On Severi varieties and Moduli spaces of curves in arbitrary characteristic

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Abstract

In the current paper we prove the irreducibility of Severi varieties on Hirzebruch surfaces in arbitrary characteristic. Our approach is of purely algebro-geometric nature, and it works in any characteristic. As a result, we obtain a deformation-theoretic proof of the irreducibility of moduli spaces $\mathcal{M}_g$ in positive characteristic, which does not involve reduction to characteristic zero.

1 Introduction

The study of Severi varieties is one of the classical problems in algebraic geometry. Given a smooth projective surface Σ, a line bundle $\mathcal{L} \in Pic(\Sigma)$, and an integer g, one defines Severi variety $V(\Sigma, \mathcal{L}, g) \subset |\mathcal{L}|$ to be the closure of the locus of irreducible nodal curves of genus g. Originally, these varieties were introduced by Severi (in the plane case) in order to prove the irreducibility of the moduli spaces of curves $\mathcal{M}_g$ in characteristic zero. In Anhang F of his famous book, *Vorlesungen über algebraische Geometrie*, F. Severi gave a false proof of the irreducibility of $V(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d), g)$. And it took

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more than sixty years, till, in 1986, Harris proved this result [6] (see also [9] for another proof due to Ran). However, the case of positive characteristic is not covered by these proofs, but it is a particular case of our result presented in this paper.

The study of various properties of Severi varieties, in particular their degrees continued, and in 1989 Ran [10] obtained the first recursive formula for the degree of varieties $V_{d,g} = V(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d), g)$. More formulas were obtained by Caporaso, Harris, Kontsevich, and Mikhalkin for other surfaces. We should mention that recent Mikhalkin's work on tropical geometry [8] provides us with a formula for the degree of $V_{\Delta,g} = V(Tor(\Delta), \mathcal{L}(\Delta), g)$, where $Tor(\Delta)$ is the projective toric surface corresponding to a convex polygon Δ, and $\mathcal{L}(\Delta)$ is the tautological line bundle on $Tor(\Delta)$. Mikhalkin's formula is not recursive, and is of combinatorial nature: namely the degree of $V_{\Delta,g}$ is computed in terms of lattice passes inside Δ.

Although the degrees of Severi varieties (in characteristic zero) have been computed for any projective toric surface, the irreducibility problem is still open in most of the cases in characteristic zero, and it is open in all the cases in positive characteristic. In the recent preprint by Shevchishin he claims the irreducibility of Severi varieties on Hirzebruch surfaces in characteristic zero [11]. Shevchishin's approach is based on the theory of Banach manifolds and involves many analytic and topological arguments. Hence in the case of positive characteristic his approach seems inapplicable.

The goal of the current paper is to give an *algebro-geometric characteristic independent* proof of the irreducibility of Severi varieties on Hirzebruch surfaces. As a result, we obtain the first direct proof of the irreducibility of moduli spaces $\mathcal{M}_{g,n}$ in positive characteristic[1]. All the previously known proofs of this fact are based on reduction of the problem to the case of characteristic zero (cf. [3, 5]). Moreover, originally these proofs involved *analytic* methods in characteristic zero, until in 1982 Fulton succeeded in replacing the analytic arguments by algebraic, using the compactification of Hurwitz schemes introduced by Mumford and Harris [7]. To conclude, we shall mention that our proof of the irreducibility of $\mathcal{M}_g$ is completely different from the previously known, and we think that the methods and techniques developed in this paper can be used in various problems in algebraic geometry.

The idea of the proof of the irreducibility of Severi varieties on Hirzebruch surfaces is as follows. Let Σ_n be a Hirzebruch surface, and let $\mathcal{L} = \mathcal{O}_{\Sigma_n}(dL_0 + kF)$ be a line bundle, where L_0 and F denote the classes generating $Pic(\Sigma_n)$,

[1]It is clear that there exist Severi varieties on Hirzebruch surfaces that dominate $\mathcal{M}_g$. For example $V_{g,d,0}$ on the blown up plane Σ_1 has this property for large d. The irreducibility of $\mathcal{M}_{g,n}$ follows immediately from the irreducibility of $\mathcal{M}_g$.

satisfying $L_0^2 = n$, $F^2 = 0$, and $L_0.F = 1$. We use the notation $V_{g,d,k}$ for the Severi variety $V(\Sigma_n, \mathcal{L}, g)$. It turns out that instead of $V_{g,d,k}$ one should consider (for technical reasons) the decorated Severi varieties $U_{d,k,\delta}$, which parameterize nodal curves with δ marked nodes in the linear system $|\mathcal{L}|$. It is important to say that the decorated Severi variety $U_{d,k,\delta}$ admits a dominant "forgetful" map to the Severi variety $V_{g,d,k}$, where $g = \frac{(d-1)(nd+2k-2)}{2} - \delta$. In these notations the first part of the proof is given by Proposition 3.2, which states the following: any irreducible component $U \subseteq U_{d,k,\delta}$ contains a pointed curve of a special type, which is denoted by $(\Gamma; p_1, ..., p_\delta)$, and, as a result, any irreducible component of $V_{g,d,k}$ contains a curve of type Γ (see Section 3 for detailed definitions and precise formulations). Now, the irreducibility follows from Proposition 3.3, claiming that if components $U, U' \subseteq U_{d,k,\delta}$ contain curves of special type $(\Gamma; p_1, ..., p_\delta)$ and $(\Gamma; p'_1, ..., p'_\delta)$ respectively, then they have the same image in the appropriate Severi variety.

The proof of Proposition 3.2 is done as follows: first, we degenerate the surface Σ_n into a fan (in Ran's terminology [9]) $X^0 \cup ... \cup X^r$, where $X^j \cong \Sigma_n$ for all j, and $X^j \cap X^{j+1}$ is L_0 on X^j and L_∞ on X^{j+1} ($L_\infty \subset \Sigma_n$ denotes the unique irreducible curve with negative self-intersection). Next we show that the degeneration of U contains the degeneration of the curve $(\Gamma; p_1, ..., p_\delta)$, which allows us to prove that U itself contains the required pointed curve, due to some transversallity and smoothness conditions.

At last, we say a few words about the proof of Proposition 3.3. In the case of characteristic zero it can be reduced to a combinatorial statement using very geometric monodromy-type arguments. However, in order to make the proof characteristic independent, we replaced these arguments by a description of explicit smooth coverings, which are shown to be connected, and hence irreducible.

To finish the introduction we should mention that there are examples of (non-rational) surfaces admitting reducible Severi varieties. Moreover, these Severi varieties can have components of different dimensions. So it is unclear how to characterize the surfaces admitting only irreducible Severi varieties. Nevertheless, we would like to state the following conjecture motivated by our result and Mikhalkin's work:

Conjecture 1.1 *The Severi varieties on any projective toric surface are irreducible in arbitrary characteristic.*

We hope that this conjecture can be approached using the rapidly developing methods of tropical geometry combined with the approach presented in the current paper.

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2 Preliminaries

2.1 Deformation theory

In this section we discuss several (basic) facts from the deformation theory of algebraic varieties and algebraic maps. Most of the statements, ideas, and proofs presented here can be found in different sources (see for example [1, 2, 6], and [12] for related topics). However, I was not able to find a characteristic free presentation of this material, and decided to write it down.

2.1.1 Deformations of maps.

Let X and Y be smooth algebraic varieties over an algebraically closed field $\mathbb{K}$, and let $f : X \longrightarrow Y$ be an algebraic map. In this section we discuss the deformation theory of the pair (X, f) - namely, we fix Y and vary X and f. Denote $\mathbb{D} = \mathrm{Spec}\mathbb{K}[\epsilon]/(\epsilon^2)$. We recall that a first-order deformation of (X, f) is a triple

- a flat family $\pi : \widetilde{X} \longrightarrow \mathbb{D}$,
- a map $F : \widetilde{X} \longrightarrow Y$,
- an isomorphism $\alpha : (\widetilde{X}_0, F_0) \longrightarrow (X, f)$, where $\widetilde{X}_0 = \widetilde{X}/(\epsilon)$, and $F_0 = F/(\epsilon)$.

Notation 2.1 *The set of first order deformations of the pair (X, f) modulo isomorphisms is denoted $Def^1(X, f)$.*

Proposition 2.2 *If $0 \longrightarrow T_X \longrightarrow f^*T_Y$ is exact, then*

$$Def^1(X, f) \cong H^0(X, \mathcal{N}_f),$$

where $\mathcal{N}_f$ denotes the normal sheaf to f, i.e. the cokernel of the map

$$df : T_X \longrightarrow f^*T_Y.$$

Proof: First, we choose affine coverings $Y = \cup_{i=1}^n Y_i$ and $X = \cup_{i=1}^n X_i$ such that $f(X_i) \subset Y_i$ for all i. Now, let $\xi \in Def^1(X, f)$ be a first-order deformation. We shall use the following well-known claim

Claim 2.3 *Let $Z = \mathrm{Spec}A$ be a smooth affine variety over an algebraically closed field $\mathbb{K}$, and let $Z_\epsilon = \mathrm{Spec}A_\epsilon$ be an infinitesimal extension of Z, i.e. a pair consisting of a flat morphism $Z_\epsilon \longrightarrow \mathbb{D}$ together with an isomorphism $Z_\epsilon/(\epsilon) \simeq Z$. Then Z_ϵ is isomorphic to the trivial extension, namely $A_\epsilon \simeq A \oplus \epsilon A$.*

Due to the claim we can fix trivializations

$$\mathcal{O}_{\widetilde{X}}(X_i) \simeq \mathcal{O}_{X_i}(X_i) \oplus \epsilon \mathcal{O}_{X_i}(X_i). \tag{2.1}$$

