ON THE GEOMETRIC AND THE ALGEBRAIC RANK OF GRAPH MANIFOLDS

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ABSTRACT. For any $n \in \mathbb{N}$ we construct graph manifolds of genus $4n$ that have $3n$ generated fundamental group.

1. INTRODUCTION

A *Heegaard surface* of an orientable closed 3-manifold M is an embedded orientable surface S such that $\overline{M - S}$ consists of 2 handlebodies V_1 and V_2. This decomposition of M is called a *Heegaard splitting* and denoted by $M = V_1 \cup_S V_2$. We say that the splitting is of *genus* g if S is of genus g. It is not difficult to see that any orientable closed 3-manifold admits a Heegaard splitting. If M admits a Heegaard splitting of genus g but no Heegaard splitting of smaller genus then we say that M has *Heegaard genus* g and write $g(M) = g$.

Clearly any curve in a handlebody can be homotoped to its boundary. It follows that for any Heegaard splitting $M = V_1 \cup_S V_2$ every curve in M can be homotoped into V_1. Thus the map induced by the inclusion of V_1 into M maps a generating set of $\pi_1(V_1)$ to a generating set of $\pi_1(M)$. As $\pi_1(V_1)$ is generated by g elements it follows that $\pi_1(M)$ is also generated by g elements. Thus $g(M) \geq r(M)$ where $r(M)$ denotes the minimal number of generators of $\pi_1(M)$. Sometimes we will refer to $g(M)$ as the *geometric rank* and to $r(M)$ as the *algebraic rank* of M.

F. Waldhausen [8] asked whether the converse inequality also holds, i.e., whether $g(M) = r(M)$. A positive answer would have implied the Poincaré conjecture. First counterexamples however were found by M. Boileau and H. Zieschang [1]. These examples were Seifert fibered manifolds with $g(M) = 3$ and $r(M) = 2$. The work of Y. Moriah and J. Schultens [4] further shows that this class extends to higher genus examples, i.e. Seifert manifolds with $g(M) = n + 1$ and $r(M) = n$. In [9] a class of graph manifolds was found for which $g(M) = 3$ and $r(M) = 2$. The original Boileau-Zieschang examples can be interpreted as a special case of these graph manifolds.

In this note we will show how the phenomenon observed in [9] generalizes and how it can occur multiple times within a single graph manifold. This yields graph manifolds where the difference between the algebraic and the geometric rank is arbitrarily high.

Key words and phrases. Graph manifolds, Heegaard genus, rank.

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2. Formulation of the main results

Let M be a closed graph manifold. We will always assume that M comes equipped with its characteristic manifold $\mathcal{T} = \mathcal{T}_M$ and a fixed Seifert fibration on every component of $\overline{M - \mathcal{T}}$. Recall that the Seifert fibrations are unique up to isotopy except for components homeomorphic to Q, the Seifert space with base orbifold the disk with two cone points of order 2. The space Q can also be fibered as the orientable circle bundle over the Möbius band. We will refer to the components of $\overline{M - \mathcal{T}}$ as the *Seifert pieces* of M. Recall the the Seifert pieces of M are up to isotopy precisely the maximal Seifert submanifolds of M. We will only work with *totally orientable* graph manifolds, i.e. orientable graph manifolds whose Seifert pieces have orientable base orbifold. This makes the Seifert fibrations unique up to isotopy on all Seifert pieces.

Let N be a Seifert piece of M. Denote the fiber of N by f. Let $T_1, \dots, T_n$ be the boundary components of N and let $\gamma_i \subset T_i$ be the curve corresponding to the fiber of the Seifert piece L_i where L_i is the Seifert piece reached by travelling from N transversely through T_i. Note that we possibly have $N = L_i$. The maximality of the Seifert piece N guarantees that for all i the intersection number of f with γ_i does not vanish.

We then define $\hat{N}$ to be the manifold $N(\gamma_1, \dots, \gamma_n)$ obtained from N by performing a Dehn filling with slope γ_i at each boundary component T_i. It is clear that the Seifert fibration of N can be extended to a Seifert fibration of $\hat{N}$ as f has non-trivial intersection number with all γ_i.

In the following we will denote the base orbifold of a Seifert piece N by $\mathcal{O}(N)$. We will denote an orbifold by its topological type with a list of the orders of cone points, where ∞ stands for a boundary component. We will denote the disc by D, the sphere by S^2, the annulus by A, the orientable surface of genus g by F_g and the projective plane by P^2.

Theorem 1. *Let M be a closed graph manifold consisting of two Seifert pieces N_1 and N_2 glued along T, where $\mathcal{O}(N_1) = F_g(r, \infty)$, $\mathcal{O}(N_2) = D(p,q)$ with $(p,q) = 1$ and $\min(p,q) \leq 2g+1$ such that the intersection number of the fibers of N_1 and N_2 equals 1.*

Then $\pi_1(M)$ is generated by $2g+1$ elements. Furthermore M admits a Heegaard splitting of genus $2g+1$ if and only if one of the following holds:

1. *N_2 is the exterior of a s-bridge knot with $s \leq 2g+1$ and the fiber of N_1 is identified with the meridian of N_2, i.e. $\hat{N}_2 = S^3$.*
2. *$\hat{N}_1$ admits a horizontal Heegaard splitting of genus $2g$.*

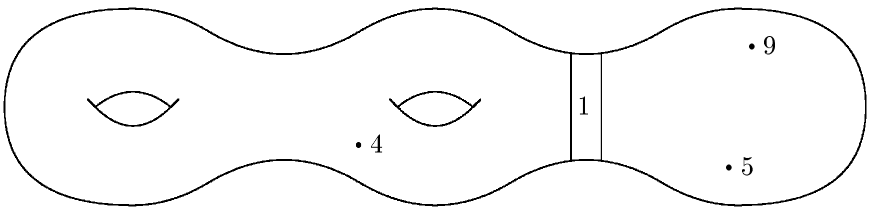

FIGURE 1. A graph manifold with 5-generated fundamental group

We will further see that all manifolds of this type admit a Heegaard splitting of genus $2g+2$. Furthermore do most do most of these manifolds not admit a Heegaard splitting of genus $2g+1$ as for any given pair of such manifolds N_1 and N_2 there are at most three glueing maps that yield a graph manifold of genus $2g+1$. It is furthermore possible to show that $\pi_1(M)$ cannot be generated by less than $2g+1$ elements, the argument however is complicated.

A careful analysis of the above examples shows that the phenomenon is of local nature, it can therefore be reproduced multiple times within a graph manifold with more complex underlying graph. This yields the following:

Theorem 2. *For any $n \in \mathbb{N}$ there exists a graph manifold M_n with $3n$-generated fundamental group that has Heegaard genus $4g$.*

This paper is organized as follows. In Section 3 we review the structure theorem for Heegaard splittings of totally orientable graph manifolds as proven in [6]. Then we study in more detail how Heegaard surfaces can intersect the Seifert pieces that are the building blocks of our examples. In Section 5 and Section 6 we will then give the proofs of Theorem 1 and Theorem 2. We conclude by describing a class orientable Seifert manifolds with $2n$-generated fundamental group which we believe to be of Heegaard genus $3n$. These manifolds are however not totally orientable.

