THE CLASSIFICATION OF HIGHLY CONNECTED MANIFOLDS IN DIMENSIONS 7 AND 15.

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CHAPTER 1

Introduction

An n-dimensional manifold M is called highly connected when all of its homotopy groups vanish below the middle dimension. That is, $\pi_i(M) = 0$ for all $i < (n-1)/2$. We begin by briefly recalling the history of the classification of highly connected, closed, smooth manifolds in dimensions greater than 4. Throughout all manifolds will be smooth and oriented and all maps orientation preserving. In [**Wa1**] Wall built on the results of Smale, most notably the h-cobordism theorem, to classify highly connected even dimensional manifolds up to the addition of homotopy spheres. Later Wall [**Wa3**] presented a classification of highly connected odd dimensional manifolds up to the addition of homotopy spheres except in dimensions 5, 7 and 15. Barden, a student of Wall, solved the problem of the classification of simply connected 5-manifolds [**Ba**]. In dimensions 7 and 15 Wilkens, another student of Wall, obtained results but they were not complete as we now explain [**Wi1, Wi2**].

Throughout this thesis P shall denote a highly connected, closed manifold of dimension $(8j-1)$ and L shall denote a handlebody by which we mean a highly connected, compact, even dimensional manifold with a nonempty and connected boundary. For every P there is a unique homotopy sphere such Σ_P (often the standard sphere) such that $P\#\Sigma_P$ is the boundary of a handlebody L and such that Σ_P bounds a Spin manifold with vanishing decomposable Pontryagin numbers and signature equal to zero (see Section 2.5). To each handlebody L, Wall associated the triple (H, λ, α), where $H = H_{4j}(L;\mathbb{Z})$ is the middle homology group of L, $\lambda : H \times H \to \mathbb{Z}$ is the intersection form of L and $\alpha \in H^{4j}(L;\mathbb{Z})$ is the primary obstruction to the stable triviality of L. We identify $H^{4j}(L;\mathbb{Z})$ with H^*, the dual group of H, via the Kronecker pairing. Two triples $(H_0, \lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ are isomorphic if there is an isomorphism $\Theta : H_0 \to H_1$ such that $\Theta^*(\alpha_1) = \alpha_0$ and $\lambda_0 = \lambda_1 \circ (\Theta \times \Theta)$. If the handlebodies L_0 and L_1 have associated triples $(H_0, \lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ then Wall proved in [**Wa1**] that an isomorphism between these triples is realized by a diffeomorphism. Wilkens later proved that if $f : \partial L_0 \to \partial L_1$ is a diffeomorphism between the boundaries of two handlebodies, then there are handlebodies V_0 and V_1 with standard spheres as boundary and a diffeomorphism $g : L_0 \natural V_0 \to L_1 \natural V_1$ which extends f (here $\natural$ denotes the boundary connected sum). Now, the intersection form of a handlebody is nonsingular if and only if the handlebody cobounds a homotopy sphere. Thus the question of classifying the manifolds P up to connected with sum homotopy spheres becomes the question of the stable classification of triples (H, λ, α). Here we say that two triples $(H_0, \lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ are stably equivalent if there are triples $(H'_0, \lambda'_0, \alpha'_0)$ and $(H'_1, \lambda'_1, \alpha'_1)$ such that λ'_0 and λ'_1 are nonsingular and $(H_0 \oplus H'_0, \lambda_0 \oplus \lambda'_0, \alpha_0 \oplus \alpha'_0)$ is isomorphic to $(H_1 \oplus H'_1, \lambda_1 \oplus \lambda'_1, \alpha_1 \oplus \alpha'_1)$.

To classify the manifolds P up to almost diffeomorphism Wilkens associated to each P the triple (G, b, β) where $G = H^{4j}(P;\mathbb{Z})$, b is the linking form of P which is a nonsingular symmetric bilinear function $b : TG \times TG \to \mathbb{Q}/\mathbb{Z}$ where TG is the torsion subgroup of G and $\beta \in G$ is the primary obstruction to the triviality of the stable tangent bundle of P. We shall call this triple the Wilkens triple or Wilkens invariants of P. Two triples (G_0, b_0, β_0) and (G_1, b_1, β_1) are isomorphic if there is an isomorphism $\theta : G_0 \to G_1$ such that $\theta(\beta_0) = \beta_1$ and $b_0 = b_1 \circ (\theta \times \theta)$. The only constraint on the triples which are realized by highly connected 7 and 15 manifolds is that $\beta = 2\gamma$ for some $\gamma \in G$. In higher dimensions all triples are realized. If $P = \partial L$ and (H, λ, α) is the triple associated to L then (H, λ, α) induces (G, b, β) in a natural way. Wilkens was able to prove that if the group G contains no 2-torsion then any two highly connected $(8j-1)$-manifolds, $j=1,2$, with isomorphic triples are diffeomorphic after the addition of homotopy spheres. When G contains 2 torsion and b is not decomposable as the sum of smaller nontrivial linking forms, Wilkens showed that up to connected sum with homotopy spheres, there were at most two highly connected $(8j-1)$-manifolds, $j=1,2$ with isomorphic triples. For a precise statement of Wilkens' results we refer the reader to Section 2.8. A $\mathbb{Z}/2$-ambiguity associated to groups of even order points to the need for a quadratic refinement of the linking form b. Indeed Wall had left the classification unsolved in dimensions 7 and 15 because the existence of bundles of Hopf-invariant one over the 4-sphere and 8-sphere meant that his definition of the quadratic refinement of the linking form of a highly connected manifold did not work. Wilkens, however, did not interpret his results on the stable algebra of triples (H,λ,α) in terms of quadratic refinements of the induced linking forms.

Now let G be a finite abelian group. A quadratic linking function on G is a function $q : G \to \mathbb{Q}/\mathbb{Z}$ such that

$$q(x+y) = q(x) + q(y) + b(x,y)$$

for every $x, y \in G$ and for some linking form, b, on G. If q is homogeneous, that is $q(-x) = q(x)$ for every $x \in G$, then q is called a quadratic linking form. We let $\mathcal{Q}(b)$ denote the set of quadratic linking functions refining a linking form b. There is a free action of G on $\mathcal{Q}(b)$ defined, for $a \in G$ and $q \in \mathcal{Q}(b)$, by $a \cdot q = q_a$ where for every $x \in G$,

$$q_a(x) = q(x) + b(x,a).$$

A nondegenerate, even bilinear form $\lambda : H \times H \to \mathbb{Z}$ on the free abelian group H induces a quadratic linking form, $q(\lambda)$, on $G = \mathrm{Cok}(\hat{\lambda})$ where $\hat{\lambda} : H \to H^*$ is the adjoint of λ (see Definition 2.32). Both Durfee [**Du1, Du2**] and Wall [**Wa4**] proved that if λ_0 and λ_1 are nondegenerate even bilinear forms then λ_0 and λ_1 are stably equivalent if and only if $q(\lambda_0) \cong q(\lambda_1)$, where stable equivalence means isomorphic after the addition of nonsingular even bilinear forms. However, they did not consider the question of the stable equivalence of the triples (H, λ, α). In particular in dimensions 8 and 16 the triples (H, λ, α) arising from handlebodies need not contain even intersection forms. So these algebraic results were confined to topological applications in dimensions other than 7 and 15.

Thus the question of the classification of 2-connected 7-manifolds and 6-connected 15-manifolds stood where Wilkens left it for thirty years.[1] Then Grove and Ziller showed that the total spaces of S^3-bundles over S^4 admit metrics of non-negative sectional curvature and they called for a classification of these total spaces [**GZ**, Problem 5.2]. Of course such total spaces are 2-connected 7-manifolds. This problem was solved in [**CE**] where we showed that the invariants s_1 and $\bar{s}_1$ defined in [**KS**] [2] resolved the ambiguity in Wilkens' classification. Specifically we showed that if P_0 and P_1 are the total spaces of S^3-bundles over S^4 which do not admit a cross section then P_0 is homeomorphic (resp. diffeomorphic) to P_1 if and only if their Wilkens invariants are isomorphic and $\bar{s}_1(P_0) = \bar{s}_1(P_1)$ (resp. $s_1(P_0) = s_1(P_1)$). We shall prove that the same statement holds for all 2-connected 7-dimensional rational homology spheres.

Now let L be a handlebody of dimension 8 or 16 with associated triple (H, λ, α). From the point of view of Durfee's and Wall's results above the difficulty in dimensions 8 and 16 is that λ need not be an even form which is an immediate consequence of the existence of bundles of Hopf invariant one. However, this is compensated by the fact that the existence of bundles of Hopf invariant one ensures that the tangential invariant, α, of L reduces mod 2 to either w_4 or w_8, the fourth or eight Stiefel-Whitney

[1] This is not quite correct. Madsen, Taylor and Williams [**MTW**] claimed that Wilkens' ambiguous pairs are homeomorphic but this is not so. However more on this in Section 6.4.

[2] We explain the invariants s_1 and $\bar{s}_1$ in detail in Section 2.12. They are defined in all dimensions $4k-1$ and in dimensions 7 and 15, s_1 coincides with the Eels-Kuiper μ-invariant [**EK**].

class. Thus α is characteristic for λ by which we mean that the quadratic function $\kappa(\lambda, \alpha)$ defined by

$$\kappa(\lambda, \alpha)(v) = \lambda(v, v) + \alpha(v)$$

takes on even values for every $v \in H$. When λ is nondegenerate this fact allows the definition of a quadratic linking function on the cokernel of $\hat{\lambda}$ as follows. For every $x \in H^*$ there is a nonzero integer r and an element $v \in H$ such that $\hat{\lambda}(v) = rx$. If $[x]$ denotes the image of x in $\mathrm{Cok}(\hat{\lambda})$ then

$$q^c(\kappa(\lambda, \alpha))([x]) := \frac{1}{2r}(x(v) + \alpha(v)) \bmod \mathbb{Z}.$$

We note that $q^c(\kappa(\lambda, \alpha))$ is not necessarily homogeneous. Let P be a highly connected $(8j-1)$-manifold, $j = 1, 2$. Recall that there is a homotopy sphere Σ_P such that $P \# \Sigma_P = \partial L$ where L is a handlebody. Suppose that L defines the triple (H, λ, α). Then, identifying $H^{4j}(P)$ and $H^{4j}(P \# \Sigma_P)$ if necessary, $\mathrm{Cok}(\hat{\lambda}) = H^{4j}(P)$. If P is a rational homology sphere then $H^{4j}(P)$ is finite and λ is nondegenerate so we may define the quadratic linking function of P by

$$q^c(P) := q^c(\kappa(\lambda, \alpha)).$$

We shall show that $q^c(P)$ is independent of the choice of handlebody and refines b, the linking form of P. Also we show that for any quadratic linking form q which refines b there is an element $a \in G$ such that $q^c(P) = q_a$ and that the Wilkens invariant of P is $(G, b, 2a)$.

As we first classify the manifolds P up connected sum with homotopy spheres we introduce the notion of an *almost diffeomorphism.* A homeomorphism $f : M_0 \to M_1$ between smooth manifolds is an almost diffeomorphism if it is a diffeomorphism except at a finite number of singular points. Two manifolds are called almost diffeomorphic when there is an almost diffeomorphism between them. Note that M_0 and M_1 are almost diffeomorphic if and only if there is a single homotopy sphere Σ and a diffeomorphism $f' : M_0 \# \Sigma \to M_1$. Given any degree one map $f : P_0 \to P_1$ between two highly connected $(8j-1)$-manifolds we define the induced Umker map on cohomology, $f_! : H^{4j}(P_0) \to H^{4j}(P_1)$, where for $x \in H^{4j}(P_0)$, $f_!(x) := f_*(x \cap [P_0])/(P_1)$. Here $[P_0] \in H_{8j}(P_0)$ and $(P_1) \in H^{8j}(P_1)$ are the orientation classes of P_0 and P_1 and $\cap$ and $/$ denote the cap and slant products respectively.

THEOREM A. *Let $j = 1$ or 2. For every finite group G, every linking form $b : G \times G \to \mathbb{Q}/\mathbb{Z}$ and every quadratic linking function $q \in \mathcal{Q}(b)$, there is a highly connected $(8j-1)$-dimensional rational homology sphere, P, such that $q^c(P) \cong q$. If P_0 and P_1 are two highly connected $(8j-1)$-dimensional rational homology spheres with quadratic linking functions $q^c(P_0)$ and $q^c(P_1)$ and Wilkens invariants (G_0, b_0, β_0) and (G_1, b_1, β_1), then,*

1. *there is an almost diffeomorphism $f : P_0 \to P_1$ such that $f_! = \theta : H^{4j}(P_0) \cong H^{4j}(P_1)$ if and only if $q^c(P_0) = q^c(P_1) \circ \theta$,*
2. *there is a diffeomorphism $f : P_0 \to P_1$ such that $f_! = \theta : H^{4j}(P_0) \cong H^{4j}(P_1)$ if and only if $q^c(P_0) = q^c(P_1) \circ \theta$, $s_1(P_0) = s_1(P_1)$ and $\Sigma_{P_0} \cong \Sigma_{P_1}$,*
3. *P_0 and P_1 are almost diffeomorphic if and only if $(G_0, b_0, \beta_0) \cong (G_1, b_1, \beta_1)$ and $\bar{s}_1(P_0) = \bar{s}_1(P_1)$,*
4. *P_0 and P_1 are diffeomorphic if and only if $(G_0, b_0, \beta_0) \cong (G_1, b_1, \beta_1)$, $s_1(P_0) = s_1(P_1)$ and $\Sigma_{P_0} \cong \Sigma_{P_1}$,*
5. *in dimension 7, "almost diffeomorphism" may be replaced by "homeomorphism" in 1,*
6. *in dimension 7 there is a homotopy equivalence $f : P_0 \to P_1$ such that $f_! = \theta : H^{4j}(P_0) \cong H^{4j}(P_1)$ if and only if $q^c(P_0)_{12a} = q^c(P_1) \circ \theta$ for some $a \in G_0$.*

REMARK 1.1. The proof of Theorem A is distributed throughout the thesis as follows. Part *1* is a special case of Theorem B which is proven in Theorem 4.2. Part *2* is a special case of Theorem 4.9. Parts *3* and *4* are equivalent to Corollary 5.24. Part *5* is proven in Theorem 6.1 and Part *6* is a special case of Theorem 6.11.

REMARK 1.2. Note that we do not follow the classical surgery path of first classifying the manifolds P up to homotopy and then up to homeomorphism. Nor do we use Kreck's [**Kr2**] modified surgery. Rather, we use Wall's [**Wa1**] classification on the handlebodies L and consider more closely the algebraic structures particular to dimensions 7/8 and 15/16 which are induced on the boundaries P.[3] In Chapter 6 we use surgery theory and the topological classification of part *5* to deduce the homotopy classification.

REMARK 1.3. Part *4* of this theorem extends Theorem 6 of [**Kr2**] to cases where $H^4(P_i)$ is not generated by β_i. After completing this thesis we discovered that F. Bermbach had already undertaken such an extension in [**Be**]. However, Bermbach's general theorem on the classification of simply-connected 7-manifolds uses a Wall style homogeneous quadratic refinement which is not easy to calculate. Moreover, he does not apply this theorem to the classification of 2-connected 7-manifolds but in applications he is exclusively concerned with certain homogeneous spaces having $\pi_2 \cong \mathbb{Z}^2$.

We now proceed to the general case where $G = H^{4j}(P)$ might have a nontrivial free subgroup. If G is itself free then the classification problem is easily solved by Wall's classification of handlebodies.

PROPOSITION 1.4. *Let P_0 and P_1 be highly connected $(8j-1)$-manifolds with Wilkens invariants (G_0, b_0, β_0) and (G_1, b_1, β_1) and suppose that G_0 and G_1 are free*

[3] This is the route suggested by Wall just after Theorem 8 in [**Wa3**] (and followed by Wilkens).

groups. Then there is an almost diffeomorphism $f : P_0 \to P_1$ such that $f_! = \theta : G_0 \cong G_1$ if and only if $\theta(\beta_0) = \beta_1$. Moreover, if $G_0 \cong \mathbb{Z}^l$ then P_0 is diffeomorphic to $P_0' \# (\#_{i=1}^{l-1}(S^{4j-1} \times S^{4j}))$ where $H^{4j}(P_0') \cong \mathbb{Z}$.

If, however, $G = H^{4j}(P)$ has a nontrivial torsion subgroup TG and a nontrival free subgroup then the classification becomes a more complicated matter because in this case a highly connected $(8j-1)$-manifold P, $(j = 1, 2)$, does not necessarily define a unique quadratic linking function on TG. We are therefore lead to the notion of a quadratic linking family which we now explain. Firstly, let τ_G denote the canonical homomorphism

$$\begin{aligned} \tau_G : \quad G &\to \quad G^{**} \\ g &\mapsto (x \mapsto x(g)) \end{aligned}$$

and let $\mathrm{Sec}(\tau_G)$ denote the set of homomorphisms $\Phi : G^{**} \hookrightarrow G$ such that $\tau_G \circ \Phi = Id_{G^{**}}$. A quadratic linking family is a triple (G, q^*, β) consisting of the finitely generated abelian group G, an element $\beta \in G$ and a function $q^* : \mathrm{Sec}(\tau_G) \to \mathcal{Q}(b)$ for some linking form on TG. The superscript $*$ takes the values c for *characteristic* and ev for *even*. When j is greater than 2, highly connected $(8j-1)$-manifolds define even quadratic linking families on their $4j$th cohomology groups. The function q^{ev} is always a constant function whose value is a quadratic linking form. Thus an even quadratic linking family is equivalent to a triple (G, q, β) where q is a quadratic linking form on TG. However, in dimensions 7 and 15, highly connected manifolds define characteristic quadratic linking families whose function q^c may be far from constant. We refer the reader to Section 2.10 for the precise definitions of quadratic linking families and isometries between them. A triple (H, λ, α) (where λ may now be a degenerate bilinear form) induces a quadratic linking family, $Q(\lambda, \alpha)$ on $\mathrm{Cok}(\hat{\lambda}) = G$. Given P, we define $Q(P)$ to be $Q(\lambda, \alpha)$ where $P \# \Sigma_P = \partial L$ for a handlebody with triple (H, λ, α) (again see Section 2.10).

THEOREM B. *Let $j = 1$ or 2. For every finitely generated abelian group G and every characteristic quadratic linking family Q defined on G, there is a highly connected $(8j-1)$-manifold P such that $Q(P) \cong Q$. The quadratic linking family $Q(P)$ of a highly connected $(8j-1)$-manifold P is a complete invariant of almost diffeomorphisms. That is, if P_0 and P_1 are two highly connected $(8j-1)$-manifolds then there is an almost diffeomorphism $f : P_0 \to P_1$ such that $f_! = \theta : H^{4j}(P_0) \cong H^{4j}(P_1)$ if and only if θ defines an isometry of quadratic linking families from $Q(P_0)$ to $Q(P_1)$.*

For comparison we present the analogous theorem in higher dimensions which follows rapidly from a compilation of Wall's results on highly connected manifolds.

THEOREM 1.5 (Wall [**Wa1, Wa2, Wa3, Wa4**]). *Let $j > 2$. For every finitely generated abelian group G and every even quadratic linking family Q on G there is a highly connected $(8j-1)$-manifold P such that $Q(P) \cong Q$. The quadratic linking family $Q(P)$ of a highly connected $(8j-1)$-manifold P is a complete invariant of almost diffeomorphisms.*

The remainder of this thesis is organized as follows. In Chapter 2 we develop the necessary preliminaries. We first summarize Wall's classification of the handlebodies L and rephrase it in terms of quadratic functions. Here a quadratic function on a free abelian group H is a function $\kappa(\lambda, \alpha) : H \to \mathbb{Z}$ such that $\kappa(v) = \lambda(v, v) + \alpha(v)$ for λ a symmetric bilinear form on H and α a linear function on H. We then show that after the addition of a homotopy sphere, every manifold P bounds a handlebody L and present Wilkens' Theorem (2.24) on the extension of diffeomorphisms of the manifolds P to cobounding handlebodies. At this stage it is clear that the almost diffeomorphism classification of the manifolds P is equivalent to the stable classification of quadratic functions where "stable" means "modulo nonsingular quadratic functions". We therefore devote the sections which follow to the algebraic invariants that a quadratic function, $\kappa(\lambda, \alpha)$, induces on its cokernel, $G = \mathrm{Cok}(\hat{\lambda})$, which are stably invariant. These invariants are, in increasing order of sophistication: a linking form on the torsion subgroup of G, a quadratic refinement of this linking form and, in the case where κ in degenerate, a quadratic linking family. In the final two sections of the chapter we provide categorical statements of our classification and define the invariants s_1 and $\bar{s}_1$.

Chapter 3 is an entirely algebraic chapter concerning the stable classification of quadratic functions. We prove that quadratic functions are stably classified by the quadratic linking families which they induce. We first consider the case of nondegenerate quadratic functions where the key idea is that quadratic linking functions behave like algebraic boundaries to the quadratic functions which induce them. In particular, the Gluing Lemma 3.3.6 states that if quadratic functions κ_0 and κ_1 induce quadratic linking functions q_0 and q_1, then an isometry, $\theta : q_0 \cong q_1$, may be used to glue κ_0 to κ_1 and form a nonsingular quadratic function which we denote $\kappa_0 \cup_\theta \kappa_1^-$. This procedure is a precise algebraic analogue of gluing two handlebodies together along a diffeomorphism of their boundaries. We are then able to mimic the proof of Wilkens' Theorem 2.24 in the algebraic setting and show that the isometry $\theta : q_0 \to q_1$ above is induced by an isometry of quadratic functions $\Theta : \kappa_0 \oplus \kappa_0 \cup_\theta \kappa_1^- \to \kappa_1 \oplus \mu$ where μ is a nonsingular quadratic function. Taking a little more care we seen that the somewhat complex notion of a quadratic linking family is the required algebraic object to generalize this result for degenerate quadratic functions. We note that our results in this chapter generalize the results of Wall and Durfee on the stable classification of nondegenerate even quadratic forms (in a natural way) to the stable classification of nondegenerate characteristic quadratic functions.

In Chapter 4 we prove our central classification results, Theorem B and Theorem 1.5, by applying Wilkens' Theorem 2.24 to the algebraic results obtained in Chapter

3. As well as completing the classification left open by Wilkens' this gives a new and simpler proof of Wall's Theorem 1.5. In Section 3 we use the h-cobordism theorem to show that the algebraic Gluing and Splitting Lemmas (3.3.6 and 3.3.10) from Chapter 3 are mimicked precisely in the world of handlebodies. In the case of rational homology spheres, this gives a second proof of Theorem B and Theorem 1.5 which is independent of Theorem 2.24. In the last section of the chapter we present the smooth classification of highly connected $(8j-1)$-manifolds.

In Chapter 5 we review the classification of linking forms and quadratic linking forms on finite abelian groups due to Kawauchi and Kojima [**KK**] and Nikulin [**Ni**] respectively and we then extend Nikulin's classification of quadratic linking forms to the classification of quadratic linking functions. Specifically, let G be a finite abelian group and let q be a quadratic linking function on G which refines the linking form $b(q)$. There is a quadratic linking form q^o such that $q = q^o_a$ for $a \in G$. If we define $\beta(q) := 2a$ then the pair $(b(q), \beta(q))$ is an invariant of q (see 2.38). In addition to the invariant $(b(q), \beta(q))$ there is a Kervaire-Arf invariant, $K(q) \in \mathbb{Q}/\mathbb{Z}$, defined for each quadratic linking function q via a Gauss sum (see Section 5.3). Our classification result is the following

Theorem 5.22. *Let q_0 and q_1 be quadratic linking functions over a finite abelian group G. Then q_0 is isometric to q_1 if and only if $(b(q_0), \beta(q_0)) \cong (b(q_1), \beta(q_1))$ and $K(q_0) = K(q_1)$.*

If $q^c(P)$ is the quadratic linking function of a highly connected $(8j-1)$-manifold $(j = 1, 2)$, we also show that $K(q(P)) = -\bar{s}_1(P)$. Hence Theorem 5.22 is the algebraic counter-part to *3* of Theorem A.

In Chapter 6 we use smoothing theory and surgery theory to deduce respectively the topological and homotopical classification of highly connected 7-manifolds from the almost diffeomorphism classification of these manifolds. We conclude by applying our results to give the classification of the total spaces of 3-sphere bundles over the 4-sphere which appears in [**CE**].

CHAPTER 2

Preliminaries

1. Manifold essentials

Throughout this thesis all manifolds shall be compact and oriented and all maps between manifolds shall be assumed orientation-preserving. If M is a manifold then $-M$ shall denote the manifold with opposite orientation. Unless otherwise noted manifolds shall also be smooth. The letter L shall always denote a $2k$-dimensional handlebody by which we mean a manifold obtained by attaching some number of k-handles to the $2k$-ball. We review the definition and theory of handlebodies in Section 2. For all $k > 2$ every highly connected $2k$-manifold with a single boundary component is diffeomorphic to a handlebody. The symbol ∂ shall be used to indicate the operation of taking the boundary of a manifold. The boundary of every handlebody is a highly connected manifold of dimension $(2k-1)$. The symbol $M_0 \# M_1$ denotes the (interior) connect sum of the manifolds M_0 and M_1. The symbol $L_0 \natural L_1$ denotes the boundary connect sum of L_0 and L_1. We refer the reader to [**KM**] for definitions. It is a consequence of these definitions that

$$\partial(L_0 \natural L_1) = \partial L_0 \# \partial L_1.$$

A simply connected h-cobordism is a simply connected compact $(n+1)$-manifold W whose boundary has two components, $\partial W = M_0 \sqcup -M_1$, such that each inclusion $M_i \hookrightarrow W$ is a homotopy equivalence for $i = 0, 1$. The h-cobordism theorem of Smale asserts that when $n > 4$ every simply connected h-cobordism is diffeomorphic to the product $M_0 \times [0,1]$ via a diffeomorphism that is the identity restricted to M_0.

A homotopy sphere is a manifold which is homotopy equivalent to the standard sphere. We shall always denote homotopy spheres by the letter Σ. In dimensions higher than 4 Kervaire and Milnor [**KM**] famously classified the groups of diffeomorphism classes of homotopy spheres under the operation of connected sum. The group of n-dimensional homotopy spheres is denoted by Θ_n and the subgroup of those spheres bounding parallelizable manifolds by bP_{n+1}. There is an exact sequence

$$0 \longrightarrow bP_{n+1} \xrightarrow{i} \Theta_n \longrightarrow \mathrm{Cok}(J_n)$$

where $\mathrm{Cok}(J_n)$ denotes the cokernel of the J-homomorphism, $J_n : \pi_n(O) \to \pi_n^S$ (see [**Wh**] pg 502-4). When $n = 4k-1$, $\Theta_{4k-1} \cong bP_{4j} \oplus \mathrm{Cok}(J_n)$ and Brumfiel defined a splitting $s : \Theta_{4k-1} \to bP_{4k}$ of i as follows. The group $bP_{4k} \cong \mathbb{Z}/|bP_{4k}|$ is cyclic (of order determined by the kth Bernoulli number). Brumfiel [**Bru**] showed that every homotopy sphere $\Sigma \in \Theta_{4k-1}$ bounds a spin manifold W with vanishing decomposable Pontryagin numbers. The homomorphism s is defined by $\Sigma \mapsto \sigma(W)(\mathrm{mod}|bP_{4k}|)$ where $\sigma(W)$ denotes the signature of the $4k$-manifold W. For the dimensions of primary concern in this thesis, namely 7 and 15, the groups of exotic spheres are:

$$\Theta_7 = bP_8 \cong \mathbb{Z}/28 \qquad\qquad \Theta_{15} \cong bP_{16} \oplus \mathbb{Z}/2 \cong \mathbb{Z}/8128 \oplus \mathbb{Z}/2.$$

Let M_0 and M_1 be smooth manifolds. A homeomorphism $f : M_0 \to M_1$ which is a diffeomorphism except at a finite number of singular points shall be called an almost diffeomorphism.

PROPOSITION 2.1. *Let $f : M_0 \to M_1$ be an almost diffeomorphism of n-dimensional manifolds. Then there is an embedded disc $D^n \hookrightarrow M_0$ containing all the singular points of f, a homotopy sphere Σ and a diffeomorphism $f' : M_0 \# \Sigma \to M_1$ such that $f'|_{M_0 - D^n} = f|_{M_0 - D^n}$. (Here it is understood that the connected sum of M_0 and Σ takes place inside D^n.)*

PROOF. Let $p_1, \ldots p_l$ be the singular points of f and let $I_i : D_i^n \hookrightarrow M_0$ be l pairwise disjoint embeddings of discs with $I_i(0) = p_i$. The image of each D_i^n under f is a subset of M_1 homeomorphic to the n-disc and restricting the smooth structure of M_1 to $f(D_i^n)$ yields a smooth manifold which is diffeomorphic to the n-disc. So let $J_i : D_{i1}^n$ be embeddings of discs in M_1 such that $J_i(D_{i1}^n) = f(I_i(D_i^n))$ and $J_i(0) = f(p_i)$. There are then diffeomorphisms $f_i : (D^n - 0) \to (D_{i1}^n - 0)$ defined by $f_i = J_i^{-1} \circ f \circ I_i|_{D_i^n - 0}$ and we construct the homotopy spheres $\Sigma_i := -D_i^n \cup_{f_i} D_{i1}^n$. Using the obvious embedding of $-D_i^n$ into Σ_i and I_i we define the connected sum $M_0 \# \Sigma_1 \# \ldots \# \Sigma_l$. There is then a diffeomorphism

$$\begin{array}{rccc} f'' : & M_0 \# \Sigma_1 \# \ldots \# \Sigma_l & \to & M_1 \\ & M_0 - \{p_0, \ldots, p_l\} \ni x & \mapsto & f(x) \\ & D_{i1}^n \ni x & \mapsto & J_i(x). \end{array}$$

Now let $\Sigma = \Sigma_1 \# \ldots \# \Sigma_l$ and let D^n be a disc in M_0 containing $\cup_{i=1}^l D_i^n$ in its interior. There is a diffeomorphism $f''' : M_0 \# \Sigma \to M_0 \# \Sigma_1 \# \ldots \# \Sigma_l$ such that $f'''|_{M^0 - D^n} = Id_{M^0 - D^n}$. We may then take f' to be the diffeomorphism $f'' \circ f'''$. $\square$

Now let $f_0 : M_0 \to M_1$ and $f_1 : M_1 \to M_2$ be a pair of almost diffeomorphisms with singular points $p_0, \ldots, p_{l_0}$ and $q_0, \ldots, q_{l_1}$ respectively. Their composition, $f_1 \circ f_0$ is a homeomorphism with singular points $p_0, \ldots, p_{l_0}, f_0^{-1}(q_0), \ldots, f_0^{-1}(q_{l_1})$ (some of which may be repeated) and hence an almost diffeomorphism. Thus, for any class of smooth manifolds there is a corresponding category whose morphisms are almost diffeomorphisms. We work primarily in the almost diffeomorphism category because this is the level at which the algebra of the manifolds L and P is the simplest.

By an almost smooth manifold we shall mean a PL n-manifold, M, with a smooth structure except at a finite number of singular points, $p_1, \ldots, p_l$. We may assume that there are mutually disjoint discs D_i^n with $p_i \in \mathrm{int}(D_i^n)$. The manifold $M - (\sqcup_{i=1}^l D_i^n)$ is a smooth manifold with boundary l disjoint homotopy spheres $\Sigma_1, \ldots, \Sigma_l$ and the

PL structure on D^n_i is obtained by coning on Σ_i. Almost smooth manifolds shall arise in this thesis in two ways. Firstly, we shall sometimes consider a handlebody L with boundary an exotic sphere Σ and form the almost smooth manifold $L \cup D^n$ with singular point at the center of the disc D^n. At other times we shall consider two handlebodies L_0 and L_1 and an almost diffeomorphism $f : \partial L_0 \to \partial L_1$. The adjunction space $L_0 \cup_f -L_1$ has the structure of an almost smooth manifold with singular points corresponding to the singular points of f. An almost smooth h-cobordism between closed simply connected smooth manifolds M_0 and M_1 is a compact almost smooth manifold W with boundary $M_0 \sqcup -M_1$, singular points contained in $W-(M_0 \sqcup M_1)$ and with the inclusions $M_i \hookrightarrow W$ homotopy equivalences. We now demonstrate that the boundaries of such an almost smooth h-cobordism are almost diffeomorphic. Let the singular points of W be $p_0, \dots, p_n$ and suppose that they are contained in pair-wise disjoint closed discs D^n_i whose boundaries are diffeomorphic to $\Sigma_i = \partial D^n_i$ with the smooth structure they inherit from W. In addition let $\gamma_i : [0,1] \times \mathrm{int}(D^{n-1}) \hookrightarrow W$ be pair-wise disjoint smooth embeddings such that $\gamma_i^{-1}(D^n_i) = 1 \times \mathrm{int}(D^{n-1}_i)$ and $\gamma_i^{-1}(\partial W) = 0 \times \mathrm{int}(D^{n-1}_i) \subset M_0$. The manifold $W - (\bigsqcup_i \mathrm{Im}(\gamma_i))$ is a smooth simply connected h-cobordism between with boundary $M_0 \#(\Sigma_1 \# \dots \# \Sigma_n) \sqcup -M_1$ and hence M_0 and M_1 are almost diffeomorphic by the h-cobordism theorem.

2. Wall's classification of the handlebodies L

We adopt the notation of [**Sm**] and consider manifolds $L = L(D^{2k}, \phi_1, \dots, \phi_l; k)$ where $\phi_i : (\partial D^k \times D^k)_i \to D^{2k}$ are embeddings and L is obtained from $D^{2k} \sqcup \sqcup_{i=1}^l (D^k \times D^k)_i$ by identifying points using the ϕ_i and then smoothing corners. The embedding

$$\phi = \sqcup_{i=1}^l \phi_i : \sqcup_{i=1}^l (S^{k-1} \times D^k)_i \hookrightarrow \partial D^{2k}$$

is called a presentation of L. Smale denoted the set of manifolds formed in this way by $\mathcal{H}(2k, l, k)$ and defined $\mathcal{H}(k) := \bigcup_{l=0}^{\infty} \mathcal{H}(2k, l, k)$. The isotopy extension theorem entails that the diffeomorphism type of $L(D^{2k}, \phi_1, \ldots, \phi_l; k)$ is determined by the isotopy class of the embedding ϕ. Two pieces of data determine the isotopy class of ϕ. Firstly if one considers ϕ restricted to the attaching spheres of each handle one obtains a link

$$\bar{\phi} = \sqcup_{i=1}^{l} \bar{\phi}_i : \sqcup_{i=1}^{l} (S^{k-1} \times 0)_i \hookrightarrow S^{2k-1}.$$

When $k > 2$, Smale proved in [**Sm**] that the isotopy class of $\bar{\phi}$ is completely determined by the linking numbers of the attaching spheres. Setting $S_i^{k-1} = (S^{k-1} \times 0)_i$, the linking number of $\bar{\phi}_i(S_i^{k-1})$ with $\bar{\phi}_j(S_j^{k-1})$ is c_{ij} where $\bar{\phi}_i(S_i^{k-1})$ is homologous to c_{ij} times $\phi(0 \times \partial D^k)_j$ in $H_{k-1}(\partial D^{2k-1} - \bar{\phi}_j(S_j^{k-1}))$. Secondly, for each ϕ_i there are framing invariants, $\bar{\alpha}_i \in \pi_{k-1}(SO(k))$, which are defined as follows. The subspace $N := \phi_i(S^{k-1} \times D^k)$ is a tubular neighborhood of $\bar{\phi}_i(S_i^{k-1})$ in S^{2k-1}. There is a framing of N, $\psi_i : N \cong \bar{\phi}_i(S_i^{k-1}) \times D^k$, uniquely defined up to homotopy, such that ψ extends to a framing of the normal bundle of a k-disc which is embedded in D^{2k} and which bounds $\bar{\phi}_i(S_i^{k-1})$. After isotopy of ϕ_i we may assume that the composition $\psi_i \circ \phi_i$ has the form

$$\begin{array}{rccc} \psi_i \circ \phi_i : & (S^{k-1} \times D^k)_i & \to & \bar{\phi}_i(S_i^{k-1}) \times D^k \\ & (x, y) & \mapsto & (\bar{\phi}_i(x), a(\phi_i)(x)(y)) \end{array}$$

for some map $a(\phi_i) : S^{k-1} \to SO(k)$. While the map $a(\phi_i)$ depends upon the choices made the homotopy class of $a(\phi_i)$, which we denote by $\bar{\alpha}_i$, does not.

Lemma 2.2 (Wall [**Wa1**]). *The invariants $c_{ij}, (i \neq j)$ and $\bar{\alpha}_i$ form a complete set of isotopy invariants of the presentation ϕ.*

Wall then proceeds to identify abstract invariants for handlebodies L which are counter parts to the invariants of a presentation but which do not depend upon giving a presentation for L. Viewed abstractly, the manifold L has invariants a free abelian group $H = H_k(L)$, an intersection pairing $\lambda : H \times H \to \mathbb{Z}$, and a map

$\bar{\alpha} : H \to \pi_{k-1}(SO(k))$. Geometrically λ is defined as follows. By the Hurewicz theorem $H_k(L) \cong \pi_k(L)$ and so every homology class $v \in H$ is represent by a map $\bar{v} : S^k \to L$. By the embedding theorems of Whitney this map may be chosen to be an embedding. Given $v, w \in H$ we may assume further that the embeddings $\bar{v} : S^k \to L$ and $\bar{w} : S^k \to L$ are in general position and so intersect at isolated points. The integer $\lambda(v, w)$ is the signed sum of these intersection points where the sign comes from comparing the orientation of L with orientation induced by $\bar{v}$ and $\bar{w}$ at a particular intersection point. The map $\bar{\alpha}$ evaluated on a homology class v is given by mapping the normal bundle of $\bar{v} : S^k \hookrightarrow L$ to the homotopy class of its clutching function which is an element of $\pi_{k-1}(SO(k))$. If $\partial\iota_k$ denotes the element of $\pi_{k-1}(SO(k))$ corresponding to the tangent bundle of the k-sphere, then Wall proves:

LEMMA 2.3 (Wall [**Wa1**]).

$$\bar{\alpha}(v + w) = \bar{\alpha}(v) + \bar{\alpha}(w) + \lambda(v, w) \cdot \partial\iota_k.$$

This allows us to translate between the abstract invariants of L and the invariants of the presentation for $L = L(D^{2k}, f_1, \ldots, f_n; k)$. If one attaches a disc D^k_i inside D^{2k} to the core of each handle $(D^k \times 0)_i$ then one obtains a k-sphere $\bar{v}_i$ which one may assume is embedded in L. The homology classes v_i corresponding to these embedded spheres form a basis for $H = H_k(L)$. Moreover the spheres $\bar{v}_i$ intersect only inside D^{2k} and their intersection number $\lambda(v_i, v_j)$ is coincides with linking number c_{ij}. The self linking numbers $\lambda(v_i, v_i)$ are given by the Euler numbers $E(\bar{\alpha}_i)$ of the framings from the presentation invariants. Finally the framings $\bar{\alpha}_i$ are evidently given by $\bar{\alpha}(v_i)$ and Lemma 2.3 ensures that the framings $\bar{\alpha}_i$ determine $\bar{\alpha}$. Wall is able to complete this translation between presentations and the abstract point of view by proving

THEOREM 2.4 (Theorem 1 [**Wa1**]). *Suppose $L \in \mathcal{H}(k)$, and choose a basis for $H_k(L)$. Then L has a presentation corresponding to this basis.*

For much of the rest of this thesis k shall be equal to $4j$ and we shall be concerned almost exclusively with $8j$-dimensional handlebodies and more particularly with the boundaries of $8j$-dimensional handlebodies. It will be algebraically convenient to consider the stabilization of the tangential invariant $\bar{\alpha}$ which is $\alpha := S \circ \bar{\alpha}$, where $S : \pi_{4j-1}(SO(4j)) \to \pi_{4j-1}(SO) \cong \mathbb{Z}$ is the stabilization map. Since $S(\partial \iota_k) = 0$, Lemma 2.3 implies that $\alpha : H \to \mathbb{Z}$ is a homomorphism. In fact, Kervaire computed that $\alpha = a_j(2j-1)!p_j$ where p_j denotes the jth Pontryagin class and $a_j = 1$ if j is even and 2 if j is odd, [**Ke**]. Note that we lose nothing by considering α over $\bar{\alpha}$ since the unstable part of $\bar{\alpha}(v)$ is given by the Euler number of the normal bundle of $\bar{v}$ and this is equal to $\lambda(v, v)$. Hence $\bar{\alpha}$ may be recovered from α and λ. We conclude this section with a summary in the form of a Corollary.

COROLLARY 2.5. *Let $k > 2$ and let L be a highly connected $2k$-manifold with a non-empty and connected boundary. Then L is diffeomorphic to a handlebody. If L_0 and L_1 are two handlebodies of dimension $2k$ with invariants $(H_0, \lambda_0, \alpha_0)$ and $(H_1, \lambda_1, \alpha_1)$ then there is diffeomorphism $f : L_0 \to L_1$ such that $f_* = \Theta$ if and only if there is an isomorphism $\Theta : H_0 \to H_1$ such that $\lambda_1(\Theta(v), \Theta(w)) = \lambda_0(v, w)$ and $\alpha_1(\Theta(v)) = \alpha_0(v)$ for all $v, w \in H_0$.*

REMARK 2.6. Note that the PL-invariance of rational Pontryagin classes (see [**MS**] §20) entails that the words words "diffeomorphic" and "diffeomorphism" in Corollary 2.5 may be replaced by "PL-homeomorphic" and "PL-homeomorphism". Novikov's result on the topological invariance of the rational Pontryagin classes entails that they may be replaced by "homeomorphic" and "homeomorphism".

3. Quadratic functions

In the previous section we saw Wall's classification of the handlebodies L via their intersection form $\lambda(L) : H_{4j}(L) \times H_{4j}(L) \to \mathbb{Z}$ and their stable tangential invariant $\alpha(L) \in H^{4j}(L)$. In this section we give an abstract summary of the algebraic invariants of L.

To begin let H be a finitely generated free abelian group. A linear form on H is simply a group homomorphism $\alpha : H \to \mathbb{Z}$. The set of all such forms with the operation of pointwise addition of homomorphisms forms a free abelian group called the dual group of H and denoted by H^*. A symmetric bilinear form on H is a bilinear map $\lambda : H \times H \to \mathbb{Z}$ such that $\lambda(v,w) = \lambda(w,v)$ for all $v,w \in H$. There are two ways one might combine a bilinear form $\lambda : H \times H \to \mathbb{Z}$ and a linear form $\alpha : H \to \mathbb{Z}$. One may follow Wall and simply consider the pair (λ, α). A morphism of pairs (λ_0, α_0) (λ_1, α_1) is a homomorphism $\Theta : H_0 \to H_1$ such that $\alpha_0 = \alpha_1 \circ \Theta$ and $\lambda_0 = \lambda_1 \circ (\Theta \times \Theta)$. Another way to combine λ and α is to take the quadratic form associated to λ and add α to it.

DEFINITION 2.7. *A function $\kappa : H \to \mathbb{Z}$ is called a* quadratic function *if there exists a bilinear form λ on H, a linear functional $\alpha \in H^*$ such that*

$$\kappa(v) = \lambda(v,v) + \alpha(v)$$

for all $v \in H$.

PROPOSITION 2.8. *The form λ and dual element α are uniquely recovered from κ using the following relations:*

$$\kappa(v+w) = \kappa(v) + \kappa(w) + 2\lambda(v,w)$$
$$\alpha(v) = \kappa(v) - \lambda(v,v).$$

PROOF. The first relation shows that λ is determined by κ and the second shows how to recover α from κ given that λ is known. □

When we wish to specify the components of a quadratic function we shall write $\kappa = \kappa(H, \lambda, \alpha)$ for a quadratic function with domain H, bilinear form λ and linear form α. On occasions we shall drop any of H, λ or α from the list when they are understood or not relevant. Note that κ is linear in the case where λ is identically zero. When $\alpha = 0$, the quadratic function κ is homogeneous, in which case κ is a *quadratic form.*

We shall see below that what distinguishes the quadratic functions which arise from handlebodies of dimension 8 or 16 is that they take on only even integer values. We therefore make the following

DEFINITION 2.9. *An element $\alpha \in H^*$ is called characteristic for λ when $\kappa(\lambda, \alpha)$ takes on only even values. That is, $\lambda(v,v) + \alpha(v)$ is even for all $v \in H$. In this case we shall call $\kappa(\lambda, \alpha)$ a characteristic quadratic function.*

Now let $\kappa_0 = \kappa(H_0, \lambda_0, \alpha_0)$ and $\kappa_1 = \kappa(H_1, \lambda_1, \alpha_1)$ be quadratic functions on H_0 and H_1 respectively and let $\Theta : H_0 \to H_1$ be a homomorphism. We define $\Theta^*\kappa_1$ by $\Theta^*\kappa_1(v) = \kappa_1(\Theta(v))$. A homomorphism Θ is called an *isometric embedding* if it is injective and $\Theta^*\kappa_1(v) = \kappa_0(v)$ for every $v \in H_0$. A bijective isometric embedding is called an *isometry.* We observe that an isometry of quadratic functions induces an isometry of both the homogeneous quadratic and linear parts and record this fact as a proposition.

PROPOSITION 2.10. *Let λ_0 and λ_1 be symmetric bilinear forms on free abelian groups H_0 and H_1 respectively and let α_0 and α_1 be elements of H_0^* and H_1^* respectively. Then the the quadratic functions $\kappa(H_0, \lambda_0, \alpha_0)$ and $\kappa(H_1, \lambda_1, \alpha_1)$ are isometric if and only if the pairs (λ_0, α_1) and (λ_1, α_1) are isomorphic.*

PROOF. This follows immediately from Proposition 2.8. $\square$

Thus the two methods for combining λ and α which we mentioned earlier are equivalent.

Now let $\kappa = \kappa(H, \lambda, \alpha)$ be a quadratic function. The adjoint of the bilinear form λ is the linear transformation induced by λ from H to its dual,

$$\begin{aligned} \hat{\lambda} : H &\rightarrow H^* \\ v &\mapsto [w \mapsto \lambda(v, w)]. \end{aligned}$$

Thus for all $v, w \in H$, $\lambda(v, w) = \hat{\lambda}(v)(w)$. We define the *radical* of λ to be the group $F := \mathrm{Ker}(\hat{\lambda}) = \{v \in H | \lambda(v, w) = 0 \ \forall w \in H\}$ and the quotient of λ to be the group $G := \mathrm{Cok}(\hat{\lambda}) = H^*/\hat{\lambda}(H)$. The groups F, H, H^* and G fit into the following exact sequence:

$$0 \longrightarrow F \longrightarrow H \xrightarrow{\hat{\lambda}} H^* \xrightarrow{\pi} G \longrightarrow 0$$

where π is the canonical projection from H^* to $H^*/\hat{\lambda}(H) = G$. We call this sequence the *fundamental sequence* of λ and κ as it will occupy us for most of the remainder of this thesis. If $F = 0$ then λ and κ are called *nondegenerate*. If in addition $G = 0$ then λ and κ are called *nonsingular*.