Then we obtain the automorphisms

$$\beta_{ij} : \mathcal{O}_{X_{ij}}(X_{ij}) \oplus \epsilon \mathcal{O}_{X_{ij}}(X_{ij}) \longrightarrow \mathcal{O}_{X_{ij}}(X_{ij}) \oplus \epsilon \mathcal{O}_{X_{ij}}(X_{ij}),$$

equal to identity modulo ϵ. Hence

$$\beta_{ij}(x + \epsilon y) = x + \epsilon(y + D_{ij}(x)),$$

where $D_{ij} : \mathcal{O}_{X_{ij}}(X_{ij}) \longrightarrow \mathcal{O}_{X_{ij}}(X_{ij})$ are derivations. The maps

$$F^* : \mathcal{O}_Y(Y_i) \longrightarrow \mathcal{O}_{X_i}(X_i) \oplus \epsilon \mathcal{O}_{X_i}(X_i)$$

are given by $F^*(x) = f^*(x) + \epsilon D_i(x)$, where

$$D_i : \mathcal{O}_Y(Y_i) \longrightarrow \mathcal{O}_{X_i}(X_i)$$

are derivations, and the following equality holds:

$$D_i - D_j = D_{ij} \circ f^*.$$

Hence the set $D^\xi = (D_1, ..., D_n)$ defines a global section of the sheaf $\mathcal{N}_f$. It is clear that D^ξ does not depend on the choice of trivializations in (2.1). Now, one can easily check that the constructed correspondence provides us with the bijection

$$Def^1(X, f) \ni \xi \longleftrightarrow D^\xi \in H^0(X, \mathcal{N}_f),$$

which in fact is an isomorphism of vector spaces. Moreover, this bijection does not depend on the choice of coverings $Y = \cup_{i=1}^n Y_i$ and $X = \cup_{i=1}^n X_i$. ■

2.1.2 Families of curves on algebraic surfaces.

Let Σ be a smooth projective algebraic surface, and let $\mathcal{L}$ be a line bundle on Σ. Consider an irreducible variety $V \subseteq |\mathcal{L}|$ whose generic element is a reduced curve. The goal of this section is to give a natural upper bound on the dimension of V. Let

$$\begin{tikzcd} \mathcal{C} \arrow[rr, hook] \arrow[dr] & & V \times \Sigma \arrow[dl] \\ & V & \end{tikzcd}$$

be the tautological family of curves over V, and let $\widetilde{\mathcal{C}} \longrightarrow \mathcal{C}$ be its normalization. Then for a generic $p \in V$ the fiber $\widetilde{\mathcal{C}}_p$ is the normalization of the fiber $\mathcal{C}_p$.

Let us choose a generic point $0 \in V$. Due to the Generic Flatness theorem and Proposition 2.2, we then have a natural map

$$\mu : T_0V \longrightarrow Def^1(C, f) \cong H^0(C, \mathcal{N}_f),$$

where $C = \widetilde{\mathcal{C}}_0$ and f is the composition of maps

$$C = \widetilde{\mathcal{C}}_0 \longrightarrow \mathcal{C} \longrightarrow \Sigma \times \{0\} = \Sigma.$$

Proposition 2.4 *(1) The map $T_0V \longrightarrow H^0(C, \mathcal{N}_f)$ is an embedding.*
(2) $Im(T_0V) \cap H^0(C, \mathcal{N}_f^{tor}) = 0$.

Proof: First, we choose a smooth projective irreducible curve $D \subset \Sigma$ intersecting $f(C)$ transversally, such that $h^0(\Sigma, \mathcal{L}(-D)) = 0$. Let

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d] & \Sigma \\ \mathbb{D} & \end{tikzcd}$$

be a first-order deformation of the pair (C, f). We define the new pair $X_D = X \times_\Sigma D$ and $F_D : X_D \longrightarrow D$.

Claim 2.5 *The natural map $X_D \longrightarrow \mathbb{D}$ is flat.*

So we constructed a map

$$\rho : Def^1(C,f) \longrightarrow Def^1(f(C)\cap D, f_D),$$

where $f_D = F_D/(\epsilon)$.

Claim 2.6 *$H^0(C,\mathcal{N}_f^{tor}) \subset Ker(\rho)$.*

To finish the proof it is enough to show that $\dim(\rho(\mu(T_0V))) = \dim(V)$. Consider the exact sequence

$$H^0(\Sigma,\mathcal{L}(-D)) \longrightarrow H^0(\Sigma,\mathcal{L}) \longrightarrow H^0(D,\mathcal{L}\otimes\mathcal{O}_D).$$

The first term is zero, hence the map $\alpha : V \hookrightarrow |\mathcal{L}\otimes\mathcal{O}_D|$ is an embedding. Now

$$\begin{aligned} T_{\alpha(0)}|\mathcal{L} &\otimes \mathcal{O}_D| = \\ & H^0\left(f(C)\cap D, \mathcal{O}_{f(C)\cap D}(f(C)\cap D)\right) = Def^1(f(C)\cap D, f_D) \end{aligned}$$

and $\rho\circ\mu = d\alpha(0)$, which implies $\dim(\rho(\mu(T_0V))) = \dim(V)$. ■
Proof of Claim 2.5: First of all we can assume that D,Σ, and C are affine. Thus X and X_D are affine as well. Next we shall use the following lemma:

Lemma 2.7 (Proposition 6.1 [4]) *Let R be a commutative ring with identity, and let M be an $R-$module. If I is an ideal of R, then $Tor^1_R(R/I,M) = 0$ if and only if the map $I\otimes_R M \longrightarrow M$ is an injection. The module M is flat if and only if this condition is satisfied for every finitely generated ideal $I\subset R$.*

In our case $R = \mathbb{K}[\epsilon]/(\epsilon^2)$ and the only non-trivial ideal we have is $I = \mathbb{K}\epsilon$. By the lemma it is enough to show that the map

$$\mathcal{O}_{X_D}\otimes_{\mathcal{O}_{\mathbb{D}}}\mathbb{K}\epsilon \xrightarrow{\gamma} \mathcal{O}_{X_D}$$

is an embedding. Since $\mathcal{O}_X$ is flat over $\mathcal{O}_{\mathbb{D}}$, we have an exact sequence

$$0\longrightarrow \mathcal{O}_X\otimes_{\mathcal{O}_{\mathbb{D}}}\mathbb{K}\epsilon \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0.$$

Tensoring this sequence with $\mathcal{O}_D$ over $\mathcal{O}_\Sigma$ we obtain

$$0\longrightarrow Tor^1_{\mathcal{O}_\Sigma}(\mathcal{O}_D,\mathcal{O}_C) \longrightarrow \mathcal{O}_{X_D}\otimes_{\mathcal{O}_{\mathbb{D}}}\mathbb{K}\epsilon \xrightarrow{\gamma} \mathcal{O}_{X_D}.$$

It remains to prove that $Tor^1_{\mathcal{O}_\Sigma}(\mathcal{O}_D,\mathcal{O}_C) = 0$. Let

$$0\longrightarrow \mathcal{O}_\Sigma \xrightarrow{\cdot j} \mathcal{O}_\Sigma \longrightarrow \mathcal{O}_D \longrightarrow 0$$

be a free resolution of $\mathcal{O}_D$, where $j \in \mathcal{O}_\Sigma$ is a function defining D. Thus

$$Tor^1_{\mathcal{O}_\Sigma}(\mathcal{O}_D, \mathcal{O}_C) = Ker\left(\mathcal{O}_C \xrightarrow{f^*(j)} \mathcal{O}_C\right) = 0,$$

since $f(C)$ intersects D transversally. ■

Proof of Claim 2.6: To prove this claim we have to construct the map ρ explicitly. Consider the fibered product diagram

$$\begin{array}{ccc} f(C)\cap D & \xrightarrow{f_D} & D \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_D} \\ C & \xrightarrow[f]{} & \Sigma \end{array}$$

The intersection $f(C) \cap D$ is transversal and its points are smooth on both $f(C)$ and D. Hence we have the following exact sequences on $f(C) \cap D$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i^*T_C & \xrightarrow{i^*(df)} & i^*f^*T_\Sigma & \longrightarrow & i^*\mathcal{N}_f & \longrightarrow & 0 \\ & & & & \| & & & & \\ 0 & \longrightarrow & f_D^*T_D & \xrightarrow{df_D} & f_D^*i_D^*T_\Sigma & \longrightarrow & f_D^*\mathcal{N}_{i_D} & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & \mathcal{N}_{f_D} & & & & & & \end{array}$$

Moreover,

$$i^*(df)(i^*T_C) \oplus df_D(f_D^*T_D) = i^*f^*T_\Sigma.$$

Hence the map $\gamma : \mathcal{N}_{f_D} \longrightarrow i^*\mathcal{N}_f$ is an isomorphism, and ρ is given by the composition

$$H^0(C, \mathcal{N}_f) \xrightarrow{i^*} H^0(f(C) \cap D, i^*\mathcal{N}_f) \xrightarrow{\gamma^{-1}} H^0(f(C) \cap D, \mathcal{N}_{f_D}).$$

To finish the proof we note that $df(p) \neq 0$ for any $p \in i(f(C) \cap D)$. Thus $i^*\mathcal{N}_f^{tor} = 0$, which implies $H^0(C, \mathcal{N}_f^{tor}) \subset Ker(\rho)$. ■

Theorem 2.8 *Let Σ be a smooth projective algebraic surface, $C' \subset \Sigma$ be a smooth curve, and let $\mathcal{L}$ be a line bundle on Σ. Consider an irreducible variety $V \subseteq |\mathcal{L}|$ whose generic element C_0 is a reduced curve. Assume that $C_0^i.K_\Sigma < -1$ for any irreducible component $C_0^i \subseteq C_0$. Then*

$$\dim(V) \leq -C_0.K_\Sigma + g - 1. \tag{2.2}$$

Furthermore, if the equality holds, and

$$C_0^i.K_\Sigma < -3 \tag{2.3}$$

for any singular component C_0^i of C_0, then C_0 has only nodes as its singularities, and C_0 intersects C' transversally.