3. Heegaard splittings of totally orientable graph manifolds

A graph manifold M is *totally orientable* if every Seifert piece N of M fibers over an orientable base space and if M itself is orientable. In [6] it is shown that the Heegaard splittings of totally orientable graph manifolds have a structure that can be completely described. We use the terms horizontal and vertical to describe the possible structure for the restriction of the Heegaard surface to the Seifert pieces of a graph manifold. Note however, that the restriction of the Heegaard surface is typically not horizontal (i.e., everywhere transverse to the fibers) or vertical (everywhere aligned with the fibers) in the true sense.

Let M be a graph manifold and $M = V \cup_S W$ be a Heegaard splitting of M. Let N be a Seifert piece of M. We say that $S \cap N$ is *h1* if it is a horizontal incompressible surface with an even number of components or if it is obtained from a such a surface by adding some number of tubes. Here adding a tube means adding an annulus that is the boundary of a regular neighborhood of a portion of a fiber and removing the two small disks bounded by the ends of the tube.

Let T be a component of $\mathcal{T}_M$ that meets N. Let $N(T)$ be a collar of T in N. Then $\partial N(T) = T_0 \cup T_1$ (where $T_0 = T$). If there are curves $c, c' \subset T$ such that $c \cap c'$ consists of a single point p, and $S \cap N(T)$ is isotopic to the boundary of a collar in $N(T)$ of $(c \times \{0\}) \cup (c' \times \{1\}) \cup (p \times I)$ and so that $S \cap (N - N(T))$ is a (two component) horizontal incompressible surface, then we say that $S \cap N$ is *h2*.

If there is a regular or exceptional fiber e in N with regular neighborhood $\eta(e)$, such that $e \subset S$ and $S \cap (N - \eta(e))$ is a (two sheeted) horizontal incompressible surface, and such that $S\eta(e)$ is an annulus that wraps once around e, then we say that $S \cap N$ is *h3*. If $S \cap N$ is either h1, h2, or h3, we say that $S \cap N$ is *horizontal*.

We say that $S \cap N$ is *vertical* if it is a separating (possibly disconnected) vertical incompressible surface or if it is obtained from such a surface by adding tubes. Here adding a tube means adding an annulus that is the boundary of a regular neighborhood of a portion of a horizontal arc that projects to an imbedded arc and removing the two small disks bounded by the ends of the tube.

We say that $M = V_1 \cup_S V_2$ is *standard* if for each component N of $M - \mathcal{T}$, $S \cap N$ is either horizontal or vertical.

The main theorem in [6] is the following:

Theorem 3. *Let M be a totally orientable graph manifold. For any Heegaard splitting $M = V_1 \cup_S V_2$, S can be isotoped so that $S \cap N$ is standard for every Seifert piece N of M.*

The most confounding example of these options appears to be h3. But this option is not as esoteric as it appears. In fact horizontal surfaces of type h3 in a Seifert piece are closely related to horizontal Heegaard surfaces of closed Seifert manifolds as studied in [4].

Consider the following: suppose that M is a closed totally orientable graph manifold and that S is a Heegaard surface that on each Seifert piece, S is either horizontal or vertical. This implies in particular that for any torus $T \subset \mathcal{T}_M$ the set $S \cap T$ consists of a family of parallel simple closed curves. To any Seifert piece N of M we associate the manifold N_S, which is the manifold obtained from N by performing a Dehn filling at every component T of ∂N along a slope represented by the curves $S \cap T$. We have the following observation:

Lemma 4. *Suppose that $S \cap N$ is of type h3. Then the Seifert manifold S_N has a horizontal Heegaard surface S' such that $S' \cap N = S \cap N$. If $S \cap N$ is planar then S' is homeomorphic to S^2.* □

Proof. There is clearly only one way to extend $S \cap N$ to a horizontal Heegaard surface of S_N namely by gluing meridional discs of the glued in solid tori to the boundary components of $S \cap N$. Clearly this surface is a horizontal Heegaard surface for S_N. If $S \cap N$ is planar then all but one boundary component get capped of which results in a disc whose regular neighborhood is a ball. The assertion follows. □

4. Some lemmata

The following lemmata will enable us to compute the Heegaard genus of certain graph manifold in the next section. We start by discussing the possible horizontal surfaces of type in h3 in the relevant Seifert manifolds. Some proofs rely on the theory of 2-dimensional orbifolds and their covering theory as discussed in [7].

Lemma 5. *Let M be a graph manifold and N be a Seifert piece with $\mathcal{O}(N) = D(p,q)$ and $(p,q) = 1$. Suppose $S \cap N$ is a planar surface that is horizontal of type h3. Then the following hold:*

1. *N_S is homeomorphic to S^3.*
2. *$S \cap \partial N$ contains exactly $2p$ or $2q$ components.*

It should be noted that N_S being homeomorphic to S^3 is equivalent to N being the exterior of an r-bridge knot with meridian μ parallel to $\partial N \cap S$ where $r = \min(p,q)$.

Proof. Possibly after exchanging p and q we can assume that S is horizontal in the space $\bar{N}$ obtained from N after removing a regular neighborhood of the exceptional fiber corresponding to the cone point of order q or by removing a neighborhood of a regular fiber. Clearly $\bar{N}$ is a Seifert space with $\mathcal{O}(\bar{N}) = A(p)$ or $\mathcal{O}(\bar{N}) = A(p,q)$. Let T_1 be the boundary component of $\bar{N}$ that bounds the drilled out solid torus and T_2 be the boundary of N. Let $\bar{S}$ be a component to $S \cap \bar{N}$. Clearly $\bar{S}$ is planar as it is a subsurface of a planar surface.

As we assume that S is of type $h3$ in N it follows that $\bar{S} \cap T_1$ consists of a single loop α. Let γ be one component of $\bar{S} \cap T_2$ and let g be an element of $\pi_1(\bar{N})$ corresponding to γ. Recall that all other components of $\bar{S} \cap T_2$ are parallel to γ. Let n be the intersection number of γ with the fiber.

As $\bar{S}$ is horizontal in $\bar{N}$ it follows that there exists a finite sheeted orbifold covering $p : \bar{S} \to O(\bar{N})$, in particular $p_*(\pi_1(\bar{S}))$ is of finite index in $\pi_1(O(\bar{N}))$. We distinguish the cases $O(\bar{N}) = A(p)$ and $O(\bar{N}) = A(p,q)$.

