen–Macaulay varieties with minimal syzygies of maximal degree. Applying our theorem 3.2 we give the following affirmative answer to this problem.

Corollary 3.5: If V is a variety with minimal syzygies of maximal degree then V is arithmetically Cohen–Macaulay.

Proof: The above definition and theorem 3.2 yield the following relation:

$$\begin{aligned}\operatorname{reg} A &= \{\text{degree } \mathfrak{p}-1/\text{codim } \mathfrak{p}\} + \Lambda(A) \\ &\leq \{\text{degree } \mathfrak{p}-1/\text{codim } \mathfrak{p}\} + \Lambda(A) - \sum_{i=1}^{d-1} \operatorname{sgn} H^i_P(A) \ .\end{aligned}$$

Hence we get $H^i_P(A) = 0$ for all $i = 1,\dots,d-1$; that is, A is Cohen–Macaulay, q.e.d.

The next application gives new Castelnuovo bounds even for curves in $\mathbb{P}^n$. Let C be a reduced, irreducible and non–degenerate curve of $\mathbb{P}^n$. In sense of our definition 1.1 such curve C is h–Buchsbaum for some $h \geq 0$. Of course, C is arithmetically Cohen–Macaulay if and only if $h = 0$. The following corollary results immediately from theorem 3.2 or 3.1.5.

Corollary 3.6: Let C be a reduced, irreducible and nondegenerate curve of $\mathbb{P}^n$. We assume that C is strictly h–Buchsbaum with $h \geq 1$. Then we have for Castelnuovo's regularity

$$r(C) \leq \{\text{degree } C-1/\text{codim } C\} + h \ .$$

In general, this bound is sharper than degree $C - \text{codim } C+1$ (see example 3.8).

Example 3.7: Consider the smooth and rational curves Γ_N in $\mathbb{P}^3$ given parametrically by

$$\{s^N, s^{N-1}t, st^{N-1}, t^N\} \ .$$

It is well–known that Γ_N is strictly $(N-3)$–Buchsbaum (see, e.g., [13], [23], [14]). Hence we get for Castelnuovo's regularity $r(\Gamma_N) \leq \{N-1/2\} + N-3$.

Hence the case $N = 5$ shows that the bound stated in theorem 3.2 and corollary 3.6 is sharp. Of course, $r(\Gamma_N) \leq N-1$.

Example 3.8: Let $a,b \geq 1$ be integers and $d \geq a+b-1$. Consider the curves C in $\mathbb{P}^{a+b-1}$ given parametrically by

$$\{s^d, s^{d-1}t, \ldots, s^{d-a}t^a, s^b t^{d-b}, \ldots, t^d\} \ .$$

We set $h = \max(\{d-b-1/a\},\{d-a-1/b\})-1$. It is then not too difficult to show that C is strictly h–Buchsbaum. Moreover, we have degree $C = d$ and codim $C = a+b$. Hence we get for $h \geq 1$ the following bound for Castelnuovo's regularity by applying corollary

3.6. $$r(C) \leq \left\{\frac{d-1}{a+b}\right\} + \max\left(\{d-b-1/a\}, \{d-a-1/b\}\right) - 1\,.$$

Using the special structure of these monomial curves we can show that
$r(C) \leq \max(\{d-b-1/a\}, \{d-a-1/b\}) + 1$, which extends results of [35], corollary 2.

§ 4. Bounds for the genus of certain Buchsbaum schemes in $\mathbb{P}^n$

In this section, using the results of section 2 together with theorem 1 and 2 of [26], we will give a bound for the geometric genus of (k,ℓ)–Buchsbaum subschemes of $\mathbb{P}^n$. Moreover, we will state an upper bound for the genus of k–Buchsbaum curves in $\mathbb{P}^3$ not contained in any surface of degree $< s$.

