GEOMETRY OF THE MODULI SPACES OF HARMONIC MAPS INTO LIE GROUPS VIA GAUGE THEORY OVER RIEMANN SURFACES

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Introduction

The aim of this paper is to investigate the geometry surrounding the moduli spaces of harmonic maps of compact Riemann surfaces into compact Lie groups or compact symmetric spaces. Our approach is to turn our attention to the gauge-theoretic equations coming from such harmonic maps and the moduli spaces of their solutions. In general the moduli space of such solutions is much larger than the moduli space of such harmonic maps itself. However, we believe that the nice features of such harmonic maps come from the character of the gauge-theoretic equations.

First of all, we shall review the formulation of harmonic map equations to the gauge-theoretic equations. Let M be a compact Riemann surface and G a compact Lie group with Lie algebra $\mathfrak{g}$. We endow G a bi-invariant Riemannian metric. Let $f : M \longrightarrow G$ be a harmonic map and set $\alpha := f^*\theta_G = f^{-1}df$, which is a 1-form pull-backed from f of the left-invariant Maurer-Cartan form θ_G of G. Then the form α satisfies the identity

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0 \tag{0.1}$$

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and the harmonic map equation

$$d * \alpha = 0. \tag{0.2}$$

These equations (0.1) and (0.2) can be formulated to the gauge-theoretic equations as follows.

Let $P = M \times G$ be a trivial principal bundle over M with the structure group G. Let $\mathcal{A}_P$ be the affine space of all smooth connections in the principal bundle P and $\Omega^i(\mathfrak{g}_P)$ be the vector space of all smooth i-forms on M with values in the adjoint bundle $\mathfrak{g}_P$. We define a connection $A \in \mathcal{A}_P$ by $d_A = d + \frac{1}{2}\alpha$ and a Higgs field $\phi \in \Omega^1(\mathfrak{g}_P)$ by $\phi = \frac{1}{2}\alpha$. Then the equations (0.1) and (0.2) can be expressed as the equations $(*)^+_G$ for a pair $(A, \phi) \in \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ (cf. [14, 15, 29, 26]). Here the gauge-theoretic equations $(*)^\pm_G$ are defined as

$$\begin{cases} F(A) \pm \dfrac{1}{2}[\phi \wedge \phi] = 0, \\ d_A * \phi = d_A \phi = 0, \end{cases} \tag*{$(*)^\pm_G$}$$

where $F(A)$ is the curvature form of the connection A. Hitchin [15] has provided the profound theory on the moduli space of solutions to the equations $(*)^+_G$ in terms of algebraic curves, so called *spectral curves*, in the case where M is a compact Riemann surface of genus one and $G = SU(2)$. We should remark that solutions to the equations $(*)^-_G$ locally corresponds to harmonic maps into noncompact symmetric space $G^{\mathbb{C}}/G$ (cf. [7, 25]).

However it is advantageous to consider these gauge-theoretic equations $(*)^\pm_G$ on general principal G-bundles P over Riemann surfaces M in the abstract setting, being independent of harmonic maps. It is well-known that the equations $(*)^-_G$ are the *self-duality equations* which were extensively investigated by Hitchin [12, 14] and others. The equations $(*)^+_G$ were investigated also by Valli [30, 31] and recently by the authors [23, 24, 25, 26, 27]. The equations $(*)^+_G$ seems to be more difficult to be studied than the equations $(*)^-_G$.

From now on we shall consider the equations $(*)^\pm_G$ on general principal G-bundles P over M. The simplest solution to the equations $(*)^\pm_G$ is $(A, 0)$, where A is a flat connection and $\phi \equiv 0$, which is called a *trivial* solution. Let

$$\widetilde{\mathcal{M}}^\pm(M, G, P) \subset \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$$

denote the spaces of all smooth solutions to the equations $(*)^\pm_G$, respectively. We denote by $[(A, \phi)]$ the gauge equivalence class of an element (A, ϕ) in $\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ under the action of the group $\mathcal{G}_P$ of gauge transformations of P. Let

$$\mathcal{M}^\pm(M, G, P) := \widetilde{\mathcal{M}}^\pm(M, G, P)/\mathcal{G}_P \subset (\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P))/\mathcal{G}_P$$

denote the moduli spaces of all gauge equivalence classes $[(A, \phi)]$ of smooth solutions to the equations $(*)^\pm_G$, respectively. We can equip the moduli space $\mathcal{M}^\pm(M, G, P)$ with a Riemannian metric defined by the L^2-inner product. We note that the moduli space of harmonic maps of M into G

$$\mathrm{Harm}(M, G)/(G \times G) \cong \mathcal{M}_{HM} \subset \mathcal{M}^+(M, G, P)$$

is characterized by the moduli space of solutions (A,ϕ) to the equations $(*)^+_G$ such that both $A+\phi$ and $A-\phi$ have the trivial holonomy. It is interesting and important to discuss the relationship and difference between moduli spaces $\mathcal{M}^+(M,G,P)$ and $\mathcal{M}^-(M,G,P)$. Both moduli spaces $\mathcal{M}^\pm(M,G,P)$ are contained in the space $(\mathcal{A}_P\times\Omega^1(\mathfrak{g}_P))/\mathcal{G}_P$ as follows:

$$\begin{array}{ccccc}
\widetilde{\mathcal{M}}^-(M,G,P) & \longrightarrow & \mathcal{A}_P\times\Omega^1(\mathfrak{g}_P) & \longleftarrow & \widetilde{\mathcal{M}}^+(M,G,P)\\
\Big\downarrow{\scriptstyle \mathcal{G}_P} & & \Big\downarrow{\scriptstyle \mathcal{G}_P} & & \Big\downarrow{\scriptstyle \mathcal{G}_P}\\
\mathcal{M}^-(M,G,P) & \longrightarrow & (\mathcal{A}_P\times\Omega^1(\mathfrak{g}_P))/\mathcal{G}_P & \longleftarrow & \mathcal{M}^+(M,G,P).
\end{array}$$

The intersection of $\mathcal{M}^+(M,G,P)$ and $\mathcal{M}^-(M,G,P)$ is

$$\begin{aligned}
&\mathcal{M}^+(M,G,P)\bigcap\mathcal{M}^-(M,G,P)\\
=&\{[(A,\phi)]\in\mathcal{A}_P\times\Omega^1(\mathfrak{g}_P)\mid A \text{ is a flat connection and}\\
&\qquad \phi \text{ is a 1-form with values in abelian subalgebras}\\
&\qquad \text{which is harmonic relative to connection } A\}.
\end{aligned}$$

If G is abelian, then we have $\mathcal{M}^+(M,G,P)=\mathcal{M}^-(M,G,P)$. If $M=S^2$ is of genus zero, then $\mathcal{M}^-(M,G,P)$ is a point. If M is of genus one, then we have

$$\begin{aligned}
&\mathcal{M}^-(M,G,P)=\mathcal{M}^+(M,G,P)\bigcap\mathcal{M}^-(M,G,P)\\
=&\{[(A,\phi)]\in\mathcal{A}_P\times\Omega^1(\mathfrak{g}_P)\mid A \text{ is a flat connection and}\\
&\qquad \phi \text{ is a 1-form with values in abelian subalgebras}\\
&\qquad \text{which is parallel relative to connection } A\}
\end{aligned}$$

(See [25]).

This article is organized as follows. In Section 1, we shall begin by discussing various geometric structures on moduli spaces $\mathcal{M}^\pm(M,G,P)$: the S^1-*actions* on $\mathcal{M}^\pm(M,G,P)$ in §1.1, the local structures of $\mathcal{M}^\pm(M,G,P)$ in §1.3, the *hyper-Kähler structure* $\{\omega_1^-,\omega_2^-,\omega_3^-\}$ on $\mathcal{M}^-(M,G,P)$ and the *hyper-presymplectic structure* $\{\omega_1^+,\omega_2^+,\omega_3^+\}$ on $\mathcal{M}^+(M,G,P)$, which is degenerate, in §1.4, the *Hitchin maps* $p^\pm$ on $\mathcal{M}^\pm(M,G,P)$ in §1.5. On the other hand, as a certain class of solutions to the equations $(*)^+_G$, we define *uniton solutions* in §1.2.

In Section 2, we shall discuss the energy functions $\mathcal{E}^\pm$ on $\mathcal{M}^\pm(M,G,P)$. The energy functions $\mathcal{E}^+$ and $\mathcal{E}^-$ coincide with the moment maps for the S^1-action on the moduli spaces relative to ω_1^+ and ω_1^-, respectively. Hence we see that all fixed points by the S^1-action on $\mathcal{M}^\pm(M,G,P)$ are critical points of $\mathcal{E}^\pm$ and that $\mathcal{F}^-(M,G,P)=\mathcal{C}^-(M,G,P)$ becuase of the non-degeneracy of ω_1^- on $\mathcal{M}^-(M,G,P)$.

In Section 3, we shall introduce a kind of a foliated structure on $\mathcal{M}^+(M,G,P)$ defined by fixing the holonomy homomorphisms ρ induced from solutions. The moduli space $\mathcal{M}^+_{[\rho]}(M,G,P)$ is regarded as the moduli space of *ρ-twisted harmonic maps*. Each $\mathcal{M}_{[\rho]}(M,G,P)$ are an integral manifold of $\mathrm{Ker}(\omega_1^+)$ and $\mathrm{Ker}(\omega_3^+)$ in the sense of $T(\mathcal{M}^+_{[\rho]}(M,G,P))\subset\mathrm{Ker}(\omega_1^+)\cap\mathrm{Ker}(\omega_3^+)$ (Proposition 3.2).

The degeneracy of ω_1^+ on $\mathcal{M}^+(M,G,P)$ seems to cause the interesting phenomena. In Section 4, We shall prove that each uniton solution is a critical point of the energy function $\mathcal{E}^+$ and hence we obtain that $\mathcal{F}^+(M,G,P) \subset \mathcal{U}(M,G,P) \subset \mathcal{C}^+(M,G,P)$ (Theorem 4.1).

In Section 5, as a further property of solutions to the equations $(*)^+_G$, we shall define a *flag transformation*, inspired by uniton transforms [29] and flag transforms [5] for harmonic maps. This is a method which transforms a given solution to a new solution by adding *flag factor*. We shall show that the flag transform for solutions preserves the values of the Hitchin map p^+ on the moduli space $\mathcal{M}^+(M,G,P)$.

In Section 6, we shall discuss the moduli spaces $\mathrm{Harm}(M,G/K)/G$ of harmonic maps of M into compact symmetric space G/K. The gauge-theoretic equations $(*)^\pm_{G/K}$ reduced to G/K and their moduli spaces $\mathcal{M}^\pm(M,G/K,Q)$ are defined. We shall show that such moduli spaces can be considered as subspaces in the moduli spaces $\mathcal{M}^\pm(M,G,P)$. Their properties will be discussed from the viewpoint of differential geometry of submanifolds.

In Section 7, we shall give a characterization of the fixed point subset $\mathcal{F}^+(M,G,P)$ by the S^1-action on the moduli space $\mathcal{M}^\pm(M,G,P)$ in terms of flag transforms (Theorem 7.1). Thus we shall see that in general $\mathcal{F}^+(M,G,P)$ does not coincide with $\mathcal{C}^+(M,G,P)$ and $\mathcal{U}(M,G,P)$ appears between them.

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1. Geometric Structures of the Moduli Spaces

1.1. **The S^1-action and the associated loop of flat connections.** We summarize the fundamental properties of solutions to the equations $(*)^\pm_G$. Let $(A,\phi) \in \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$, decomposing a 1-form ϕ with values in $\mathfrak{g}$ into $(1,0)$- and $(0,1)$-parts as $\phi = \phi' + \phi''$. For each $\lambda \in S^1$, we set

$$\phi_\lambda := \lambda^{-1}\phi' + \lambda\phi'' \tag{1.1}$$

and define a loop of connections

$$A_\lambda := A - \phi_\lambda \in \mathcal{A}_P. \tag{1.2}$$

Proposition 1.1 ([30, 31])**.** (i) *If (A,ϕ) is a solution to the equations $(*)^\pm_G$, then for each $\lambda \in S^1$, (A,ϕ_λ) is also a solution to the equations $(*)^\pm_G$.*

(ii) *A pair (A,ϕ) is a solution to the equations $(*)^+_G$ if and only if A_λ is a flat connection for each $\lambda \in S^1$. In particular, if (A,ϕ) is a solution to the equations $(*)^+_G$, then both $A+\phi$ and $A-\phi$ are flat connections.*

For each solution $(A,\phi) \in \widetilde{\mathcal{M}}^+(M,G,P)$, we call A_λ the *associated loop (circle) of flat connections*. From Proposition 1.1(i), for a solution (A,ϕ) to the equations

$(*)^{\pm}_{G}$, we obtain an S^1-family (A, ϕ_λ) of solutions to the equations $(*)^{\pm}_{G}$. Then the S^1-*actions* on $\widetilde{\mathcal{M}}^{\pm}(M, G, P)$ and hence $\mathcal{M}^{\pm}(M, G, P)$ have been well-defined by

$$\lambda^{\sharp}([(A, \phi)]) := [(A, \phi_\lambda)] \quad \text{for each } \lambda \in S^1. \tag{1.3}$$

1.2. **Uniton solutions.** Calabi, Chern, Barbosa, Bryant, Eells-Wood, Wolfson and others investigated a certain class of weakly conformal harmonic maps (branched minimal immersions) of Riemann surfaces into the standard spheres and the complex projective spaces, which includes all harmonic maps of Riemann spheres into such spaces. Harmonic maps belonging to the class were called pseudo-holomorphic maps, superminimal immersions, or isotropic harmonic maps and were well investigated by so many differential geometers. Uhlenbeck [29] introduced the notion of harmonic maps of finite uniton number and proved that any harmonic map of the Riemann sphere into the unitary group is a harmonic map of finite uniton number. It has been clarified that pseudo-holomorphic maps, superminimal immersions and isotropic harmonic maps are harmonic maps of finite uniton number.

The notion of these harmonic maps can be extended to the solution to the equations $(*)^{+}_{G}$ as follows.

Definition 1.1. A solution $(A, \phi) \in \widetilde{\mathcal{M}}^{+}(M, G, P)$ to the equations $(*)^{+}_{G}$ is called a *uniton solution* if $\{A_\lambda \mid \lambda \in S^1\}$ is contained in a single orbit of the gauge transformation group $\mathcal{G}_P$ on $\mathcal{A}_P$. Denote the subspace of moduli space consisting of all uniton solutions by

$$\mathcal{U}(M, G, P) := \{[(A, \phi) \in \mathcal{M}^{+}(M, G, P) \mid (A, \phi) \text{ is a uniton solution}\}.$$

Remark 1.1. Any solution to the equations $(*)^{+}_{G}$ on a Riemann sphere $M = S^2$ is always a uniton solution, because any flat connection on a Riemann sphere is a trivial flat connection.

