A certain Dirichlet series attached to Siegel modular forms of degree two

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MPI/88-22

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Introduction

In this paper we shall study a certain Dirichlet series $D_{F,G}(s)$ attached to two Siegel cusp forms F and G of integral weight k on $Sp_2(\mathbb{Z})$, which formally could be viewed as an analogue of the Rankin convolution series in the theory of elliptic modular forms. By definition, its N^{th} coefficient equals $\langle\phi_N,\psi_N\rangle$, where ϕ_N and ψ_N are the N^{th} coefficients of the Fourier-Jacobi expansions of F and G, respectively, and $\langle\ ,\ \rangle$ denotes the Petersson scalar product on Jacobi cusp forms of weight k and index N.

By applying the Rankin-Selberg method with a certain non-holomorphic Eisenstein series on $Sp_2(\mathbb{Z})$ of Klingen-Siegel type, we shall prove that

$$D^{*}_{F,G}(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)\zeta(2s-2k+4)D_{F,G}(s)$$

has a meromorphic continuation to $\mathbb{C}$ and is invariant under $s\mapsto 2k-2-s$ (§1). Since for k even the Γ-factor and type of functional equation is the same as that of the spinor zeta function of a Hecke eigenform of weight k and degree 2, one might ask if in this case there is any connection between $D^*_{F,G}(s)$ and linear combinations of functions $Z^*_{F_i}(s)$ ($\{F_i\}$ a basis of Hecke eigenforms, $Z^*_{F_i}(s)$ = spinor zeta function of F_i completed with its natural Γ-factor).

Although in general this question remains unanswered here, we can prove two special results (§2). First, it will be shown that if F is a non-zero eigenfunction in the Maass space then $D^*_{F,F}(s)$ coincides up to the factor $\langle \phi_1,\phi_1 \rangle$ with $Z^*_F(s)$, in other words

$$D_{F,F}(s) = \langle \phi_1,\phi_1 \rangle \frac{\zeta(s-k+1)\zeta(s-k+2)}{\zeta(2s-2k+4)} L_f(s),$$

where f is the normalized Hecke eigenform of weight 2k-2 on $SL_2(\mathbb{Z})$ which corresponds to F under the Saito-Kurokawa lifting, and $L_f(s)$ denotes its Hecke L-function. As corollary we shall obtain a simple proof (and even a more precise statement) of a formula obtained previously by one of the authors [5] relating the quotient of Petersson products $\frac{\langle F,F \rangle}{\langle \phi_1,\phi_1 \rangle}$ to the special value $L_f(k)$. Secondly, if F is an arbitrary non-zero Hecke eigenform of weight k and $P_{k,D}$ (D<0 a fundamental discriminant) is the Maass lift of the D^{th} Jacobi-Poincaré series of weight k and index 1, then we shall prove that $D^*_{F,P_{k,D}}(s)$ is proportional to $Z^*_F(s)$. In particular, if the constant of proportionality is non-zero for some D, one obtains a new proof of the meromorphic continuation and functional equation of $Z^*_F(s)$.

Certainly some of our results can be generalized to higher genus n, however, a more detailed study of Jacobi forms of genus n-1 is then required. We hope to come back to this in a future paper.

Notations

We let $\mathfrak{H}$ be the upper half-plane. The symbol $\mathfrak{H}_2$ denotes the Siegel upper half-space of degree 2 consisting of complex 2×2 matrices

Z with positive definite imaginary part. We often write $Z=\begin{pmatrix}\tau & z\\ z & \tau'\end{pmatrix}$, $X=\mathrm{Re}(Z)=\begin{pmatrix}u & x\\ x & u'\end{pmatrix}$ and $Y=\mathrm{Im}(Z)=\begin{pmatrix}v & y\\ y & v'\end{pmatrix}$. We usually set $|Y|=\det Y$.

We let $\Gamma_1=SL_2(\mathbb{Z})$ be the modular group, $\Gamma_2=Sp_2(\mathbb{Z})$ the group of integral symplectic 4×4-matrices and $\Gamma_1^J=\Gamma_1\ltimes\mathbb{Z}^2$ be the Jacobi modular group ([2]). These groups act on $\mathfrak{H}$, $\mathfrak{H}_2$ and $\mathfrak{H}\times\mathbb{C}$, respectively, by

$$\tau\mapsto\begin{pmatrix}a & b\\ c & d\end{pmatrix}\langle\tau\rangle=\frac{a\tau+b}{c\tau+d}\qquad\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_1\right),$$

$$Z\mapsto M\langle Z\rangle=(AZ+B)(CZ+D)^{-1}\qquad\left(M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_2\right)$$

and

$$(\tau,z)\mapsto\left(\frac{a\tau+b}{c\tau+d},\frac{z+\lambda\tau+\mu}{c\tau+d}\right)\qquad\left(\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix},(\lambda,\mu)\right)\in\Gamma_1^J\right).$$

The letter k denotes a positive integer. We write $S_k(\Gamma_n)$ for the space of cusp forms of weight k on Γ_n. By $J_{k,N}^{cusp}$ we understand the space of Jacobi cusp forms of weight k and index N ([2]). The Petersson products on these spaces are normalized by

$$\langle f,g\rangle=\int_{\Gamma_1\backslash\mathfrak{H}}f(\tau)\overline{g(\tau)}v^{k-2}du\,dv\qquad(f,g\in S_k(\Gamma_1))$$

$$\langle F,G\rangle=\int_{\Gamma_2\backslash\mathfrak{H}_2}F(Z)\overline{G(Z)}|Y|^{k-3}dX\,dY\qquad(F,G\in S_k(\Gamma_2))$$

and

$$\langle\phi,\psi\rangle=\int_{\Gamma_1^J\backslash\mathfrak{H}\times\mathbb{C}}\phi(\tau,z)\overline{\psi(\tau,z)}v^{k-3}e^{-4\pi Ny^2/v}du\,dv\,dx\,dy\qquad(\phi,\psi\in J_{k,N}^{cusp}).$$

1. Meromorphic continuation and functional equation of $D_{F,G}(s)$

Let F be a Siegel cusp form of weight k on Γ_2. Then F has a Fourier-Jacobi expansion

$$F(Z) = \sum_{N\geq 1} \phi_N(\tau,z)e^{2\pi iN\tau'},$$

where ϕ_N is a Jacobi cusp form of weight k and index N ([2],§6).