Theorem 4.1: Let $X \subset \mathbb{P}^n$ be an irreducible, reduced, non–degenerate subscheme of codimension r. Assume that X is (k,ℓ)–Buchsbaum, $0 \leq \ell \leq n-r$. Then we obtain

$$g = g(X) \leq \begin{bmatrix} M \\ n-r \end{bmatrix} d(X) - \sum_{j=n-r}^{M} \begin{bmatrix} j-1 \\ n-r-1 \end{bmatrix} h'_X(j) ,$$

where $M := \min\{t \in \mathbb{N} \mid h'_X(t) = d(X)\} - 1$, $s' := \max(2, s-(\ell-1)k-2^{n-\ell-r}k)$, and $h'_X(i(s'-1)+t) = \min\{d(X), i\left[\begin{bmatrix} r+s'-1 \\ s'-1 \end{bmatrix} -1\right] + \begin{bmatrix} t+r \\ r \end{bmatrix}\}$ for all $0 \leq t \leq s'-1$.

Proof: It follows from [26] Theorem 1 and the fact (see Corollary 2.3) that a generic section $X' := X \cap L_r$ of X with a sufficiently general linear space L_r of dimension r is not lying on a hypersurface of degree $< s'$.

For the next theorem we introduce some notations. Let $X \subset \mathbb{P}^3$ be a reduced, irreducible, non–degenerate curve and $M(X) = \oplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^3, I_X(t))$ its Hartshorne–Rao module. For a linear form $L \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, let $\Phi_L : M(X) \longrightarrow M(X)(1)$ be the induced homomorphism; we set $K_L(X) := \ker \Phi_L$. Notice that $M(X)$, and hence $K_L(X)$, is a graded S–module of finite length.

Furthermore, from the exact sequence:

$$0 \longrightarrow I_X(-1) \longrightarrow I_X \longrightarrow I_{X\cap H} \longrightarrow 0$$

we get

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & I(X)(-1) \arrow[r] & I(X) \arrow[r] & I(X\cap H) \arrow[rr] \arrow[dr] & & M(X)(-1) \arrow[r] & M(X) \\
& & & & K_L(X)(-1) \arrow[ur] \arrow[dr] & & \\
& & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

In particular,

$$H_{X\cap H}(t+1) = H_X(t+1) - H_X(t) - \dim K_L(X)_t = \Delta H_X(t+1) - \dim K_L(X)_t \ .$$

Finally, given a reduced, irreducible, non–degenerate curve X in $\mathbb{P}^3$, we denote $b := \min\{t \mid H^1(\mathbb{P}^3, I_X(t)) \neq 0\}$ and $c := \max\{t \mid H^1(\mathbb{P}^3, I_X(t)) \neq 0\}$ ($c = b = -\infty$, if C is arithmetically Cohen–Macaulay).

Lemma 4.2. Let $X \subset \mathbb{P}^3$ be a reduced, irreducible, non–degenerate curve. It holds:

(a) X is arithmetically Cohen–Macaulay if and only if $M(X) = 0$ if and only if $K_L(X) = 0$ for every $L \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$.

(b) X is 1–Buchsbaum if and only if $M(X) = K_L(X)$ for every $L \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$.

(c) If X is k–Buchsbaum, $k \geq 2$, then $\dim K_L(X) \geq h^1 I_X(c) + \gamma(X)$, where $\gamma(X) := \#\{i \geq 1 \mid c-ik \geq b$ and there exists $j, c-ik \leq j < c-(i-1)k$, with $H^1 I_X(j) \neq 0\}$.

Proof: (a), (b) are well known. (c) follows from lemma 2.1 (a).

Now, we will improve Theorem 4.1 in the case of space curves.

Theorem 4.3. Let $X \subset \mathbb{P}^3$ be a reduced, irreducible, non–degenerate curve. Assume that X is k–Buchsbaum, $k \geq 2$. Then,

$$g = g(X) \leq Md(X) - \sum_{j=1}^{M} h'_X(j) - h^1 I_X(c) - \gamma(X)$$

where $M := \min\{t \in \mathbb{N} \mid h'_X(t) = d(X)\} - 1$, $s' := \max(2, s-k)$, $h'_X(i(s'-1)+t) = \min\{d(X), i\left[\binom{2+s'-1}{2}-1\right] + \binom{t+2}{2}\}$ for all $0 \leq t \leq s'-1$, and
$\gamma(X) := \#\{i \geq 1 \mid c-ik \geq b$ and there exists $j, c-ik \leq j < c-(i-1)k$, with $H^1 I_X(j) \neq 0\}$.