1.3. **Local structure.** We shall review the local structure of the moduli spaces $\mathcal{M}^{\pm}(M, G, P)$ ([12, 23, 24]). The linearized equations for the equations $(*)^{\pm}_{G}$ are

$$\begin{cases} d_A\alpha \pm [\phi \wedge \eta] = 0, \\ d_A * \eta + [*\phi \wedge \alpha] = 0, \\ d_A\eta + [\phi \wedge \alpha] = 0, \end{cases} \tag{1.4}$$

for $(\alpha, \eta) \in \Omega^1(\mathfrak{g}_P) \oplus \Omega^1(\mathfrak{g}_P)$. The elliptic cochain complex associated with deformations for solutions to the equations $(*)^{\pm}_{G}$ is

$$0 \longrightarrow \Omega^0(\mathfrak{g}_P) \xrightarrow{d^0} \oplus^2\Omega^1(\mathfrak{g}_P) \xrightarrow{d^{1,\pm}} \oplus^3\Omega^2(\mathfrak{g}_P) \longrightarrow 0. \tag{$C^{\pm}_{(A,\phi)}$}$$

Here the linear differential operators $d^0_{(A,\phi)}$ and $d^{1,\pm}_{(A,\phi)}$ are defined by

$$d^0 = d^0_{(A,\phi)}(\xi) := (d_A\xi, [\phi, \xi]), \tag{1.5}$$

$$d^{1,\pm} = d^{1,\pm}_{(A,\phi)}(\alpha, \eta) := (d_A\alpha \pm [\phi \wedge \eta], d_A * \eta + [*\phi \wedge \alpha], d_A\eta + [\phi \wedge \alpha]). \tag{1.6}$$

We define the cohomology groups $H^0_{(A,\phi)}$, $H^{1,\pm}_{(A,\phi)}$ and $H^{2,\pm}_{(A,\phi)}$ associated with the cochain complex $(C^{\pm}_{(A,\phi)})$ as

$$\begin{aligned} H^0_{(A,\phi)} &= \mathrm{Ker}(d^0_{(A,\phi)}), \\ H^{1,\pm}_{(A,\phi)} &= \mathrm{Ker}(d^{1,\pm}_{(A,\phi)}) / \mathrm{Im}(d^0_{(A,\phi)}), \\ H^{2,\pm}_{(A,\phi)} &= \oplus^3 \Omega^2(\mathfrak{g}_P) / \mathrm{Im}(d^{1,\pm}_{(A,\phi)}). \end{aligned}$$

By using the method of Kuranishi, we obtain the local description of the moduli spaces in terms of the cohomology groups. Each the first cohomology group $H^{1,\pm}_{(A,\phi)}$ can be identified with the formal tangent space $T_{[(A,\phi)]}(\mathcal{M}^{\pm}(M,G,P))$ of the moduli space $\mathcal{M}^{\pm}(M,G,P)$ at the point $[(A,\phi)]$.

For each $(A,\phi) \in \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$, we define two kinds of the *Jacobi differential operator* $\mathcal{J}^{\pm}_{(A,\phi)} : \Omega^0(\mathfrak{g}_P) \longrightarrow \Omega^0(\mathfrak{g}_P)$ by

$$\mathcal{J}^{\pm}_{(A,\phi)}(\xi) := d_A^* d_A \xi \mp *[[\phi, \xi] \wedge *\phi], \tag{1.7}$$

where $d_A^* = - * d_A *$ on $\Omega^1(\mathfrak{g}_P)$. We call ξ a *Jacobi field* if $\mathcal{J}^{\pm}_{(A,\phi)}(\xi) = 0$. We note that $\mathrm{Ker}(d^0_{(A,\phi)}) \subset \mathrm{Ker}(\mathcal{J}^{\pm}_{(A,\phi)})$. We see that in the case of $(A,\phi) \in \widetilde{\mathcal{M}}^-(M,G,P)$, $\mathrm{Ker}(d^0_{(A,\phi)}) = \mathrm{Ker}(\mathcal{J}^-_{(A,\phi)})$. On the other hand, in the case of $(A,\phi) \in \widetilde{\mathcal{M}}^+(M,G,P)$, the equality does not satisfy necessarily. In [23, 24], the first named author showed that the second cohomology group $H^{2,+}_{(A,\phi)}$ can be described in terms of $\mathrm{Ker}(\mathcal{J}^+_{(A,\phi)})$ and then it never vanishes in general.

In order to use later, we shall list some operators associated with $(C^+_{(A,\phi)})$ for $(A,\phi) \in \widetilde{\mathcal{M}}^+(M,G,P)$ (see [23, 24]). The formal adjoint operators with respect to the L^2-inner product are

$$(d^0_{(A,\phi)})^*(\alpha, \eta) = - * d_A * \alpha - *[\phi \wedge *\eta], \tag{1.8}$$

$$(d^{1,+}_{(A,\phi)})^*(u_1, u_2, u_3) = (\beta(u_1,u_2,u_3), \zeta(u_1,u_2,u_3)), \tag{1.9}$$

where we put

$$\begin{cases} \beta(u_1,u_2,u_3) := - * d_A * u_1 + [\phi, *u_2] - [*\phi, *u_3], \\ \zeta(u_1,u_2,u_3) := -[*\phi, *u_1] - d_A * u_2 - *d_A * u_3. \end{cases}$$

Moreover, we obtain a nice description of $(d^{1,+}_{(A,\phi)})^*$ in terms of complex structures $\{F_1, F_2, F_3\}$ defined in Subsection §1.4, (1.18).

Lemma 1.2. *For each solution $(A,\phi) \in \widetilde{\mathcal{M}}^+(M,G,P)$ to the equations $(*)^+_G$, the operator $(d^{1,+}_{(A,\phi)})^*$ can be described in terms of $\{F_1, F_2, F_3\}$ as*

$$(d^{1,+}_{(A,\phi)})^*(u_1,u_2,u_3) = F_1(d^0_{(A,\phi)}(*u_1)) + F_2(d^0_{(A,\phi)}(*u_2)) + F_3(d^0_{(A,\phi)}(*u_3)).$$

Moreover, the self-adjoint elliptic operator $P^0_{(A,\phi)} = (d^0_{(A,\phi)})^* \circ d^0_{(A,\phi)}$ is

$$P^0_{(A,\phi)}(\xi) = \mathcal{J}^+_{(A,\phi)}(\xi) + 2 * [[\phi, \xi] \wedge *\phi] = \mathcal{J}^-_{(A,\phi)}(\xi). \tag{1.10}$$

See [23, 24] for explicit descriptions of the other operators

1.4. **Hyper-Kähler structure and hyper-presymplectic structure.** The moduli spaces $\mathcal{M}^{\pm}(M,G,P)$ of solutions to the equations $(*)^{\pm}_{G}$ have the expressions from the viewpoint of moment maps ([12, 16, 14, 23, 24]).

The infinite dimensional affine space $\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ has two kinds of a triple set of symplectic forms; hyper-Kähler structure $\{\widetilde{\omega}_1^-, \widetilde{\omega}_2^-, \widetilde{\omega}_3^-\}$ and hyper-symplectic structure $\{\widetilde{\omega}_1^+, \widetilde{\omega}_2^+, \widetilde{\omega}_3^+\}$:

$$\begin{cases} \widetilde{\omega}_1^-((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad := -\langle *\alpha_1, \alpha_2\rangle_{L^2} + \langle *\eta_1,\eta_2\rangle_{L^2}, \\ \widetilde{\omega}_2^-((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad := \langle \eta_1, \alpha_2\rangle_{L^2} - \langle \alpha_1,\eta_2\rangle_{L^2}, \\ \widetilde{\omega}_3^-((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad := -\langle *\eta_1, \alpha_2\rangle_{L^2} - \langle *\alpha_1,\eta_2\rangle_{L^2}, \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{\omega}_1^+((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad := -\langle *\alpha_1, \alpha_2\rangle_{L^2} - \langle *\eta_1,\eta_2\rangle_{L^2}, \\ \widetilde{\omega}_2^+((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad := \langle \eta_1, \alpha_2\rangle_{L^2} - \langle \alpha_1,\eta_2\rangle_{L^2}, \\ \widetilde{\omega}_3^+((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad := -\langle *\eta_1, \alpha_2\rangle_{L^2} - \langle *\alpha_1,\eta_2\rangle_{L^2}, \end{cases} \tag{1.11}$$

for $(\alpha_1,\eta_1),(\alpha_2,\eta_2) \in \oplus^2\Omega^1(\mathfrak{g}_P)$. Note that $\widetilde{\omega}_2^- = \widetilde{\omega}_2^+$ and $\widetilde{\omega}_3^- = \widetilde{\omega}_3^+$. The structure $\{\widetilde{\omega}_1^-, \widetilde{\omega}_2^-, \widetilde{\omega}_3^-\}$ has a underlying quaternionic structure $\{J_1, J_2, J_3\}$ and the positive definite L^2-inner product $\langle\ ,\ \rangle_{+,+}$ where

$$\langle(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,+} := \langle \alpha_1,\alpha_2\rangle_{L^2} + \langle \eta_1,\eta_2\rangle_{L^2},$$

which is preserved by the S^1-action, and the structure $\{\widetilde{\omega}_1^+, \widetilde{\omega}_2^+, \widetilde{\omega}_3^+\}$ has a underlying split-quaternionic structure $\{E_1, E_2, E_3\}$ and the indefinite L^2-inner product $\langle\ ,\ \rangle_{+,-}$ where

$$\langle(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,-} := \langle \alpha_1,\alpha_2\rangle_{L^2} - \langle \eta_1,\eta_2\rangle_{L^2},$$

which is preserved by the S^1-action, respectively. Namely,

$$\begin{cases} \widetilde{\omega}_1^-((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad = \langle J_1(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,+}, \\ \widetilde{\omega}_2^-((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad = \langle J_2(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,+}, \\ \widetilde{\omega}_3^-((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad = \langle J_3(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,+}, \end{cases} \quad \text{and} \quad \begin{cases} \widetilde{\omega}_1^+((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad = \langle E_1(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,-}, \\ \widetilde{\omega}_2^+((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ \quad = \langle E_2(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,-}, \\ \widetilde{\omega}_3^+((\alpha_1,\eta_1),(\alpha_2,\eta_2)) \\ = \langle E_3(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,-}. \end{cases} \tag{1.12}$$

Here the complex structures J_1, J_2, J_3 and E_1 and para-complex structures E_2, E_3 on $\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ are defined by

$$\begin{cases} J_1(\alpha,\eta) := (-*\alpha, *\eta), \\ J_2(\alpha,\eta) := (\eta, -\alpha), \\ J_3(\alpha,\eta) := (-*\eta, -*\alpha), \end{cases} \quad \text{and} \quad \begin{cases} E_1(\alpha,\eta) := (-*\alpha, *\eta), \\ E_2(\alpha,\eta) := (\eta, \alpha), \\ E_3(\alpha,\eta) := (-*\eta, *\alpha), \end{cases} \tag{1.13}$$

and they satisfy the following conditions:

$$\begin{cases} J_1^2 = J_2^2 = J_3^2 = -\mathrm{Id}, \\ J_1J_2 = -J_2J_1 = J_3, \\ J_2J_3 = -J_3J_2 = J_1, \\ J_3J_1 = -J_1J_3 = J_2 \end{cases} \quad \text{and} \quad \begin{cases} E_1^2 = -\mathrm{Id}, E_2^2 = E_3^2 = \mathrm{Id}, \\ E_1E_2 = -E_2E_1 = E_3, \\ E_2E_3 = -E_3E_2 = -E_1, \\ E_3E_1 = -E_1E_3 = E_2. \end{cases}$$

Then the three corresponding moment maps for the action of $\mathcal{G}_P$ on $\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ relative to $\{\widetilde{\omega}_1^-, \widetilde{\omega}_2^-, \widetilde{\omega}_3^-\}$ and $\{\widetilde{\omega}_1^+, \widetilde{\omega}_2^+, \widetilde{\omega}_3^+\}$ exist and are given by

$$\begin{cases} \mu_1^-(A,\phi) = F(A) - \frac{1}{2}[\phi \wedge \phi], \\ \mu_2^-(A,\phi) = d_A * \phi, \\ \mu_3^-(A,\phi) = d_A\phi, \end{cases} \quad \text{and} \quad \begin{cases} \mu_1^+(A,\phi) = F(A) + \frac{1}{2}[\phi \wedge \phi], \\ \mu_2^+(A,\phi) = d_A * \phi, \\ \mu_3^+(A,\phi) = d_A\phi, \end{cases}$$

respectively. The solution spaces $\widetilde{\mathcal{M}}^{\pm}(M,G,P)$ can be expressed as

$$\widetilde{\mathcal{M}}^-(M,G,P) = \bigcap_{i=1}^{3} \{\mu_i^-\}^{-1}(0) \quad \text{and} \quad \widetilde{\mathcal{M}}^+(M,G,P) = \bigcap_{i=1}^{3} \{\mu_i^+\}^{-1}(0)$$

and thus moduli space $\mathcal{M}^{\pm}(M,G,P)$ is given by

$$\mathcal{M}^-(M,G,P) = \bigcap_{i=1}^{3} \{\mu_i^-\}^{-1}(0)/\mathcal{G}_P \quad \text{and} \quad \mathcal{M}^+(M,G,P) = \bigcap_{i=1}^{3} \{\mu_i^+\}^{-1}(0)/\mathcal{G}_P,$$

respectively. Via the description in terms of moment maps, $\{\widetilde{\omega}_1^-, \widetilde{\omega}_2^-, \widetilde{\omega}_3^-\}$ and $\{\widetilde{\omega}_1^+, \widetilde{\omega}_2^+, \widetilde{\omega}_3^+\}$ induce three closed 2-forms $\{\omega_1^-, \omega_2^-, \omega_3^-\}$ and $\{\omega_1^+, \omega_2^+, \omega_3^+\}$ on the moduli spaces $\mathcal{M}^-(M,G,P)$ and $\mathcal{M}^+(M,G,P)$, respectively.

For each solution $(A,\phi) \in \widetilde{\mathcal{M}}^-(M,G,P)$, let denote the tangent space of the gauge orbit at (A,ϕ) and its orthogonal complement with respect to $\langle\ ,\ \rangle_{+,+}$ by

$$\begin{aligned} {}_{(A,\phi)}(\mathcal{G}) &:= T_{(A,\phi)}\mathcal{G}^*(A,\phi) = \mathrm{Im}(d^0_{(A,\phi)}), \\ T^{\perp}_{(A,\phi)}(\mathcal{G}) &:= \{X \in \mathrm{Ker}(d^{1,-}_{(A,\phi)}) \mid \langle X, Y\rangle_{+,+} = 0 \text{ for all } Y \in T_{(A,\phi)}(\mathcal{G})\}, \end{aligned}$$

respectively. Similarly, for each solution $(A,\phi) \in \widetilde{\mathcal{M}}^+(M,G,P)$, let denote the tangent space of the gauge orbit at (A,ϕ) and its orthogonal complement with respect to $\langle\ ,\ \rangle_{+,-}$ by

$$\begin{aligned} T_{(A,\phi)}(\mathcal{G}) &:= T_{(A,\phi)}\mathcal{G}^*(A,\phi) = \mathrm{Im}(d^0_{(A,\phi)}), \\ T^{\vee}_{(A,\phi)}(\mathcal{G}) &:= \{X \in \mathrm{Ker}(d^{1,+}_{(A,\phi)}) \mid \langle X, Y\rangle_{+,-} = 0 \text{ for all } Y \in T_{(A,\phi)}(\mathcal{G})\}, \end{aligned}$$

respectively. First we obtain the following lemma.