Given two quadratic functions $\kappa_0 = \kappa(H_0, \lambda_0, \alpha_0)$ and $\kappa_1 = \kappa(H_1, \lambda_1, \alpha_1)$, the direct sum $\kappa_0 \oplus \kappa_1$ is defined on $H_0 \oplus H_1$ via the equation

$$(\kappa_0 \oplus \kappa_1)[(v_0, v_1)] = \kappa_0(v_0) + \kappa_1(v_1).$$

We conclude this section by making the relation between handlebodies and quadratic functions explicit.

DEFINITION 2.11. *Let $L \in \mathcal{H}(4j)$ be a handlebody with intersection form $\lambda(L)$ and tangential invariant $\alpha(L)$ defined on $H_{4j}(L)$, then $\kappa(L) := \kappa(H_{4j}(L), \lambda(L), \alpha(L))$ shall be called the* quadratic function of L. *Conversely, if $\kappa = \kappa(H, \lambda, \alpha)$ is a quadratic function such that there is an $8j$-dimensional handlebody L with $\kappa(L) \cong \kappa(H, \lambda, \alpha)$ then $L^{8j}(\kappa)$ shall denote the $8j$-dimensional handlebody defined up to diffeomorphism by κ.*

Now let P denote the boundary of L, let $PD : H_{4j}(L,P) \cong H^{4j}(L)$ denote the Poincaré duality isomorphism and let $j : H_{4j}(L) \to H_{4j}(L,P)$ denote the map induced by the map of pairs $(L,\phi) \to (L,P)$. The following fact is fundamental and well known. If we use the Kronecker pairing to identify $H^{4j}(L)$ with $\mathrm{Hom}(H_{4j}(L),\mathbb{Z})$ then $PD \circ j : H_{4j}(L) \to H^{4j}(L)$ is the adjoint of the intersection pairing on L. Thus, given Definition 2.11 we have the following pair of isomorphic sequences

$$\begin{tikzcd} 0 \arrow[r] & H_{4j}(P) \arrow[r, "i_*"] \arrow[d, leftrightarrow] & H_{4j}(L) \arrow[r, "PD\circ j"] \arrow[d, leftrightarrow] & H^{4j}(L) \arrow[r, "i^*"] \arrow[d, leftrightarrow] & H^{4j}(P) \arrow[r] \arrow[d, leftrightarrow] & 0 \\ 0 \arrow[r] & F \arrow[r] & H \arrow[r, "\hat{\lambda}"] & H^* \arrow[r, "\pi"] & G \arrow[r] & 0 \end{tikzcd}$$

where i_* and i^* denote the maps induced by the inclusion $i : P \hookrightarrow L$. This gives a topological interpretation of the fundamental sequence of $\kappa(L)$ which we shall use throughout this thesis.

It is apparent that the boundary connected sum of handlebodies gives rise to the direct sum of the associated quadratic functions so that $\kappa(L_0 \natural L_1) = \kappa(L_0) \oplus \kappa(L_1)$. If we reverse the orientation on L then this will change the intersection form of L by reversing its sign. However, reversing the orientation on L does not affect the tangential invariant. This is because $\alpha(L)$ is the primary obstruction to trivializing the tangent bundle and we noted above that this class is a multiple of the Pontryagin class which is not affected by a change of orientation. Thus if $\kappa = \kappa(H,\lambda,\alpha)$ we define κ^- to be the quadratic function $\kappa(H,-\lambda,\alpha)$ and with this definition $\kappa(-L) = \kappa(L)^-$.

4. Wall's classification restated

We may now restate Wall's theorem 2.4 as follows: for every isometry $\Theta : \kappa(L_0) \cong \kappa(L_1)$ there corresponds a diffeomorphism $f(\Theta)$ whose induced map on homology is Θ. We shall now dress this statement up in categorical terms by defining the following categories. Let $\mathcal{H}^{8j} := \mathcal{H}(4j)$ be the category whose objects are highly connected $8j$-manifolds L and whose morphisms are equivalence classes of diffeomorphisms of handlebodies. Two diffeomorphisms $g, h : L_0 \to L_1$ are equivalent if they induce the same isomorphism on middle homology, $g_* = h_* : H_{4j}(L_0) \to H_{4j}(L_1)$. We define $\mathcal{F}$ to be the category whose objects are quadratic functions on free abelian groups and whose morphisms are isometries of quadratic functions. Finally we define $\mathcal{F}^{8j}$ to be the full sub-category of $\mathcal{F}$ whose objects are quadratic functions which are isomorphic to the quadratic function of some handlebody L. A symmetric monoidal category is one with an associative and commutative addition operation on its objects that is functorially well behaved (see [**MacL**] VII 7). We shall not distract the reader with the precise definitions but assure him that $\mathcal{H}^{8j}$ and $\mathcal{F}^*$ are symmetric monoidal categories with respect to the operations of boundary connected sum of handlebodies and of direct sum of quadratic functions.

THEOREM 2.12 (Restatement of [**Wa1**] Theorem 1). *There is an equivalence of (symmetric monoidal) categories*

$$\begin{array}{rccc} \kappa^j : & \mathcal{H}^{8j} & \cong & \mathcal{F}^{8j} \\ & \{L, [g], \natural\} & \to & \{\kappa(L), g_*, \oplus\}. \end{array}$$

PROOF. Recall that a skeleton of a category $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ which contains precisely one object for every isomorphic class of objects in $\mathcal{C}$. Every category $\mathcal{C}$ is equivalent to every one of its skeletons ([**MacL**] IV 4). So let $\underline{\mathcal{H}^{8j}}$ be a skeleton of $\mathcal{H}^{8j}$. By Corollary 2.5, $\kappa^j(\underline{\mathcal{H}^{8j}})$ is a skeleton of $\mathcal{F}^{8j}$ and therefore that κ is an equivalence of categories. Finally, for any pair of handlebodies L_0 and L_1 and any pair of diffeomorphisms f_0 and f_1, $\kappa(L_0 \natural L_1) = \kappa(L_0) \oplus \kappa(L_1)$ and $(f_0 \natural f_1)_* = f_{0*} \oplus f_{1*}$. $\square$

We now come to a fundamental observation which Wall and Wilkens did not fully exploit.

PROPOSITION 2.13. *When $j \neq 1, 2$ the category $\mathcal{F}^{8j}$ has as objects all quadratic functions $\kappa(\lambda, \alpha)$ with even bilinear forms λ (and no restriction on α). When $j = 1, 2$*

the category $\mathcal{F}^{8j}$ has objects all characteristic quadratic functions (with no restriction on the type of the bilinear form).

Proof. We have seen that a handlebody L is completely described by a presentation $\{(v_1, \ldots, v_l), \{c_{ij}\}, \bar{\alpha}(1), \ldots, \bar{\alpha}(l)\}$ consisting of a basis of handles, linking numbers between the handles and the normal bundle of each handle. With respect to this basis the intersection matrix of L has components λ_{ij} where

$$\lambda_{ij} = \left\{ \begin{array}{ll} c_{ij} & i \neq j \\ E(\alpha(i)) & i = j \end{array} \right\}$$

and where $E(\alpha(i))$ is the Euler number of $\bar{\alpha}(i) \in \pi_{4j-1}(SO((4j))$. When $j \neq 1, 2$ Adams' result on the non-existence of Hopf invariant one bundles ensures that each $E(\bar{\alpha}(i))$ is even and hence that λ is even. However, since every set of linking numbers may be realized by a link of $(4j-1)$-spheres in S^{4j-1} we see that any even form may be produced. The stable tangential invariant is determined by the stabilizations $S(\bar{\alpha}(i)) \in \pi_{4j-1}(SO)$ which are the values of α on the basis elements v_i. That these can be chosen arbitrarily follows from the result (see [**Wa1**, 171]) that the image of the homomorphism

$$S \oplus E : \pi_{4j-1}(SO(4j)) \to \pi_{4j-1}(SO) \oplus \mathbb{Z}$$

has index 2. Since E takes only even values for $j \neq 1, 2$ the map S must be onto $\pi_{4j-1}(SO)$. When $j = 1, 2$ then there are bundles with every possible Euler number and thus any intersection form one desires may be produced. Moreover, for every $\bar{\alpha} \in \pi_{4j-1}(SO(4j))$, we now have that $S(\bar{\alpha})$ and $E(\bar{\alpha})$ are of the same parity. Hence for $i = 1, \ldots, l$,

$$\lambda(v_i, v_i) = E(\bar{\alpha}(i)) \equiv S(\bar{\alpha}(i)) \pmod 2 \equiv \alpha(v_i) \pmod 2.$$

As $\{v_1, \ldots, v_l\}$ is a basis for $H_{4j}(L)$ it follows that $\lambda(v, v) \equiv \alpha(v) \bmod (2)$ for every $v \in H_{4j}(L)$. $\square$

As a consequence of Proposition 2.13 we define three categories of quadratic functions as follows.

- $\mathcal{F}^c$ shall denote the full sub-category of $\mathcal{F}$ whose objects are characteristic quadratic functions with arbitrary bilinear forms.
- $\mathcal{F}^{ev}$ shall denote the full sub-category of $\mathcal{F}$ whose objects are quadratic functions with even bilinear forms and arbitrary linear terms.
- $\mathcal{F}^o$ shall denote the full sub-category of $\mathcal{F}$ whose objects are even quadratic forms.

Proposition 2.13 states that $\mathcal{F}^{8j} = \mathcal{F}^c$ when $j = 1, 2$ and $\mathcal{F}^{8j} = \mathcal{F}^{ev}$ when $j > 2$. For all values of j the category $\mathcal{F}^o$ corresponds to the category of stably parallelizable handlebodies. Within each category $\mathcal{F}^*$ there are full subcategories of nondegenerate and nonsingular quadratic functions. They correspond respectively to the categories of handlebodies cobounding rational homology spheres and handlebodies cobounding homotopy spheres. As we are primarily concerned with the classification of the boundaries of handlebodies up to connected sum with homotopy spheres we define the notion of stable $\mathcal{F}^*$-equivalence of quadratic functions.

DEFINITION 2.14. *We call two quadratic functions κ_0 and κ_1 belonging to the category $\mathcal{F}^*$, stably $\mathcal{F}^*$-equivalent if there are nonsingular quadratic functions μ_0 and μ_1 also belonging to $\mathcal{F}^*$ and an isometry*

$$\Theta : \kappa_0 \oplus \mu_0 \cong \kappa_1 \oplus \mu_1.$$

We shall denote the stable $\mathcal{F}^*$-equivalence class of $\kappa \in \mathcal{F}^*$ by $[\kappa]_{\mathcal{F}^*}$. Note the direct sum of stable $\mathcal{F}^*$-equivalence classes of quadratic functions is well defined,

$$[\kappa_0]_{\mathcal{F}^*} \oplus [\kappa_1]_{\mathcal{F}^*} := [\kappa_0 \oplus \kappa_1]_{\mathcal{F}^*}.$$

5. Highly connected bordism

Our perspective on highly connected $(8j-1)$-manifolds is to view them as the boundaries of handlebodies. However, not every highly connected $(8j-1)$-manifold P bounds a handlebody. In this section we digress somewhat to introduce an invariant of P which is the obstruction to P bounding a handlebody. With this invariant we shall show that every P bounds a handlebody after the addition of the right homotopy sphere. We remind the reader that an n-manifold is highly connected if all of its homotopy groups vanish below its middle dimension which is $[n/2]$. Now let Ω_n^{HC} be the set of highly connected bordism classes of highly connected, closed (smooth, oriented) manifolds. A highly connected bordism between highly connected n-manifolds M_0 and M_1 is a compact highly connected $(n+1)$-manifold W and a diffeomorphism $\partial W \cong M_0 \sqcup -M_1$. The relation of being bordant via a highly connected manifold defines an equivalence relation. We shall write $[M]$ for the highly connected bordism class of a highly connected manifold M and 0 for the class of the empty manifold. Note that $[P]=0$ if and only if P is the boundary of a handlebody L.

Lemma 2.15. *When n is even or when $n = 2k-1$ is odd and $k \equiv 0, 4, 6, 7 \pmod 8$ the operations of connected sum and reversing orientation give rise to well defined operations on Ω_n^{HC} under which Ω_n^{HC} is an abelian group.*

Proof. Given highly connected bordism classes $[M_0]$ and $[M_1]$ we define

$$[M_0] + [M_1] := [M_0 \# M_1] \qquad \text{and} \qquad -[M_0] := [-M_0].$$

We now verify that these operations are well defined and that $[M \# -M] = 0$. If W is a highly connected bordism from M_0 to M_0' then $-W$ is a bordism from $-M_0$ to $-M_0'$ and so reversing orientation is well defined on highly connected bordism. Now, $M \# -M = \partial((M - D^n) \times [0,1])$ and if the dimension of M is even, $(M-D^n)\times[0,1]$ is a highly connected zero bordism of $M \# -M$. When $n = 2k-1$ is odd the additional hypothesis on k entails that $\pi_{k-2}(SO) = 0$. This ensures that every stable bundle over S^{k-1} is trivial and hence that $(M - D^n) \times [0,1]$ is $(k-1)$-parallelizable in the language of [**Mi**]. We may therefore perform surgeries in the interior of $(M - D^n) \times [0,1]$ to kill π_{k-1}. The resulting manifold is a highly connected zero bordism of $M \# -M$.

Suppose that W_0 is a highly connected bordism from M_0 to M_0'. To show that connected sum is well defined over highly connected bordism we must show that there is a highly connected bordism from $M_0 \# M_1$ to $M_0' \# M_1$. We first note that there is a highly connected bordism W_1 from M_1 to $-M_1$. When n is even we take $M_1 \times [0,1]$ and when $n = 2k-1$ we argue as before that surgery may be performed in the interior of $M_1 \times [0,1]$ to produce a highly connected bordism. The band connected sum of W_0 and W_1 (see [**KM**] §2) is then a highly connected bordism from $M_0 \# M_1$ to $M_0' \# M_1$. It is clear that the connected sum operation on highly connected bordism is associative and commutative. It follows that Ω_n^{HC} is an abelian group in the indicated dimensions. $\square$

LEMMA 2.16. *Let $k > 1$ and $k \equiv 0, 4, 6, 7 \pmod 8$. Then every member of Ω_{2k-1}^{HC} is represented by a homotopy sphere and*

$$\Omega_{2k-1}^{HC} \cong \Theta_{2k-1}/bP_{2k}.$$

PROOF. We let

$$\Omega : \Theta_{2k-1} \to \Omega_{2k-1}^{HC}$$

be the homomorphism we sends a homotopy sphere to the highly connected bordism class which it represents. It follows from calculations of Wall in [**Wa1**] that the kernel of Ω is precisely bP_{2k}. It remains to show that Ω is onto but this is a simple consequence of the success of simply connected surgery in odd dimensions as we now show. One may always take the $(k-1)$-skeleton of a closed highly connected $(2k-1)$-manifold M to be a wedge of $(k-1)$-spheres. From the assumption on k it follows that that $\pi_{k-2}(SO) = 0$ and thus every stable bundle over a wedge of $(k-1)$-spheres is trivial. In particular the tangent bundle of M restricted to the $(k-1)$-skeleton of M is trivial. Thus M is $(k-1)$-parallelizable and we may perform a series of surgeries on $(k-1)$ spheres in M to kill $\pi_{k-1}(M)$. The resulting $(2k-1)$-manifold is a homotopy sphere Σ and the trace of the surgeries performed is a highly connected bordism from M to Σ. ☐

COROLLARY 2.17. *Let P be a highly connected $(8j-1)$-manifold. There is a unique homotopy sphere Σ_P such that $P\#\Sigma_P$ is the boundary of a handlebody and $s(\Sigma_P) = 0 \in bP_{8j-1}$. Moreover, if P_0 and P_1 are two highly connected $(8j-1)$-manifolds then $\Sigma_{P_0\#P_1} = \Sigma_{P_0}\#\Sigma_{P_1}$.*

PROOF. We note first that $8j-1$ is a dimension to which both Lemmas 2.15 and 2.16 apply. By Lemma 2.16, $-P$ is highly connected bordant to a homotopy sphere Σ which is determined up to connected sum with homotopy spheres in bP_{8j}. There is thus a unique homotopy sphere Σ_P such that $s(\Sigma_P) = 0$ and $[\Sigma_P] = -[P] \in \Omega^{HC}_{8j-1}$. Now by Lemma 2.15, $[P\#\Sigma_P] = 0$ and therefore $P\#\Sigma_P$ bounds a handlebody. To prove the second assertion we need only note that if L_i are handlebodies such that $P_i\#\Sigma_{P_i} = \partial L_i$ for $i = 0, 1$ then

$$(P_0\#P_1)\#(\Sigma_{P_0}\#\Sigma_{P_1}) = \partial(L_0\natural L_1) \text{ and } s(\Sigma_{P_0}\#\Sigma_{P_1}) = s(\Sigma_{P_0}) + s(\Sigma_{P_1}) = 0.$$

☐

We remind the reader that in the dimensions of primary concern in this thesis $\Theta_7/bP_8 = 0$ and $\Theta_{15}/bP_{16} = \mathbb{Z}/2$. In particular, in dimension 7, the homotopy sphere Σ_P is always the standard sphere and every manifold P bounds a handlebody.

6. Decomposition of quadratic functions

Quadratic functions are of two essentially different types. There are linear or "totally degenerate" quadratic functions at one extreme and at the other there are nondegenerate quadratic functions. In this section we consider the decomposition of a general quadratic function into the sum of linear and nondegenerate quadratic functions.

Let $\kappa = \kappa(H, \lambda, \alpha)$ be a quadratic function and recall the fundamental sequence of κ,

$$0 \longrightarrow F \longrightarrow H \xrightarrow{\hat{\lambda}} H^* \xrightarrow{\pi} G \longrightarrow 0.$$

By definition F is the subspace of H on which κ is linear. It is clear that F is a summand of H for suppose that $r.v \in F$ for some non-zero integer r. This means that $\lambda(rv, w) = r\lambda(v, w) = 0$ for every $w \in H$. Since r is non-zero we conclude that $\lambda(v, w) = 0$ for every $w \in H$ and that v belongs to F. Thus F is a summand of H and H/F is a free group. The bilinear form λ descends to define a nondegenerate bilinear form on H/F in the familiar way:

$$\begin{array}{rccc} \lambda_{/F}: & H/F \times H/F & \to & \mathbb{Z} \\ & (v+F, w+F) & \mapsto & \lambda(v, w). \end{array}$$

The function $\lambda_{/F}$ is well defined because different coset representatives differ by an element of F but $\lambda(u, v) = 0$ for any $u \in F$ and any $v \in H$. Moreover, if $\lambda/F(v + F, w + F) = 0$ for every $w + F \in H/F$ then $v \in F$, $v + F = 0$ and so $\lambda_{/F}$ is nondegenerate. Now let $\rho : H \to H/F$ denote the canonical projection from H onto H/F and let

$$\mathrm{Sec}(\rho) := \{\Psi : H/F \to H | \rho \circ s = Id_{H/F}\}$$

denote the set of sections of ρ. The set $\mathrm{Sec}(\rho)$ is nonempty and there is a free and transitive action of $\mathrm{Hom}(H/F, F)$ on $\mathrm{Sec}(\rho)$ defined by

$$(\psi \cdot \Psi)(v+F) = \Psi(v+F) + \psi(v+F)$$

where $\psi \in \mathrm{Hom}(H/F, F)$, $\Psi \in \mathrm{Sec}(\rho)$ and $v + F \in H/F$. For every $\Psi \in \mathrm{Sec}(\rho)$, we define the projections

$$\begin{array}{rccc} p_F(\Psi): & H & \to & F \\ & v & \mapsto & v - \Psi(\rho(v)) \end{array} \quad \text{and} \quad \begin{array}{rccc} p(\Psi): & H & \to & \Psi(H/F) \\ & v & \mapsto & \Psi(\rho(v)) \end{array}$$

and the subgroups

$$\begin{array}{cc} H(\Psi) := \Psi(H/F) & H^*(\Psi) := p(\Psi)^*(H(\Psi)^*) \\ F^*(\Psi) := p_F(\Psi)^*(F^*) & G(\Psi) = \pi(F^*(\Psi)) \subset G. \end{array}$$

It is straightforward to check that there are direct sum decompositions $H = F \oplus H(\Psi)$, $H^* = F^*(\Psi) \oplus H^*(\Psi)$ and $G = G(\Psi) \oplus TG$ where TG denotes the torsion subgroup of G. Moreover, we show below that $H^*(\Psi) \subset H^*$ does not depend upon Ψ and so we shall henceforth identify $H(\Psi)^*$ and $H^*(\Psi)$ via $p(\Psi)^*$. The reason for doing this is that it is technically convenient to think of $H(\Psi)^*$ as a subgroup of H^*. We denote the restrictions of α, λ and κ as follows

$$\alpha(\Psi) := \alpha|_{H(\Psi)}, \ \lambda(\Psi) := \lambda|_{H(\Psi)\times H(\Psi)} \text{ and } \kappa(\Psi) := \kappa|_{H(\Psi)}.$$

With this notation $\kappa(\Psi) = \kappa(H(\Psi), \lambda(\Psi), \alpha(\Psi))$, $\alpha = \alpha(\Psi) + p_F(\Psi)^*(\alpha|_F)$. We also obtain the following decomposition of the fundamental sequence of κ:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F & \xrightarrow{=} & F & \xrightarrow{0} & F^*(\Psi) & \longrightarrow & G(\Psi) & \longrightarrow & 0 \\ & & & & \bigoplus & & \bigoplus & & \bigoplus & & \\ & & 0 & \longrightarrow & H(\Psi) & \xrightarrow{\hat{\lambda}(\Psi)} & H^*(\Psi) & \longrightarrow & TG & \longrightarrow & 0 \end{array}$$

where $\hat{\lambda}(\Psi)$ denotes the adjoint of $\lambda(\Psi)$.

PROPOSITION 2.18. *Let $\kappa(H, \lambda, \alpha)$ be a quadratic function and $\lambda_{/F}$, ρ and $Sec(\rho)$ be as above. Then, for every section $\Psi \in \mathrm{Sec}(\rho)$ and for every $\psi \in \mathrm{Hom}(H/F, F)$,*

1. $\Psi^*(\lambda) = \lambda_{/F}$,
2. Ψ *and* $\rho|_{H(\Psi)}$ *define inverse isometries* $\Psi : \Psi^*(\kappa) \cong \kappa(\Psi)$ *and* $\rho|_{H(\Psi)} : \kappa(\Psi) \cong \Psi^*(\kappa)$,
3. $\kappa(\Psi)$ *is nondegenerate and* $\kappa = \kappa(\Psi) \oplus \alpha|_F$,
4. $H^*(\Psi) = \rho^*((H/F)^*)$ *and* $TG = H^*(\Psi)/\hat{\lambda}(\Psi)(H(\Psi))$,
5. $\alpha(\psi \cdot \Psi) = \alpha(\Psi) + \rho^*(\psi^*(\alpha|_F))$.

PROOF. *(1.)* Let $v, w \in H$ be arbitrary and let $v' = \Psi(v+F)$ and $w' = \Psi(w+F)$ so that $v' + F = v + F$ and $w' + F = w + F$. We calculate that

$$\begin{aligned}\Psi^*(\lambda)(v+F, w+F) &= \lambda(\Psi(v+F), \Psi(w+F)) \\ &= \lambda(v', w') \\ &= \lambda_{/F}(v+F, w+F).\end{aligned}$$

(2.) This is true by definition. Certainly $\Psi \circ \rho|_{H(\Psi)} = Id_{H(\Psi)}$ and $\rho|_{H(\Psi)} \circ \Psi = Id_{H/F}$. Moreover, for $v \in H(\Psi)$ we calculate that

$$\begin{aligned}\kappa(\Psi)(\Psi(v+F)) &= \kappa(\Psi(v+F)) \\ &= \Psi^*(\kappa)(v+F)\end{aligned} \quad \text{and} \quad \begin{aligned}\Psi^*(\kappa)(\rho|_{H(\Psi)}(v)) &= \kappa(\Psi(\rho|_{H(\Psi)}(v)) \\ &= \kappa(v) \\ &= \kappa(\Psi)(v).\end{aligned}$$

(3.) We showed above the that $\lambda_{/F}$ is nondegenerate and so by *1* we see that $\Psi^*(\kappa)$ is nondegenerate and then by *2* we have that $\kappa(\Psi)$ is nondegenerate. For each $v \in H$, we decompose v as $v = v - \Psi(\rho(v)) + \Psi(\rho(v))$ where $v - \Psi(\rho(v)) \in F$ and $\Psi(\rho(v)) \in H(\Psi)$. Therefore

$$\begin{aligned}\kappa(v) &= \lambda(\Psi(\rho(v)), \Psi(\rho(v))) + \alpha(\Psi(\rho(v))) + \alpha(v - \Psi(\rho(v))) \\ &= \kappa(\Psi)(\Psi(\rho(v)) + \alpha|_F(v - \Psi(\rho(v))) \\ &= (\kappa(\Psi) \oplus \alpha|_F)[(\Psi(\rho(v)), v - \Psi(\rho(v)))].\end{aligned}$$

(4.) Since this result justifies our convention identifying $H^*(\Psi)$ and $H(\Psi)^*$ we briefly drop that convention. For every Ψ, $\rho^* : (H/F)^* \to H^*$ factors as

$$(H/F)^* \overset{(\rho|_{H(\Psi)})^*}{\longrightarrow} H(\Psi)^* \overset{p(\Psi)^*}{\longrightarrow} H^*(\Psi) \subset H^*$$

as the following calculation demonstrates. For any $z \in (H/F)^*$ and any $v \in H$

$$\begin{aligned} p(\Psi)^* \circ (\rho|_{H(\Psi)})^*(z)(v) &= (\rho|_{H(\Psi)})^*(z)((\Psi \circ \rho)(v)) \\ &= z((\rho|_{H(\Psi)} \circ \Psi \circ \rho)(v)) \\ &= z(\rho(v)) \\ &= \rho^*(z)(v). \end{aligned}$$

As $(\rho|_{H(\Psi)})^*$ and $p(\Psi)^*$ are isomorphisms it follows that $H^*(\Psi) = p(\Psi)^*(H(\Psi)^*) = \rho^*((H/F)^*)$. The second assertion follows from the splitting of the fundamental sequence of κ just prior to the statement of this proposition. Equivalently it may demonstrated as follows. For every $v \in H$, $\hat{\lambda}(v) = \hat{\lambda}(\Psi(\rho(v)))$ and for every $w \in H$,

$$\begin{aligned} \lambda(v, w) &= \lambda(v - \Psi(\rho(v)) + \Psi(\rho(v)), w) \\ &= \lambda(\Psi(\rho(v)), w). \end{aligned}$$

It follows that $\hat{\lambda}(H) = \hat{\lambda}(\Psi)(H(\Psi))$ and that

$$\mathrm{Cok}(\hat{\lambda}) = F^*(\Psi) \oplus H^*(\Psi)/\hat{\lambda}(\Psi)(H(\Psi)) = F^*(\Psi) \oplus \mathrm{Cok}(\hat{\lambda}(\Psi)).$$

Moreover, as $\lambda(\Psi)$ is nondegenerate, $\mathrm{Cok}(\hat{\lambda}(\Psi))$ is a torsion group and thus it is the torsion subgroup of $\mathrm{Cok}(\hat{\lambda})$.

(5.) Given $v \in H$, $\alpha(\Psi)(v) = \alpha(\Psi(\rho(v)))$ and thus

$$\begin{aligned} \alpha(\psi \cdot \Psi)(v) &= \alpha(\Psi(\rho(v)) + \psi(\rho(v))) \\ &= \alpha(\Psi(\rho(v))) + \alpha|_F(\psi(\rho(v))) \\ &= \alpha(\Psi)(v) + \rho^*(\psi^*(\alpha|_F))(v). \end{aligned}$$

□

7. Linking forms and Kronecker pairings

In this section we review the algebraic structures that a quadratic function $\kappa = \kappa(H, \lambda, \alpha)$ induces on its quotient G and radical F. The most important of these is the linking form induced on the torsion subgroup of G. By a linking form on a finite abelian group TG we shall mean a function

$$b : TG \times TG \to \mathbb{Q}/\mathbb{Z}$$

with the following properties.

1. Symmetry: $b(x, y) = b(y, x)$.
2. Bilinearity: $b(x, y + z) = b(x, y) + b(x, z)$.
3. Nonsingularity: $b(x, y) = 0 \ \forall x \in TG \Rightarrow y = 0$.

In Chapter 5 we shall describe in detail the classification of linking forms on finite abelian groups but for now we report some fundamental facts concerning linking forms. If (TG_0, b_0) and (TG_1, b_1) are linking forms then an isomorphism $\theta : TG_0 \to TG_1$ is an isometry from b_0 to b_1 if $b_0 = b_1 \circ (\theta \times \theta)$. There is a linking form $(TG_0 \oplus TG_1, b_0 \oplus b_1)$ defined by

$$(b_0 \oplus b_1)[(x_0, x_1), (y_0, y_1)] = b_0(x_0, y_0) + b_1(x_1, y_1).$$

If (TG, b) is a linking form and TG_0 is a subgroup of TG then the annihilator of TG_0 is the set

$$TG_0^{\perp} := \{x \in TG : b(x, y) = 0 \ \forall y \in TG_0\}.$$

LEMMA 2.19. *Let (TG, b) be a linking form. If TG_0 is a finite subgroup of TG such that*
$b|_{TG_0 \times TG_0} : TG_0 \times TG_0 \to \mathbb{Q}/\mathbb{Z}$ *is nonsingular then* $b|_{TG_0^{\perp} \times TG_0^{\perp}}$ *is nonsingular and*

$$b \cong b|_{TG_0 \oplus TG_0} \oplus b|_{TG_0^{\perp} \times TG_0^{\perp}}.$$

PROOF. We postpone the proof which is straight forward until chapter 5. □

A linking form (TG, b) is called *indecomposable* if it cannot be written as a non-trivial direct sum of sublinking forms. In [**Wa2**] Wall proved that the groups which arise as the domains of indecomposable linking forms are $\mathbb{Z}/p^k$ and $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$ for p any prime and k any positive integer.

Every quadratic function $\kappa(H, \lambda, \alpha)$ defines a linking form on the torsion subgroup of its quotient group $TG \subset G = \mathrm{Cok}(\hat{\lambda})$ via the bilinear form λ. We begin by assuming that λ is nondegenerate in which case the fundamental sequence of λ is

$$0 \longrightarrow H \xrightarrow{\hat{\lambda}} H^* \xrightarrow{\pi} TG \longrightarrow 0.$$

If we tensor this sequence with $\mathbb{Q}$ and $\hat{\lambda}$ with $Id_{\mathbb{Q}}$ then $\hat{\lambda} \otimes Id_{\mathbb{Q}}$ is an isomorphism. We let $\hat{\lambda}^{-1}$ denote the restriction of the inverse of $\hat{\lambda} \otimes Id_{\mathbb{Q}}$ to H^*,

$$\hat{\lambda}^{-1} := (\hat{\lambda} \otimes Id_{\mathbb{Q}})^{-1}|_{H^*} : H^* \to H \otimes \mathbb{Q}.$$

The linear map $\hat{\lambda}^{-1} : H^* \to H \otimes \mathbb{Q}$ defines a bilinear pairing

$$\begin{array}{rccc} \lambda^{-1}: & H^* \times H^* & \to & \mathbb{Q} \\ & (x, y) & \mapsto & y(\hat{\lambda}^{-1}(x)) \end{array}$$

where $y(v \otimes \frac{p}{q}) = \frac{p}{q} y(v)$. The symmetry of $\hat{\lambda}^{-1}$ follows immediately from the symmetry of λ. The pairing $\hat{\lambda}^{-1}$ may be defined alternatively as follows. For any $x, y \in H^*$ there is a non-zero $r \in \mathbb{Z}$ and some $v \in H$ such that $rx = \hat{\lambda}(v)$. We have the following identity since $\hat{\lambda}^{-1}(x) = \frac{1}{r} v$,

$$\lambda^{-1}(x, y) = \frac{1}{r} y(v).$$

The bilinear pairing λ^{-1} induces a nonsingular linking form on TG as follows. Let $[x] = \pi(x)$ and $[y] = \pi(y) \in TG$.

$$\begin{array}{rccc} b = b(\lambda): & TG \times TG & \to & \mathbb{Q}/\mathbb{Z} \\ & ([x], [y]) & \mapsto & \lambda^{-1}(x, y) \mod \mathbb{Z} \end{array}$$

The linking form b is well defined in $\mathbb{Q}/\mathbb{Z}$, for suppose that $x + \hat{\lambda}(v)$ and $y + \hat{\lambda}(w)$ are different lifts of $[x]$ and $[y]$. Then

$$\begin{aligned} \lambda^{-1}(x + \hat{\lambda}(v), y + \hat{\lambda}(w)) &= \lambda^{-1}(x, y) + \lambda^{-1}(x, \hat{\lambda}(w)) + \lambda^{-1}(\hat{\lambda}(v), y) \\ &\quad + \lambda^{-1}(\hat{\lambda}(v), \hat{\lambda}(w)) \bmod \mathbb{Z} \\ &= \lambda^{-1}(x, y) + x(w) + y(v) + \lambda(v, w) \bmod \mathbb{Z} \\ &= \lambda^{-1}(x, y) \bmod \mathbb{Z}. \end{aligned}$$

The function $b(\lambda)$ is evidently bilinear and symmetric. It is also nonsingular as we now show. We may choose an indivisible $v \in H$ and an integer r such that $rx = \hat{\lambda}(v)$ for if $v' \in H$ and $r' \in \mathbb{Z}$ satisfy $x = r'\hat{\lambda}(v')$ then if $v' = s.v$ for $s \in \mathbb{Z}$ and v indivisible then $x = r's\hat{\lambda}(v)$. Now, if If $b([x],[y]) = 0$ for every $[y] \in TG$ then $\frac{1}{r}y(v) \in \mathbb{Z}$ for every $y \in H^*$. Since v is indivisible there is a $y \in H^*$ such that $y(v) = 1$. Thus $r = \pm 1$ and hence $[x] = 0 \in TG$. We may also define b by its adjoint map to the torsion dual of TG, $TG^\wedge = \mathrm{Hom}(TG, \mathbb{Q}/\mathbb{Z})$,

$$\begin{array}{rccc} \hat{b}: & TG & \to & TG^\wedge \\ & [x] & \mapsto & ([y] \mapsto b([x],[y])). \end{array}$$

As TG and $TG^\wedge$ are finite groups of the same order $\hat{b}$ is an isomorphism if and only if b is nonsingular.

If the quadratic function κ is degenerate then by Proposition 2.18, each section Ψ defines a nondegenerate quadratic function $\kappa(\Psi)$ where $TG = H^*(\Psi)/\hat{\lambda}(\Psi)(H(\Psi))$ and $H^*(\Psi) = \rho^*((H/F)^*)$ are is independent of Ψ. In fact the following is true.

LEMMA 2.20. *The bilinear pairing* $\lambda(\Psi)^{-1} : H^*(\Psi) \times H^*(\Psi) \to \mathbb{Q}$ *is independent of* Ψ.

PROOF. Suppose that Ψ' is another section of ρ, that $x, y \in H^*(\Psi') = H^*(\Psi)$ and that $rx = \hat{\lambda}(\Psi)(v)$ for $v \in H(\Psi)$ and a non-zero integer r. If we set $v' := \Psi'(\rho(v))$, then $\hat{\lambda}(\Psi')(v) = rx$, $v' - v \in F$ and hence $y(v' - v) = 0$. We calculate that

$$\begin{aligned} \lambda^{-1}(\Psi)(x,y) &= \tfrac{1}{r}y(v) \\ &= \tfrac{1}{r}y(v + v' - v) \\ &= \tfrac{1}{r}y(v') \\ &= \lambda^{-1}(\Psi')(x,y). \end{aligned}$$

□

It follows from Lemma 2.20 that $b(\lambda(\Psi)) = b(\lambda(\Psi'))$ and so given a degenerate bilinear form λ, we define

$$b(\lambda) := b(\lambda(\Psi))$$

for any section Ψ.

In addition to the linking form $b(\lambda)$, a bilinear form λ also defines a Kronecker pairing between $F = \mathrm{Ker}(\hat{\lambda})$ and $G = \mathrm{Cok}(\hat{\lambda})$ by

$$\begin{array}{rccc} \bar{\lambda}: & F \times G & \to & \mathbb{Z} \\ & (v, [x]) & \mapsto & x(v). \end{array}$$

The pairing $\bar{\lambda}$ is well defined because any two choices of lifts of $[x]$ differ by $\hat{\lambda}(w)$ for some $w \in H$ and $\hat{\lambda}(w)(v) = 0$. We label the adjoint homomorphisms associated to $\bar{\lambda}$ by $\bar{\lambda}_F : F \to G^*$ and $\bar{\lambda}_G : G \to F^*$ where $\bar{\lambda}_F(v)([x]) = \bar{\lambda}(v, [x])$ and $\bar{\lambda}_G([x])(v) = \bar{\lambda}(v, [x])$. Here we have extended the * notation to any abelian group G, so that $G^* := \mathrm{Hom}(G, \mathbb{Z})$ denotes the group of homomorphism from G to $\mathbb{Z}$. Recall the canonical homomorphism

$$\begin{array}{rccc} \tau_G: & G & \to & G^{**} \\ & g & \mapsto & (x \mapsto x(g)). \end{array}$$

LEMMA 2.21. *The composition $\bar{\lambda}_F^* \circ \tau_G : G \to F^*$ is equal to $\bar{\lambda}_G$. The homomorphisms and $\bar{\lambda}_G : G \to F^*$ and $\bar{\lambda}_F : F \to G^*$ are respectively surjective and bijective.*

PROOF. The first assertion is a simple matter to check. Let $[x] \in G$ and $v \in F$,

$$\begin{aligned} \bar{\lambda}_F^*(\tau_G([x]))(v) &= \tau_G([x])(\bar{\lambda}_F(v)) \\ &= \bar{\lambda}_F(v)([x]) \\ &= x(v) \\ &= \bar{\lambda}_G([x])(v). \end{aligned}$$

In Section 6 we noted that choosing a section $\Psi \in \mathrm{Sec}(\rho)$ defines a splitting $H^* = F^*(\Psi) \oplus H^*(\Psi)$ and the inclusion $p_F(\Psi)^* : F^* \cong F^*(\Psi) \hookrightarrow H^*$. Given $x \in F^*$ it is a matter of definitions that $\bar{\lambda}_G(\pi(p_F(\Psi)^*(x))) = x$ and therefore $\bar{\lambda}_G$ is onto. Since $\bar{\lambda}_G = \bar{\lambda}_F^* \circ \tau_G$, it follows that $\bar{\lambda}_F^*$ in onto. Now note that F and G have the same rank by the Rank-Nullity Theorem applied to $\hat{\lambda}$. Thus G^{**} and F^* have the same rank and so $\bar{\lambda}_F^*$ is a surjective homomorphism between free abelian groups of the same rank and thus an isomorphism. This entails that $\bar{\lambda}_F$ is itself an isomorphism. □

We now review and relate the linking form and Kronecker pairing of a bilinear form from a slightly more abstract perspective. Let A and B be free abelian groups and $f : A \to B$ a homomorphism with kernel F and cokernel G. Again we let $\rho : A \to A/F$ denote the canonical projection. Then there is a map $\hat{b}(f) : \rho^*((A/F)^*) \to TG^\wedge$ constructed in a similar manner as before. If $[b] \in TG$ then $[b] = \pi(b)$ for some $b \in B$ and $rb = f(a)$ for some non-zero integer r and some $a \in A$. For $x \in \rho^*((A/F)^*)$, $x(a) = 0$ for every $a \in F$, and so we define may

$$\begin{array}{rccc} \hat{b}(f): & \rho^*((A/F)^*) & \to & TG^\wedge \\ & x & \mapsto & ([b] \mapsto \frac{1}{r}x(a) \in \mathbb{Q}/\mathbb{Z}). \end{array}$$

Now choose any section $\Psi : A/F \to A$ so that we have the decomposition $A^* = \rho^*((A/F)^*) \oplus F^*(\Psi)$. With respect to this splitting define $\hat{\pi} : A^* \to F^* \oplus TG^\wedge$ by $\hat{\pi}(x, y) = (y|_F, \hat{b}(x))$. It is easy to check that we now have the following pair of exact sequences:

$$0 \longrightarrow F \longrightarrow A \xrightarrow{f} B \xrightarrow{\pi} G \longrightarrow 0$$

$$0 \longrightarrow G^* \xrightarrow{\pi^*} B^* \xrightarrow{f^*} A^* \xrightarrow{\hat{\pi}} F^* \oplus TG^\wedge \longrightarrow 0$$

Observe also that $\hat{b}(f)$ is identically zero on the image of f and thus defines a function which we also denote $\hat{b}(f) : (A/F)^*/f(A) \to TG^\wedge$. If we return to the case where $B = A^*$ and $f = \hat{\lambda}$ is the adjoint of a bilinear form on A then $\hat{b}(\hat{\lambda})$ is the adjoint map of the linking form $b(\lambda)$ which we defined above. Setting $G(\Psi) := \pi(F^*(\Psi)) \subset G$ the above pair of sequences becomes linked into the following commutative diagram where each vertical arrow denotes and isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & F \arrow[r] \arrow[d, "\bar{\lambda}_F"'] & A \arrow[r, "\hat{\lambda}"] \arrow[d, "\tau_A"'] & A^* \arrow[r, "\pi"] \arrow[d, "Id"'] & G(\Psi) \oplus TG \arrow[r] \arrow[d, "\begin{pmatrix} \bar{\lambda}_G|_{G(\Psi)} & 0 \\ 0 & \hat{b}(\hat{\lambda}) \end{pmatrix}"] & 0 \\
0 \arrow[r] & G^* \arrow[r, "\pi^*"] & A^{**} \arrow[r, "\hat{\lambda}^*"] & A^* \arrow[r, "\hat{\pi}"] & F^* \oplus TG^{\wedge} \arrow[r] & 0
\end{tikzcd}$$

Recall that if a handlebody L has quadratic function $\kappa = \kappa(H, \lambda)$ and boundary P then the topological correlates of $F = \mathrm{Ker}(\hat{\lambda})$ and $G = \mathrm{Cok}(\hat{\lambda})$ are $H_{4j}(P)$ and $H^{4j}(P)$ respectively. The pairing $\bar{\lambda}$ corresponds to the Kronecker pairing $H_{4j}(P) \times H^{4j}(P) \to \mathbb{Z}$ and this gives an alternate explanation of Lemma 2.21. The linking form $b(\lambda)$ is the linking form of P which is a defined on the torsion subgroup of $H^{4j}(P)$. Recall also that for every manifold P there is a specific homotopy Σ_P such that $P \# \Sigma_P$ bounds a handlebody, L say. Now there is a canonical homeomorphism $P \to P \# \Sigma_P$ and therefore a canonical identification of $H^{4j}(P)$ with $H^{4j}(P \# \Sigma_P)$.

Definition 2.22. *Let P be a highly connected $(8j-1)$-manifold and L a handlebody cobounding $P \# \Sigma_P$ with quadratic function $\kappa(L) = \kappa(\lambda)$. The linking form of P is defined to be the linking form induced by λ;*

$$b(P) := b(\lambda).$$

Remark 2.23. The linking form of P has at least two other definitions: by way of Poincaré duality with $\mathbb{Q}/\mathbb{Z}$ coefficients ([**KM**]) and by letting r times a cycle representing a homology class bound and computing intersection (mod r) ([**Wa3**]). We leave it as an exercise to verify the well known fact that all three definitions give the same linking form. That our definition of $b(P)$ is independent of the choice of handlebody will follow from results of Wilkens' thesis which we review in Section 8.

Now let λ_0 and λ_1 be a pair of bilinear forms. It is clear from the definitions that $b(\lambda_0 \oplus \lambda_1) = b(\lambda_0) \oplus b(\lambda_1)$. We claim for two highly connected manifolds P_0 and P_1

that

$$b(P_0\#P_1) = b(P_0) \oplus b(P_1).$$

For firstly by Corollary 2.17, $\Sigma_{P_0\#P_1} = \Sigma_{P_0}\#\Sigma_{P_1}$ and thus if $\partial L_i = P_i\#\Sigma_{P_i}$ for $i = 0, 1$, then $(P_0\#P_1)\#\Sigma_{P_0\#P_1} = \partial(L_0 \natural L_1)$. But now $\lambda(L_0\natural L_1) = \lambda(L_0) \oplus \lambda(L_1)$ and our claim follows.

8. Wilkens' classification of the manifolds P

The fundamental topological result in Wilkens' thesis is the following theorem (whose proof we shall present in Chapter 4).