Case 1: $O(\bar{N}) = A(p)$**.** We have $\pi_1(A(p)) = \langle x, y | x^p \rangle$ where the generator y corresponds to the boundary curve corresponding to T_2. This implies in particular that $p_*(g)$ is conjugate to y^n.

As $\bar{S}$ is planar this implies that $\pi_1(\bar{S})$ is generated by homotopy classes that correspond to the components of $\bar{S} \cap T_2$, i.e. $p_*(\pi_1(\bar{S}))$ is generated by conjugates of the element y^n. Let $N(y^n)$ be the normal closure of y in $\pi_1(A(p))$. Clearly $\pi_1(A(p))/N(y^n) \cong \mathbb{Z}_n * \mathbb{Z}_p$ is infinite unless $n = 1$. As $p_*(\pi_1(\bar{S})) \subset N(y^n)$ this implies that $n = 1$ as otherwise $p_*(\pi_1(\bar{S}))$ is contained in a subgroup of infinite index in $\pi_1(A(p))$ and is therefore of infinite index itself. Thus we can assume that $n = 1$ and that $p_*(\pi_1(\bar{S})) \subset N(y)$.

Note first that the orbifold covering space $\tilde{S}$ corresponding to $N(y)$ is a orbifold without cone points and is homeomorphic to the $(p+1)$-punctured sphere. Denote the corresponding covering map by $\tilde{p}$.

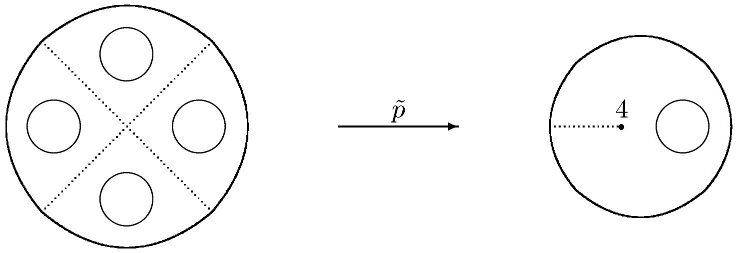

FIGURE 2. The 4-sheeted covering of $A(4)$ by a 5-punctured sphere

As $p_*(\pi_1(\bar{S})) \subset N(y)$ it follows that there exists a covering $p' : \bar{S} \to \tilde{S}$ such that $p = \tilde{p} \circ p'$.

Claim p' is a homeomorphism. As for both $\bar{S}$ and $\tilde{S}$ all but one boundary component map onto a curve corresponding to the element y it follows that p' is a homeomorphism when restricted to any of these boundary components. In particular p' extends to a covering $p'_\# : \bar{S}_\# \to \tilde{S}_\#$ where $\bar{S}_\#$ and $\tilde{S}_\#$ are the spaces obtained from $\bar{S}$ and $\tilde{S}$ by gluing discs to these boundary components. As $\bar{S}_\#$ and $\tilde{S}_\#$ are discs it follows that the obtained map is a homeomorphism. Thus the original p' was a homeomorphism which proves the claim.

The second assertion is now immediate as $S \cap \bar{N}$ is obtained from 2 copies of $\bar{S}$ and identifying two boundary components. All resulting boundary components lie in T_2. The first assertion follows from Lemma 4.

Case 2: $O(\bar{N}) = A(p,q)$**.** We have $\pi_1(A(p,q)) = \langle x, y, z | y^p, z^q \rangle$ where the generator x corresponds to the boundary curve corresponding to T_2. We see as in the first case that $p_*(O(\bar{N}))$ lies in the kernel of the map $\phi : \pi_1(A(p,q)) \to \pi_1(A(p,q))/N(x^n)$. As $\pi_1(A(p,q))/N(y^n) \cong \mathbb{Z}_n * \mathbb{Z}_p * \mathbb{Z}_q$ is infinite for all $n \in \mathbb{N}$ this implies that $p_*(O(\bar{N}))$ is of infinite index in $\pi_1(A(p,q))$ which contradicts our assumption. $\square$

Lemma 6. *Let M be a graph manifold and let N be a Seifert piece with $\mathcal{O}(N) = F_g(p,\infty)$ or $\mathcal{O}(N) = F_g(p,\infty,\infty)$. Suppose that $S \cap N$ is horizontal of type h3 and $\chi(S \cap N) > -8g$ or $\chi(S \cap N) > -8g - 4$, respectively. Then the following hold*

1. *$S \cap T$ has two components for every component T of ∂N.*
2. *$S \cap N$ extends to a horizontal Heegaard surface of genus $2g$ of N_S.*

Proof. We only deal with the case $\mathcal{O}(N) = F_g(p,\infty)$ the other case is analogous.

Suppose that $S \cap N$ is of type h3 with respect to the exceptional fiber or a regular fiber and let $\bar{N}$ be the space obtained by drilling out the neighborhood of this fiber. Let $\bar{S}$ be a component of $\bar{N} \cap S$. Recall that $S \cap N$ is obtained from two copies of $\bar{S}$ by identifying them along a boundary component. In particular we have that $\chi(S \cap N) = 2\chi(\bar{S})$.

Now $\bar{S}$ is a finite sheeted covering of $\mathcal{O}(\bar{N})$ where $\mathcal{O}(\bar{N}) = F_g(\infty,\infty)$ or $\mathcal{O}(\bar{N}) = F_g(p,\infty,\infty)$ depending on what kind of fiber was drilled out. Suppose that the covering is n-sheeted. Note that in the case $\mathcal{O}(\bar{N}) = F_g(p,\infty,\infty)$ it must hold that $n \geq p$ as otherwise the covering space must be a orbifold with singularities. Thus we have

$$\chi(S \cap N) = 2\chi(\bar{S}) = 2n\chi(\mathcal{O}(\bar{N})).$$

As $\chi(\mathcal{O}(\bar{N})) = -2g$ or $\chi(\mathcal{O}(\bar{N})) = -2g - 1 + 1/p$ it follows immediately from the hypothesis on the Euler characteristic that $n = 1$. Thus $\mathcal{O}(\bar{N}) = F_g(\infty,\infty)$, i.e. the exceptional fiber was drilled out. Assertion (1) is now immediate and (2) follows from the proof of Lemma 4 $\square$

It will be important that many Seifert manifolds do not admit a horizontal surface of type h3 of small genus indiscriminately of what graph manifold they belong to.