For two functions F and G in $S_k(\Gamma_2)$ we define a formal Dirichlet series by

$$D_{F,G}(s) = \sum_{N\geq 1} \langle\phi_N,\psi_N\rangle N^{-s}.$$

Here ϕ_N and ψ_N denote the N^{th} Fourier-Jacobi coefficients of F and G, respectively, and $\langle\ ,\ \rangle$ is the Petersson product on $J_{k,N}^{cusp}$.

Lemma 1. <u>The coefficients</u> $\langle\phi_N,\psi_N\rangle$ of $D_{F,G}(s)$ <u>satisfy</u>

$$\langle\phi_N,\psi_N\rangle = \mathcal{O}(N^k),$$

<u>where the $\mathcal{O}$-constant depends only on</u> F <u>and</u> G. <u>Hence</u> $D_{F,G}(s)$ <u>is absolutely convergent for</u> Re(s)>k+1 <u>and represents a holomorphic function in this domain</u>.

Proof. We use a variant of the classical Hecke argument. For fixed $(\tau,z)\in\mathfrak{H}\times\mathbb{C}$ we write

$$\phi_N(\tau,z) = \int_{iC}^{iC+1} F(Z)e^{-2\pi iN\tau'}d\tau',$$

where C is any real constant greater than $\frac{y^2}{v}$. Observing that $|Y|^{k/2}|F(Z)|$ is bounded on $\mathfrak{H}_2$ and choosing $C=\frac{y^2}{v}+\frac{1}{N}$ we obtain

$$\phi_N(\tau,z) = \mathcal{O}((\tfrac{v}{N})^{-k/2}e^{2\pi Ny^2/v})$$

with the $\mathcal{O}$-constant independent of τ and z. From this Lemma 1 follows immediately.

We define

$$D^{*}_{F,G}(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)\zeta(2s-2k+4)D_{F,G}(s).$$

The main result of this section is the following

Theorem 1. <u>The function</u> $D^{*}_{F,G}(s)$ <u>has a meromorphic continuation to</u> $\mathbb{C}$ <u>and satisfies the functional equation</u> $D^{*}_{F,G}(s) = D^{*}_{F,G}(2k-2-s)$. <u>It is holomorphic except for at most two simple poles at</u> s=k <u>and</u> s=k-2. <u>The residue at</u> s=k <u>equals</u> $\pi^{-k+2}\langle F,G\rangle$.

The rest of this section is devoted to the <u>proof</u> of Theorem 1. According to the Rankin-Selberg method we shall write $D_{F,G}(s)$ as the Petersson product of $F(Z)\overline{G(Z)}|Y|^{k}$ against a certain non-holomorphic Eisenstein series $E_s(Z)$ of Klingen-Siegel type. The analytic properties of $D_{F,G}(s)$ and the functional equation then follow from the corresponding properties of E_s.

Denote (for the moment) the upper left entry of $Z\in\mathfrak{H}_2$ by Z_1 and let $C=C_{2,1}$ be the subgroup of Γ_2 consisting of matrices whose last rows have the form (0,0,0,1). For $Z\in\mathfrak{H}_2$ and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>2$ we put

$$E_s(Z) = \sum_{M\in C\backslash\Gamma_2} \left(\frac{\det \mathrm{Im}\, M\langle Z\rangle}{\mathrm{Im}\, M\langle Z\rangle_1}\right)^{s}.$$

This series is well-defined, converges absolutely and uniformly on compact sets and is invariant under Γ_2. Indeed, if $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in C$ and (if) we denote by a,b,c,d the upper left entries of A,B,C,D, respectively, then $M_1=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ is in Γ_1 and the formula $M\langle Z\rangle_1 = M_1\langle Z_1\rangle$ holds. From this, the formula

$$\det \mathrm{Im}\, M\langle Z\rangle = |\det(CZ+D)|^{-2}\det \mathrm{Im}\, Z \qquad \left(M=\begin{pmatrix}*&*\\C&D\end{pmatrix}\in\Gamma_2\right),$$

the corresponding formula for matrices in Γ_1 and the well-known fact that $\begin{pmatrix} * & * \\ C & D \end{pmatrix} \in C$ implies $C = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$ it follows that

$$\frac{\det \operatorname{Im} M\langle Z\rangle}{\operatorname{Im} M\langle Z\rangle_1}$$

is invariant under left-multiplication by elements of C. The absolute and uniform convergence on compact sets of the series $E_s(Z)$ for Re(s) >2 can be checked by the same arguments as used in [4],pp.33,34. The invariance of $E_s(Z)$ under Γ_2 is then clear.

We define

$$E_s^*(Z) = \pi^{-s}\Gamma(s)\zeta(2s)E_s(Z).$$

Main Lemma. <u>The function</u> $E_s^*(Z)$ <u>has a meromorphic continuation to all</u> s, <u>the only singularities being simple poles at</u> s=2 <u>and</u> s=0 <u>of residues</u> 1 <u>and</u> -1, <u>respectively. It satisfies the functional equation</u> $E_s^*(Z) = E_{2-s}^*(Z)$.

Although this result certainly is implicitly contained in the general theory of Eisenstein series, we repeat, for the reader's convenience, a <u>proof</u> in this special case.