THEOREM 2.24 (Wilkens' Thesis 3.2). *Let L_0 and L_1 be $2k$-dimensional handlebodies ($k > 2$) with boundaries $\partial L_0 = P_0$ and $\partial L_1 = P_1$. Suppose that $f : P_0 \to P_1$ is a diffeomorphism. Then there are handlebodies V_0, V_1 with boundaries the standard sphere and a diffeomorphism g which makes the following diagram commute.*

$$\begin{tikzcd} P_0 \arrow[r, "f"] \arrow[d] & P_1 \arrow[d] \\ L_0\natural V_0 \arrow[r, "g"] & L_1\natural V_1 \end{tikzcd}$$

In the light of Theorem 2.24 and recalling Definition 2.14 which defined the stable $\mathcal{F}^*$-equivalence class of a quadratic function, we make the following

DEFINITION 2.25. *Let P be a highly connected $(8j-1)$-manifold and L a handlebody such that $\partial L = P\#\Sigma_P$. We define the stable quadratic function of P, $[\kappa(P)]$, to be the stable $\mathcal{F}^*$-equivalence class of $\kappa(L)$.*

COROLLARY 2.26 (To Wilkens' 2.24 and Wall's 2.12). *The stable quadratic function of a highly connected manifold P, $[\kappa(P)]$, is a well defined and complete almost diffeomorphism invariant of P.*

PROOF. We show that $[\kappa(P)]$ is both well defined and invariant under almost diffeomorphisms at the same time. Let P_0 and P_1 be highly connected manifolds, Σ a homotopy sphere and $f : P_0 \to P_1\#\Sigma$ a diffeomorphism. Then $\Sigma_{P_0} = \Sigma_{P_1\#\Sigma} = \Sigma_{P_1}\#\Sigma_\Sigma$ and there is a diffeomorphism

$$f' : P_0\#\Sigma_{P_0} \to P_1\#\Sigma\#\Sigma_{P_0} = P_1\#\Sigma_{P_1}\#\Sigma_\Sigma\#\Sigma.$$

Let L_0, L_1 and L_Σ be handlebodies bounding respectively $P_0\#\Sigma_{P_0}$, $P_1\#\Sigma_{P_1}$ and $\Sigma\#\Sigma_\Sigma$. Applying Theorem 2.24 to f', L_0 and $L_1\natural L_\Sigma$ we deduce that there are handle bodies V_0 and V_1 bounding standard spheres and a diffeomorphism

$$g : L_0\natural V_0 \to L_1\natural L_\Sigma\natural V_1$$

which extends f'. So by Wall's classification of handlebodies there is an isometry

$$\Theta(g) : \kappa(L_0) \oplus \kappa(V_0) \to \kappa(L_1) \oplus \kappa(L_\Sigma) \oplus \kappa(V_1).$$

But the boundaries of V_0, V_1 and L_Σ are all homotopy spheres, so $\kappa(V_0)$, $\kappa(V_1)$ and $\kappa(L_\Sigma)$ are all nonsingular quadratic functions in $\mathcal{F}^*$ and thus $\Theta(g)$ defines a stable $\mathcal{F}^*$-equivalence from $\kappa(L_0)$ to $\kappa(L_1)$. That is, $[\kappa(P_0)] = [\kappa(L_0)]_{\mathcal{F}^*} = [\kappa(L_1)]_{\mathcal{F}^*} = [\kappa(P_1)]$. If we take $P_0 = P_1 = P$, $\Sigma = S^{8j-1}$ and $f = Id_P$ then this argument shows that $[\kappa(P)]$ is independent of the handlebody chosen and hence well defined.

Finally we show that $[\kappa(P)]$ is a complete invariant. Suppose that $[\kappa(P_0)] = [\kappa(P_1)]$. Then there are nonsingular quadratic functions μ_0 and μ_1 belonging to $\mathcal{F}^*$ and an isometry

$$\Theta : \kappa(L_0) \oplus \mu_0 \to \kappa(L_1) \oplus \mu_1.$$

Again by Wall's classification of handlebodies there are handlebodies $L(\mu_0)$ and $L(\mu_1)$ which cobound homotopy spheres Σ_0 and Σ_1 and there is a diffeomorphism implementing Θ

$$h(\Theta) : L_0\natural L(\mu_0) \to L_1\natural L(\mu_1).$$

When restricted to the boundary of $L_0 \natural L(\mu_0)$, $h(\Theta)$ yields a diffeomorphism

$$h' : P_0 \# \Sigma_{P_0} \# \Sigma_0 \to P_1 \# \Sigma_{P_1} \# \Sigma_1$$

and therefore P_0 and P_1 are almost diffeomorphic. $\square$

As Wilkens worked with triples (H, λ, α) as opposed to quadratic functions $\kappa(H, \lambda, \alpha)$ he would not have stated Corollary 2.26 as it is stated here. However, the equivalent statement for triples (H, λ, α) was known to Wilkens and was indeed central to his thesis. In the language of this thesis, the algebraic task which Wilkens undertook was to find further algebraic invariants of stable quadratic functions in $\mathcal{F}^c$ which would allow him to determine when two quadratic functions in $\mathcal{F}^c$ were stably equivalent. He was partially successful. Given a triple (H, λ, α) Wilkens extracted the triple (G, b, β). Where $G = \mathrm{Cok}(\hat{\lambda})$, b is the linking form induced on G and β is the image of $\alpha \in H^*$ under the canonical map $H^* \to H^*/\hat{\lambda}(H) = \mathrm{Cok}(\hat{\lambda})$. The topological interpretation of this triple is by now largely familiar. We suppose that L is a 8 or 16 dimensional handlebody with quadratic function $\kappa(H, \lambda, \alpha)$ and with boundary P. Then $G = H^{4j}(P)$, $b = b(P)$ and if $i : P \hookrightarrow L$ denotes the inclusion then $\beta = i^*(\alpha)$ is the tangential invariant of P which is the primary obstruction to the stable triviality of the tangent bundle of P.

Wilkens' first achievement was to show that the manifolds P split via connected sum according to the splitting of their linking forms. To be precise, Wilkens defined P to be *decomposable* if P can be expressed as a connected sum of manifolds neither of which is a homotopy sphere and he defined P to be *indecomposable* if is it not decomposable.

PROPOSITION 2.27 (Wilkens Thesis). *A manifold P is indecomposable if and only if either $H^{4j}(P) \cong \mathbb{Z}$ or $H^{4j}(P)$ is finite and the linking form of P is indecomposable. To every decomposition of $H^{4j}(P)$ as a sum of infinite cyclic groups and finite groups* *over which the linking form of P is indecomposable there corresponds a decomposition of P by indecomposable manifolds.*

Wilkens' second achievement was to largely classify the indecomposable manifolds.

THEOREM 2.28 (Wilkens' Thesis). *Let P be indecomposable and let P have invariants (G, b, β). If G contains no 2-torsion then the triple (G, b, β) is a complete almost diffeomorphism invariant of P. If G contains 2-torsion then there are at most two almost diffeomorphism classes of manifolds with the same invariants. A triple (G, b, β) with b indecomposable is ambiguous (i.e. is realized by a pair of not almost diffeomorphic manifolds) if and only if $|G| \leq 4$ and $\beta = 0$ or if $|G| > 4$ and β is not divisible by 4.*

9. Quadratic linking functions

In this section we define the quadratic linking functions which are the invariants required to settle the ambiguity in Wilkens classification. We also prove elementary lemmas concerning $\mathcal{Q}(b)$, the set of quadratic linking functions which refine a given linking form b, and then describe how a nondegenerate quadratic function induces a quadratic linking function on its quotient group.

DEFINITION 2.29. *A quadratic linking function $q : TG \to \mathbb{Q}/\mathbb{Z}$ on a finite abelian group TG is a function such that the function $b(q) : TG \times TG \to \mathbb{Q}/\mathbb{Z}$ defined by*

$$b(q)(x, y) := q(x + y) - q(x) - q(y) \in \mathbb{Q}/\mathbb{Z}$$

is a linking form on TG. We say that q is a quadratic refinement of $b(q)$. If q satisfies $q(x) = q(-x)$ for all $x \in TG$ then we say that q is homogeneous and we call homogeneous quadratic linking functions quadratic linking forms.

LEMMA 2.30. *Let (TG, b) be a bilinear form. Then*

1. *there is a quadratic linking form $q : TG \to \mathbb{Q}/\mathbb{Z}$ which refines b,*
2. *if the order of G is odd then q is unique.*

PROOF. Choose a basis (minimal set of generators each of order p^k, p a prime and k a positive integer) $\{[x_1] \dots, [x_n]\}$ for TG. If the order of $[x_i]$ is a power of 2 then let $q([x_i])$ be either of the members of $\mathbb{Q}/\mathbb{Z}$ such that $2q([x_i]) = b([x_i], [x_i])$. If the order of $[x_i]$ is odd choose the unique fraction with an odd denominator from amongst the two choices. The values of $q([x_i])$ and $b([x_i], [x_j])$ determine q by Definition 2.32. The constraint placed on the choice of q for odd order elements arises because $q(nx) = n^2x$ (as we show below). If $[x_i]$ is of odd order then $p^k[x_i] = 0$ for p an odd prime and some positive integer k. Thus $0 = p^{2k}q([x_i])$ and hence the denominator of $q([x_i])$ must be odd. Now we show $q(nx) = n^2q(x)$ since firstly

$$\begin{aligned} b([x],[x]) &= -b([x],-[x]) \\ &= -(q(0) - q([x]) - q(-[x])) \\ &= 2q([x]). \end{aligned}$$

From this it follows that $q((n+1)[x]) - q(n[x]) - q([x]) = b(n[x],[x]) = 2nq([x])$ and so by induction $q(n[x]) = n^2q([x])$. □

LEMMA 2.31. *Let $b : TG \times TG \to \mathbb{Q}/\mathbb{Z}$ be a linking form, let $\mathcal{Q}(b)$ be the set of all quadratic linking functions refining b and let $q \in \mathcal{Q}(b)$. For any $a \in TG$ define*

$$q_a(x) := q(x) + b(x,a) = q(x+a) - q(a).$$

Then,

1. *q_a is a quadratic linking function refining b,*
2. *$q_a(x) - q_b(x) = b(x, a - b)$,*
3. *TG acts freely and transitively on $\mathcal{Q}(b)$ by the action $a \cdot q = q_a$,*
4. *if q is homogeneous then q_a is also homogeneous if and only if $2a = 0$,*

5. *if $q = q^o_a$ where q^o is a quadratic linking form, then $\beta(q) := 2a$ is independent of the choice of q_o and a and thus an invariant of q.*

PROOF. *(1.)* Let x, y be elements of TG.

$$\begin{aligned} q_a(x+y) - q_a(x) - q_a(y) \quad &= q(x+y) + b(x+y,a) \\ &\quad -q(x) - b(x,a) - q(y) - b(y,a) \\ &= q(x) + q(y) + b(x,y) - q(x) - q(y) \\ &= b(x,y) \end{aligned}$$

Thus q_a is a refinement of b.

(2.) The second statement is clear.

(3.) It is evident for $a, b \in G$ that $q_{a+b} = (q_a)_b$ and hence we have an action. Suppose now that q' is another quadratic refinement of b. We must show that $q' = q_a$ for some $a \in TG$ hence that the action is transitive. Consider then the function on TG defined by $r(x) = q'(x) - q(x)$. Clearly $r(0) = 0$ and

$$\begin{aligned} r(x+y) \quad &= q'(x+y) - q(x+y) \\ &= q'(x) + q'(y) + b(x,y) - q(x) - q(y) - b(x,y) \\ &= (q'(x) - q(x)) + (q'(y) - q(y)) \\ &= r(x) + r(y). \end{aligned}$$

Thus r is linear and since b is nonsingular there exists an element $a \in TG$ such that $r(x) = b(x,a)$. So we see that

$$\begin{aligned} q_a(x) \quad &= q(x) + b(x,a) \\ &= q(x) + r(x) \\ &= q'(x). \end{aligned}$$

Finally note that *2* and the nonsingularity of b entail that $q_a = q_b$ if and only if $a = b$ and hence the action is free.

(4.) Assume that q is homogeneous. Then

$$\begin{aligned} q_a(x) - q_a(-x) &= q(x) + b(x,a) - (q(-x) + b(-x,a)) \\ &= b(2x,a). \end{aligned}$$

Now $b(2a,x) = 0$ for all $x \in G$ if and only if $2a = 0$ and hence q_a is homogeneous if and only if $2a = 0$.

(5.) By Lemma 2.30, there is a quadratic form q^o which refines b and so by *3*, there is an a such that $q = q^o_a$. Now let $q^{o'}$ be another quadratic form refining b and let a' be such that $q = q^{o'}_{a'}$. By *3* and *4*, $q^{o'} = q^o_\epsilon$ where $2\epsilon = 0$ and by *3*, $a = a' + \epsilon$. Thus $2a' = 2a$ as required.

□

We now describe how a nondegenerate quadratic function $\kappa(H, \lambda, \alpha)$ induces a quadratic linking function on the torsion subgroup of $G = \mathrm{Cok}(\hat{\lambda})$. Recall the fundamental sequence of κ

$$0 \longrightarrow H \xrightarrow{\hat{\lambda}} H^* \xrightarrow{\pi} TG \longrightarrow 0$$

and that for $x \in H^*$ we denote $\pi(x)$ by $[x]$.

Definition 2.32. *Let $\kappa = \kappa(H, \lambda, \alpha)$ be a nondegenerate quadratic function. If $\kappa \in \mathcal{F}^{ev}$ then λ is even and we define*

$$\begin{aligned} q^{ev}(\kappa) : \ TG &\rightarrow \mathbb{Q}/\mathbb{Z} \\ [x] &\mapsto \lambda^{-1}(x,x)/2 \bmod \mathbb{Z}. \end{aligned}$$

If $\kappa \in \mathcal{F}^c$ then we define

$$\begin{aligned} q^{c}(\kappa) : \ TG &\rightarrow \mathbb{Q}/\mathbb{Z} \\ [x] &\mapsto (\lambda^{-1}(x,x) + \lambda^{-1}(x,\alpha))/2 \bmod \mathbb{Z}. \end{aligned}$$

Remark 2.33. I am grateful to Stephan Stolz who first suggested the definition of q^c to me and also use of Gauss sums (see Chapter 5) in the classification of quadratic functions. The definition of q^{ev} has been know at least since [**Wa3**] and its properties have been thoroughly investigated by Wall [**Wa3**], Durfee [**Du1**] and [**Du2**] and Nikulin [**Ni**]. Much of this thesis is devoted to proving that the results of these three authors concerning even quadratic forms and quadratic linking forms generalize to characteristic quadratic functions and quadratic linking functions.

LEMMA 2.34. *Let $\kappa = \kappa(H, \lambda, \alpha)$ be a nondegenerate quadratic function in $\mathcal{F}^*$.*

1. *If $\kappa \in \mathcal{F}^{ev}$ then $q^{ev}(\kappa)$ is a homogeneous refinement of $b(\lambda)$.*
2. *If $\kappa \in \mathcal{F}^{c}$ then $q^{c}(\kappa)$ is a quadratic refinement of $b(\lambda)$.*

PROOF. The calculations are entirely routine. Since definition the of $q^{ev}(\kappa)$ is well known, we verify only that $q^c(\kappa)$ is well defined. Thus let $\kappa \in \mathcal{F}^*$, let $x \in H^*$ and let $x + \hat{\lambda}(v)$ be another representative of $[x]$. We calculate that

$$\begin{aligned}
&\lambda^{-1}(x+\hat{\lambda}(v), x+\hat{\lambda}(v)+\alpha)/2 \bmod \mathbb{Z} \\
&\quad = \lambda^{-1}(x, x+\alpha)/2 + [2\lambda^{-1}(x, \hat{\lambda}(v)) + \lambda^{-1}(\hat{\lambda}(v), \hat{\lambda}(v)) + \lambda^{-1}(\hat{\lambda}(v), \alpha)]/2 \bmod \mathbb{Z} \\
&\quad = \lambda^{-1}(x, x+\alpha)/2 + [2x(v) + \lambda(v,v) + \alpha(v)]/2 \bmod \mathbb{Z} \\
&\quad = \lambda^{-1}(x, x+\alpha)/2 \bmod \mathbb{Z},
\end{aligned}$$

where the last equality holds because α is characteristic for λ.

Next, given $x, y \in H^*$ and $[x], [y] \in TG$ we calculate that

$$\begin{aligned}
q^{ev}(\kappa)([x]+[y]) &= \lambda^{-1}(x+y, x+y)/2 && \bmod \mathbb{Z} \\
&= \lambda^{-1}(x,x)/2 + \lambda^{-1}(y,y)/2 + \lambda^{-1}(x,y) && \bmod \mathbb{Z} \\
&= q^{ev}(\kappa)([x]) + q^{ev}([y]) + b(\lambda)([x],[y]).
\end{aligned}$$

Hence q^{ev} refines $b(\lambda)$. Moreover it is clear that $q^{ev}(-[x]) = q^{ev}([x])$ and hence we have proven *1.* In the characteristic case we calculate that

$$\begin{aligned}
q^{c}(\kappa)([x]+[y]) &= \lambda^{-1}(x+y, x+y)/2 + \lambda^{-1}(x+y, \alpha)/2 && \bmod \mathbb{Z} \\
&= (\lambda^{-1}(x,x) + \lambda^{-1}(x,\alpha))/2 \\
&\quad +(\lambda^{-1}(y,y) + \lambda^{-1}(y,\alpha))/2 + \lambda^{-1}(x.y) && \bmod \mathbb{Z} \\
&= q^{c}(\kappa)([x]) + q^{c}([y]) + b(\lambda)([x],[y])
\end{aligned}$$

and this proves *2.* □

Given two quadratic linking functions $q_0 : TG_0 \to \mathbb{Q}/\mathbb{Z}$ and $q_1 : TG_1 \to \mathbb{Q}/Z$ an isomorphism $\theta : TG_0 \to TG_1$ is called an isometry if $q_0 = q_1 \circ \theta$. In this situation we shall write that $q_0 \cong q_1$. We consider next the question of which quadratic linking functions are isometric to $q^*(\kappa)$ for some κ.

THEOREM 2.35 (Wall [**Wa3**]). *Let $q : TG \to \mathbb{Q}/\mathbb{Z}$ be a quadratic linking form on a finite abelian group. Then there is a nondegenerate even quadratic form $\kappa(\lambda, 0)$ such that $q \cong \delta^{ev}(\kappa)$.*

COROLLARY 2.36.
1. *Let $b : TG \times TG \to \mathbb{Q}/\mathbb{Z}$ be a linking form on a finite abelian group, then there is a bilinear form λ such that $b \cong b(\lambda)$.*
2. *Let $q : TG \to \mathbb{Q}/\mathbb{Z}$ be a quadratic linking function of a finite abelian group, then there is a characteristic quadratic function κ such that $q \cong \delta^c(\kappa)$.*

PROOF. *(1.)* By Lemma 2.30 there is a quadratic linking form q which refines b. Now apply Theorem 2.35 to deduce that there is a nondegenerate quadratic form $\kappa(\lambda, 0)$ such that $q \cong \delta^{ev}(\kappa)$. From Lemma 2.34 we see that $b \cong b(\lambda)$.
(2.) We begin by stating and proving a simple lemma which will be useful elsewhere.

LEMMA 2.37. *Let $\kappa = \kappa(H, \lambda, \alpha)$ be a nondegenerate characteristic quadratic function and let $\epsilon \in H^*$ be divisible by 2. Then*

$$q^c(\kappa(H, \lambda, \alpha + \epsilon)) = q^c(\kappa)_{[\epsilon/2]}.$$

PROOF. Let $[x]$ be any element in $TG = \mathrm{Cok}(\hat{\lambda})$.

$$\begin{aligned} q^c(\kappa(\lambda, \alpha + \epsilon))([x]) &= \hat{\lambda}^{-1}(x, x + \alpha + \epsilon)/2 && \mod \mathbb{Z} \\ &= \hat{\lambda}^{-1}(x, x + \alpha)/2 + \hat{\lambda}^{-1}(x, \epsilon/2) && \mod \mathbb{Z} \\ &= q^c(\kappa)([x]) + b(\lambda)([x], [\epsilon/2]) \\ &= q^c(\kappa)_{[\epsilon/2]}([x]). \end{aligned}$$

$\square$

Now let b be the linking form induced by q and let q' be a homogeneous refinement of b. By Theorem 2.35 there is an even quadratic form $\kappa(H, \lambda, 0)$ and an isomorphism $\theta : \mathrm{Cok}(\hat{\lambda}) \to TG$ such that $q' \circ \theta = \delta^{ev}(\kappa)$. By Lemma 2.31 there is an element $a \in TG$ such that $q = q'_a$. So choose $\epsilon \in H^*$ such that $\theta([\epsilon/2]) = a$. As λ is even, $\kappa(\lambda, 0)$ is characteristic and we may apply Lemma 2.37 to $\kappa(\lambda, 0)$ and ϵ to conclude that $\delta^c(\kappa(\lambda, \epsilon)) = q'_a \circ \theta = q \circ \theta$. $\square$

Observe that any quadratic linking function q over a finite abelian group TG, has for invariants the bilinear for it refines, $b(q)$, and the "linear element", $\beta(q) \in TG$. We shall call the pair $(b(q), \beta(q))$ the "Wilkens invariants" of q. Chapter 5 deals with question of when quadratic functions q_0 and q_1 with isomorphic Wilkens invariants are isometric. (The answer is that there is one extra invariant, the Kervaire-Arf Gauss sum invariant K, such that $q_0 \cong q_1$ if and only if $(b(q_0), \beta(q_0) \cong (b(q_1), \beta(q_1))$ and $K(q_0) = K(q_1)$). We conclude this section with the following obvious but important remark concerning $\beta(q)$.

Remark 2.38. Let q be a quadratic function on a finite abelian group TG. The element $\beta(q) \in TG$ is even. If $q = \delta^c(\kappa)$, where κ is the nondegenerate quadratic function $\kappa = \kappa(\lambda, \alpha)$, then $\beta(q) = [\alpha] \in \mathrm{Cok}(\hat{\lambda}) = TG$.

10. Quadratic linking families

In this section we define the notion of a quadratic linking family on a finitely generated abelian group G and describe how a quadratic function $\kappa(\lambda, \alpha)$ induces a quadratic linking family on its quotient group $G = \mathrm{Cok}(\hat{\lambda})$.

We begin with some terminology and preliminary definitions before our main definition. An element β in the abelian group G shall be called *n-divisible* for a natural number n if $\beta = n\gamma$ for some $\gamma \in G$. If in addition γ is indivisible β shall be called *precisely n-divisible.* We shall call 2-divisible elements *even.* Recall the homomorphism $\tau_G : G \to G^{**}$ from G to its double dual. Of course the torsion subgroup of G, TG, is the kernel of τ_G. The set of sections of τ_G is the set of homomorphisms

$$\mathrm{Sec}(\tau_G) := \{\Phi : G^{**} \to G | \tau_G \circ \Phi = Id_{G^{**}}\}.$$

There is a free and transitive action of $\mathrm{Hom}(G^{**}, TG)$ on $\mathrm{Sec}(\tau_G)$ defined for $\phi \in \mathrm{Hom}(G^{**}, TG)$, $\Phi \in Sec(\tau_G)$ and $z \in G^{**}$ as follows:

$$(\phi \cdot \Phi)(z) = \Phi(z) - \phi(z).$$

Now for any linking form b defined on TG recall from Lemma 2.31 that TG acts freely and transitively on $\mathcal{Q}(b)$ which is the set of all quadratic linking functions refining b. Given an even element $\beta \in G$ we let $\mathrm{Hom}(G^{**}, TG)$ act on $\mathcal{Q}(b)$ by $\phi \cdot q = q_{\phi(\tau_G(\beta)/2)}$.

DEFINITION 2.39. *Let G be a finitely generated abelian group with torsion subgroup TG. A quadratic linking family on G is a triple (G, q^*, β) where $q^* : \mathrm{Sec}(\tau_G) \to \mathcal{Q}(b)$ is a function for some linking form b defined on TG and β is an element of G.*

1. *A characteristic quadratic linking family on G is a quadratic linking family $Q^c = (G, q^c, \beta)$ with β even and satisfying the following properties:*
   (A) *q^c is equivariant with respect to the actions of* $\mathrm{Hom}(G^{**}, TG)$*, that is, $q^c(\phi \cdot \Phi) = q^c(\Phi)_{\phi(\tau_G(\beta)/2)}$ for every $\phi \in \mathrm{Hom}(G^{**}, TG)$ and every $\Phi \in$* $\mathrm{Sec}(\tau_G)$*,*
   (B) *$\beta = \Phi(\tau_G(\beta)) + \beta(q^c(\Phi))$ for every $\Phi \in \mathrm{Sec}(\tau_G)$, (recall that $\beta(q^c(\Phi))$ is the even element in TG defined in Lemma 2.31,* 5*).*
2. *An even quadratic linking family on G is a quadratic linking family $Q^{ev} = (G, q^{ev}, \beta)$ where q^{ev} is a* constant *function whose value is a quadratic linking form refining b and β may be any element of G.*

In order to motivate the somewhat complicated definition of a characteristic quadratic linking family we proceed now to show how a quadratic function induces a quadratic linking family on its quotient group. So suppose that $\kappa(H,\lambda,\alpha)$ is a possibly degenerate quadratic function with $\pi : H^* \to G = \mathrm{Cok}(\hat{\lambda})$ the projection onto the quotient of λ and with $F = \mathrm{Ker}(\hat{\lambda})$ the radical of λ. In Section 6 we defined the canonical projection $\rho : H \to H/F$, the sections of ρ, $\mathrm{Sec}(\rho) = \{\Psi : H/F \to H | \rho \circ \Psi = Id_{H/F}\}$ and the projections $p_F(\Psi) : H \to F$. Every section $\Psi \in \mathrm{Sec}(\rho)$ defines a section $\Phi(\Psi) \in \mathrm{Sec}(\tau_G)$ which is the composition

$$\Phi(\Psi) := \pi \circ p_F(\Psi)^* \circ \bar{\lambda}_F^* : G^{**} \xrightarrow{\bar{\lambda}_F^*} F^* \xrightarrow{p_F(\Psi)^*} F^*(\Psi) \xrightarrow{\pi|_{F^*(\Psi)}} G$$

where $\bar{\lambda}_F$ was defined in Section 7. Now by Lemma 2.21 every $x \in G^*$ is of the form of $\bar{\lambda}_F(v)$ for some $v \in F$ and for $z \in G^{**}$ we calculate that

$$\begin{aligned}
\tau_G(\Phi(\Psi)(z))[x] &= x[\Phi(\Psi)(z)] \\
&= \bar{\lambda}_F(v)[(\pi|_{F^*(\Psi)} \circ p_F(\Psi)^* \circ \bar{\lambda}_F^*)(z)] \\
&= (p_F(\Psi)^* \circ \bar{\lambda}_F^*(z))[v] \\
&= \bar{\lambda}_F^*(z)[p_F(\Psi)(v)] \\
&= \bar{\lambda}_F^*(z)[v] \\
&= z[\bar{\lambda}_F(v)] \\
&= z(x).
\end{aligned}$$

Hence $\tau_G \circ \Phi(\Psi) = Id_{G^{**}}$ and $\Phi(\Psi)$ is indeed a section of τ_G. There is a free and transitive action of $\mathrm{Hom}(H/F, F)$ on $\mathrm{Sec}(\rho)$ and for $\psi \in \mathrm{Sec}(\rho)$ we define $\phi(\psi) \in \mathrm{Sec}(\tau_G)$ to be the composition

$$\phi(\psi) : G^{**} \xrightarrow{\bar{\lambda}_F^*} F^* \xrightarrow{\psi^*} (H/F)^* \xrightarrow{\pi \circ \rho^*} TG.$$

LEMMA 2.40. *Consider the mappings*

$$\begin{array}{rccc}
\delta_H : & \mathrm{Hom}(H/F, F) & \to & \mathrm{Hom}(G^{**}, TG) \\
& \psi & \mapsto & \phi(\psi)
\end{array}
\qquad
\begin{array}{rccc}
\delta_S : & \mathrm{Sec}(\rho) & \to & \mathrm{Sec}(\tau_G) \\
& \Psi & \mapsto & \Phi(\Psi).
\end{array}$$

1. *δ_H is a surjective homomorphism,*
2. *for every $\Psi \in \mathrm{Sec}(\rho)$ and every $\psi \in \mathrm{Hom}(H/F, F)$, $\Phi(\psi \cdot \Psi) = \phi(\psi) \cdot (\Phi(\Psi))$,*
3. *δ_S is onto and $\delta_S^{-1}(\Phi(\Psi)) = \{\psi \cdot \Psi | \psi \in \mathrm{Ker}(\delta_H)\}$.*

PROOF. *(1.)* We show first that δ_H is onto. Let $\{z_1, \ldots, z_l\}$ be a basis for G^{**}. Every section $\phi \in \mathrm{Sec}(\tau_G)$ is determined by $\phi(z_1), \ldots, \phi(z_l)$. We choose $y_1, \ldots, y_l \in (H/F)^*$ so that $\pi \circ \rho^*(y_i) = \phi(z_i)$ for each i which we may do since $\pi \circ \rho^*$ is onto TG. Now as $\bar{\lambda}_F^* : G^{**} \to F^*$ is an isomorphism $\{\bar{\lambda}_F^*(z_1), \ldots, \bar{\lambda}_F^*(z_l)\}$ is a basis for F^* and we choose $\psi : H/F \to F$ so that $\psi^*(\bar{\lambda}_F^*(z_i)) = y_i$ for each i. It follows that

$$\phi(\psi)(z_i) = (\pi \circ \rho^*)(\psi^*(\bar{\lambda}_F^*(z_i)) = \phi(z_i)$$

for each i. Thus $\phi(\psi) = \phi$ and so δ is onto.

The group operation in both $\mathrm{Hom}(H/F, F)$ and $\mathrm{Hom}(G^{**}, TG)$ is pointwise addition. For $\psi_0, \psi_1 \in \mathrm{Hom}(H/F, F)$,

$$\begin{aligned} \phi(\psi_0 + \psi_1) &= \pi \circ \rho^* \circ (\psi_0^* + \psi_1^*) \circ \bar{\lambda}_F^* \\ &= \pi \circ \rho^* \circ \psi_0^* \circ \bar{\lambda}_F^* + \pi \circ \rho^* \circ \psi_1^* \circ \bar{\lambda}_F^* \\ &= \phi(\psi_0) + \phi(\psi_1). \end{aligned}$$

(2.) Observe first that for any $y \in F^*$ and any $v \in H$

$$\begin{aligned} p_F(\psi \cdot \Psi)^*(y)(v) &= y(v - (\Psi + \psi)(\rho(v))) \\ &= y(v - \Psi(\rho(v))) - y(\psi(\rho(v))) \\ &= p_F(\Psi)^*(y)(v) - \rho^*(\psi^*(y))(v). \end{aligned}$$

We therefore calculate that for $z \in G^{**}$

$$\begin{aligned} \Phi(\psi.\Psi)(z) &= (\pi|_{F(\psi.\Psi)})[p_F(\psi.\Psi)^*(\bar{\lambda}_F^*(z))] \\ &= \pi[p_F(\Psi)^*(\bar{\lambda}_F^*(z)) - \rho^*(\psi^*(\bar{\lambda}_F^*(z)))] \\ &= \Phi(\Psi)(z) - \phi(\psi)(z) \\ &= (\phi(\psi) \cdot \Phi(\Psi))(z). \end{aligned}$$

(3.) This follows immediately from the *1*, *2* and the fact that $\mathrm{Hom}(H/F, F)$ and $\mathrm{Hom}(G^{**}, TG)$ act freely and transitively on $\mathrm{Sec}(\rho)$ and $\mathrm{Sec}(\tau_G)$ respectively. $\square$

Now recall from proposition Proposition 2.18 that every section $\Psi \in \mathrm{Sec}(\rho)$ defines a splitting $\kappa = \alpha|_F \oplus \kappa(\Psi)$ where $\kappa(\Psi) = \kappa|_{\Psi(H)}$ is nondegenerate. Recall also Definition 2.32 which defined, for a nondegenerate quadratic function $\kappa'(\lambda', \alpha') \in \mathcal{F}^*$, a quadratic linking function $q^*(\kappa')$ on $\mathrm{Cok}(\hat{\lambda}')$.

LEMMA 2.41. *Let $\kappa = \kappa(H, \lambda, \alpha)$ be a quadratic function and let $\beta = [\alpha] \in \mathrm{Cok}(\hat{\lambda})$. Then, for every $\psi \in \mathrm{Hom}(H/F, F)$ and every $\Psi \in \mathrm{Sec}(\rho)$,*

1. *if $\kappa \in \mathcal{F}^{ev}$, $q^{ev}(\kappa(\psi \cdot \Psi)) = q^{ev}(\kappa(\Psi))$,*
2. *if $\kappa \in \mathcal{F}^c$, $q^c(\kappa(\psi \cdot \Psi)) = q^c(\kappa(\Psi))_a$, where $a = \phi(\psi)(\tau_G(\beta))/2 \in TG$.*

PROOF. Each $q^*(\kappa(\Psi))$ is defined on $TG = \mathrm{Cok}(\hat{\lambda}(\Psi)) = H^*(\Psi)/\hat{\lambda}(\Psi)(H(\Psi))$ and by Proposition 2.18 both $H^*(\Psi)$ and $\mathrm{Cok}(\hat{\lambda}(\Psi)) = TG$ are independent of Ψ. By Lemma 2.20 the bilinear pairing $\lambda(\Psi)^{-1} : H^*(\Psi) \times H^*(\Psi) \to \mathbb{Q}$ is also independent of Ψ. Now for $x \in H^*(\Psi)$ and $[x] = \pi(x) \in TG$,

$$q^{ev}(\kappa(\Psi))([x]) = \lambda(\Psi)^{-1}(x, x)/2 \bmod \mathbb{Z}$$

and so $q^{ev}(\kappa(\Psi))$ is independent of Ψ and *1* is proven.

(2.) By the preceding remarks the only effect that altering Ψ may have on $q^c(\kappa(\Psi))$ comes via altering the linear part of $\kappa(\Psi)$ which is $\alpha(\Psi)$. Now by *5* of Proposition 2.18 $\alpha(\psi \cdot \Psi) = \alpha(\Psi) + \rho^*(\psi^*(\alpha|_F))$. Since κ is characteristic $\alpha|_F$ must be even for if $v \in F$, $0 = \lambda(v, v) = \alpha(v) \pmod 2$. We may therefore apply Lemma 2.37 to conclude that

$$q^c(\kappa(\psi \cdot \Psi)) = q^c(\kappa(\lambda(\Psi), \alpha(\Psi) + \rho^*(\psi^*(\alpha|_F))) = q^c(\kappa(\Psi))_a$$

where $a = \pi(\rho^*(\psi^*(\alpha|_F)))/2$ and π, as usual, is the projection $H^* \to \mathrm{Cok}(\hat{\lambda})$. Now since $\pi(\alpha) = \beta$, $\alpha|_F = \bar{\lambda}_G(\beta)$ and by Lemma 2.21, $\bar{\lambda}_G = \bar{\lambda}_F^* \circ \tau_G$. Therefore

$$\begin{aligned} a &= \pi(\rho^*(\psi^*(\alpha|_F)))/2 \\ &= (\pi \circ \rho^* \circ \psi^* \circ \bar{\lambda}_F^*)(\tau_G(\beta))/2 \\ &= \phi(\psi)(\tau_G(\beta))/2. \end{aligned}$$

□

PROPOSITION 2.42. *Let $\kappa = \kappa(H, \lambda, \alpha)$ be a quadratic function in $\mathcal{F}^*$ with $G = \mathrm{Cok}(\hat{\lambda})$ and $F = \mathrm{Ker}(\hat{\lambda})$. Let $\tau_G : G \to G^{**}$, $\mathrm{Sec}(\tau_G)$, $\rho : H \to H/F$ and $\mathrm{Sec}(\rho)$ be as above and let $\beta := [\alpha] \in G$. If $\kappa \in \mathcal{F}^*$ and $q^*(\kappa)$ is defined by*

$$\begin{array}{rccc} q^*(\kappa): & \mathrm{Sec}(\tau_G) & \to & \mathcal{Q}(b(\lambda)) \\ & \Phi & \mapsto & q^c(\kappa(\Psi)) \end{array}$$

for any $\Psi \in \mathrm{Sec}(\rho)$ such that $\Phi(\Psi) = \Phi$. Then the triple

$$\delta^*(\kappa) := (G, q^*(\kappa), \beta)$$

is a characteristic (resp. even) quadratic linking family on G for $* = c$ (resp. $* = ev$).

PROOF. Suppose that $\kappa \in \mathcal{F}^c$. We show first the $q^c(\kappa)$ is well defined. By Lemma 2.34 each $q^c(\kappa(\Psi))$ refines $b(\lambda(\Psi))$ which is independent of Ψ and equal, by definition, to $b(\lambda)$. Secondly, by Lemma 2.40, if Ψ and Ψ' are such that $\Phi(\Psi) = \Phi(\Psi')$ then $\Psi' = \psi \cdot \Psi'$ for some ψ such that $\phi(\psi) = 0$. If follows now from *2* of Lemma 2.41 that $q^c(\kappa(\Psi')) = q^c(\kappa(\Psi))$ and so $q^c(\kappa)$ is well defined. Now let $\Phi \in \mathrm{Sec}(\tau_G)$ and $\phi \in \mathrm{Hom}(G^{**}, TG)$ and let $\Psi \in \mathrm{Sec}(\rho)$ and $\psi \in \mathrm{Hom}(H/F, F)$ be such that $\Phi(\Psi) = \Phi$ and $\phi(\psi) = \phi$.

$$\begin{aligned} q^c(\phi \cdot \Phi) &= q^c(\kappa(\psi.\Psi)) && \text{by Lemma 2.40 } \mathit{2} \\ &= q^c(\Psi)_{\phi(\tau_G(\beta))/2} && \text{by Lemma 2.41 } \mathit{1}. \end{aligned}$$

Hence $q^c(\kappa)$ is equivariant with respect to the action of $\mathrm{Hom}(G^{**}, TG)$. With Φ and Ψ as before we must show that $\beta = \Phi(\tau_G(\beta)) + \beta(q^c(\Phi))$. Recall from the proof of Lemma

2.41 that $\alpha|_F = \bar{\lambda}_F^*(\tau_G(\beta))$ and also from Section 6 that $\alpha = p_F(\Psi)^*(\alpha|_F) + \alpha(\Psi)$. We calculate that

$$\begin{aligned}
\beta &= \pi(\alpha) \\
&= \pi(p_F(\Psi)^*(\alpha|_F))) + \pi(\alpha(\Psi)) \\
&= (\pi \circ p_F(\Psi)^* \circ \hat{\lambda}_F^*)(\tau_G(\beta)) + \beta(q^c(\kappa(\Psi))) \\
&= \Phi(\tau_G(\beta)) + \beta(q^c(\Phi)).
\end{aligned}$$

Finally, this decomposition shows that β is even for by Remark 2.38, $\beta(q^c(\Psi))$ is even and, as we showed in the proof of Lemma 2.41, $\alpha|_F$ (and hence $\tau_G(\beta)$) is even.

If $\kappa \in \mathcal{F}^{ev}$ then by Lemma 2.41, $q^{ev}(\kappa(\Psi))$ is independent of Ψ and hence $q^{ev}(\kappa)$ is a constant function. By Lemma 2.34 $q^{ev}(\kappa)$ is a homogeneous refinement of $b(\lambda)$ and so the triple $(G, q^{ev}(\kappa), \beta)$ defines an even quadratic linking family on G. $\square$

REMARK 2.43. We note that we have been using the symbol q^* in two different but closely related ways. When $\kappa = \kappa(\lambda)$ is a degenerate quadratic function the symbol $q^*(\kappa)$ denotes a function which is defined over $\mathrm{Sec}(\tau_G)$ (or equivalently the splittings of $G = \mathrm{Cok}(\hat{\lambda})$) which values in the quadratic linking functions defined on TG. When $\kappa = \kappa(\lambda)$ is a nondegenerate characteristic quadratic function the symbol $q^*(\kappa)$ denotes a single quadratic linking function define on $TG = \mathrm{Cok}(\hat{\lambda})$. Now κ is nondegenerate if and only if $G = TG$ is torsion. In this case $G^{**} = \{e\}$ and $\mathrm{Sec}(\tau_G) = \{0\}$ is a singleton set containing the zero homomorphism from the trivial group to TG. Hence our two uses of q^* coincide provided we are prepared to identify a function on a singleton set with its range. In the characteristic case observe also that when $G = TG$ is a torsion group $\beta = \beta(q^c(0))$ is determined by $q^c(0)$ and so in the torsion case a characteristic quadratic linking family is completely determined by $q^c(0)$.

REMARK 2.44. Note that the equivariance property (B) of characteristic quadratic linking families ensures that the function q^c is determined by its value on any single section $\Phi \in \mathrm{Sec}(\tau_G)$ and the divisibility of $\tau_G(\beta)$. In particular if $\tau_G(\beta)$ is precisely $2r$-divisible then a simple argument concerning the action of $\mathrm{Hom}(G^{**}, TG)$ on $\mathcal{Q}(b)$ shows for any $\Phi \in \mathrm{Sec}(\tau_G)$ that $\mathrm{Im}(q^c) = \{q^c(\Phi)_{ra} | a \in TG\}$.

REMARK 2.45. When G is a free group then $TG = \{e\}$, $\mathrm{Sec}(\tau_G) = \{Id_G\}$ and $q^c(Id)$ is the trivial quadratic linking function.

We now define isometries between and sums of quadratic linking functions. Let $Q_0 = (G_0, q_0^*, \beta_0)$ and $Q_1 = (G_1, q_1^*, \beta_1)$ be a pair of quadratic linking families. An isomorphism $\theta : G_0 \to G_1$ defines a bijection

$$\begin{array}{rccc} \tau_\theta : & \mathrm{Sec}(\tau_{G_0}) & \to & \mathrm{Sec}(\tau_{G_1}) \\ & \Phi & \mapsto & \theta \circ \Phi \circ (\theta^{**})^{-1} \end{array}$$

which is illustrated in the following commutative diagram.

$$\begin{tikzcd} G_0 \arrow[r, "\theta"] & G_1 \\ G_0^{**} \arrow[u, "\Phi"] & G_1^{**} \arrow[l, "(\theta^{**})^{-1}"'] \arrow[u, "\tau_\theta(\Phi)"'] \end{tikzcd}$$

We define θ to be an isometry from Q_0 to Q_1 if $\theta(\beta_0) = \beta_1$ and if for all $\Phi \in \mathrm{Sec}(\tau_{G_0})$,

$$q_0^*(\Phi) = q_1^*(\tau_\theta(\Phi)) \circ \theta_{TG_0}.$$

REMARK 2.46. If If Q_0 and Q_1 are characteristic quadratic linking families then Q_0 and Q_1 are isometric if and only if there is an isomorphism $\theta : G_0 \to G_1$ such that $\theta(\beta_0) = \beta_1$ and $q_1^c(\Phi_1) \circ \theta_{TG_0} = q_0^c(\Phi_0)$ for some sections $\Phi_0 \in \mathrm{Sec}(\tau_{G_0})$ and $\Phi_1 \in \mathrm{Sec}(\tau_{G_1})$. However, there is no guarantee that θ itself is an isometry from Q_0 to Q_1. But there is an automorphism $\theta' : G_1 \cong G_1$ such that $\theta'|_{TG_1} = Id_{TG_1}$ and such that $\theta' \circ \theta$ is an isometry from Q_0 to Q_1.

Given homomorphisms $f_0 : G_0 \to F_0$ and $f_1 : G_1 \to G_0$ we let $f_0 \oplus f_1$ denote the obvious homomorphism from $G_0 \oplus G_1$ to $F_0 \oplus F_1$. It is clear that $\tau_{G_0 \oplus G_1} = \tau_{G_0} \oplus \tau_{G_1}$

and that there is an inclusion

$$\begin{array}{rcl} \mathrm{Sec}(\tau_{G_0}) \oplus \mathrm{Sec}(\tau_{G_1}) : & \to & \mathrm{Sec}(\tau_{G_0 \oplus G_1}) \\ (\Phi_0, \Phi_1) & \mapsto & \Phi_0 \oplus \Phi_1. \end{array}$$

With the quadratic linking families Q_0 and Q_1 as above we define $Q_0 \oplus Q_1$ to be the quadratic linking family $(G_0 \oplus G_1, q_0^* \oplus q_1^*, (\beta_0, \beta_1))$. Here

$$q_0^* \oplus q_1^* : \mathrm{Sec}(\tau_{G_0 \oplus G_1}) \to \mathcal{Q}(b_0 \oplus b_1)$$

is the unique function such that $(q_0^* \oplus q_1^*)(\Phi_0 \oplus \Phi_1) = q_0^*(\Phi_0) \oplus q_1^*(\Phi_1)$ for every $\Phi_i \in \mathrm{Sec}(\tau_{G_i})$, $i = 0, 1$ and which is equivariant with respect to the action of $\mathrm{Hom}(G_0^{**} \oplus G_1^{**}, TG_0 \oplus TG_1)$. Note that in the even case we take this action to be trivial so that $q_0^{ev} \oplus q_1^{ev}$ is again a constant function.

We now show that the isometries and directs sums of quadratic functions induce isometries and direct sums of the associated quadratic linking families. Let $\kappa_0 = (H_0, \lambda_0, \alpha_0)$ and $\kappa_1 = (H_1, \lambda_1, \alpha_1)$ be a pair of quadratic functions with canonical projections $\rho_i : H_0 \to H_0/F_0$, $i = 0, 1$. For every isometry $\Theta : H_0 \to H_1$, $\Theta(F_0) = F_1$ and hence Θ induces an isomorphism $\Theta_{/F} : H_0/F_0 \to H_1/F_1$. The isometry Θ defines a bijection

$$\begin{array}{rrcl} \rho_\Theta : & \mathrm{Sec}(\rho_0) & \to & \mathrm{Sec}(\rho_1) \\ & \Psi & \mapsto & \Theta \circ \Psi \circ (\Theta_{/F})^{-1} \end{array}$$

which is illustrated in the following commutative diagram.

$$\begin{tikzcd} H_0 \arrow[r, "\Theta"] & T_1 \\ H_0/F_0 \arrow[u, "\Psi"] & H_1/F_1 \arrow[l, "(\Theta_{/F})^{-1}"'] \arrow[u, "\rho_\Theta(\Psi)"'] \end{tikzcd}$$

LEMMA 2.47. *Let $\Theta : H_0 \to H_1$ be an isometry between quadratic functions $\kappa_0 = \kappa(H_0, \lambda_0, \alpha_0)$ and $\kappa_1 = \kappa(H_1, \lambda_1, \alpha_1)$. Then,*

1. *the following diagram commutes,*

$$\begin{tikzcd} 0 \arrow[r] & F_0 \arrow[r] \arrow[d, "\Theta|_{F_0}"'] & H_0 \arrow[r, "\hat{\lambda}_0"] \arrow[d, "\Theta"'] & H_0^* \arrow[r, "\pi_0"] \arrow[d, "(\Theta^*)^{-1}"] & G_0 \arrow[r] & 0 \\ 0 \arrow[r] & F_1 \arrow[r] & H_1 \arrow[r, "\hat{\lambda}_1"] & H_1^* \arrow[r, "\pi_1"] & G_1 \arrow[r] & 0 \end{tikzcd}$$

2. *for $x \in H_0^*$ and $[x] = \pi_0(x) \in G_0$, the map*

$$\begin{aligned} \delta^*(\Theta): \ G_0 &\to G_1 \\ [x] &\mapsto [(\Theta^*)^{-1}x] \end{aligned}$$

   is a well-defined isomorphism from G_0 to G_1,

3. *for every $\Psi \in \mathrm{Sec}(\rho_0)$, $\Phi(\rho_\Theta(\Psi)) = \tau_{\delta^*(\Theta)}(\Phi(\Psi)) \in \mathrm{Sec}(\tau_{G_1})$,*

4. *for every $\Psi \in \mathrm{Sec}(\rho_0)$, $\Theta|_{H(\Psi)} : \kappa_0(\Psi) \to \kappa_1(\rho_\Theta(\Psi))$ is an isometry,*

5. *if κ_0 and κ_1 are nondegenerate then $\hat{\lambda}_0^{-1} = (\Theta^{-1} \otimes Id_{\mathbb{Q}}) \circ \hat{\lambda}_1^{-1} \circ (\Theta^*)^{-1}$.*

PROOF. *(1.)* Let v_0, w_0 be elements of H_0.

$$\begin{aligned} \Theta^*(\hat{\lambda}_1(\Theta(v_0)))(w_0) &= \hat{\lambda}_1(\Theta(v_0))(\Theta(v_1)) \\ &= \lambda_1(\Theta(v_0), \Theta(v_1)) \\ &= \lambda_0(v_0, w_0) \\ &= \hat{\lambda}_0(v_0)(w_0) \end{aligned}$$

Thus $\hat{\lambda}_0 = \Theta^* \circ \hat{\lambda}_1 \circ \Theta$ and so $(\Theta^*)^{-1} \circ \hat{\lambda}_0 = \hat{\lambda}_1 \circ \Theta$ and the diagram commutes.