Lemma 7. *Let N be a Seifert manifold with $\mathcal{O}(N) = F_g(p,\infty)$ such that the exceptional fiber has invariant (α,β) with $1 \leq \beta < \alpha$. Then the following hold:*

1. *If $\alpha = 2$ then there exist two slopes γ on ∂N such that $N(\gamma)$ admits a horizontal Heegaard splitting of genus $2g$.*
2. *If $\alpha \neq 2$ and $\beta \in \{1, \alpha - 1\}$ then exists one slope γ on ∂N such that $N(\gamma)$ admits a horizontal Heegaard splitting of genus $2g$.*
3. *In all other cases $N(\gamma)$ has no Heegaard splitting of genus $2g$ if $\gamma \neq f$.*

Proof. If γ is the fiber then $N(\gamma)$ is not a Seifert manifold. In particular $N(\gamma)$ admits no horizontal Heegaard splitting as those are only defined for Seifert manifolds. If the intersection number m of γ with the fiber is greater than 1 then $M(\gamma)$ is a Seifert space with base orbifold $F_g(p,m)$ which has no Heegaard splitting of genus $2g$ by (i) of Proposition 1.4 of [1]. Suppose now that $m = 1$. Let $e \in \mathbb{Z}$ be the Euler class of the Seifert space. By (iii) of Proposition 1.4 of [1] it follows that $N(\gamma)$ admits no Heegaard splitting of genus $2g$ unless $\beta - e\alpha = \pm 1$. It is clear that there exists two values for e such that the equation holds if $\beta = 1$ and $\alpha = 2$, that there exists one solution if $\beta \in \{1, \alpha - 1\}$ and none otherwise. The corresponding Heegaard splittings are constructed in Section 1.10 of [1]. This proves the assertion. $\square$

Lemma 8. *Let N be a Seifert manifold with $\mathcal{O}(N) = D(p,q)$ and $(p,q) = 1$. Then N contains no compact planar horizontal surface.*

Proof. Suppose that S is a compact planar horizontal surface in N. Then there exists a finite sheeted orbifold covering $p : S \to D(p,q)$. As all components of ∂S are parallel on ∂N it follows that there exists a number $n \in \mathbb{N}$ such the restriction of p to any component of ∂S is a n-sheeted covering. This implies that we can extend p to a orbifold covering $p : S^2 \to S^2(p,q,n)$ by gluing a disc to any component of ∂S and a disc with a cone point of order n to $D(p,q)$. If $n = 1$ this yields a contradiction as $S^2(p,q,1) = S^2(p,q)$ is a bad orbifold which admits no covering by a manifold. If $n \neq 1$, then $S^2(p,q,n)$ must be a spherical orbifold with universal cover the sphere. Moreover, N_S is a Seifert manifold with $\mathcal{O}(N_S) = S(p,q,n)$. As such it is irreducible. This yields a contradiction, as $S \subset N$ extends to a horizontal, hence incompressible, sphere in N_S. $\square$

Lemma 9. *Let M be a graph manifold and let N be a Seifert fibered submanifold with $\mathcal{O}(N) = F_g(p,\infty)$ or $\mathcal{O}(N) = F_g(p,\infty,\infty)$. If $S \cap N$ is vertical, then $\chi(S \cap N) - |S \cap \partial N| \leq -4g-2$ or $\chi(S \cap N) - |S \cap \partial N| \leq -4g-4$, respectively.*

Proof. If $S \cap N$ is vertical, then $S \cap N$ is obtained from saturated annuli $A_1, \ldots, A_n$ and tori $T_1, \ldots, T_k$ by performing ambient 1-surgery along arcs $\beta_1, \ldots, \beta_m$ that project to disjoint imbedded arcs disjoint from the projection of $A_1, \ldots, A_n$ and $T_1, \ldots, T_k$ except at their endpoints. Thus the projections $a_1, \ldots, a_n$ of $A_1, \ldots, A_n$, the projections $t_1, \ldots, t_k$ of $T_1, \ldots, T_k$ and the projections $b_1, \ldots, b_m$ of $\beta_1, \ldots, \beta_m$ form a graph Γ in $F_g(p,\infty)$ or $F_g(p,\infty,\infty)$. By isotoping $\beta_1, \ldots, \beta_m$, if necessary, we may assume that Γ is trivalent. We denote the set of vertices of Γ by $V\Gamma$ and the set of edges by $E\Gamma$. A simple calculation shows that $\#V\Gamma = 2n + k + 2m$ and that $\#E\Gamma = n + k + 3m$.

Recall that the result of cutting a handlebody along annuli is still a handlebody or an union of handlebodies. The preimage under the projection map of $F_g(p,\infty) - \Gamma$ or $F_g(p,\infty,\infty) - \Gamma$ is the result of cutting this union of handlebodies along disks. Thus, this preimage must also be a union of handlebodies. In particular, each component of this preimage must be handlebody. A component of this preimage is just the preimage of a component of $F_g(p,\infty) - \Gamma$ or $F_g(p,\infty,\infty) - \Gamma$. The preimage of such a component can only be a handlebody if the component is a disk with at most one cone point. Thus Γ cuts $F_g(p,\infty)$ or $F_g(p,\infty,\infty)$ into a collection of disks with at most one cone point which we denote by $D\Gamma$.

This induces a cell decomposition of $F = F_g(p,\infty)$ and $F = F_g(p,\infty,\infty)$, respectively, where $D\Gamma$ provides the 2-cells, $V\Gamma$ the 0 cells and $E\Gamma$ together with the $2n$ components of $\partial F - V\Gamma$ the 1-cells. This implies that for H the underlying surface of $F = F_g(p,\infty)$ and $F = F_g(p,\infty,\infty)$

$$\chi(H) = -2g + 1 = \#V\Gamma - (\#E\Gamma + 2n) + \#D\Gamma$$

and $\chi(H) = -2g = \#V\Gamma - (\#E\Gamma + 2n) + \#D\Gamma$, respectively. Clearly $|S \cap \partial N| = 2n$ and $\chi(S \cap N) = -2m$.

Here $A_1, \ldots, A_n, T_1, \ldots, T_k$ is a separating collection of annuli and tori. Thus $a_1, \ldots, a_n, t_1, \ldots, t_k$ is a separating collection of arcs. Each component of the complement of Γ in H is contained in a component of the complement of $a_1, \ldots, a_n$, $t_1, \ldots, t_k$ in H. It follows that the number of complementary disks for Γ is at least two. Thus,

$$-\frac{|S \cap \partial N|}{2} + \frac{\chi(S \cap N)}{2} + 2 =$$

$$-n - m + 2 = (2n + k + 2m) - (3n + k + 3m) + 2 \leq -2g + 1$$

or

$$-\frac{|S \cap \partial N|}{2} + \frac{\chi(S \cap N)}{2} + 2 =$$

$$-n - m + 2 = (2n + k + 2m) - (3n + k + 3m) + 2 \leq -2g.$$

□

Lemma 10. *Let M be a graph manifold and let N be a Seifert piece with $\mathcal{O}(N) = F_g(p, \infty)$ or $\mathcal{O}(N) = F_g(p, \infty, \infty)$. If $S \cap N$ is horizontal of type h1, then $\chi(S \cap N) \leq -4g - 2$ or $\chi(S \cap N) \leq -4g - 2 - 2p$, respectively.*