For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2$ we have

$$\operatorname{Im} M\langle Z\rangle = |\det(CZ+D)|^{-2} \cdot (CZ+D)^{*\,t}Y(CZ+D)^*,$$

where for a 2×2-matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we denote by $A^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ its adjoint and by A^t its transpose. From this we see that

$$\operatorname{Im} M\langle Z\rangle_1 = |\det(CZ+D)|^{-2} \cdot Y\left[Z^*\begin{pmatrix} c_4 \\ -c_3 \end{pmatrix} + \begin{pmatrix} d_4 \\ -d_3 \end{pmatrix}\right].$$

(Notation: $C = \begin{pmatrix} * & * \\ c_3 & c_4 \end{pmatrix}$, $D = \begin{pmatrix} * & * \\ d_3 & d_4 \end{pmatrix}$, $Y\begin{bmatrix} a \\ b \end{bmatrix} = (\bar{a}, \bar{b})Y\begin{pmatrix} a \\ b \end{pmatrix}$ for $a, b \in \mathbb{C}$.).

Hence

$$\frac{\det \operatorname{Im} M\langle Z\rangle}{\operatorname{Im} M\langle Z\rangle_1} = \frac{|Y|}{Y[Z^*\binom{c_4}{-c_3}+\binom{d_4}{-d_3}]} ,$$

where (c_3, c_4, d_3, d_4) denotes the last row of M.

It is well-known and can easily be checked that the map $\Gamma_2 \to \mathbb{Z}^4$, $M \mapsto (0,0,0,1)M$ induces a bijection between $C\backslash\Gamma_2$ and the set of primitive vectors in $\mathbb{Z}^4$. Thus

$$\zeta(2s)E_s(Z) = \sum_{c,d\in\mathbb{Z}^2}{}' \frac{|Y|^s}{Y[Z^* c+d]^s}$$

where the sum extends over all vectors c and d in $\mathbb{Z}^2$ with $(c,d) \neq (0,0)$

Now for positive real t define a theta series

$$\theta_t(Z) = \sum_{c,d\in\mathbb{Z}^2} e^{-\pi t\cdot |Y|^{-1} Y[Z^* c+d]} .$$

Then by Mellin's formula we have for s in the region of absolute convergence

$$E_s^*(Z) = \int_0^\infty (\theta_t(Z)-1)t^s \frac{dt}{t}.$$

Splitting the integral into the sum of the corresponding integrals from 1 to ∞ and from 0 to 1 and then making the substitution $t \mapsto \frac{1}{t}$ in the latter integral we deduce for $\mathrm{Re}(s) \gg 0$

$$E_s^*(Z) = \int_1^\infty (\theta_t(Z)-1)t^s \frac{dt}{t} + \int_1^\infty (\theta_{1/t}(Z)-1)t^{-s} \frac{dt}{t}.$$

For Z fixed write

$$f_t(c,d) = e^{-\pi t\cdot |Y|^{-1} Y[Z^* c+d]},$$

so that

$$\theta_t(Z) = \sum_{c,d\in\mathbb{Z}^2} f_t(c,d).$$

By the Poisson summation formula we have

$$\theta_{1/t}(Z) = \sum_{c,d\in\mathbb{Z}^2} \hat{f}_{1/t}(c,d),$$

where

$$\hat{f}_{1/t}(c,d) = \int_{\mathbb{R}^4} e^{-2\pi i(c^t,d^t)\cdot(v,w)} f_{1/t}(v,w)\ dvdw$$

is the Fourier transform and the dot denotes the usual scalar product on $\mathbb{R}^4$.

Lemma 2. One has

$$\hat{f}_{1/t}(c,d) = t^{-2}\cdot f_t(d,-c).$$

Proof. For any symmetric positive definite 4×4-matrix F the identity

$$\int_{\mathbb{R}^4} e^{-2\pi i x\cdot y} e^{-\pi y^t F y}\ dy = |F|^{-1/2}\ e^{-\pi x^t F^{-1} x}$$

holds. Setting

$$F = \begin{pmatrix} Y^{*t} & X^{*t} \\ O_2 & E_2 \end{pmatrix} \begin{pmatrix} t^{-1}|Y|^{-1}Y & O_2 \\ O_2 & t^{-1}|Y|^{-1}Y \end{pmatrix} \begin{pmatrix} Y^* & O_2 \\ X^* & E_2 \end{pmatrix},$$

where O_2 and E_2 denote the zero and unit matrix, respectively, and observing

$$|F| = t^{-4}$$

and (as is easily checked)

$$(c^t,d^t)F^{-1}\binom{c}{d} = t\cdot|Y|^{-1}Y[Z^*d-c],$$

our assertion follows.

Lemma 2 implies the transformation formula

$$\theta_{1/t}(Z) = t^2\theta_t(Z)$$

and hence the identity

$$E_s^*(Z) = \int_1^\infty (\theta_t(Z)-1)t^s \frac{dt}{t} + \int_1^\infty (t^2\theta_t(Z)-1)t^{-s} \frac{dt}{t}$$

$$= \int_1^\infty (\theta_t(Z)-1)(t^s+t^{2-s}) \frac{dt}{t} - (\frac{1}{s} + \frac{1}{2-s}),$$

from which the meromorphic continuation and the functional equation of $E_s^*(Z)$ are obvious. This proves our Main Lemma.