(2.) This follows from part *1* and an elementary diagram chase.

(3.) This is a particularly complicated but routine diagram chase. We include the commutative diagram and leave the reader to check that it commutes.

$$\begin{tikzcd} G_0^{**} \arrow[r, "\bar{\lambda}_{F_0}^*"] & F_0^* \arrow[r, "p_{F_0}(\Psi)^*"] & F_0^*(\Psi) \arrow[r, "\pi_0|_{F_0^*(\Psi)}"] \arrow[d, "(\Theta^*)^{-1}|_{F_0^*(\Psi)}"] & G_0 \arrow[d, "\delta^*(\Theta)"] \\ G_1^{**} \arrow[r, "\bar{\lambda}_{F_1}^*"] \arrow[u, "\delta^*(\Theta)^{**}"] & F_1^* \arrow[r, "p_{F_1}(\rho_\Theta(\Psi))^*"] \arrow[u, "(\Theta|_{F_0})^*"'] & F_1^*(\rho_\Theta(\Psi)) \arrow[r, "\pi_1|_{F_1^*(\rho_\Theta(\Psi))}"] & G_1 \end{tikzcd}$$

The homomorphism $\Phi(\rho_\Theta(\Psi))$ is the composition of the bottom three horizontal arrows whereas the homomorphism $\tau_{\delta^*(\Theta)}(\Phi(\Psi))$ moves up $\delta^*(\Theta)^{**}$ along the top of the diagram and then down $\delta^*(\Theta)$.

(4.) It is only necessary to observe that

$$\Theta(H_0(\Psi)) = \Theta(\Psi(H_0/F_0)) = \rho_\Theta(\Psi)(\Theta_{/F}(H_0/F_0)) = \rho_\Theta(\Psi)(H_1/F_1) = H_1(\rho_\Theta(\Psi)),$$

because the restriction of an isometry remains an isometry.

(5.) If κ_0 and κ_1 are nondegenerate then $\hat{\lambda}_i \otimes Id_{\mathbb{Q}}$ is invertible and the homomorphism $\hat{\lambda}_i^{-1} = (\hat{\lambda}_i \otimes Id_{\mathbb{Q}})^{-1}|_{H_i^*}$, $i = 0, 1$. We calculate that

$$\begin{aligned} \hat{\lambda}_0^{-1} &= ((\Theta^* \circ \hat{\lambda}_1 \circ \Theta) \otimes Id_{\mathbb{Q}})^{-1}|_{H_0^*} \\ &= (\Theta^{-1} \otimes Id_{\mathbb{Q}}) \circ (\hat{\lambda}_1^{-1} \otimes Id_{\mathbb{Q}})|_{H_1^*} \circ ((\Theta^*)^{-1} \otimes Id_{\mathbb{Q}})|_{H_0^*} \\ &= (\Theta^{-1} \otimes Id_{\mathbb{Q}}) \circ \hat{\lambda}_1^{-1} \circ (\Theta^*)^{-1}. \end{aligned}$$

□

Lemma 2.48. *Let $\kappa_0 = \kappa(H_0, \lambda_0, \alpha_0)$ and $\kappa_1 = \kappa(H_1, \lambda_1, \alpha_1)$ be a pair of quadratic functions in $\mathcal{F}^*$.*

1. *If $\Theta : \kappa_0 \to \kappa_1$ is an isometry then $\delta^*(\Theta) : \delta^*(\kappa_0) \to \delta^*(\kappa_1)$ is an isometry of quadratic linking families.*
2. $\delta^*(\kappa_0 \oplus \kappa_1) = \delta^*(\kappa_0) \oplus \delta^*(\kappa_1)$.

Proof. *(1.)* Let $\delta^*(\kappa_0) = (G_0, q_0^*, \beta_0)$ to $\delta^*(\kappa_1) = (G_1, q_1^*, \beta_1)$. As $(\Theta^*)^{-1}(\alpha_0) = \alpha_1$ it follows that

$$\delta^*(\Theta)(\beta_0) = \delta^*(\Theta)([\alpha_0]) = [\alpha_1] = \beta_1.$$

Suppose now that κ_0 and κ_1 are nondegenerate. Another diagram chase shows that $\delta^*(\Theta)$ is an isometry from $\delta^*(\kappa_0)$ to $\delta^*(\kappa_1)$. We include the computation for the characteristic case with $x \in H^*$ a lift of $[x] \in TG_0$.

$$
\begin{aligned}
q_1(\delta^c(\Theta)([x])) &= \lambda_1^{-1}((\Theta^*)^{-1}(x), (\Theta^*)^{-1}(x) + \alpha_1)/2 && \bmod \mathbb{Z} \\
&= (\Theta^*)^{-1}(x+\alpha_0)[\hat{\lambda}_1^{-1}((\Theta^*)^{-1}(x))]/2 && \bmod \mathbb{Z} \\
&= (\Theta^*)^{-1}(x+\alpha_0)[(\Theta \otimes Id_{\mathbb{Q}})(\hat{\lambda}_0^{-1}(x))]/2 && \bmod \mathbb{Z} \quad \text{by Lemma 2.48, } \mathit{5} \\
&= (x+\alpha_0)[\hat{\lambda}_0^{-1}(x)]/2 && \bmod \mathbb{Z} \\
&= \lambda_0^{-1}(x, x+\alpha_0)/2 && \bmod \mathbb{Z} \\
&= q_0([x])
\end{aligned}
$$

Now consider the general case. Let $\Psi \in \mathrm{Sec}(\rho_0)$ and $\Phi \in \mathrm{Sec}(\tau_{G_0})$ be such that $\Phi = \Phi(\Psi)$. By part *4* of Lemma 2.47, $\Theta|_{H(\Psi)}$ defines an isometry form $\kappa_0(\Psi)$ to $\kappa_1(\rho_\Theta(\Psi))$. Moreover, $\delta^*(\Theta)|_{TG_0} = \delta^*(\Theta|_{H(\Psi)})$ and so by the calculation for the nondegenerate case we conclude that $q^*(\kappa_1(\rho_\Theta(\Psi))) \circ \delta^*(\Theta)|_{TG_0} = q^*(\kappa_0(\Psi))$. We calculate that

$$
\begin{aligned}
q_1^*(\tau_{\delta^*(\Theta)}(\Phi)) \circ \delta^*(\Theta)|_{TG_0} &= q_1^*(\Phi(\rho_\Theta(\Psi))) \circ \delta^*(\Theta)|_{TG_0} \quad \text{by Lemma 2.47 } \mathit{3} \\
&= q^*(\kappa_1(\rho_\Theta(\Psi))) \circ \delta^*(\Theta)|_{TG_0} \\
&= q^*(\kappa_0(\Psi)) \\
&= q_0^*(\Phi)
\end{aligned}
$$

and so $\delta^*(\Theta)$ defines an isometry from $\delta^*(\kappa_0)$ to $\delta^*(\kappa_1)$.

(2.) It is clear that $\mathrm{Cok}(\hat{\lambda}_0 \oplus \hat{\lambda}_1) = \mathrm{Cok}(\hat{\lambda}_0) \oplus \mathrm{Cok}(\hat{\lambda}_1)$ and that $[(\alpha_0, \alpha_1)] = ([\alpha_0], [\alpha_1])$. It is also clear that the quadratic linking function induced by the direct sum of nondegenerate quadratic functions is the direct sum of the induced quadratic linking functions. Now we note that for $\Psi_0 \in \mathrm{Sec}(\rho_0)$ and $\Psi_1 \in \mathrm{Sec}(\rho_1)$ that $\Phi(\Psi_0 \oplus \Psi_1) =$

$\Phi(\Psi_0) \oplus \Phi(\Psi_1)$. Setting $\Phi_0 = \Phi(\Psi_0)$ and $\Phi_1 = \Phi(\Psi_1)$ we calculate that

$$\begin{aligned} q^*(\kappa_0 \oplus \kappa_1)(\Phi_0 \oplus \Phi_0) &= q^*(\kappa_0 \oplus \kappa_1)(\Psi_0 \oplus \Psi_1) \\ &= q^*((\kappa_0 \oplus \kappa_1)(\Psi_0 \oplus \Psi_1)) \\ &= q^*(\kappa_0(\Psi_0) \oplus \kappa_1(\Psi_1)) \\ &= q^*(\kappa_0(\Psi_0)) \oplus q^*(\kappa_1(\Psi_1)) \\ &= q^*(\kappa_0)(\Phi_0) \oplus q^*(\kappa_0)(\Phi_1) \end{aligned}$$

which shows that $\delta^*(\kappa_0 \oplus \kappa_1) = \delta^*(\kappa_0) \oplus \delta^*(\kappa_1)$. □

REMARK 2.49. For $i = 0, 1$, let $L_i = L(\kappa_i)$ be handlebodies corresponding to quadratic functions κ_i and let $P_i = \partial L_i$. The topological correlates of H_i, H_i^* and $G_i = \mathrm{Cok}(\hat{\lambda}_i)$ are respectively $H_{4j}(L_i)$, $H^{4j}(L_i)$ and $H^{4j}(P_i)$. The isometry $\Theta : H_0 \to H_1$ therefore defines an isomorphism $H_{4j}(L_0) \to H_{4j}(L_1)$ which, by Wall's classification of handlebodies, is realized as the map induced on homology by a diffeomorphism $g : L_0 \to L_1$. If $f := g|_{P_0}$ is restriction of g to the boundary of L_0, then $\delta^*(\Theta)$ corresponds to the the "wrong way" map $f_! : H^{4j}(P_0) \to H^{4j}(P_1)$. Recall that $f_!$ was defined in the introduction as follows. For $x \in H^{4j}(P_0)$, $f_!(x) := f_*(x \cap [P_0])/(P_1)$, where $[P_0] \in H_{8j}(P_0)$ and $(P_1) \in H^{8j}(P_1)$ are the orientation classes of P_0 and P_1 and $\cap$ and $/$ denote the cap and slant products respectively.

DEFINITION 2.50. *Let P be a highly connected $(8j-1)$-manifold and let L be a handlebody with boundary $P\#\Sigma_P$. We define the quadratic linking family of P, $Q(P)$, to be the quadratic linking family induced by $\kappa(L)$,*

$$Q(P) := \delta^*(\kappa(L)).$$

LEMMA 2.51. *The quadratic linking family of a highly connected manifold P is well defined and if P_0 and P_1 are two highly connected $(8j-1)$-manifolds then*

$$Q(P_0\#P_1) = Q(P_0) \oplus Q(P_1).$$

PROOF. Note that the proof is completely analogous to the proof of Corollary 2.26. Suppose that L and L' are two handlebodies with boundary $P\#\Sigma_P$. Then by Wilkens' Theorem 2.24 applied to $Id_{P\#\Sigma_P}$ there are handlebodies V and V' with boundaries the standard sphere and a diffeomorphism

$$g : L\natural V \to L'\natural V'$$

which extends $Id_{P\#\Sigma_P}$. The induced map $g_* : \kappa(L) \oplus \kappa(V) \to \kappa(L') \oplus \kappa(V')$ is an isometry. So by Lemma 2.48

$$\delta^*(g_*) : \delta^*(\kappa(L)) \oplus \delta^*(\kappa(V)) = \delta^*(\kappa(L')) \oplus \delta^*(\kappa(V'))$$

is an isometry. Now by the preceding remark, $\delta^*(g_*) = (g|_{\partial(L\natural V)})_! = (Id_{P\#\Sigma_P})_! = Id$. Moreover, both $\delta^*(\kappa(V))$ and $\delta^*(\kappa(V'))$ are the trivial quadratic linking family and hence $\delta^*(\kappa(L)) = \delta^*(\kappa(L'))$ and so $Q(P)$ is well defined.

Now suppose that $P_i\#\Sigma_i = \partial L_i$ for $i = 0, 1$ then $(P_0\#P_1)\#\Sigma_{P_0\#P_1} = \partial(L_0\natural L_1)$ and we see that

$$\begin{aligned} Q(P_0\#P_1) &= \delta^*(\kappa(L_0\natural L_1)) \\ &= \delta^*(\kappa(L_0) \oplus \kappa(L_1)) \\ &= \delta^*(\kappa(L_0)) \oplus \delta^*(\kappa(L_1)) \\ &= Q(P_0) \oplus Q(P_1). \end{aligned}$$

□

REMARK 2.52. In dimensions $(8j-1)$ with $j > 2$, Wall gave an intrinsic definition of the quadratic form associated to a highly connected manifold P [**Wa3**]. At this stage I am still unable to give an intrinsic definition of $Q(P)$ in dimensions 7 and 15 although this would be most desirable.

To conclude this section we summarize the algebraic invariants arising from a pair (L, P) which we have described in previous sections.

- A free abelian group $H = H_{4j}(L) \cong H^{4j}(L, P)$ and its dual $H^* = H^{4j}(L) \cong H_{4j}(L, P)$.
- A symmetric intersection pairing $\lambda : H \times H \to \mathbb{Z}$ with adjoint map $\hat{\lambda} : H \to H^*$, $\lambda(v, w) = \lambda(w, v) = \hat{\lambda}(w)(v) = \hat{\lambda}(v)(w)$.
- An element $\alpha \in H^*$ which is the stable tangential invariant of L. When $j = 1, 2$ α is characteristic for λ, that is $\lambda(v, v) + \alpha(v)$ is even for all $v \in H$. When $j > 2$ there is no constraint upon α.
- A quadratic function $\kappa = \kappa(H, \lambda, \alpha) = \kappa(L)$ which is a synthesis of the above invariants.
- A finitely generated abelian group $G = H^{4j}(P) \cong H_{4j-1}(P)$.
- A nonsingular, symmetric, bilinear (linking) form b on TG, $b : TG \times TG \to \mathbb{Q}/\mathbb{Z}$.
- An element $\beta \in G$ which is the stable tangential invariant of P. When $j = 2, 4$ β is even.
- A function $q^* : \mathrm{Sec}(\tau_G) \to \mathcal{Q}(b)$ which associates a quadratic linking function to every section of the canonical map $\tau_G : G \to G^{**}$ (in the even case q^{ev} is a constant function with value an quadratic linking form).
- A quadratic linking family, $Q(P) = (G, q^*, \beta) = \delta^*(\kappa(L))$.

11. Categorical statements

Let $\mathcal{HC}^{8j-1}$ be the category whose objects are closed smooth highly connected $(8j-1)$-manifolds. An almost diffeomorphism is a homeomorphism which fails to be a diffeomorphism at at most finitely many points (see Section 1). The morphisms of $\mathcal{HC}^{8j-1}$ shall be equivalence classes of almost diffeomorphisms $f : P_0 \to P_1$. Two almost diffeomorphisms f and f' are equivalent if they induce the same isomorphism on cohomology, $f_! = f'_! : H^{4j}(P_0) \to H^{4j}(P_1)$. We let $\mathcal{Q}^c$ (resp. $\mathcal{Q}^{ev}$) denote the categories whose objects are characteristic (resp. even) quadratic linking families. The morphisms of $\mathcal{Q}^*$ are isometries of quadratic linking families. The categories $\mathcal{HC}^{8j-1}$ and $\mathcal{Q}^*$ are symmetric monoidal categories with respect to the operations of connected sum of manifolds and direct sum of quadratic linking families respectively.

THEOREM 2.53. *The functors Q^j which associate to every highly connected $(8j-1)$-manifold its quadratic linking family and to every almost diffeomorphism the induced isomorphism on cohomology are equivalences of (symmetric monoidal) categories.*

$$\begin{array}{rcl} Q^j \ : \mathcal{HC}^{8j-1} & \to & \mathcal{Q}^c \\ \{P,[f]\} & \mapsto & \{Q(P), f_!\} \end{array} \quad (j=1,2) \qquad \begin{array}{rcl} Q^j \ : \mathcal{HC}^{8j-1} & \to & \mathcal{Q}^{ev} \\ \{P,[f]\} & \mapsto & \{Q(P), f_!\} \end{array} \quad (j>2)$$

PROOF. Let $\underline{\mathcal{HC}}^{8j-1}$ be a skeleton of $\mathcal{HC}^{8j-1}$. By Theorem B, $Q^j(\underline{\mathcal{HC}}^{8j-1})$ is a skeleton of $\mathcal{Q}^*$ and hence Q^j defines an equivalence of categories. Moreover, for any pair of highly connected $(8j-1)$-manifolds, P_0 and P_1 and any pair of almost diffeomorphisms f_0 and f_1, $Q(P_0\#P_1) = Q(P_0) \oplus Q(P_1)$ and $(f_0\#f_1)_! = (f_0)_! \oplus (f_1)_!$. $\square$

Of course the equivalences of categories in the restatement of Wall's classification of handlebodies (2.12) are related to the equivalences of categories in Theorem 2.53. Recall that the functor κ^j from Theorem 2.12 associates to every $8j$-dimensional handlebody its quadratic function and to every equivalence class of diffeomorphisms the induced map on homology. We let $\partial^j : \mathcal{H}^{8j} \to \mathcal{HC}^{8j-1}$ be the functor which associates to every handlebody its boundary and to the every equivalence class of diffeomorphisms the equivalence class of diffeomorphisms restricted to the boundary.

PROPOSITION 2.54. *The functors $\partial^j, \delta^*, \kappa^j$ and Q^j fit into the following commuting squares.*

$$\begin{array}{ccc} \mathcal{H}^{8j} & \xrightarrow{\kappa^j} & \mathcal{F}^c \\ \downarrow \partial^j & & \downarrow \delta^c \quad (j = 1, 2) \\ \mathcal{HC}^{8j-1} & \xrightarrow{Q^j} & \mathcal{Q}^c \end{array} \qquad \begin{array}{ccc} \mathcal{H}^{8j} & \xrightarrow{\kappa^j} & \mathcal{F}^{ev} \\ \downarrow \partial^j & & \downarrow \delta^{ev} \quad (j > 2). \\ \mathcal{HC}^{8j-1} & \xrightarrow{Q^j} & \mathcal{Q}^{ev} \end{array}$$

PROOF. This follows from the definition of $Q(\partial L)$ since $Q(\partial L) = \delta^*(\kappa(L))$. □

12. The invariants s_1, $\bar{s}_1$ and μ

Up until this point our preliminary remarks have been preparing for the almost diffeomorphism classification of highly connected $(8j-1)$-manifolds by their quadratic linking families. In this section we define the smooth invariant s_1 of highly connected rational homology spheres in preparation for their smooth classification. We also define the almost diffeomorphism invariant $\bar{s}_1$ whose algebraic analogue is crucial for the classification of quadratic linking functions.

If a handlebody L has a nondegenerate quadratic function then the boundary of L, P, is a rational homology sphere bounding the a spin manifold in which case the Eels-Kuiper μ-invariant [**EK**] is well defined. In fact, Stolz [**St**] has defined a stronger invariant but only for homotopy spheres. However, there is nothing which prevents the definition of this invariant, which I shall call s_1, for simply connected rational homology spheres and so we now present a generalization of Stolz's s_1 invariant. Suppose then that M is a simply connected spin rational homology sphere of dimension $4k-1$ bounding a spin manifold W. We let $\mathcal{L}(W)$, $\hat{A}(W)$ and $\mathrm{ph}(W)$ denote respectively the $\mathcal{L}$-genus, the $\hat{A}$-genus and the Pontryagin character of W which are rational power series in the Pontryagin classes p_i of W, [**Hi**] and [**St**]. For any power series T in the Pontryagin classes we let T_k denote its $4k$-dimensional component. So for example, the first four terms of the power series of $\mathcal{L}(W)$, $\hat{A}(W)$ and $\mathrm{ph}(W)$ are

$$\begin{array}{llll}
\mathcal{L}_0 = 1, & \mathcal{L}_1 = \frac{1}{3}p_1, & \mathcal{L}_2 = \frac{1}{45}(7p_2 - p_1^2), & \mathcal{L}_3 = \frac{1}{3^2\cdot 5\cdot 7}(62p_3 - 13p_2p_1 + 2p_1^3), \\
\hat{A}_0 = 1, & \hat{A}_1 = -\frac{2}{3}p_1, & \hat{A}_2 = \frac{2}{45}(-4p_2 + 7p_1^2), & \hat{A}_3 = \frac{4}{3^2\cdot 5\cdot 7}(-16p_3 + 44p_2p_1 - 31p_1^3), \\
\mathrm{ph}_0 = 0, & \mathrm{ph}_1 = p_1, & \mathrm{ph}_2 = \frac{1}{24}(-4p_2 + p_1 62), & \mathrm{ph}_3 = \frac{1}{2^7\cdot 3^3\cdot 5}(144p_3 - 28p_2p_1 + p_1^3).
\end{array}$$

Now following Stolz we define

$$S_k(W) = \frac{1}{8}\mathcal{L}_k(W) + \frac{|bP_{4k}|}{a_k}\left(c_k\hat{A}_k(W) + (-1)^k d_k(\hat{A}(W)\mathrm{ph}(W))_k\right)$$

where $a_k = 1$ for k even and 2 for k odd, and c_k and d_k are integers such that

$$c_k \mathrm{num}(B_k/4k) + d_k \mathrm{denom}(B_k/4k) = 1.$$

The symbol B_k denotes the kth Bernoulli number and $\mathrm{num}(B_k/4k)$ and $\mathrm{denom}(B_k/4k)$ denote respectively the numerator and denominator of the fraction $B_k/4k$ when expressed in lowest terms. As M is a rational homology sphere $H^*(W;\mathbb{Q}) \cong H^*(W,M;\mathbb{Q})$ and we use this isomorphism to view each Pontryagin class $p_i(W)$ as an element of $H^*(W,M;\mathbb{Q})$. Hence we may evaluate $S_k(W)$ against on the fundamental class of W, $[W,\partial W] \in H_{4k}(W)$, to obtain the rational number $< S_k(W), [W,\partial W] >$.

Theorem 2.55 (c.f. [**St**] Theorem). *Let M be a $(4k-1)$-dimensional spin rational homotopy sphere bounding a spin manifold W and let $\sigma(W)$ denote the signature of W. Then the coefficient of $p_k(W)$ in $S_k(W)$ is zero and*

$$s_1(M) := \frac{1}{|bP_{4k}|}\left(< S_k(W), [W,\partial W] > -\frac{1}{8}\sigma(W)\right) \mod \mathbb{Z}$$

is a well defined smooth invariant of M. If $M = \Sigma$ is a homotopy sphere then

$$-|bP_{4k}| \cdot s_1(\Sigma) = s(\Sigma) \in \mathbb{Z}/|bP_{4k}|$$

computes Brumfiel's splitting invariant of Σ.

Proof. The first and third assertions of the Theorem are proven by Stolz. Moreover, Stolz's proof that the quantity $\frac{1}{8}\sigma(W) - < S_k(W), [W,\partial W >$ $(\mathrm{mod}|bP_{4k}|)$ is independent of the choice of spin coboundary W makes no essential use of the fact that M is a homotopy sphere as opposed to a rational homology sphere. We present this proof again to reassure the reader of this claim. Suppose that W and W' are two spin manifolds bounding M. Let X be the closed spin manifold obtained by gluing W to $-W'$ along M and let $i : W \hookrightarrow X$ and $i' : -W' \hookrightarrow X$ denote the inclusions. As M is a rational homology sphere, $i^* \oplus i^{'*} : H^{2k}(X;\mathbb{Q}) \to H^{2k}(W;\mathbb{Q}) \oplus H^{2k}(W'\mathbb{Q})$ is an isomorphism and $(i^* \oplus i^{'*})p_i(X) = (p_i(W), p_i(-W'))$ for $i < k$. It follows that

$$\sigma(X) = \sigma(W) - \sigma(W')$$

and

$$< S_k(X), [X] >=< S_k(W), [W, \partial W] > - < S_k(W'), [W', \partial W'] > .$$

We now calculate that

$$\begin{aligned}
&(1/|bP_{4k}|)\left(< S_k(W), [W, \partial W] > -\tfrac{1}{8}\sigma(W)\right)\\
&\qquad\qquad -(1/|bP_{4k}|)\left(< S_k(W'), [W', \partial W'] > -\tfrac{1}{8}\sigma(W')\right)\\
&= (1/|bP_{4k}|)\left(< S_k(X), [W, \partial X] > \tfrac{1}{8}\sigma(X)\right)\\
&= (1/|bP_{4k}|)(L_k(X) - \tfrac{1}{8}\sigma(X))\\
&+ \Big(c_k(1/a_k) < \hat{A}(X), [X] > +(-1)^k d_k(1/a_k) < (\hat{A}(X)\mathrm{ph}(X))_k, [X] >\Big)\\
&= 0 \bmod \mathbb{Z}
\end{aligned}$$

because $\sigma(X) =< \mathcal{L}_k(X), [X] >$ by Hirzebruch's signature theorem the quantities $(1/a_k) < \hat{A}(X), [X] >$ and $(1/a_k) < (\hat{A}(X)\mathrm{ph}(X))_k, [X] >$ are integers by the Hirzebruch-Riemann-Roch Theorem. $\square$

REMARK 2.56. We mentioned above that Stolz's s_1-invariant generalizes the Eels-Kuiper μ-invariant and we now explain this remark. For W and M as before Eels and Kuiper made the following definitions:

$$N_k(W) := \frac{1}{8 \cdot |bP_{4k}|}\mathcal{L}_k(W) + \frac{1}{a_k}\hat{A}_k(W)$$

and

$$\mu(M) := < N_k(W), [W, \partial W] > - \frac{1}{8 \cdot |bP_{4k}|}\sigma(W).$$

Now when $\mathrm{num}(B_{4k}/4k) = 1$ (which is the case for $k = 2$ and when $k = 4$) then we may take $c_k = 1$ and $d_k = 0$ in the definition of $S_k(W)$ and $s_1(M)$ and then $s_1(M) = \mu(M)$. However, when $\mathrm{num}(B_{4k}/4k) \neq 1$ the s_1-invariant is strictly shaper than the μ-invariant.

Observe that for two closed spin $(4k-1)$-manifolds M_0 and M_1 bounding spin manifolds W_0 and W_1 that $M_0 \# M_1 = \partial(W_0 \natural W_1)$ and hence $s_1(M_0 \# M_1) = s_1(M_0) + s_1(M_1)$. Observe also that

$$|bP_{4k}| = \max\{\mathrm{denom}(s_1(\Sigma)) | \Sigma \in \Theta_{4k-1}\}$$

and then define

$$\bar{s}_1(M) := |bP_{4j}| s_1(M) \in \mathbb{Q}/\mathbb{Z}.$$

It follows immediately that $\bar{s}_1(\Sigma) = 0$ for any homotopy sphere and therefore that $\bar{s}_1$ is an almost diffeomorphism invariant. In fact, when $k = 2$ Kreck and Stolz show that $\bar{s}_1$ is a topological invariant [**KS**]. We shall later identify the $\bar{s}_1$-invariant of a highly connected rational homology sphere in dimensions 7 and 15 with the Kervaire-Arf invariant of its quadratic linking function. To conclude this section we present the calculations of s_1 for 2-connected 7-manifolds and 6-connected 15-manifolds. If L is an $8j$-dimensional handlebody with tangential invariant α then the Pontryagin classes of L are as follows:

$$p_i(L) = \left\{ \begin{array}{cc} (2j-1))! a_k \alpha & i = j \\ 0 & i \neq j \end{array} \right\}.$$

It follows that the only possibly non-zero term in $S_{2j}(L)$ is that containing $p_j(L)^2$. Now let $\mathcal{L}_{(j,2j)}$ and $\hat{A}_{(j,2j)}$ denote the coefficient of p_j^2 in respectively $\mathcal{L}_{2j}$ and $\hat{A}_{2j}$. We also need to know the coefficients of p_j in $\mathrm{ph}(L)_j$ and p_j^2 in $\mathrm{ph}(L)_{2j}$ but these may be computed using Newton's formula ([**MS**] Ex 16.A) and the fact that all but one Pontryagin class vanish. We calculate that

$$S_{2j}(L) = S_{(j,2j)}(L)p_j^2 = \tilde{S}_{(j,2j)}(L)\alpha^2$$

where $\tilde{S}_{(j,2j)}(L) = ((2j-1)!)^2 S_{(j,2j)}(L)$ and

$$S_{(j,2j)}(L) := \frac{1}{8}L_{(j,2j)} + |bP_{8j}| \left(c_{2j}\hat{A}_{(j,2j)} + d_{2j}\left(\hat{A}_{(j,2j)} \frac{(-1)^j}{2j!(2j-1)!} + \frac{1}{(2j-1)!} \right) \right).$$

PROPOSITION 2.57. *Let P be a highly connected manifold of dimension 7 or 15 and let $P\#\Sigma_P = \partial L$ where the handlebody L has quadratic function $\kappa(L) = \kappa(H, \lambda, \alpha)$. Then*

$$\dim(P) = 8j-1 : s_1(P) = (1/|bP_{8j}|)\left(\tilde{S}_{(j,2j)}\lambda^{-1}(\alpha,\alpha) - \frac{1}{8}\sigma(\lambda) \right),$$

$$\dim(P) = 7 : \quad s_1(P) = \tfrac{1}{28}\cdot\tfrac{1}{8}(\lambda^{-1}(\alpha,\alpha) - \sigma(\lambda)), \quad \bar{s}_1(P) = \tfrac{1}{8}(\lambda^{-1}(\alpha,\alpha) - \sigma(\lambda)),$$

$$\dim(P) = 15 : \quad s_1(P) = \tfrac{1}{8{,}128}\cdot\tfrac{1}{8}(\lambda^{-1}(\alpha,\alpha) - \sigma(\lambda)), \quad \bar{s}_1(P) = \tfrac{1}{8}(\lambda^{-1}(\alpha,\alpha) - \sigma(\lambda)).$$

PROOF. By definition $s(\Sigma_P) = 0$. Thus $s_1(\Sigma_P) = 0$ and $s_1(P) = s_1(\partial L)$. The term $\lambda^{-1}(\alpha,\alpha)$ appears since to compute $< \alpha^2, [L, \partial L] >$ we must transport $\alpha \in H^{4j}(L;\mathbb{Q})$ to $\hat{\lambda}^{-1}(\alpha) \in H^{4j}(L, \partial L; \mathbb{Q})$ and $\hat{\lambda}^{-1}(\alpha)^2 = \lambda^{-1}(\alpha,\alpha)$. The specific computations for dimensions 7 and 15 come from from knowledge of $\mathcal{L}_{(1,2)}$, $\mathcal{L}_{(2,4)}$, $\hat{A}_{(1,2)}$, $\hat{A}_{(2,4)}$, B_2 and B_4 which may all be found in [**Hi**]. Alternatively, since $B_2 = B_4 = 1/30$, $s_1 = \mu$ in dimensions 7 and 15 and these calculations are in [**EK**]. $\square$

CHAPTER 3

Stable algebra of quadratic functions

In this chapter we establish the algebraic results we shall need to classify the manifolds P up to almost diffeomorphism. In Chapter 4 we shall apply these results. Recall that two quadratic functions κ_0 and κ_1 belonging to a category $\mathcal{F}^*$ are stably $\mathcal{F}^*$-equivalent if there are nonsingular quadratic functions μ_0 and μ_1 also belonging to $\mathcal{F}^*$ such that $\kappa_0 \oplus \mu_0 \cong \kappa_1 \oplus \mu_1$. In this chapter we establish that stable $\mathcal{F}^*$-equivalence is closely related to two further relations between quadratic functions. The first of these is that of isometry between their induced quadratic linking families and the second is orthogonality which we shall define presently.

DEFINITION 3.1. *An isometric embedding $i_0 : \kappa_0(H_0) \hookrightarrow \kappa(H)$ between nondegenerate quadratic functions is said to be* primitive *if $i_0(H_0)$ is a summand of H. The* orthogonal complement *of a primitive embedding i_0, denoted by $i_0(\kappa_0)^\perp$, is defined to be $\kappa|_{i_0(H_0)^\perp}$ where*

$$i_0(H_0)^\perp := \{v \in H | \lambda(v, i_0(v_0)) = 0 \ \forall v_0 \in H_0\}.$$

EXAMPLE 3.2. *Let $\kappa = \kappa(H, \lambda, 0)$ and $\kappa_0 = \kappa(H_0, \lambda, 0)$ be quadratic forms defined on $H = \mathbb{Z}^2[v, w]$ and $H_0 = \mathbb{Z}[w_0]$, by the following equations: $\lambda(v, v) = 0, \lambda(v, w) = 1, \lambda(w, w) = n$ and $\lambda_0(w_0, w_0) = n$, where $n \in \mathbb{Z} - \{-1, 0, 1\}$. Then $i_0 : H_0 \hookrightarrow H, w_0 \mapsto w$, is a primitive embedding of a nondegenerate but singular form into a nonsingular form. In this case $i_0(H_0)^\perp = \mathbb{Z}[nv - w]$ and $i_0(\kappa_0)^\perp$ is defined by*

$$i_0(\kappa_0)^\perp(nv - w) = n.$$

DEFINITION 3.3. *Let* $\kappa_0 = \kappa(H_0, \lambda_0, \alpha_0)$ *and* $\kappa_1 = \kappa(H_1, \lambda_1, \alpha_1)$ *be quadratic functions belonging to the category* $\mathcal{F}^*$.

1. *If* κ_0 *and* κ_1 *are nondegenerate then they are* $\mathcal{F}^*$-orthogonal *if there exists a nonsingular quadratic function* κ *belonging to* $\mathcal{F}^*$ *and primitive embeddings* $i_0 : \kappa_0 \hookrightarrow \kappa$ *and* $i_1 : \kappa_1 \hookrightarrow \kappa$ *such that* $i_0(\kappa_0)^\perp = i_1(\kappa_1)$.
2. *In general,* κ_0 *and* κ_1 *are* $\mathcal{F}^*$-orthogonal *if there exits splittings (see Section 9)*
$$\kappa_0 = \alpha_0|_{F_0} \oplus \kappa_0(\Psi_0) \qquad \textit{and} \qquad \kappa_1 = \alpha_1|_{F_1} \oplus \kappa_1(\Psi_1)$$
such that $\alpha_0|_{F_0} \cong \alpha_1|_{F_1}$ *and such that* $\kappa_0(\Psi_0)$ *is* $\mathcal{F}^*$*-orthogonal to* $\kappa_1(\Psi_1)$.

In both cases we write $\kappa_0 \perp_{\mathcal{F}^*} \kappa_1$.

We now state the theorem whose proof is the subject of the chapter.

THEOREM 3.4. *Let* κ_0 *and* κ_1 *be two quadratic functions in the same family* $\mathcal{F}^*$. *Then the following are equivalent.*

1. κ_0 *is stably* $\mathcal{F}^*$*-equivalent to* κ_1, $\kappa_0 \sim_{\mathcal{F}^*} \kappa_1$.
2. *The quadratic linking families induced by* κ_0 *and* κ_1 *are isometric,* $\delta^*(\kappa_0) \cong \delta^*(\kappa_1)$.
3. κ_0 *and* κ_1^- *are* $\mathcal{F}^*$*-orthogonal,* $\kappa_0 \perp_{\mathcal{F}^*} \kappa_1^-$.

REMARK 3.5. In the nondegenerate case when $\mathcal{F}^* = \mathcal{F}^o$, the equivalence of *1* and *2* was proven by Durfee [**Du1**], and Wall [**Wa4**]. Though he did not state it this way, Wilkens [**Wi1**] proved the equivalence of *1* and *2* for $\mathcal{F}^c$ in the case where the induced quadratic linking family is indecomposable. Nikulin proved the equivalence of *2* and *3* ([**Ni**] Corollary 1.6.2) in the nondegenerate case where $\mathcal{F}^* = \mathcal{F}^o$. Of course, the family $\mathcal{F}^c$ is of primary concern for the classification of manifolds in dimensions 7 and 15, however the proofs given below may shed some light on the results when $\mathcal{F}^* = \mathcal{F}^o$ or $\mathcal{F}^{ev}$. In particular, Durfee's methods use quite delicate p-adic techniques, Wall's proof is somewhat opaque (to this reader) though similar to our own and Nikulin's proofs are extremely brief. Our perspective on these results is that they are algebraic counterparts to the topology of handlebodies and their boundaries.

1. Gluing and splitting nondegenerate quadratic functions

The gluing lemma which follows is inspired by the following topological idea: suppose that $f : P_0 \to P_1$ is an almost diffeomorphism of highly connected rational homology spheres with coboundaries L_0 and L_1. We could then form the almost smooth manifold $M = L_0 \cup_f -L_1$ which would be a highly connected $8j$-manifold and there would be a Mayer-Vietoris decomposition for $H_{4j}(W)$ relating the domains of the quadratic functions of M, L_0 and L_1.

$$\begin{tikzcd}
0 \arrow[r] & H_{4j}(L_0) \oplus H_{4j}(-L_1) \arrow[r] \arrow[d] & H_{4j}(M) \arrow[r] \arrow[d] & H_{4j-1}(P) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_0 \oplus H_1 \arrow[r] & H \arrow[r] & TG \arrow[r] & 0
\end{tikzcd}$$

The idea then is to manufacture the algebraic data for a manifold M from the algebraic data corresponding to the almost diffeomorphism f. If $\kappa_0 = \kappa(L_0)$ and $\kappa_1 = \kappa(L_1)$ are the quadratic functions of the handlebodies L_0 and L_1 then this data is an isomorphism $\theta : \delta^*(L_0) \cong \delta^*(L_1)$ of the induced quadratic linking families. In the following lemma we show how to define a quadratic function in the class $\mathcal{F}^*$ from θ, κ_0 and κ_1 which we shall denote by $\kappa_0 \cup_\theta \kappa_1^-$. All of the notation follows the conventions established in Chapter 2. In particular the fundamental sequence of a non-degenerate quadratic function $\kappa_i = \kappa(H_i, \lambda_i, \alpha_i)$ is

$$0 \longrightarrow H_i \xrightarrow{\hat{\lambda}_i} H_i^* \xrightarrow{\pi_i} TG_i \longrightarrow 0$$

and for $x_i \in H_i^*$ we shall write $[x_i]$ for $\pi_i(x_i)$.

LEMMA 3.6 (Gluing Lemma). *Let $\mathcal{F}^*$ be one of the families $\mathcal{F}^c$, $\mathcal{F}^{ev}$ or $\mathcal{F}^o$ and let $\kappa_0 = \kappa(H_0, \lambda_0, \alpha_0)$ and $\kappa_1 = \kappa(H_1, \lambda_1, \alpha_1)$ be nondegenerate quadratic functions* *in the same family $\mathcal{F}^*$. Suppose that $\theta : \mathrm{Cok}(\hat{\lambda}_0) \to \mathrm{Cok}(\hat{\lambda}_1)$ is an isometry of the induced quadratic linking families;*

$$\delta^*(\kappa_0) \xrightarrow{\theta} \delta^*(\kappa_1).$$

Let $H \subset H_0^* \oplus H_1^*$ be the set $\{(x_0, x_1) : \theta([x_0]) = [x_1])\}$ and λ the function

$$\begin{array}{rccc} \lambda : & H \times H & \to & \mathbb{Q} \\ & [(x_0,x_1),(y_0,y_1)] & \mapsto & \lambda_0^{-1}(x_0,y_0) - \lambda_1^{-1}(x_1,y_1). \end{array}$$

Then

1. *λ is a nonsingular integral bilinear form on the free abelian group H,*
2. *there is a canonical isometric embedding $(\lambda_0 \oplus -\lambda_1) \hookrightarrow \lambda$,*
3. *the element $(\alpha_0, \alpha_1) \in H$,*
4. *if $\alpha := \hat{\lambda}((\alpha_0, \alpha_1))$, then the quadratic function $\kappa_0 \cup_\theta \kappa_1^- := \kappa(H, \lambda, \alpha)$ is of type $\mathcal{F}^*$,*
5. *there is a canonical isometric embedding $\kappa_0 \oplus \kappa_1^- \hookrightarrow \kappa_0 \cup_\theta \kappa_1^-$.*

Proof. *(1. & 2.)* Observe that H is the kernel of the homomorphism $H_0^* \oplus H_1^* \to TG_1$ which takes (x_0, x_1) to $\theta([x_0]) - [x_1]$. Thus H is a subgroup of the free abelian group $H_0^* \oplus H_1^*$ so is itself a free abelian group. Now let (x_0, x_1) and $(y_0, y_1) \in H$. We show first that $\lambda[(x_0, x_1), (y_0, y_1)]$ is an integer.

$$\begin{aligned} \lambda[(x_0,x_1),(y_0,y_1)] \bmod \mathbb{Z} &= \lambda_0^{-1}(x_0,y_0) - \lambda_1^{-1}(x_1,y_1) && \bmod \mathbb{Z} \\ &= b_0([x_0],[y_0]) - b_1([x_1],[y_1]) && \bmod \mathbb{Z} \\ &= b_0([x_0],[y_0]) - b_1(\theta([x_0]),\theta([y_0])) && \bmod \mathbb{Z} \\ &= 0 && \bmod \mathbb{Z} \end{aligned}$$

The last equality holds as $q^*(\kappa_0) = \theta^* q^*(\kappa_1)$ entails that $b_0 = b_1 \circ (\theta \times \theta)$. We next show that λ is nonsingular. Consider first the inclusions

$$\begin{array}{rccc} i_0 : & H_0 & \hookrightarrow & H \\ & v_0 & \mapsto & (\hat{\lambda}_0(v_0), 0) \end{array} \qquad \begin{array}{rccc} i_1 : & H_1 & \hookrightarrow & H \\ & v_1 & \mapsto & (0, -\hat{\lambda}_1(v_1)). \end{array}$$

The sum of i_0 and i_1, $i_0 \oplus i_1 : H_0 \oplus H_1 \hookrightarrow H$ defines an embedding $(\lambda_0 \oplus -\lambda_1) \hookrightarrow \lambda$ for we calculate that

$$\begin{aligned}
\lambda[(\hat{\lambda}_0(v_0), -\hat{\lambda}_1(v_1)), (\hat{\lambda}_0(w_0), -\hat{\lambda}_1(w_1))] &= \lambda_0^{-1}(\hat{\lambda}_0(v_0), \hat{\lambda}_0(w_0)) \\
&\qquad -\lambda_1^{-1}(-\hat{\lambda}_1(v_1), -\hat{\lambda}_1(w_1)) \\
&= \lambda_0(v_0, w_0) - \lambda_1(v_1, w_1) \\
&= (\lambda_0 \oplus -\lambda_1)[(v_0, w_0), (v_1, w_1)].
\end{aligned}$$

We now define TG to be the subgroup of $TG_0 \oplus TG_1$ consisting of pairs $([x_0], \theta[x_0])$ and we let $\pi : H \to TG$ be the map $(x_0, x_1) \to ([x_0], [x_1])$. The linking form $b := b(\lambda_0) \oplus b(-\lambda_1)$ is defined on $TG_0 \oplus TG_1$ with adjoint map $\hat{b} : TG_0 \oplus TG_1 \to (TG_0 \oplus TG_1)^\wedge$. Setting $R_{TG} : (TG_0 \oplus TG_1)^\wedge \to TG^\wedge$ to be the restriction map we define $\Pi : H_0^* \oplus H_1^* \to TG^\wedge$ to be the composition $R_{TG} \circ \hat{b} \circ \pi_0 \oplus \pi_1$. It is clear that $i(H) \subset \mathrm{Ker}(\Pi)$ where $i : H \to H_0^* \oplus H_1^*$ is the inclusion $i(x_0, x_1) = (x_0, x_1)$. Moreover, $\Pi|_{TG_0 \oplus 0}$ is onto and since the groups involved are finite we conclude that $\mathrm{Ker}(\Pi) = i(H)$. We shall show that the following diagram is commutative with exact rows and columns (where we have identified $(H_0 \oplus H_1) = (H_0 \oplus H_1)^{**}$.)

Diagram 3.7.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & H_0 \oplus H_1 \arrow[r, "i_0 \oplus i_1"] \arrow[d, "i^*"'] & H \arrow[r, "\pi"] \arrow[d, "i"] \arrow[dl, "\hat{\lambda}"'] & TG \arrow[r] & 0 \\
0 \arrow[r] & H^* \arrow[r, "i_0^* \oplus i_1^*"'] \arrow[d, "\hat{\Pi}"'] & H_0^* \oplus H_1^* \arrow[r, "\hat{\pi}"] \arrow[d, "\Pi"] & TG^\wedge \arrow[r] & 0 \\
 & TG^{\wedge\wedge} \arrow[d] & TG^\wedge \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

It is evident that $i_0 \oplus i_1$ is injective, that $\mathrm{Im}(i_0 \oplus i_1) = \mathrm{Ker}(\pi)$ and that π is onto. It is similarly clear that i is injective and we have already observed that Π is onto and that $\mathrm{Ker}(\Pi) = \mathrm{Im}(i)$. Hence the first row and second column are exact. The exactness of first column and second row follow from the duality we reviewed in section 7. We must show that $i^* : (H_0 \oplus H_1)^{**} \to H^*$ is the same as $\hat{\lambda} \circ (i_0 \oplus i_1)$. For $v_i \in H_i$ let $v_i^{**} \in H_i^{**}$ denote the element $\phi \mapsto \phi(v_i)$ for $\phi \in H_i^*$. We compute that for $(x_0, y_0) \in H$

$$\begin{aligned} i^*(v_0^{**}, v_1^{**})[(x_0, x_1)] &= (v_0^{**}, v_1^{**})[(x_0, x_1)] \\ &= x_0(v_0) + x_1(v_1) \\ &= \lambda_0^{-1}(\hat{\lambda}(v_0), x_0) - \lambda_1^{-1}(-\hat{\lambda}_1(v_1), x_1) \\ &= \lambda[(\hat{\lambda}(v_0), -\hat{\lambda}(v_1)), (x_0, x_1)] \\ &= \hat{\lambda}[(i_0 \oplus i_1)(v_0, v_1)][(x_0, x_1)] \end{aligned}$$

and thus $i^* = \hat{\lambda} \circ (i_0 \oplus i_1)$ after we have identified $H_i = H_i^{**}$ by $v_i \mapsto v_i^{**}$. A similar computation shows that $i = (i_0^* \oplus i_1^*) \circ \hat{\lambda}$. We now require the following lemma whose proof is left to the reader.