Proof. Suppose that H is a horizontal incompressible surface in N that covers regular points of $F_g(p, \infty)$ k times. Note that here $k \geq p \geq 2$. By the Riemann-Hurwitz formula, $\chi(H) = k(-2g + \frac{1}{p}) \leq p(-2g + \frac{1}{p}) = -2pg + 1 \leq -4g + 1$ or $\chi(H) = k(-2g - 1 + \frac{1}{p}) \leq p(-2g - 1 + \frac{1}{p}) = -2pg - p + 1 \leq -4g - p + 1$, respectively. Here $S \cap N$ is obtained from a separating horizontal incompressible surface by ambient 1-surgery. A connected horizontal incompressible surface is non separating, thus $S \cap N$ is obtained from two parallel horizontal incompressible surfaces by ambient 1-surgery. Thus $\chi(S \cap N) \leq 2(-4g + 1) = -8g + 2 \leq -4g - 4 + 2 = -4g - 2$ or $\chi(S \cap N) \leq 2(-4g - p + 1) = -8g - 2p + 2 \leq -4g - 4 - 2p + 2 = -4g - 2 - 2p$, respectively. It follows that $g(S \cap N) \geq 2g + 2 - \frac{|S \cap N|}{2}$ or $g(S \cap N) \geq 2g + 2 + p - \frac{|S \cap N|}{2}$, respectively. □

Lemma 11. *Let M be a graph manifold and let N be a Seifert piece with $\mathcal{O}(N) = F_g(p, \infty)$ or $\mathcal{O}(N) = F_g(p, \infty, \infty)$. If $S \cap N$ is horizontal of type h2, then $\chi(S \cap N) \leq -4g - 4$ or $\chi(S \cap N) \leq -4g - 4 - 2p$, respectively.*

Proof. If $S \cap N$ is horizontal of type h2, then there is a boundary component T of N such that $S \cap (N - \eta(T))$ is a separating horizontal incompressible surface. The computation in Lemma 10 applies to this portion of S. Furthermore, $S \cap \eta(T)$ consists of two annuli tubed together. Thus $\chi(S \cap N) \leq -4g - 4$ or $\chi(S \cap N) \leq -4g - 4 - 2p$, respectively. Whence $genus(S \cap N) > 2g + 3 - \frac{|S \cap \partial N|}{2}$ or $genus(S \cap N) > 2g + 3 + p - \frac{|S \cap \partial N|}{2}$, respectively. □

Lemma 12. *Let M be a graph manifold and let N be a Seifert piece with $\mathcal{O}(N) = D(p, q)$ with $(p, q) = 1$.*

1. *If $S \cap N$ is horizontal of type h1, then for some $l \geq 1$, $k \geq pq$, $|S \cap N| \leq 2l$, $\chi(S \cap N) \leq 2kl(-1 + \frac{1}{p} + \frac{1}{q})$ and $genus(S \cap N) \geq 2l$.*
2. *If $S \cap N$ is horizontal of type h2, then $\chi(S \cap N) \leq 2k(-1 + \frac{1}{p} + \frac{1}{q}) - 2$ and $genus(S \cap N) \geq 3$.*
3. *If $S \cap N$ is horizontal of type h3, then $\chi(S \cap N) \leq -2min(p, q) + 2$. Furthermore, either $S \cap N$ is as in Lemma 5, or $genus(S \cap N) \geq 2$.*

Proof. Suppose that H is a horizontal incompressible surface in N that covers regular points of $D(p,q)$ k times. Clearly $k \geq pq$ as p and q are coprime. By the Riemann-Hurwitz formula, $\chi(H) = k(-1+\frac{1}{q}+\frac{1}{p})$. If $S \cap N$ is horizontal of type h1, then $S \cap N$ is obtained from $2l$ parallel surfaces H via ambient 1-surgery. Thus $\chi(S \cap N) \leq 2lk(-1+\frac{1}{p}+\frac{1}{q})$. By Lemma 8, H is non planar, so $genus(H \cap N) \geq 1$. Whence $genus(S \cap N) \geq 2l$.

If $S \cap N$ is horizontal of type h2, then by the same reasoning as in Lemma 11, $\chi(S \cap N) \leq 2k(-1+\frac{1}{p}+\frac{1}{q}) - 2 \leq -2pq+2p+2q-2$ and $genus(S \cap N) \geq 3$.

Suppose that e is the exceptional fiber of index q and let $N' = N - \eta(e)$. Suppose that H' is a horizontal incompressible surface in N' that covers regular points k times. Here $k \geq p$. Then $\chi(H') = k(-1+\frac{1}{p}) \leq -p+1$. Thus if $S \cap N$ is horizontal of type h3 with respect to e, then $\chi(S \cap N) \leq 2k(-1+\frac{1}{p}) \leq -2q+2$. The last comment follows immediately from Lemma 5. $\square$

5. The proof of Theorem 1

In order to give the proof of Theorem 1 we will first show that the fundamental groups can in fact be generated by $2g+1$ elements end then that only the manifolds listed admit a Heegaard splitting of genus $2g+1$.

Lemma 13. *The manifolds described in Theorem 1 have* $2g+1$*-generated fundamental groups.*

Proof. We first recall the presentations of the fundamental groups of N_1 and N_2:

$$\pi_1(N_1) = \langle a_1, b_1, \ldots, a_g, b_g, s, t, f_1 \mid R \rangle \text{ with}$$
$$R = \{[a_1, f_1], \ldots, [a_g, f_1], [b_1, f_1], \ldots, [b_g, f_1], [s, f_1], [t, f_1],$$
$$s^r = f_1^{\beta}, [a_1, b_1] \cdot \ldots \cdot [a_g, b_g] st = f_1^e\}$$

and

$$\pi_1(N_2) = \langle x, y, f_2 \mid [x, f_2], [y, f_2], x^p = f_2^{\beta_1}, y^q = f_2^{\beta_2} \rangle.$$

As the manifold M is obtained from the manifold N_1 and N_2 by identifying their boundary it follows from van Kampen's theorem that

$$\pi_1(M) = \pi_1(N_1) *_C \pi_1(N_2) \text{ with } C \cong \mathbb{Z}^2.$$

Note that $f_1 = xyf_2^l$ for some $l \in \mathbb{Z}$ as we assume that the intersection number between f_1 and f_2 is 1. A simple calculation (see [5]) shows that $n = \min(p,q)$ conjugates of f_1 generate a subgroup of $\pi_1(N_2)$ that maps surjectively onto the orbifold group $\pi_1(D(p,q))$. We do however need something slightly stronger:

Claim: We can choose elements $h_2, \ldots, h_n \in \pi_1(N_2)$ such that

$$U = \langle f_1, h_2 f_1 h_2^{-1}, \ldots, h_n f_1 h_n^{-1} \rangle$$

maps surjectively onto the base group and that additionally $h_i \in U$ for $2 \leq i \leq n$.