From the Main Lemma we shall now deduce the assertions of Theorem 1. Let $F,G \in S_k(\Gamma_2)$ with Fourier-Jacobi coefficients ϕ_N and ψ_N, respectively. Then by the usual unfolding argument

$$\langle FE_s,G\rangle = \int_{\Gamma_2\backslash \mathfrak{H}_2} F(Z)E_s(Z)\overline{G(Z)}\,|Y|^{k-3}\,dXdY$$

$$= \int_{C\backslash \mathfrak{H}_2} F(Z)\overline{G(Z)}v^{-s}|Y|^{k-3+s}\,dXdY \qquad (\mathrm{Re}(s)>2).$$

Now note that the group C is the centralizer of the element $\begin{pmatrix} E_2 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ 0_2 & E_2 \end{pmatrix}$ in Γ_2 and hence we have an isomorphism

$$\Gamma_1 \ltimes \mathbb{H}(\mathbb{Z}) \xrightarrow{\sim} C,$$

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix},(\lambda,\mu),\kappa\right) \longmapsto \begin{pmatrix} a & 0 & b & \mu' \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & -\lambda' \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \left((\lambda,\mu) = (\lambda',\mu')\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right),$$

where $\mathbb{H}(\mathbb{Z}) = \{((\lambda,\mu),\kappa) \mid (\lambda,\mu)\in \mathbb{Z}^2, \kappa\in\mathbb{Z}\}$ is the Heisenberg group (cf. [2], §6; recall that $\mathbb{H}(\mathbb{Z})$ is a group under the law $((\lambda,\mu),\kappa)((\lambda',\mu'),\kappa')$ $= ((\lambda+\lambda',\mu+\mu',\kappa+\kappa'+\lambda\mu'-\lambda'\mu)$, and that Γ_1 acts on $\mathbb{H}(\mathbb{Z})$ on the right by $(X,\kappa)\circ M = (XM,\kappa)$).

From this we see that a fundamental domain for the action of C on $\mathfrak{H}_2$ is given by $\{\begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \mid (\tau,z)\in F, v'>\frac{y^2}{v}, 0\le u'\le 1\}$, where F is a fundamental domain for the action of $\Gamma_1^J=\Gamma_1\ltimes\mathbb{Z}^2$ on $\mathfrak{H}\times\mathbb{C}$. Therefore we obtain after inserting the Fourier-Jacobi expansions of F and G

$$\langle FE_s,G\rangle = \int_F \Big[\int_{v'>\frac{y^2}{v},\,0\le u'\le 1} \sum_{M,N\ge 1} \phi_M(\tau,z)\overline{\psi_N(\tau,z)}\,e^{-2\pi(M+N)v'}$$

$$\cdot\, e^{2\pi i(M-N)u'}\cdot v^{k-3}\big(v'-\tfrac{y^2}{v}\big)^{k-3+s}\,du'dv'\Big]\ dudvdxdy.$$

Carrying out the integration over u' and making the substitution $t=v'-\frac{y^2}{v}$ we deduce

$$\langle FE_s,G\rangle = \int_F \Big[\sum_{N\ge 1} \phi_N(\tau,z)\overline{\psi_N(\tau,z)}\,e^{-4\pi Ny^2/v}\ v^{k-3}$$

$$\cdot\Big(\int_0^\infty e^{-4\pi Nt}t^{k-3+s}\,dt\Big)\Big]\ dudvdxdy$$

$$= (4\pi)^{-(s+k-2)}\Gamma(s+k-2)\sum_{N\ge 1}\langle\phi_N,\psi_N\rangle N^{-(s+k-2)} \qquad (\mathrm{Re}(s)>3),$$

where in the last line we have used the standard integral representatio of the Γ-function and have interchanged the order of summation and integration.

Hence we obtain the identity

$$\pi^{-k+2}\langle E^*_{s-k+2}F,G\rangle = D^*_{F,G}(s)$$

from which the assertions of Theorem 1 are obvious.

§2. Relations to spinor zeta functions

In this section we shall give a relation between the Dirichlet series constructed in the preceding paragraph and spinor zeta functions We shall assume throughout that k is even.

For $F \in S_k(\Gamma_2)$ a non-zero Hecke eigenform with $T(n)F = \lambda_F(n)F$ $(n \in \mathbb{N})$ we denote by

$$Z_F(s) = \prod_p (1-\lambda_F(p)p^{-s}+(\lambda_F(p)^2-\lambda_F(p^2)-p^{2k-4})p^{-2s}+\lambda_F(p)p^{2k-3-2s}$$
$$+p^{4k-6-4s})^{-1} \qquad (\mathrm{Re}(s)>>0)$$

the associated spinor zeta function. According to Andrianov [1], $Z_F(s)$ has a meromorphic continuation to all s with at most one simple pole at s=k, and the modified function

$$Z_F^*(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

is invariant under $s \mapsto 2k-2-s$.

Recall that for $N \in \mathbb{N}$ we have a linear operator

$$V_N: J_{k,1}^{cusp} \to J_{k,N}^{cusp},$$

$$\sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv r^2(4)}} c(D,r)\,e(\tfrac{r^2-D}{4}\tau+rz) \longmapsto \sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv r^2(4N)}} \Bigg(\sum_{\substack{d\mid(r,N)\\ D\equiv r^2(4Nd)}} d^{k-1}c(\tfrac{D}{d^2},\tfrac{r}{d}) \Bigg)$$
$$\cdot\, e(\tfrac{r^2-D}{4N}\tau+rz) \qquad (e(z)=e^{2\pi iz})$$

([2],§4). We shall use the following result whose proof will be postponed until the end of this section:

Proposition. <u>Let $V_N^*: J_{k,N}^{cusp} \to J_{k,1}^{cusp}$ be the adjoint of V_N with respect to the Petersson products. Then:</u>

i) <u>The action of V_N^* on Fourier coefficients is given by</u>

$$\sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv r^2(4N)}} c(D,r)\,e(\tfrac{r^2-D}{4N}\tau+rz) \longmapsto \sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv r^2(4)}} \Bigg(\sum_{d\mid N} d^{k-2} \sum_{\substack{s\,(2d)\\ s^2\equiv D(4d)}} c(\tfrac{N^2}{d^2}D,\tfrac{N}{d}s) \Bigg)$$
$$\cdot\, e(\tfrac{r^2-D}{4}\tau+rz).$$

ii) <u>One has</u>

$$V_N^* V_N = \sum_{t|N} \Big(\sum_{\substack{s|t \\ \frac{t}{s} \text{ squarefree}}} s \Big) t^{k-2}\, T(\tfrac{N}{t}),$$

<u>where</u> $T(n)$ <u>denotes the Hecke operator on</u> $J_{k,1}^{cusp}$.