Lemma 1.3. (i) *For each solution $(A,\phi) \in \widetilde{\mathcal{M}}^-(M,G,P)$ to the equations $(*)^-_G$,*

$$T_{(A,\phi)}(\mathcal{G}) \cap T^{\perp}_{(A,\phi)}(\mathcal{G}) = d^0{}_{(A,\phi)}(\mathrm{Ker}\,\mathcal{J}^-_{(A,\phi)}) = 0.$$

(ii) *For each solution $(A,\phi) \in \widetilde{\mathcal{M}}^+(M,G,P)$ to the equations $(*)^+_G$,*

$$T_{(A,\phi)}(\mathcal{G}) \cap T^{\vee}_{(A,\phi)}(\mathcal{G}) = d^0{}_{(A,\phi)}(\mathrm{Ker}\,\mathcal{J}^+_{(A,\phi)}).$$

Proof. For all $\xi, \widetilde{\xi} \in \Omega^0(\mathfrak{g}_P)$, we have

$$\begin{aligned} \langle (d_A\xi, [\phi,\xi]), (d_A\widetilde{\xi}, [\phi,\widetilde{\xi}])\rangle_{+,\pm} &= \langle d_A\xi, d_A\widetilde{\xi}\rangle_{L^2} \pm \langle [\phi,\xi], [\phi,\widetilde{\xi}]\rangle_{L^2} \\ &= \langle d_A^* d_A\xi, \widetilde{\xi}\rangle_{L^2} \pm \langle *[[\phi,\xi] \wedge *\phi], \widetilde{\xi}\rangle_{L^2} \\ &= \langle \mathcal{J}^{\mp}_{(A,\phi)}(\xi), \widetilde{\xi}\rangle_{L^2}. \end{aligned}$$

Then the statements follow from these formulas. $\square$

Now let us return to the structures on the moduli spaces. We shall review in the case of $(A,\phi) \in \widetilde{\mathcal{M}}^-(M,G,P)$. It is easy to show that each null space $\mathrm{Ker}(\widetilde{\omega}_i^-)$ of $\widetilde{\omega}_i^-$ $(i=1,2,3)$ at (A,ϕ) is given by

$$\begin{aligned}
\mathrm{Ker}(\widetilde{\omega}_1^-)_{(A,\phi)} &= J_2\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\perp}_{(A,\phi)}(\mathcal{G})\Big) \\
&\qquad + J_3\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\perp}_{(A,\phi)}(\mathcal{G})\Big) + T_{(A,\phi)}(\mathcal{G}), \\
\mathrm{Ker}(\widetilde{\omega}_2^-)_{(A,\phi)} &= J_3\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\perp}_{(A,\phi)}(\mathcal{G})\Big) \\
&\qquad + J_1\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\perp}_{(A,\phi)}(\mathcal{G})\Big) + T_{(A,\phi)}(\mathcal{G}), \\
\mathrm{Ker}(\widetilde{\omega}_3^-)_{(A,\phi)} &= J_1\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\perp}_{(A,\phi)}(\mathcal{G})\Big) \\
&\qquad + J_2\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\perp}_{(A,\phi)}(\mathcal{G})\Big) + T_{(A,\phi)}(\mathcal{G}).
\end{aligned}$$

Because of Lemma 1.3(i), we see that

$$\mathrm{Ker}(\widetilde{\omega}_1^-)_{(A,\phi)} = \mathrm{Ker}(\widetilde{\omega}_2^-)_{(A,\phi)} = \mathrm{Ker}(\widetilde{\omega}_3^-)_{(A,\phi)} = T_{(A,\phi)}(\mathcal{G}),$$

and hence we obtain that each null space of ω_i^- $(i=1,2,3)$ at $[(A,\phi)]$ is

$$\mathrm{Ker}(\omega_i^-)_{[(A,\phi)]} = \mathrm{Ker}(\widetilde{\omega}_i^-)_{(A,\phi)}/T_{(A,\phi)}(\mathcal{G}) = 0.$$

It means that $\{\omega_1^-, \omega_2^-, \omega_3^-\}$ becomes nondegenerate closed 2-forms on $\mathcal{M}^-(M,G,P)$. By combining three complex structures induced from J_1, J_2, J_3 and the L^2-inner product with them, we can endow a *hyper-Kähler structure* on the moduli space $\mathcal{M}^-(M,G,P)$ ([18, 12]).

In contrast, we obtain the following in the case of $(A,\phi) \in \widetilde{\mathcal{M}}^+(M,G,P)$.

Lemma 1.4 ([16, 23, 24])**.** *Each null space* $\mathrm{Ker}(\widetilde{\omega}_i^+)$ *of* $\widetilde{\omega}_i^+$ $(i=1,2,3)$ *at* $(A,\phi) \in \widetilde{\mathcal{M}}^+(M,G,P)$ *is given by*

$$\mathrm{Ker}(\widetilde{\omega}_1^+)_{(A,\phi)} = E_2\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\vee}_{(A,\phi)}(\mathcal{G})\Big) + E_3\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\vee}_{(A,\phi)}(\mathcal{G})\Big) + T_{(A,\phi)}(\mathcal{G}), \tag{1.14}$$

$$\mathrm{Ker}(\widetilde{\omega}_2^+)_{(A,\phi)} = E_3\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\vee}_{(A,\phi)}(\mathcal{G})\Big) + E_1\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\vee}_{(A,\phi)}(\mathcal{G})\Big) + T_{(A,\phi)}(\mathcal{G}), \tag{1.15}$$

$$\mathrm{Ker}(\widetilde{\omega}_3^+)_{(A,\phi)} = E_1\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\vee}_{(A,\phi)}(\mathcal{G})\Big) + E_2\Big(T_{(A,\phi)}(\mathcal{G}) \cap T^{\vee}_{(A,\phi)}(\mathcal{G})\Big) + T_{(A,\phi)}(\mathcal{G}). \tag{1.16}$$

Hence each null space of ω_i^+ $(i=1,2,3)$ *at* $[(A,\phi)] \in \mathcal{M}^+(M,G,P)$ *is given by*

$$\mathrm{Ker}(\omega_i^+)_{[(A,\phi)]} = \mathrm{Ker}(\widetilde{\omega}_i^+)_{(A,\phi)}/T_{(A,\phi)}(\mathcal{G}). \tag{1.17}$$

Before showing Lemma 1.4, we shall define another three complex structures $\{F_1, F_2, F_3\}$ on $\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ by

$$\begin{cases} F_1(\alpha,\eta) := (-*\alpha, -*\eta), \\ F_2(\alpha,\eta) := (\eta, -\alpha), \\ F_3(\alpha,\eta) := (-*\eta, -*\alpha). \end{cases} \tag{1.18}$$

They satisfy

$$\begin{cases} F_1^2 = F_2^2 = F_3^2 = -\mathrm{Id}, \\ F_1F_2 = F_2F_1 = E_3 \neq F_3, \\ F_2F_3 = -F_3F_2 = E_1 \neq F_1, \\ F_3F_1 = F_1F_3 = -E_2 \neq F_2 \end{cases} \tag{1.19}$$

and preserve the Riemannian metric $\langle\ ,\ \rangle_{+,+}$. Then we have the another expressions of $\{\tilde{\omega}_1^+, \tilde{\omega}_2^+, \tilde{\omega}_3^+\}$ as

$$\begin{cases} \tilde{\omega}_1^+((\alpha_1,\eta_1),(\alpha_2,\eta_2)) = \langle F_1(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,+}, \\ \tilde{\omega}_2^+((\alpha_1,\eta_1),(\alpha_2,\eta_2)) = \langle F_2(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,+}, \\ \tilde{\omega}_3^+((\alpha_1,\eta_1),(\alpha_2,\eta_2)) = \langle F_3(\alpha_1,\eta_1),(\alpha_2,\eta_2)\rangle_{+,+}. \end{cases} \tag{1.20}$$

By the simple computation, we have the following lemma which is convenient to prove Lemma 1.4 and Proposition 3.1.

Lemma 1.5. *Let $(A,\phi) \in \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$. Suppose that $(\alpha,\eta) \in \oplus^2\Omega^1(\mathfrak{g}_P)$ is of the form*

$$(\alpha,\eta) = d^0_{(A,\phi)}(\xi_1) + E_2(d^0_{(A,\phi)}(\xi_2)) + E_3(d^0_{(A,\phi)}(\xi_3)) \tag{1.21}$$

for some $\xi_1, \xi_2, \xi_3 \in \Omega^0(\mathfrak{g}_P)$. Then we have

$$\begin{cases} d_A\alpha + [\phi\wedge\eta] = [F(A) + \frac{1}{2}[\phi\wedge\phi], \xi_1] + [d_A\phi, \xi_2] + [d_A*\phi, \xi_3], \\ d_A*\eta + [*\phi\wedge\alpha] = [d_A*\phi, \xi_1] - *\mathcal{J}^+_{(A,\phi)}(\xi_2) - [F(A) + \frac{1}{2}[\phi\wedge\phi], \xi_3], \\ d_A\eta + [\phi\wedge\alpha] = [d_A\phi, \xi_1] + [F(A) + \frac{1}{2}[\phi\wedge\phi], \xi_2] - *\mathcal{J}^+_{(A,\phi)}(\xi_3). \end{cases}$$

In particular, if (A,ϕ) is a solution to the equations $(*)^+_G$, then we obtain

$$\begin{cases} d_A\alpha + [\phi\wedge\eta] \equiv 0, \\ d_A*\eta + [*\phi\wedge\alpha] = -*\mathcal{J}^+_{(A,\phi)}(\xi_2), \\ d_A\eta + [\phi\wedge\alpha] = -*\mathcal{J}^+_{(A,\phi)}(\xi_3). \end{cases} \tag{1.22}$$

Proof of Lemma1.4. Suppose that $(A,\phi) \in \widetilde{\mathcal{M}}^+(M,G,P)$ is a solution to the equations $(*)^+_G$. It is sufficient to show the equation (1.14), because we can show similarily the equations (1.15) and (1.16). We note that

$$\begin{aligned} \mathrm{Ker}(\tilde{\omega}_1^+)_{(A,\phi)} &= \{X \in T_{(A,\phi)}(\widetilde{\mathcal{M}}^+(M,G,P)) \mid \langle F_1X, Y\rangle_{+,+} = 0 \text{ for all } Y \in T_{(A,\phi)}(\mathcal{G})\} \\ &= \{X \in \oplus^2\Omega^1(\mathfrak{g}_P) \mid d^{1,+}_{(A,\phi)}X = 0 \text{ and } F_1X \in \mathrm{Im}(d^{1,+}_{(A,\phi)})^*\}. \end{aligned}$$

Suppose that $X =: (\alpha,\eta) \in \mathrm{Ker}(\widetilde{\omega}_1^+)_{(A,\phi)}$. By Lemma 1.2, there must exists $(u_1,u_2,u_3) \in \oplus^3\Omega^2(\mathfrak{g}_P)$ such that

$$F_1(\alpha,\eta) = F_1(d^0_{(A,\phi)}(*u_1)) + F_2(d^0_{(A,\phi)}(*u_2)) + F_3(d^0_{(A,\phi)}(*u_3)).$$

Then by using (1.19),

$$(\alpha,\eta) = d^0_{(A,\phi)}(*u_1) - E_3(d^0_{(A,\phi)}(*u_2)) + E_2(d^0_{(A,\phi)}(*u_3)).$$

By replacing $\xi_1 = *u_1$, $\xi_2 = *u_3$ and $\xi_3 = - * u_2$ in Lemma 1.5, we have

$$\begin{cases} d_A\alpha + [\phi \wedge \eta] \equiv 0, \\ d_A * \eta + [*\phi \wedge \alpha] = - * \mathcal{J}^+_{(A,\phi)}(*u_3), \\ d_A\eta + [\phi \wedge \alpha] = *\mathcal{J}^+_{(A,\phi)}(*u_2). \end{cases} \tag{1.23}$$

Since (α,η) must satisfy the linearized equations (1.4) and by using Lemma 1.3(ii), we conclude that (α,η) belongs to the right-hand side of (1.14).

Conversely, assume that (α,η) is an arbitrary element of the right-hand side of (1.14). Our claim is that

$$\widetilde{\omega}_1^+((\alpha,\eta),(\hat{\alpha},\hat{\eta})) = \langle F_1(\alpha,\eta),(\hat{\alpha},\hat{\eta})\rangle_{+,+} = 0 \tag{1.24}$$

for each element $(\hat{\alpha},\hat{\eta})$ of $T_{(A,\phi)}(\widetilde{\mathcal{M}}^+(M,G,P))$. By the assumption, (α,η) must be written as

$$\begin{aligned}(\alpha,\eta) &= (d_A\xi_1,[\phi,\xi_1]) + E_2(d_A\xi_2,[\phi,\xi_2]) + E_3(d_A\xi_3,[\phi,\xi_3]) \\ &= (d_A\xi_1 + [\phi,\xi_2] - [*\phi,\xi_3], [\phi,\xi_1] + d_A\xi_2 + *d_A\xi_3)\end{aligned}$$

for an element ξ_1 and some Jacobi fields $\xi_2,\xi_3 \in \mathrm{Ker}\,\mathcal{J}^+_{(A,\phi)}$. By using the equations (1.23) again, we see that such (α,η) satisfies the linearized equations (1.4). Since

$$F_1(\alpha,\eta) = (- * d_A\xi_1 - [*\phi,\xi_2] - [\phi,\xi_3], -[*\phi,\xi_1] - *d_A\xi_2 + d_A\xi_3),$$

we compute

$$\begin{aligned}&\langle F_1(\alpha,\eta),(\hat{\alpha},\hat{\eta})\rangle_{+,+} \\ =&\langle - * d_A\xi_1,\hat{\alpha}\rangle_{L^2} + \langle -[*\phi,\xi_2],\hat{\alpha}\rangle_{L^2} + \langle -[\phi,\xi_3],\hat{\alpha}\rangle_{L^2} \\ &\quad + \langle -[*\phi,\xi_1],\hat{\eta}\rangle_{L^2} + \langle - * d_A\xi_2,\hat{\eta}\rangle_{L^2} + \langle d_A\xi_3,\hat{\eta}\rangle_{L^2} \\ =&\langle \xi_1, *(d_A\hat{\alpha} + [\phi \wedge \hat{\eta}])\rangle_{L^2} \\ &\quad + \langle \xi_2, *(d_A\hat{\eta} + [\phi \wedge \hat{\alpha}])\rangle_{L^2} + \langle \xi_3, - * (d_A * \hat{\eta} + [*\phi \wedge \hat{\alpha}])\rangle_{L^2} = 0.\end{aligned}$$

□

Finally, we mention that the relationship of the S^1-action (1.3) and the structures $\{\omega_1^-,\omega_2^-,\omega_3^-\}$ and $\{\omega_1^+,\omega_2^+,\omega_3^+\}$ are given as follows: For $\lambda \in S^1$,

$$\begin{cases}(\lambda^\sharp)^*\omega_1^\pm = \omega_1^\pm, \\ (\lambda^\sharp)^*\omega_2^\pm = \cos(2\pi t)\omega_2^\pm - \sin(2\pi t)\omega_3^\pm, \\ (\lambda^\sharp)^*\omega_3^\pm = \cos(2\pi t)\omega_3^\pm + \sin(2\pi t)\omega_2^\pm,\end{cases}$$

and $(\lambda^\sharp)^*(\omega_2^\pm + \sqrt{-1}\omega_3^\pm) = \lambda(\omega_2^\pm + \sqrt{-1}\omega_3^\pm)$.

1.5. **Hitchin maps.** We choose a basis $\{p_1, \cdots, p_k\}$ of the ring consisting of $\mathrm{Ad}G^{\mathbb{C}}$-invariant polynomials on $\mathfrak{g}^{\mathbb{C}}$. We denote by d_i the degree of the polynomial p_i. The Hitchin maps

$$p^{\pm} : \mathcal{M}^{\pm}(M, G, P) \longrightarrow \bigoplus_{i=1}^{k} H^0(M; K_M^{\otimes d_i}) \tag{1.25}$$

are defined by $p^{\pm}([(A, \phi)]) := (p_1(\phi'), \cdots, p_k(\phi'))$ for each $[(A, \phi)] \in \mathcal{M}^{\pm}(M, G, P)$, respectively. Here K_M denotes the canonical line bundle of M.

It is known that the map p^- is surjective ([13, 17]). On the other hand, is the map p^+ surjective? It is a fundamental problem on the existence of solutions to the gauge-theoretic equations $(*)^+_G$. Though the theory for harmonic maps of compact Riemann surfaces of genus greater than or equal to two into compact symmetric spaces is not a well-developed area yet, this problem can be considered as the first step on the research.

2. Energy Functions

We shall introduce the *energy functions* $\mathcal{E}^{\pm}$ on the moduli space $\mathcal{M}^{\pm}(M, G, P)$ by

$$\mathcal{E}^{\pm}([(A, \phi)]) := ||\phi||^2_{L^2}, \quad \text{for } [(A, \phi)] \in \mathcal{M}^{\pm}(M, G, P), \tag{2.1}$$

respectively.