Proof: Let C be the normalization of C_0 and let $f : C \longrightarrow \Sigma$ be the natural map. Then $\dim(h^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor})) \leq -C_0.K_\Sigma + g - 1$ implies (2.2), by Proposition 2.4. So it is enough to show the analogous inequality for every irreducible component of C. Thus we can assume that C is irreducible.

Choose an invertible sheaf $\mathcal{F}$ on C such that the sequence

$$0 \longrightarrow \mathcal{N}_f/\mathcal{N}_f^{tor} \longrightarrow \mathcal{F} \longrightarrow \mathcal{N}_f^{tor} \longrightarrow 0$$

is exact (the existence of such $\mathcal{F}$ is completely obvious). Then

$$c_1(\mathcal{F}) = c_1(\mathcal{N}_f) = c_1(f^*T_\Sigma) - c_1(T_C) = 2g - 2 + c_1(f^*T_\Sigma) > 2g - 1. \tag{2.4}$$

Hence

$$h^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor}) \leq h^0(C, \mathcal{F}) = c_1(\mathcal{F}) + 1 - g = -C_0.K_\Sigma + g - 1$$

by the Riemann-Roch theorem, and the equality holds if and only if $\mathcal{N}_f^{tor} = 0$.

For the second part we note that if the dimension of V equals $-C_0.K_\Sigma + g - 1$, then $\mathcal{N}_f^{tor} = 0$, and hence $df \neq 0$ everywhere. So it remains to prove that C_0 has no triple points, and that all its double points have two different tangent directions. If $p \in C_0$ is a triple point and $q_1, q_2, q_3 \in C$ are three points mapped to p, then any $\xi \in T_0V$ vanishing at q_1 and q_2, must vanish at q_3 as well. However, due to the Riemann-Roch theorem, inequality (2.4), and condition (2.3), there exists $\eta \in H^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor})$ such that $\eta(q_1) = \eta(q_2) = 0 \neq \eta(q_3)$. Thus

$$\dim T_0V < h^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor}) = -C_0.K_\Sigma + g - 1,$$

which contradicts the equality in (2.2). If $p \in C_0$ is a double point with a unique tangent direction and $q_1, q_2 \in C$ are the two pre-images of p, then any $\xi \in T_0V$ vanishing at q_1, must also vanish at q_2. However, applying Riemann-Roch theorem, inequality (2.4), and condition (2.3), we can find $\eta \in H^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor})$, such that $\eta(q_1) = 0 \neq \eta(q_2)$. Hence

$$\dim T_0V < h^0(C, \mathcal{N}_f/\mathcal{N}_f^{tor}) = -C_0.K_\Sigma + g - 1,$$

which is a contradiction.

It remains to prove that if $\dim(V) = -C_0.K_\Sigma + g - 1$, then C_0 intersects C' transversally. Assume that $\dim(V) = -C_0.K_\Sigma + g - 1$. Then the linear system V has no fixed components; hence C_0 does not contain C'. If the intersection $C_0 \cap C'$ is not transversal at some point p, then either p has at least two pre-images $q_1, q_2 \in C$ or any $\xi \in T_0V$ vanishes at q, where $q \in C$ is the unique pre-image of p. In the first case any $\xi \in T_0V$ vanishing at q_1 must also vanish at q_2. So both cases contradict the Riemann-Roch theorem, due to (2.4) and (2.3). ■

2.2 Severi varieties on Hirzebruch surfaces

Let $\Sigma_n = \mathbb{P}roj(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ be the Hirzebruch surface and let $\pi : \Sigma_n \longrightarrow \mathbb{P}^1$ be the natural projection $\pi : \Sigma_n \longrightarrow \mathbb{P}^1$. Consider two sections $(1,0), (0,\sigma) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. They define the maps $(\mathbb{P}^1 \backslash Z(\sigma)) \longrightarrow \Sigma_n$. We denote the closures of the images of these maps by L_0 and L_∞, respectively. It is clear that L_∞ is independent of the choice of σ. The following facts will be useful

- The Picard group $Pic(\Sigma_n)$ is a free abelian group generated by the classes F and L_∞, where F denotes the fiber of the projection π. It is important to mention that $L_0 \equiv nF + L_\infty$.
- The intersection form on $NS(\Sigma_n) = Pic(\Sigma_n)$ is given by $F^2 = 0$, $L_\infty^2 = -n$, and $F.L_\infty = 1$.
- Any effective divisor $M \in Div(\Sigma_n)$ is linearly equivalent to a linear combination of F and L_∞ with non-negative coefficients. Moreover, if M does not contain L_∞, then it is linearly equivalent to a combination of F and L_0 with non-negative coefficients.
- The canonical class is $K_{\Sigma_n} \equiv -(2L_\infty + (2+n)F)$.
- Any smooth curve $C \equiv dL_0 + kF$ has genus $g(C) = \frac{(d-1)(dn+2k-2)}{2}$.

Now let us define the Severi varieties on Σ_n.

Definition 2.9 *(1) Let*

$$\breve{\Sigma}_n^\delta = \left\{(p_1, ..., p_\delta) \in \Sigma_n^\delta \;\mid\; p_i \neq p_j \, \textit{for any } i \neq j\right\}$$

be the configuration space of δ points in Σ_n. For non-negative integers d, k, δ, we define the decorated Severi variety

$$U_{d,k,\delta} \subset |\mathcal{O}_{\Sigma_n}(dL_0 + kF)| \times \breve{\Sigma}_n^\delta$$

to be

$$\{(C; p_1, ..., p_\delta) \mid C \textit{ is reduced, } L_\infty \nsubseteq C, \textit{ and} \, p_1, ..., p_\delta \in C \textit{ are nodes}\}.$$

(2) Let g, d, k be non-negative integers. We define the Severi variety $V_{g,d,k} \subseteq |\mathcal{O}_{\Sigma_n}(dL_0 + kF)|$ to be the closure of the locus of reduced nodal curves of genus g which do not contain L_∞.

Next, we establish the basic properties of (decorated) Severi varieties:

Claim 2.10 *(1) Let d, k, δ be non-negative integers. Then either $U_{d,k,\delta} = \emptyset$ or $U_{d,k,\delta}$ is a smooth equidimensional variety of dimension*

$$\frac{(d+1)(nd+2k+2)}{2} - 1 - \delta = \dim |\mathcal{O}_{\Sigma_n}(dL_0 + kF)| - \delta.$$

(2) Let $\psi : U_{d,k,\delta} \longrightarrow |\mathcal{O}_{\Sigma_n}(dL_0 + kF)|$ be the projection to the first factor. Then for $g = \frac{(d-1)(nd+2k-2)}{2} - \delta$

$$V_{g,d,k} = \begin{cases} \overline{\psi\left(U_{d,k,\delta}\right)} & \textit{if} \quad \delta \geq 0; \\ \emptyset & \textit{otherwise.} \end{cases}$$

Proof: (1) Let us choose arbitrary $(C^0; p_1^0, ..., p_\delta^0) \in U_{d,k,\delta}$. We can find an open subset $U \subset \Sigma_n$ isomorphic to $\mathbb{A}^2$ and containing all the points $p_1^0, ..., p_\delta^0$. Fix a trivialization of $\mathcal{O}_{\Sigma_n}(dL_0 + kF)(U) \simeq \mathbb{K}[x, y]$. Then in a neighborhood of $(C^0; p_1^0, ..., p_\delta^0)$ $U_{d,k,\delta}$ is given by the (homogeneous in the coefficients) equations $f(p_l) = f'_x(p_l) = f'_y(p_l) = 0$, $1 \leq l \leq \delta$, where $f \in H^0(\Sigma_n, \mathcal{O}_{\Sigma_n}(dL_0 + kF))$. We denote $(x_l^0, y_l^0) = p_l^0$ and define a_{ij}^l to be the coefficients of $f(x + x_l^0, y + y_l^0)$, $f = \sum a_{ij} x^i y^j \in \mathbb{K}[x, y]$. Let $f_0(x, y) = \sum \beta_{ij} x^i y^j$ be an equation defining C^0. Then

$$d(f(p_l))_{|_{(f_0, p_1^0, ..., p_\delta^0)}} = da_{00}^l,$$

$$d(f'_x(p_l))_{|_{(f_0, p_1^0, ..., p_\delta^0)}} = da_{10}^l + 2\beta_{20}^l dx_l + \beta_{11}^l dy_l,$$

and

$$d(f'_y(p_l))_{|(f_0,p_1^0,...,p_\delta^0)} = da_{01}^l + \beta_{11}^l dx_l + 2\beta_{02}^l dy_l.$$

The points $p_1^0, ..., p_\delta^0$ are nodes of C^0, hence the matrices

$$\begin{pmatrix} 2\beta_{20}^l & \beta_{11}^l \\ \beta_{11}^l & 2\beta_{02}^l \end{pmatrix}$$

are invertible (even in characteristic 2). So, it remains to prove that $da_{00}^1 = ... = da_{00}^\delta = 0$ defines a subspace of codimension δ in the tangent space to $|\mathcal{O}_{\Sigma_n}(dL_0 + kF)|$ at C^0. In other words we have to prove that

$$h^0(C^0, \mathcal{J}(dL_0 + kF)) = h^0(C^0, \mathcal{O}_{C^0}(dL_0 + kF)) - \delta, \tag{2.5}$$

where $\mathcal{J}$ denotes the ideal of the zero-dimensional reduced subscheme

$$X = \cup_{i=1}^\delta p_i \subset C^0.$$

$H^1(C^0, \mathcal{J}(dL_0 + kF)) = 0$ by Claim 2.12, which implies (2.5).