We will first prove a result on eigenforms in the Maass space $S_k^*(\Gamma_2) \subset S_k(\Gamma_2)$. Recall that $S_k^*(\Gamma_2)$ consists of those forms

$$F(Z) = \sum_{\substack{n,r\in\mathbb{Z},N\in\mathbb{N} \\ r^2<4Nn}} A(n,r,N)\ e(n\tau+rz+N\tau')$$

whose Fourier coefficients $A(n,r,N)$ depend only on the discriminant r^2-4Nn and the content $\gcd(n,r,N)$, and that it is stable under all Hecke operators. If F is a non-zero Hecke eigenform in $S_k^*(\Gamma_2)$ then there exists a unique normalized Hecke eigenform f in $S_{2k-2}(\Gamma_1)$ such that

(1) $$Z_F(s) = \zeta(s-k+1)\zeta(s-k+2)L_f(s),$$

where

$$L_f(s) = \prod_p (1-\lambda_f(p)p^{-s}+p^{2k-3-2s})^{-1} \quad (\mathrm{Re}(s)>>0, T(n)f=\lambda_f(n)f \ \ (n\in\mathbb{N}))$$

is the Hecke L-function associated to f (Saito-Kurokawa correspondence, loc. cit.). More precisely , there exist isomorphisms

$$\begin{array}{ccc} S_k^*(\Gamma_2) & \overset{\sim}{\longleftrightarrow} & S_{2k-2}(\Gamma_1) \\ & \nwarrow\!\!\searrow^{\sim} \quad {}^{\sim}\!\swarrow\!\!\nearrow & \\ & J_{k,1}^{cusp} & \end{array}$$

which are compatible with Hecke operators in the following sense: $T(p)$ on $J_{k,1}^{cusp}$ corresponds to $T(p)$ on $S_{2k-2}(\Gamma_1)$ and to $T(p)-p^{k-1}-p^{k-2}$

on $S_k^*(\Gamma_2)$. (Note that on $S_k^*(\Gamma_2)$ the relation $T(p^2) = T(p)^2+(p^{k-1}+p^{k-2})\cdot(p^{k-1}+p^{k-2}-T(p))-2p^{2k-3}-p^{2k-4}$ holds.) Moreover, when suitably normalized, the isomorphism

$$J_{k,1}^{cusp} \xrightarrow{\sim} S_k^*(\Gamma_2)$$

is given explicitly by

$$\phi(\tau,z) \mapsto \sum_{N\geq 1} V_N\phi(\tau,z)\ e(N\tau'). \tag{2}$$

By results of Evdokimov [3] and Oda [7] the Hecke eigenforms F in $S_k^*(\Gamma_2)$ are characterized among all Hecke eigenforms in $S_k(\Gamma_2)$ by the fact that their zeta functions $Z_F(s)$ have a pole at s=k.

Theorem 2. <u>Let</u> $F\in S_k^*(\Gamma_2)$ <u>be a non-zero Hecke eigenform, and let</u> $\phi\in J_{k,1}^{cusp}$ <u>be its first Fourier-Jacobi coefficient. Then</u>

$$D_{F,F}^*(s) = \langle\phi,\phi\rangle\ Z_F^*(s). \tag{3}$$

By comparing residues at s=k on both sides of (3) and using (1) we obtain

Corollary. <u>Denote by</u> $f\in S_{2k-2}(\Gamma_1)$ <u>the normalized Hecke eigenform corresponding to</u> F <u>under the Saito-Kurokawa correspondence (1). Then the formula</u>

$$\pi^k c_k \frac{\langle F,F\rangle}{\langle\phi,\phi\rangle} = L_f(k) \tag{4}$$

<u>holds, where</u> $c_k = \frac{3\cdot 2^{2k+1}}{(k-1)!}$.

Formula (4) was first proved by one of the authors ([5],Thm.) by a different method, however, without giving the exact rational value of the constant c_k. Note that $\langle\phi,\phi\rangle = 2^{2k-3}\langle g,g\rangle$, where g is the cusp form of weight $k-\frac{1}{2}$ on $\Gamma_0(4)$ which corresponds to ϕ under the natural map $J_{k,1}^{cusp} \xrightarrow{\sim} M_{k-1/2}$ ([2],Thm. 5.4 and Cor. 4).

Proof of Theorem 2. We have

$$F(Z) = \sum_{N\geq 1} V_N\phi(\tau,z)\ e(N\tau')$$

and hence

$$D_{F,F}(s) = \sum_{N\geq 1} \langle V_N\phi, V_N\phi\rangle N^{-s} \qquad (\mathrm{Re}(s)>>0).$$

By the Proposition, ii)

$$\begin{aligned}\langle V_N\phi, V_N\phi\rangle &= \langle V_N^* V_N\phi, \phi\rangle \\ &= \langle \sum_{t\mid N} t^{k-2}\Big(\sum_{s\mid t}\mu(\tfrac{t}{s})^2 s\Big) T(\tfrac{N}{t})\phi, \phi\rangle .\end{aligned}$$

Since $T(n)\phi = \lambda_f(n)\phi$ for all n, where $\lambda_f(n)$ is the eigenvalue of f under T(n) and f corresponds to F by (1),

$$\langle V_N\phi, V_N\phi\rangle = \sum_{t\mid N} t^{k-2}\Big(\sum_{s\mid t}\mu(\tfrac{t}{s})^2 s\Big)\lambda_f(\tfrac{N}{t})\cdot\langle\phi,\phi\rangle.$$

From the identity

$$\sum_{N\geq 1}\Big(\sum_{s\mid t}\mu(\tfrac{t}{s})^2 s\Big) N^{-s} = \frac{\zeta(s-1)\zeta(s)}{\zeta(2s)}$$

we find

$$D_{F,F}(s) = \frac{\zeta(s-k+1)\zeta(s-k+2)}{\zeta(2s-k+4)} L_f(s),$$

and this by (1) is equivalent to the statement of Theorem 2.