2.1. **Weak compactness theorem.** For each $c \geq 0$, we set

$$\mathcal{M}^{\pm}_c(M, G, P) := \{[(A, \phi)] \in \mathcal{M}^{\pm}(M, G, P) \mid \mathcal{E}^{\pm}([A, \phi]) \leq c\}.$$

We shall review the results in [25]. In order to prove the *weak compactness theorem* (Theorem 2.2) on sequences of solutions to the equations $(*)^+_G$ with bounded energy, we showed the *removability theorem* (Theorem 2.1) for isolated singularities, which are analogue to Sacks-Uhlenbeck's theorem.

Theorem 2.1 ([25]). *Let $P = D \times G$ be a trivial principal G-bundle over $D = \{z = x + \sqrt{-1}y \mid |z| < 1\}$. Let $(A, \phi) \in \mathcal{A}_{P|D\setminus\{0\}} \times \Omega^1(\mathfrak{g}_{P|D\setminus\{0\}})$ be a solution to the equations $(*)^+_G$ defined on $D \setminus \{0\}$ such that $A - \phi$ is a trivial flat connection with respect to $P = D \times G$. Assume that*

$$\int_{D\setminus\{0\}} |\phi|^2 dx \wedge dy < \infty.$$

Then (A, ϕ) extends smoothly to the whole disk D.

Theorem 2.2 ([25]). *For each $0 \leq c < \infty$, any sequence $[(A_i, \phi_i)] \in \mathcal{M}^+_c(M, G, P)$ has a subsequence $[(A_j, \phi_j)] \in \mathcal{M}^+_c(M, G, P)$ and a point $[(A_\infty, \phi_\infty)] \in \mathcal{M}^+_c(M, G, P)$ such that the sequence (A_j, ϕ_j) converges to (A_∞, ϕ_∞) on each compact subset $K \subset M \setminus \{x_1, \cdots, x_s\}$ in the C^k-topology for each k. Moreover, at each singular point x_α, a nontrivial solution $(\tilde{A}_\alpha, \tilde{\phi}_\alpha) \in \mathcal{M}^+_c(S^2, G, S^2 \times G)$ appears as the phenomena of bubbling off.*

Since we see that $\lim_{j\to\infty} p^+([(A_j, \phi_j)]) = p^+([(A_\infty, \phi_\infty)])$, we obtain the following.

Corollary 2.3. *For each $c \geq 0$, the image $p^+(\mathcal{M}_c^+(M,G,P))$ is a compact subset of $\oplus_{i=1}^k H^0(M;K^{\otimes d_i})$.*

2.2. **Critical points of the energy functions.**

Proposition 2.4 ([12, 23, 24]). *The energy functions $\mathcal{E}^+$ and $\mathcal{E}^-$ are moment maps of the S^1-action on the moduli spaces $\mathcal{M}^+(M,G,P)$ and $\mathcal{M}^-(M,G,P)$ with respect to ω_1^+ and ω_1^-, respectively. That is,*

$$d\mathcal{E}^+(\cdot) = \omega_1^+(V^+,\cdot) \quad \text{and} \quad d\mathcal{E}^-(\cdot) = \omega_1^-(V^-,\cdot), \tag{2.2}$$

where V^+ and V^- are the vector fields on $\mathcal{M}^+(M,G,P)$ and $\mathcal{M}^-(M,G,P)$ induced by the S^1-actions, respectively.

Now, let

$$\begin{aligned}\mathcal{F}^\pm(M,G,P) :=&\{[(A,\phi)] \in \mathcal{M}^\pm(M,G,P) \mid \\ &\qquad\qquad \lambda^\sharp[(A,\phi)] = [(A,\phi)] \text{ for each } \lambda \in S^1\}\end{aligned}$$

denote the subsets of all points fixed by the S^1-actions on $\mathcal{M}^\pm(M,G,P)$, respectively, and

$$\mathcal{C}^\pm(M,G,P) := \{[(A,\phi)] \in \mathcal{M}^\pm(M,G,P) \mid \ [(A,\phi)] \text{ is a critical point of } \mathcal{E}^\pm\}$$

denote the subsets of all critical points of the energy function $\mathcal{E}^\pm$ on $\mathcal{M}^\pm(M,G,P)$, respectively. Then it follows from Proposition 2.4 that

$$\mathcal{F}^+(M,G,P) \subset \mathcal{C}^+(M,G,P) \text{ and } \mathcal{F}^-(M,G,P) \subset \mathcal{C}^-(M,G,P). \tag{2.3}$$

Moreover, we see that in $\mathcal{M}^-(M,G,P)$

$$\mathcal{F}^-(M,G,P) = \mathcal{C}^-(M,G,P), \tag{2.4}$$

because ω_1^- is nondegenerate at each point $[(A,\phi)] \in \mathcal{M}^-(M,G,P)$ ([12]).

In contrast, the degeneracy of ω_1^+ at each point $[(A,\phi)] \in \mathcal{M}^+(M,G,P)$ will cause the interesting phenomena as we shall prove in Theorem 4.1.

3. The Foliated Structure and Twisted Harmonic Maps

In this section, we shall discuss a kind of a foliated structure on the moduli space $\mathcal{M}^+(M,G,P)$.

Let P be a principal G-bundle over M. We embed G into the product Lie group $G \times G$ as a diagonal subgroup. Then we define a larger principal bundle $\bar{P}$

$$\bar{P} := P \times_G (G \times G)$$

over M with structure group $G \times G$ by the fiber bundle associated with P with respect to the left action of G on $G \times G$. It contains P as a reduced subbundle.

Fix a solution $[(A,\phi)] \in \mathcal{M}^+(M,G,P)$. Then we can associate $[(A,\phi)]$ to a holonomy homomorphism $\rho_{(A,\phi)} : \pi_1(M) \longrightarrow G \times G$ of the flat connection

$$(A_{-1}, A_1) = (A+\phi, A-\phi) \in \mathcal{A}_{\bar{P}}$$

on principal bundle $(G \times G)$-bundle $\bar{P}$. Here $\pi_1(M)$ is the fundamental group of M with some reference point $x_0 \in M$. More generally, we have a kind of a foliated structure

$$\mathcal{M}^+(M, G, P) = \coprod_{[\rho]} \mathcal{M}^+_{[\rho]}(M, G, P), \tag{3.1}$$

where $[\rho] \in \mathrm{Hom}(\pi_1(M), G \times G)/(G \times G)$ is a conjugacy class of a homomorphism of $\pi_1(M)$ into $G \times G$. Here we define

$$\mathcal{M}^+_{[\rho]}(M, G, P) := \{[(A, \phi)] \in \mathcal{M}^+(M, G, P) \mid \rho_{(A,\phi)} \in [\rho]\}. \tag{3.2}$$

We shall give another characterization of $\mathcal{M}^+_{[\rho]}(M, G, P)$. Fix a solution $[(A, \phi)] \in \mathcal{M}^+(M, G, P)$ again. Let $\widetilde{M} \longrightarrow M$ be the universal covering of M, which can be considered as a principal bundle with structure group $\pi_1(M)$. The fundamental group $\pi_1(M)$ acts on $(G \times G)/G \cong G$ via the holonomy homomorphism $\rho_{(A,\phi)} : \pi_1(M) \longrightarrow G \times G$. Then we can construct a flat fiber bundle

$$E_\rho := \widetilde{M} \times_{\pi_1(M)} ((G \times G)/G)$$

with the fiber $(G \times G)/G \cong G$ over M. Note that E_ρ depends on $[(A, \phi)]$. We know that a section s of E_ρ is a *ρ-twisted map* of $\widetilde{M}$ into $(G \times G)/G \cong G$, that is, a map $s : \widetilde{M} \longrightarrow (G \times G)/G \cong G$ with the condition

$$s(\tilde{x} \cdot \gamma) = \rho(\gamma)^{-1} s(\tilde{x}) \quad \text{for each } \tilde{x} \in \widetilde{M} \text{ and each } \gamma \in \pi_1(M). \tag{3.3}$$

Then we can show that $\mathcal{M}^+_{[\rho]}(M, G, P)$ coincides with the moduli space of *harmonic sections* of E_ρ or *ρ-twisted harmonic maps* of $\widetilde{M}$ into G:

$$\begin{aligned}\mathcal{M}^+_{[\rho]}(M, G, P) &\cong \{ \text{ harmonic sections of } E_\rho\}/\cong \\ &\cong \{ \rho\text{-twisted harmonic maps of } \widetilde{M} \text{ into } G\}/(G \times G).\end{aligned}$$

See also [7]. Note that when ρ is a trivial homomorphism, $\mathcal{M}^+_{[\rho]}(M, G, P)$ coincides with the moduli space $\mathcal{M}_{HM}$ of harmonic maps of M into G.

Next we shall give the description of the formal tangent space of $\mathcal{M}^+_{[\rho]}(M, G, P)$.

Proposition 3.1. *Let $[(A, \phi)] \in \mathcal{M}^+_{[\rho]}(M, G, P)$. Then the formal tangent space $T_{[(A,\phi)]}(\mathcal{M}^+_{[\rho]}(M, G, P))$ at $[(A, \phi)]$ consists of elements $(\alpha, \eta) \in \oplus^2 \Omega^1(\mathfrak{g}_P)$ of the form*

$$(\alpha, \eta) = \left(\frac{1}{2}(d_A \xi + [\phi, \xi_J]), \frac{1}{2}(d_A \xi_J + [\phi, \xi])\right) \tag{3.4}$$

for some Jacobi field $\xi_J \in \mathrm{Ker}\, \mathcal{J}^+_{(A,\phi)}$ and an element $\xi \in \Omega^0(\mathfrak{g}_P)$ with the condition

$$* P^0_{(A,\phi)}(\xi) + 2[*\phi \wedge d_A \xi_J] = 0. \tag{3.5}$$

Here the operator $P^0_{(A,\phi)}$ is defined by (1.10).

Proof. Let $[(A^t,\phi^t)] \in \mathcal{M}^+_{[\rho]}(M,G,P)$ be a deformation of $[(A^0,\phi^0)] = [(A,\phi)]$ and set

$$(\alpha,\eta) = \frac{d}{dt}(A^t,\phi^t)|_{t=0} \in \Omega^1(\mathfrak{g}_P) \oplus \Omega^1(\mathfrak{g}_P),$$

which is its infinitesimal deformation at $t=0$. Since two flat connections $(A^t + \phi^t, A^t - \phi^t)$ and (A_{-1}, A_1) in $\bar{P}$ belong to the same conjugacy class of their holonomy homomorphisms, each $(A^t+\phi^t, A^t-\phi^t)$ is gauge-equivalent to (A_{-1},A_1) in $\bar{P}$, that is, there exist a gauge transformation $\bar{g}^t := (g_1^t, g_2^t) \in \mathcal{G}_{\bar{P}}$ on $\bar{P}$ such that

$$(A^t+\phi^t, A^t-\phi^t) = (\bar{g}^t)^*(A_{-1},A_1) \tag{3.6}$$

for each t. Set

$$(s_1,s_2) = \frac{d}{dt}\bar{g}^t|_{t=0} \in \Omega^0((\mathfrak{g}\oplus\mathfrak{g})_{\bar{P}}) = \Omega^0(\mathfrak{g}_P)\oplus\Omega^0(\mathfrak{g}_P).$$

By taking the derivative of the both-hand of the equation (3.6), we have

$$\alpha+\eta = d_{A_{-1}}s_1, \quad \alpha-\eta = d_{A_1}s_2. \tag{3.7}$$

By putting $\xi := s_1+s_2$ and $\xi_J := s_1 - s_2$, we have

$$\alpha = \frac{1}{2}(d_A\xi + [\phi,\xi_J]), \quad \eta = \frac{1}{2}(d_A\xi_J + [\phi,\xi]).$$

We note that such (α,η) just becomes of the form of 1.21, by replacing $\xi_1 = \frac{1}{2}\xi$, $\xi_2 = \frac{1}{2}\xi_J$ and $\xi_3 = 0$. Since $(\alpha,\eta) \in T_{[(A,\phi)]}(\mathcal{M}^+_{[\rho]}(M,G,P)) \subset T_{[(A,\phi)]}(\mathcal{M}^+(M,G,P))$, (α,η) must satisfy the linearized equations $d^{1,+}_{(A,\phi)}(\alpha,\eta)=0$ and moreover the equation $(d^0{}_{(A,\phi)})^*(\alpha,\eta) = 0$. By the equations (1.22), we have

$$\begin{cases} d_A\alpha + [\phi\wedge\eta] \equiv 0, \\ d_A * \eta + [*\phi\wedge\alpha] = - * \mathcal{J}^+_{(A,\phi)}(\xi_J), \\ d_A\eta + [\phi\wedge\alpha] \equiv 0, \end{cases} \tag{3.8}$$

and moreover

$$d_A * \alpha + [\phi \wedge *\eta] = *P^0_{(A,\phi)}(\xi) + 2[*\phi\wedge d_A\xi_J]. \tag{3.9}$$

Hence we obtain the statement. $\square$

We obtain the relationship between $\mathcal{M}^+_{[\rho]}(M,G,P)$ and two structures ω_1^+ and ω_3^+ as follows.

Proposition 3.2 (See also [16]). *Each $\mathcal{M}^+_{[\rho]}(M,G,P)$ is an integral manifold of $\mathrm{Ker}(\omega_1^+)$ and $\mathrm{Ker}(\omega_3^+)$ in the sense of*

$$T_{[(A,\phi)]}(\mathcal{M}^+_{[\rho]}(M,G,P)) \subset \mathrm{Ker}(\omega_1^+)_{[(A,\phi)]} \cap \mathrm{Ker}(\omega_3^+)_{[(A,\phi)]}. \tag{3.10}$$

Proof. Let $(\alpha,\eta)\in T_{[(A,\phi)]}(\mathcal{M}^+_{[\rho]}(M,G,P))$ which is of the form of (3.4) in Proposition 3.1. In terms of the para-complex structure E_2 in (1.13), we can rewrite (α,η) as

$$\begin{aligned}(\alpha,\eta) &= \Big(\frac{1}{2}(d_A\xi+[\phi,\xi_J]),\frac{1}{2}(d_A\xi_J+[\phi,\xi])\Big)\\ &= \frac{1}{2}\{d^0_{(A,\phi)}(\xi)+E_2(d^0_{(A,\phi)}(\xi_J))\}.\end{aligned} \tag{3.11}$$

It follows from Lemma 1.3(ii) that

$$(\alpha,\eta)\in T_{(A,\phi)}(\mathcal{G})+E_2\Big(T_{(A,\phi)}(\mathcal{G})\cap T^\vee_{(A,\phi)}(\mathcal{G})\Big).$$

By vietue of the equations (1.14) and (1.16), we obtain

$$(\alpha,\eta)\in \mathrm{Ker}(\omega_1^+)_{[(A,\phi)]}\cap \mathrm{Ker}(\omega_3^+)_{[(A,\phi)]}.$$

□

Remark 3.1. We can define maps

$$\tilde{F}_{\pm1}:\widetilde{\mathcal{M}}^+(M,G,P)\longrightarrow \mathcal{A}^{fl}_P$$

as $\tilde{F}_{\pm1}(A,\phi):=A\mp\phi$, where we set

$$\mathcal{A}^{fl}_P:=\{A\in\mathcal{A}_P \mid \ A \text{ is a flat connection on } P\}.$$

If we denote by ω the standard Kähler form on the moduli space $\mathcal{M}^{fl}(M,G,P):=\mathcal{A}^{fl}_P/\mathcal{G}_P$ of flat connections defined as in Subsection 1.4, then maps

$$F_{\pm1}:\mathcal{M}^+(M,G,P)\longrightarrow\mathcal{M}^{fl}(M,G,P)$$

induced by the maps $\tilde{F}_{\pm1}$ satisfy

$${F_{-1}}^*\omega=\omega_1^++\omega_3^+ \quad\text{ and }\quad {F_1}^*\omega=\omega_1^+-\omega_3^+.$$

For each $\rho=(\rho_A,\rho_B)\in\mathrm{Hom}(\pi_1(M),G\times G)$ with corresponding flat connections $A,B\in\mathcal{A}^{fl}_P$, we have

$$\mathcal{M}^+_{[\rho]}(M,G,P)=\{F_{-1}\}^{-1}([A])\cap\{F_1\}^{-1}([B]). \tag{3.12}$$

On the other hand, the relationship between $\mathcal{M}^+_{[\rho]}(M,G,P)$ and another structure ω_2^+ is given as follows.