(2) The inclusion

$$V_{g,d,k} \subseteq \begin{cases} \overline{\psi\,(U_{d,k,\delta})} & \text{if} \quad \delta \geq 0; \\ \emptyset & \text{otherwise.} \end{cases}$$

is obvious. Let us prove the opposite direction. If $g \geq \frac{(d-1)(nd+2k-2)}{2}$, then there is nothing to prove. Otherwise let $U \subseteq U_{d,k,\delta}$ be an irreducible component, and let $(C; p_1, ..., p_\delta) \in U$ be a generic point. Then by part (1) and Theorem 2.8

$$-C.K_{\Sigma_n} + g(C) - 1 \geq \dim \psi\,(U_{d,k,\delta}) = \dim |\mathcal{O}_{\Sigma_n}(dL_0 + kF)| - \delta. \tag{2.6}$$

Hence $g(C) \geq \frac{(d-1)(nd+2k-2)}{2} - \delta = g_{ar}(C) - \delta$, which is possible only if the equality holds. Thus the equality holds in (2.6) as well, and by Theorem 2.8 this implies the nodality of the curve C. So

$$\psi(C; p_1, ..., p_\delta) \in V_{g,d,k}$$

and we are done. ■

It is easy to see that if $(C; p_1, ..., p_\delta) \in U_{d,k,\delta}$ and C has exactly δ nodes, then the map $\psi : U_{d,k,\delta} \longrightarrow V_{g,d,k}$ for $g = \frac{(d-1)(nd+2k-2)}{2} - \delta$, is étale in a neighborhood of $(C; p_1, ..., p_\delta)$.

Corollary 2.11 *If $V_{g,d,k} \neq \emptyset$ then it has pure dimension $nd+2k+2d+g-1$, and for any $C \in V_{g,d,k}$ having exactly $\delta = \frac{(d-1)(nd+2k-2)}{2} - g$ nodes, $V_{g,d,k}$ is smooth at C, and $T_C(V_{g,d,k}) \simeq H^0(C, I(C))$, where $I \subset \mathcal{O}_C$ is the conductor ideal.*

Claim 2.12 *Let $C \in |\mathcal{O}_{\Sigma_n}(dL_0 + kF)|$ be a reduced curve. Assume that the restriction of the canonical class to any irreducible component of C is negative*[2]. *Let $q_1, ..., q_r \in C$ be some of the nodes of C, and let $\mathcal{J} \subset \mathcal{O}_C$ be the ideal of $\cup_{i=1}^r q_i \subset C$. Then $H^1(C, \mathcal{J}(C)) = 0$.*

Proof: We prove this claim by induction on the number of irreducible components of C. Assume that C is irreducible. Let $p : \widetilde{C} \longrightarrow C$ be its normalization. Then

$$H^1(C, \mathcal{J}(dL_0 + kF)) = H^1(\widetilde{C}, p^*\mathcal{J}(dL_0 + kF)) = 0$$

by the Riemann-Roch theorem, since

$$\deg(\mathcal{J}(dL_0 + kF)) = C^2 - 2r = C.(C + K_{\Sigma_n}) - 2r - C.K_{\Sigma_n} > 2g(\widetilde{C}) - 2.$$

We assume now that C has $l > 1$ irreducible components and that the statement is correct for any curve with the number of components less than l. Let C_1 be an irreducible component. Then $C = C_1 \cup C_2$ and without loss of generality assume $q_1, ..., q_{r_1} \in C_1 \backslash C_2$, $q_{r_1+1}, ..., q_{r_2} \in C_1 \cap C_2$, $q_{r_2+1}, ..., q_r \in C_2 \backslash C_1$. Let $\mathcal{J}_2 \subset \mathcal{O}_{C_2}$ be the ideal of $\cup_{i=r_2+1}^r q_i$, and let $\mathcal{J}_1 \subset \mathcal{O}_{C_1}$ be the ideal of $\cup_{i=1}^{r_2} q_i$. Then the following sequence

$$0 \longrightarrow \mathcal{J}_2(C_2) \xrightarrow{\times C_1} \mathcal{J}(C) \longrightarrow \mathcal{J}_1(C) \longrightarrow 0$$

is exact. Hence, by the induction hypothesis it is enough to prove that $H^1(C_1, \mathcal{J}_1(C)) = 0$. Let $p : \widetilde{C_1} \longrightarrow C_1$ be the normalization of C_1. Then

$$H^1(C_1, \mathcal{J}_1(C)) = H^1(\widetilde{C_1}, p^*\mathcal{J}_1(C)) = 0$$

by the Riemann-Roch theorem, since

$$\begin{aligned} \deg(p^*\mathcal{J}_1(C)) &= C_1^2 - 2r_1 + C_1.C_2 - (r_2 - r_1) \\ &\geq C_1.(C_1 + K_{\Sigma_n}) - 2r_1 - C_1.K_{\Sigma_n} > 2g(\widetilde{C_1}) - 2. \ \blacksquare \end{aligned}$$

3 The Result

Theorem 3.1 *Let g, k, d be non-negative integers. If $V_{g,d,k} \neq \emptyset$, then it has a unique component, whose generic point corresponds to an irreducible curve.*

Proof: Let L_i, $1 \leq i \leq d$, be generic curves in the linear system $|\mathcal{O}_{\Sigma_n}(L_0)|$, and let F_i, $1 \leq i \leq k$, be generic curves in the linear system $|\mathcal{O}_{\Sigma_n}(F)|$. Define

$$\Gamma = L_1 \cup ... \cup L_d \cup F_1 \cup ... \cup F_k \in |\mathcal{O}_{\Sigma_n}(dL_0 + kF)|.$$

[2]If $n < 2$ then this condition is empty, otherwise it is equivalent to C not containing L_∞ as its component.

Proposition 3.2 *Consider an arbitrary component V of the Severi variety $V_{g,d,k}$. Then $\Gamma \in V$.*

Proposition 3.3 *There exists a unique component $V \subset V_{g,d,k}$ containing Γ, whose generic point corresponds to an irreducible curve.*

The theorem now follows. ∎

3.1 Proof of Proposition 3.2

To prove the proposition we will need the following lemma

Lemma 3.4 *Let $(\Sigma, \mathcal{L})$ be a smooth rational surface equipped with a line bundle, and let $L \subset \Sigma$ be a smooth curve. Let $p_1, ..., p_r \in L$ be arbitrary points, $k_1, ..., k_r$ be non-negative integers, and let*

$$R \subset \left\{ C \in |\mathcal{L}| \,:\, L \cap C = \sum_{i=1}^{r} k_i p_i \right\}$$

be a non-empty subvariety. Choose a generic curve $C \in R$. If C is reduced and $C_i.(K_\Sigma + L) < -1$ for any irreducible component $C_i \subset C$, then

$$\dim(R) \leq -C.K_\Sigma + g(C) - 1 - L.C.$$

Moreover, if the equality holds, and for any singular component C_i of C we have $C_i.(K_\Sigma + L) < -3$, then C has only nodes as its singularities outside of L, and for any smooth irreducible curve C' that does not contain any of the p_i, a generic curve $D \in R$ intersects C' transversally.

Proof: The proof is by induction on $L.C$. If $L.C = 0$, then the lemma follows from Theorem 2.8, since $L.C_i = 0$ for any irreducible component $C_i \subseteq C$.

Assume now that $k_i > 0$ for some i. Without loss of generality $k_1 > 0$. Consider the blow up $\widetilde{\Sigma} = Bl_{p_1}(\Sigma)$ with its natural projection $\pi : \widetilde{\Sigma} \longrightarrow \Sigma$. We denote the proper transform of L by $\widetilde{L}$. Then $\mathcal{O}_{\widetilde{\Sigma}}(\widetilde{L}) \simeq \pi^* \mathcal{O}_\Sigma(L) \otimes \mathcal{O}_{\widetilde{\Sigma}}(-E)$, where E denotes the exceptional divisor $\pi^{-1}(p_1)$. Let $\widetilde{C} \subset \widetilde{\Sigma}$ be the proper transform of C. Then $\mathcal{O}_{\widetilde{\Sigma}}(\widetilde{C}) \simeq \pi^* \mathcal{O}_\Sigma(C) \otimes \mathcal{O}_{\widetilde{\Sigma}}(-m_1 E)$, where $m_1 = mult_{p_1}(C)$. Define $p'_1 = E \cap \widetilde{L}$ and $p'_i = p_i$ for any $i > 1$, and consider the family

$$R_1 = \left\{ X \in |\mathcal{O}_{\widetilde{\Sigma}}(\widetilde{C})| \,:\, g(X) = g(C) \,\text{and}\, X \cap \widetilde{L} = \sum_{i=1}^{r} k'_i p'_i \right\},$$

where $k_1 = m_{p'_1}(\widetilde{C}, \widetilde{L})$ and $k'_i = k_i$ for all $i > 1$. It is clear that $\dim(R_1) \geq \dim(R)$, since $C \in R$ is generic. Since $K_{\widetilde{\Sigma}} \equiv \pi^* K_\Sigma + E$,

$$\widetilde{C}_i.(K_{\widetilde{\Sigma}} + \widetilde{L}) = C_i.(K_\Sigma + L) < -1$$

for any irreducible component $\widetilde{C}_i \subset \widetilde{C}$. Since $\widetilde{L}.\widetilde{C} = L.C - m_1 < L.C$,

$$\begin{aligned} \dim(R) &\leq \dim(R_1) \\ &\leq -\widetilde{C}.K_{\widetilde{\Sigma}} + g(\widetilde{C}) - 1 - \widetilde{L}.\widetilde{C} = -C.K_\Sigma + g(C) - 1 - L.C, \end{aligned}$$

by the induction hypothesis. Moreover, if the equalities hold and for any singular component C_i of C we have $C_i.(K_\Sigma + L) < -3$, which implies $\widetilde{C}_i.(K_{\widetilde{\Sigma}} + \widetilde{L}) < -3$, then C has only nodes as its singularities outside of L, and for any irreducible curve C' satisfying $C' \cap \{p_1, ..., p_r\} = \emptyset$, a generic curve $D \in R$ intersects C' transversally. ■

Corollary 3.5 *Let n, d, k be non-negative integers and let $p_1, ..., p_r \in L_\infty \subset \Sigma_n$ be arbitrary points. Choose $k_1, ..., k_r$ positive integers such that $\sum_{i=1}^r k_i = k$. Let*

$$R \subset \left\{ C \in |\mathcal{O}_{\Sigma_n}(dL_0 + kF)| \ : \ L\infty \cap C = \sum_{i=1}^{r} k_i p_i \right\},$$

be a non-empty subvariety, and let $C \in R$ be generic element. If C is reduced, then

$$\dim(R) \leq -C.K_{\Sigma_n} + g(C) - 1 - k. \tag{3.1}$$

Moreover, if the equality holds, then C has only nodes as its singularities outside of L_∞, and C intersects L_0 transversely.