Choose k_i such that $\langle f_1, k_2 f_1 k_2^{-1}, \ldots, k_n f_1 k_n^{-1} \rangle$ maps surjectively. For any k_i choose $h_i \in U$ and $z_i \in \mathbb{Z}$ such that $k_i = h_i f_2^{z_i}$. Clearly such h_i and z_i exist as we assume that U maps surjectively one $\pi_1(D(p,q))$ and as the kernel is generated by f_2. As f_1 and f_2 commute it follows that $k_i f_1 k_i^{-1} = h_i f_2^{z_i} f_1 f_2^{-z_i} h_i^{-1} = h_i f_1 h_i^{-1}$. This clearly implies that $U = \langle f_1, h_2 f_1 h_2^{-1}, \ldots, h_n f_1 h_n^{-1} \rangle$. The claim follows.

Note that U is a subgroup of finite index in $\pi_1(N_2)$ and that we can choose the elements h_i such that $\pi_1(N_2) = U$ if and only if N_2 is the exterior of a torus knot with meridian f_1. It is however always true that $\pi_1(N_2) = \langle U, C\rangle$ as $f_2 \in C$.

Note further that the subgroup $\langle s, f_1\rangle$ of $\pi_1(N_1)$ is generated by a single element g_0 which corresponds to the core of the solid torus corresponding to the exceptional fiber of N_1. It follows that $g_0^k = f_1$ for some $k \in \mathbb{Z}$. In order to prove the lemma we describe elements $g_1, \dots, g_{2g} \in \pi_1(M)$ such that $\pi_1(M) = \langle g_0, \dots, g_{2g}\rangle$.

Recall that by assumption $n + 1 \leq 2g$. Put $h_i = 1$ for $n + 1 \leq i \leq 2g$. We define

- $g_i = h_i a_i$ for $1 \leq i \leq g$
- $g_i = h_i b_{i-g}$ for $g + 1 \leq i \leq 2g$

Claim: $U \subset \langle g_0, \dots, g_{2g}\rangle$.

To see this it clearly suffices to show that f_1 and the elements $h_i f_1 h_i^{-1}$ lie in $\langle g_0, \dots, g_{2g}\rangle$ for $1 \leq i \leq 2g$. Clearly $f_1 \in \langle g_0, \dots, g_{2g}\rangle$ as $f_1 = g_0^k$. Furthermore $h_i f_1 h_i^{-1} \in \langle g_0, \dots, g_{2g}\rangle$ for $1 \leq i \leq g$ as $g_i g_0^k g_i^{-1} = h_i a_i f_1 a_i^{-1} h_i^{-1} = h_i f_1 h_i^{-1}$. The same argument shows that $g_i g_0^k g_i^{-1} = h_i f_1 h_i^{-1}$ for $g + 1 \leq i \leq 2g$. Thus the claim is proven.

As $h_i \in U$ for $1 \leq i \leq 2g$ this implies that $h_i \in \langle g_0, \dots, g_{2g}\rangle$ for $1 \leq i \leq 2g$ and therefore $h_i^{-1} g_i \in \langle g_0, \dots, g_{2g}\rangle$ for $1 \leq i \leq 2g$. As $h_i^{-1} g_i = a_i$ for $1 \leq i \leq g$ and $h_i^{-1} g_i = b_{i-g}$ for $g + 1 \leq i \leq 2g$ it follows that all a_i and b_i lie in $\langle g_0, \dots, g_{2g}\rangle$. Furthermore both f_1 and s are powers of g_0 and lie in $\langle g_0, \dots, g_{2g}\rangle$. The last generator t can be written as a product in the remaining generators by the last relation. Thus all generators of $\pi_1(N_1)$ lie in $\langle g_0, \dots, g_{2g}\rangle$ which shows that $\pi_1(N_1) \subset \langle g_0, \dots, g_{2g}\rangle$. Thus $C \subset \langle g_0, \dots, g_{2g}\rangle$ and therefore $\pi_1(N_2) = \langle U, C\rangle \subset \langle g_0, \dots, g_{2g}\rangle$. This shows that $\pi_1(M) = \langle g_0, \dots, g_{2g}\rangle$. $\square$

Lemma 14. *Let M be a closed graph manifold consisting of two Seifert manifolds N_1 and N_2 glued along T, where $\mathcal{O}(N_1) = F_g(p, \infty)$, $\mathcal{O}(N_2) = D(p, q)$ with $(p, q) = 1$ such that the intersection number of the fibers of N_1 and N_2 equals 1. Let S be a Heegaard surface of M and suppose that S is in standard position.*

1. *Suppose that $S \cap N_1$ is vertical. Further suppose that $S \cap N_2$ is as in Lemma 5 and that it is horizontal with respect to the exceptional fiber e of index p. If $q \leq 2g + 1$, then $S \cap N_2$ can be extended to a Heegaard surface $\tilde{S}$ such that $\tilde{S} \cap N_1$ is vertical and such that genus$(\tilde{S}) = 2g + 1$.*
2. *If $S \cap N_1$ is as in Lemma 6 and $S \cap N_2$ consists of a single (vertical) annulus, then genus$(S) = 2g + 1$.*
3. *In all other cases, genus$(S) \geq 2g + 2$.*

Note that despite the fact that $S \cap N_1$ is vertical, it may have lower Euler characteristic than $\tilde{S}$. So it could be the case that *genus*$(S) > 2g + 1$. (Note also that although distinct irreducible vertical Heegaard splittings of Seifert fibered spaces all have the same genus, this is not the case for Seifert fibered submanifolds of graph manifolds. This necessitates the more careful phrasing in option 2.)

Proof. Case 1: $S \cap N_1$ is vertical.

If $S \cap N_2$ is horizontal of type h1, then by Lemma 9

$$\chi(S) = \chi(S \cap N_1) + \chi(S \cap N_2) \leq -4g - 2 + |S \cap T| + \chi(S \cap N_2) =$$

$$-4g-2+(-\chi(S\cap N_2)+2|S\cap N_2|-2genus(S\cap N_2))+\chi(S\cap N_2)=$$

$$-4g-2+2|S\cap N_2|-2genus(S\cap N_2)).$$

By Lemma 12,

$$-4g-2+2|S\cap N_2|-2genus(S\cap N_2))\leq$$

$$-4g-2+4l-2(2l)=-4g-2.$$

Thus $genus(S)\geq 2g+2$.

If $S\cap N_2$ is horizontal of type h2, we similarly obtain $genus(S)\geq 2g+3$.

Suppose that $S\cap N_2$ is horizontal of type h3 with respect to the exceptional fiber e and $S\cap(N_2-\eta(e))$ consists of two planar surfaces. I.e., suppose that $S\cap N_2$ is as in Lemma 5. Then we obtain

$$\chi(S)\leq -4g-2+|S\cap T|+\chi(S\cap N_2)=$$

$$-4g-2+|S\cap\partial N_2|+(-|S\cap\partial N_2|+2)=-4g.$$

Thus $genus(S)\geq 2g+1$.