Proposition 3.3. *Each $\mathcal{M}^+_{[\rho]}(M,G,P)$ is isotropic in the moduli space $\mathcal{M}^+(M,G,P)$ with respect to ω_2^+, that is,*

$$\omega_2^+|_{\mathcal{M}^+_{[\rho]}(M,G,P)}=0. \tag{3.13}$$

Proof. Let $(\alpha,\eta),(\hat\alpha,\hat\eta)\in T_{(A,\phi)}(\mathcal{M}^+_{[\rho]}(M,G,P))$, putting as

$$(\alpha,\eta)=\frac{1}{2}\{d^0_{(A,\phi)}(\xi)+E_2(d^0_{(A,\phi)}(\xi_J))\},\ (\hat\alpha,\hat\eta)=\frac{1}{2}\{d^0_{(A,\phi)}(\hat\xi)+E_2(d^0_{(A,\phi)}(\hat\xi_J))\},$$

where $\xi_J,\hat\xi_J\in\mathrm{Ker}\,\mathcal{J}^+_{(A,\phi)}$ and $\xi,\hat\xi\in\Omega^0(\mathfrak{g}_P)$ as Proposition 3.1. Then we note

$$\omega_2^+(E_2(d^0_{(A,\phi)}(\xi_J)),d^0_{(A,\phi)}(\hat\xi))=\langle d^0_{(A,\phi)}(\xi_J),d^0_{(A,\phi)}(\hat\xi)\rangle_{+,-}=0, \tag{3.14}$$

because of Lemma 1.3(ii). Then we compute

$$\begin{aligned}\omega_2^+((\alpha,\eta),(\hat\alpha,\hat\eta)) &= \frac{1}{4}\omega_2^+(E_2(d^0_{(A,\phi)}(\xi_J)), E_2(d^0_{(A,\phi)}(\hat\xi_J)))\\ &= \frac{1}{4}\langle d^0_{(A,\phi)}(\xi_J), E_2(d^0_{(A,\phi)}(\hat\xi_J))\rangle_{+,-}\\ &= \frac{1}{4}\omega_2^+(d^0_{(A,\phi)}(\hat\xi_J), d^0_{(A,\phi)}(\xi_J)) = 0,\end{aligned}$$

by using (3.14) again. □

4. Each Uniton Solution is a Critical Point of the Energy Function

We shall state our main theorem.

Theorem 4.1. *Each uniton solution $[(A,\phi)] \in \mathcal{M}^+(M,G,P)$ is a critical point of the energy function $\mathcal{E}^+$ on $\mathcal{M}^+(M,G,P)$, and hence we have*

$$\mathcal{F}^+(M,G,P) \subset \mathcal{U}(M,G,P) \subset \mathcal{C}^+(M,G,P) \tag{4.1}$$

in the moduli space $\mathcal{M}^+(M,G,P)$.

Remark 4.1. In Section 7 we shall show that in general $\mathcal{F}^+(M,G,P)$ does not coincide with $\mathcal{C}^+(M,G,P)$ and hence $\mathcal{U}(M,G,P)$ appear between them.

In order to prove Theorem 4.1, we shall show the following.

Lemma 4.2. *Each point $[(A,\phi)] \in \mathcal{M}^+(M,G,P)$ fixed by the S^1-action is a uniton solution.*

Proof. If $[(A,\phi)] \in \mathcal{M}^+(M,G,P)$ is a point fixed by S^1-action, then there exists a smooth one-parameter family of gauge transformations $g_t \in \mathcal{G}_P$ with $g_0 = \mathrm{Id}$ such that $(A,\phi_\lambda) = g_t^*(A,\phi)$ with $\lambda = e^{2\pi\sqrt{-1}t}$. It implies that $A_\lambda = g_t^* A_1$ for each $\lambda = e^{2\pi\sqrt{-1}t} \in S^1$. □

Proposition 4.3. *Let $[(A,\phi)] \in \mathcal{M}^+(M,G,P)$ be a uniton solution and $[\rho]$ a conjugacy class of $\rho_{(A,\phi)}$. Then the solution $\lambda^\sharp[(A,\phi)]$ obtained by S^1-action also belongs to $\mathcal{M}^+_{[\rho]}(M,G,P)$.*

Proof. Let $[(A,\phi)] \in \mathcal{U}(M,G,P)$ and $[\rho]$ a conjugacy class of $\rho_{(A,\phi)}$. If $A_\lambda = g_t^* A_1$, then we have

$$A_{-\lambda} = g^*_{t+\frac{1}{2}} A_1 = g^*_{t+\frac{1}{2}} (g^{-1}_{\frac{1}{2}})^* A_{-1} = (g^{-1}_{\frac{1}{2}} g_{t+\frac{1}{2}})^* A_{-1}.$$

If we define a gauge transformation $\bar h_t$ on $\bar P$ as $\bar h_t = (g^{-1}_{\frac{1}{2}} g_{t+\frac{1}{2}}, g_t)$, then we have $\bar h_t^*(A_{-1}, A_1) = (A_{-\lambda}, A_\lambda)$ for $\lambda = e^{2\pi\sqrt{-1}t}$. It means that $\lambda^\sharp[(A,\phi)] = [(A,\phi_\lambda)] \in \mathcal{M}^+_{[\rho]}(M,G,P)$. □

Therefore we obtain Theorem 4.1, by combining with Propositions 2.4, 3.2, 4.3 and Lemma 4.2.

5. Flag Transforms for Solutions

Uhlenbeck [29] introduced the "uniton transform" for harmonic maps of Riemann surfaces into $G = U(n)$, and more generally Burstall and Rawnsley [5] introduced the "flag transform" for harmonic maps of Riemann surfaces into compact Lie group G, which constructs a new harmonic map by multiplying a "flag factor" to a given harmonic map. In this section we shall show that the notion of the flag transform can be extended to the property of the transform on the moduli space $\mathcal{M}^+(M, G, P)$.

Let M be a compact Riemann surface, G a compact Lie group with Lie algebra $\mathfrak{g}$ and P a principal G-bundle over M. Let $\Gamma = \Gamma(\mathfrak{g})$ denote the set of all canonical elements of $\mathfrak{g}$ for parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$ (see [5]). Since the structure group G acts on Γ, we have an associated fiber bundle $\Gamma_P := P \times_G \Gamma$. Let $C^\infty(\Gamma_P)$ denote the space of all smooth sections of Γ_P. We can identify each smooth section Ξ of Γ_P with a smooth function Ξ on P with values in Γ satisfying $\Xi(ua) = \mathrm{Ad}(a)^{-1}\Xi(u)$ for each $u \in P$ and each $a \in G$. For each $x \in M$, we define the eigenspace of $\mathrm{ad}(\Xi(x))$ as

$$\mathfrak{g}_i(\Xi(x)) := \{X \in (\mathfrak{g}_P^{\mathbb{C}})_x \mid \mathrm{ad}(\Xi(x))X = \sqrt{-1}iX\}$$

and we let

$$\mathfrak{g}_P^{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i(\Xi)$$

the orthogonal direct sum decomposition into eigenspace subbundles of $\mathrm{ad}\Xi$. For each i, we denote by $\pi_i : \mathfrak{g}_P^{\mathbb{C}} \longrightarrow \mathfrak{g}_i(\Xi)$ the Hermitian projection. Then we have $\mathrm{Id} = \sum_{i \in \mathbb{Z}} \pi_i$.

Let $(A, \phi) \in \widetilde{\mathcal{M}}^+(M, G, P)$ be a given solution to the equations $(*)^+_G$. For each smooth section $\Xi \in C^\infty(\Gamma_P) \subset \Omega^0(\mathfrak{g}_P)$, we define a one-parameter subgroup

$$f_t := \exp(2\pi t\Xi) \in \mathcal{G}_P, \tag{5.1}$$

for each $t \in \mathbb{R}$. Since $f_t \in C(G)$ for each integer $t \in \mathbb{Z}$ and $C(G) \cap \{\exp(2\pi t\Xi)\}$ is a finite subgroup, we see that f_t is a constant function on G for each integer $t \in \mathbb{Z}$. Note that $\{f_t \mid t \in \mathbb{R}\}$ induces an S^1-action on $\mathcal{A}_P$ and $\Omega^1(\mathfrak{g}_P)$. Thus a loop of flat connections $f_t^* A_\lambda$ with $\lambda = e^{2\pi\sqrt{-1}t} \in S^1$ is well-defined, where A_λ denotes the circle of flat connections corresponding to the solution (A, ϕ).

Then we have the following theorem.

Theorem 5.1. *Let (A, ϕ) be a solution to the equations $(*)^+_G$. Then $f_t^* A_\lambda$ is of the form $f_t^* A_\lambda = B - \psi_\lambda$ for some solution (B, ψ) to the equations $(*)^+_G$ if and only if the smooth section $\Xi \in C^\infty(\Gamma_P)$ satisfies the following conditions :*

(i) *$d''_A\Xi$ has values in $\mathfrak{g}_{-1}(\Xi)$,*
(ii) *ϕ' has values in $\mathfrak{g}_0(\Xi) \oplus \mathfrak{g}_{-1}(\Xi)$.*

We shall give a new proof of Theorem 5.1 by using results on geometry of the based loop group $\Omega G^\star$. Here $G^\star := G/C(G) = \mathrm{Int}(\mathfrak{g})$ denotes the adjoint group with Lie algebra $\mathfrak{g}^\star$ which is the semi-simple part of $\mathfrak{g}$. We define an embedding

$$j : \Gamma \longrightarrow \Omega G^\star$$

by $j(\xi) := \gamma_\xi$, where $\gamma_\xi(\lambda) = \exp(2\pi t\xi)$. For each ξ, we let F a connected component of Γ containing ξ. The Lie group G acts transitively on F via the adjoint action and F is the orbit $\mathrm{Ad}(G)\xi$ of G through ξ and hence is diffeomorphic to a homogeneous space $G/C_G(\xi)$. Here $C_G(\xi)$ denotes the centralizer of G at ξ.

Suppose that a compact Lie group G acts transitively on F. Denote by $[\mathfrak{g}] := F\times\mathfrak{g}$ the trivial vector bundle over F with fiber $\mathfrak{g}$ equipped with the fiber metric induced from the $\mathrm{Ad}G$-invariant inner product of $\mathfrak{g}$. For each point $\xi \in F$, the action of G on F induces a linear map $\ell_\xi : \mathfrak{g} \longrightarrow T_\xi F$ and set $[\mathfrak{k}]_\xi = \mathrm{Ker}(\ell_\xi) \subset [\mathfrak{g}]_\xi$. Thus we define a vector subbundle $[\mathfrak{k}]$ of $[\mathfrak{g}]$. We take the subbundle $[\mathfrak{k}]^\perp$ orthogonally complemetary to $[\mathfrak{k}]$ so that we have an orthogonal direct sum $[\mathfrak{g}] = [\mathfrak{k}] \oplus [\mathfrak{k}]^\perp$. The subbundle $[\mathfrak{k}]^\perp$ can be identified with the tangent bundle TF of F. The 1-form θ_F on M with values in $\mathfrak{g}$ is defined as $\theta_F : TF \longrightarrow [\mathfrak{k}]^\perp \subset [\mathfrak{g}] \longrightarrow \mathfrak{g}$.

Let $\theta_{\Omega G^\star}$ denote the left-invariant Maurer-Cartan form of group $\Omega G^\star$, which is a 1-form on $\Omega G^\star$ with values in the based loop algebra $\Omega\mathfrak{g}^\star$. Then the differential dj of the map j is given by the following formula: For each $v \in T_\xi F$,

$$(\theta_{\Omega G^\star} \circ (dj)_\xi)(v) = (\mathrm{Exp}(-2\pi t\mathrm{ad}\xi) - \mathrm{Id})\theta_F(v) \tag{5.2}$$

(cf. [10]). Denote by $i : F \longrightarrow \mathfrak{g}$ the inclusion map of $F \subset \Gamma$ into $\mathfrak{g}$. We should note that $di : TF \longrightarrow \mathfrak{g}$ is given by

$$(di)_\xi(v) = [\theta_F(v), \xi]$$

for each $v \in T_\xi F$. For a map $f : P \longrightarrow F$, set $\Xi := i \circ f : P \longrightarrow \mathfrak{g}$. Then we have

$$d\Xi = [\theta_F \circ df, \Xi]. \tag{5.3}$$

Now, if we consider $f_\lambda := f_t$ with $\lambda = e^{2\pi\sqrt{-1}t} \in S^1$ in (5.1) as a section of $P \times_{\mathrm{Ad}} \Omega G^\star$, then we obtain

$$f_\lambda = j \circ f. \tag{5.4}$$

Then by the equation (5.4) we obtain the following formula.

$$\begin{aligned} f_t^* A_\lambda =& f_t^*\theta_{\Omega G^\star} + \mathrm{Ad}(f_t^{-1})A - \mathrm{Ad}(f_t^{-1})\phi_\lambda \\ =&(\theta_{\Omega G^\star} \circ (dj)_f)(df) + (\mathrm{Exp}(-2\pi t\mathrm{ad}\Xi))A - (\mathrm{Exp}(-2\pi t\mathrm{ad}\Xi))\phi_\lambda, \end{aligned}$$

by using the formulas (5.2) and (5.3),

$$\begin{aligned} &= \sum_i (\mathrm{Exp}(-2\pi t\mathrm{ad}\Xi) - \mathrm{Id})\pi_i(\theta_F \circ df) + \sum_i \lambda^{-i}\pi_i(A) - \sum_i \lambda^{-i}\pi_i(\phi_\lambda) \\ &= \sum_{i\neq 0} (\lambda^{-i} - 1)\pi_i(\theta_F \circ df) + \sum_i \lambda^{-i}\pi_i(A) - \sum_i \lambda^{-i-1}\pi_i(\phi') - \sum_i \lambda^{-i+1}\pi_i(\phi''). \end{aligned}$$

Thus the $(1,0)$-part of $f_t^* A_\lambda$ is given by

$$
\begin{aligned}
(f_t^* A_\lambda)^{1,0} =& \sum_{i\neq 0}(\lambda^{-i}-1)\pi_i(\theta_F \circ d'f) + \sum_i \lambda^{-i}\pi_i(A') - \sum_i \lambda^{-i-1}\pi_i(\phi') \\
=& -\sum_{i\neq 0}\pi_i(\theta_F\circ d'f) + \pi_0(A') - \pi_{-1}(\phi') \\
& \quad + \lambda^{-1}(\pi_1(\theta_F\circ d'f) + \pi_1(A') - \pi_0(\phi')) \\
& \quad + \sum_{i\neq 0,1}\lambda^{-i}(\pi_i(\theta_F\circ d'f) + \pi_i(A') - \pi_{i-1}(\phi')) \\
=& -\theta_F\circ d'f + \pi_0(\theta_F\circ d'f + A') - \pi_{-1}(\phi') \\
& \quad + \lambda^{-1}(\pi_1(\theta_F\circ d'f + A') - \pi_0(\phi')) \\
& \quad + \sum_{i\neq 0,1}\lambda^{-i}(\pi_i(\theta_F\circ d'f + A') - \pi_{i-1}(\phi')),
\end{aligned}
$$

where $d'f : T^{1,0}M \longrightarrow TF^{\mathbb{C}}$ denotes the restriction of the complex extension $df : TM^{\mathbb{C}} \longrightarrow TF^{\mathbb{C}}$ to the (1,0)-tangent space $T^{1,0}M$. Hence $(f_t^* A_\lambda)^{1,0}$ has the form $B' - \lambda^{-1}\psi'$ for some $(B,\psi) \in \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ if and only if

$$\pi_i(\theta_F\circ d'f + A') = 0 \quad \text{and} \quad \pi_{i-1}(\phi') = 0$$

for each $i \neq 0, 1$, or equivalently,

$$\theta_F\circ d'f + A' \in \mathfrak{g}_0(\Xi)\oplus\mathfrak{g}_1(\Xi), \tag{5.5}$$

$$\phi' \in \mathfrak{g}_0(\Xi)\oplus\mathfrak{g}_{-1}(\Xi). \tag{5.6}$$

The first condition (5.5) is equivalent to

$$\theta_F\circ d''f + A'' \in \mathfrak{g}_0(\Xi)\oplus\mathfrak{g}_{-1}(\Xi),$$

which is equivalent to

$$
\begin{aligned}
-d''_A\Xi &= -(d''\Xi + [A'',\Xi]) \\
&= [\Xi, \theta_F\circ d''f + A''] \in \mathfrak{g}_{-1}(\Xi),
\end{aligned} \tag{5.7}
$$

by using the formula (5.3). Therefore we obtain Theorem 5.1.