Proof: By definition $k = C.L_\infty$; hence it is enough to prove (3.1) only for irreducible C. If $C = F$, then there is nothing to prove, otherwise $d > 0$. Thus $C.(K_{\Sigma_n} + L_\infty) = -k - (2+n)d < -1$ and if C is singular, then $d \geq 2$, and $C.(K_{\Sigma_n} + L_\infty) = -k - (2+n)d < -3$. Now, (3.1) follows from the lemma.

Assume that the equality holds in (3.1). Let $C_1, ..., C_s$ be the irreducible components of type F, and let $C_{s+1}, ..., C_r$ be the irreducible components of other types. Then $C_1, ..., C_s$ are *generic* elements of $|\mathcal{O}_{\Sigma_n}(F)|$, and $C_{s+1} \cup ... \cup C_r$ is a nodal curve intersecting L_0 transversally, due to the lemma. This implies the last part of the corollary. ■

Claim 3.6 *Let $g, k, n, d \geq 0$ be natural numbers such that $V_{g,d,k} \neq \emptyset$, and let $V \subset V_{g,d,k}$ be any component. Then given $g + nd + 2k + 2d - 1$ points $p_1, ..., p_{g+nd+2k+2d-1} \in \Sigma_n$ in general position, there exists a finite positive number of curves $C \in V$ passing through these points.*

Proof: The claim follows from Corollary 2.11 and Riemann-Roch Theorem. ■

Proof of Proposition 3.2:

Step 0: Prelude. First, we would like to explain the idea of the proof. The proof goes by induction on d. For $d = 0, 1$ it is easy. To prove the induction step, we degenerate the surface Σ_n into a fan (in Ran's terminology [9]) $X^0 \cup ... \cup X^r$, where $X^j \cong \Sigma_n$ for all j, and $X^j \cap X^{j+1}$ is L_0 on X^j and L_∞ on X^{j+1}. Next we show that the degeneration of V contains the degeneration of the curve Γ, which allows us to prove that V itself contains the required curve.

The following remark is important: it is enough to prove the statement for V whose generic element is irreducible, since if the generic curve $C \in V$ is reducible, then $C = C_1 \cup ... \cup C_l$, where $C_i \in V_i \subseteq V_{g(C_i),d_i,k_i}$ for some d_i, k_i. So we have a natural map $\prod_{i=1}^{l} V_i \longrightarrow V$ and if the proposition is true for all V_i, then it is true for V as well.

Step 1: The base of induction. If $d = 0$, then $k = 1, g = \delta = 0$ and the proposition is obvious. If $d = 1$, then any nodal curve $C \in V$ contains fibers $F_1, ..., F_\delta$ passing through the nodes of C, since $C.F = 1$. Thus $\delta = 0$, since the generic $C \in V$ is irreducible. So

$$V = V_{g,1,k} = |\mathcal{O}_{\Sigma_n}(L_0 + kF)|$$

and $\Gamma \in V$. Next we show the induction step.

Step 2: The construction of the fan. Our goal is to construct a flat family of surfaces $\rho : Z \longrightarrow \mathbb{A}^1$ and a flat family of curves $\mathcal{C} \subset Z \longrightarrow \mathbb{A}^1$ such that

1. $Z_t = \Sigma_n$ for all $t \neq 0$
2. $Z_0 = X^0 \cup ... \cup X^r$ where $X^j \cong \Sigma_n$ for all j, and $X^j \cap X^{j+1}$ is $L_0 = L_0^j$ on X^j and $L_\infty = L_\infty^{j+1}$ on X^{j+1}
3. $\mathcal{C}_t \in V$ for almost all $t \neq 0$
4. $\mathcal{C}_0$ does not contain L_∞^j for all j
5. There are at least two indices $j_1 \neq j_2$, such that $\mathcal{C}_0 \cap X^{j_i}$ is not a collection of fibers of $X^{j_i} \cong \Sigma_n$.

Consider the trivial family $Z^0 = \Sigma_n \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$. Let

$$\sigma : \mathbb{A}^1 \longrightarrow \Sigma_n^{g+nd+2k+2d-1}$$

be a generic algebraic map for which $\sigma(0) \in \Sigma_n^{g+dn+k+2d-2} \times L_\infty^{k+1}$ is in general position. Then σ takes the generic point of $\mathbb{A}^1$ to the generic point of $\Sigma_n^{g+nd+2k+2d-1}$; hence we can find a flat family of curves $\mathcal{C}^0 \subset \Sigma_n \times \mathbb{A}^1$ such that $\mathcal{C}^0_t \in V$ for almost all t. The curve $\mathcal{C}^0_0$ contains L_∞ as its component, so $\mathcal{C}^0_0 = lL_\infty \cup C'$ and $l > 0$. We claim that $l < d$, since otherwise $\mathcal{C}^0_0$ would be a union of dL_∞ with $dn + k$ fibers, but such a curve can not pass through $g + dn + k + 2d - 2$ points in general position, since $d > 1$ and $g \geq 0$ due to Step 0. Thus

$$0 < l < d. \tag{3.2}$$

Next we shall modify the pair $(Z^0, \mathcal{C}^0)$ in order to obtain the required family $(Z, \mathcal{C})$. Namely, we shall blow up Z^0 along $L_\infty \times \{0\}$, replace $\mathcal{C}^0$ by its strict transform, and continue this process till we get $(Z, \mathcal{C})$ satisfying conditions 1-5.

Let $F \subset \Sigma_n$ be a generic fiber and let $L \subset \Sigma_n$ be a generic curve in the system $|\mathcal{O}_{\Sigma_n}(L_0)|$. Then $W = \Sigma_n \backslash (F \cup L) \simeq \mathbb{A}^2$. We choose coordinates (x, y) on W such that $x = 0$ defines $L_\infty \cap W$ and for any $c \in \mathbb{K}$ $y = c$ defines $F_c \cap W$ for some fiber $F_c \subset \Sigma_n$. Let $f(x, y, t) \in \mathbb{K}[x, y, t]$ be a polynomial defining $\mathcal{C}^0 \cap (W \times \mathbb{A}^1)$. Consider $f_y(x, t) = f(x, y, t) \in \mathbb{F}[x, t]$, where $\mathbb{F} = \overline{\mathbb{K}(y)}$.

Claim 3.7 *Let $\mathbb{F}$ be an algebraically closed field, and let*

$$S_0 = \operatorname{Spec} \mathbb{F}[x, t].$$

We define

$$S_r = Bl_{p_r}(Bl_{p_{r-2}}(...(Bl_{p_1})..),$$

where $p_1 = (0, 0)$ and p_{i+1} is the intersection of the exceptional divisor E_i with the strict transform of the t-axis. Let $g(x, t) \in \mathbb{F}[x, t]$ be a polynomial not divisible by t and x, but vanishing at $(0, 0)$. Then there exist N and r such that

$$\widehat{C} \cap \{p_{r+1}\} = \widehat{C} \cap E_i \cap E_{i-1} = \emptyset$$

for any $1 \leq i \leq r$, where E_0 is the strict transform of the x-axis and $\widehat{C}$ denotes the strict transform on S_r of the curve given by $g(x, t^N) = 0$ if N is not divisible by $cahr(\mathbb{F})$, and given by $g(x, t^N + t^{N+1}) = 0$ if N is a multiple of $cahr(\mathbb{F})$.

Since $f_y(x, t)$ is not divisible by t and x, there exist numbers N and r as in the claim. Without loss of generality we can assume that $N = 1$, since this will be the case after replacing σ by σ' defined by either $\sigma'(t) = \sigma(t^N)$ or $\sigma'(t) = \sigma(t^N + t^{N+1})$, respectively.

Now we can construct the pair $(Z, \mathcal{C})$ satisfying conditions 1-5. To do this, we first blow up Z^0 along $L_\infty \times \{0\}$. Then the exceptional divisor X^1

is isomorphic to Σ_n. Next, we blow up the resulting family along $L_\infty \subset X^1$, and we proceed this way r times till, finally, we get the required family of surfaces $\rho : Z \longrightarrow \mathbb{A}^1$. The family of curves $\mathcal{C}$ is defined to be the strict transform of $\mathcal{C}^0$. It is clear that conditions 1, 2, 3 are satisfied for the pair $(Z, \mathcal{C})$. Condition 4 follows immediately from Claim 3.7, and finally (3.2) implies condition 5.