Proof: Consider a general plane section $\Gamma := X \cap H$ such that Γ is a set of d points with the UPP. Let $\ell >> 0$. Then, by Riemann–Roch, we have:

$$d\ell - g + 1 = h^0(X, \mathcal{O}_X(\ell)) = H_X(\ell) = 1 + \sum_{t=1}^{\ell} \Delta H_X(t) = 1 + \sum_{t=1}^{\ell} H_\Gamma(t) + \sum_{t=1}^{\ell} \dim K_L(X)_{t-1} .$$

Therefore,

$$g = d\ell - \sum_{t=1}^{\ell} H_\Gamma(t) - \dim K_L(X) \leq$$

$$\sum_{t=1}^{\ell} (d - H_\Gamma(t)) - \dim K_L(X) \leq$$

$$\sum_{t=1}^{\ell} (d - H_\Gamma(t)) - h^1 I_X(c) - \gamma(X) \text{, by lemma 4.2.}$$

On the other hand, since Γ is not lying on a curve of degree $< s$, we have that $h'_X(t)$ is a lower estimation of $H_\Gamma(t)$ (lemma 1.6). So we get:

$$g = g(X) \leq M\, d(X) - \sum_{j=1}^{M} h'_X(j) - h^1 I_X(c) - \gamma(X)$$

Remark 4.3.1: For the case $k = 1$,

$$g = g(X) \leq M\, d(X) - \sum_{j=1}^{M} h'_X(j) - \sum_{\ell \in \mathbb{Z}} h^1(\mathbb{P}^3, I_X(\ell))$$

(See [26], Theorem 6).

References

[1] **Achilles, R.., Schenzel, P.**: On bounds for Castelnuovo's index of regularity. J. Math. Kyoto Univ. 29, 91–104 (1989)

[2] **Amasaki, M.**: On the structure of arithmetically Buchsbuam curves in $\mathbb{P}^3_k$. Publ. Res. Inst. Math. Sci. 20, 793–837 (1984)

[3] **Ballico, E., Bolondi, G., Miró–Roig, R.M.**: Numerical invariants of rank 2 arithmetically Buchsbaum sheaves. J. Pure Appl. Algebra 58, 107–125 (1989)

[4] **Bolondi, G., Migliore, J.**: Classification of maximal rank curves in the liaison class L_n. Math. Ann. 277, 585–603 (1987)

[5] **Bolondi, G., Migliore, J.**: Buchsbaum liaison classes. J. Algebra 123, 426–456 (1989)

[6] **Bolondi, G., Migliore, J.**: The structure of an even liaison class. Trans. Am. Math. Soc. 316, 1–37 (1989)

[7] **Bolondi, G., Miró–Roig, R.M.**: Two codimensional Buchsbaum subschemes of $\mathbb{P}^n$ via their hyperplane sections. Commun. Algebra 17, 1989–2016 (1989)

[8] **Bresinsky, H., Schenzel, P., Vogel, W.**: On liaison, arithmetically Buchsbaum curves and monomial curves in $\mathbb{P}^3$. J. Algebra 86, 283–301 (1984)

[9] **Brodmann, M.**: Generalized Castelnuovo bounds. Preprint, University of Zürich, May 1990.

[10] **Chang, M.C.**: Characterization of arithmetically Buchsbaum subschemes of codimension 2 in $\mathbb{P}^n$. J. Diff. Geom. 31, 323–341 (1990)

[11] **Ellia, Ph.**: Ordres et cohomologies des fibrés de rang deux sur l'espace projectif. Preprint, Univ. Nice 170 (1987)

[12] **Ellia, Ph., Fiorentini, M.**: Courbes arithmetiquement Buchsbaum de l'espace projectif. Ann. Univ. Ferrara Sez. VII (N.S.) 33, 89–111 (1987)

[13] **Fiorentini, M., Lascu, A.T.**: Projective embeddings and linkage. Rend. Sem. Mat. Fis. Milano 57, 161–182 (1987)

[14] **Fiorentini, M., Vogel, W.**; On Buchsbaum rings, old and new. In preparation

[15] **Flenner, H.**: Die Sätze von Bertini für lokale Ringe. Math. Ann. 229, 97–11 (1977)

[16] **Gaeta, F.**: A geometrical characterization of smooth k–Buchsbaum curves. To appear in Springer Lecture Notes in Math.