Thus we obtain a new solution $(B,\psi) \in \widetilde{\mathcal{M}}^+(M,G,P)$ such that $f_t^* A_\lambda = B - \psi_\lambda$. A section Ξ satisfying the conditions (i) and (ii) in Theorem 5.1 is called a *flag factor*. This transformation from a given solution (A,ϕ) to a new solution (B,ψ) defined by a flag factor Ξ is called the *flag transform* (or *uniton transform* in the case of unitary group $G = U(n)$ and F is a complex Grassmann manifold) of (A,ϕ) by Ξ. From the proof of Theorem 5.1, we also obtain the formulae

$$
\begin{cases}
B' = -\theta_F\circ d'f + \pi_0(\theta_F\circ d'f + A') - \pi_{-1}(\phi'), \\
\psi' = -\pi_1(\theta_F\circ d'f + A') + \pi_0(\phi').
\end{cases} \tag{5.8}
$$

Corollary 5.2. *Suppose that Ξ has height one, and hence F is an Hermitian symmetric space. The subbundle $\mathfrak{u}^\Xi$ of parabolic subalgebras of $\mathfrak{g}_P^{\mathbb{C}}$ relative to Ξ is a flag factor of (A,ϕ) if and only if $\mathfrak{u}^\Xi$ is a holomorphic subbundle relative to d''_A and invariant under the action of ϕ'.*

The following is a fundamental result.

Theorem 5.3 ([29, 23, 24]). *In the case of $G = U(n)$ any uniton solution on a compact Riemann surface with based gauges f_λ, i.e., $f_\lambda(z_0) = Id$ at the based point $z_0 \in M$, such that $f_\lambda^* A_1 = A_\lambda$ can be transformed into a trivial solution by a finite number of uniton transforms.*

Question 5.1. Without the assumption of having such based gauges f_λ, is it possible that any uniton solution on a compact Riemann surface be transformed into a trivial solution by a finite number of uniton transforms ?

We call the minimum m of such numbers of uniton transforms the *minimal uniton number* of uniton solution (A, ϕ), and then we say that the uniton solution is an *m-uniton solution.* Valli [31] showed the following.

Theorem 5.4 ([31]). *In the case of $G = U(n)$ any solution to the equations $(*)_G^+$ on a compact Riemann surface M can be transformed into a solution to the equations $(*)_G^+$ on M with semistable holomorphic Higgs pair by a finite number of uniton transforms.*

Question 5.2. How about in the case of any compact Lie group G?

Theorem 5.5 ([5], see also [26]). *In the case of a compact Lie group G of the type except for E_8, F_4, G_2, any solution to the equations $(*)_G^+$ on a Riemann shpere $M = S^2$ can be transformed into a trivial solution by a finite number of flag transforms.*

As one of important properties of this transform, we shall show that the values of the Hitchin map p^+ are preserved by flag transforms on the moduli space $\mathcal{M}^+(M, G, P)$.

Theorem 5.6. *If a given solution (A, ϕ) to the equations $(*)_G^+$ is transformed to a new solution (B, ψ) to the equations $(*)_G^+$ by a flag transform, then*

$$p^+([(A, \phi)]) = p^+([(B, \psi)]).$$

Lemma 5.7. *Suppose that $X_1 \in \mathfrak{g}_{i_1}(\xi), \cdots, X_{d_\alpha} \in \mathfrak{g}_{i_{d_\alpha}}(\xi)$. If $\sum_{a=1}^{d_\alpha} i_a \neq 0$, then we have*

$$p_\alpha(X_1, \cdots, X_{d_\alpha}) = 0.$$

Proof. Since p_α is invariant by the adjoint action of $\mathfrak{g}^{\mathbb{C}}$, we have

$$\sum_{a=1}^{d_\alpha} p_\alpha(X_1, \cdots, \mathrm{ad}\xi(X_a), \cdots, X_{d_\alpha}) = 0.$$

Thus we have

$$(\sum_{a=1}^{d_\alpha} \sqrt{-1} i_a) p_\alpha(X_1, \cdots, X_{d_\alpha}) = 0.$$

□

Proof of Theorem 5.6. By using Lemma 5.7, we compute

$$\begin{aligned}p_\alpha(\psi',\cdots,\psi') &= p_\alpha\Big(-\pi_1(\theta_F\circ d'f+A')+\pi_0(\phi'),\cdots,-\pi_1(\theta_F\circ d'f+A')+\pi_0(\phi')\Big)\\ &= p_\alpha(\pi_0(\phi'),\cdots,\pi_0(\phi')).\end{aligned}$$

On the other hand, since ϕ' has values in $\mathfrak{g}_0(\Xi)\oplus\mathfrak{g}_{-1}(\Xi)$, we have

$$\begin{aligned}p_\alpha(\phi',\cdots,\phi') &= p_\alpha(\pi_0(\phi')+\pi_1(\phi'),\cdots,\pi_0(\phi')+\pi_1(\phi'))\\ &= p_\alpha(\pi_0(\phi'),\cdots,\pi_0(\phi')).\end{aligned}$$

□

Corollary 5.8. *If $[(A,\phi)]\in\mathcal{M}^+(M,G,P)$ is a uniton solution which can be transformed into a trivial solution by a finite number of flag transforms, then we have $[(A,\phi)]\in(p^+)^{-1}(0)$.*

6. Subspaces in the Moduli Spaces

6.1. **Gauge-theoretic equations reduced to symmetric spaces.** In [25], generalizing the equations $(*)^{\pm}_{G}$, we introduced certain gauge-theoretic equations for each compact symmetric space G/K and discussed the local structure of the moduli spaces. Now let (G,K,σ) be a compact symmetric pair, more generally (G,K) be a pair of a compact Lie group G and its compact subgroup K associated with an orthogonal symmetric Lie algebra $(\mathfrak{g},\mathfrak{k},\sigma)$ (see [11]). Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ be the canonical decomposition of $\mathfrak{g}$ with respect to a symmetric space G/K. Let Q be a principal K-bundle over M. Define associated vector bundles over M as

$$\mathfrak{k}_Q:=Q\times_K\mathfrak{k},\quad \mathfrak{m}_Q:=Q\times_K\mathfrak{m},\quad \mathfrak{g}_Q:=Q\times_K\mathfrak{g}.$$

Then we have $\mathfrak{g}_Q=\mathfrak{k}_Q\oplus\mathfrak{m}_Q$. Note that $\mathfrak{g}_Q\cong\mathfrak{g}_P$ as vector bundles. Let $\mathcal{A}_Q$ denote the space of all smooth connections in Q. For each $i=0,1,2$, let $\Omega^i(\mathfrak{k}_Q)$ and $\Omega^i(\mathfrak{m}_Q)$ denote the vector spaces of all smooth i-forms with values in $\mathfrak{k}_Q$ and $\mathfrak{m}_Q$ on M, respectively. The space $\mathcal{A}_Q$ is an infinite dimensional affine space with the associated vector space $\Omega^1(\mathfrak{k}_Q)$.

For $(A,\phi)\in\mathcal{A}_Q\times\Omega^1(\mathfrak{m}_Q)$, we shall consider the gauge-theoretic equations

$$\begin{cases}F(A)\pm\dfrac{1}{2}[\phi\wedge\phi]=0,\\ d_A*\phi=d_A\phi=0.\end{cases}\qquad (*)^{\pm}_{G/K}$$

A solution (A,ϕ) to the equations $(*)^{\pm}_{G/K}$ with $\phi\not\equiv 0$ is called *nontrivial.* A connection $A\in\mathcal{A}_Q$ is called *irreducible* if the holonomy algebra of the connection A is equal to the Lie algebra $\mathfrak{k}$ of the structure group K. Set

$$\begin{aligned}\widetilde{\mathcal{M}}^{\pm}(M,G/K,Q):=&\{(A,\phi)\in\mathcal{A}_Q\times\Omega^1(\mathfrak{m}_Q)\mid\\ &\qquad (A,\phi)\text{ is a solution to the equations }(*)^{\pm}_{G/K}\}\end{aligned}$$

Denote by $\mathcal{K}_Q$ the group of gauge transformations of the principal K-bundle Q. The right action of $\mathcal{K}_Q$ on $\mathcal{A}_Q\times\Omega^1(\mathfrak{m}_Q)$ is defined by

$$k^*(A,\phi)=(k^*A,\mathrm{Ad}(k)^{-1}\phi).$$

The tangent space of the gauge orbit through (A, ϕ) is given by

$$\{(d_A\xi, [\phi, \xi]) \mid \xi \in \Omega^0(\mathfrak{k}_Q)\}.$$

The moduli spaces of all gauge equivalence classes $[(A, \phi)]$ of smooth solutions to the equations $(*)^{\pm}_{G/K}$ are defined by

$$\mathcal{M}^{\pm}(M, G/K, Q) := \widetilde{\mathcal{M}}^{\pm}(M, G/K, Q)/\mathcal{K}_Q.$$

For $(A, \phi) \in \mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q)$, we set

$$\Gamma_{(A,\phi)}(\mathcal{K}_Q) := \{k \in \mathcal{K}_Q \mid k^*(A, \phi) = (A, \phi)\}.$$

A point $(A, \phi) \in \mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q)$ is called *regular* if $\Gamma_{(A,\phi)}(\mathcal{K}_Q) = C(G) \cap K$. Set

$$\widetilde{\mathcal{M}}^{\pm,*}(M, G/K, Q) := \{(A, \phi) \in \widetilde{\mathcal{M}}^{\pm}(M, G/K, Q) \mid (A, \phi) \text{ is regular}\}$$

and define $\mathcal{M}^{\pm,*}(M, G/K, Q) := \widetilde{\mathcal{M}}^{\pm,*}(M, G/K, Q)/\mathcal{K}_Q$.

We showed [25] that the moduli space of harmonic maps of M into a compact symmetric space G/K by the left translations of G on G/K can be embedded in the above moduli spaces as follows:

$$\mathrm{Harm}(M, G/K)/G \subset \coprod_Q \mathcal{M}^{+}(M, G/K, Q),$$

where Q runs through all equivalence classes of principal K-bundles over M.

The local structure of the moduli spaces $\mathcal{M}^{\pm}(M, G/K, Q)$ were discussed in [25]. The linearized equations for the equations $(*)^{\pm}_{G/K}$ are

$$\begin{cases} d_A\alpha \pm [\phi \wedge \eta] = 0, \\ d_A * \eta + [*\phi \wedge \alpha] = 0, \\ d_A\eta + [\phi \wedge \alpha] = 0, \end{cases} \tag{6.1}$$

for $(\alpha, \eta) \in \Omega^1(\mathfrak{k}_Q) \oplus \Omega^1(\mathfrak{m}_Q)$. The elliptic cochain complex associated with deformations for solutions to the equations $(*)^{\pm}_{G/K}$ is

$$\begin{aligned} 0 \longrightarrow \Omega^0(\mathfrak{k}_Q) &\longrightarrow \Omega^1(\mathfrak{k}_Q) \oplus \Omega^1(\mathfrak{m}_Q) \\ &\longrightarrow \Omega^2(\mathfrak{k}_Q) \oplus \Omega^2(\mathfrak{m}_Q) \oplus \Omega^2(\mathfrak{m}_Q) \longrightarrow 0, \end{aligned}$$

where the linear differential operators

$$\begin{aligned} d_0 = d_{0(A,\phi)} &: \Omega^0(\mathfrak{k}_Q) \longrightarrow \Omega^1(\mathfrak{k}_Q) \oplus \Omega^1(\mathfrak{m}_Q), \\ d_1 = d_{1(A,\phi)} &: \Omega^1(\mathfrak{k}_Q) \oplus \Omega^1(\mathfrak{m}_Q) \longrightarrow \Omega^2(\mathfrak{k}_Q) \oplus \Omega^2(\mathfrak{m}_Q) \oplus \Omega^2(\mathfrak{m}_Q) \end{aligned}$$

are defined by

$$\begin{aligned} &d_{0(A,\phi)}(\xi) := (d_A\xi, [\phi, \xi]), \\ &d_{1(A,\phi)}(\alpha, \eta) := (d_A\alpha \pm [\phi \wedge \eta], d_A * \eta + [*\phi \wedge \alpha], d_A\eta + [\phi \wedge \alpha]). \end{aligned}$$

Set $P_0 := d_0^* d_0$, $P_1^{\pm} := d_0 d_0^* + d_1^* d_1$ and $P_2^{\pm} := d_1 d_1^*$. Then the i-th cohomology group $H^i_{(A,\phi)}(M, G/K, Q)$ of the cochain complex is isomorphic to $\mathrm{Ker}\, P_i$ for each

$i = 0, 1, 2$. In [25], we showed that for each $(A, \phi) \in \widetilde{\mathcal{M}}^{\pm}(M, G/K, Q)$, there exists a real analytic $\Gamma_{(A,\phi)}(\mathcal{K}_Q)$-equivariant map

$$\psi_{(A,\phi)} : U \longrightarrow H^2_{(A,\phi)}(M, G/K, Q)$$

defined on a neighborhood U of the origin in $H^1_{(A,\phi)}(M, G/K, Q)$ with $\psi_{(A,\phi)}(0) = 0$ and $(d\psi_{(A,\phi)})_0 = 0$ such that a neighborhood of $[(A, \phi)]$ in $\mathcal{M}^{\pm}(M, G/K, Q)$ is homeomorphic to $\psi^{-1}_{(A,\phi)}(0)/\Gamma_{(A,\phi)}(\mathcal{K}_Q)$. We can show that the image of $\psi_{(A,\phi)}$ is contained in a vector subspace of $H^2_{(A,\phi)}(M, G/K, Q)$ orthogonally complementary to $(\mathfrak{c}(\mathfrak{g}) \cap \mathfrak{k}) \oplus (\mathfrak{c}(\mathfrak{g}) \cap \mathfrak{m}) \oplus (\mathfrak{c}(\mathfrak{g}) \cap \mathfrak{m})$. We define subsets of $\mathcal{M}^{\pm}(M, G/K, Q)$ as follows:

$$\begin{aligned}
\mathcal{M}^{\pm,sm}(M, G/K, Q) &:= \{[(A, \phi)] \in \mathcal{M}^{\pm,*}(M, G/K, Q) \mid \\
&\qquad H^2_{(A,\phi)}(M, G/K, Q) = (\mathfrak{c}(\mathfrak{g}) \cap \mathfrak{k}) \oplus (\mathfrak{c}(\mathfrak{g}) \cap \mathfrak{m}) \oplus (\mathfrak{c}(\mathfrak{g}) \cap \mathfrak{m})\}, \\
\mathcal{M}^{\pm}(M, G/K, Q)^{irr} &:= \{[(A, \phi)] \in \mathcal{M}^{\pm}(M, G/K, Q) \mid A \text{ is irreducible}\}, \\
\mathcal{M}^{\pm,*}(M, G/K, Q)^{irr} &:= \mathcal{M}^{\pm,*}(M, G/K, Q) \bigcap \mathcal{M}^{\pm}(M, G/K, Q)^{irr}.
\end{aligned}$$

Then we see that a neighborhood of each point $[(A, \phi)]$ in $\mathcal{M}^{\pm,sm}(M, G/K, Q)$ becomes a smooth manifold of dimension equal to $\dim H^1_{(A,\phi)}(M, G/K, Q)$. If we specify the case of moduli space $\mathcal{M}^{-}(M, G/K, Q)$, then we obtain more results on the vanishing theorem for the second cohomology group and the manifold structure of the moduli space (cf. [27]).

