Wiener criterion on metric spaces: Boundary regularity in axiomatic and Poincaré-Sobolev spaces

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Abstract

We obtain a Wiener-type criterion for the Hölder continuity of extremal functions on general metric spaces in an abstract setting. We use then this result to establish the boundary regularity of quasi-minimizers of the p-energy integral in the axiomatic framework of Gol'dshtein-Troyanov and also for extremal functions from the class of Poincaré-Sobolev functions.

Introduction

The problem of minimizing a variational integral in a set of functions with prescribed boundary values is closely related to solving the corresponding Dirichlet problem for its Euler-Lagrange equation. In particular, for the Dirichlet p-energy integral

$$\int_{\Omega\subset\mathbb{R}^n} |\nabla u(x)|^p dx$$

the corresponding Euler-Lagrange equation, for $1 < p < \infty$, is

$$\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = 0,$$

and the minimizers of the former functional are solutions of the latter equation. In the case when $p = 2$, this equation reduces to the well-known Laplace equation $\Delta u = 0$, whose solutions are called harmonic functions. A fundamental problem in the potential theory is to study boundary behavior of the solutions of such equations and the minimizers of the corresponding variational problems.

In 1924, N. Wiener [23] established a criterion to characterize continuity at the boundary for harmonic functions. For the more general case of elliptic equations the first steps to find a similar criterion were made by W. Littman, G. Stampacchia and H.F. Weinberger [14] who proved that a point in the boundary of an arbitrary domain was simultaneously regular for harmonic functions and weak solutions of linear equations with bounded, measurable coefficients. In his work concerning the local behavior of solutions of quasilinear equations, J. Serrin [20] discovered that a capacity, now known as the p-capacity, was the appropriate measurement for describing removable sets for weak solutions. Later, V.G. Maz'ya [16], [17] discovered a Wiener-type expression involving this capacity which provided a sufficient condition for continuity at the boundary of weak solutions of equations whose structure is similar to that of the p-Laplacian. Utilizing different techniques, R. Gariepy and W.P. Ziemer [3] showed that the Maz'ya's condition was also sufficient for boundary continuity for solutions of a large class of quasilinear equations in divergence form. After some time, Ziemer [24] generalized this result for quasi-minimizers (minimizers up to a multiplicative constant), a concept generalizing the notion of solutions of elliptic equations and variational problems.

When one wants to treat similar questions on general metric spaces, where a version of the p-energy integral can be defined, and one considers the problem of boundary regularity for the (quasi-)minimizers of this energy, of a particular relevance is the famous method of De Giorgi [2]. There are several ways to generalize the notion of p-energy (or equivalently the (length of) gradient) on a general metric spaces and to introduce the corresponding Sobolev-type spaces. One of the first is the notion of Sobolev space considered by P. Hajłasz [8]. Among other known approaches are the Sobolev spaces via the upper gradients [11], [21] well adapted to the length spaces, the axiomatic Sobolev spaces of Gol'dshtein-Troyanov [6],[7] and the Sobolev spaces based on a Poincaré inequality, first considered in [13] and later developed in [9]. Let us stress here that the last two concepts are quite different from the upper gradients Sobolev spaces in their methods and spirit. In [1] J. Björn has applied the De Giorgi's method to study the Hölder continuity at the boundary for the quasi-minimizers of the p-Dirichlet integral on an abstract metric space. She has obtained a (sufficiency part of) Wiener-type criterion for boundary regularity using the notion of upper gradients.

The main goal of the present paper is to show that the De Giorgi's method can also be applied to establish the boundary regularity within the axiomatic or the Poincaré inequality framework, and even in more general situations.

Our strategy is close in spirit to that of [22], where we prove the interior regularity of extremal functions in the same context. In particular, we follow the same as in [22] pattern, reducing the De Giorgi argument to a system of three hypotheses H1-H3 (hypothesis H3 in the present case is different from that in [22]) that a function u defined on a measure metric space may satisfy. These hypotheses are very general, they can be formulated for any function u in L^p, and in particular we do not assume that a notion of Sobolev space has been defined. The essence of the De Giorgi argument for the boundary regularity is first to show that a function satisfying the three hypotheses plus an additional assumption, which is an abstract form of the Wiener condition, is Hölder continuous at a boundary point, and then to show that a function minimizing an appropriate energy must satisfy these hypotheses. The technique leaves us a lot of freedom to choose various kind of energies. We show in the second part of the paper how this technique can be applied to prove the boundary Hölder regularity of extremal functions in an axiomatic Sobolev space, and in the third part we consider the case of functions satisfying a Poincaré inequality and the De Giorgi condition. The results are stated in Theorems 3.2 and 4.1.