[17] **Geramita, A., Migliore, J.**: On the ideal of an arithmetically Buchsbaum curve. J. Pure Appl. Algebra 54, 215–247 (1988)

[18] **Geramita, A., Migliore, J.**: Generators for the ideal of a Buchsbaum curve. J. Pure Appl. Algebra 58, 147–167 (1989)

[19] **Griffiths, P., Harris, J.**: Residues and zero–cycles on algebraic varieties. Ann. Math. 108, 461–505 (1978)

[20] **Harris, J.**: Curves in projective space. Les presses de l'Université, Montreal, 1982.

[21] **Maroscia, P., Stückrad, J., Vogel, W.**: Upper bounds for the degrees of the equations defining locally Cohen–Macaulay schemes. Math. Ann. 277, 53–65 (1987)

[22] **Migliore, J.**: Buchsbaum curves in $\mathbb{P}^3$. In: Algebraic geometry, Proc. Berlin Conference 1985, Tuebner–Texte zur Math. 92 (1986), 259–266

[23] **Migliore, J., Miró–Roig, R.M.**: On k–Buchsbaum curves. Commun. Algebra (to appear)

[24] **Miró–Roig, R.M.**: On the theorem of Castelnuovo for Buchsbaum cruves. Arch. Math. 52, 513–518 (1989)

[25] **Miyazaki, C.**: Graded Buchsbaum algebras and Segre products. Tokyo J. Math. 12, 1–20 (1989)

[26] **Nagel, U., Vogel, W.**: Bounds for Castelnuovo's regularity and the genus of projective varieties. In: Topics in Algebra. Banach Center Publ., Vol. 26, Part 2. Polish Scientific Publishers, 1990 (to appear)

[27] **Ooishi, A.**: Castelnuovo's regularity of graded rings and modules. Hiroshima Math. J. 12, 627–644 (1982)

[28] **Stückrad, J., Vogel, W.**: Buchsbaum rings and applications. Berlin, Heidelberg, New York: Springer 1986

[29] **Stückrad, J., Vogel, W.**: Castelnuovo bounds for certain subvarieties in $\mathbb{P}^n$. Math. Ann. 276, 341–352 (1987)

[30] **Stückrad, J., Vogel, W.**: Castelnuovo bounds for locally Cohen–Macaulay schemes. Math. Nachr. 136, 307–320 (1988)

[31] **Stückrad, J., Vogel, W.**: Castelnuovo's regularity and multiplicity. Math. Ann. 281, 355–368 (1988)

[32] **Stückrad, J., Vogel, W.**: Castelnuovo's regularity and cohomological properties of sets of points in $\mathbb{P}^n$. Math. Ann. 284, 487–501 (1989)

[33] Suzuki, N.: On quasi–Buchsbaum modules. In: Commutative Algebra and Combinatorics, 215–243. Advanced Studies in Pure Mathematics 11, 1987

[34] Trung, N.V.: Toward a theory of generalized Cohen–Macaulay modules. Nagoya Math. J. 102, 1–49 (1986)

[35] Trung, N.V.: Degree bound for the defining equations of projective monomial curves. Acta Math. Vietnam 9, 157–163 (1984)

Le Tuan Hoa
Institute of Mathematics
Viên toán hoc
Box 631
Bò' Hô, Hanoi
VIETNAM

Rosa M. Mirò–Roig
Departament d'Àlgebra
i Geometria
Universitat de Barcelona
Gran Via 585
08007 Barcelona
SPAIN

Wolfgang Vogel
Dep. of Mathematics
Martin–Luther–University
4010 Halle
GERMAN DEMOCRATIC REPUBLIC