Note that equality is obtained if and only if for Γ, H as in the proof of Lemma 9, $H-\Gamma$ consists of exactly two disks. It is easy to construct examples for which this is the case as long as $|S\cap T|\leq 4g+2$. If $|S\cap T|>4g+2$, then this is impossible. It follows that this Heegaard splitting of genus $2g+1$ exists if and only if $min(p,q)\leq 2g+1$.

Suppose that $S\cap N_2$ is horizontal of type h3 with respect to the exceptional fiber e and $S\cap(N_2-\eta(e))$ consists of two non planar surfaces. Then we obtain

$$\chi(S)\leq -4g+|S\cap T|+\chi(S\cap N_2)=$$

$$-4g+|S\cap\partial N_2|+(2-2genus(S\cap N_2)-|S\cap\partial N_2|)\leq -4g-2.$$

Thus $genus(S)\geq 2g+2$.

Case 2: $S\cap N_1$ is horizontal.

If $S\cap N_2$ is vertical, then $\chi(S\cap N_2)\leq 0$ by construction. If $S\cap N_2$ is horizontal, then $\chi(S\cap N_2)\leq 0$ by Lemma 12. Thus if $S\cap N_1$ is horizontal of type h1, then $\chi(S)=\chi(S\cap N_1)+\chi(S\cap N_2)\leq -4g-2$ by Lemma 10. Whence $genus(S)\geq 2g+2$. Similarly, if $S\cap N_1$ is horizontal of type h2, then $\chi(S)\leq -4g-4$. Whence $genus(S)\geq 2g+3$. Similarly, if $S\cap N_1$ is horizontal of type h3 and not as in Lemma 6, then $\chi(S)\leq -8g-4$ by Lemma 11. Whence $genus(S)\geq 4g+3$.

If $S\cap N_1$ is horizontal of type then $S\cap N_1$ must for Euler characteristic reasons be as in the conclusion of Lemma 6, in particular it must be horizontal of type h3 with respect to the exceptional fiber e. Furthermore, it must be constructed from two parallel surfaces homeomorphic to $F_g(\infty,\infty)$.

If the boundary component of each of these two surfaces meets T in the fiber f_2 of N_2, then $S\cap N_2$ can be vertical. By adjoining a single vertical annulus that cuts N_2 into regular neighborhoods of the two exceptional fibers, we obtain a Heegaard splitting of genus $2g+1$.

As all essential properly embedded annuli in N_2 are vertical and all properly embedded discs in N_2 are boundary parallel it follows that all other possibilities for $N \cap N_2$ yield a Heegaard surface of genus more than $2g + 2$. $\square$

To complete the proof of Theorem 1 we merely need to put the pieces together:

Lemma 15. *If option 1 occurs in Lemma 14, then N_2 is a q-bridge knot complement and the fiber of N_1 is identified with the meridian of N_2. If option 2 occurs in Lemma 14, then $\hat{N}_1$ admits a horizontal Heegaard splitting of genus $2g$.*

Proof. This follows from Lemma 5 and Lemma 6 respectively. $\square$

6. The proof of Theorem 2

In this section we construct for any $n \in \mathbb{N}$ a graph manifold M_n such that $\pi_1(M_n)$ is $3n$-generated but that the Heegaard genus of M_n is $4n$. We denote the graph underlying M_n by Γ_n. Γ_n is a tree on $2n+1$ vertices $z, c_1, \ldots, c_n, d_1, \ldots, d_n$ and $2n$ edges $e_1, \ldots, e_n, f_1, \ldots, f_n$ such that c_i and d_i are the endpoints of e_i and that d_i and z are the endpoints of f_i.

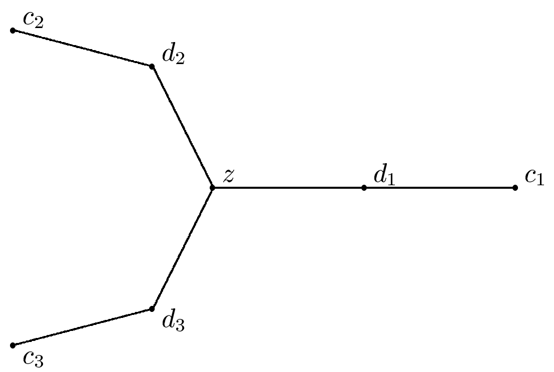

Figure 3. The tree Γ_3

The closed graph manifold M_n is then constructed as follows, where we denote the Seifert piece corresponding to a vertex v by N_v.

1. The intersection number between the fibers of the adjacent Seifert spaces is 1 at any torus of the JSJ decomposition.
2. $\mathcal{O}(N_z)$ is a n-punctured sphere with at most one cone point and $\hat{N}_z = S^3$.
3. $\mathcal{O}(N_{d_i}) = T^2(\infty, \infty, 5n)$ and N_{d_i} admits no horizontal surface of type h3 that has genus 2.
4. $\mathcal{O}(N_{c_i})$ is of type $D(p,q)$ with $p \in \{2,3\}$ and $(p,q) = 1$ but N_{c_i} is not homeomorphic to the exterior of a torus knot in S^3.

Remark 16. Note that (2) is equivalent to stating that N_z is the exterior of a Seifert fibered n component n-bridge link in S^3, in particular $\pi_1(N_z)$ is generated by the fibers of the N_{d_i}. The existence of the spaces N_{d_i} satisfying (3) is an immediate consequence of Lemma 7.

The first part of the proof of Theorem 2 is again a simple calculation:

Lemma 17. *$\pi_1(M_n)$ can be generated by $3n$ elements.*

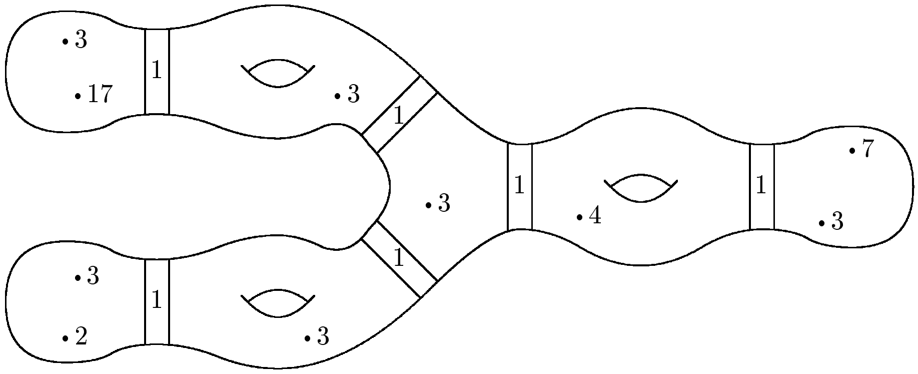

FIGURE 4. A graph manifold M with $g(M) = 12$ and $r(M) \leq 9$

Proof. The proof is almost identically to the proof of Lemma 13 and we frequently omit explicit calculations if they are identical. Recall that

$$\pi_1(N_{d_i}) = \langle a_i, b_i, s_i, t_{i1}, t_{i2}, f_i \,|\, R_i \rangle \text{ with}$$

$$R_i = \{[a_i, f_i], [b_i, f_i], [s_i, f_i], [t_{i1}, f_i], [t_{i2}, f_i], s_i^{5n} = f_i^{\beta_i}, [a_i b_i] t_{i1} t_{i2} s_i = f_i^{e_i}\}$$

where t_{i1} corresponds the the boundary component between N_{d_i} and N_z and t_{i2} corresponds to the boundary component between N_{d_i} and N_{c_i}.