The paper is organized as follows. In the first section we formulate the hypotheses H1-H3 we work with later and we show in Theorem 1.5 that a function u satisfying these hypotheses together with the abstract Wiener condition (4) is Hölder continuous at a boundary point. The arguments in the proof of Theorem 1.5 are similar to those in [1]. Essentially, they reproduce the De Giorgi argument for the regularity at the boundary, revisited in our abstract setting. For the sake of completeness we give the details of the proof in Section 1.2. Section 2 gives necessary preliminaries on the axiomatic and Poincaré-Sobolev spaces. The third section is devoted to the boundary regularity of a quasi-minimizer of the energy functional in the axiomatic setting. We show that all three hypotheses H1-H3 are satisfied by the quasi-minimizer and its minimal pseudo-gradient and, therefore, that the quasi-minimizer is Hölder continuous at a boundary point. In the forth section we verify that functions from the Poincaré-Sobolev space, which have an additional property (De Giorgi condition), satisfy the hypotheses H1-H3 and, thus, are also Hölder continuous at a boundary point.

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1 Boundary Regularity in an Abstract Setting

Throughout the paper (X, d) will be a metric space equipped with a Borel regular outer measure μ. For any ball $B = B(R) = B(z, R) = \{x \in X : d(x, z) < R\}$ and $\sigma > 0$ we denote by σB the ball $B(z, \sigma R)$.

Our standing assumption for the measure μ is that it is *doubling*, i.e. that there exists a constant $C_d \geq 1$ such that for all balls $B \subset X$ we have

$$\mu(2B) \leq C_d \mu(B) .$$

C_d is called the *doubling constant*. Note that this property of the measure μ implies, in particular, that the measure of a ball of positive radius is strictly greater than zero.

We will suppose for convenience that the space X is locally compact and separable. For $1 \leq p < \infty$, $L^p_{loc}(X) = L^p_{loc}(X, d, \mu)$ is the space of measurable functions on X which are p-integrable on every relatively compact subset of X.

At the beginning of this section we want to underline that in the sequel the notation $g_{(u)}$ for a function from $L^p(X)$ means no a priori dependence of this function on the given function $u \in L^p_{loc}(X)$, whereas g_u stands for the minimal pseudo-gradient of the function u (see Section 2.1).

Let Ω be an open subset of X and $x_0 \in \partial\Omega$ be a boundary point of the set Ω. Suppose also that functions $u \in L^p_{loc}(X)$ and $\vartheta \in L^p(X)$ are such that $u = \vartheta$ a.e. on $X \setminus \Omega$, and that the function ϑ is Hölder continuous at the point x_0. In this section we prove that if the functions u and $-u$ satisfy at the point x_0 Hypotheses H1-H3 stated below in the pairs with some functions $g_{(u)}, g_{(-u)} \in L^p(X)$ respectively, then the function u is Hölder continuous at the boundary point x_0.

Unless otherwise stated, C denotes a positive constant whose exact value is unimportant, can change even within a line and depends only on fixed parameters, such as X, d, μ, p and others.

1.1 List of hypotheses

Let x_0, a boundary point of Ω, be fixed. The hypotheses for two functions $u, g_{(u)} \in L^p(\Omega)$, which we shall need are the following:

Hypothesis H1 (De Giorgi condition) *There exist constants $C > 0$ and $k^* \in \mathbb{R}$, such that for all $k \geq k^*$ and $0 < \rho < R \leq \operatorname{diam}(X)/3$, the following Caccioppoli type inequality on the "upper-level" sets of the function u holds*

$$\int_{A(k,\rho)} g_{(u)}^p d\mu \leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p d\mu, \tag{1}$$

where $A(k,r) = A_z(k,r) = \{x \in B(x_0,r) = B(r) \colon u(x) > k\}$.

Let η be a $\frac{C}{(R-\rho)}$-Lipschitz (cutoff) function for some $C > 0$, such that $0 \leq \eta \leq 1$, the support of η is contained in $B(\frac{R+\rho}{2})$ and $\eta = 1$ on $B(\rho)$.

Hypothesis H2 *There exists a constant $C > 0$ such that for functions $v = \eta\,(u-k)_+$ and $g_{(v)} = g_{(u)}\,\chi_{A(k,\frac{R+\rho}{2})} + \frac{C}{R-\rho}(u-k)_+$ and for some t and q, $t > p > q$, we have*

$$\left(\fint_{B(\frac{R+\rho}{2})} v^t d\mu \right)^{\frac{1}{t}} \leq CR \left(\fint_{B(\frac{R+\rho}{2})} g_{(v)}^q d\mu \right)^{\frac{1}{q}}, \tag{2}$$

where k, ρ and R are as in Hypothesis H1. *Here, as usual, $(u-k)_+ = \max\{u-k, 0\}$, $\chi_{A(k,\frac{R+\rho}{2})}$ is the characteristic function of the set $A(k,\frac{R+\rho}{2})$.*