The L^2-metric and a complex structure on $\mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q)$ are defined by

$$\langle (\alpha_1, \eta_1), (\alpha_2, \eta_2) \rangle_{+,+} := \langle \alpha_1, \alpha_2 \rangle_{L^2} + \langle \eta_1, \eta_2 \rangle_{L^2},$$
$$J(\alpha, \eta) := (- * \alpha, * \eta)$$

for each $(\alpha_1, \eta_1), (\alpha_2, \eta_2), (\alpha, \eta) \in \Omega^1(\mathfrak{k}_Q) \oplus \Omega^1(\mathfrak{m}_Q)$. If we define a Kähler form $\widetilde{\omega}^-$ on $\mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q)$ by

$$\widetilde{\omega}^-((\alpha_1, \eta_1), (\alpha_2, \eta_2)) := \langle J(\alpha_1, \eta_1), (\alpha_2, \eta_2) \rangle_{+,+},$$

then it induces a Kähler structure ω^- on moduli space $\mathcal{M}^-(M, G/K, Q)$ ([27]). On the other hand, using the indefinite L^2-metric

$$\langle (\alpha_1, \eta_1), (\alpha_2, \eta_2) \rangle_{+,-} := \langle \alpha_1, \alpha_2 \rangle_{L^2} - \langle \eta_1, \eta_2 \rangle_{L^2}$$

for each $(\alpha_1, \eta_1), (\alpha_2, \eta_2) \in \Omega^1(\mathfrak{k}_Q) \oplus \Omega^1(\mathfrak{m}_Q)$, and a complex structure $E = J$ on $\mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q)$, we define another Kähler form $\widetilde{\omega}^+$ on $\mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q)$ by

$$\widetilde{\omega}^+((\alpha_1, \eta_1), (\alpha_2, \eta_2)) := \langle E(\alpha_1, \eta_1), (\alpha_2, \eta_2) \rangle_{+,-}$$

Then it induces a presymplectic structure ω^+ on moduli space $\mathcal{M}^+(M, G/K, Q)$.

6.2. **Maps between moduli spaces.** We assume that ρ is a Lie group homomorphism from a compact Lie group G' to another compact Lie group G and $P' \longrightarrow P$ is a bundle homomorphism from a principal G'-bundle P' over M to another principal G-bundle P over M. We have the induced map

$$\mathcal{A}_{P'} \times \Omega^1(\mathfrak{g}_{P'}) \longrightarrow \mathcal{A}_P \times \Omega^1(\mathfrak{g}_P).$$

This map induces maps

$$\tilde{i} = \tilde{i}_{P'} : \widetilde{\mathcal{M}}^{\pm}(M, G', P') \longrightarrow \widetilde{\mathcal{M}}^{\pm}(M, G, P),$$

which are equivariant under the actions of $\mathcal{G}_{P'}$ and $\mathcal{G}_P$, and hence maps between the moduli spaces

$$i = i_{P'} : \mathcal{M}^{\pm}(M, G', P') \longrightarrow \mathcal{M}^{\pm}(M, G, P). \tag{6.2}$$

Let (G, K, σ) be a compact symmetric pair. A subbundle Q of P with a structure group K which is a compact subgroup of G is called a *reduced subbundle* of the structure group G of P to K. It is known (cf. [20]) that

$$\{Q \mid \text{reduced subbundles of } P \text{ to } K\} \cong C^{\infty}(M; P \times_G (G/K)),$$

where $P \times_G (G/K)$ is a fiber bundle associated to P with respect to the left action of G on the compact symmetric space G/K. Note that any principal K-bundle Q over M becomes a reduced subbundle of a principal G-bundle P to K. Indeed, the structure group K of the principal bundle Q acts on Lie group G on the left hand side. We define a principal bundle

$$P = Q \times_K G$$

over M with the structure group G associated with the left action of K on G. Then the principal K-bundle Q is embedded into the principal G-bundle P as a reduced bundle. Conversely, if Q is a reduced subbundle of P to K, then P is canonically isomorphic to $Q \times_K G$ as principal G-bundles.

Since Q is a reduced subbundle of the structure group G of P to K, we have natural embeddings

$$\mathcal{A}_Q \longrightarrow \mathcal{A}_P \quad \text{and} \quad \Omega^1(\mathfrak{m}_Q) \longrightarrow \Omega^1(\mathfrak{g}_P).$$

Since $k \in \mathcal{K}_Q$ extends to a gauge transformation of $P \cong Q \times_K G$, we have an injective homomorphism

$$\mathcal{K}_Q \longrightarrow \mathcal{G}_P.$$

These embeddings induce embeddings between the solution spaces

$$\tilde{j} = \tilde{j}_Q : \widetilde{\mathcal{M}}^{\pm}(M, G/K, Q) \longrightarrow \widetilde{\mathcal{M}}^{\pm}(M, G, P), \tag{6.3}$$

which are equivariant under the actions of $\mathcal{K}_Q$ and $\mathcal{G}_P$, and hence we obtain a map between the moduli spaces

$$j = j_Q : \mathcal{M}^{\pm}(M, G/K, Q) \longrightarrow \mathcal{M}^{\pm}(M, G, P) \tag{6.4}$$

for each compact symmetric pair (G, K, σ) with the same G and each $Q \in C^{\infty}(M; P \times_G (G/K))$. We should note that $j = j_Q$ is not necessarily injective. However we showed ([27]) that if $N_G(\mathfrak{k}) = K$, then the map

$$j_Q : \mathcal{M}^{\pm}(M, G/K, Q)^{irr} \longrightarrow \mathcal{M}^{\pm}(M, G, P)$$

is injective, where $N_G(\mathfrak{k}) = \{a \in G \mid \mathrm{Ad}(a)\mathfrak{k} = \mathfrak{k}\}$, the normalizer of $\mathfrak{k}$ in G.

The group $\mathcal{G}_P$ of gauge transformations acts on $C^\infty(M; P\times_G (G/K))$ by transforming Q to $g(Q)$. Then we have $j_{g(Q)}\circ g = j_Q$ and thus

$$j_Q(\mathcal{M}^\pm(M, G/K, Q)) = j_{g(Q)}(\mathcal{M}^\pm(M, G/K, g(Q))).$$

Therefore, when we discuss the image of the map j, we may consider the quotient space $C^\infty(M; P\times_G (G/K))/\mathcal{G}_P$ by the group $\mathcal{G}_P$ of gauge transformations rather than $C^\infty(M; P\times_G (G/K))$.

The local properties of the maps

$$j = j_Q : \mathcal{M}^\pm(M, G/K, Q) \longrightarrow \mathcal{M}^\pm(M, G, P)$$

such as smoothness, immersions and embeddings were discussed in [27].

Then these subspaces in the moduli spaces have the following properties:

Theorem 6.1 ([27])**.** *Let G be a compact Lie group and P a principal G-bundle over a compact Riemann surface M. For each compact symmetric pair (G, K, σ) with the same G and each $Q \in C^\infty(M; P\times_G (G/K))/\mathcal{G}_P$, the map $j = j_Q$: $\mathcal{M}^+(M, G/K, Q) \longrightarrow \mathcal{M}^+(M, G, P)$ satisfies*

$$j_Q^*\omega_1^+ = \omega^+, \quad j_Q^*\omega_2^+ = 0 \quad \textit{and} \quad j_Q^*\omega_3^+ = 0,$$

and the map $j = j_Q : \mathcal{M}^-(M, G/K, Q) \longrightarrow \mathcal{M}^-(M, G, P)$ satisfies

$$j_Q^*\omega_1^- = \omega^-, \quad j_Q^*\omega_2^- = 0 \quad \textit{and} \quad j_Q^*\omega_3^- = 0.$$

Moreover, concerning to the moduli space $\mathcal{M}^-(M, G, P)$,

(i) *$\mathcal{M}^-(M, G/K, Q)$ is a complex submanifold of $\mathcal{M}^-(M, G, P)$ with respect to J_1.*
(ii) *$\mathcal{M}^-(M, G/K, Q)$ is a totally real submanifold of $\mathcal{M}^-(M, G, P)$ with respect to J_2 and J_3.*
(iii) *$2\dim\mathcal{M}^-(M, G/K, Q) = \dim\mathcal{M}^-(M, G, P)$.*

That is, $\mathcal{M}^-(M, G/K, Q)$ is a complex Lagrangian submanifold of a hyper-Kähler moduli space $\mathcal{M}^-(M, G, P)$.

Theorem 6.2 ([27])**.** *Let G be a compact Lie group and P a principal G-bundle over a compact Riemann surface M. For each compact symmetric pair (G, K, σ) with the same G and each $Q \in C^\infty(M; P\times_G (G/K))/\mathcal{G}_P$, the maps*

$$\mathcal{M}^\pm(M, G/K, Q) \longrightarrow \mathcal{M}^\pm(M, G, P)$$

are totally geodesic.

Remark 6.1. In [27], we gave the similar construction also for k-symmetric spaces.

6.3. **Automorphisms on moduli spaces.** The automorphisms on moduli spaces induced from the automorphisms of the structure group were discussed in [19, 21, 22]. In this subsection, we shall mention the relationship of their works with geometry of submanifolds in moduli spaces constructed here.

Suppose that (G, K, σ) is a compact symmetric pair. For each reduction $Q \in C^\infty(M; P\times_G (G/K))$, we can define a diffeomorphism

$$\sigma_Q : P = Q\times_K G \longrightarrow Q\times_K G = P$$

by sending ua to $u\sigma(a)$. If we denote also by A the connection form on P corresponding to a connection $A \in \mathcal{A}_P$, then

$$A^{\sigma_Q} := (\sigma_Q^{-1})^*(\sigma \circ A)$$

is also a connection form on P and we denote also by $A^{\sigma_Q} \in \mathcal{A}_P$ its corresponding connection. If $\phi \in \Omega^1(\mathfrak{g}_P)$, then

$$\phi^{\sigma_Q} := (\sigma_Q^{-1})^*(\sigma \circ \phi)$$

is also a 1-form with values in $\mathfrak{g}_P$ on M. Thus we obtain a transformation on $\mathcal{A}_P \times \Omega^1(\mathfrak{g}_P)$ by sending (A, ϕ) to $(A^{\sigma_Q}, \phi^{\sigma_Q})$, which induces a transformation on $\mathcal{M}^{\pm}(M, G, P)$. We should note that if σ is of inner type, then the induced transformation σ_Q on the moduli space $\mathcal{M}^{\pm}(M, G, P)$ is the identity transformation ([19]).

Now we consider the composition $(-1)^\sharp \circ \sigma_Q$ of the action of an element $-1 \in S^1$ by the S^1-action and the transformation σ_Q on the moduli space $\mathcal{M}^{\pm}(M, G, P)$.

Theorem 6.3. *Let $\mathcal{F}(\mathcal{M}^{\pm}(M, G, P), (-1)^\sharp \circ \sigma_Q)$ denote the subsets of all points fixed by the automorphism $(-1)^\sharp \circ \sigma_Q$ on the moduli spaces $\mathcal{M}^{\pm}(M, G, P)$. Then we have*

$$j_Q(\mathcal{M}^{\pm}(M, G/K, Q)) \subset \mathcal{F}(\mathcal{M}^{\pm}(M, G, P), (-1)^\sharp \circ \sigma_Q).$$

Moreover, $j_Q(\mathcal{M}^{-,sm}(M, G/K, Q))$ is an open submanifold of $\mathcal{F}(\mathcal{M}^{-}(M, G, P), (-1)^\sharp \circ \sigma_Q)$, and thus it is totally geodesic.

Proof. Suppose that $[(A, \phi)] \in \mathcal{M}^{\pm}(M, G/K, Q)$. On Q we have $A^{\sigma_Q}|_Q = A$ and $\phi^{\sigma_Q}|_Q = -\phi$, and thus $A^{\sigma_Q} = A$ and $\phi^{\sigma_Q} = -\phi$. Hence $((-1)^\sharp \circ \sigma_Q)(A, \phi) = (A, \phi)$, in particular we have $((-1)^\sharp \circ \sigma_Q)[(A, \phi)] = [(A, \phi)]$. □

Moreover we obtain the following description of the set of points fixed by the $(-1)^\sharp$-action on the moduli space.

Theorem 6.4.

$$\mathcal{F}(\mathcal{M}^{\pm}(M, G, P), (-1)^\sharp) = \bigcup_{G', G'/K', Q' \subset P'} (i_{P'} \circ j_{Q'})(\mathcal{M}^{\pm}(M, G'/K', Q')),$$

where G' runs through compact Lie subgroups of G, G'/K' runs through compact symmetric spaces with involutions given by $\sigma = \mathrm{Ad}(c)$ for some $c \in G$, Q' runs through reduced subbundles of P to K', and $P' := Q' \times_{K'} G'$ denotes a principal subbundle of P with structure group G'.

Proof. Suppose that $[(B, \psi)] \in \mathcal{M}^{\pm}(M, G'/K', Q')$ and set $(A, \phi) = (\tilde{i}_{P'} \circ \tilde{j}_{Q'})(B, \psi)$. If we define a map $g|_{Q'} : Q' \longrightarrow G$ by $g|_{Q'}(u) := c$ for each $u \in Q'$, then we see that $g|_{Q'}$ extends to $g \in \mathcal{G}_P$ in the natural way, because of $P' = Q' \times_{K'} G'$ and $P := Q \times_{K'} G$. Then we obtain $g^*(A, \phi) = (A, -\phi)$, and hence we have $[(A, \phi)] \in \mathcal{F}(\mathcal{M}^{\pm}(M, G, P), (-1)^\sharp)$.

Conversely, suppose that $[(A, \phi)] \in \mathcal{F}(\mathcal{M}^{\pm}(M, G, P), (-1)^\sharp)$. Then we have

$$(-1)^\sharp[(A, \phi)] = [(A, -\phi)] = [(A, \phi)],$$

and hence there is a gauge transformation $g \in \mathcal{G}_P$ such that $g^*(A,\phi) = (A,-\phi)$. Thus we have

$$d_A g = 0, \tag{6.5}$$
$$g^*\phi = -\phi. \tag{6.6}$$

Fix an arbitrary point $x \in M$ and $u \in P$ with $\pi(u) = x$. Denote by H_A and $\mathfrak{h}_A$ the holonomy group and the holonomy algebra of the connection A at the point u. We take a holonomy subbundle $Q_A = Q_A(u)$ of P with respect to the connection A, which is a principal subbundle of P with structure group H_A. Then, since g is parallel with respect to the connection A, when the gauge transformation g is considered as a AdG-equivariant map $P \longrightarrow G$, the restriction $g|_{Q_A}$ of g to Q_A is constant, which is denoted by $c \in G$. Denote by G' the identity component of a compact Lie subgroups of G defined by

$$\{a \in G \mid (\mathrm{Ad}(c))^2(a) = a\},$$

and define

$$K' := \{a \in G' \mid (\mathrm{Ad}(c))(a) = a\}. \tag{6.7}$$

Then (G', K') is a compact symmetric space with an involutive automorphism $\sigma = \mathrm{Ad}(c)$. Denote by $\mathfrak{g}'$ the Lie algebra of G' and define

$$\mathfrak{k}' := \{X \in \mathfrak{g} \mid \mathrm{Ad}(c)X = X\}, \tag{6.8}$$
$$\mathfrak{m}' := \{X \in \mathfrak{g} \mid \mathrm{Ad}(c)X = -X\}. \tag{6.9}$$

Then we have a decomposition $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{m}'$ as a symmetric Lie algebra with the involutive automorphism $\sigma = \mathrm{Ad}(c)$. By (6.5), we see that $H_A(x) \subset K'$ and thus $\mathfrak{h}_A(x) \subset \mathfrak{k}'$. The associated principal bundle $Q' := Q_A \times_{H_A(x)} K'$ is a principal subbundle of P with structure group K' and moreover its associated principal bundle $P' := Q' \times_{K'} G'$ is a principal subbundle of P with structure group G'. Then we have $P = Q_A \times_{H_A(x)} G$ and the connection A reduces to a connection in Q_A and thus a connection in Q'. Setting

$$\begin{aligned} \mathfrak{g}'_Q &:= Q_A \times_{H_A} \mathfrak{g}' = Q' \times_{K'} \mathfrak{g}' = P' \times_{G'} \mathfrak{g}', \\ \mathfrak{k}'_Q &:= Q_A \times_{H_A} \mathfrak{k}' = Q' \times_{K'} \mathfrak{k}', \\ \mathfrak{m}'_Q &:= Q_A \times_{H_A} \mathfrak{m}' = Q' \times_{K'} \mathfrak{m}', \end{aligned} \tag{6.10}$$

by (6.6) we see that ϕ has image in $\mathfrak{m}'_Q$.