We shall now consider Hecke eigenforms not necessarily in the Maass space. For a fundamental discriminant D<0 we let $P_{k,D}$ be the D^{th} Poincaré series in $J_{k,1}^{cusp}$ characterized by

(5) $\langle \phi, P_{k,D} \rangle = c_\phi(D,r) \quad (\forall\, \phi(\tau,z) = \sum_{\substack{D<0,\, r\in\mathbb{Z} \\ D\equiv r^2 (4)}} c_\phi(D,r)\, e(\frac{r^2-D}{4}\tau + rz) \in J_{k,1}^{cusp})$.

We let $\mathcal{P}_{k,D}$ be the image of $P_{k,D}$ in $S_k^*(\Gamma_2)$ under the map (2), i.e.

$$\mathcal{P}_{k,D}(\tau,z) = \sum_{N\geq 1} V_N P_{k,D}(\tau,z)\, e(N\tau').$$

We denote integral binary quadratic forms by $Q(x,y) = [\alpha,\beta,\gamma](x,y) = \alpha x^2 + \beta xy + \gamma y^2$. Recall that the group Γ_1 acts on such forms by

$$Q \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}(x,y) = Q(ax+by, cx+dy).$$

We ocassionally write $A(Q)$ instead of $A(\alpha,\beta,\gamma)$ for the Fourier coefficients of Siegel modular forms.

Theorem 3. <u>Let</u> F <u>be a non-zero Hecke eigenform in</u> $S_k(\Gamma_2)$. <u>Then</u>

$$D^*_{F,\mathcal{P}_{k,D}}(s) = A(Q)\, Z_F^*(s),$$

<u>where</u> Q <u>denotes any quadratic form of discriminant</u> D <u>representing</u> 1 <u>and</u> $A(Q)$ <u>is the</u> Q<u>-th coefficient of</u> F.

Remark. If $A(Q) \neq 0$ for some D, then by combining Theorems 1 and 3 we obtain a new proof for the meromorphic continuation and functional equation of $Z_F^*(s)$, and also for the fact that for F in the orthogonal complement of the Maass space the zeta function $Z_F^*(s)$ is holomorphic for all s (cf. [3,7]). The smallest weight k for which $S_k^*(\Gamma_2)^\perp \neq \{0\}$ is $k=20$, and in this case we have $A(Q)\neq 0$ for $D=-4$, cf. [6], p. 157.

Proof of Theorem 3. Let ϕ_N be the N^{th} Fourier-Jacobi coefficient of F, and write $F(Z) = \sum A(n,r,N)\, e(n\tau + rz + N\tau')$. The N^{th} coefficient of $D_{F,\mathcal{P}_{k,D}}(s)$ equals

$$\langle \phi_N, V_N P_{k,D} \rangle = \langle V_N^* \phi_N, P_{k,D} \rangle$$

$$= \sum_{d|N} d^{k-2} \sum_{\substack{s(2d)\\ s^2\equiv D(4N)}} A\left(\frac{N}{d}\cdot\frac{s^2-D}{4N}, \frac{N}{d}\cdot s, \frac{N}{d}\cdot d\right)$$

by (5) and the Proposition, i).

Let $\{Q_i\}_{i=1,\ldots,h}$ be a set of representatives of binary quadratic forms of discriminant D. Then the above sum can be written as

$$\sum_{i=1}^{h} \sum_{d|N} d^{k-2} n(Q_i;d) A\left(\tfrac{N}{d}Q_i\right),$$

where $n(Q_i;d)$ is the number of s(mod 2d) such that $s^2\equiv D(\mathrm{mod}\ 4d)$ and $[\frac{s^2-D}{4d}, s, d]$ is equivalent to Q_i.

Observing that

$$\sum_{N\geq 1} n(Q_i;N)\ N^{-s} = \zeta_{Q_i}(s)\zeta(2s)^{-1},$$

where $\zeta_{Q_i}(s)$ is the zeta function of the ideal class of $\mathbb{Q}(\sqrt{D})$ corresponding in the usual way to the Γ_1-class of Q_i (cf. [8],Propos.3), we obtain

(6) $$\zeta(2s-2k+4)D_{F,P_{k,D}}(s) = \sum_{i=1}^{h} \zeta_{Q_i}(s-k+2)R_{Q_i}(s)$$

with

$$R_{Q_i}(s) = \sum_{N\geq 1} A(NQ_i)N^{-s}.$$

Identity (6) so far is true for any form F in $S_k(\Gamma_2)$. We shall now rewrite the right-hand side of (6) in terms of $Z_F(s)$, if F is an eigenform. In this case we have the fundamental identity

(7) $$A_\chi Z_F(s) = L(s-k+2,\chi) \sum_{i=1}^{h} \chi(Q_i)R_{Q_i}(s)$$

valid for any ideal class character χ, where $L(s,\chi)$ is the L-function attached to χ and $A_\chi = \sum_{i=1}^{h} \chi(Q_i)A(Q_i)$ ([1],Thm. 2.4.1). Inverting (7) we find

$$(8) \qquad R_{Q_i}(s) = \frac{1}{h}Z_F(s)\sum_{\chi}\overline{\chi}(Q_i)A_\chi\, L(s-k+2,\chi)^{-1} \qquad (i=1,\ldots,h).$$

Inserting (8) into (6) and using the fact that $L(s,\chi)=L(s,\overline{\chi})$ we obtain after a short calculation

$$\zeta(2s-2k+4)D_{F,P_{k,D}}(s) = A(Q)Z_F(s),$$

where Q represents the trivial class. This proves Theorem 3.