Step 3: The construction of the degeneration. First we shall prove that $\rho_* \mathcal{O}_Z(\mathcal{C})$ is locally free. By the semi-continuity theorems, it is enough to show that $H^1(Z_t, \mathcal{O}_{Z_t}(\mathcal{C}_t)) = 0$ for all $t \in \mathbb{A}^1$. For $t \neq 0$ this is obvious. Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{Z_0}(\mathcal{C}_0) \longrightarrow \oplus_{i=0}^r \mathcal{O}_{X^i}(\mathcal{C}_0) \longrightarrow \oplus_{i=0}^{r-1} \mathcal{O}_{X^i \cap X^{i+1}}(\mathcal{C}_0) \longrightarrow 0$$

and define $\mathcal{C}_0^i = \mathcal{C}_0 \cap X_i$. Then $\mathcal{C}_0^i \in |\mathcal{O}_{X^i}(d_i L_0^i + k_i F^i)|$ for some $d_i, k_i \geq 0$, satisfying $\sum_{i=0}^r d_i = d$, $k_i = n d_{i+1} + k_{i+1}$, and $k_r = k$. Thus we have: $\oplus_{i=1}^r H^1(X^i, \mathcal{O}_{X^i}(\mathcal{C}_0)) = 0$, and in order to prove $h^1(Z_0, \mathcal{O}_{Z_0}(\mathcal{C}_0)) = 0$, it is enough to show that the map

$$\oplus_{i=0}^r H^0(X^i, \mathcal{O}_{X^i}(\mathcal{C}_0)) \longrightarrow \oplus_{i=0}^{r-1} H^0(X^i \cap X^{i+1}, \mathcal{O}_{X^i \cap X^{i+1}}(\mathcal{C}_0)) \qquad (3.3)$$

is surjective. It is clear that the map

$$H^0(\Sigma_n, \mathcal{O}_{\Sigma_n}(aL_0 + bF)) \longrightarrow H^0(L_\infty, \mathcal{O}_{L_\infty}(aL_0 + bF))$$

is surjective for any $a, b \geq 0$. This implies the surjectivity in (3.3).

Now we can construct the degeneration of V: $\rho_* \mathcal{O}_Z(\mathcal{C})$ is locally free, and therefore the natural map

$$\rho_* \mathcal{O}_Z(\mathcal{C}) \otimes k(t) \longrightarrow H^0(Z_t, \mathcal{O}_{Z_t}(\mathcal{C}_t))$$

is an isomorphism for any $t \in \mathbb{A}^1$. Thus $V = V_t \subset |\mathcal{O}_{Z_t}(\mathcal{C}_t)|$ for $t \neq 0$, degenerates into a variety $V_0 \subset |\mathcal{O}_{Z_0}(\mathcal{C}_0)|$, which is defined to be the zero fiber of

$$\overline{V \times \mathbb{K}^*} \subset \mathbb{P}\mathrm{roj}(\rho_* \mathcal{O}_Z(\mathcal{C})) \longrightarrow \mathbb{A}^1.$$

Step 4: The count of dimensions. The goal of this step is to prove that V_0 contains a nodal curve. Let $W \subseteq V_0$ be the irreducible component containing $\mathcal{C}_0$. We can assume that $\mathcal{C}_0$ is a generic element of W. Denote $C_0^i = (\mathcal{C}_0^i)^{red}$. Then by Corollary 3.5 and by the genus formula[3]

$$\begin{array}{rcl} \dim(W) & \leq & \sum_{i=0}^r (-C_0^i.K_{X^i} + g(C_0^i) - 1) - \sum_{i=0}^{r-1} C_0^i.L_\infty^i \\ & \leq & -(dL_0 + kF).K_{\Sigma_n} + \sum_{i=0}^r (g(C_0^i) - 1) + \sum_{i=0}^{r-1} C_0^i.L_\infty^i \\ & \leq & -(dL_0 + kF).K_{\Sigma_n} + g - 1 \;\; = \;\; \dim(V). \end{array}$$

[3]We leave it as an exercise to the reader to check that $\sum_{i=0}^r \left(g(C_0^i) - 1\right) + \sum_{i=0}^{r-1} C_0^i.L_\infty^i \leq g - 1$ (see [9], proof of Lemma 2.1, for similar calculation).

Moreover, the second inequality becomes an equality if and only if $\mathcal{C}_0^i = C_0^i$. Thus if the first and second inequalities are equalities, then C_0^i is nodal and intersects L_0^i transversally for any i, due to Corollary 3.5.

Since V_0 is a flat limit of V_t, all its components have the same dimension as V_t, in particular $\dim(W) = \dim(V)$. Thus a generic element of W is a nodal curve, and for any i the image of W under the natural projection $|\mathcal{O}_{Z_0}(\mathcal{C}_0)| \longrightarrow |\mathcal{O}_{X^i}(\mathcal{C}_0^i)|$ is a component of some Severi variety.

Step 5: Final accord. By the induction hypothesis we can find a curve $\Gamma_0 \in V_0$ such that all the curves $\Gamma_0^i = \Gamma_0 \cap X^i$ are of the form Γ. Then Γ_0 has $\delta' = \sum_{i=0}^r (d_i k_i + n\frac{d_i(d_i-1)}{2})$ nodes outside of $\cup_{i=1}^r L_0^i$. We denote these nodes by $p_0^1, ..., p_0^{\delta'}$.

Next, we shall construct a deformation $\Gamma_t \in V_t$ of $\Gamma_0 \in W$ preserving the δ' nodes. To do this, consider the relative decorated Severi varieties $\mathcal{U}_{d,k,\delta} \longrightarrow \mathbb{A}^1$ and $\mathcal{U}_{d,k,\delta'} \longrightarrow \mathbb{A}^1$, where $\delta = \frac{(d-1)(nd+2k-2)}{2} - g$. Let $\mathcal{U} \subseteq \mathcal{U}_{d,k,\delta}$ be the irreducible component whose projection to $\mathbb{P}\mathrm{roj}(\rho_* \mathcal{O}_Z(\mathcal{C}))$ is contained in $\overline{V \times \mathbb{K}^*}$. Then $(\Gamma_0; p_0^{i_1}, ..., p_0^{i_\delta}) \in \mathcal{U} \otimes k(0)$ for some $i_1, ..., i_\delta$.

Similarly to Claim 2.10, one can easily check that $\mathcal{U}_{d,k,\delta'}$ and $\mathcal{U}_{d,k,\delta}$ are smooth at the points $(\Gamma_0; p_0^1, ..., p_0^{\delta'})$ and $(\Gamma_0; p_0^{i_1}, ..., p_0^{i_\delta})$ respectively, and $\mathcal{U}_{d,k,\delta'}$ is irreducible. Consider the natural projection $\phi : \mathcal{U}_{d,k,\delta'} \longrightarrow \mathcal{U}_{d,k,\delta}$, sending $(C; p^1, ..., p^{\delta'})$ to $(C; p^{i_1}, ..., p^{i_\delta})$. Then $\phi_0(\mathcal{U}_{d,k,\delta'} \otimes k(0)) \subset \mathcal{U} \otimes k(0)$, which implies

$$\phi_t(\mathcal{U}_{d,k,\delta'} \otimes k(t)) \subset \mathcal{U} \otimes k(t)$$

for almost all $t \in \mathbb{A}^1$. This allows us to construct a deformation Γ_t over a (formal) neighborhood of $0 \in \mathbb{A}^1$.

Using the relations $\sum_{i=0}^r d_i = d$, $k_i = nd_{i+1} + k_{i+1}$, and $k_r = k$, it is easy to see that

$$\begin{aligned} g(\Gamma_t) &= \tfrac{(d-1)(nd+2k-2)}{2} - \textstyle\sum_{i=0}^r \left(d_i k_i + n\frac{d_i(d_i-1)}{2}\right) \\ &= \tfrac{(d-1)(nd+2k-2)}{2} - dk + n\tfrac{d(d-1)}{2} \;=\; 1 - k - d. \end{aligned}$$

Thus Γ_t has at least $k + d$ components. Since $L_\infty^r \not\subseteq \Gamma_0$, we know that $L_\infty \not\subseteq \Gamma_t$ for almost all t, and hence $\Gamma_t = \Gamma$ for generic t. So we are done. ■

Proof of Claim 3.7: Let $\mathbb{K} = \overline{\mathbb{F}((t))}$ and let $\xi_1, ..., \xi_l \in \mathbb{K}$ be the roots of $g_t(x) = g(x,t) = 0$. Let ν be the natural valuation on the field $\mathbb{K}$. We define N to be the common denominator of $\nu(\xi_1), ..., \nu(\xi_l)$. Then the valuation has integral values on the roots of $h_t(x) = g(t^N, x)$ ($h_t(x) = g(t^N + t^{N+1}, x)$ respectively). Since $g(x,t)$ is not divisible by t and x, there exist $m_1, ..., m_l \in \mathbb{Z}_{\geq 0}$ and $\alpha \in \mathbb{F}(t) \subset \mathbb{K}$ such that

$$h_t(x) = \alpha \prod_{i=1}^{l} (t^{m_i} x - \xi_i'),$$

$\nu(\alpha) = 0$, $\nu(\xi_i') \geq 0$, and if $m_i > 0$ then $\nu(\xi_i') = 0$. Thus without loss of generality, we can assume that $m_1 = ... = m_l = 0$ and $\nu(\xi_i') > 0$ for all i. Now we define $r = \max\{\nu(\xi_i')\}$, and one can easily check that it satisfies the required property. ∎

3.2 Proof of Proposition 3.3

We start with some combinatorics.

Definition 3.8 *(1) An ordered subset $\mu \subseteq \Gamma^{sing}$ consisting of r nodes is called an $r-$marking on the curve Γ.*

(2) An $r-$marking μ is called irreducible if and only if the curve $\Gamma \backslash \mu$ is connected.

(3) We define D-moves on the set of $r-$markings as follows: let $\mu = \{p_1, ..., p_r\}$ be an $r-$marking, let $D, D' \subset \Gamma$ be two different irreducible components, and let $q, q' \in D \cap D'$ be two nodes, such that $q = p_i$ for some i. Then

$$D_{q,q'}(\mu) = \begin{cases} \{p_1, ..., p_{i-1}, q', p_{i+1}, ..., p_r\} & \text{if } q' \notin \mu, \\ \{p_{\tau_{ij}(1)}, ..., p_{\tau_{ij}(r)}\} & \text{if } q' = p_j \text{ for some } j, \end{cases}$$

where $\tau_{ij} \in \mathfrak{S}_r$ denotes the elementary transposition $\tau_{ij} = (i\, j)$.