Lemma 3.8. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be the composition of a pair of injective homomorphisms of abelian groups. Then there is a short exact sequence*

$$0 \to \mathrm{Cok}(f) \to \mathrm{Cok}(g \circ f) \to \mathrm{Cok}(g) \to 0.$$

If we apply the lemma to $\hat{\lambda} \circ (i_0 \oplus i_1)$ we see that

$$0 \to TG^{\wedge\wedge} \to TG \to \mathrm{Cok}(\hat{\lambda}) \to 0$$

is exact, that $\mathrm{Cok}(\hat{\lambda}) = 0$ and therefore that λ is nonsingular.

(3.) By the definition of an isometry of a quadratic linking family $\theta([\alpha_0]) = [\alpha_1]$ and hence $(\alpha_0, \alpha_1) \in H$.

(4.) Suppose that $\mathcal{F}^* = \mathcal{F}^c$. We must show that $\alpha = \hat{\lambda}((\alpha_0, \alpha_1))$ is characteristic for

λ given that each α_i is characteristic for λ_i. Let $x = (x_0, x_1) \in H$ and observe that

$$\begin{aligned}[\lambda(x,x) + \alpha(x)]/2 \bmod \mathbb{Z} &= [\lambda(x,x) + \hat{\lambda}(\alpha_0, \alpha_1)(x)]/2 && \bmod \mathbb{Z} \\ &= [\lambda_0^{-1}(x_0, x_0 + \alpha_0) - \lambda_1^{-1}(x_1, x_1 + \alpha_1)]/2 && \bmod \mathbb{Z} \\ &= q^c(\kappa_0)([x_0]) - q^c(\kappa_1)(\theta([x_0])) && \bmod \mathbb{Z} \\ &= 0 && \bmod \mathbb{Z}.\end{aligned}$$

Thus $\hat{\lambda}(\alpha)$ is characteristic for λ. If $\mathcal{F}^* = \mathcal{F}^{ev}$ then we must show that λ is even. We again let $x = (x_0, x_1) \in H$ and compute

$$\begin{aligned}\lambda(x,x)/2 \bmod \mathbb{Z} &= [\lambda_0^{-1}(x_0, x_0) - \lambda_1^{-1}(x_1, x_1)]/2 && \bmod \mathbb{Z} \\ &= q^{ev}(\kappa_0)([x_0]) - q^{ev}(\kappa_1)(\theta([x_0])) && \bmod \mathbb{Z} \\ &= 0 && \bmod \mathbb{Z}.\end{aligned}$$

Thus λ is even. If $\mathcal{F}^* = \mathcal{F}^o$ then we must show that λ is even and that $\alpha = 0$. However, $\alpha = \hat{\lambda}((\alpha_0, \alpha_1)) = \hat{\lambda}((0,0)) = 0$ and λ is even by the computation for the case $\mathcal{F}^* = \mathcal{F}^{ev}$.

(5.) We have already proven this result for the quadratic forms $(\lambda_0 \oplus -\lambda_1) \hookrightarrow \lambda$ so we need only show that the inclusions i_0 and i_1 also commute with respect to α_0, α_1 and $\alpha := \hat{\lambda}((\alpha_0, -\alpha_1))$. We do this now for $(v_0, v_1) \in H_0 \oplus H_1$.

$$\begin{aligned}\alpha[(i_0 \oplus i_1)(v_0, v_1)] &= \lambda[(\alpha_0, \alpha_1), (\hat{\lambda}_0(v_0), -\hat{\lambda}_1(v_1))] \\ &= \hat{\lambda}_0^{-1}(\alpha_0, \hat{\lambda}_0(v_0)) - \hat{\lambda}_1^{-1}(\alpha_1, -\hat{\lambda}_1(v_1)) \\ &= \alpha_0(v_0) + \alpha_1(v_1)\end{aligned}$$

□

REMARK 3.9. When $\kappa_0 = \kappa_1$ and $\theta = \mathrm{Id}$ then we shall simply write $\kappa_0 \cup \kappa_0^-$ for the "double" of κ.

The Splitting Lemma which follows shows that every primitive embedding $i_0 : \kappa_0 \hookrightarrow \kappa$ of a nondegenerate quadratic function into a nonsingular quadratic function gives rise to a decomposition of κ as in the Gluing Lemma: $\kappa \cong i_0(\kappa_0) \cup_\theta i_0(\kappa_0)^{\perp -}$.

It is topologically motivated by considering the embedding of a handlebody L_0 with boundary a rational homology sphere into a closed highly connected $8j$-manifold L. The complement of this embedding is itself a handlebody with boundary the same rational homology sphere (though the orientation is reversed) and L is obtained by gluing L_0 to $(L - L_0)$ along their common boundary.

LEMMA 3.10 (Splitting Lemma). *Let $\kappa = \kappa(H, \lambda, \alpha)$ be a nonsingular quadratic function in the family $\mathcal{F}^*$ and let H_0 be a summand of H such that $\kappa_0 := \kappa|_{H_0}$ is nondegenerate. Then*

1. *the group $H_1 := H_0^{\perp} = \{v \in H : \lambda(v, v_0) = 0 \ \forall v_0 \in H_0\}$ is a summand of H,*
2. *the quadratic functions $\kappa_1 := (\kappa|_{H_1})^-$ and κ_0 are nondegenerate members of $\mathcal{F}^*$,*
3. *there is an isomorphism $\theta_\kappa : \delta^*(\kappa_0) \cong \delta^*(\kappa_1)$,*
4. *κ and $\kappa_0 \cup_{\theta_\kappa} \kappa_1^-$ are isometric.*

PROOF. *(1.)* If rv belongs to H_1 for a nonzero integer r then rv pairs trivially with all of H_0 but this entails that v does also. Thus $v \in H_1$ and so H_1 is a summand. *(2.)* By definition a quadratic function is nondegenerate if and only if the associated bilinear form is nondegenerate so we consider the bilinear forms λ, $\lambda_0 := \lambda|_{H_0 \times H_0}$ and $\lambda_1 := -\lambda|_{H_1 \times H_1}$. We let a subscript $\mathbb{Q}$ denote the forms obtained after tensoring with the rationals so that for example $\lambda_{\mathbb{Q}} : (H \otimes \mathbb{Q}) \times (H \otimes \mathbb{Q}) \to \mathbb{Q}$. As $\lambda_{0\mathbb{Q}}$ is a nonsingular subform of the nonsingular form $\lambda_{\mathbb{Q}}$ it follows that $\lambda_{\mathbb{Q}} = \lambda_{0\mathbb{Q}} \oplus -\lambda_{1\mathbb{Q}}$ and therefore that $\lambda_{1\mathbb{Q}}$ is nonsingular. Thus λ_1 and κ_1 are nondegenerate. Now let $\alpha_i = \alpha|_{H_i}$, $i = 0, 1$. That both $\kappa_0 = \kappa(H_0, \lambda_0, \alpha_0)$ and $\kappa_1 = \kappa(H_1, \lambda_1, \alpha_1)$ belong to $\mathcal{F}^*$ follows from the fact that the properties of being homogeneous ($\mathcal{F}^o$), having an even quadratic form ($\mathcal{F}^{ev}$) and being even ($\mathcal{F}^c$) are all preserved upon restricting to a summand of the domain of a quadratic function.

(3.) We shall present the proof in the case where $\mathcal{F}^* = \mathcal{F}^c$ as the other cases are similar and easier. From the fact that $\lambda_{\mathbb{Q}} = \lambda_{0\mathbb{Q}} \oplus -\lambda_{1\mathbb{Q}}$, we deduce that the rational vector space $H \otimes \mathbb{Q} = (H_0 \oplus H_1) \otimes \mathbb{Q}$ and thus $H_0 \oplus H_1$ is a subgroup of H with finite index. We therefore define π to be the projection from H onto this finite group $TG := H/(H_0 \oplus H_1)$. We now define a series of groups and homomorphisms which shall fit into an extensive commutative diagram. Let π' be the projection from H^* to the finite group $\hat{\lambda}(TG) := H^*/\hat{\lambda}(H_0 \oplus H_1))$ which is canonically isomorphic to TG via the map induced by $\hat{\lambda}$. Let $i_0 : H_0 \hookrightarrow H$ and $i_1 : H_1 \hookrightarrow H$ denote the obvious inclusion maps and as usual let π_i denote the projection from H_i^* to $H_i^*/\hat{\lambda}(H_i) = TG_i$ for $i = 0, 1$. In addition define $J : H \to H_0^* \oplus H_1^*$ to be the map which sends $v \in H$ to $(\hat{\lambda}(v)|_{H_0}, \hat{\lambda}(v)|_{H_1})$, I to be the map which sends $\pi(v)$ to $(\pi_0 \oplus \pi_1)(J(v))$ and $K : \hat{\lambda}(TG) \to TG_0 \oplus TG_1$ be the map which sends $\pi'(x)$ to the pair $(\pi_0(i_0^*(x)), \pi_1(i_1^*(x)))$. That I and K are well defined follows from standard diagram chases in Diagram 3.11. Note that Diagram 3.11 corresponds to Diagram 3.7 of Lemma 3.6 and extends that diagram. It is commutative and all sets of collinear arrows represent exact sequences of homomorphisms.

Diagram 3.11.

$$
\begin{tikzcd}
0 \arrow[dr] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & H_0 \oplus H_1 \arrow[r, "i_0\oplus i_1"] \arrow[d, "\hat\lambda\circ(i_0\oplus i_1)"'] \arrow[dr, "\hat\lambda_0\oplus-\hat\lambda_1"] & H \arrow[r, "\pi"] \arrow[d, "J"] \arrow[dl, "\hat\lambda"'] & TG \arrow[r] \arrow[dd, "I"] & 0 \arrow[d] & \\
0 \arrow[r] & H^* \arrow[r, "i_0^*\oplus i_1^*"] \arrow[d, "\pi'"'] & H_0^* \oplus H_1^* \arrow[rr, "\pi^\wedge"] \arrow[dr, "\pi_0\oplus\pi_1"] \arrow[dd] & & TG^\wedge \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & \hat\lambda(TG) \arrow[rr, "K"] \arrow[d] & & TG_0 \oplus TG_1 \arrow[r] \arrow[d] \arrow[dr] & TG^\wedge \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & \hat\lambda(TG)^\wedge \arrow[r, "Id"] \arrow[d] & \hat\lambda(TG)^\wedge \arrow[r] \arrow[d] & 0 & \\
& & 0 & 0 & &
\end{tikzcd}
$$

Let $\mathrm{pr}_i : TG_0 \oplus TG_1 \to TG_i$ is the projection onto the indicated factor and for $v \in H$ suppose that $\mathrm{pr}_0(I(\pi(v))) = 0$. By definition this means that there is some $v_0 \in H_0$ such that $\hat\lambda(v)|_{H_0} = \hat\lambda_0(v_0)$. That is,

$$\lambda(v_0, w) = \lambda(v, w)\ \forall w \in H_0.$$

It follows that $\lambda(v - v_0, w) = 0$ for all $w \in H_0$ and hence that $(v - v_0) \in H_0^\perp = H_1$. Therefore $v = v_0 + (v - v_0) = (i_0 \oplus i_1)(v_0, v - v_0)$ and hence $\pi(v) = 0 \in TG$. Hence $\mathrm{pr}_0 \circ I$ is injective and a completely similar argument shows the same for $\mathrm{pr}_1 \circ I$. Now by Lemma 3.8 applied to $\hat\lambda_0 \oplus -\hat\lambda_1 = J \circ (i_0 \oplus i_1)$ we see that there is a short exact sequence

$$0 \to TG \to TG_0 \oplus TG_1 \to \hat\lambda(TG)^\wedge \to 0.$$

Now $|\hat\lambda(TG)^\wedge| = |TG|$ and so $|TG|^2 = |TG_0| \cdot |TG_1|$. But since $\mathrm{pr}_i \circ I$ is injective $|TG_i| \geq |TG|$ and so $|TG| = |TG_i|$. We conclude that that $\mathrm{pr}_i \circ I$ is an isomorphism.

We show next that $\delta^c(\kappa_0) \oplus -\delta^c(\kappa_1)$ is identically zero on $I(TG) \subset TG_0 \oplus TG_1$. Given $v \in H$ there is a non-zero integer r and elements $v_i \in H_i$ for $i = 0, 1$ such that $v = r(v_0 + v_1)$. Thus, for any $w \in H_0$, $\lambda(v, w) = \lambda(rv_0, w) = r\lambda_0(v_0, w)$ and hence

$\hat{\lambda}(v)|_{H_0} = r\hat{\lambda}_0(v_0)$. Where as, for any $w \in H_1$, $\lambda(v,w) = \lambda(rv_1, w) = -r\lambda_1(v_1, w)$ and $\hat{\lambda}(v)|_{H_1} = r\hat{\lambda}_1(-v_1)$. We now calculate

$$\begin{aligned}(\delta^c(\kappa_0) \oplus -\delta^c(\kappa_1))[I(\pi(v))] &= [\hat{\lambda}_0^{-1}(\hat{\lambda}(v)|_{H_0}, \hat{\lambda}(v)|_{H_0} + \alpha_0) \\ &\qquad -\hat{\lambda}_1^{-1}(\hat{\lambda}(v)|_{H_1}, \hat{\lambda}(v)|_{H_1} + \alpha_1)]/2 \\ &= [\lambda(v, v_0) + \alpha_0(v_0) - \lambda(v, -v_1) + \alpha_1(-v_1)]/2r \\ &= [\lambda(v, v_0 + v_1) + \alpha(v_0 + v_1)]/2r \\ &= [\lambda(v,v) + \alpha(v)]/2.\end{aligned}$$

Now this last quantity is integral as $\kappa \in \mathcal{F}^c$ and α is characteristic for λ. Thus

$$(\delta^c(\kappa_0) \oplus -\delta^c(\kappa_1))[I(\pi(v))] = 0 \in \mathbb{Q}/\mathbb{Z}$$

for every $\pi(v) \in TG$. So it follows that

$$\theta_\kappa := (I \circ \mathrm{pr}_1) \circ (I \circ \mathrm{pr}_0)^{-1} : TG_0 \to TG_1$$

is an isometry from $\delta^c(\kappa_0)$ to $\delta^c(\kappa_1)$. In fact $I(TG) = \{([x_0], \theta_\kappa([x_0])) | [x_0] \in TG_0\} \subset TG_0 \oplus TG_1$ and since $\delta^c(\kappa_0) \oplus -\delta^c(\kappa_1)$ vanishes identically on $I(TG)$ we see that for all $[x_0] \in TG_0$

$$\delta^c(\kappa_0)([x_0]) = \delta^c(\kappa_1)(\theta_\kappa([x_0])).$$

(4.) Observe that $J(H) = \{(x_0, x_1) \in H_0^* \oplus H_1^* | \theta_\kappa([x_0]) = [x_1]\}$ which is the underlying abelian group of $\kappa_0 \cup_{\theta_\kappa} \kappa_1^-$. As above, for any $v \in H$ we obtain a non-zero integer r and $v_i \in H_i$, $i = 0, 1$, such that $v = r(v_0 + v_1)$, $\hat{\lambda}(v)|_{H_0} = r\hat{\lambda}_0(v_0)$ and $r\hat{\lambda}(v)|_{H_1} = r\hat{\lambda}_1(-v_1)$.

A similar calculation to preceding one yields

$$\begin{aligned}(\kappa_0 \cup_{\theta_\kappa} \kappa_1^-)(J(v)) &= (\kappa_0 \cup_{\theta_\kappa} \kappa_1^-)((\hat\lambda(v)|_{H_0}, \hat\lambda(v)|_{H_1}))\\ &= \hat\lambda_0^{-1}(\hat\lambda(v)|_{H_0}, \hat\lambda(v)|_{H_0} + \alpha_0) - \hat\lambda_1^{-1}(\hat\lambda(v)|_{H_1}, \hat\lambda(v)|_{H_1} + \alpha_1)\\ &= [\lambda(v_0, v) + \alpha_0(v_0)]/r - [\lambda(-v_1, v) + \alpha(-v_1)]/r\\ &= [\lambda(v_0, v) + \alpha(v_0) + \lambda(v_1, v) + \alpha(v_1)]/r\\ &= \lambda(v_0 + v_1, v)/r + \alpha(v_0 + v_1)/r\\ &= \lambda(v, v) + \alpha(v)\\ &= \kappa(v).\end{aligned}$$

So J is an isometry from κ to $\kappa_0 \cup_{\theta_\kappa} \kappa_1^-$. □

2. Stable equivalence of quadratic functions

In this section we shall establish that the induced family of quadratic linking functions is a complete stable invariant of quadratic functions. Our method of proof is inspired by Wilkens' proof of the analogous topological result (see Section 4.1).

LEMMA 3.12. *Let $\kappa_0(H_0, \lambda_0, \alpha_0)$ and $\kappa(H_1, \lambda_1, \alpha_1)$ be nondegenerate quadratic functions in the family $\mathcal{F}^*$. Suppose that θ is an isometry of the induced quadratic linking families, $\theta : \delta^*(\kappa_0) \cong \delta^*(\kappa_1)$. Then, there exists a nonsingular quadratic function μ_0 and an isometry $\Theta : \kappa_0 \oplus \mu_0 \cong \kappa_1 \oplus (\kappa_0 \cup_\theta \kappa_1^-)$ such that $\delta^*(\Theta) = \theta$.*

PROOF. Consider the four isometric embeddings:

$$\begin{array}{rcll} j_0 : \kappa_0 & \hookrightarrow & (\kappa_0 \cup \kappa_0^-) \\ v & \mapsto & (\hat\lambda_0(v), 0) \end{array} \qquad \begin{array}{rcll} j_1 : \kappa_0^- & \hookrightarrow & (\kappa_0 \cup \kappa_0^-) \\ v & \mapsto & (0, -\hat\lambda_0(v)) \end{array}$$

$$\begin{array}{rcll} i_0 : \kappa_0 & \hookrightarrow & (\kappa_0 \cup_\theta \kappa_1^-) \\ v & \mapsto & (\hat\lambda_0(v), 0) \end{array} \qquad \begin{array}{rcll} i_1 : \kappa_1^- & \hookrightarrow & (\kappa_0 \cup_\theta \kappa_1^-) \\ v & \mapsto & (0, -\hat\lambda_1(v)). \end{array}$$

We shall prove that there is a further isometric embedding $I : (\kappa_0 \cup \kappa_0^-) \hookrightarrow (\kappa_0 \cup_\theta \kappa_1^-) \oplus (\kappa_0 \cup_\theta \kappa_1^-)^-$, a nonsingular quadratic function μ_0 and an isometry $\Phi : (\kappa_0 \cup \kappa_0^-) \oplus \mu_0^- \cong$

$(\kappa_0 \cup_\theta \kappa_1^-) \cup (\kappa_0 \cup_\theta \kappa_1^-)^-$ such that the following diagram commutes.

$$\begin{tikzcd}
\kappa_0 \arrow[r, "j_0"] \arrow[dr, "i_0"] & (\kappa_0 \cup \kappa_0^-) \arrow[r, "\mathrm{Id}\oplus 0"] \arrow[dr, "I"] & (\kappa_0 \cup \kappa_0^-) \oplus \mu_0^- \arrow[d, "\Phi"] \\
 & (\kappa_0 \cup_\theta \kappa_1) \arrow[r, "\mathrm{Id}\oplus 0"] & (\kappa_0 \cup_\theta \kappa_1^-) \oplus (\kappa_0 \cup_\theta \kappa_1^-)^-
\end{tikzcd}$$

Let $\theta' : H_0^* \to H_1^*$ be any homomorphism extending θ. Recall that

$$(\kappa_0 \cup \kappa_0^-)(x_0, y_0) = \lambda_0^{-1}(x_0, x_0 + \alpha_0) - \lambda_0^{-1}(y_0, y_0 + \alpha_0)$$

on $\{(x_0, y_0) \in H_0^* \oplus H_0^* : \pi_0(x_0) = \pi_0(y_0)\}$. We define

$$I((x_0, y_0)) = [(x_0, \theta'(y_0)), (y_0, \theta'(y_0))]$$

and verify that $I : \kappa_0 \cup \kappa_0^- \hookrightarrow (\kappa_0 \cup_\theta \kappa_1^-) \oplus (\kappa_0 \cup_\theta \kappa_1^-)^-$ is an isometric embedding.

$$\begin{aligned}
&(\kappa_0 \cup_\theta \kappa_1^-) \oplus (\kappa_0 \cup_\theta \kappa_1^-)^-[I(x_0, y_0)] \\
&\qquad = \lambda_0^{-1}(x_0, x_0 + \alpha_0) - \lambda_0^{-1}(y_0, y_0 + \alpha_0) \\
&\qquad\qquad\qquad + \lambda_1^{-1}(\theta'(y_0), \theta'(y_0) + \alpha_1) - \lambda_1^{-1}(\theta'(y_0), \theta'(y_0) + \alpha_1) \\
&\qquad = (\kappa_0 \cup \kappa_0^-)([(x_0, y_0)]
\end{aligned}$$

Since $I(j_0(v)) = [(\hat{\lambda}_0(v), 0), (0, 0)] = (\mathrm{Id} \oplus 0)(i_0(v))$ the left side of the diagram commutes. Now $I(\kappa_0 \cup \kappa_0^-)$ is a nonsingular subform of $(\kappa_0 \cup_\theta \kappa_1^-) \oplus (\kappa_0 \cup_\theta \kappa_1^-)^-$ which is also nonsingular and hence there is a nonsingular form μ_0 and an isometry

$$\Phi : (\kappa_0 \cup \kappa_0^-) \oplus \mu_0^- \cong (\kappa_0 \cup_\theta \kappa_1^-) \oplus (\kappa_0 \cup_\theta \kappa_1^-)^-$$

which extends I. Thus the right side of the diagram commutes.

Now as Φ is an isometry $\Phi(j_0(\kappa_0)^\perp) = (\Phi(j_0(\kappa_0)))^\perp$. But $j_0(\kappa_0)^\perp = j_1(\kappa_0^-) \oplus \mu_0^-$ and $(\Phi(j_0(\kappa_0)))^\perp = (i_0(\kappa_0))^\perp = i_1(\kappa_1^-) \oplus (\kappa_0 \cup_\theta \kappa_1^-)^-$. We therefore define $\Theta_1 :=$

$\Phi|_{j_0(\kappa_0)^\perp}$ and then

$$\Theta_1 : j_1(\kappa_0^-) \oplus \mu_0^- \cong i_1(\kappa_1^-) \oplus (\kappa_0 \cup_\theta \kappa_1^-)^-$$

is an isometry. It follows that $\Theta := (i_1^{-1} \oplus \mathrm{Id}_{\kappa_0 \cup_\theta \kappa_1^-}) \circ \Theta_1 \circ (j_1 \oplus \mathrm{Id}_{\mu_0})$ is an isometry, $\Theta : \kappa_0 \oplus \mu_0 \cong \kappa_1 \oplus (\kappa_0 \cup_\theta \kappa_1^-)$. It remains to show that Θ induces θ. Let the underlying modules and forms of μ_0 and $\kappa_0 \cup_\theta \kappa_1^-$ be $(H_{\mu_0}, \lambda_{\mu_0})$ and $(H_\kappa, \lambda_\kappa)$ respectively. We must determine the right most vertical homomorphism in the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & H_0 \oplus H_{\mu_0} \arrow[r, "\hat\lambda_0 \oplus \hat\lambda_{\mu_0}"] \arrow[d, "\Theta"] & H_0^* \oplus H_{\mu_0}^* \arrow[r, "\pi_0"] \arrow[d, "(\Theta^*)^{-1}"] & TG_0 \arrow[r] \arrow[d, dotted, "?"] & 0 \\
0 \arrow[r] & H_1 \oplus H_\kappa \arrow[r, "\hat\lambda_1 \oplus \hat\lambda_\kappa"] & H_1^* \oplus H_\kappa^* \arrow[r, "\pi_1"] & TG_1 \arrow[r] & 0.
\end{tikzcd}$$

As both μ_0 and $\kappa_0 \cup_\theta \kappa_1^-$ are nonsingular $\pi_0|_{H_{\mu_0}^*}$ and $\pi_1|_{H_\kappa^*}$ are identically zero and so it suffices to consider $\Theta|_{H_0 \oplus 0} \circ \mathrm{pr}_{H_1}$. But $\Theta|_{H_0 \oplus 0} = (i_1^{-1} \oplus Id_{\kappa_0 \cup_\theta \kappa_1^-}) \circ I \circ j_1$ and hence $\Theta|_{H_0 \oplus 0} \circ \mathrm{pr}_{H_1} = \hat\lambda_1^{-1} \circ \theta' \circ \hat\lambda_0$. But now one verifies that $(\hat\lambda_1^{-1} \circ \theta' \circ \hat\lambda_0)^{*-1} = \theta'$. Hence Θ induces the same map on TG_0 as θ' which is θ as θ' was chosen to extend θ. We summarize this argument with the following subdiagram of the previous diagram

$$\begin{tikzcd}
0 \arrow[r] & H_0 \arrow[r, "\hat\lambda_0"] \arrow[d, "\Theta|_{H_0 \oplus 0}"'] & H_0^* \arrow[r, "\pi_0"] \arrow[d, "\theta'"] & TG_0 \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & H_1 \arrow[r, "\hat\lambda_1"] & H_1^* \arrow[r, "\pi_1"] & TG_1 \arrow[r] & 0.
\end{tikzcd}$$

□

Corollary 3.13. *Let $\kappa_0(H_0, \lambda_0, \alpha_0)$ and $\kappa(H_1, \lambda_1, \alpha_1)$ be quadratic functions in the family $\mathcal{F}^*$. Suppose that θ is an isometry of the induced quadratic linking families,*

$\theta : \delta^*(\kappa_0) \cong \delta^*(\kappa_1)$. *Then, there exist nonsingular quadratic functions* μ_0 *and* μ_1 *and an isometry* $\Theta : \kappa_0 \oplus \mu_0 \cong \kappa_1 \oplus \mu_1$ *which induces* θ.

PROOF. Choose a section $\Phi_0 \in \mathrm{Sec}(\tau_{G_0})$ and let $\Psi_i \in \mathrm{Sec}(\rho_i)$, $i = 0, 1$, be sections such that $\Phi(\Psi_0) = \Phi_0$ and $\Phi(\Psi_1) = \Phi_1 := \tau_\theta(\Phi_0)$. Then since θ is an isometry of quadratic linking families, $\theta|_{TG_0}$ is an isometry from $q^*(\kappa_0)(\Phi_0) = q^*(\kappa_0(\Psi_0))$ to $q^*(\kappa_1)(\Phi_1) = q^*(\kappa_1(\Psi_1))$. Applying Lemma 3.12 to the $\theta|_{TG_0}$ and setting $\mu_1 := (\kappa_0(\Psi_0) \cup_{\theta|_{TG_0}} \kappa_1(\Psi_1)^-)$ we conclude that there is a nonsingular quadratic function μ_0 and an isometry

$$\Theta_N : \kappa_0(\Psi_0) \oplus \mu_0 \to \kappa_1(\Psi_1) \oplus \mu_1$$

such that $\delta^*(\Theta_N) = \theta|_{TG_0}$. Now $\kappa_0 = \alpha|_{F_0} \oplus \kappa(\Psi_0)$ and $\kappa_1 = \alpha_1|_{F_1} \oplus \kappa(\Psi_1)$ and the isometry Θ_N is defined on $H(\Psi_0) \oplus H_{\mu_0}$ where H_{μ_0} is the domain of μ_0. Thus it remains to use θ to define an isometry from $\alpha_0|_{F_0}$ to $\alpha_1|_{F_1}$. By the definition of Φ_i, $\pi_i|_{F_i^*(\Psi_i)} : F_i^*(\Psi_i) \to \Phi_i(G_i^{**})$ is an isomorphism for $i = 0, 1$. Also, by the definition of τ_θ and the fact that $\Phi_1 = \tau_\theta(\Phi_0)$, $\theta(\Phi_0(G_0^{**})) = \Phi_1(G_1^{**})$. Moreover, there are isomorphisms, $R_{F_i} : F_i^*(\Psi_i) \to F_i^*$, $i = 0, 1$ defined by restricting the domain of homomorphisms in $F_i^*(\Psi_i)$

$$\begin{array}{rccc} R_{F_i}: & F_i^*(\Psi_i) & \to & F_i^* \\ & z & \mapsto & z|_{F_i}. \end{array}$$

We therefore define $\Theta_D : F_0 \to F_1$ so that Θ_D^{*-1} makes the following diagram (in which all morphisms are isomorphisms) commute.

$$\begin{tikzcd} F_0^* \arrow[d, dotted, "\Theta_D^{*-1}"] & F_0^*(\Psi_0) \arrow[l, "R_{F_0}"'] \arrow[r, "\pi_0|_{F_0^*(\Psi_0)}"] & \Phi_0(G_0^{**}) \arrow[d, "\theta|_{\Phi_0(G_0^{**})}"] \\ F_1^* & F_1^*(\Psi_1) \arrow[l, "R_{F_1}"'] \arrow[r, "\pi_1|_{F_1(\Psi_1)^*}"] & \Phi_1(G_1^{**}) \end{tikzcd}$$

Under the splittings $G_i = \Phi_i(G_i^{**}) \oplus TG_i$ and $H_i^* = F_i^*(\Psi_i) \oplus H_i^*(\Psi_i)$ the elements β_i and α_i are mapped to $(\Phi_i(\tau_{G_i}(\beta_i)), \beta_i - \Phi_i(\tau_{G_i}(\beta_i)))$ and $(p_{F_i}(\Psi)^*(\alpha|_{F_i}), \alpha(\Psi_i))$ respectively (see Sections 2.6 and 2.10). Now $\pi_i(\alpha_i) = \beta_i$, and so $\pi_i|_{F_i^*(\Psi_i)}(p_{F_i}(\Psi)^*(\alpha|_{F_i})) = \Phi_i(\tau_{G_i}(\beta_i))$. Since $\theta(\beta_0) = \beta_1$ and θ respects the splittings of G_0 and G_1, $\theta(\Phi_0(\tau_{G_0}(\beta_0))) = \Phi_1(\tau_{G_1}(\beta_1))$. Finally, as $R_{F_i}(p_{F_i}(\Psi)^*(\alpha|_{F_i})) = \alpha|_{F_i}$ it follows by chasing the diagram defining Θ_D that $\Theta_D^*(\alpha_1|_{F_1}) = \alpha_0|_{F_0}$. That is, Θ_D defines an isometry from $\alpha_0|_{F_0}$ to $\alpha_1|_{F_1}$. By definition $\delta^*(\Theta_D) = \theta|_{\Phi_0(G_0^{**})}$. We therefore define the isometry

$$\Theta := \Theta_D \oplus \Theta_N : \kappa_0 = [\alpha_0|_{F_0} \oplus \kappa_0(\Psi_0) \oplus \mu_0] \to [\alpha_1|_{F_1} \oplus \kappa_1(\Psi_1) \oplus \mu_1] = \kappa_1$$

and note that

$$\delta^*(\Theta) = \delta^*(\Theta_D) \oplus \delta^*(\Theta_N) = \theta|_{\Phi_0(G_0^{**})} \oplus \theta|_{TG_0} = \theta.$$

$\square$

Having assembled the necessary results we conclude this chapter with the proof of Theorem 3.4 which we restate for the reader's convenience.

THEOREM 3.4. *Let κ_0 and κ_1 be two quadratic functions in the same family $\mathcal{F}^*$. Then the following are equivalent.*

1. *κ_0 is stably $\mathcal{F}^*$-equivalent to κ_1, $\kappa_0 \sim_{\mathcal{F}^*} \kappa_1$.*
2. *The quadratic linking families induced by κ_0 and κ_1 are isometric, $\delta^*(\kappa_0) \cong \delta^*(\kappa_1)$.*
3. *κ_0 and κ_1^- are $\mathcal{F}^*$-orthogonal, $\kappa_0 \perp_{\mathcal{F}^*} \kappa_1^-$.*

PROOF. *(1.) $\Rightarrow$ (2.)* The argument should be familiar by now. Let μ_0 and $\mu_1 \in \mathcal{F}^*$ be nonsingular and $\Theta : \kappa_0 \oplus \mu_0 \to \kappa_1 \oplus \mu_1$ be an isometry. Then by Lemma 2.48, $\delta^*(\Theta)$ induces an isometry from $\delta^*(\kappa_0 \oplus \mu_0)$ to $\delta^*(\kappa_1 \oplus \mu_1)$. But for $i = 0, 1$, $\delta^*(\kappa_i \oplus \mu_i) = \delta^*(\kappa_i) \oplus \delta^*(\mu_i) = \delta^*(\kappa_i)$. The last equality holding because μ_i induces the trivial quadratic linking family because it is nonsingular. Hence $\delta^*(\Theta)$ defines an isometry from $\delta^*(\kappa_0)$ to $\delta^*(\kappa_1)$.

(2.) $\Rightarrow$ *(3.)* If $\theta : \delta^*(\kappa_0) \cong \delta^*(\kappa_1)$ is an isometry then by the proof of Corollary 3.13 there are sections $\Psi_0 \in \mathrm{Sec}(\rho_0)$ and $\Psi_1 \in \mathrm{Sec}(\rho_1)$ such that $\theta_{TG_0} : \kappa_0(\Psi_0) \cong \kappa_1(\Psi_1)$ and such that there is an isometry $\Theta_D : \kappa_0|_{F_0} \cong \kappa_1|_{F_1}$. We apply Lemma 3.6 to θ_{TG_0} and observe that $\kappa_0(\Psi_0)$ and $\kappa_1(\Psi_1)^-$ are orthogonal inside $\kappa_0(\Psi_0) \cup_{\theta|_{TG_0}} \kappa_1(\Psi_1)^-$. Hence κ_0 and κ_1 are orthogonal.

(3.) $\Rightarrow$ *(1.)* Suppose that $\kappa_0 \perp_{\mathcal{F}^*} \kappa_1^-$. This means firstly there is an isometry $\Theta_D : \kappa_0|_{F_0} \cong \kappa_1|_{F_1}$. Secondly it means that there are sections $\Psi_0 \in \mathrm{Sec}(\rho_0)$ and $\Psi_1 \in \mathrm{Sec}(\rho_1)$, a nonsingular quadratic function κ and embeddings $i_0 : \kappa_0(\Psi_0) \hookrightarrow \kappa$ and $i_1 : \kappa_1(\Psi_1)^- \hookrightarrow \kappa$ such that $i_1(\kappa_1(\Psi_1)^-) = i_0(\kappa_0(\Psi_0))^\perp$. We calculate that

$$\begin{aligned}
\delta^*(\kappa_0(\Psi_0)) &\cong \delta^*(i_0(\kappa_0(\Psi_0))) \\
&\cong \delta^*(i_0(\kappa_0(\Psi_0))^{\perp -}) \quad \text{By Lemma 3.10 } \mathit{3} \\
&\cong \delta^*(i_1(\kappa_1(\Psi_1)^-)^-) \\
&\cong -\delta^*(i_1(\kappa_1(\Psi_1)^-)) \\
&\cong -\delta^*(\kappa(\Psi_1)^-) \\
&\cong \delta^*(\kappa(\Psi_1)).
\end{aligned}$$

So by Lemma 3.12 $\kappa_0(\Psi_0)$ and $\kappa_1(\Psi_1)$ are stably $\mathcal{F}^*$-equivalent. Since $\kappa_0 = \kappa_0|_{F_0} \oplus \kappa_0(\Psi_0)$, $\kappa_1 = \kappa_1|_{F_1} \oplus \kappa_1(\Psi_1)$ and $\kappa_0|_{F_0} \cong \kappa_1|_{F_0}$ we deduce that κ_0 and κ_1 are stably $\mathcal{F}^*$-equivalent. $\square$

CHAPTER 4

Classification of the manifolds P

It is an immediate consequence of Corollary 2.26 and Theorem 3.4 that two highly connected manifolds P_0 and P_1 are almost diffeomorphic if and only if the quadratic linking families $Q(P_0)$ and $Q(P_1)$ are isometric. Corollary 2.26 asserts that P_0 and P_1 are almost diffeomorphic if and only if the stable quadratic functions $[\kappa(P_0)]$ and $[\kappa(P_1)]$ are equal and Theorem 3.4 asserts (in part) that $[\kappa(P_0)] = [\kappa(P_1)]$ if and only if $Q(P_0) \cong Q(P_1)$. In Section 2 of this chapter we prove the sharper result that every isometry $\theta : Q(P_0) \to Q(P_1)$ is realized as $f_!$ for some almost diffeomorphism $f : P_0 \to P_1$. Section 3 is an aside which explains the topology behind the key algebraic Lemmas 3.10 and 3.6. In Section 4 we consider the smooth classification of highly connected $(8j-1)$-manifolds.

1. Extensions of diffeomorphisms

In Section 8 we reported the following

THEOREM 2.24 (Wilkens' Thesis 3.2). *Let L_0 and L_1 be $2k$-dimensional handlebodies ($k > 2$) with boundaries $\partial L_0 = P_0$ and $\partial L_1 = P_1$. Suppose that $f : P_0 \to P_1$ is a diffeomorphism. Then there are handlebodies V_0, V_1 with boundaries the standard sphere and a diffeomorphism g which makes the following diagram commute.*

$$\begin{array}{ccc} P_0 & \xrightarrow{f} & P_1 \\ \downarrow & & \downarrow \\ L_0 \natural V_0 & \xrightarrow{g} & L_1 \natural V_1 \end{array}$$

We give Wilkens' proof of this result both for completeness and because the steps in the proof of Lemma 3.12 are algebraic analogues inspired by the geometric steps in Wilkens' proof.

Proof. To begin form the manifold $M = L_0 \cup_f -L_1$ by identifying points in $\partial L_0 = P_0$ and $\partial L_1 = P_1$ via f. Using Van Kampen's theorem and the Mayer-Vietoris sequence for $M = L_0 \cup -L_1$ we see that M is a closed highly connected $2k$-manifold. We wish to show that $M\# - M$ is the connected sum of the double of $L_0 = L_0 \cup_{Id} -L_0$ and some other highly connected manifold which will be called M_0. To do this we embed a 2^k-disc D^{2k} in L_1 which itself is embedded in M and let $N = (M - D^{2k}) \times [0,1]$. Then there is a diffeomorphism $F_1 : \partial N \cong M\# - M$. Now if we consider the pair (N, L_0) where L_0 is embedded as $L_0 \times 0 \hookrightarrow (M \times 0)$ then

$$\begin{aligned} H_l(N, L_0) &\cong H_l(M - D^{2k}, L_0) \text{ as } N \simeq M - D^{2k} \\ &\cong H_l(L_1 - D^{2k}, P_1) \text{ by excision} \\ &\cong \left\{ \begin{array}{cc} H_k(L_1, P_1) & l = k \\ 0 & \text{else} \end{array} \right\}. \end{aligned}$$

Thus N has a handlebody decomposition based on $L_0 \times [0,1]$ with k-handles attached, one for each element of a basis of $H_k(M - D^{2k}, L_0)$. The attaching maps for the cores of these handles are embeddings $S^{k-1} \hookrightarrow L_0 \times 1$ and as $L_0 \times 1$ is $(k-1)$-connected we may assume after isotopy that the images of the attaching maps all lie in another $2k$-disc $D_1^{2k} \hookrightarrow L_0 \times 1$. If follows that there is diffeomorphism $F_0 : \partial N \cong (L_0 \cup_{Id} -L_0)\# M_0$ where M_0 is the boundary of the manifold obtained by attaching the j-handles to $D_1^{2k} \subset D^{2k+1}$. Notice that since all the necessary isotopies occur in $L_0 \times 1$ that we have a fixed embedding of $L_0 = (L_0 \times 0) \hookrightarrow \partial N$. Hence if we define G to be the diffeomorphism $G := F_0^{-1} \circ F_1$ we have a commutative diagram of embeddings and diffeomorphisms

$$\begin{tikzcd}
 & ((L_0 \cup_{Id} -L_0) - D^{2j}) \arrow[r] & (L_0 \cup_{Id} -L_0)\#M_0 \arrow[dd, "G"] \\
L_0 \arrow[ur, "i_0"] \arrow[dr, "i_1"'] & & \\
 & ((L_0 \cup_f -L_1) - D^{2j}) \arrow[r] & (L_0 \cup_f -L_1)\# - M
\end{tikzcd}$$

such that $G|_{i_0(L_0)} = Id$. If follows that G defines a diffeomorphism

$$G|_{-L_0\#M_0} : -L_0\#M_0 \cong -L_1\# - M$$

of the complements of $i_0(L_0)$ and $i_1(L_0)$. Therefore $G|_{-L_0\#M_0}$ induces a diffeomorphism on the boundary which must be f since we have the following commutative square.

$$\begin{tikzcd}
\partial L_0 \arrow[r, "Id"] \arrow[d, "Id"'] & -\partial L_0 \arrow[d, "G|_{-\partial L_0}"] \\
\partial L_0 \arrow[r, "f"] & -\partial L_1
\end{tikzcd}$$

□

REMARK 4.1. Wilkens' theorem is, in particular, a result about bordism of diffeomorphisms which is a topic that has since received significant attention. In particular see Kreck [**Kr1**] and Quinn [**Q**]. One might hope to prove this theorem from a more general point of view by arguing first that the bordism of diffeomorphisms over P are trivial and then performing surgery on the cobounding diffeomorphism to make it a diffeomorphism of highly connected manifolds.

2. Almost smooth classification of the manifolds P

The following theorem combines the statements of Theorem B and 1.5.

THEOREM 4.2. *For every finitely generated abelian group G and every characteristic (resp. even) quadratic linking family Q defined on G, there is a highly connected $(8j-1)$-manifold P, $j = 1, 2$ (resp $j > 2$), such that $Q(P) \cong Q$. The quadratic linking family $Q(P)$ of a highly connected manifold P is a complete invariant of almost diffeomorphisms. That is, if P_0 and P_1 are two highly connected $(8j-1)$-manifolds then there is an almost diffeomorphism $f : P_0 \to P_1$ such that $f_! = \theta : H^{4j}(P_0) \cong H^{4j}(P_1)$ if and only if θ defines an isometry of quadratic linking families from $Q(P_0)$ to $Q(P_1)$.*

PROOF. Let $Q = (G, q^*, \beta)$ be a quadratic linking family and let Φ be any section in $\mathrm{Sec}(\tau_G)$. Then $G = \Phi(G^{**}) \oplus TG$ where TG is the torsion subgroup of G. In the characteristic case Corollary 22.36 ensures that there is a nondegenerate quadratic function $\kappa' = \kappa(H', \lambda', \alpha') \in \mathcal{F}^c$ and an isomorphism $\theta' : \mathrm{Cok}(\hat{\lambda}') \to TG$ such that $q^c(\kappa') = q^c(\Phi) \circ \theta$. We define $\kappa := \kappa' \oplus \tau_G(\beta) : H' \oplus G^* \to \mathbb{Z}$ and leave the reader to verify that $\theta \oplus \Phi : \mathrm{Cok}(\hat{\lambda}) \oplus G^{**} \to G$ defines an isometry from $\delta^c(\kappa)$ to Q. The argument in the even case is similar but applies instead Theorem 2.35 to deduce the existence of an nondegenerate even quadratic function such κ such that $q^{ev}(\kappa) \cong q^{ev}(\Phi)$.

Now let L_i be handlebodies such that $\partial L_i = P_i \# \Sigma_{P_i}$. Suppose that there is an isometry

$$\theta : Q(P_0) = \delta^*(\kappa(L_0)) \to \delta^*(\kappa(L_1)) = Q(P_1).$$

By Corollary 3.13 there is a nonsingular quadratic function μ_0 and an isometry

$$\Theta : \kappa(L_0) \oplus \mu_0 \to \kappa(L_1) \oplus \kappa_0 \cup_\theta \kappa_1^-$$

which induces θ, $\delta^*(\Theta) = \theta$. Since μ_0 is nonsingular the boundary of $L(\mu_0)$ is a homotopy sphere which we shall denote by Σ_0. Now, by Wall's classification of handlebodies Θ is realized by a diffeomorphism

$$g(\Theta) : L_0 \natural L(\mu_0) \to L_1 \natural L(\kappa_0 \cup_\theta \kappa_1^-).$$

Restricting g to the boundary of $L_0 \natural L(\mu_0)$ we obtain a diffeomorphism

$$f : P_0 \# \Sigma_{P_0} \# \Sigma_0 \to P_1 \# \Sigma_{P_1}$$

which defines an almost diffeomorphism $f' : P_0 \to P_1$ such that $f'_! = f_! = \theta$.

Conversely, suppose that $f : P_0 \to P_1$ be an almost diffeomorphism. By Proposition 2.1 there is a homotopy sphere Σ and a diffeomorphism $f' : P_0 \# \Sigma \to P_1$ which agrees with f outside a disc. Then $\Sigma_{P_1} = \Sigma_{P_0 \# \Sigma} = \Sigma_{P_0} \# \Sigma_\Sigma$ and there is a diffeomorphism

$$f'' : P_0 \# \Sigma \# \Sigma_{P_1} = P_0 \# \Sigma_{P_0} \# \Sigma \# \Sigma_\Sigma \to P_1 \# \Sigma_{P_1}.$$

Let L_Σ be a handlebody bounding $\Sigma \# \Sigma_\Sigma$. Applying Theorem 2.24 to $f'' : \partial(L_0 \natural L_\Sigma) \to \partial L_1$ we deduce that there are handlebodies V_0 and V_1 cobounding standard spheres and a diffeomorphism

$$h : L_0 \natural L_\Sigma \natural V_0 \to L_1 \natural V_1$$

which extends f''. The map induced by h, $\Theta(h) : \kappa(L_0) \oplus \kappa(L_\Sigma \natural V_0) \to \kappa(L_1) \oplus \kappa(V_1)$, is an isometry. By Lemma 2.48 it follows that

$$f''_! : \delta^*(\kappa(L_0)) \oplus \delta^*(\kappa(L_\Sigma \natural V_0)) \to \delta^*(\kappa(L_1)) \oplus \delta^*(\kappa(V_1))$$

is an isometry. However as V_0 and $L_\Sigma \natural V_0$ are both bounded by homotopy spheres, $\delta^*(\kappa(V_0))$ and $\delta^*(\kappa(L_\Sigma \natural V_1))$ are both the trivial quadratic linking family and therefore $f''_!$ defines an isometry from $\delta^*(\kappa(L_0)) = Q(P_0)$ to $\delta^*(\kappa(L_1)) = Q(P_1)$. Finally, $f''_! = f_!$ since we may assume that the two maps differ only on the interior of a disc. $\square$

3. The manifolds $L(\kappa_0 \cup_\theta \kappa_1^-)$ and $L_0 \cup_{f^o} -L_1$

Let P_0 and P_1 be the boundaries of handlebodies and let $f : P_0 \to P_1$ be an almost diffeomorphism. Recall that by Proposition 2.1 that there is a homotopy sphere Σ, a disc $D^{8j-1} \hookrightarrow P_0$ and a diffeomorphism $f' : P_0 \# \Sigma \to P_1$ such that $f'|_{P_0 - D^{8j-1}} = f|_{P_0 - D^{8j-1}}$ (where the connected sum of Σ and P_0 is understood to occur inside D^{8j-1}). We define f^o to be the restriction $f^o := f|_{P_0 - D^{8j-1}}$ which is a diffeomorphism onto its image.