We still have to prove the Proposition.

Proof of Proposition, i). We identify Γ_1 with its canonical image in Γ_1^J. Let G be a Γ_1-conjugate of a subgroup of finite index of Γ_1^J. Then G contains a subgroup of finite index in Γ_1^J, say H. We define the Petersson product of two cusp forms ϕ and ψ of weight k and index N on G by

$$\langle\phi,\psi\rangle = [\Gamma_1^J:H]^{-1} \int_{G\backslash \mathfrak{H}\times\mathbb{C}} \phi(\tau,z)\overline{\psi(\tau,z)}\, v^{k-3}e^{-4\pi Ny^2/v}\, du\,dv\,dx\,dy.$$

This normalization of the scalar product does not depend on the choice of the subgroup H, and we have the formula

$$(9) \qquad \langle\phi|\eta,\psi\rangle = \langle\phi,\psi|\eta^{-1}\rangle$$

for all $\eta\in J(\mathbb{Q}):=SL_2(\mathbb{Q})\ltimes\mathbb{Q}^2\cdot S^1$ (S^1 the circle group). Here we use the notation "$\phi|\eta = \phi|_{k,N}\eta$" for the usual "$|_{k,N}$"-action of elements $\eta\in J(\mathbb{Q})$ on functions $\phi(\tau,z)$ (cf. [2],§1). The above assertions can easily be checked using standard techniques as in the case of ordinary modular forms.

By [2],§6 we have for $\phi\in J_{k,1}^{cusp}$

$$V_N\phi = N^{k/2-1} \sum_{A\in\Gamma_1\backslash M_2(\mathbb{Z})_N} \phi_{\sqrt{N}} \,|_{k,N}\left(\tfrac{1}{\sqrt{N}}\, A\right),$$

where

$$M_2(\mathbb{Z})_N = \{(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix})\in M_2(\mathbb{Z}) \mid ad-bc=N\}$$

and where $\phi_c(\tau,z) = \phi(\tau,cz)$ $(c\in\mathbb{C})$. Denoting by $M_2^*(\mathbb{Z})_N$ the primitive elements in $M_2(\mathbb{Z})_N$ and using the notation "$\frac{N}{N'}=\square$" to mean that $\frac{N}{N'}$ is a perfect square we can rewrite the above formula as

$$V_N\phi = N^{k/2-1} \sum_{N'|N,N/N'=\square}\ \sum_{A\in\Gamma_1\backslash M_2^*(\mathbb{Z})_{N'}} \phi_{\sqrt{N}}|_{k,N}\left(\tfrac{1}{\sqrt{N'}}\, A\right)$$

$$= N^{k/2-1} \sum_{N'|N,N/N'=\square}\ \sum_{A\in\Gamma^0(N')\backslash\Gamma_1} \phi_{\sqrt{N}}|_{k,N} \begin{pmatrix}\sqrt{N'}^{-1} & 0\\ 0 & \sqrt{N'}\end{pmatrix} A,$$

where in the last line $\Gamma^0(N')$ is the subgroup $\{(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix})\in\Gamma_1 \mid N'|b\}$ and we have made use of the fact that the map $\Gamma_1 \to M_2^*(\mathbb{Z})_{N'}$, $(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}) \mapsto (\begin{smallmatrix} 1 & 0\\ 0 & N'\end{smallmatrix})(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix})$ induces a bijection $\Gamma^0(N')\backslash\Gamma_1 \xrightarrow{\sim} \Gamma_1\backslash M_2^*(\mathbb{Z})_{N'}$. Observe that the function

$$\phi_{\sqrt{N}}|_{k,N} \begin{pmatrix}\sqrt{N'}^{-1} & 0\\ 0 & \sqrt{N'}\end{pmatrix}(\tau,z) = \phi\left(\tfrac{\tau}{N'},\sqrt{\tfrac{N}{N'}}\, z\right)$$

is a Jacobi cusp form of weight k and index N on $\Gamma^0(N)\ltimes\mathbb{Z}^2$.

The above discussion gives for $\phi\in J_{k,1}^{cusp}$, $\psi\in J_{k,N}^{cusp}$ the formula

$$\langle V_N\phi,\psi\rangle = N^{k/2-1} \sum_{N'|N,N/N'=\square}\ \sum_{A\in\Gamma^0(N')\backslash\Gamma_1} \left\langle \phi_{\sqrt{N}}|_{k,N}\begin{pmatrix}\sqrt{N'}^{-1} & 0\\ 0 & \sqrt{N'}\end{pmatrix}A,\psi\right\rangle$$

$$= N^{k/2-1} \sum_{N'|N,N/N'=\square} [\Gamma_1:\Gamma^0(N')] \left\langle \phi_{\sqrt{N}}|_{k,N}\begin{pmatrix}\sqrt{N'}^{-1} & 0\\ 0 & \sqrt{N'}\end{pmatrix},\psi\right\rangle$$

(by (9)).