Recall from the proof of Lemma 13 that there exist elements $h_{i1}, h_{i2} \in \pi_1(N_{c_i})$ such that $U_i = \langle f_i, h_{i1} f_i h_{i1}^{-1}, h_{i2} f_i h_{i2}^{-1} \rangle$ is a subgroup of finite index in $\pi_1(N_{c_i})$ that maps surjectively onto the fundamental group of $\mathcal{O}(N_{c_i})$ and that $h_{i1}, h_{i2} \in U_i$.

We will show that $\pi_1(M_n)$ is generated by the generators $g_1, \ldots, g_{3n}$ defined as follows:

1. g_i is the generator of the cyclic group $\langle f_i, s_i \rangle$ for $1 \leq i \leq n$.
2. $g_{n+i} = h_{i1} a_i$ for $1 \leq i \leq n$.
3. $g_{2n+i} = h_{i2} b_i$ for $1 \leq i \leq n$.

Let $H = \langle g_1, \ldots, g_{3n} \rangle$. We show that $H = \pi_1(M_n)$.

Note first that $\pi_1(N_z) \subset H$ as $g_i \in H$ implies $f_i \in H$ for $1 \leq i \leq n$ and $\pi_1(N_z)$ is generated by the f_i. This implies that $t_{i1} \in H$ for $1 \leq i \leq n$.

The same calculation as in the proof of Lemma 13 further shows that $U_i \subset H$ for $1 \leq i \leq n$. It follows that $a_i, b_i \in H$ for $1 \leq i \leq n$. Thus $\pi_1(N_{d_i}) \subset H$ as $\pi_1(N_{d_i})$ is generated by $a_i, b_i, s_i, f_i, t_{i_1}$ and s_i and f_i are powers of g_i.

It now further follows that $\pi_1(N_{c_i}) \subset H$ as $\pi_1(N_{c_i})$ is generated by U_i and C_i where $C_i = \pi_1(N_{c_i}) \cap \pi_1(N_{d_i})$. □

To conclude it clearly suffices to establish the following:

Proposition 18. *The Heegaard genus of M_n is $4n$.*

Proof. A Heegaard splitting $M_n = V_n \cup_{S_n} W_n$ of genus $4n$ for M_n exists that is structured as follows: $S_n \cap N_z$ is a planar surface that is horizontal of type h3. It meets each component of ∂N_z twice. $S_n \cap N_{d_i}$ is vertical and consists of two annuli that are parallel into the component of ∂N_{d_i} corresponding to the edge between z and d_i, one annulus that is parallel to the component of ∂N_{d_i} corresponding to the edge between d_i and c_i. Two nonparallel tubes are attached to these annuli.

$S_n \cap N_{c_i}$ is also vertical and consists of one annulus parallel into ∂N_{c_i} with one tube attached.

To see that M_n admits no Heegaard splitting of genus less than $4n$, proceed analogously to the proof of Lemma 14. The restrictions in the setting here show that in all possible cases,

$$\chi(S \cap (N_{d_i} \cup N_{c_i})) \leq -4(1) - 4 + |S \cap T_{z_i}|$$

where T_{z_i} is the torus that meets N_z and N_{d_i}. Furthermore,

$$\chi(S \cap N_z) \leq 2 - 2 \sum_i |S \cap T_{z_i}|.$$

Thus $\chi(S) \leq 2 - 8n$. Whence $genus(S) \geq 4n$. □

7. Some comments on non totally orientable graph manifolds

During the proofs of Theorem 1 and Theorem 2 me make extensive use of the structure theorem for Heegaard splittings of totally orientable graph manifolds [6]. We believe however that similar statements are true for graph manifolds in general. This suggest a more straight forward generalization of the example provided in [9] which are not totally orientable.

It should be noted that the verification that the manifolds constructed in [9] are not of Heegaard genus 2 relies on the classification of 3-manifolds with non-empty characteristic submanifold that have a genus 2 Heegaard splitting as given by T. Kobayashi [3].

Thus we conjecture that the manifolds M_n constructed below are of Heegaard genus $3n$, the same argument as above show that they can be generated by $2n$ elements.

The graph underlying the manifold M_n is again Γ_n, the Seifert piece corresponding to the vertex v is again denoted by N_v and the following hold:

1. The intersection number between the fibers of the adjacent Seifert spaces is 1 at any torus of the JSJ decomposition.
2. $\mathcal{O}(N_z)$ is a n-punctured sphere with at most one cone point and $\hat{N}_z = S^3$.
3. $\mathcal{O}(N_{d_i}) = P^2(\infty, \infty, 5n)$ and N_{d_i} admits no horizontal surface of type h3 that has genus 2.
4. $\mathcal{O}(N_{c_i})$ is of type $D(2,q)$ with odd q but N_{c_i} is not homeomorphic to the exterior of a 2-bridge knot in S^3.

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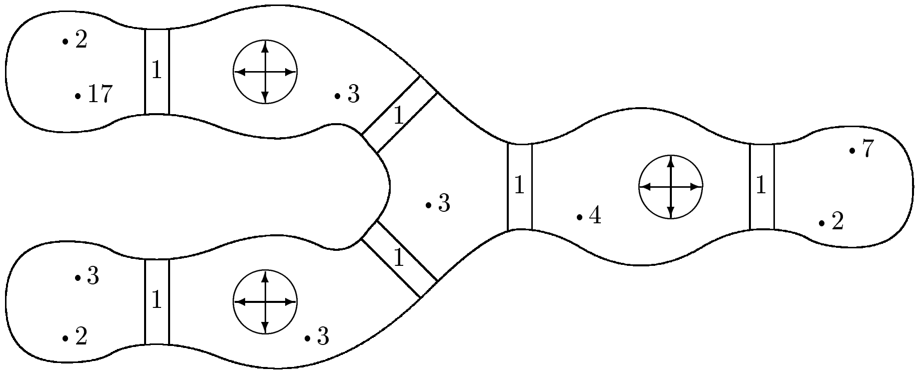

FIGURE 5. A graph manifold M with $g(M) = 9$ and $r(M) \leq 6$?

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