LEMMA 4.3. *Let $L_0 = L(\kappa_0)$ and $L_1 = L(\kappa_1)$ be handlebodies with associated nondegenerate quadratic functions $\kappa_0 = \kappa(H_0, \lambda_0)$ and $\kappa_1 = \kappa(H_1, \lambda_1)$. Let P_0 and P_1 be the boundaries of L_0 and L_1, let $f : \partial P_0 \to \partial P_1$ be an almost diffeomorphism which induces $f_! = \theta : H^{4j}(P_0) \to H^{4j}(P_1)$ and let Σ and f^o be as above. Then $\theta : Q(P_0) \cong Q(P_1)$ is an isometry of quadratic linking families and $L_0 \cup_{f^o} -L_1$ is diffeomorphic to $\kappa_0 \cup_\theta \kappa_1^-$.*

PROOF. The map f^o is a diffeomorphism onto its image and, after perhaps rounding corners, the adjunction space $L := L_0 \cup_{f^o} -L_1$ is a smooth manifold with boundary. The boundary of L is obtained by gluing the disc $D^{8j-1} \subset P_0$ to a disc D_1^{8j-1} which is the complement of the range of f^0. Thus ∂L is a homotopy sphere. Now let P^o be the subset $P_0 - D^{8j-1} = f^o(P_0 - D^{8j-1}) \subset P_1$ contained in $L := L_0 \cup_{f^o} -L_1$. Applying Van-Kampen's theorem and the Mayer-Vietoris sequence to the decomposition of L into the triple (L_0, L_1, P^o) we conclude that L is $(4j-1)$ connected. It follows that L falls under Wall's classification of handlebodies and we let $\kappa = \kappa(H, \lambda, \alpha)$ be the quadratic function of L. We will show that κ is isometric to $\kappa_0 \cup_\theta \kappa_1^-$. There are embeddings of $i_0 : H_0 \hookrightarrow H$ and $i_1 : H_1 \hookrightarrow H$ embedding κ_0 and κ_1^- respectively into κ corresponding to the obvious embeddings of L_0 and $-L_1$ into $L_0 \cup_{f^o} -L_1$. We claim that that $\kappa_0^\perp = \kappa_1^-$ because firstly it is clear that spheres in L_0 representing homology classes do not intersect spheres in L_1 and vice versa. Secondly, suppose that $\bar{v} : S^{4j} \hookrightarrow L$ represents $v \in H_{4j}(L)$ and that $\lambda(v, v_0) = 0$ for every $v_0 \in i_0(H_0)$. It follows via the Whitney trick that $\bar{v}(S^{4j})$ may be isotoped away from all the handles of L_0 and thence into $-L_1$. Therefore $v \in i_1(H_1)$. Applying Lemma 3.10 we conclude that $\kappa \cong \kappa_0 \cup_{\theta_\kappa} \kappa_1^-$ and that

$$\theta_\kappa : Q(P_0) = \delta^*(\kappa_0) \to \delta^*(\kappa_1) = Q(P_1)$$

is an isometry of quadratic linking families. It remains therefore to demonstrate that $\theta_\kappa = \theta$.

From the proof of Lemma 3.10 recall the projections $\pi_i : H_i^* \to \mathrm{Cok}(\hat{\lambda}_0)TG_0 = H^{4j}(P_0)$, $i = 0, 1$ and recall that $\theta_\kappa : TG_0 \to TG_1$ is the unique isomorphism such that $(\pi_0 \oplus \pi_1)(i_0^* \oplus i_1^*)(H^*) \subset TG_0 \oplus TG_1$ is the graph of θ_κ. Now let $j_0 : P^o \hookrightarrow L_0$ and $j_1 : P^o \hookrightarrow -L_1$ denote the inclusions. The usual Mayer-Vietoris arguments show that there is a long exact sequence

$$0 \longrightarrow H^{4j}(L) \xrightarrow{i_0^* \oplus i_1^*} H^{4j}(L_0) \oplus H^{4j}(-L_1) \xrightarrow{j_0^* - j_1^*} H_{4j-1}(P^o) \longrightarrow 0.$$

Identifying $H^{4j}(P^o) = H^{4j}(P_0 - D^{8j-1})$ we see that $(j_0^* - j_1^*)(x_0, x_1) = [x_0] - f^*[x_1]$. Hence

$$\begin{aligned}(\pi_0 \oplus \pi_1)(i_0^* \oplus i_1^*)(H^*) &= \{([x_0], [x_1]) | f^{*-1}([x_0]) = [x_1]\} \\ &= \{([x_0], [x_1]) | \theta([x_0]) = [x_1]\}\end{aligned}$$

and $\theta = \theta_\kappa$ as required. □

Given the manifold $L(\kappa_0 \cup_\theta \kappa_1^-)$ with boundary Σ we let D^{8j} denote the almost smooth disc obtained by taking the cone of Σ. Hence D^{8j} is a PL manifold equipped with a smooth structure except at the cone point which we denote by p_0. Now form $M(\kappa_0 \cup_\theta \kappa_1^-) := L(\kappa_0 \cup_\theta \kappa_1^-) \cup_\Sigma D^{8j}$ which is an almost smooth manifold with singular point p_0.

LEMMA 4.4. *Let $L_0 = L(\kappa_0)$ and $L_1 = L(\kappa_1)$ be handlebodies associated to non-degenerate quadratic functions $\kappa_0 = \kappa(H_0, \lambda_0, \alpha_0)$ and $\kappa_1 = \kappa(H_1, \lambda_1, \alpha_1)$. If $\theta : \delta^*(\kappa_0) \cong \delta^*(\kappa_1)$ is an isomorphism of the induced quadratic linking families then*

1. *there is an embedding of the disjoint union $L(\kappa_0) \sqcup -L(\kappa_1) \hookrightarrow M(\kappa_0 \cup_\theta \kappa_1^-)$ with complement an almost smooth h-cobordism from ∂L_0 to $-\partial L_1$,*

2. *there is an almost diffeomorphism* $f(\theta) : \partial L_0 \cong \partial L_1$ *realizing* θ *on homology,*
3. *the manifold* $M(\kappa_0 \cup_\theta \kappa_1^-)$ *is almost diffeomorphic to the manifold* $L_0 \cup_{f(\theta)} -L_1$ *formed by gluing* L_0 *to* $-L_1$ *with* $f(\theta)$.

PROOF. *1.* Recall that the group $H = \{(x_0, x_1) \in H_0^* \oplus H_1^* | \theta([x_0]) = [x_1]\}$ is the domain $\kappa_0 \cup_\theta \kappa_1^-$ and that there are isometric embeddings

$$\begin{array}{rccc} i_0 : & H_0 & \hookrightarrow & H \\ & v_0 & \mapsto & (\hat{\lambda}_0(v_0), 0) \end{array} \qquad \begin{array}{rccc} i_1 : & H_1 & \hookrightarrow & H \\ & v_1 & \mapsto & (0, -\hat{\lambda}_1(v_1)). \end{array}$$

from Lemma 3.6. For $j = 0, 1$, the group $i_j(H_j)$ is a summand of H for it is the kernel of the homomorphism $H \to H_j^*$ which sends a pair $(x_0, x_1) \in H$ to $x_j \in H_j^*$. Thus if $\{v_{01}, \ldots, v_{0m}\}$ and $\{v_{11}, \ldots, v_{1n}\}$ are bases for H_0 and H_1 respectively, then there are bases $\{y_1, \ldots, y_N\}$ and $\{z_1, \ldots, z_N\}$ for H such that $i_0(v_{0j}) = y_j$ for $1 \le j \le l$ and $i_1(v_{1j}) = z_j$ for $1 \le j \le m$. According to Wall's Theorem 2.4 the handlebody $L := L(\kappa_0 \cup_\theta \kappa_1^-)$ has a presentation for each of the bases $\{y_i\}$ and $\{z_i\}$. In fact, Wall proved the following. Let $D^{8j} \hookrightarrow L$ be a fixed disc in the interior of L and let K be the closure of $L - D^{8j}$. Then

$$H_{4j}(L) \cong H_{4j}(L, D^{8j}) \cong H_{4j}(K, \partial D^{8j}) \cong \pi_{4j}(K, \partial D^{8j}).$$

If $\{z_1', \ldots, z_n'\}$ is any base for $H_{4j}(L)$ then for each z_i' there are pairwise disjoint embeddings of $\bar{h}_i : (D^{4j}, \partial D^{4j}) \hookrightarrow (K, \partial D^{8j})$ which are transverse to ∂D^{8j} along $\bar{h}_i(\partial D^{4j})$ and which represent the element in $\pi_{4j}(K, \partial D^{8j})$ corresponding to z_i'. These embeddings can be thickened to handles $h_i : (D^{4j} \times D^{4j}, \partial D^{4j} \times D^{4j}) \hookrightarrow (K, \partial D^{8j})$ and the resulting handle body $L' \subset L$ obtained from the union of D^{8j} and the union of the images of the h_i is diffeomorphic to L. In fact $L - L'$ is an h-cobordism from ∂L to $\partial L'$. We now choose disjoint discs $D_0^{8j} \hookrightarrow L$ and $D_1^{8j} \hookrightarrow L$ one for each basis $\{y_i\}$ and $\{z_i\}$ and attach handles h_{0i} and h_{1j} to these discs as above but only for basis elements $\hat{\lambda}_0(v_{0j})$ and $\hat{\lambda}_1(v_{1j})$. The manifolds $D_0^{8j} \cup (\cup_{i=1}^l h_{0i})$ and $D^{8j} \cup (\cup_{j=1}^m h_{1j})$ thus define embeddings of L_0 and $-L_1$ into L. Recall that λ denotes the intersection form of L and that

$$\lambda[(\hat{\lambda}_0(v_0), 0), (0, -\hat{\lambda}_1(v_1)] = 0$$

for every $v_0 \in H_0$ and $v_1 \in H_1$. It follows that the handles h_{0i} and h_{1j} have algebraic intersection zero and so by the Whitney trick they may be isotoped to have empty geometric intersection. We may therefore assume that $L(\kappa_0)$ and $-L(\kappa_1)$ are disjointly embedded into $L(\kappa_0 \cup_\theta \kappa_1^-) \subset M(\kappa_0 \cup_\theta \kappa_1^-)$ and we use the same symbols to denote the disjointly embedded manifolds.

Let $N := M(\kappa_0 \cup_\theta \kappa_1^-) - \mathrm{Int}(L_0 \sqcup -L_1)$ be the complement of the interior of $L_0 \sqcup -L_1$ inside $M(\kappa_0 \cup_\theta \kappa_1^-)$ which is a codimension zero submanifold of M. We orient N compatibly with M and then $\partial N = \partial L_0 \sqcup -\partial L_1$. Let $J : L_0 \sqcup -L_1 \hookrightarrow M$, $i_N : N \hookrightarrow M$ and $i_{\partial N} : \partial N \hookrightarrow L_0 \sqcup -L_1$ denote the obvious inclusions and $E^* : H^*(M, L_0 \sqcup -L_1) = H^*(N, \partial N)$ the excision isomorphisms. The long exact cohomology sequences of the pairs $(N, \partial N)$ and $(M, L_0 \sqcup -L_1)$ are linked in the following commutative diagram (where we abbreviate $L_0 \sqcup -L_1$ to $L_{0\sqcup 1}$).

$$\begin{tikzcd}
H^0(N,\partial N) \arrow[r] & H^0(N) \arrow[r] & H^0(\partial N) \arrow[r] & H^1(N,\partial N) \arrow[r] & H^1(N) \\
H^0(M, L_{0\sqcup 1}) \arrow[r] \arrow[u, "E^0"] & H^0(M) \arrow[r, "J^0"] \arrow[u, "i^0_N"] & H^0(L_{0\sqcup 1}) \arrow[r] \arrow[u, "i^0_{\partial N}"] & H^1(M, L_{0\sqcup 1}) \arrow[r] \arrow[u, "E^1"] & H^1(M) \arrow[u, "i^0_N"] \\
0 \arrow[r] \arrow[u, "="] & \mathbb{Z} \arrow[r] \arrow[u, "="] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[u, "="] & \mathbb{Z} \arrow[r] \arrow[u, "="] & 0 \arrow[u, "="]
\end{tikzcd}$$

As E^0, $I^0_{\partial N}$ and E^1 are all isomorphisms we conclude that i^0_N is an isomorphism and hence that N is connected. A simple application of Van Kampen's theorem shows that N is also simply connected. Applying Poincaré duality to the manifold N we have that $H_{8j-*}(N) \cong H^*(N, \partial N) \cong H^*(M, L_0 \sqcup -L_1)$. We may calculate the groups $H^*(M, L_0 \sqcup -L_1)$ again using the long exact cohomology sequence of the pair

$(M, L_0 \sqcup -L_1)$. This sequence is zero below dimension $(8j-1)$ except in dimensions $4j$ and $4j+1$. The following diagram shows the relative cohomology sequences for the pairs $(M, L_0 \sqcup -L_1)$ and $(N, \partial N)$ on the left. On the right it shows an isomorphic sequence extending the right hand side of Diagram 3.7 in Lemma 3.6 from where we recall the projections $\pi_0 : H_0^* \to TG_0$, $\pi_1 : H_1^* \to TG_1$, $\pi : H \to TG = \{([x_0], \theta[x_0]\} \subset TG_0 \oplus TG_1$ and $\Pi : H_0^* \oplus H_1^* \to TG^\wedge$.

Diagram 4.5.

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
H^{4j}(M) \arrow[r, "i_N^*"] \arrow[d, "J^*"] & H^{4j}(N) \arrow[d, "I^*"] & H \arrow[r, "\pi"] \arrow[d, "i"] & TG \arrow[d] \\
H^{4j}(L_0 \sqcup -L_1) \arrow[r, "J_0^* \oplus J_1^*"] \arrow[d, "\delta_1"'] & H^{4j}(\partial N) \arrow[d, "\delta_2"'] & H_0^* \oplus H_1^* \arrow[r, "\pi_0 \oplus \pi_1"] \arrow[d, "\Pi"] & TG_0 \oplus TG_1 \arrow[d] \\
H^{4j+1}(M, L_0 \sqcup -L_1) \arrow[r, "E^{4j+1}"] \arrow[d] & H^{4j+1}(N, \partial N) \arrow[d] & TG^\wedge \arrow[r] \arrow[d] & TG^\wedge \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}
$$

Applying Poincaré duality and the universal coefficient theorem for torsion groups we conclude that

$$
H^*(N) \cong \left\{ \begin{array}{ll} \mathbb{Z} & * = 0, 8j-1 \\ TG & * = 4j \\ 0 & \text{else.} \end{array} \right\}
$$

With the identifications $H^{4j}(\partial L_0) = TG_0$, $H^{4j}(-\partial L_1) = TG_1$, $H^{4j}(N) = TG$ and $H^{4j}(\partial N)TG_0 \oplus TG_1$ and the notations $I_0 : \partial L_0 \hookrightarrow N$ and $I_1 : -\partial L_1 \hookrightarrow N$ for the standard inclusions we conclude also that for $x = ([x_0], \theta[x_0]) \in H^{4j}(N)$, $I_0^*(x) = [x_0]$ and $I_1^*(x) = \theta[x_0]$. Given our calculation of $H^*(N)$ it follows that both I_0 and I_1 are I_0 and I_1 are (co)homology equivalences of simply connected manifolds. So by Whitehead's theorem both I_0 and I_1 are homotopy equivalences. Thus N is an h-cobordism from ∂L_0 to $-\partial L_1$. It is an almost smooth manifold since the singular point of M in contained in N.

(2.) The almost smooth manifold N has one singular point with associated exotic sphere $\Sigma := \partial L(\kappa_0 \cup_\theta \kappa_1^-)$. As described in Section 1 we may construct a smooth h-cobordism from $\partial L_0 \# \Sigma$ to ∂L_1 by thickening a path from a point on Σ to ∂L_0 and then cutting out this thickened path to produce $N' \subset N$. We note that cutting out this path has no effect on the $4j$th cohomology groups of the manifolds involved. We denote the inclusions of the boundary components of the modified manifold by $I_0' : \partial L_0 \# \Sigma \hookrightarrow N'$ and $I_1' : -\partial L_1 \hookrightarrow N'$. Now let $F : N' \to (\partial L_0 \# \Sigma) \times [0, 1]$ be a diffeomorphism which restricts to the identity on $\partial L_0 \# \Sigma$. The diffeomorphism at the other end of N' is a diffeomorphism $F_1 := F \circ I_1' : -\partial L_1 \to -(L_0 \# \Sigma)$. Given $x \in H^{4j}(N')$ we have from above that $I_1'^*(x) = \theta I_0'^*(x)$. But since $I_0' \circ F|_{\partial L_0 \# \Sigma} = Id_{\partial L_0 \# \Sigma}$ we calculate that

$$\begin{aligned}
F_1^*(I_0'^*(x)) &= (F \circ I_1')^*((I_0')^*(x)) \\
&= I_1'^*(F^*(I_0'^*(x))) \\
&= I_1'^*((I_0' \circ F_{\partial L_0 \# \Sigma})^*(x)) \\
&= I_1'^*(x) \\
&= \theta(I_0^{*'}(x)).
\end{aligned}$$

Thus F_1 induces θ on cohomology. Therefore we take $f(\theta)$ to be the map $F_1^{-1} : \partial L_0 \# \Sigma \to \partial L_1$ which we may consider as an almost diffeomorphism from ∂L_0 to

∂L_1 with one singular point whose associated exotic sphere is Σ. The isomorphism $f(\theta)_! = (F_1^{-1})^{*-1} = F_1^* = \theta$.

(3.) The almost diffeomorphism $f(\theta)$ has one singular point so let $f^o = f(\theta)|_{\partial L_0 - D^{8j-1}}$. By Lemma 4.3 the adjunction space $L_0 \cup_{f^o} -L_1$ space is diffeomorphic to $L(\kappa_0 \cup_\theta \kappa_1^-)$. If we now form the adjunction space $L_0 \cup_{f(\theta)} -L_1$ then the effect is to identify the two discs from which the homotopy sphere $\Sigma = \partial(L_0 \cup_{f^o} -L_1)$ is formed. But this has the same effect as coning off Σ to produce $M(\kappa_0 \cup_\theta \kappa_1^-)$. Hence $M(\kappa_0 \cup_\theta \kappa_1^-)$ and $L_0 \cup_{f(\theta)} -L_1$ are diffeomorphic except at one singular point. □

Remark 4.6. Note that for highly connected rational homology spheres bounding handlebodies Lemmas 4.3 and 4.4 together give a proof that the quadratic linking function is a well defined and complete invariant of almost diffeomorphisms. This second proof is independent of Theorem 2.24. Its engine is the h-cobordism theorem which is the heart of Lemma 4.3. We note also that with a little care this alternate proof could be extended to all highly connected $(8j-1)$-manifolds and thus give a second proof of Theorem 4.2 which is independent of Theorem 2.24.

4. Smooth classification of the manifolds P

Recall Corollary 2.26 which states that a highly connected $(8j-1)$-manifold P is classified up to almost diffeomorphism by its stable quadratic function $[\kappa(P)]$. We begin this section by defining a smooth stable quadratic function for P and showing that it is a complete diffeomorphism invariant of those P which bound handlebodies. We shall call a quadratic function $\mu \in \mathcal{F}^{8j}$ *spherical* if and only if the boundary of $L^{8j}(\mu)$ is diffeomorphic to the standard sphere. Now we have already noted in the proof of Lemma 2.16 that every homotopy sphere Σ bounding a handlebody lies in bP_{8j} and by Theorem 2.55, Σ is standard if and only if $s_1(\Sigma) = 0 \in \mathbb{Q}/\mathbb{Z}$. We can therefore characterize spherical quadratic functions completely algebraically as follows.

LEMMA 4.7. *A quadratic function $\mu(H, \lambda, \alpha) \in \mathcal{F}^{8j}$ is spherical if and only if it is nonsingular and*

$$s_1(\partial L(\mu)) = (1/|bP_{8j}|)\left(\tilde{S}_{(j,2j)}\lambda^{-1}(\alpha, \alpha) - \frac{1}{8}\sigma(\mu)\right) = 0 \in \mathbb{Q}/\mathbb{Z}.$$

We shall call two quadratic functions κ_0 and κ_1 in $\mathcal{F}^{8j}$ *smoothly* equivalent if there are spherical quadratic functions μ_0 and μ_1 in $\mathcal{F}^{8j}$ and an isometry $\Theta : \kappa_0 \oplus \mu_0 \cong \kappa_1 \oplus \mu_1$. It is not difficult to see that smooth equivalence is in fact an equivalence relation. So given a highly connected $(8j-1)$-manifold we define the smooth quadratic function of P, $[\kappa(P)]^s_{\mathcal{F}^{8j}}$, to be the smooth equivalence class of $\kappa(L)$ where L is any handlebody such that $\partial L = P \# \Sigma_P$.

THEOREM 4.8. *Two highly connected $(8j-1)$-manifolds P_0 and P_1 are diffeomorphic if and only if*

$$\Sigma_{P_0} \cong \Sigma_{P_1} \quad \textit{and} \quad [\kappa(P_0)]^s_{\mathcal{F}^{8j}} = [\kappa(P_1)]^s_{\mathcal{F}^{8j}}.$$

PROOF. That the manifolds $P_0 \# \Sigma_{P_0}$ and $P_1 \# \Sigma_{P_1}$ are diffeomorphic if and only if $[\kappa(P_0)]^s_{\mathcal{F}^{8j}} = [\kappa(P_1)]^s_{\mathcal{F}^{8j}}$ follows from Wilkens' Theorem 2.28 and Wall's classification of handlebodies in a manner completely analogous to Corollary 2.26. The theorem now follows immediately. □

The determination of smooth equivalence for nondegenerate quadratic functions is a straightforward matter and this allows us to give a more explicit diffeomorphism classification for rational homology spheres.

THEOREM 4.9. *Highly connected $(8j-1)$-dimensional rational homology spheres P_0 and P_1 are diffeomorphic if and only if*

$$Q(P_0) \cong Q(P_1), \quad s_1(P_0) = s_1(P_1) \quad \textit{and} \quad \Sigma_{P_0} \cong \Sigma_{P_1}.$$

Moreover, if P_0 and P_1 are diffeomorphic then every isometry $\theta : Q(P_0) \to Q(P_1)$ is realized by a diffeomorphism.

Proof. By Theorem B, P_0 and P_1 are almost diffeomorphic if and only if $Q(P_0) \cong Q(P_1)$. So we may suppose that there is a homotopy sphere Σ such that P_0 is diffeomorphic to $P_1 \# \Sigma$. But then

$$s_1(\Sigma) = s_1(P_0) - s_1(P_1) \text{ and } \Sigma_\Sigma \cong \Sigma_{P_0} - \Sigma_{P_1}$$

and the first assertion follows since Σ is diffeomorphic to the standard sphere if and only if $s_1(\Sigma) = 0$ and $\Sigma_\Sigma \cong S^{8j-1}$.

Now suppose that P_0 and P_1 are diffeomorphic and that $\theta : Q(P_0) \to Q(P_1)$ is an isometry. By Lemma 3.12 there is a nonsingular quadratic function $\mu_0 \in \mathcal{F}^{8j}$ and an isometry

$$\Theta : \kappa_0 \oplus \mu_0 \to \kappa_1 \oplus \kappa_0 \cup_\theta \kappa_1^-$$

where $\kappa_i = \kappa(L_i)$ and L_i is a handlebodies with $\partial L_i = P_i \# \Sigma_{P_i}$, $i = 0, 1$. Of course by Wall's classification of handlebodies, Θ is realized by a diffeomorphism $g : L_0 \natural L(\mu_0) \cong L_1 \natural L(\kappa_0 \cup_\theta \kappa_1^-)$ and so it suffices to show that μ_0 and $\kappa_0 \cup_\theta \kappa_1^-$ are spherical. By Lemma 4.7 this means showing that $s_1(\partial L(\mu_0)) = 0 = s_1(\partial L(\kappa_0 \cup_\theta \kappa_1^-))$. But now if $\kappa_0 \cup_\theta \kappa_1^- = \kappa(H, \lambda, \alpha)$ and $\kappa_i = \kappa(H_i, \lambda_i, \alpha_i)$ then

$$\sigma(\kappa_0 \cup_\theta \kappa_1^-) = \sigma(\kappa_0) - \sigma(\kappa_1) \text{ and } \lambda^{-1}(\alpha, \alpha) = \lambda_0^{-1}(\alpha_0, \alpha_0) - \lambda_1^{-1}(\alpha_1, \alpha_1).$$

It follows immediately that

$$s_1(\partial L(\kappa_0 \cup_\theta \kappa_1^-)) = s_1(\partial L_0) - s_1(\partial L_1) = s_1(P_0) - s_1(P_1) = 0$$

and then we have

$$s_1(\partial L(\mu_0)) = s_1(\partial L_1) + s_1(\partial(L(\kappa_0 \cup_\theta \kappa_1^-)) - s_1(\partial L_0) = 0.$$

□

The question of the smooth equivalence of degenerate quadratic functions is a difficult algebraic problem. It is equivalent to calculating the the inertia group of the boundaries of the corresponding handlebodies. The inertia group of a manifold M, $I(M)$ is the subgroup of the group of homotopy spheres for which $\Sigma \# M$ is diffeomorphic to M. If P is a highly connected $(8j-1)$-manifold then $I(P) \subset bP_{8j}$ for if $[\Sigma] \neq 0 \in \Omega^{HC}_{8j-1}$ then $[P] \neq [P \# \Sigma]$ and so P and $P \# \Sigma$ cannot be diffeomorphic. If in addition P is a rational homology sphere then $I(P) = 0$ for in this case s_1 is defined for P and detects the addition of any exotic sphere in bP_{8j}. In cite [**Wi1**] and [**Wi3**] Wilkens investigated the inertia groups of highly connected manifolds of dimension 7 and 15 and discovered them to be quite subtle. In particular, he found examples where $I(P_0) = 0 = I(P_1)$ but $I(P_0 \# P_1) \neq 0$. Wilkens concluded [**Wi3**] with a conjecture concerning the inertia groups of highly connected manifolds of dimension 7 and 15 which we have been unable to prove or disprove.

CHAPTER 5

Linking forms and linking functions

We have now classified highly connected manifolds in dimension 7 and 15 up to almost diffeomorphism by their quadratic linking functions which need not be homogeneous. Thus the existence of bundles of Hopf invariant one leads us to a somewhat more general class of quadratic object in order to classify manifolds. In this chapter we review the classifications of symmetric bilinear forms on finite abelian groups started by Wall and finished by Kawauchi and Kojima. We next recall Nikulin's classification of quadratic linking forms refining a given bilinear form and then generalize Nikulin's result to all quadratic linking functions. Throughout this chapter G shall denote a finite abelian group.

1. Bilinear forms on finite abelian groups

Recall that a nondegenerate symmetric bilinear form b on a finite abelian group G is called a *linking form* (G, b). Two linking forms (G_0, b_0) and (G_1, b_1) are isometric if there is an isomorphism of abelian groups $\theta : G_0 \cong G_1$ such that $b_0 = b_1 \circ (\theta \times \theta)$. The block sum of two linking forms (G_1, b_1) and (G_2, b_2) is defined in the obvious way to be $(G_1 \oplus G_2, b_1 \oplus b_2)$. If H is a sub-group of G we let $H^{\perp}$ denote the annihilator of H,

$$H^{\perp} = \{g \in G | b(g, h) = 0 \ \forall h \in H\}.$$

In Section 7 we promised a proof of the following fundamental lemma which (according to Wall [**Wa3**]) is well know and may be found in [**Wa3**].

LEMMA 5.1. *Let (G, b) be a linking form and H a subgroup of G such that $b|_{H\times H} : H \times H \to \mathbb{Q}/\mathbb{Z}$ is nonsingular. Then $(H^{\perp}, b|_{H^{\perp}\times H^{\perp}})$ is nonsingular and (G, b) splits as the block sum $(H, b|_{H\times H}) \oplus (H^{\perp}, b|_{H^{\perp}\times H^{\perp}})$.*

PROOF. Let $R_H : G^{\wedge} \to H^{\wedge}$ be the map which restricts the domain of an element in the torsion dual of G to the subgroup H. Now consider the composition of the adjoint map of b with R_H,

$$R_H \circ \hat{b} : G \to H^{\wedge}.$$

By definition $H^{\perp}$ is the kernel of $R_H \circ \hat{b}$ and by hypothesis $(R_H \circ \hat{b})|_H$ is an isomorphism. Thus $H \cap H^{\perp} = \{e\}$ and H and $H^{\perp}$ generate G which is therefore the direct sum of H and $H^{\perp}$. As $b(h, h') = 0$ for all $h \in H$ and $h' \in H^{\perp}$ the form b splits over $H \oplus H^{\perp}$ and it follows that $b|_{H^{\perp}}$ must be nonsingular for if it were singular then b would also be singular. □

Let p be a prime. Every linking form splits as a block sum over the p-primary components of its domain. We therefore let $\mathcal{N}_p$ denote the semi-group of linking forms on p-groups under block sum and $\mathcal{N}$ denote abelian semi-group of all linking forms under block sum. Hence $\mathcal{N} \cong \bigoplus_p \mathcal{N}_p$. The structure of linking forms for p an odd prime is relatively simple. In this case every linking form splits as the block sum of linking forms over cyclic groups $\mathbb{Z}/p^k$. The following notation comes from Wall [**Wa3**] and Kawauchi and Kojima [**KK**]. The form, $A^k_p(\theta)$ denotes the unique (up to isometry) linking form $(\mathbb{Z}/p^k, b)$ with $b(e, e) = \theta$ where e is a generator of $\mathbb{Z}/p^k$. When p is understood we write simply $A^k(\theta)$. If p is an odd prime there are two isometry classes of linking form over $\mathbb{Z}/p^k$ depending on whether θ is a quadratic residue (mod p) or not. The semi-group $\mathcal{N}_p$ is generated by the linking forms $A^k(1)$ and $A^k(\theta), \theta$ a quadratic non-residue where k ranges over the natural numbers. All relations amongst the linking forms are generated by the following isometry

$$A^k(1) \oplus A^k(1) \cong A^k(\theta) \oplus A^k(\theta).$$

Thus for odd primes the Grothendieck group of $\mathcal{N}_p$, $\mathcal{G}(\mathcal{N}_p)$, is isomorphic to the group $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2)^\infty$, where the infinite summands are generated by $[A^k(1)]$ and the summands of order two are generated by $[A^k(1)] - [A^k(\theta)]$.

The main result of [**KK**] was the computation of the abelian semi-group of 2-primary linking forms, $\mathcal{N}_2$. The generators had already been discovered by Wall [**Wa3**] however the precise relations between them were unknown. We now review the generators of $\mathcal{N}_2$. There are the cyclic linking forms $A^k(n)$ for n odd. In this case $A^1(n) = A^1(1)$ for all n, $A^2(n) = A^2((-1)^{(n-1)/2})$ and $A^k(n) \cong A^k(n')$ if and only if $n \equiv n'(8)$ for all $k \geq 3$. There are also the hyperbolic and pseudo-hyperbolic linking forms with matrices

$$E^{k,0} = \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix} \;(k \geq 1) \qquad E^{k,1} = \begin{pmatrix} 2^{1-k} & 2^{-k} \\ 2^{-k} & 2^{1-k} \end{pmatrix} \;(k \geq 2).$$

THEOREM 5.2 (Kawauchi and Kojima [**KK**] Theorem 0.1). *$\mathcal{N}_2$ has a presentation with generators $A^k(n)$, where n is $1(k=1), \pm 1(k=2), \pm 1$ or $\pm 5(k \geq 3)), E^{k,0}(k \geq 1)$ and $E^{k,1}(k \geq 2)$ and relations*

$$A^k(n_1) \oplus A^k(n_2) = A^k(n_1+4) \oplus A^k(n_2+4) \qquad (k \geq 3) \qquad (0.1)$$

$$A^k(n) \oplus 2A^k(-n) = A^k(-n) \oplus E^{k,0} \qquad (k \geq 1) \qquad (0.2)$$

$$3A^k(n) = A^k(-n+4) \oplus E^{k,0} \qquad (k \geq 2) \qquad (0.3)$$

$$2E^{k,0} = 2E^{k,1} \qquad (k \geq 2) \qquad (0.4)$$

$$A^k(n_1) \oplus A^{k+1}(n_2) = A^k(n_1+2n_2) \oplus A^{k+1}(n_2+2n_1) \quad (k \geq 1) \qquad (1.1)$$

$$A^k(n) \oplus E^{k+1,1} = A^k(n+4) \oplus E^{k+1,0} \qquad (k \geq 1) \qquad (1.2)$$

$$E^{k,1} \oplus A^{k+1}(n) = E^{k,0} \oplus A^{k+1}(n+4) \qquad (k \geq 2) \qquad (1.3)$$

$$A^k(n_1) \oplus A^{k+2}(n_2) = A^k(n_1+4) \oplus A^{k+2}(n_2+4) \qquad (k \geq 1) \qquad (2.1)$$

We call a finite abelian group G, as opposed to a linking form, *homogeneous* if G is isomorphic to $(\mathbb{Z}/p^k)^l$ for some integer l. It is implicit in the statement of the statement of Theorem 5.2 that every linking form b is isomorphic to a direct sum of linkings over homogeneous groups which is fact that was already proven in [**Wa3**, Lemma 8]. A complete list of homogeneous linking forms is given in [**KK**, 34-35]. Kawauchi and Kojima also provide a complete set of invariants which classify linking forms and we review these now. First let $\bar{G}_p^k$ be the subgroup of G generated by elements of order p^s, $s \leq k$. Then let $\widetilde{G}_p^k := \bar{G}_p^k/(\bar{G}_p^{k-1} + p\bar{G}_p^{k+1})$. The group $\widetilde{G}_p^k$ is a vector space over $\mathbb{Z}/p$ and we define $r_p^k(G) = \dim_{\mathbb{Z}/p}(\widetilde{G}_p^k)$. Thus each linking form (G,b) is isomorphic to the direct sum $\bigoplus_{k\geq 1}(G_p^k, b_p^k)$ with each $G_p^k \cong (\mathbb{Z}/p^k)^{r_p^k(G)}$. The function

$$\begin{array}{rrcl} \widetilde{b}_p^k: & \widetilde{G}_p^k \times \widetilde{G}_p^k & \to & \mathbb{Q}/\mathbb{Z} \\ & ([x],[y]) & \mapsto & p^{k-1}b(x,y). \end{array}$$

is well defined and indeed defines a nonsingular linking form on $\widetilde{G}_p^k$. For each linking form (G,b) we make the following

Definition 5.3 ([**KK**] Definition 1.1). *Let $c^k(b) \in \widetilde{G}_2^k$ be the characteristic element of $\widetilde{b}_2^k$ which is uniquely defined by the identity*

$$\widetilde{b}_2^k(c^k(b),[x]) = \widetilde{b}_2^k([x],[x]).$$

Observe that $c^k(b) \neq 0$ if and only if (G,b) contains an indecomposable cyclic linking form $A^k(n_k)$ as a summand. We now wrap up a lemma, corollary and definition of Kawauchi and Kojima into a single proposition.

Proposition 5.4 ([**KK**] §2). *Let (G, b) be a linking form with $c^k(b) = 0$ and let q_k be the (well defined) function*

$$\begin{array}{rccc} q_k: & G_2/\bar{G}_2^k & \to & \mathbb{Q}/\mathbb{Z} \\ & [x] & \mapsto & 2^{k-1}b(x,x). \end{array}$$

The complex number

$$GS_k(b) := \sum_{[x]\in G_2/\bar{G}_2^k} \exp(2\pi i q_k([x]))$$

(where exp *denotes the standard exponential and $i = \sqrt{-1}$) is non-zero and there is an integer σ defined uniquely* mod 8 *such that $GS_k(b) = |GS_k(b)| \cdot \exp(2\pi i\sigma/8)$. The invariant $\sigma_k(b)$ is defined to be $\sigma_k(b) \in \mathbb{Z}/8$ if $c^k(b) = 0$ and ∞ if $c^k(b) \neq 0$.*

Theorem 5.5 ([**KK**] 4.1). *Two linking forms (G_2, b) and (G_2', b') are isometric if and only if $r_2^k(G_2) = r^k(G_2')$ and $\sigma_k(b) = \sigma_k(b')$ for all $k \geq 1$.*

Corollary 5.6. *Let $r_p^k : \mathcal{G}(\mathcal{N}_2) \longrightarrow \mathbb{Z}$ be the kth-rank homomorphism defined on the Grothendieck group of $\mathcal{N}_2$ by*

$$\begin{array}{rcc} r_2^k : \mathcal{G}(\mathcal{N}_2) & \longrightarrow & \mathbb{Z}[k] \\ ([b_0],[b_1]) & \mapsto & r_2^k(b_0) - r_2^k(b_1). \end{array}$$

Then the homomorphism r_2 which is the sum of r_2^k for all positive integers k gives an isomorphism

$$r_2 : \bigoplus_{k=1}^{\infty} r_2^k : \mathcal{G}(\mathcal{N}_2) \cong \bigoplus_{k=1}^{\infty} \mathbb{Z}[k].$$

Proof. The homomorphism r_2 is clearly surjective. Let b_0 and b_1 be linking forms for which $([b_0],[b_1])$ lies in the kernel of r_2. If b is the linking form

$$b = \bigoplus_{\{k: r_2^k(b_0)\neq 0\}} A^k(1)$$

then $\sigma_k(b_0 \oplus b) = \sigma_k(b_1 \oplus b) = \infty$ for every positive integer k. By Theorem 5.5, $b_0 \oplus b \cong b_1 \oplus b$ and hence the kernel of r_2 is trivial. $\square$

REMARK 5.7. Notice that cancellation fails in $\mathcal{N}_2$ and hence information is lost in passing to $\mathcal{G}(\mathcal{N}_2) \cong (\mathcal{N}_2 \oplus \mathcal{N}_2)/\Delta$, (where $\Delta = \{([q],[q]) \in \mathcal{N}_2 \times \mathcal{N}_2\}$). In particular, if

$$\bar{\Delta} = \{([q_0],[q_1]) : (j[q_0], j[q_1]) \in \Delta \text{ for some } j \in \mathbb{Z}^+\},$$

then there is an isomorphism of *sets*

$$\bar{\Delta}/\Delta \cong \mathbb{Z}/2^\infty \cup \mathbb{Z}/4^\infty.$$

2. Quadratic forms on finite abelian groups

Recall from Section 9 that a homogeneous quadratic refinement of a linking form (G, b) is a function $q : G \to \mathbb{Q}/\mathbb{Z}$ such that $q(-x) = q(x)$ for all $x \in G$ and such that

$$b(x,y) = q(x+y) - q(x) - q(y) \ \in \mathbb{Q}/\mathbb{Z}$$

for all $x, y \in G$. We note that many authors including Wall consider quadratic refinements in $\mathbb{Q}/2\mathbb{Z}$ in which case the defining equation is

$$\bar{q}(x+y) = \bar{q}(x) + \bar{q}(y) + 2b(x,y) \ \in \mathbb{Q}/2\mathbb{Z}$$

where multiplication by 2 induces an isomorphism $\cdot 2 : \mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}/2\mathbb{Z}$. The two definitions are equivalent with $\bar{q} = (\cdot 2) \circ q$. We call any such q a quadratic linking form. Two quadratic linking forms (G_0, q_0) and (G_1, q_1) are isometric if there exists and isomorphism of abelian groups $\theta : G_0 \cong G_1$ such that $q_0 = q_1 \circ \theta$. Note that any isometry of quadratic refinements is necessarily an isometry of the associated bilinear forms. If G contains no 2-torsion then every bilinear form on G has a unique quadratic refinement and so we shall be concerned with bilinear forms on 2-groups. Given that Kawauchi and Kojima have classified bilinear forms on 2-groups we shall focus on the question of when two homogeneous quadratic refinements of the same linking form are isometric. Recall from Lemma 2.31 *4*, that if q is a fixed quadratic linking form refining b, then all other quadratic linking forms refining b are of the form q_a, where

$a \in G$ and $2a = 0$. Thus our goal can be restated as follows: given $a, c \in \bar{G}^1_2$ when is $q_a \cong q_c$? We shall discover a most satisfactory answer to this question in Corollary 5.13. We turn now to the definition of the invariants which resolve this question. Recall Wall's theorem that every quadratic form (G, q) is the boundary of some even quadratic form $\kappa(H, \lambda, 0)$,

$$0 \longrightarrow H \xrightarrow{\hat{\lambda}} H^* \to G \longrightarrow 0.$$

LEMMA 5.8. *Let q be a quadratic linking form and $\kappa(\lambda, 0)$ any nondegenerate even quadratic function such that $q^{ev}(\kappa) = q$. The signature of $\lambda \pmod 8$ is independent of the choice of $\kappa(\lambda, 0)$ and is thus an invariant of q:*

$$\sigma(q) := \sigma(\lambda) \pmod 8 \ \in \mathbb{Z}/8.$$

PROOF. We use the technique of algebraic gluing from Lemma 3.6. Let $\kappa_0 = \kappa(\lambda_0)$ and $\kappa_1 = \kappa(\lambda_1)$ be two nondegenerate quadratic forms inducing q, then the quadratic form $(\kappa_0 \cup_q \kappa_1^-)$ defined in Lemma 3.6 is nonsingular and even and hence has signature divisible by 8 according to [**MH**, Theorem 5.1]. If λ is the bilinear form associated to $(\kappa_0 \cup_q \kappa_1^-)$ then $\lambda \otimes Id_{\mathbb{Q}} = (\lambda_0 \otimes Id_{\mathbb{Q}}) \oplus (-\lambda_1 \otimes Id_{\mathbb{Q}})$ and hence $\sigma(\lambda) = \sigma(\lambda_0) - \sigma(\lambda_1)$ and $\sigma(q)$ is well-defined. $\square$

THEOREM 5.9 (Nikulin [**Ni**]). *Two quadratic linking forms q and q' are isometric if and only if $b(q) \cong b(q')$ and $\sigma(q) \equiv \sigma(q') \pmod 8$.*

The invariant $\sigma(q)$ may also be computed by a Gauss-sum formula as follows.

DEFINITION 5.10. *Let (G, q) be a quadratic linking form on a finite abelian group. Define the Gauss sum invariant of q by*

$$GS(q) = 1/\sqrt{|G|} \cdot \sum_{x \in G} \exp(2\pi i q(x)) \in \mathbb{C}.$$

THEOREM 5.11 (Milgram [**Milg**]). *Let (G,q) be a quadratic linking form and let $\kappa(\lambda,0)$ be a nondegenerate quadratic function such that $q^{ev}(\kappa)=q$. Then*

$$GS(q)=\exp(2\pi i\sigma(\lambda)/8).$$

Theorem 5.11 gives another proof that $\sigma(q):=\sigma(\lambda) \pmod 8$ is well defined. It follows in particular from Theorem 5.11, that $GS(q)$ is never zero. Moreover, the important part of this complex number for our purposes is its argument, so we make the following

DEFINITION 5.12. *Let (G,q) be a quadratic linking form. Define the Kervaire-Arf invariant of q, $K(q)\in\mathbb{Q}/\mathbb{Z}$, to be the argument of $GS(q)$ divided by 2π.*

COROLLARY 5.13. *Two quadratic linking forms q and q' are isometric if and only if $b(q)\cong b(q')$ and $K(q)=K(q')$. For a fixed quadratic linking form q and for any $a,c\in\bar{G}_2^1$, q_a and q_c are isometric if and only if $q(a)=q(c)$.*

PROOF. The first statement is a trivial consequence of Theorems 5.9 and 5.11. The second statement follows from the following simple calculation.

$$\begin{aligned} GS(q_a) &= \textstyle\sum_{x\in G}\exp(2\pi i q_a(x)) \\ &= \textstyle\sum_{x\in G}\exp(2\pi i(q(x+a)-q(a))) \\ &= \left[\textstyle\sum_{x\in G}\exp(2\pi i q(x))\right].\exp(-2\pi i q(a)) \\ &= GS(q).\exp(-2\pi i q(a)). \end{aligned}$$

Thus $K(q_a)=K(q)-q(a)$ and so $q_a\cong q_c$ if and only if and only if $q(a)=q(c)$. □

We shall not give Nikulin's proof of Theorem 5.9 here and mention only that it is achieved by induction over the number of generators of G and the classification of bilinear forms on 2-groups (which Nikulin seems to have obtained independently from Kawauchi and Kojima). To conclude this section we mention another result of Wall's which reduces the problem of classifying quadratic forms to the classification of bilinear forms. If G is a finite 2-group choose a basis for G, that is a set of generators $\{x_i\}$ for cyclic groups of which G is a direct sum. Let $G_{\times 2}$ be the group with generators $\{x_i'\}$ where the order of x_i' is twice the order of x_i. There is a short exact sequence of groups

$$0 \to \bar{G}^1_{\times 2} \to G_{\times 2} \to G \to 0.$$

THEOREM 5.14 (Wall [**Wa3**] Theorem 5). *A quadratic linking form $q : G \to \mathbb{Q}/\mathbb{Z}$ determines and is determined by a linking form $B(q) : G_{\times 2} \times G_{\times 2} \to \mathbb{Q}/\mathbb{Z}$.*

With this result of Wall it is not difficult to use the classification of Kawauchi and Kojima to prove Nikulin's theorem. In fact, let $b(q) : G \times G \to \mathbb{Q}/\mathbb{Z}$ be the bilinear form which q refines. By definition $GS_k(B(q)) = GS_{k-1}(b(q))$ for $k > 1$ and $GS_1(B(q)) = GS(q)$. (Note that since $G_{\times 2}$ has no elements of order two $c^1(B(q)) = 0$ and the results of [**KK**] give an alternate proof that $GS(q)$ is non-zero.) Theorem 5.9 now follows from Theorems 5.14 and 5.5 applied to $B(q)$ and the fact that linking forms on groups of odd order have unique quadratic refinements. In fact we may now state the following extension of Theorem 5.5 to quadratic linking forms.