Since $\psi_{\sqrt{N}^{-1}}|_{k,N}\begin{pmatrix}\sqrt{N'} & 0\\ 0 & \sqrt{N'}^{-1}\end{pmatrix}$ has index 1 and is on $\Gamma_0(N)\ltimes\mathbb{Z}^2$ ($\Gamma_0(N) = \{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_1 \mid N\mid c\}$), and since

$$<\phi_{\sqrt{N}}|_{k,N}\begin{pmatrix}\sqrt{N'}^{-1} & 0\\ 0 & \sqrt{N'}\end{pmatrix},\psi> = <\phi,\psi_{\sqrt{N}^{-1}}|_{k,N}\begin{pmatrix}\sqrt{N'} & 0\\ 0 & \sqrt{N'}^{-1}\end{pmatrix}>,$$

we have

$$<\phi_{\sqrt{N}}|_{k,N}\begin{pmatrix}\sqrt{N'}^{-1} & 0\\ 0 & \sqrt{N'}\end{pmatrix},\psi> = N'^{-2}[\Gamma_1:\Gamma_0(N')]^{-1}\sum_{X \bmod N'\mathbb{Z}^2}\ \sum_{A\in\Gamma_0(N')\backslash\Gamma_1}$$
$$<\phi,\psi_{\sqrt{N}^{-1}}|_{k,N}\begin{pmatrix}\sqrt{N'} & 0\\ 0 & \sqrt{N'}^{-1}\end{pmatrix}A|_{k,N}X>,$$

hence by a similar argument as above

$$<V_N\phi,\psi> = <\phi,N^{k/2-3}\sum_{X \bmod N\mathbb{Z}^2}\ \sum_{A\in\Gamma_1\backslash M_2(\mathbb{Z})_N}\psi_{\sqrt{N}^{-1}}|_{k,N}\left(\tfrac{1}{\sqrt{N}}A\right)|_{k,N}X>$$

As the function standing on the right-hand side in the Petersson product in the above formula is, in fact, in $J_{k,1}^{cusp}$ (immediate verification!), we have proved that

$$J_{k,N}^{cusp}\longrightarrow J_{k,1}^{cusp},$$

$$\psi\longmapsto N^{k/2-3}\sum_{X \bmod N\mathbb{Z}^2}\ \sum_{A\in\Gamma_1\backslash M_2(\mathbb{Z})_N}\psi_{\sqrt{N}^{-1}}|_{k,N}A|_{k,N}X$$

is the operator V_N^* adjoint to V_N.

We must now compute the Fourier expansion of $V_N^*\psi$. Write

$$\psi(\tau,z) = \sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv r^2(4N)}} c(D,r)\,e\left(\frac{r^2-D}{4N}\tau+rz\right).$$

Choosing as a set of representatives for $\Gamma_1\backslash M_2(\mathbb{Z})_N$ the matrices $\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\in M_2(\mathbb{Z})$ with $ad=N$, $b\,(\mathrm{mod}\ d)$ we obtain from the above formula

$$V_N^*\psi(\tau,z) = N^{k/2-3}\sum_{\lambda,\mu(N)}\ \sum_{\substack{ad=N\\ b(d)}}\left(\frac{d}{\sqrt{N}}\right)^{-k}\psi\left(\frac{a\tau+b}{d},\frac{z+\lambda\tau+\mu}{d}\right)e(\lambda^2\tau+2\lambda z)$$

$$= N^{k/2-3} \sum_{\lambda,\mu(N)} \sum_{\substack{ad=N\\ b(d)}} \left(\frac{d}{\sqrt{N}}\right)^{-k} \sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv r^2(4N)}} c(D,r)\, e\Big(\big(\tfrac{r^2-D}{4N}\cdot\tfrac{a}{d}+\tfrac{\lambda r}{d}+\lambda^2\big)\tau$$

$$+\big(\tfrac{r}{d}+2\lambda\big)z+\tfrac{r^2-D}{4N}\cdot\tfrac{b}{d}+\tfrac{r\mu}{d}\Big).$$

The sum

$$\sum_{b(d),\,\mu(N)} e\left(\frac{r^2-D}{4N}\cdot\frac{b}{d}+\frac{r\mu}{d}\right)$$

has the value Nd or zero according as both the conditions $d\,|\,\frac{r^2-D}{4N}$ and $d\,|\,r$ are satisfied or not. Hence replacing r by rd and D by Dd^2 we obtain

$$V_N^*\psi(\tau,z) = N^{k-2} \sum_{\lambda(N)} \sum_{d|N} d^{1-k} \sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv r^2(4N/d)}} c(d^2D,dr)$$

$$\cdot\, e\left(\frac{(r+2\lambda)^2-D}{4}\tau+(r+2\lambda)z\right)$$

$$= N^{k-2} \sum_{d|N} d^{1-k} \sum_{\lambda(N)} \sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv (r-2\lambda)^2(4N/d)}} c(d^2D,dr)\, e\left(\frac{r^2-D}{4}\tau+rz\right).$$

Now set $\lambda\equiv s+\frac{N}{d}s'\pmod N$ with s running over $\mathbb{Z}/\frac{N}{d}\mathbb{Z}$ and s' over $\mathbb{Z}/d\mathbb{Z}$. Then

$$d(r-2\lambda)\equiv d(r-2s) \pmod{2N},\quad D\equiv (r-2s)^2 \pmod{4\tfrac{N}{d}}.$$

Since the coefficients c(D,r) depend only on the pair (D,r) with r(mod 2N) and $D\equiv r^2$(mod 4N) we obtain

$$V_N^*\psi(\tau,z) = N^{k-2} \sum_{d|N} d^{2-k} \sum_{s(N/d)} \sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv (r-2s)^2(4N/d)}} c(d^2D,d(r-2s))$$

$$\cdot\, e\left(\frac{r^2-D}{4}\tau+rz\right)$$

$$= \sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv r^2(4)}} \left(\sum_{d|N} d^{k-2} \sum_{\substack{s(2d)\\ s^2\equiv D(4d)}} c(d^2D,ds)\right) e\left(\frac{r^2-D}{4}\tau+rz\right),$$

where in the last line we have replaced d by $\frac{N}{d}$ and r-2s by s.

Proof of Proposition,ii). The identity claimed can be checked using the explicit formulas for the action of V_N, V_N^* and $T(n)$ on Fourier coefficients. In fact, it is sufficient to check it on Fourier coefficients indexed by fundamental discriminants, since $V_N^* V_N$ and $T(n)$ commute and $J_{k,1}^{cusp}$ has a basis of Hecke eigenforms whose Fourier coefficients are determined by those indexed by fundamental discriminants. This simplifies the calculations considerably. We leave the details to the reader.

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