(4) Assume that $n > 0$. We define T-moves on the set of $r-$markings as follows: let $\mu = \{p_1, ..., p_r\}$ be an $r-$marking, let $D, D', D'' \subset \Gamma$ be three different irreducible components, and let $q \in D' \cap D''$, $q' \in D \cap D''$, $q'' \in D \cap D'$ be three nodes satisfying $q' \notin \mu$ and $q = p_i$ for some i. Then

$$T_{q,q',q''}(\mu) = \begin{cases} \{p_1, ..., p_{i-1}, q'', p_{i+1}, ..., p_r\} & \text{if } q'' \notin \mu, \\ \{p_{\tau_{ij}(1)}, ..., p_{\tau_{ij}(r)}\} & \text{if } q'' = p_j \text{ for some } j. \end{cases}$$

(5) Assume that $n = 0$. We define Q^h-moves and Q^v-moves as follows: let $\mu = \{p_1, ..., p_r\}$ be an $r-$marking, let $X, X', Y, Y' \subset \Gamma$ be four different irreducible components satisfying $X \equiv X' \equiv L_0$ and $Y \equiv Y' \equiv F$, and let $q \in X \cap Y$, $q' \in X \cap Y'$, $q'' \in X' \cap Y$, $q''' \in X' \cap Y'$ be nodes. Assume that $q = p_i$ for some i, and $q''' \notin \mu$. If $q'' \notin \mu$, we define

$$Q^h_{q,q',q'',q'''}(\mu) = \begin{cases} \{p_1, ..., p_{i-1}, q', p_{i+1}, ..., p_r\} & \text{if } q' \notin \mu, \\ \{p_{\tau_{ij}(1)}, ..., p_{\tau_{ij}(r)}\} & \text{if } q' = p_j \text{ for some } j. \end{cases}$$

If $q' \notin \mu$, we define

$$Q^v_{q,q',q'',q'''}(\mu) = \begin{cases} \{p_1, .., p_{i-1}, q'', p_{i+1}, .., p_r\} & \text{if } q'' \notin \mu, \\ \{p_{\tau_{ij}(1)}, .., p_{\tau_{ij}(r)}\} & \text{if } q'' = p_j \text{ for some } j. \end{cases}$$

(6) Two $r-$markings, μ and μ', are called equivalent if and only if they can be joined by a sequence of T-moves, Q-moves, and D-moves.

Notation 3.9 *Let μ be any $r-$marking and let $C, C' \subset \Gamma$ be two different components. The following notation will be useful:*

$$\mu_{C,C'} = |\mu \cap C \cap C'|, \;\; \mu_C = |\mu \cap C|, \;\; \textit{and} \;\; \mu_i = p_i.$$

Claim 3.10 *Let $r > 0$ be such an integer, that the set of irreducible $r-$markings on the curve Γ is not empty. Then for any pair of distinct irreducible components $C, C' \subset \Gamma$ satisfying $C.C' > 0$ and for any $q \in C \cap C'$, there exist irreducible $r-$markings μ and μ' such that $q \in \mu$ and $q \notin \mu'$.*

Proof: Obvious. ■

¿From now on we will assume that $n > 0$. The remaining case, $n = 0$, is much easier, and the proof in this case can be obtained via the same lines as in the case we consider. Thus, we leave it to the reader.

Lemma 3.11 *If $n > 0$ then any two irreducible $r-$markings on the curve Γ are equivalent.*

Proof: It is enough to prove the lemma for the case $r = k(d-1) + n\frac{d(d-1)}{2} - (d-1)$, since any irreducible $r-$marking can be extended to an irreducible marking of order $k(d-1) + n\frac{d(d-1)}{2} - (d-1)$. We prove the lemma by induction on $d+k$. If either $d = 2 = k+2$ or $d \leq 1$, then the lemma is obvious. Assume now that the statement is true for all $d+k \leq m$ and let us prove it for $d+k = m+1$. We can assume that $m \geq 2$ and $d \geq 2$, and let μ be an irreducible $r-$marking.

Step 1: The goal of this step is to prove that there exists a marking $\mu' \sim \mu$ such that

$$\mu'_{C,C'} = \begin{cases} C.C' - 1 & \text{if } L_d \in \{C, C'\}, \\ C.C' & \text{otherwise.} \end{cases} \tag{3.4}$$

Choose a component $D \subset \Gamma$ in the following way: if $k \neq 0$ then $D = F_k$, otherwise $D = L_1$. Then any irreducible component $D' \subset \Gamma$ different from D and satisfying $D.D' > 0$ belongs to the linear system $|\mathcal{O}_{\Sigma_n}(L_0)|$. Moreover, there are at least two such components. Now, let us choose $\widetilde{\mu} \sim \mu$, such that

$$\widetilde{\mu}_D = \max_{\mu' \sim \mu} \{\mu'_D\}.$$

Then $\widetilde{\mu}_{|_{\Gamma \setminus D}}$ is an irreducible marking on $\Gamma \setminus D$, since otherwise, due to the choice of D, we would be able to find two distinct irreducible components $D', D'' \subset \Gamma$ different from D such that $\widetilde{\mu}_{D,D'} < D.D'$, $\widetilde{\mu}_{D,D''} < D.D''$, and

$\widetilde{\mu}_{D',D''} = D.D'' > 0$. Hence, there would exist $q'' \in D \cap D'$, $q' \in D \cap D''$, and $q \in D' \cap D''$, such that $q', q'' \notin \mu$ and $q = \widetilde{\mu}_i$ for some i. Thus $\widetilde{\mu}_D < \mu'_D$, where $\mu' = T_{q,q',q''}(\widetilde{\mu}) \sim \mu$, which would be a contradiction.

Next, we shall prove that

$$\widetilde{\mu}_D = D.(\Gamma - D) - 1. \tag{3.5}$$

Consider two cases: $k > 0$ and $k = 0$.
Case 1: $k > 0$. In this case $D = F_k$, and

$$\widetilde{\mu}_D \geq r - \left((k-1)(d-1) + n\frac{d(d-1)}{2} - (d-1)\right),$$

since $\widetilde{\mu}_{|_{\Gamma \backslash D}}$ is an irreducible marking on $\Gamma \backslash D$. However,

$$r - \left((k-1)(d-1) + n\frac{d(d-1)}{2} - (d-1)\right) = d - 1 = D.(\Gamma - D) - 1,$$

and we are done.
Case 2: $k = 0$. In this case $D = L_1$, and

$$\widetilde{\mu}_D \geq r - \left(n\frac{(d-1)(d-2)}{2} - (d-2)\right),$$

since $\widetilde{\mu}_{|_{\Gamma \backslash D}}$ is irreducible. However,

$$r - \left(n\frac{(d-1)(d-2)}{2} - (d-2)\right) = n(d-1) - 1 = D.(\Gamma - D) - 1.$$

Thus, $\widetilde{\mu}_D = D.(\Gamma - D) - 1$ in both cases.

Now we can complete the proof of the first step. By (3.5) there exists a unique component $D_1 \subset \Gamma$ satisfying $\widetilde{\mu}_{D,D_1} = D.D_1 - 1$, and $\widetilde{\mu}_{D,D'} = D.D'$ for all $D' \subset \Gamma$ different from D_1. By the induction hypothesis it remains to prove that there exists $\mu' \sim \widetilde{\mu}$ such that $\mu'_{D,L_d} = D.L_d - 1$ and $\mu'_D = \widetilde{\mu}_D$. If $D_1 = L_d$ then there is nothing to prove. Otherwise, by the induction assumption and Claim 3.10, we can find an irreducible marking $\widehat{\mu} \sim \widetilde{\mu}$ with the following properties: $\widehat{\mu}_{D,D'} = \widetilde{\mu}_{D,D'}$ for all $D' \subset \Gamma$, and $\widehat{\mu}_{D_1,L_d} < D_1.L_d$. Then there exist $q'' \in D \cap D_1$, $q \in D \cap L_d$, and $q' \in D_1 \cap L_d$ such that $q', q'' \notin \widehat{\mu}$, and $q = \widehat{\mu}_i$ for some i. Thus $\mu' = T_{q,q',q''}(\widehat{\mu})$ satisfies the required condition.

Step 2: The goal of this step is to prove that any two markings satisfying (3.4) are equivalent. Let μ, μ' be two such markings. Applying several D-moves we can find a marking $\mu'' \sim \mu'$ such that μ'' differs from μ only by the order of marked points. Namely, there exists $\tau \in \mathfrak{S}_r$ such that $\mu''_i = \mu_{\tau(i)}$ for all i. We use the notation $\tau(\mu)$ for such μ''. It remains to prove that $\mu \sim \tau(\mu)$. Without loss of generality we can assume that $\tau = \tau_{ij}$ is a simple transposition; moreover we can assume that $i = 1$ and $j = 2$. Let D_1, D_2, D_3, D_4 be components such that $\mu_1 \in D_1 \cap D_2$ and $\mu_2 \in D_3 \cap D_4$. If $\{D_1, D_2\} = \{D_3, D_4\}$, then we apply D_{μ_1,μ_2} to finish the proof. So we can assume that $D_1 \neq D_3, D_4$ and $D_2 \neq D_3$.

If $\mu_1 \in L_d \cap D'$ for some D', then $D' \equiv L_0$ due to (3.4). Thus there exists a component $D'' \neq D, L_d$ such that $D''.D' > 0$. Let $\mu_i \in D'' \cap D'$ be any node, and let $q \in L_d \cap D''$ be the node not belonging to μ. Then $\mu \sim T_{\mu_1,q,\mu_i}(\mu)$, and hence we can reduce the statement to the case when $\mu_1 \notin L_d$. Similarly, we can assume that $\mu_2 \notin L_d$, and thus that $D_i \neq L_d$ for all i. Now we shall consider several cases:

Case 1: $D_2 = D_4 \equiv L_0$. Let $q_1 \in L_d \cap D_1$, $q_2 \in L_d \cap D_2$, and $q_3 \in L_d \cap D_3$ be the nodes not belonging to μ. Then

$$\tau_{12}(\mu) = T_{\mu_2,q_3,q_2}(T_{\mu_1,q_1,q_2}(T_{\mu_2,q_3,q_2}(\mu))) \sim \mu.$$

Case 2: $D_2 = D_4 \equiv F$. In this case $D_1 \equiv D_3 \equiv L_0$. Let $\mu_i \in D_1 \cap D_3$ be any node. Then $\tau_{12} = \tau_{1i} \circ \tau_{i2} \circ \tau_{1i}$, and thus $\tau_{12}(\mu) \sim \mu$. So we reduce to the previous case.