COROLLARY 5.15. *Two quadratic linking forms (G_2, q) and (G_2', q') are isometric if and only if $r_2^k(G_2) = r^k(G_2')$ and $\sigma_k(B(q)) = \sigma_k(B(q'))$ for all $k \geq 1$.*

PROOF. By Theorem 5.14 q and q' are isometric if and only if $B(q)$ and $B(q')$ are isometric and this occurs according to Theorem 5.5 if and only if $r_2^k(G_2) = r^k(G_2')$ and $\sigma_k(B(q)) = \sigma_k(B(q'))$ for all $k \geq 1$. $\square$

REMARK 5.16. Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic linking form defined on the 2-group G and let the range of q, $R(q)$, be the images of q in $\mathbb{Q}/\mathbb{Z}$ counted with multiplicity. It follows from Corollary 5.15 that if (G, q') is another quadratic liking form then q is isometric to q' if and only if $R(q) = R(q')$.

3. Quadratic functions on finite abelian groups

A quadratic linking functions is a function $q : G \to \mathbb{Q}/\mathbb{Z}$ which refines a bilinear form b but which need not be homogeneous. Just as in the homogeneous case we may define the Gauss-sum invariant and the associated Kervaire-Arf invariant.

$$GS(q) = \sum_{x \in G} \exp(2\pi i q(x)) \in \mathbb{C} \qquad\qquad K(q) = Arg(GS(q))/2\pi$$

However the signature invariant of a general quadratic linking function must be somewhat modified as follows. Recall that for every quadratic linking function q, Corollary 2.36 guarantees a characteristic quadratic function κ such that $q^c(\kappa) \cong q$.

Proposition 5.17. *Let (G, q) be a quadratic linking function and let $\kappa_0 = \kappa(\lambda_0, \alpha_0)$ be a nondegenerate quadratic function such that $\delta^c(\kappa_0) = q$. Then the quantity*

$$\bar{s}_1(q) := (\lambda_0^{-1}(\alpha_0, \alpha_0) - \sigma(\lambda_0))/8 \ \in \mathbb{Q}/\mathbb{Z}.$$

is independent of the choice of κ_0.

Proof. Suppose that κ_0 and κ_1 are two quadratic functions such that $\delta^c(\kappa_0) = q = \delta^c(\kappa_1)$. Then by Lemma 3.6 we may form the nonsingular characteristic quadratic function $\kappa_0 \cup_{Id} \kappa_1^-$. We let λ and α denote respectively the bilinear and linear parts of $\kappa_0 \cup_{Id} \kappa_1^-$. Since α is characteristic for λ it is well known that $\lambda(\alpha, \alpha) \equiv \sigma(\lambda)$ (mod 8) (see for example [**MH**]). However

$$\begin{aligned} [\lambda(\alpha,\alpha) - \sigma(\lambda)]/8 \bmod \mathbb{Z} &= [\lambda_0^{-1}(\alpha_0,\alpha_0) - \lambda_1^{-1}(\alpha_1,\alpha_1) - (\sigma(\lambda_0) - \sigma(\lambda_1))]/8 \bmod \mathbb{Z} \\ 0 \qquad \bmod \mathbb{Z} &= [\lambda_0^{-1}(\alpha_0,\alpha_0) - \sigma(\lambda_0)]/8 - [\lambda_1^{-1}(\alpha_1,\alpha_1) - \sigma(\lambda_1)]/8 \bmod \mathbb{Z}. \end{aligned}$$

As the right hand side is the difference of the possible values for $\bar{s}_1(q)$ we have proven the proposition. $\square$

Remark 5.18. We have called this invariant of a quadratic linking functions $\bar{s}_1$ because when $q = \delta^c(\kappa_0)$ then $\bar{s}_1(q) = \bar{s}_1(\partial L(\kappa_0))$ where $\bar{s}_1$ is the almost diffeomorphism invariant of ∂L_0 which was defined in Section 2.12.

PROPOSITION 5.19. *Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic linking function on a finite abelian group. Then*

$$\bar{s}_1(q) = -K(q)$$

PROOF. By the proof of Corollary 2.36 there is a nondegenerate characteristic quadratic function $\kappa = \kappa(H, \lambda, \alpha)$ with λ an even bilinear form such that $q = \delta^c(\kappa)$. It follows that 2 divides α and so we let $a = [\alpha/2] \in G = H^*/\hat{\lambda}(H)$. Moreover, if we let $q^o = \delta^c(\kappa(\lambda, 0))$ be the quadratic linking form defined by λ alone then $q = q^o_a$ by Lemma 2.37. By Proposition 5.17

$$\begin{aligned}\bar{s}_1(q) &= [\lambda^{-1}(\alpha,\alpha) - \sigma(\lambda)]/8 \bmod \mathbb{Z}\\ &= \lambda^{-1}(\alpha/2, \alpha/2)/2 - K(q^o)\\ &= q'(a) - K(q^o)\\ &= -K(q).\end{aligned}$$

The last step follows by the arithmetic of Corollary 5.13 which showed that $K(q^o_a) = K(q^o) - q^o(a)$ (and which made no use of the fact operating then that $2a = 0$). ∎

COROLLARY 5.20. *If q_0 and q_0 are two quadratic linking functions, then $K(q_0 \oplus q_1) = K(q_0) + K(q_1)$.*

PROOF. This will follow from the simple fact that $\bar{s}_1$ is additive. For let $\kappa_0(\lambda_0, \alpha_0)$ and $\kappa_1(\lambda_1, \alpha_1)$ be two quadratic functions such that $q_i = \delta^c(\kappa_i)$. Then

$$\begin{aligned}\bar{s}_1(q_0 \oplus q_1) &= [(\lambda_0 \oplus \lambda_1)^{-1}(\alpha_0, \alpha_1) - \sigma(\lambda_0 \oplus \lambda_1)]/8 \bmod \mathbb{Z}\\ &= [\lambda_0^{-1}(\alpha_0,\alpha_0) - \sigma(\lambda_0)]/8 + [\lambda_1^{-1}(\alpha_1,\alpha_1) - \sigma(\lambda_1)]/8 \bmod \mathbb{Z}\\ &= \bar{s}_1(q_0) + \bar{s}_1(q_1))\end{aligned}$$

∎

Now recall the Wilkens data $(b(q), \beta(q))$ of a quadratic linking function q which is the pair composed of the linking form refined by q and the even element $\beta(q)$ defined in 2.31, *5*. The element $\beta(q)$ is twice the linear part of q.

LEMMA 5.21. *Let $q : G \to \mathbb{Q}/\mathbb{Z}$ be a quadratic linking function. If the order of G is odd then $(b(q), \beta(q))$ is a complete invariant of q.*

PROOF. In a group of odd order there is a unique quadratic linking form q^o refining $b(q)$. As there is a unique $\gamma \in G$ such that $q = q^o_\gamma$ and $\gamma = (1/2)\beta(q)$ is determined by $\beta(q)$, we see that $(b(q), \beta(q))$ determines q. $\square$

Before stating and proving the classification of quadratic linking functions we establish notation for quadratic linking functions parallel to the notation for linking forms in Section 1. Let $q^k(n)$, $q^{k,0}$ and $q^{k,1}$ denote homogeneous quadratic refinements of $A^k(n)$, $E^{k,0}$ and $E^{k,1}$ respectively. We define homogeneous quadratic refinements $q^k(n)$, $q^{k,0}$, and $q^{k,1}$ of $A^k(n)$, $E^{k,0}$, and $E^{k,1}$ as follows. For $r, s \in \mathbb{Z}$,

$$(k \geq 1)\ A^k(n) = (\mathbb{Z}/2^k[e], A^k_n(e,e) = n2^{-k})$$
$$q^k(n)(re) = r^2 n 2^{-k-1}$$

$$(k \geq 1)\ E^{k,0} = (\mathbb{Z}/2^k[e_1] \oplus \mathbb{Z}/2^k[e_2],\ E^{k,0}(e_i,e_i) = 0,\ E^{k,0}(e_1,e_2) = 2^{-k})$$
$$q^{k,0}(re_1 + se_2) = rs2^{-k}$$

$$(k \geq 2)\ E^{k,1} = (\mathbb{Z}/2^k[e_1] \oplus \mathbb{Z}/2^k[e_2],\ E^{k,1}(e_i,e_i) = 2^{1-k},\ E^{k,1}(e_1,e_2) = 2^{-k})$$
$$q^{k,1}(re_1 + se_2) = (r^2 + rs + s^2)2^{-k}.$$

We shall continue to with our convention of using a subscript to denote a linear perturbation of a quadratic function, $q_a(x) = q(x) + b(x,a)$. So for example

$$[q^k(n)_e](re) = q^k(n)(re) + A^k(n)(re,e) = r^2 n 2^{-k-1} + rn2^{-k}.$$

THEOREM 5.22. *Let q_0 and q_1 be quadratic linking functions over a finite abelian group G. Then q_0 is isometric to q_1 if and only if $(b(q_0), \beta(q_0)) \cong (b(q_1), \beta(q_1))$ and $K(q_0) = K(q_1)$.*

PROOF. By Lemma 5.21, $(b(q), \beta(q))$ is an invariant of the isometry class of any quadratic linking function q. Moreover, $K(q)$ is manifestly an invariant. Therefore, it remains to show that q_0 and q_1 are isometric if $(b(q_0), \beta(q_0)) \cong (b(q_1), \beta(q_1))$ and $K(q_0) = K(q_1)$. By Lemma 5.21 (or by the results of Wilkens' thesis) the theorem is true if $b(q_0)$ and $b(q_1)$ are linking forms on a group of odd order. We therefore assume that the linking forms are defined on 2-primary groups. Let q have Wilkens data $(b(q), \beta(q))$ and a fixed Kervaire-Arf invariant $K(q) \in \mathbb{Q}/\mathbb{Z}$. The strategy of the proof shall be to show that q is uniquely determined up to isometry by these data using induction on the number of indecomposable summands in $b(q)$.

We first observe that if $\theta : (b(q_0), \beta_0)) \cong (b(q_1), \beta(q_1))$ then we may replace $(b(q_1), \beta(q_1))$ with $(b(q_1) \circ (\theta \times \theta), \theta^{-1}(\beta(q_1))$. We shall therefore always assume that $(b(q_0), \beta(q_0)) = (b(q_1), \beta(q_1))$. In order to prove the existence of the isometries of quadratic linking functions predicted by the theorem we shall constantly use the following lemma.

Lemma 5.23. *Two quadratic linking functions (G_0, q_0) and (G_1, q_1) are isometric if and only if there are bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ for G_0 and G_1 respectively such that and $q_0(e_i) = q_1(f_i)$ and $b(q_0)(e_i, e_j) = b(q_1)(f_i, f_j)$ for all i and j.*

Proof of Lemma. Clearly the condition given is necessary for q_0 and q_1 to be isometric. It is sufficient because the relation $q_i(x + y) = q_i(x) + q_i(y) + b_i(x, y)$, $i = 0, 1$, means that q_i is determined by $b(q_i)$ and the values q_i takes on a basis. □

The base case of our induction is when $b(q)$ is indecomposable. In this case Wilkens' thesis implies that there are at most two quadratic linking functions with invariants (G, b, β) and one may check using Wilkens' calculations that in the cases where there are two quadratic linking functions then they have distinct Kerviare-Arf invariant. We now give a proof of the theorem in the indecomposable case which does not rely on Wilkens' results. Consider the case where $(b(q), \beta(q)) = (A^k(n), \beta)$ is a linking form on the cyclic group $\mathbb{Z}/2^k$. We must therefore consider $q^k(n)_a$ and

$q^k(n)_{a+2^{k-1}e}$ where $2a = \beta$. We have already noted that the proof of Corollary 5.13 shows that $K(q_a) = K(q) - q(a)$ for any quadratic linking function. Hence

$$\begin{aligned} K(q^k(n)_{a+2^{k-1}e}) - K(q^k(n)_a) &= q^k(n)(a+2^{k-1}e) - q^k(n)(a) \\ &= (na^2 + 2^k na + n2^{2k-2} - na^2).2^{-k-1} \\ &= na2^{-1} + n2^{k-3}. \end{aligned}$$

We see that $K(q^k(n)_{a+2^{k-1}e}) \neq K(q^k(n)_a$ when $k=1$, when $k=2$ and $2|a$ and when $k \geq 3$ and $2 \not| a$ (recall that n is odd). In these cases the two quadratic refinements are not isometric, this is detected by the Kervaire-Arf invariant and there is nothing further to prove. In the remaining cases ($k = 2$ and $2 \not| a$ or $k \geq 3$ and $2|a$) we must show that $q^k(n)_{a+2^{k-1}e}$ and $q^k(n)_a$ are isometric. We let $\theta : \mathbb{Z}/2^k \to \mathbb{Z}/2^k$ be multiplication by $(1+2^{k-1})$ and verify that θ is the desired isometry. Firstly let $k=2$ and $2 \not| a$.

$$\begin{aligned} q^2(n)_a(3e) &= q^2(n)(e+2e) + A^2(n)(e+2e, a) \\ &= q^2(n)(e) + q^2(2e) + A^2(n)(e, 2e) + A^2(n)(2e, a) + A^2(n)(e, a) \\ &= q^2(n)(e) + 2^{-1} + A^2(n)(e, 2e) + 2^{-1} + A^2(n)(e, a) \\ &= q^2(n)(e) + A^2(n)(e, a+2e) \\ &= q^2(n)_{a+2e}(e) \end{aligned}$$

Hence by Lemma 5.23 applied to $(q^k(n)_{a+2^{k-1}e}, e)$ and $(q^k(n)_a, 3e)$ we have the desired isometry. In the second case $k \geq 3$ and $2|a$.

$$\begin{aligned} q^k(n)_a((1+2^{k-1})e) &= q^k(n)(e + 2^{k-1}e) + A^k(n)(e + 2^{k-1}e, a) \\ &= q^k(n)(e) + q^k(2^{k-1}e) + A^k(n)(e, 2^{k-1}e) + A^k(n)(a, e+2^{k-1}e) \\ &= q^k(n)(e) + 0 + A^k(n)(e, 2^{k-1}e) + A^k(n)(a, e) \\ &= q^k(n)(e) + A^k(n)(e, a+2^{k-1}e) \\ &= q^k(n)_{a+2^{k-1}e}(e) \end{aligned}$$

Again by Lemma 5.23 applied to $(q^k(n)_{a+2^{k-1}e}, e)$ and $(q^k(n)_a, (1+2^k)e)$ we have the desired isometry. We now consider the hyperbolic and pseudo-hyperbolic linking forms. When $(b(q), \beta(q)) = (E_0^1, 0)$ the classification of quadratic refinements via the Kervaire-Arf invariant is well known: $K(q^1) = K(q^1_{e_1}) = K(q^1_{e_2}) = 0$ and $K(q^1_{e_1+e_2}) = 1/2$. It is perhaps most instructive simply to list the table of values for all four quadratic refinements of E^1.

	q^1	$q^1_{e_1}$	$q^1_{e_2}$	$q^1_{e_1+e_2}$
0	0	0	0	0
e_1	0	0	$1/2$	$1/2$
e_2	0	$1/2$	0	$1/2$
e_1+e_2	$1/2$	0	0	$1/2$

The first three quadratic linking functions are evidently isometric via permutations of e_1, e_2 and e_1+e_2. We now consider Wilkens data $(b(q), \beta(q)) = (E^{k,\epsilon}, \beta)$ where $\epsilon = 0, 1$, $k \geq 2$, $\beta = 2a$ and $a = a_1e_1 + a_2e_2$ is any element in $\mathbb{Z}/2^k \oplus \mathbb{Z}/2^k$. We let $c = c_1e_1 + c_2e_2$ denote any of the elements e_1, e_2 or e_1+e_2 so that $c_i = 0$ or 1. Again we apply Corollary 5.13 to calculate that

$$\begin{aligned}
K(q^{k,\epsilon}_{a+2^{k-1}c}) - K(q^{k,\epsilon}_a) &= q^{k,\epsilon}(a+2^{k-1}c) - q^{k,\epsilon}(a) \\
&= q^{k,\epsilon}(a) + q^{k,\epsilon}(2^{k-1}c) + E^{k,\epsilon}(a, 2^{k-1}c) - q^{k,\epsilon}(a) \\
&= E^{k,\epsilon}(a, 2^{k-1}c) \\
&= (a_1c_1 + a_2c_2)2^{-1} \\
&= \left\{ \begin{array}{cl} 0 & \text{if } (a_1c_1+a_2c_2) \text{is even} \\ 1/2 & \text{if } (a_1c_1+a_2c_2) \text{is odd} \end{array} \right\}.
\end{aligned}$$

Hence we see Wilkens' criteria that when the order of the group is greater than 4, $(k \geq 2)$, that $(E^{k,\epsilon}, \beta)$ is ambiguous if and only if 4 does not divide $\beta = 2a_1e_1 + 2a_2e_1$. More importantly we must exhibit an isometry θ from $q^{k,\epsilon}_a$ to $q^{k,\epsilon}_{a+2^{k-1}f}$ when $a_1c_1 + a_2c_2$ is odd. Now either both a_i are odd in which case we assume (as we may interchange bases elements at will) that $c_2 = 0$ or precisely one a_i is odd in which case we assume it to be a_1 and hence $c_1 = 1$. We describe the necessary isometry θ in these two cases and present the matrices which by Lemma 5.23 ensures that θ is an isometry.

$$(a_i \text{ odd}, c_1 = 1, c_2 = 0) \left\{ \begin{array}{ccc} q_a^{k,\epsilon} & \xrightarrow{\theta} & q_{a+2^{k-1}f}^{k,\epsilon} \\ e_1 & \mapsto & e_1 \\ e_2 & \mapsto & e_2 + 2^{k-1}e_1. \end{array} \right\}$$

$$\begin{bmatrix} q_a^{k,\epsilon}(e_1) & E^{k,\epsilon}(e_1, e_2) \\ * & q_a^{k,\epsilon}(e_2) \end{bmatrix} = \begin{bmatrix} q_a^{k,\epsilon}(e_1) & 2^{-k} \\ * & q_a^{k,\epsilon}(e_2) \end{bmatrix}$$

$$\begin{bmatrix} q_{a+2^{k-1}c}^{k,\epsilon}(e_1) & E^{k,\epsilon}(e_1, e_2 + 2^{k-1}e_1) \\ * & q_{a+2^{k-1}c}^{k,\epsilon}(e_2 + 2^{k-1}e_1) \end{bmatrix} = \begin{bmatrix} q_a^{k,\epsilon}(e_1) + c_2 & 2^{-k} \\ * & q_a^{k,\epsilon}(e_2) + 2^{-1}(c_1 + a_2) \end{bmatrix}$$

$$(a_1 \text{ odd}, a_2 \text{ even}, c_1 = 1) \left\{ \begin{array}{ccc} q_a^{k,\epsilon} & \xrightarrow{\theta} & q_{a+2^{k-1}f}^{k,\epsilon} \\ e_1 & \mapsto & e_1 + 2^{k-1}c_2e_2 \\ e_2 & \mapsto & e_2 + 2^{k-1}e_2. \end{array} \right\}$$

$$\begin{bmatrix} q_{a+2^{k-1}c}^{k,\epsilon}(e_1 + 2^{k-1}c_2e_2) & E^{k,\epsilon}(e_1, e_2 + 2^{k-1}c_2e_1) \\ * & q_{a+2^{k-1}c}^{k,\epsilon}(e_2 + 2^{k-1}e_1) \end{bmatrix} =$$

$$\begin{bmatrix} q_a^{k,\epsilon}(e_1) + 2^{-1}(c_2 + c_2a_1) & 2^{-k} \\ * & q_a^{k}(e_2) + 2^{-1}(c_1 + a_1) \end{bmatrix}$$

This completes the classification of quadratic linking forms refining indecomposable pairs $(b(q), \beta)$. We call the Wilkens data (b, β) ambiguous if there is more than one isometry class of quadratic linking function with Wilkens data (b, β). We call the Wilkens data (b, β) unambiguous if there is a unique quadratic linking function with invariants (b, β). We have just reproven Wilkens' result that an indecomposable pair (b, β) is ambiguous if and only if the order of the group is less than or equal to 4 and $\beta = 0$ or the order of the group is greater than 4 and β is not divisible by 4. Moreover we note that in all ambiguous cases except for $(A^1(1), 0)$ that the $K(q_0) - K(q_1) = 1/2$ where q_0 and q_1 are the two possible refinements. When $(b(q), \beta(q)) = (A^1(1), 0)$ then there are two possible quadratic refinements $q^1(1)$ and $q^1(1)_e$ and these have distinct Kervaire-Arf invariants:

$$K(q^1(1)) = \frac{1}{8} \quad \text{and} \quad K(q^1(1)_e) = \frac{7}{8}.$$

The inductive step requires us to verify the theorem in a large number of special cases. We start with a simple case by way of illustration. Suppose that $(b(q), \beta(q)) = (b_0, \beta_0) \oplus (b', \beta')$ where (b_0, β_0) is an unambiguous pair with quadratic refinement q_0. It follows that $q \cong q_0 \oplus q'$ where q' refines (b', β'). But since the Kervaire-Arf invariant is additive over block sum $K(q') = K(q) - K(q_0)$ is determined and hence by induction q' is uniquely determined. If suffices therefore to restrict our attention to pairs $(b(q), \beta(q))$ consisting entirely of ambiguous irreducible components. If these all have $\beta(q) = 0$ then all the quadratic linking functions are homogeneous and hence quadratic linking forms in which case Nikulin has proven the result. We thus check the cases in which at least one of the ambiguous indecomposable pairs has $\beta(q) \neq 0$ and therefore $4 \not| \beta(q)$. We label that the ambiguous indecomposable inhomogeneous Wilkens data (I, β) and recall that they are as follows:

$$(A^k(n), \beta)\ k > 2,\ 4 \not| \beta, \quad (E^{k,\epsilon}, \beta)\ k > 1,\ 4 \not| \beta.$$

Suppose that $(b(q), \beta(q)) = (A^1(1), 0) \oplus (I, \beta)$ then there are four possible quadratic refinements and four different values of $K(q)$ so the result is proved. In the remaining cases we consider Wilkens data of the form $(I, \beta_0) \oplus (J, \beta_1) \oplus (b', \beta')$ where (I, β_0) is ambiguous, indecomposable and inhomogeneous and (J, β_1) is ambiguous and either indecomposable or $(A^1(1) \oplus A^1(1), 0)$. Let q^I and $*q^I$ denote the two isometry classes of quadratic refinements of (I, β_0) and let q^J and $*q^J$ be quadratic refinements of (J, β_1) such that $K(q^J) - K(*q^J) = 1/2$. We shall show that $*q^I \oplus *q^J \cong q^I \oplus q^J$. It follows that every quadratic refinement of $(I,\beta_0)\oplus(J,\beta_1)\oplus(b',\beta')$ is isometric to $q^I\oplus q''$ for some q'' which refines $(J,\beta_1)\oplus(b',\beta')$ and hence by induction that the value of $K(q)$ determines the isometry class of any quadratic refinement of $(I,\beta_0)\oplus(J,\beta_1)\oplus(b',\beta')$.

We now proceed to the necessary computations. Let a_i be elements of the domain of I or J such that $2a_i=\beta_i$. From our computations in the indecomposable case we know that if I or J is isomorphic to $E^{k,\epsilon}$ with $k>1$ then, by swapping e_1 and e_2 if necessary, we may assume that $K(q^{k,\epsilon}_{a_i})-K(q^{k,\epsilon}_{a_i+2^{k-1}e_1})=1/2$. Finally, when considering pairs of similar linking forms we shall distinguish generators by the place they hold. So, for example, we shall denote generators of $A^k(n_0)\oplus A^l(n_1)$ by $(e,0)$ and $(0,e)$.

$$(b(q),\beta(q))=(A^k(n_0),\beta_1)\oplus(A^l(n_1),\beta_2)\quad(k\geq 3,l\geq 2)$$

$$\begin{array}{ccccccccc}
q^k(n_0)_{a_0} & \oplus & q^l(n_1)_{a_1} & \to & q^k(n_0)_{a_0+2^{k-1}e} & \oplus & q^l(n_1)_{a_1+2^{l-1}e} \\
(e & , & 0) & \mapsto & (e & , & 2^{l-1}e) \\
(0 & , & e) & \mapsto & (2^{k-1}e & , & e)
\end{array}$$

$$(b(q),\beta(q))=(A^k(n_0),\beta_1)\oplus(E^{l,\epsilon},\beta_2)\quad(k,l\geq 2).$$

$$\begin{array}{ccccccccc}
q^k(n_0)_{a_0} & \oplus & q^{l,\epsilon}_{a_1} & \to & q^k(n_0)_{a_0+2^{k-1}e} & \oplus & q^{l,\epsilon}_{a_1+2^{l-1}e_1} \\
(e & , & 0) & \mapsto & (e & , & 2^{l-1}e_1) \\
(0 & , & e_1) & \mapsto & (0 & , & e_1) \\
(0 & , & e_2) & \mapsto & (2^{k-1}e & , & e_2)
\end{array}$$

$$(b(q), \beta(q)) = (E^{k,\epsilon}, \beta_1) \oplus (E^{l,\epsilon}, \beta_2)) \quad (k, l \geq 2).$$

$$\begin{array}{ccccccccc}
q^{k,\epsilon}_{a_0} & \oplus & q^{l,\epsilon}_{a_1} & \to & q^{k,\epsilon}_{a_0+2^{k-1}e} & \oplus & q^{l,\epsilon}_{a_1+2^{l-1}e_1} \\
(e_1 & , & 0) & \mapsto & (e_1 & , & 0) \\
(e_2 & , & 0) & \mapsto & (e_2 & , & 2^{l-1}e_1) \\
(0 & , & e_1) & \mapsto & (0 & , & e_1) \\
(0 & , & e_2) & \mapsto & (2^{k-1}e_1 & , & e_2)
\end{array}$$

$$(b(q), \beta(q)) = (A^k(n_0), \beta_1) \oplus (E^{1,0}, 0) \quad (k \geq 3, 4 \not| \beta_1).$$

$$\begin{array}{ccccccc}
q^k(n_0)_{a_0} & \oplus & q^1_{e_1+e_2} & \to & q^k(n_0)_{a_0+2^{k-1}e} & \oplus & q^1 \\
(e & , & 0) & \mapsto & (e & , & e_1+e_2) \\
(0 & , & e_1) & \mapsto & (2^{k-1}e & , & e_1) \\
(0 & , & e_2) & \mapsto & (2^{k-1}e & , & e_2)
\end{array}$$

$$(b(q), \beta(q)) = (A^k(n_0), \beta_1) \oplus (A^1(1) \oplus A^1(1), 0)) \quad (k \geq 3, 4 \not| \beta_1).$$

$$\begin{array}{ccccccccccc}
q^k(n_0)_{a_0} & \oplus & q^1(1)_{a_1} & \oplus & q^1(1)_{a_2} & \to & q^k(n_0)_{a_0+2^{k-1}e} & \oplus & q^1(1)_{a_1+e} & \oplus & q^1(1)_{a_2+e} \\
(e & , & 0 & , & 0) & \mapsto & (e & , & e & , & e) \\
(0 & , & e & , & 0) & \mapsto & (2^{k-1}e & , & e & , & 0) \\
(0 & , & 0 & , & e) & \mapsto & (2^{k-1}e & , & 0 & , & e)
\end{array}$$

$$(b(q), \beta(q)) = (E^{k,\epsilon}, \beta_1) \oplus (E^{1,0}, 0)) \quad (k \geq 2, 4 \not| \beta_1).$$

$$\begin{array}{ccccccc}
q^{k,\epsilon}_{a_0} & \oplus & q^1_{e_1+e_2} & \to & q^{k,\epsilon}_{a_0+2^{k-1}e_1} & \oplus & q^1 \\
(e_1 & , & 0) & \mapsto & (e_1 & , & 0) \\
(e_2 & , & 0) & \mapsto & (e_2 & , & e_1+e_2) \\
(0 & , & e_1) & \mapsto & (2^{k-1}e_1 & , & e_1) \\
(0 & , & e_2) & \mapsto & (2^{k-1}e_1 & , & e_2)
\end{array}$$

$$(b(q), \beta(q)) = (E^{k,\epsilon}, \beta_1) \oplus (A^1(1) \oplus A^1(1), 0)) \quad (k \geq 2, 4 \not| \beta_1).$$

$$\begin{array}{ccccccc}
q^{k,\epsilon}_{a_0} & \oplus & q^1_{a_1} & \to & q^{k,\epsilon}_{a_0+2^{k-1}e} & \oplus & q^1_{a_1+2^{l-1}e_1} \\
(e_1 & , & 0) & \mapsto & (e_1 & , & 0) \\
(e_2 & , & 0) & \mapsto & (e_2 & , & 2^{l-1}e_1) \\
(0 & , & e_1) & \mapsto & (0 & , & e_1) \\
(0 & , & e_2) & \mapsto & (2^{k-1}e_1 & , & e_2)
\end{array}$$

This concludes the proof of Theorem 5.22. □

Corollary 5.24. *Let P_0 and P_1 be highly connected rational homology spheres of dimension 7 or 15 with isometric Wilkens invariants (G_0, b_0, β_0) and (G_1, b_1, β_1). Then P_0 and P_1 are almost diffeomorphic (resp. diffeomorphic) if and only if $\bar{s}_1(P_0) = \bar{s}_1(P_1)$ (resp. $s_1(P_0) = s_1(P_1)$ and $\Sigma_{P_0} = \Sigma_{P_1}$).*

Proof. By Theorem B, P_0 and P_1 are almost diffeomorphic if and only if $Q(P_0) = (G_0, q^c_0, \beta_0)$ is isometric to $Q(P_1) = (G_1, q^c_1, \beta_1)$. As P_0 and P_1 are rational homology spheres $q^c_0 = q_0$ and $q^c_1 = q_1$ are quadratic linking functions which by hypothesis define isometric Wilkens data. Now by Theorem 5.22, q_0 and q_1 are isometric if and only if $K(q_0) = K(q_1)$ but by Proposition 5.19, $K(q_i) = -\bar{s}_1(q_i)$, $i = 0, 1$, and by

Remark 5.18, $\bar{s}_1(q_i) = \bar{s}_1(P_i)$. Hence P_0 and P_1 are almost diffeomorphic if and only if $\bar{s}_1(P_0) = \bar{s}_1(P_1)$. By Theorem 4.9, P_0 and P_1 are diffeomorphic if and only if q_0 is isometric to q_1, $s_1(P_0) = s_1(P_1)$ and $\Sigma_{P_0} = \Sigma_{P_1}$. As $\bar{s}_1$ is a multiple of s_1, we conclude that P_0 and P_1 are diffeomorphic if and only if $s_1(P_0) = s_1(P_1)$ and $\Sigma_{P_0} = \Sigma_{P_1}$. $\square$

CHAPTER 6

Classification up to homeomorphism and homotopy

In this final chapter we shall restrict our attention to highly connected manifolds of dimension 7 and thus the pair (L, P) shall always denote an 8-dimensional handlebody L with boundary P. We remind the reader that we consider only smooth manifolds P. We shall extend the almost diffeomorphism classification obtained in Chapter 4 to a classification up to PL-homeomorphism, homeomorphism and homotopy. The classification of manifolds is often presented in the opposite direction, moving from the homotopy classification to the homeomorphism classification via the surgery exact sequence and then upwards via smoothing theory to the PL-classification and the smooth classification. However, in the case of highly connected manifolds it has proven easier to classify up to almost diffeomorphism first and then descend to the other categories using smoothing theory and the surgery exact sequence in the opposite direction.

1. Smoothing theory

We begin by briefly recalling the theory of smoothing manifolds. Suppose that M is a PL manifold of dimension greater than 4. If we embed M in Euclidean space of sufficiently highly dimension then the embedding we choose is unique up to isotopy and hence M has a well defined stable normal bundle which we denote by $\nu_M^{PL} : M \to BPL$. Now there is a forgetful map from $\pi_{O,PL} : BO \to BPL$ which classifies the universal smooth bundle $\gamma : EO \to BO$ as a PL-bundle. It is a remarkable fact (proven in [**HM**]) that the question of whether M may be equipped with a smooth structure reduces to the question of whether the map stable PL-normal bundle of M lifts through $\pi_{O,PL}$.

$$\begin{tikzcd} & BO \arrow[d, "\pi_{O,PL}"] \\ M \arrow[ur, dotted] \arrow[r, "\nu_M^{PL}"'] & BPL \end{tikzcd}$$

In fact, if ν is a lift of ν_M^{PL} then there is a smooth structure on M_ν for which ν classifies the stable normal bundle. We would now like to determine the number of distinct smooth structures which the PL manifold M admits. It amounts to the same thing, but gives us a little more freedom, to determine the smoothings of M where we define a smoothing of M to be a PL-homeomorphism $f : M_\nu \to M$ where M_ν is a smooth manifold. Such a smoothing defines the smooth structure $f^{-1*}M_\nu$ on M. Two smoothings $f_0 : M_{\nu_0} \to M$ and $f_1 : M_{\nu_1} \to M$ are called concordant if there is a smooth structure on $M \times I$ which restricts to be $f_0^{-1*}M_{\nu_0}$ on $M \times 0$ and $f_1^{-1*}M_{\nu_1}$ on $M \times 1$. Another theorem of [**HM**] states that the concordance classes of smoothings of M are in bijective correspondence with $[M, PL/O]$ where PL/O denotes the fiber of $\pi_{O,PL}$. We shall call two smoothings $f_i : M_{\nu_i} \to M$ diffeomorphic if there is a diffeomorphism $g : M_{\nu_0} \to M_{\nu_1}$ such that $f_0 = f_1 \circ g$. Of course one often would rather know the classification of the smoothings of M up to diffeomorphism and these are the quotient of the concordance classes of smoothings of M under the action of the PL-homeomorphisms of M. A PL-homeomorphism $h : M \to M$ acts on a smoothing $M_\nu :\to M$ by composition, $h \cdot f = h \circ f$. If h is PL-isotopic to the identity then it acts trivially on any smoothing. In the case where $M \cong (S^n)^{PL}$ it is known that all PL-homeomorphisms which preserve orientation are PL isotopic to the identity and hence $[S^n, PL/O] = \pi_n(PL/O)$ corresponds bijectively with the set of orientation preserving diffeomorphism classes of smoothings of S^n, otherwise known as Θ_n (for $n \neq 3, 4$). For $i \leq 6$ it is known that $\pi_i(PL/O) = 0$. In general the computation of $[M, PL/O]$ for an arbitrary manifold is a (not necessarily easy) task in homotopy theory.

The problem of classifying the PL structures on a given topological manifold was solved by Kirby and Siebenmann [**KiSi**] who showed that the problem again reduced to the lifting question for the classifying maps of the stable tangent bundles. Specifically every compact topological manifold M has a stable topological normal bundle which classified by a map ν_M^{Top} from M to the classifying space $BTop$. There is a universal forgetful map $\pi_{PL,Top} : BPL \to BTop$ and a topological manifold M admits a PL-structure if and only if the following lifting problem can be solved.

$$\begin{tikzcd} & BPL \arrow[d, "\pi_{PL,Top}"] \\ M \arrow[ur, dotted] \arrow[r, "\nu_M^{Top}"'] & BTop \end{tikzcd}$$

Moreover, the homotopy fiber of $\pi_{PL,Top}$, Top/PL, has but one non-trivial homotopy group and this is $\mathbb{Z}/2$ in dimension 3. There is thus a single obstruction to placing a PL structure on a topological manifold M which lies in $H^4(M;\mathbb{Z}/2)$ and it is called the Kirby-Siebenmann invariant. If M is a topological manifold which admits a PL structure then the concordance classes of PL structures on M are in bijective correspondence with $[M, Top/PL] \cong H^3(M;\mathbb{Z}/2)$.

Having reviewed the fundamental results concerning the smoothing of manifolds we now consider the PL and topological classification of highly connected manifolds of dimension 7. We shall write P^{PL} and PL^{Top} for the PL and topological manifolds underlying the smooth manifold P.

THEOREM 6.1. *Let P_0 and P_1 be highly connected, closed, smooth 7-manifolds with quadratic linking families $Q(P_0)$ and $Q(P_1)$. The following are equivalent:*

1. *P_0 is homeomorphic to P_1,*

2. *P_0 is PL-homeomorphic to P_1,*

3. *P_0 is almost diffeomorphic to P_1,*

4. *$Q(P_0)$ is isometric to $Q(P_1)$.*

PROOF. Evidently *3* implies *2* and *2* implies *1*. The main result of Chapter 4 was the equivalence of *3* and *4*. In dimension 7 we see that $[P, PL/O] \cong H^7(P; \pi_7(PL/O)) \cong \Theta_7$. It follows that all possible smooth structures on P_0^{PL} are obtained by taking the connected sum of P_0 with a homotopy sphere and thus *2* implies *3*. Finally we show that *1* implies *2*.

Let $f : P_0 \to P_1$ be a homeomorphism. Suppose, to begin with, that P_0 and P_1 have torsion free third homology. Then $Q(P_i) \cong (\mathbb{Z}^l, 0, \beta_i)$ where l is the rank of $H^4(P_i)$, 0 indicates the trivial quadratic linking function and β_i is the tangential invariant of P_i. By Novikov's theorem on the topological invariance of the rational Pontryagin classes, $f_!(\beta_0) = \beta_1$ and hence f induces an isometry of the quadratic linking families. Hence by Theorem B, P_0 and P_1 are almost diffeomorphic and in particular PL-homeomorphic. If P_0 and P_1 have finite fourth cohomology then by Corollary 5.24 the quadratic linking functions $Q(P_i) = (TG, q_i, \beta_i)$ are determined by their Wilkens data $(b(q_i), \beta_i)$ and Kervaire-Arf invariants $K(q_i)$. Now the homeomorphism f preserves the linking form since b and it also preserves the tangential invariant since by [**KS**] $\beta(P_i) = \frac{p_1}{2}(P_i)$ is a topological invariant. But by Lemma 5.19 $K(q_i) = -\bar{s}_1(q_i) = -\bar{s}_1(P_i)$ and again by [**KS**] the invariant $\bar{s}_1$ is a topological invariant. Thus P_0 and P_1 have isometric quadratic linking functions and are therefore almost diffeomorphic and in particular PL-homeomorphic.

Thus if P is a highly connected 7-manifold with either finite homology or torsion free homology, then every element of $H^3(P; \pi_3(PL/O))$ is realized by a self-smoothing $f : P^{PL} \to P^{Top}$. Finally suppose that $H^4(P_0) \cong F \oplus TG$ where F is a free group and TG is a finite group. We choose a splitting of P_0 as the connected sum of manifolds

$P_0 = P_0(F)\#P_0((TG)$ where $F = H^4(P(F))$ and $TG = H^4(P(TG))$. Given any $x \in H^3(P_0; \pi_3(PL/O))$ we may write $x = y \oplus z$ where $y \in H^3(P_0(F); \pi_3(PL/O))$ and $z \in H^3(P_0(TG)^{Top}; \pi_3(PL/O))$. Then by the previous arguments y and z are realized by self PL smoothings $f_y : P_0(TG)^{PL} \to P_0(TG)^{Top}$ and $f_z : P_0(F)^{PL} \to P_0(F)^{Top}$. If we now form the connected sum $f_y\#f_z : P_0^{PL} \to P_0^{Top}$ then it realizes x and is manifestly a self smoothing. Thus every element of $H^3(P_0; \pi_3(PL/O))$ is realized by a self PL smoothing and hence every PL manifold homeomorphic to P_0 is PL-homeomorphic to P_0 and in particular P_1^{PL} is PL-homeomorphic to P_0. □

2. The surgery exact sequence

We present now a rapid review of the surgery exact sequence which is an important tool for the classification of manifolds. We begin with a pair of finite CW-complexes (Y, X) which we assume to be simply connected and where we allow X to be empty. Let Cat denote either of the smooth (O), piecewise linear (PL) or topological (Top) categories. We wish to know if there is a homotopy equivalence $f : (W, M) \to (Y, X)$ from a compact pair of Cat-manifolds (W, M) to (Y, X). A necessary condition for such a homotopy equivalence is of course that the pair (Y, X) satisfy Poincaré duality. That is, for some n there is a class $[Y, X] \in H_n(Y, X; \mathbb{Z})$ such that $\cap[Y, X] : H^k(Y, X) \cong H_{n-k}(Y)$. As the pair (Y, X) is a finite complex there a large N and an embedding $(Y, X) \hookrightarrow D^N, S^{N-1})$ where (D^N, S^{N-1}) is N-disc with boundary the $(N-1)$-sphere S^{N-1}. Inside D, the complex Y has a regular neighborhood N. The manifold N is a smooth manifold and retracts onto Y via a retraction we denote by $r : N \to Y$. It is a consequence of Poincaré duality for the pair (Y, X) that the map $r|_{\partial N}$, when considered as a fibration, has homotopy fiber the homotopy type of a sphere. The stable fiber homotopy type of the fibration defined by $r|_{\partial N}$ is independent of the embedding and is a homotopy invariant of the pair (Y, X) ([**Sp**]). It is called the Spivak normal bundle of (Y, X) and is classified by a map $\nu : Y \to BG$ where BG is the classifying space for spherical fibrations. Now if Y were a Cat manifold then r would be a stable normal bundle of Y and $r|_{\partial N}$ would be the sphere bundle of the stable normal bundle. Hence a further necessary condition for (Y, X) to have the homotopy type of a compact manifold with boundary is that its Spivak normal bundle have a Cat bundle reduction. Assuming that the dimension of (Y, X) as a Poincaré space is greater than 5 then it is a theorem of Browder [**Bro**] and Novikov [**No**] (in the smooth and PL categories) and Kirby and Siebenmann [**KiSi**] (in the Top category)[1] that a Cat reduction of the Spivak normal bundle of a simply connected Poincaré space is in fact a sufficient condition for it to have the homotopy type of a Cat manifold.

With an answer to the question of when a homotopy type (Y, X) contains the homotopy type of a manifold, surgery proceeds to the task of classifying Cat-manifolds homotopy equivalent to (Y, X) up to Cat-isomorphism. However, the primary objects which surgery theory seeks to classify are in fact smoothings of a finite CW-pair (Y, X). The Cat-structure set of Y, $\mathcal{S}^{\mathrm{Cat}}(Y)$, consists of equivalence classes of Cat-smoothings in Y where a Cat smoothing in Y is an orientation preserving homotopy equivalences of pairs $f : (M, \partial M) \to (Y, X)$ from a compact Cat-manifold with boundary to (Y, X). Two homotopy equivalences (M_1, f_1) and (M_2, f_2) are equivalent in $\mathcal{S}^{\mathrm{Cat}}(Y)$ if there is a Cat-isomorphism $g : M_1 \to M_2$ such that f_1 is homotopy equivalent to $f_2 \circ g$. In order to compute the set of Cat smoothings in Y one maps $\mathcal{S}^{\mathrm{Cat}}(Y)$ to the set $\mathcal{N}^{Cat}(Y)$ of degree one normal maps in Y modulo normal bordism.

[1] By their extension of transversality methods to Top [**KiSi**, III])

We now define this set. A degree one normal map to Y is a bundle map $(f,\hat{f})$

$$\begin{array}{ccc} \nu_M & \xrightarrow{\hat{f}} & \nu \\ \downarrow & & \downarrow \\ (M,\partial M) & \xrightarrow{f} & (Y,X) \end{array}$$

from the stable normal bundle ν_M of a Cat-manifold M to a Cat-bundle over Y such that map induced on homology by f carries the fundamental class of $(M,\partial M)$ to $[Y,X]$. The equivalence between degree one normal maps is simply that of bordism with boundary over the bundle $\nu \to Y$. Specifically, two degree one normal maps

$$\begin{array}{ccc} \nu_{M_0} & \xrightarrow{\hat{f}_0} & \nu_0 \\ \downarrow & & \downarrow \\ (M_0,\partial M_0) & \xrightarrow{f_0} & (Y,X) \end{array} \qquad\qquad \begin{array}{ccc} \nu_{M_1} & \xrightarrow{\hat{f}_1} & \nu_1 \\ \downarrow & & \downarrow \\ (M_1,\partial M_1) & \xrightarrow{f_1} & (Y,X) \end{array}$$

are normally bordant if there exists a compact manifold W and a degree one normal map $(F,\hat{F})$

$$\begin{array}{ccc} \nu_W & \xrightarrow{\hat{F}} & \nu \times I \\ \downarrow & & \downarrow \\ W & \xrightarrow{F} & Y \times I \end{array}$$

with the following two properties: *(i)* $\partial W = M_1 \ \cup \ M_2 \ \cup \ N$ where N is a compact manifold with $\partial N = N \cap (M_1 \cup M_2) = \partial(M_1 \cup M_2)$, and *(ii)* $(F|_{M_i}, \hat{F}|_{M_i}) = (f_i, \hat{f}_i)$.

To report the essential results of simply connected surgery theory we first define the following abelian groups.

$$L_n(\mathbb{Z}) = \left\{ \begin{array}{ll} \mathbf{Z}, & \text{if } n = 4\,k; \\ \mathbf{Z}_2, & \text{if } n = 4\,k+2; \\ 0, & \text{otherwise.} \end{array} \right\}, \qquad\qquad L_n(\mathbb{Z} \to \mathbb{Z}) = 0.$$

The following two facts are well-known, see [**Wa5**] or [**MM**].

FACT 6.2. *(1) Let M be a simply connected manifold and let G/Cat denote the fiber of the map $B\mathrm{Cat} \to \mathrm{BG}$, then $\mathcal{N}^{Cat}(M) \cong [M, G/\mathrm{Cat}]$.*

(2) If $k > 5$ there is a pair of surgery exact sequences of sets

$$\begin{array}{ccccccc}
L_{k+1}(\mathbb{Z} \to \mathbb{Z}) & \longrightarrow & \mathcal{S}^{\mathrm{Cat}}(W^k) & \overset{\eta^{\mathrm{Cat}}}{\longrightarrow} & \mathcal{N}^{\mathrm{Cat}}(W^k) & \longrightarrow & L_k(\mathbb{Z} \to \mathbb{Z}) \\
\downarrow & & i^* \downarrow & & i^* \downarrow & & \downarrow \\
L_k(\mathbb{Z}) & \longrightarrow & \mathcal{S}^{\mathrm{Cat}}(\partial W^{k-1}) & \overset{\eta^{\mathrm{Cat}}}{\longrightarrow} & \mathcal{N}^{\mathrm{Cat}}(\partial W^{k-1}) & \longrightarrow & L_{k-1}(\mathbb{Z})
\end{array}$$

where exactness means that each set has a preferred element $*$ and that at every stage the pre-image of $*$ is the image of the previous map. The base point of both $\mathcal{S}^{\mathrm{Cat}}$ and $\mathcal{N}^{\mathrm{Cat}}$ is the the class of the identity map $Id : W \to W$ or $Id : \partial W \to \partial W$. The base point of the L-groups is the identity element. In fact there is an action of the left-most L-group on $\mathcal{S}^{\mathrm{Cat}}$ and two smoothings in $\mathcal{S}^{\mathrm{Cat}}$ are in the same orbit if and only if they have the same image under η^{Cat}. In the smooth category this action is connect sum in the domain of a smoothing with a homotopy sphere bounding a parallelizable manifold and in either the PL or Top categories the action is trivial and η is injective. Finally, the vertical maps i^* are given by restricting a smoothing or degree one normal map to its boundary.