Therefore we conclude that $(A,\phi) \in \tilde{j}_{Q'}(\widetilde{\mathcal{M}}^{\pm}(M, G'/K', Q'))$ and hence $[(A,\phi)] \in j_{Q'}(\mathcal{M}^{\pm}(M, G'/K', Q'))$. □

7. 1-uniton Solutions and Fixed Point Subset by the S^1-Actions

Let $\xi \in \Gamma(\mathfrak{g})$ be a canonical element. Then we take the eigen-space decomposition of $\mathfrak{g}^{\mathbb{C}}$ by the endomorphism adξ:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_0(\xi) + \sum_{i \in \mathbb{Z}\setminus\{0\}} \mathfrak{g}_i(\xi),$$

where

$$\mathfrak{g}_i(\xi) := \{X \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}\xi(X) = \sqrt{-1}iX\}.$$

If we denote by $H = C_G(\xi)$ the centralizer of G at ξ, then its Lie algebra $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}}(\xi)$ is a real form of $\mathfrak{g}_0(\xi)$. Set $F := G/H$, which is a generalized flag manifold. If the element ξ satisfies the equation $(\mathrm{ad}\xi)^3 + \mathrm{ad}\xi = 0$, then $F = G/H$ is a compact Hermitian symmetric space. Conversely, all compact Hermitian symmetric spaces can be obtained in this way. We have the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that

$$\mathfrak{m}^{\mathbb{C}} = \sum_{i \in \mathbb{Z}\setminus\{0\}} \mathfrak{g}_i(\xi).$$

Let P be a principal G-bundle over M and Q a reduced subbundle of P with structure group H. Denote by $\mathcal{H}_Q$ the group of gauge transformations of the principal bundle Q. Further we take associated vector bundles

$$\mathfrak{g}_Q = \mathfrak{h}_Q + \mathfrak{m}_Q \quad \text{and} \quad \mathfrak{m}_Q^{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}\setminus\{0\}} \mathfrak{g}_i(\xi)_Q. \tag{7.1}$$

We define the following gauge-theoretic equations for $(A, \phi) \in \mathcal{A}_Q \times \Omega^1(\mathfrak{m}_Q)$:

$$\begin{cases} F(A) \pm \dfrac{1}{2}[\phi \wedge \phi] = 0, \\ d_A\phi = 0, \\ \phi' \in \Omega^{1,0}(\mathfrak{g}_{-1}(\xi)_Q). \end{cases} \tag*{$(*)^{\pm}_{G/H,1}$}$$

Let $\widetilde{\mathcal{M}}_1^{\pm}(M, G/H, Q)$ denote the spaces of all smooth solutions to the equations $(*)^{\pm}_{G/H,1}$ and

$$\mathcal{M}_1^{\pm}(M, G/H, Q) := \widetilde{\mathcal{M}}_1^{\pm}(M, G/H, Q)/\mathcal{H}_Q$$

denote these moduli spaces. It is easy to show that if (A, ϕ) is a solution to the equations $(*)^{\pm}_{G/H,1}$, then (A, ϕ) extends to a solution to the equations $(*)^{\pm}_{G}$. Therefore we have the canonical injections

$$\tilde{j}_{1,Q} : \widetilde{\mathcal{M}}_1^{\pm}(M, G/H, Q) \longrightarrow \widetilde{\mathcal{M}}^{\pm}(M, G, P),$$

which are equivariant with respect to $\mathcal{H}_Q$ and $\mathcal{G}_P$. These injections induce maps between moduli spaces

$$j_{1,Q} : \mathcal{M}_1^{\pm}(M, G/H, Q) \longrightarrow \mathcal{M}^{\pm}(M, G, P). \tag{7.2}$$

Now for each $t \in \mathbb{R}$ we define

$$g_t := \exp(-2\pi t\xi) \in C(H) \subset H, \tag{7.3}$$

which is a constant function on Q, where $C(H)$ denotes the center of H. Then g_t can be considered as a gauge transformation of the principal H-bundle Q, that is,

$g_t \in \mathcal{H}_Q$. For $[(A,\phi)] \in \mathcal{M}_1^{\pm}(M, G/H, Q)$, we obtain

$$\begin{aligned} g_t^*(A,\phi) &= (g_t^*A, g_t^*\phi) \\ &= (g_t^{-1}dg_t + \mathrm{Ad}(\exp(-t\xi)^{-1})A, \mathrm{Ad}(\exp(-t\xi)^{-1})\phi) \\ &= (A, \mathrm{Exp}(\mathrm{tad}\xi)(\phi' + \phi'')) \\ &= (A, \lambda^{-1}\phi' + \lambda\phi'') \\ &= (A, \phi_\lambda) = \lambda^\sharp(A,\phi) \end{aligned}$$

for each $\lambda = e^{2\pi\sqrt{-1}t} \in S^1$. This means that each $[(A,\phi)] \in \mathcal{M}_1^{\pm}(M, G/H, Q)$ is a point fixed by the S^1-action on the moduli spaces $\mathcal{M}^{\pm}(M, G, P)$. Namely,

$$j_{1,Q}(\mathcal{M}_1^{\pm}(M, G/H, Q)) \subset \mathcal{F}^{\pm}(M, G, P). \tag{7.4}$$

Moreover, concerning to the moduli space $\mathcal{M}_1^{+}(M, G/H, Q)$, we shall obtain the following. If we denote by $\mathrm{SHH}(M, G/H)$ the space of all super-horizontal holomorphic maps of M into G/H ([5]), then we have

$$\mathrm{SHH}(M, G/H)/G \subset \coprod_Q \mathcal{M}_1^{+}(M, G/H, Q).$$

On the other hand, we have

$$g_\lambda^{-1*} A_\lambda = A_1$$

and $\Xi := \xi$ is a flag factor for (A,ϕ), because $d''_A\Xi = d''\xi + [A'', \xi] = 0$ and ϕ' has values in $\mathfrak{g}_{-1}(\Xi)$. This means that each solution $(A,\phi) \in \mathcal{M}_1^{+}(M, G/H, Q)$ can be transformed into a trivial solution $(A_1, 0)$ by a flag transform defined by a flag factor $\Xi = \xi$.

So in the case when the element ξ satisfies the equation $(\mathrm{ad}\xi)^3 + \mathrm{ad}\xi = 0$, and hence G/H is a compact Hermitian symmetric space with pair (G, H, τ), we call the solution (A,ϕ) an 1-*uniton solution.*

Let $\mathrm{Hol}(M, G/H)$ be the space of all holomorphic maps of M into a compact Hermitian symmetric space G/H. Then we have

$$\mathrm{Hol}(M, G/H)/G \subset \coprod_Q \mathcal{M}_1^{+}(M, G/H, Q). \tag{7.5}$$

Finally, we obtain the following.

Theorem 7.1.

$$\mathcal{F}^{\pm}(M, G, P) = \bigcup_{G' \subset G, G'/H', Q' \subset P'} (i_{P'} \circ j_{1,Q'})(\mathcal{M}_1^{\pm}(M, G'/H', Q')),$$

where G' runs through compact Lie subgroups of G, G'/H' runs through generalized flag manifolds with a canonical element ξ of G', Q' runs through reduced subbundles of P to the structure group H', and $P' := Q' \times_{H'} G'$ denotes a principal subbundle of P with structure group G'.

Moreover the subspace $\mathcal{F}^{+}(M, G, P)$ of all points fixed by S^1-action is that of $\mathcal{M}^{+}(M, G, P)$ each element of which can be obtained as a flag transform of a trivial solution.

Proof. Assume that $[(A,\phi)] \in \mathcal{M}^{\pm}(M,G,P)$ is a point fixed by the S^1-action on the moduli space, that is

$$\lambda^{\sharp}[(A,\phi)] = [(A,\phi)]$$

for each $\lambda \in S^1$. There is a one-parameter family $g_t \in \mathcal{G}_P$ with $g_0 = e$ such that

$$(A, \phi_\lambda) = {g_t}^*(A,\phi) \tag{7.6}$$

for each $\lambda = e^{2\pi\sqrt{-1}t} \in S^1$. Then we have

$$d_A g_t = 0 \quad \text{ and } \quad \mathrm{Ad}({g_t}^{-1})\phi = \phi_\lambda. \tag{7.7}$$

Setting $\lambda = e^{2\pi\sqrt{-1}t} \in S^1$, we define

$$-2\pi\Xi := \frac{\partial}{\partial t} g_\lambda|_{t=0} \in \Omega^0(\mathfrak{g}_P). \tag{7.8}$$

It follows from (7.7) that

$$d_A\Xi = 0 \tag{7.9}$$

$$[\Xi,\phi'] = -\sqrt{-1}\phi', \quad [\Xi,\phi''] = \sqrt{-1}\phi''. \tag{7.10}$$

We fix $x \in M$ and $u \in P$ with $x = \pi(u)$ and we use the identification $u : \mathfrak{g} \longrightarrow (\mathfrak{g}_P)_x$. We see that the subbundle of holonomy subalgebras of $\mathfrak{g}_P$ relative to the connection A is contained in $\mathfrak{g}_0(\Xi)$. Let Q_A be the holonomy subbundle of P at the point $u \in P$ relative to the connection A with structure group $H_A \subset G$. By (7.9), we see that the restriction of Ξ to Q_A is constant, which is denoted by $\xi \in \mathfrak{g}$. Then we have the eigen-space decomposition of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathrm{ad}\xi$ as

$$\mathfrak{g}^{\mathbb{C}} = (\mathfrak{g}_0(\xi) \oplus \mathfrak{g}_{-1}(\xi) \oplus \mathfrak{g}_1(\xi) \oplus \cdots \oplus \mathfrak{g}_{-k}(\xi) \oplus \mathfrak{g}_k(\xi)) \oplus W^{\mathbb{C}}, \tag{7.11}$$

where we set

$$\{\text{ eigenvalues of } \mathrm{ad}\xi\} \cap \sqrt{-1}\mathbb{Z} := \{0, \sqrt{-1}, -\sqrt{-1}, \cdots, \sqrt{-1}k, -\sqrt{-1}k\}$$

and W is a vector subspace of $\mathfrak{g}$. The section Ξ induces an orthogonal direct sum decomposition of $\mathfrak{g}_P^{\mathbb{C}}$ relative to the induced Hermitian inner product which is preserved by the parallel translatons of the connection A:

$$\mathfrak{g}_P^{\mathbb{C}} = (\mathfrak{g}_0(\Xi) \oplus \mathfrak{g}_{-1}(\Xi) \oplus \mathfrak{g}_1(\Xi) \oplus \cdots \oplus \mathfrak{g}_{-k}(\Xi) \oplus \mathfrak{g}_k(\Xi)) \oplus W_A^{\mathbb{C}}, \tag{7.12}$$

and W_A is a vector subbundle of $\mathfrak{g}_P$. Set

$$\mathfrak{g}'^{\mathbb{C}} := \mathfrak{g}_0(\xi) \oplus \mathfrak{g}_{-1}(\xi) \oplus \mathfrak{g}_1(\xi) \oplus \cdots \oplus \mathfrak{g}_{-k}(\xi) \oplus \mathfrak{g}_k(\xi), \tag{7.13}$$

$$\mathfrak{h}'^{\mathbb{C}} := \mathfrak{g}_0(\xi) \tag{7.14}$$

which are subalgebras of $\mathfrak{g}^{\mathbb{C}}$ with real forms $\mathfrak{g}'$ and $\mathfrak{h}'$ in $\mathfrak{g}$. We denote by G' the identity component of a compact Lie subgroup of G defined by

$$\{a \in G \mid \mathrm{Ad}(\exp(2\pi\xi))a = a\}, \tag{7.15}$$

and set

$$\begin{aligned} H' :&= \{a \in G' \mid \mathrm{Ad}(\exp(2\pi t\xi))a = a \text{ for each } t \in \mathbb{R}\} \\ &= C_{G'}(\xi). \end{aligned} \tag{7.16}$$

Then G' is a compact connected Lie subgroup with Lie algebra $\mathfrak{g}'$ and H' is a compact connected Lie subgroup with Lie algebra $\mathfrak{h}'$, and there is a real subspace $\mathfrak{m}'$ of $\mathfrak{g}'$ with direct sum $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$ and

$$\mathfrak{m}'^{\mathbb{C}} = \mathfrak{g}_{-1}(\xi) \oplus \mathfrak{g}_1(\xi) \oplus \cdots \oplus \mathfrak{g}_{-k}(\xi) \oplus \mathfrak{g}_k(\xi). \tag{7.17}$$

Moreover ξ becomes a canonical element of $\mathfrak{g}'$. Setting $Q' := Q_A \times_{H_A} H'$ and $P' := Q' \times_{H'} G'$, we have reduced subbundles $Q_A \subset Q' \subset P' \subset P$ with structure groups $H_A \subset H' \subset G' \subset G$, and

$$\begin{aligned}
{\mathfrak{g}'_{Q'}}^{\mathbb{C}} &= Q' \times_{H'} \mathfrak{g}'^{\mathbb{C}} \\
&= \mathfrak{g}_0(\Xi) \oplus \mathfrak{g}_{-1}(\Xi) \oplus \mathfrak{g}_1(\Xi) \oplus \cdots \oplus \mathfrak{g}_{-k}(\Xi) \oplus \mathfrak{g}_k(\Xi), \\
{\mathfrak{h}'_{Q'}}^{\mathbb{C}} &= Q' \times_{H'} \mathfrak{h}'^{\mathbb{C}} = \mathfrak{g}_0(\Xi), \\
{\mathfrak{m}'_{Q'}}^{\mathbb{C}} &= Q' \times_{H'} \mathfrak{m}'^{\mathbb{C}} \\
&= \mathfrak{g}_{-1}(\Xi) \oplus \mathfrak{g}_1(\Xi) \oplus \cdots \oplus \mathfrak{g}_{-k}(\Xi) \oplus \mathfrak{g}_k(\Xi).
\end{aligned} \tag{7.18}$$

Then the connection A reduces to a connection in Q_A and thus in Q', and ϕ' has image in $\mathfrak{g}_{-1}(\Xi) = \mathfrak{g}_{-1}(\xi)_{Q'}$. Thus the solution (A, ϕ) reduces to a solution $(B, \psi) \in \mathcal{A}_{Q'} \times \Omega^1(\mathfrak{m}_{Q'})$. Hence we conclude that $[(B, \psi)] \in \mathcal{M}_1^{\pm}(M, G'/H', Q')$ and $(i_{P'} \circ j_{1,Q'})([(B, \psi)]) = [(A, \phi)]$. $\square$

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