Case 3: $\{D_1, D_2\} \cap \{D_3, D_4\} = \emptyset$. Without loss of generality $D_1 \equiv D_3 \equiv L_0$. Let $\mu_i \in D_1 \cap D_3$ be any node. Then $\tau_{12} = \tau_{1i} \circ \tau_{i2} \circ \tau_{1i}$, so $\tau_{12}(\mu) \sim \mu$, by the first case, and we are done, since cases 1, 2, 3 cover all the possibilities. ■

Lemma 3.12 *Let $X = X_1 \cup X_2 \cup ... \cup X_r$ be a nodal curve. Assume that $X_2, ..., X_r$ are generic curves of types L_0 and F, and assume that X_1 is a generic nodal rational curve whose type is one of L_0, L_0+F or $2L_0$. Consider the variety $W(X, \mathcal{L}) \subset (X^{smooth})^s \times |\mathcal{L}|$ given by*

$$\{(p_1, ..., p_s; \xi) \,:\, p_i \neq p_j \, \text{for all } i \neq j, \text{ and } \xi(p_1) = ... = \xi(p_s) = 0\},$$

where $s = \deg(\mathcal{L} \otimes \mathcal{O}_X)$ and either $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(L_0)$ or $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(F)$. Let $(p_1, ..., p_s; \xi) \in W(X, \mathcal{L})$ be an arbitrary point, and let $W \subseteq W(X, \mathcal{L})$ be the irreducible component containing $(p_1, ..., p_s; \xi)$. Assume that $n > 0$ and there exist $a \neq b$ satisfying $p_a, p_b \in X_1$. Then

$$(p_{\tau_{ab}(1)}, ..., p_{\tau_{ab}(s)}; \xi) \in W,$$

where $\tau_{ab} \in \mathfrak{S}_s$ denotes the elementary transposition $\tau_{ab} = (a\, b)$.

Proof: If $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(F)$, we can assume that $X_1 \equiv 2L_0$. Consider the variety $W(X_1, \mathcal{L})$, which is irreducible since X_1 is irreducible and $s = 2$. The natural forgetful map $f : W \longrightarrow W(X_1, \mathcal{L})$ is dominant, and since $F.X_i \leq 1$ for all $i > 1$ it is also one-to-one. This implies the statement. Therefore we can assume that $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(L_0)$.

It is easy to see that $W(X, \mathcal{L})$ is smooth. Let $(x_1, ..., x_s; \eta) \in \overline{W(X, \mathcal{L})}$ be any point with the following properties: $x_i \in X^{smooth}$ for all i, $x_a = x_b$, $\eta(x_a) = d(\eta_{|X})(x_a) = 0$, and $x_i \neq x_j$ for all $\{i, j\} \neq \{a, b\}$. Then $(x_1, ..., x_s; \eta)$ is a smooth point of $\overline{W(X, \mathcal{L})}$. Thus to prove the lemma it suffices to show that $\overline{W}$ contains a point with such properties. Consider the forgetful map $f : \overline{W} \longrightarrow X_1^2$ given by $f(x_1, ..., x_s; \eta) = (x_a, x_b)$. This map is surjective, since $\mathcal{L} \simeq \mathcal{O}_{\Sigma_n}(L_0)$ and $n > 0$, and it is clear that a generic $\alpha \in f^{-1}(\Delta_{X_1})$ satisfies the properties mentioned above. ∎

Proof of Proposition 3.3: We start with the following remark: let $U \subset U_{d,k,\delta}$ be an irreducible component containing $(\Gamma; p_1, ..., p_\delta)$. We define a $\delta-$marking $\mu^{\Gamma;p_1,...,p_\delta}$ on the curve Γ in the following way:

$$\mu^{\Gamma;p_1,...,p_\delta} = \{p_1, ..., p_\delta\}.$$

Then the generic curve $C \in \overline{\psi(U)}$ is irreducible if and only if the marking $\mu^{\Gamma;p_1,...,p_\delta}$ is irreducible. The collection of all $\delta-$markings corresponding to U is denoted by

$$\mathcal{M}(U) = \{\mu^{\Gamma;p_1,...,p_\delta} \,|\, (\Gamma; p_1, ..., p_\delta) \in U\}.$$

Thus Proposition would be proven once we show that $\mathcal{M}(U)$ is closed under T-moves and under D-moves. This is what we proceed to proving now under the assumption $n > 0$.

Step 1: First, we shall prove that $\mathcal{M}(U)$ is closed under T-moves. We choose an arbitrary marking $\mu = \mu^{\Gamma;p_1,...,p_\delta} \in \mathcal{M}(U)$ and label the rest of the nodes of Γ by $p_{\delta+1}, ..., p_{\delta'}$, where $\delta' = dk + n\frac{d(d-1)}{2}$. Let $D, D', D'' \subset \Gamma$ be three different irreducible components, and let $q \in D' \cap D''$, $q' \in D \cap D''$, $q'' \in D \cap D'$ be three nodes such that $q' \notin \mu$ and $q \in \mu$. We shall show that $T_{q,q',q''}(\mu) \in \mathcal{M}(U)$. Without loss of generality, $q = p_1$, $q'' = p_{\delta'-1}$, and $q' = p_{\delta'}$.

Consider the irreducible component $U' \subset U_{d,k,\delta'-1}$ containing the pointed curve $(\Gamma; p_1, ..., p_{\delta'-1})$, and let $f : U' \longrightarrow U$ be the natural forgetful map. It is sufficient to prove that

$$(\Gamma; p_{\delta'-1}, p_2, ..., p_{\delta'-2}, p_1) \in U',$$

since $T_{q,q',q''}(\mu) = \mu^{f(\Gamma;p_{\delta'-1},p_2,...,p_{\delta'-2},p_1)}$.

Let $(C; x_1, ..., x_{\delta'-1}) \in U'$ be a generic element. Then C has a unique component C_2 of type $D + D''$ among its $d + k - 1$ irreducible components. Moreover, there exists another irreducible component C_1 such that $x_1, x_{\delta'-1} \in$

$C_1 \cap C_2$. We denote $C^{fix} = \cup_{i=2}^{d+k-1} C_i$, where $C_1, ..., C_{d+k-1}$ are the irreducible components of C.

Consider the locus $U'' \subset U'$ of pointed curves $(C'; x'_1, ..., x'_{\delta'-1})$ with the following property: $C' = C'_1 \cup C^{fix}$, where $C'_1 \equiv C_1$ is generic. Let $U''' \subset U''$ be the irreducible component containing the pointed curve $(C; x_1, ..., x_{\delta'-1})$, and let

$$\phi : U''' \longrightarrow W(C^{fix}, \mathcal{O}_{C^{fix}}(C_1))$$

be the natural forgetful map (cf. Lemma 3.12). Then ϕ is dominant and one-to-one. Hence

$$(\Gamma; p_{\delta'-1}, p_2, ..., p_{\delta'-2}, p_1) \in U''' \subset U'$$

by Lemma 3.12.

Step 2: The goal of this step is to prove that $\mathcal{M}(U)$ is closed under D-moves. We choose an arbitrary marking $\mu = \mu^{\Gamma; p_1, ..., p_\delta} \in \mathcal{M}(U)$ and label the rest of the nodes of Γ by $p_{\delta+1}, ..., p_{\delta'}$, where $\delta' = dk + n\frac{d(d-1)}{2}$. Let $D, D' \subset \Gamma$ be two different irreducible components, and let $q, q' \in D \cap D'$ be two nodes such that $q \in \mu$. We shall show that $D_{q,q'}(\mu) \in \mathcal{M}(U)$.

Consider the irreducible component $U' \subset U_{d,k,\delta'}$ containing the pointed curve $(\Gamma; p_1, ..., p_{\delta'})$, and let $f : U' \longrightarrow U$ be the natural forgetful map. Then

$$D_{q,q'}(\mu) = \mu^{f(\Gamma; p_{\tau_{ij}(1)}, ..., p_{\tau_{ij}(\delta')})},$$

where $q = p_i$, $q' = p_j$, and $\tau_{ij} \in \mathfrak{S}_r$ denotes the elementary transposition $\tau_{ij} = (i\, j)$. Hence it is sufficient to prove that

$$(\Gamma; p_{\tau_{ij}(1)}, ..., p_{\tau_{ij}(\delta')}) \in U'.$$

Let $\Gamma_1, ..., \Gamma_{d+k}$ be the irreducible components of Γ. We can assume that $D = \Gamma_1$, and we denote $\Gamma^{fix} = \cup_{i=2}^{d+k} \Gamma_i$.

Consider the locus $U'' \subset U'$ of pointed curves $(C; x_1, ..., x_{\delta'})$ with the following property: $C = C_1 \cup \Gamma^{fix}$, where $C_1 \equiv \Gamma_1$ is generic. Let $U''' \subset U''$ be the irreducible component containing $(\Gamma; p_1, ..., p_{\delta'})$, and let

$$\phi : U''' \longrightarrow W(\Gamma^{fix}, \mathcal{O}_{\Gamma^{fix}}(\Gamma_1))$$

be the natural forgetful map (cf. Lemma 3.12). Then ϕ is dominant and one-to-one. Hence

$$(\Gamma; p_{\tau_{ij}(1)}, ..., p_{\tau_{ij}(\delta')}) \in U''' \subset U'$$

by Lemma 3.12. ■

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