We now illustrate this theory in the PL case for the manifolds $(W, \partial W) = (L, P)$. In this case $n = 8$, $L_7(e) = 0$ and η^{PL} is a bijection. Thus both PL-structure sets and PL-normal invariant sets coincide. Moreover the normal invariants are extremely easy to calculate since $[L, G/PL]$ and $[P, G/PL]$ can be identified, via the the primary obstruction to null homotopy, with $H^4(L; \pi_4(G/PL)) \cong H^4(L; \mathbb{Z})$ and $H^4(P; \pi_4(G/PL)) \cong H^4(P; \mathbb{Z})$ respectively. The surgery exact sequences become the following commutative squares.

$$\left\{\begin{array}{ccc}
\mathcal{S}^{PL}(L) & \overset{\eta}{\cong} & \mathcal{N}^{PL}(L) \\
i^* \downarrow & & i^* \downarrow \\
\mathcal{S}^{PL}(P) & \overset{\eta}{\cong} & \mathcal{N}^{PL}(P)
\end{array}\right\} \cong \left\{\begin{array}{ccc}
H^4(L) & \cong & H^{4j}(L) \\
i^* \downarrow & & i^* \downarrow \\
H^4(P) & \cong & H^{4j}(P)
\end{array}\right\}$$

We emphasize that the vertical maps i^* are given by simply restricting a PL-structure or PL-normal invariant to the structure or normal invariant on the boundary. Since the map $i^* : H^4(L) \to H^4(P)$ is always onto it follows that every degree one normal map in P is normally bordant to the restriction of a degree one normal map in L and thus that every equivalence class of PL-smoothings in P has a representative which is the restriction of a PL-smoothing in L.

3. Constructing smoothings in L and P

Let $SG(n)$ denote the H-space of orientation preserving self homotopy equivalences of S^{n-1}. In what follows the groups $SG(4)$ and $SO(4)$, and in particular $\pi_3(SG(4))$ and $\pi_3(SO(4))$, shall play a crucial role so we begin by digressing somewhat to describe these groups and the relationships between them.

We start with $SO(4)$. Let $|\ |$ denote the standard metric on $\mathbb{R}^4$ and identify $\mathbb{R}^4$ with the quaternions, $\mathbb{H}$. Then $S^3 = \{x \in \mathbb{H} : |x| = 1\}$ is the set of unit quaternions. For $y \in \mathbb{H}$ and $x \in S^3$ we define the function

$$\begin{array}{rccc} a(m,n) : & S^3 & \to & SO(4) \\ & x & \mapsto & [y \mapsto x^{n+m} \cdot y \cdot x^{-m}] \end{array}$$

where $\cdot$ indicates quaternionic multiplication. The mapping $a(m,n)$ is manifestly linear and orientation preserving and since $|x^{n+m} \cdot y \cdot x^{-m}| = |x|^{n+m}|y||x|^{-m|} = |y|$ and it follows that $a(m,n)(x)$ is an element of $SO(4)$ for each $x \in S^3$. We define $\bar{\alpha}(m,n)$ to be the homotopy class of $a(m,n)$. It is well known that the mapping

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z} & \to & \pi_3(SO(4)) \\ (m,n) & \mapsto & \bar{\alpha}(m,n) \end{array}$$

defines an isomorphism of groups. Given the function $a(m,n)$ we define the associated twist $A(m,n)$ as follows

$$\begin{array}{rccc} A(m,n): & S^3 \times D^4 & \to & S^3 \times D^4 \\ & (x,y) & \mapsto & (x, a(m,n)(y)). \end{array}$$

Identifying $S^3 \times D^4$ with $\partial D^4 \times D^4$ we may use $A(m,n)$ to glue two copies of $D^4 \times D^4$ together thereby form a D^4-bundle over S^4, $\xi(m,n) : L(m,n) \to S^4$. We define the Euler number of $\bar{\alpha}(m,n)$ to be the Euler number of $\xi(m,n)$, $E(\bar{\alpha}(m,n)) := E(\xi(m,n))$. The following facts are also well known and may be found in [**Hu**] ($r = \bar{\alpha}(1,0)$ and $s = \bar{\alpha}(0,1)$).

1. The Euler number of $\bar{\alpha}(m,n)$, $E(\bar{\alpha}(m,n))$, is n.
2. The stabilization of $\bar{\alpha}(m,n)$, $S(\bar{\alpha}(m,n))$, is $2m+n \in \mathbb{Z} = \pi_3(SO)$.
3. If $i : SO(3) \hookrightarrow SO(4)$ denotes the standard inclusion with induced map $i_* : \pi_3(SO(3)) \hookrightarrow \pi_3(SO(4))$, and if $m \in Z = \pi_3(SO(3))$, then $i_*(m) = \bar{\alpha}(m,0)$.

By restricting an element of $SO(n)$ to S^{n-1} we obtain an orientation preserving homotopy equivalence of S^{n-1} and this defines an inclusion $i_n : SO(n) \hookrightarrow SG(n)$.

Lemma 6.3. *There is an isomorphism $\pi_3(SG(4)) \cong \mathbb{Z}/12 \oplus \mathbb{Z}$ such that*

$$i_{4*}(\bar{\alpha}(m,n)) = (m \,(\mathrm{mod}\ 12), n).$$

Proof. Let $SF(n)$ denote the sub-H-space of $SG(n+1)$ consisting of orientation preserving homotopy equivalences of S^n which fix a point p. By the adjoint correspondence there is an isomorphism $A_{n,j} : \pi_j(SF(n)) \cong \pi_{j+n}(S^n)$. We relate the homotopy groups of $SF(n)$ to $SG(n+1)$ by noting that $SF(n)$ is the fiber of the fibration

$$\begin{array}{rccc} ev: & SG(n+F) & \to & S^n \\ & f & \mapsto & f(p). \end{array}$$

Moreover, $ev|_{SO(n+1)} : SO(n+1) \to S^n$ is the standard fibration with fiber $SO(n) \subset SF(n)$. When $n = 3$, $ev|_{SO(4)}$ has a section $s : S^3 \to SO(4)$ given by $s(x) = a(0,n)(x)$. Applying the long exact homotopy sequence to this pair of fibrations at π_3 we obtain the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi_3(SO(3)) \arrow[r] \arrow[d, "j_*"'] & \pi_3(SO(4)) \arrow[r, "ev|_{SO(4)*}"] \arrow[d, "i_{4*}"] & \pi_3(S^3) \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & \pi_3(SF(3)) \arrow[r] \arrow[d, "\cong"] & \pi_3(SG(4)) \arrow[r, "ev_*"] \arrow[d, "\cong"] & \pi_3(S^3) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{Z}/12 \arrow[r] & \mathbb{Z}/12 \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

To conclude the proof we note the following. Firstly, if $n \in \pi_3(S^3)$ then $s_*(n) = \bar{\alpha}(0,n) \in \pi_3(SO(4))$. Secondly, if $m \in \pi_3(SO(3))$ then $i_*(m) = \bar{\alpha}(m,0)$. Finally, if $j_n : SO(n) \hookrightarrow SF(n)$ denotes the inclusion of one fiber into the other then the homomorphism

$$A_{n,j} \circ j_{n*} : \pi_j(SO(n)) \to \pi_{n+j}(S^n)$$

is the usual J-homomorphism and the J-homomorphism $J_{3,3} : \pi_3(SO(3)) \to \pi_6(S^3)$ is known to be the surjection $\mathbb{Z} \to \mathbb{Z}/12$ ([**JW2**]). $\square$

Now, let L be a 8-dimensional handlebody and let the quadratic function of L be $\kappa(L) = \kappa(H, \lambda, \alpha)$. Recall from Section 2.2 that for every choice of basis, $\{v_1 \dots v_l\}$, for H there is a corresponding presentation for the handlebody L,

$$\phi = \sqcup_{i=1}^{l} \phi_i : \sqcup_{i=1}^{l} (S^3 \times D^4)_i \hookrightarrow \partial D^8,$$

and that L is diffeomorphic to the adjunction space $D^8 \cup_\phi \sqcup_{i=1}^{l} (D^4 \times D^4)_i$. The isotopy invariants of ϕ are the linking numbers c_{ij} of the attaching spheres $\bar{\phi}(S_i^3)$ where $S_i^3 = (S^3 \times 0)_i$, and the framing invariants $\bar{\alpha}(i) \in \pi_3(SO(4))$ which we denote now by ba_i. They are related to λ and α as follows:

$$c_{ij} = \lambda(v_i, v_j)\ (i \neq j), \qquad E(\bar{\alpha}_i) = \lambda(v_i, v_i), \qquad S(\bar{\alpha}_i) = \alpha(v_i).$$

LEMMA 6.4. *Suppose that the handlebody L has a presentation*

$$\phi = \sqcup_{i=1}^{l} \phi_i : \sqcup_{i=1}^{n} (S^3 \times D^4)_i \hookrightarrow \partial D^8$$

with invariants $c_{ij}, (i \neq j)$, and $\bar{\alpha}_i$, and let $\bar{\gamma}_i = \bar{\alpha}(m_i, n_i) \in \pi_3(SO(4))$. Then

$$\phi(\bar{\gamma}_1, \dots, \bar{\gamma}_l) := \sqcup_{i=1}^{l} \phi_i \circ A(m_i, n_i)$$

is a presentation with invariants $c_{ij}, (i \neq j)$, and $\bar{\alpha}_i + \bar{\gamma}_i$.

PROOF. As $A(m_i, n_i)(x, 0) = (x, 0)$ for every $x \in S^3$, $\phi_i \circ A(m_i, n_i)|_{S_i^3} = \phi_i|_{S_i^3}$. Hence the attaching spheres and linking numbers c_{ij} are unchanged. Recall that the framing invariant $\bar{\alpha}_i$ is defined as the homotopy class of $a(\phi_i)$ where

$$\psi_i : \phi_i((S^3 \times S^4)_i) \to \bar{\phi}_i(S_i^3) \times D^4$$

is a framing of $\phi_i((S^3 \times D^4)_i)$ which extends to a framing of the normal bundle of a 4-disc embedded in D^8 bounding $\phi_i((S^3 \times 0)_i)$ and where

$$\begin{array}{rccc} \psi_i \circ \phi_i : & (S^3 \times D^4)_i & \to & \bar{\phi}_i(S_i^3) \times D^4 \\ & (x, y) & \mapsto & (\bar{\phi}_i(x), a(\phi_i)(x)(y)). \end{array}$$

After isotopy we may assume that $a(\phi_i) = a(p_i, q_i)$ for some p_i and q_i. For the presentation $\phi(\bar{\gamma}_1, \dots, \bar{\gamma}_l)$ we consider instead the diffeomorphism $\psi_i \circ \phi_i \circ A(m_i, n_i)$ which is

$$\begin{array}{rccc} A(p_i, q_i) \circ A(m_i, n_i) : & (S^3 \times D^4)_i & \to & \bar{\phi}_i(S^3) \times D^4 \\ & (x, y) & \mapsto & (\bar{\phi}_i(x), x^{m_i+n_i+p_i+q_i}.y.x^{-m_i-p_i}). \end{array}$$

It follows that the framing functions for $a(\phi(\bar{\gamma}_1, \dots, \bar{\gamma}_l)_i)$ are $a(n_i + p_i, m_i + q_i)$ and that the framing invariants for $\phi(\bar{\gamma}_1, \dots, \bar{\gamma}_l)$ are

$$\bar{\alpha}(n_i + p_i, m_i + q_i) = \bar{\alpha}(p_i, q_i) + \bar{\alpha}(n_i, m_i) = \bar{\alpha}_i + \bar{\gamma}_i.$$

$\square$

We shall abbreviate $\bar{\gamma}_1, \dots, \bar{\gamma}_l$ to $\bar{\gamma}$ and denote the handlebody obtained from the twisted presentation by $L(\bar{\gamma})$.

LEMMA 6.5. *Let L, ϕ, $L(\bar{\gamma})$ and $\phi(\bar{\gamma})$ be as in the previous lemma and let* $\bar{\gamma}_i \in \operatorname{Ker}[i_{4*} : \pi_3(SO(4)) \to \pi_3(SG(4))]$ *for $i = 1, \dots, l$. Then there is a homotopy equivalence*

$$g(\bar{\gamma}) : (L(\bar{\gamma}), \partial L(\bar{\gamma})) \to (L, \partial L).$$

PROOF. We make the identifications

$$L = D^8 \cup_\phi \sqcup_{i=1}^{l} (D^4 \times D^4)_i \text{ and } L(\bar{\gamma}) = D^8 \cup_{\phi(\bar{\gamma})} \sqcup_{i=1}^{l} (D^4 \times D^4)_i.$$

The map $g(\bar{\gamma})$ shall take each handle of L to the corresponding handle of $L(\bar{\gamma})$. On the zero handle, D^8, we define $g(\bar{\gamma})$ to be the identity. Now consider the ith handles of L and $L(\bar{\gamma})$ which we denote by $(D^4 \times D^4)_i$. The map $g(\bar{\gamma})$ has been defined only on $(\partial D^4 \times D^4)_i$ where it is $\phi_i^{-1} \circ \phi(\bar{\gamma})_i$. Now $\phi(\bar{\gamma})_i = \phi_i \circ A(m_i, n_i)$ where $A(m_i, n_i)$ is the twisting corresponding to $a(m_i, n_i)$. It follows that $g(\bar{\gamma})|_{(\partial D^4 \times D^4)_i} = A(m_i, n_i)$. As $a(m_i, n_i)$ represents $\bar{\gamma}_i$ and $i_{4*}(\bar{\gamma}_i) = 0 \in \pi_3(SG(4))$ we conclude from Lemma 6.3 firstly that each $n_i = 0$ and that 12 divides m_i. Secondly, for every j, $i_4 \circ a(12j, 0) : S^3 \to SG(4)$ is null homotopic so we let $b(12j, 0) : D^4 \to SG(4)$ be such a null homotopy and we define an extension of $A(12j, 0)$ as follows. For $x \in D^4$, $y \in S^3$, $t \in [0, 1]$ and $ty \in D^4$ let

$$\begin{array}{rccc} B(12j,0): & D^4 \times D^4 & \to & D^4 \times D^4 \\ & (x, ty) & \mapsto & (x, tb(12j,0)(x)(y)). \end{array}$$

On each handle $(D^4 \times D^4)_i$ we define $g(\bar{\gamma})$ to be $B(m_i, 0)$.

We now show that $g(\bar{\gamma})$ is a homotopy equivalence of pairs. The boundary of L is obtained by performing surgeries on ∂D^8 described by the embeddings ϕ,

$$\partial L = (\partial D^8 - [\sqcup_{i=n}^{l} \phi_i((\partial D^4 \times D^4)_i)] \cup_{\phi|_{\sqcup_{i=1}^{l} \partial D^4 \times \partial D^4}} \sqcup_{i=1}^{l} (D^4 \times \partial D^4)_i.$$

A similar statement holds true for $\partial L(\bar{\gamma})$. Since $g(\bar{\gamma})|_{\partial D^8} = Id_{\partial D^8}$ and for each i, $B(m_i,0)((D^4 \times \partial D^4)_i) = (D^4 \times \partial D^4)_i$, we see that $g(\bar{\gamma})$ maps the boundary of $L(\bar{\gamma})$ to the boundary of L. The generators of $H_4(L)$ and $H_4(L(\bar{\gamma}))$ may be represented by embedded spheres constructed by gluing the cores of each handle to 4-discs embedded in D^8 and bounding the corresponding attaching spheres. Since $B(m_i,0)$ is the identity map when restricted to the core of each handle and since $g(\bar{\gamma})$ is the identity on the zero handle we see that $g(\bar{\gamma})$ carries the generators of $H_4(L)$ to the generators of $H_4(L(\bar{\gamma}))$. Since L and $L(\bar{\gamma})$ have non-zero homology only in dimensions 0 and 4 we conclude that $g(\bar{\gamma})_* : H_*(L(\bar{\gamma})) \cong H_*(L)$. Since all the groups involved are free we conclude that $g(\bar{\gamma})^* : H^*(L) \cong H^*(L(\bar{\gamma}))$. We now apply the naturality of the Poincaré duality isomorphisms $H^{8-*}(L) \cong H_*(L, \partial L)$ and $H^{8-*}(L(\bar{\gamma})) \cong H_*(L(\bar{\gamma}), \partial L(\bar{\gamma}))$ to conclude that $g(\bar{\gamma})_* : H_*(L(\bar{\gamma}), \partial L(\bar{\gamma})) \cong H_*(L, \partial L)$. Applying the five lemma to the long exact homology sequences of $(L, \partial L)$ and $(L(\bar{\gamma}), \partial L(\bar{\gamma}))$, it follows that $(g(\bar{\gamma})|_{\partial L(\bar{\gamma})})_* : H_*(\partial L(\bar{\gamma})) \cong H_*(\partial L)$. Thus $g(\bar{\gamma})$ induces a homology equivalence of simply connected pairs and so by Whitehead's Theorem $g(\bar{\gamma})$ induces a homotopy equivalence of pairs. □

COROLLARY 6.6. *Let L be a handlebody with quadratic function $\kappa(H, \lambda, \alpha)$ and for each $\epsilon \in 24 \cdot H^*$ let L_ϵ be the handlebody with quadratic function $\kappa(L_\epsilon) = \kappa(H, \lambda, \alpha+\epsilon)$. Then there is a homotopy equivalence*

$$g_\epsilon : (L_\epsilon, \partial L_\epsilon) \to (L, \partial L)$$

such that $g_{\epsilon *} = Id : H \to H$.

PROOF. Let $\{v_1, \dots, v_l\}$ be a basis for H and let ϕ and ϕ_ϵ be presentations for L and L_ϵ respectively which correspond to this basis. If the invariants of ϕ are c_{ij} and $\bar{\alpha}_i$ then the invariants for ϕ_ϵ are c_{ij} and $\bar{\alpha}_i + \bar{\gamma}_i$ where $\bar{\gamma}_i = \bar{\alpha}(\epsilon(v_i)/2, 0)$. As 24 divides ϵ, 12 divides $\epsilon(v_i)/2$ for each i and so by Lemma 6.3, $\bar{\gamma}_i \in \mathrm{Ker}(i_{4*})$. Thus we apply Lemma 6.5 and set $f_\epsilon = g(\bar{\gamma})$. The induced map g_{ϵ_*} is the identity because, as we noted in the proof of Lemma 6.5, g_ϵ maps the embedded sphere representing v_i in L_ϵ to the corresponding sphere in L. Hence for each i, $g_{\epsilon *}(v_i) = v_i$. Since $\{v_1, \ldots, v_n\}$ is a basis for H, $g_{\epsilon *} = Id_H$. $\square$

LEMMA 6.7. *Let* $g_\epsilon : L(\kappa(H, \lambda, \alpha + \epsilon)) \to L(\kappa(H, \lambda, \alpha))$ *be the homotopy equivalence of handle bodies defined in Corollary 6.6. Then* g_ϵ *has PL-normal invariant* $\eta^{PL}(g_\epsilon) = \epsilon/24 \in \mathcal{N}^{PL}(L) \cong H^{4j}(L) = H^*$.

PROOF. Having previously identified $\mathcal{N}^{PL}(L)$ with $H^4(L; \pi_4(G/PL))$ we consider the change of coefficient map to $U : H^4(L; \pi_4(G/PL)) \to H^4(L; \pi_4(BPL))$. If $[f] \in H^4(L; \pi_4(G/PL))$ corresponds to a degree one normal map

$$\begin{array}{ccc} \nu_M & \xrightarrow{\hat{f}} & \nu \\ \downarrow & & \downarrow \\ (M, \partial M) & \xrightarrow{f} & (L, P), \end{array}$$

then $U([f])$ corresponds to the difference of bundles $\nu_L - \nu \in [L, BPL]$. Now the map $\pi_4(G/PL) \to \pi_4(BPL)$ is $\cdot 24 : \mathbb{Z} \to \mathbb{Z}$ as it has cokernel $\pi_3^S \cong \mathbb{Z}/24$ and $\pi_4(BPL) \cong \pi_4(BO) \cong \mathbb{Z}$. Thus U is injective and all normal invariants in L are detected by the bundle data. We wish to calculate $U(\eta(g_\epsilon))$. Now, as g_ϵ is a homotopy equivalence $\eta(g_\epsilon)$ is the degree on normal map

$$\begin{array}{ccc} \nu_{L_\epsilon} & \xrightarrow{\hat{g_\epsilon}} & (g_\epsilon^{-1})^* \nu_{L_\epsilon} \\ \downarrow & & \downarrow \\ (L_\epsilon, \partial L_\epsilon) & \xrightarrow{g_\epsilon} & (L, \partial L) \end{array}$$

where $L = L(\kappa(\lambda, \alpha))$ and $L_\epsilon = L(\kappa(\lambda, \alpha + \epsilon))$. Note that $\nu_L = -\alpha$ is the inverse of the stable tangent bundle of L and that g_ϵ induces the identity on cohomology.

$$\begin{aligned} U(\eta(g_\epsilon)) &= \nu_L - (g_\epsilon^{-1})^* \nu_{L_\epsilon} \\ &= (-\alpha) - (g_\epsilon^{-1})^*(-\alpha - \epsilon) \\ &= \epsilon \end{aligned}$$

Since U is the coefficient homomorphism which is multiplication by 24, $\eta(g_\epsilon) = \epsilon/24$.

□

COROLLARY 6.8. *Let L be an 8-dimensional handlebody with quadratic function $\kappa = \kappa(H, \lambda, \alpha)$ and boundary $P = \partial L$. If $g_\epsilon : L_\epsilon := L_(\lambda, \alpha + \epsilon) \to L(\lambda, \alpha)$ is the homotopy equivalence of Corollary 6.6 and let $f_\epsilon = \partial L_\epsilon$, then,*

1. $\mathcal{S}^{PL}(L) = \{[g_\epsilon : L_\epsilon \to L] | \, \epsilon \in 24 \cdot H^*\}$,
2. $\mathcal{S}^{PL}(P) = \{[f_\epsilon : \partial L_\epsilon \to \partial L] | \, \epsilon \in 24 \cdot H^*\}$.

PROOF. The first assertions follows immediately from Lemma 6.7 and the isomorphism $\mathcal{S}^{PL}(L) \cong H^4(L)$ which was discussion in Section 2 following Fact 6.2. The second assertion follows from the fact that the map $i^* : \mathcal{S}^{PL}(L) \to \mathcal{S}^{PL}(P)$ is onto, which was demonstrated in the same discussion. □

It is a simple task to use Corollary 6.8 to descend from the PL classification of handlebodies L to their homotopy classification. We introduce some extra notation so that the theorem we state is of the same form as Wall's corresponding result for closed manifolds ([**Wa1**] Lemma 8). We let $J : \pi_3(SO) \to \pi_3^S$ denote the stable J-homomorphism which is known to be the surjection $\mathbb{Z} \to \mathbb{Z}/24$. The J-homomorphism induces a coefficient homomorphism for any space X, $J_* : H^*(X; \pi_3(SO(4)) \to H^*(X; \pi_3^s)$.

THEOREM 6.9. *Two handlebodies $L(\kappa_0)$ and $L(\kappa_1)$ with quadratic functions $\kappa_0 = \kappa(H_0, \lambda_0, \alpha_0)$ and $\kappa_1 = \kappa(H_1, \lambda_1, \alpha_1)$ are homotopy equivalent if and only if there is* *an isometry of their intersection forms,* $\Theta : H_0 \to H_1, \lambda_1 \circ (\Theta \times \Theta) = \lambda_0$, *such that* $J_*(\alpha_1) = J_*(\Theta^*(\alpha_0))$. *Moreover, every such isomorphism* Θ *is induced by a homotopy equivalence* $f : L_0 \to L_1$.

PROOF. From Corollary 6.8 we deduce that the handlebody L_1 is homotopic to L_0 if and only if L_1 is PL-homeomorphic to $L(\kappa(H_0, \lambda_0, \alpha_0 + \epsilon))$ for some $\epsilon \in 24 \cdot H^*$. Now by the classification of handlebodies (see Remark 2.6) this is the case if and only if there is an isometry

$$\Theta : \kappa(H_1, \lambda_1, \alpha_1) \cong \kappa(H_0, \lambda_0, \alpha_0 + \epsilon))$$

for some $\epsilon \in 24 \cdot H^*$. But by Proposition 2.10 there is such an isometry if and only if there is an isomorphism $\Theta : H_1 \to H_0$ such that $\lambda_1 \circ (\Theta \times \Theta) = \lambda_0$ and $\Theta^*(\alpha_0 + \epsilon) = \alpha_1$ for some $\epsilon \in 24 \cdot H^*$. But $\mathrm{Ker}(J_*) = 24 \cdot H^*$ and thus there is an $\epsilon \in 24 \cdot H^*$ such that $\Theta^*(\alpha_0 + \epsilon) = \alpha_1$ if and only if $J_*(\Theta^*(\alpha_0)) = J_*(\alpha_1)$.

Finally, suppose $\Theta : H_1 \to H_0$ is as in the statement of this theorem. Then by the remarks above there is an $\epsilon \in 24 \cdot H_0^*$ such that Θ defines an isometry $\kappa_1 \cong \kappa(H_0, \lambda_0, \alpha_0 + \epsilon)$. This isometry is implemented by a diffeomorphism g which we compose with g_ϵ from Corollary 6.6. Since $g_{\epsilon *} = Id$, $(g_\epsilon \circ g)_* = g_* = \Theta$. □

In order to apply Corollary 6.8 to the homotopy classification of highly connected 7-manifolds we first remind the reader of the definition of the characteristic quadratic linking family induced by a characteristic quadratic function and define the related concept of a J-quadratic linking family. Suppose that $\kappa(H, \lambda, \alpha)$ is a (characteristic) quadratic function. Recall that G and F denote respectively $\mathrm{Cok}(\hat{\lambda})$ and $\mathrm{Ker}(\hat{\lambda})$ ant that $\tau_G : G \to G^{**}$ and $\rho : H \to H/F$ denote the canonical maps whose sets of sections are $\mathrm{Sec}(\rho)$ and $\mathrm{Sec}(\tau_G)$. Each section $\Psi \in \mathrm{Sec}(\rho)$ induces a splitting $H^* = F^*(\Psi) \oplus H^*(\Psi)$ and every $x \in H^*$ splits as $x = (p_F(\Psi)^*(x|_F), x(\Psi))$ where $p_F(\Psi) : H \to F$. In addition, each Ψ defines a section $\Phi(\Psi)$ in $\mathrm{Sec}(\tau_G)$ and also a nondegenerate quadratic function $\kappa(\Psi)$. The quadratic linking function of κ, $\delta^c(\kappa)$, is the triple $(G, q^c(\kappa), \beta)$ where $\beta = [\alpha]$ is the image of α in G and $q^c(\kappa) : \mathrm{Sec}(\tau) \to \mathcal{Q}(b(\lambda))$ is the function defined by $q^c(\Phi(\Psi)) = q^c(\kappa(\Psi))$. Here $\mathcal{Q}(b)$ denotes the set of quadratic linking functions which refine the linking form induced by λ, $b(\lambda)$. A crucial property of $q^c(\kappa)$ is that it is equivariant with respect to the actions of $\mathrm{Hom}(G^{**}, TG)$ on $\mathrm{Sec}(\tau_G)$ and $\mathcal{Q}(b(\lambda))$ which are defined, for $\phi \in \mathrm{Hom}(G^{**}, TG)$, $\Phi \in \mathrm{Sec}(\tau_G)$ and $q \in \mathcal{Q}(b(\lambda))$ by $\phi \cdot \Phi = \Phi - \phi$ and $\phi \cdot q = q_{\phi(\tau_G(\beta)/2)}$. We now make the following definitions for an abelian group G containing the even element β and with torsion subgroup TG. Firstly, given a linking form b on TG, let $\mathcal{Q}_J(b)$ be the subset of the power set of $\mathcal{Q}(b)$ consisting of sets of the form

$$S(q) := \{q_{12a} | a \in TG, q \in \mathcal{Q}(b)\}.$$

We let $\mathrm{Hom}(G^{**}, TG)$ act on $\mathcal{Q}_J(b)$ by setting $\phi \cdot S(q) := S(q_{\phi(\tau(\beta))/2})$ where $\phi \in \mathrm{Sec}(\tau_G)$ and $S(q) \in \mathcal{Q}_J(b)$.

DEFINITION 6.10. 1. *A* J-quadratic linking family *is a triple* (G, q^c_J, β_J) *where* G *is a finitely generated abelian group,* β_J *is an even element of* $G \otimes \mathbb{Z}/24$ *and* $q^c_J : \mathrm{Sec}(\tau_G) \to \mathcal{Q}_J(b)$ *is a function for some linking form* b *on* TG *and which satisfies the following properties.*

(A) q^c_J *is equivariant with respect to the actions of* $\mathrm{Hom}(G^{**}, TG)$ *on* $\mathrm{Sec}(\tau_G)$.

(B) *If* $q^c_J(\Phi) = S(q)$ *then* $\beta_J = (\Phi(\tau(\beta)) + \beta(q)) \otimes 1 \in G \otimes \mathbb{Z}/24$.

2. *For any characteristic quadratic linking family,* $Q = (G, q^c, \beta)$*, the of* J*-quadratic linking family of* Q *is the triple* $Q_J = (G, q^c_J, \beta_J)$*, where* $\beta_J = \beta \otimes 1 \in G \otimes \mathbb{Z}/24$ *and*

$$q^c_J(\Phi) := S(q^c(\Phi)) = \{q^c(\Phi)_{12a} | \, a \in TG\}.$$

3. *Let* P *be a highly connected* 7*-dimensional manifold. The* J*-quadratic linking family of* P*,* $Q_J(P)$*, is the* J*-quadratic linking family of* $Q(P)$*.*

Recall from Section 2.10 that an isomorphism of abelian groups $\theta : G_0 \to G_1$ induces a bijection $\tau_\theta : \mathrm{Sec}(\tau_{G_0}) \to \mathrm{Sec}(\tau_{G_1})$. If $Q_{J0} = (G_0, q^c_{J0}, \beta_{J0})$ and $Q_{J1} = (G_1, q^c_{J1}, \beta_{J1})$ are J-quadratic linking families then an isomorphism $\theta : G_0 \cong G_1$ shall be called a J-isometry if $(\theta \otimes Id)(\beta_{J0}) = \beta_{J1}$ and if for every $\Phi \in \mathrm{Sec}(\tau_{G_0})$,

$$q^c_{J1}(\tau(\theta(\Phi)) \circ \theta|_{TG_0} = q^c_{J0}(\Phi)$$

where, if $q^c_{J1}(\tau(\theta(\Phi)) = S(q)$, then $q^c_{J1}(\tau(\theta(\Phi)) \circ \theta|_{TG_0} = S(q \circ \theta|_{TG_0})$.

THEOREM 6.11. *Let P_0 and P_1 be a pair of 2-connected 7-manifolds with J-quadratic linking families $Q_J(P_0) = (G_0, q^c_{J0}, \beta_{J0})$ and $Q_J(P_1) = (G_1, q^c_{J1}, \beta_{J1})$. Then there is a homotopy equivalence $f : P_0 \to P_1$ such that $f_! = \theta : H^4(P_0) \cong H^4(P_1)$ if and only if θ defines a J-isometry of J-quadratic linking families from $Q_J(P_0)$ to $Q_J(P_1)$.*

PROOF. In dimension 7 every highly connected manifold bounds a handlebody. So for $i = 0, 1$, let $P_i = \partial L_i$ and let the handlebody L_i have quadratic function $\kappa_i = \kappa(H_i, \lambda_i, \alpha_i)$. Suppose that $f : P_0 \to P_1$ is a homotopy equivalence. As there are no obstructions to smoothing 7-dimensional PL-manifolds, $\mathcal{S}^{PL}(P_1) \cong \mathcal{S}^O(P_1)/\Theta_7$, where Θ_7 acts by connected sum with a homotopy sphere in the domain of a homotopy equivalence. Applying Corollary 6.8 we conclude that for some $\epsilon \in 24 \cdot H_1^*$ there is an almost diffeomorphism $f' : P_0 \to \partial L_\epsilon$ such that $f = f_\epsilon \circ f'$, where $g_\epsilon : L_\epsilon := L(\kappa(H_1, \lambda_1, \alpha_1 + \epsilon)) \to L_1$ is the homotopy equivalence of Lemma 6.6 and $f_\epsilon = g_\epsilon|_{L_\epsilon}$. By Theorem B, $f'_!$ defines an isometry of quadratic linking families $f'_! : Q^c(P_0) \cong \delta^c(\kappa(\lambda_1, \alpha_1 + \epsilon))$. It is clear from the definitions that an isometry of quadratic linking families induces an isometry of the induced J-quadratic linking families. Moreover, $f_! = f_{\epsilon!} \circ f'_! = f'_!$ since $f_{\epsilon!} = Id_{G_1}$. So it remains to check that Id_{G_1} defines a J-isometry from $Q_J(\partial(L_\epsilon))$ to $Q_J(\partial L_1) = Q_J(P_1)$. Clearly since 24 divides ϵ, $\beta_1 \otimes 1 = (\beta_1 + [\epsilon]) \otimes 1 \in G_1 \otimes \mathbb{Z}/24$. For any $\Phi \in \mathrm{Sec}(\tau_{G_1})$, let $\Psi \in \mathrm{Sec}(\rho_1)$ be such that

$\Phi(\Psi) = \Phi$. Recalling that $\epsilon(\Psi)$ denotes the component of ϵ in $H_1^*(\Psi)$ we calculate that

$$\begin{aligned} q_J^c(\kappa(\lambda_1, \alpha_1 + \epsilon))(\Phi) &= S[q^c(\kappa(\lambda_1, \alpha_1 + \epsilon))(\Psi)] \\ &= S[q^c(\kappa(\lambda_1(\Psi), \alpha_1(\Psi) + \epsilon(\Psi))] \\ &= S[q^c(\kappa(\lambda_1(\Psi), \alpha_1(\Psi))_{[\epsilon(\Psi)/2]}] \quad \text{by Lemma 2.37} \\ &= S[q^c(\kappa(\lambda_1(\Psi), \alpha_1(\Psi))] \quad \text{since 12 divides } [\epsilon(\Psi)/2] \\ &= S[q^c(\kappa_1)(\Psi)] \\ &= S[q^c(\kappa_1)(\Phi)] \\ &= q_{J1}^c(\Phi). \end{aligned}$$

Hence Id_{G_1} defines a J-isometry from $Q_J(L_\epsilon)$ to $Q_J(P_1)$.

Conversely, suppose that θ is a J-isometry from $Q_J^c(P_0)$ to $Q_J^c(P_1)$. Let $\Phi_0 \in \mathrm{Sec}(\tau_{G_0})$ be any section and choose $\Psi_0 \in \mathrm{Sec}(\rho_0)$ and $\Psi_1 \in \mathrm{Sec}(\rho_1)$ such that $\Phi_0 = \Phi(\Psi_0)$ and $\tau_\theta(\Phi_0) = \Phi(\Psi_1)$. Then for some $a \in TG_1$,

$$q^c(\kappa_1(\Psi_1))_{12a} \circ \theta|_{TG_0} = q^c(\kappa_1)(\tau_\theta(\Phi_0))_{12a} \circ \theta|_{TG_0} = q^c(\kappa_0)(\Phi_0).$$

Applying Lemma 2.37 in reverse, choose an $\epsilon' \in H_1^*(\Psi_1)$ which is divisible by 24 and so that $q^c(\kappa(\lambda_1(\Psi_1), \alpha_1(\Psi_1) + \epsilon')) = q^c(\kappa_1(\Psi_1))_{12a}$. It follows that

$$q^c(\kappa(\lambda_1(\Psi_1), \alpha_1(\Psi_1) + \epsilon')) \circ \theta|_{TG_0} = q^c(\kappa_0)(\Phi_0)$$

and in particular that $\theta(\beta(q^c(\Phi_0)) = [\alpha(\Psi_1) + \epsilon']$. Now $\beta_0 = \Phi_0(\tau_{G_0}(\beta_0)) + \beta(q^c(\Phi_0))$ and so $\theta(\beta_0) - [\alpha(\Psi_1) + \epsilon'] = \theta(\Phi_0(\tau_{G_0})(\beta_0)) = \tau_\theta(\Phi_0)(\beta_1)$. Now $\alpha_1 - \alpha_1(\Psi_1) \in F_1^*(\Psi_1)$ and so we may choose $\epsilon'' \in F_1^*(\Psi_1)$ such that $\theta(\beta_0) = [\alpha_1 + \epsilon' + \epsilon'']$ and ϵ'' is necessarily divisible by 24 as $(\theta(\beta_0) \otimes 1) = \beta_1 \otimes 1 \in G_1 \otimes \mathbb{Z}/24$. Setting $\epsilon = \epsilon' + \epsilon'' \in 24 \cdot H_1^*$, we see that $(\alpha_1 + \epsilon)(\Psi_1) = \alpha_1(\Psi_1) + \epsilon'$ and we calculate that

$$\begin{aligned} q^c(\kappa(\lambda_1, \alpha_1 + \epsilon))(\tau_\theta(\Phi_0)) \circ \theta|_{TG_0} &= q^c(\kappa(\lambda_1, \alpha_1 + \epsilon))(\Psi_1) \circ \theta|_{TG_0} \\ &= q^c(\kappa(\lambda_1(\Psi_1), \alpha_1(\Psi_1) + \epsilon')) \circ \theta|_{TG_0} \\ &= q^c(\kappa_0)(\Phi_0). \end{aligned}$$

By the equivariance property of characteristic quadratic linking families, θ defines an isometry from $Q^c(P_0)$ to $\delta^c(\kappa(\lambda_1, \alpha_1 + \epsilon))$. By Theorem B, there is an almost diffeomorphism $f' : P_0 \to \partial L(\kappa(\lambda_1, \alpha_1 + \epsilon))$ such that $f'_! = \theta$. Composing f' with f_ϵ we obtain a homotopy equivalence $f_\epsilon \circ f'$ such that $(f_\epsilon \circ f')_! = f_{\epsilon !} \circ f'_! = Id_{G_1} \circ \theta = \theta$. □

4. Smooth S^3 bundles over S^4

As we remarked in the introduction, the recent interest in 2-connected 7-manifolds was generated by Grove and Ziller's geometric result that the total space of every S^3-bundle over S^4 carries a metric of non-negative sectional curvature [**GZ**]. These manifolds have a lengthy history in differential topology which is covered in [**CE**] where the manifolds are classified up to diffeomorphism, PL-homeomorphism, homeomorphism and homotopy. The proofs in [**CE**] used the results of Wilkens [**Wi1, Wi2**] and calculations of the invariants s_1 and $\bar{s}_1$. By Hatcher's result on the homotopy type of Diff(S^3) [**Ha**] the classification holds equally well for smooth bundles over S^4 with fiber a differentiable 3-sphere. We now present the classification of these total spaces using the results and notation of this thesis.

Recall from Section 3 the oriented $\mathbb{R}^4$-bundles over S^4, $\xi(m,n)$. Let $L_{m,n}$ be the handle body which is the total space of the disc bundle of $\xi(m,n)$ and let $P_{m,n}$ be the boundary of $L_{m,n}$. We orient $L_{m,n}$ and $P_{m,n}$ by fixing an orientation on S^4 and using the orientation of the vector bundle on the fiber.[2] There is a diagram of commuting

[2] We warn the reader that this orientation convention differs from the one use in [**CE**]. There, following James and Whitehead, we fixed an orientation on S^4 and oriented the fiber so that the Euler number of $\xi(m,n)$ was positive. The two conventions agree for $n \geq 0$. The later one seems more natural.

fiber bundles

$$\begin{tikzcd} D^4 \arrow[r] & L_{m,n} \arrow[r, "\pi_L"] & S^4 \\ S^3 \arrow[r] \arrow[u] & P_{m,n} \arrow[r, "\pi_P"] \arrow[u, "i"] & S^4. \arrow[u, "Id"] \end{tikzcd}$$

Let s_4 and s^4 denote the preferred generators of $H_4(S^4)$ and $H^4(S^4)$ respectively. The group $H_4(L_{m,n})$ is infinite cyclic with a preferred generator s such that $\pi_{L*}(s) = s_4$. The bundle $\xi(m,n)$ has Euler number n and stable invariant $n+2m$, thus $s \cdot s = n$ and the tangential invariant of $L_{m,n}$, $\alpha_{m,n}$, satisfies $\alpha_{m,n}(s) = (n+2m)$.[3] The group $H^4(P_{m,n})$ is isomorphic to $\mathbb{Z}/n$ with preferred generator $e = \pi_P^*(s^4)$. By applying Proposition 1.4 and Remark 2.6 it is easy to see that $L_{m_0,0}$ (resp. $P_{m_0,0}$) is diffeomorphic to $L_{m_1,0}$ (resp. $P_{m_1,0}$) if and only if $m_0 = \pm m_1$. We assume now that $n \neq 0$. The algebraic invariants of $L_{m,n}$ and $P_{m,n}$ are as follows (where j and k are integers, we have used Proposition 2.57 to compute $s_1(P_{m,n})$ and $\bar{s}_1(P_{m,n})$ and $\epsilon(n) = n/|n|$):

$$\begin{array}{cl}
\alpha_{m,n} : \mathbb{Z} \to \mathbb{Z} & j \cdot s = (n+2m)j, \\
\lambda_{m,n} = \lambda_n : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} & (j \cdot s, k \cdot s) \mapsto njk, \\
\kappa_{m,n} : \mathbb{Z} \longrightarrow \mathbb{Z} & j \cdot s \mapsto nj^2 + (n+2m)j, \\
\beta_{m,n} \in \mathbb{Z}/n & \beta_{m,n} = 2m \cdot e \\
b_{m,n} = b_n : \mathbb{Z}/n \times \mathbb{Z}/n \longrightarrow \mathbb{Q}/\mathbb{Z} & (j \cdot e, k \cdot e) \mapsto \frac{1}{n}(jk) \bmod \mathbb{Z} \\
q_{m,n} : \mathbb{Z}/n \longrightarrow \mathbb{Q}/\mathbb{Z} & je \mapsto \frac{1}{2n}(j^2 + 2mj) \bmod \mathbb{Z} \\
s_1(P_{m,n}) = \frac{1}{28} \cdot \frac{1}{8n}((n+2m)^2 - n\epsilon(n)) & s_1(P_{m,n}) = \frac{1}{8n}((n+2m)^2 - n\epsilon(n)).
\end{array}$$

When $n \neq 0$, $P_{m,n}$ is a rational homology sphere which falls under the classification Theorem A. There (*1* and *5*) it is stated that the isometry class of $q_{m,n}$ determines the topological classification and (*2*) that the isometry class of $q_{m,n}$ along with $s_1(P_{m,n})$ determines the smooth classification. Moreover by (*4*), the quadratic function $q_{m,n}$ is

[3] The cohomology class $\alpha_{m,n} \in H^4(L_{m,n})$ is also equal to the Spin characteristic class, $\frac{p_1}{2}(L_{m,n})$.

determined by the pair $(b(n), \beta_{m,n})$ and $\bar{s}_1(P_{m,n}) \in \mathbb{Q}/\mathbb{Z}$. Lastly, by Theorem A (*6*), P_{m_0,n_0} is homotopy equivalent to P_{m_1,n_1} if and only if $q_{m_0,n_0} \cong (q_{m_1,n_1})_{12je}$ for some integer j. It is then a matter of definitions and some simple algebra that the following classification holds.

THEOREM 6.12 ([**CE**]). *Let $n \neq 0$ and $\epsilon = \pm 1$. The total spaces $P_{m_0,n}$ and $P_{m_1,\epsilon n}$ are (1.) diffeomorphic, (2.) homeomorphic or (3.) homotopic if and only if the following (appropriately numbered) condition holds:*

1. *(a)* $4m_0(n+m_0) + (n^2 - n) \equiv \pm[4m_1(n_1 + \epsilon m_1) + \epsilon(n^2 - n)] \pmod{224n}$,

   (b) $2m_0 \equiv 2\alpha m_1 \pmod{n}$ *for some* $\alpha \in \mathbb{Z}/n$ *such that* $\alpha^2 \equiv \pm\epsilon \pmod{n}$.
2. *(a)* $4m_0(n+m_0) + (n^2 - n) \equiv \pm[4m_1(n_1 + \epsilon m_1) + \epsilon(n^2 - n)] \pmod{8\ n}$,

   (b) $2m_0 \equiv 2\alpha m_1 \pmod{n}$ *for some* $\alpha \in \mathbb{Z}/n$ *such that* $\alpha^2 \equiv \pm\epsilon \pmod{n}$.
3. *(a)* $m_0 \equiv \alpha m_1 \pmod{(n, 12)}$ *where* $\alpha^2 \equiv \pm\epsilon \pmod{(n, 12)}$.

The diffeomorphism/homeomorphism/homotopy equivalence is orientation preserving or reversing as the relevant equations hold with either a $+$ sign or $-$ sign.

There are more detailed arithmetical calculations in [**CE**] concerning how many of the smooth structures within a given topological type are realized by S^3-bundles over S^4 and of the numbers $\alpha \in \mathbb{Z}/n$ such that $\alpha^2 = \pm 1 \pmod{n}$. We conclude by noting that the manifolds $P_{1,8}$ and $P_{5,8}$ are tangentially homotopy equivalent but not homeomorphic as we now show. One computes that $\bar{s}_1(P_{1,8}) = 9/16 \neq 1/16 = \bar{s}_1(P_{5,8})$ and hence $P_{1,8}$ and $P_{5,8}$ are not homeomorphic by Theorem A *3*. The quadratic functions $P_{1,8}$ and $P_{5,8}$ are respectively $q^3(1)_{5e}$ and $q^3(1)_e$. Since, $q^3(1)_{5e+12e} = q^3(1)_e$, it follows by Theorem A *6* that $P_{1,8}$ and $P_{5,8}$ are homotopy equivalent via a homotopy equivalence f which induces the identity on $H^4(P_{1,8}) = \mathbb{Z}/8 = H^4(P_{5,8})$. Since

$\beta(P_{1,8}) = \beta(P_{5,8}) = 2 \in \mathbb{Z}/8$, it follows that f is tangential.[4] This pair of manifolds yields a counter example to Theorems C and Theorem 5.10 of [**MTW**] where it is claimed, amongst other things, that all tangentially equivalent 2-connected 7-manifolds are homeomorphic.

[4]Note that the same conclusion holds for every pair of manifolds $(P^7, *P^7)$ for which $(b(P), \beta(P)) \cong (b(*P), \beta(*P))$ but $\bar{s}_1(P) \neq \bar{s}_1(*P)$. There is such a pair for every finite abelian group with order divisible by 8.

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