Hypothesis H3 *There exists a function $\Phi : 2^X \times X \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for fixed $\Omega \subset X$ and $x_0 \in X$, $\frac{\Phi(\Omega,x_0,R)}{R}$ is bounded for all $R \in \mathbb{R}_+$, Φ is not constant for all of its three arguments, and for all $h, k \in \mathbb{R}$, $h > k \geq k^*$, for the functions $w = u_k^h$ and $g_{(w)} = g_{(u)}\,\chi_{\{k<u\leq h\}}$ we have*

$$\left(\int_{B(R)} w^q d\mu \right)^{\frac{1}{q}} \leq \Phi(\Omega, x_0, R) \left(\int_{B(\sigma R)} g_{(w)}^q d\mu \right)^{\frac{1}{q}}, \tag{3}$$

where $\sigma > 1$ is a constant and q is as in Hypothesis H2.

Remark 1.1. *In* [22], *where we studied the regularity in the interior of Ω, the function $\Phi(\Omega, x_0, R)$ of Hypothesis* H3 *is replaced by the term $C\,R$ in the inequality* (3), *where $C > 0$ is a constant. Hypotheses* H1 *and* H2 *stay unchanged.*

Remark 1.2. *Hypotheses* H2 *and* H3 *are the characteristics of the Sobolev space of functions we will work with in the next sections, whereas Hypothesis* H1 *is the property of some particular functions, the functions whose regularity we want to establish. Hypotheses* H2 *and* H3 *are Sobolev-type inequalities* *that are typically true for pairs $(u, g_{(u)})$ in a sufficiently nice metric measure space: they essentially assert that the associated Poincaré inequality remains stable under cutoffs and truncations.*

Remark 1.3. *The introduction of the auxiliary function Φ in Hypothesis* H3 *quantifies the validity of a Sobolev-type inequality near a boundary point. Appearing later in a Wiener-type criterion for the boundary regularity, it unifies various possible types of the Winer condition like the measure density condition or capacitary conditions. In particular, a typical example of the function Φ is given (up to a constant) by the following quantity*

$$\Phi(\Omega, x_0, R) = \left(\frac{\mu(B(x_0, R))}{C_q\left(B(x_0, \frac{1}{2}R) \setminus \Omega\right)} \right)^{\frac{1}{q}},$$

where C_q is the Sobolev q-capacity of the axiomatic setting (see Proposition 3.3 *and Remark* 3.4*). Another example of Φ is (up to a constant) the function*

$$\Phi(\Omega, x_0, R) = R\left(\frac{\mu(B(x_0, R))}{\mu(B(x_0, R) \setminus \Omega)} \right)^{1-\frac{1}{q}},$$

(see Proposition 4.2*).*

1.2 Boundedness and Hölder continuity

In this subsection we show that Hypotheses H1-H3 for a function $u \in L^p_{loc}(\Omega)$ at a boundary point x_0 combined with the abstract Wiener-type condition (4) guarantee the Hölder continuity of u at x_0.

Theorem 1.4. *Suppose that a pair of functions $(u, g_{(u)})$ satisfies Hypotheses* H1 *and* H2 *at the boundary point x_0. Then for all $k \geq k^*$ there exists a constant $C > 0$ such that*

$$\operatorname*{ess\,sup}_{B(x_0, \frac{R}{2})} u \leq k + C\left(⨍_{B(x_0,R)} (u-k)_+^p \, d\mu \right)^{\frac{1}{p}}.$$

Proof See the proof of Theorem 1.1 in [22].

Theorem 1.5. *If the pairs $(u, g_{(u)})$ and $(-u, g_{(-u)})$ satisfy Hypotheses* H1-H3 *(with some $g_{(-u)} \in L^p(X)$), the function $\vartheta \in L^p(X)$ is Hölder continuous at $x_0 \in \partial\Omega$ and the following condition is satisfied*

$$\liminf_{\rho \to 0} \frac{1}{|\log \rho|} \int_\rho^1 \exp\left(-C\left(\frac{\Phi(R)}{R} \right)^{\frac{pq}{p-q}} \right) \frac{dR}{R} > 0\,, \tag{4}$$

for some constant $C > 0$, then the function u is Hölder continuous at x_0.

With $\Omega \subset X$ and $x_0 \in \partial\Omega$ being fixed, here and in the sequel we indicate for simplicity the dependance of the function $\Phi : (\Omega, x_0, R) \mapsto \Phi(\Omega, x_0, R)$ only on its third argument, i.e. instead of $\Phi(\Omega, x_0, R)$ we write $\Phi(R)$.

Proof Using the inequality (3) for our auxiliary functions w and $g_{(w)}$ with some h and k, $h > k \geq k^*$, we obtain

$$\begin{aligned}
(h-k)\mu(A(h,R)) = \int_{A(h,R)} w d\mu \ &\leq \int_{B(R)} w d\mu \\
&\leq \left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&\leq \Phi(R)\left(\int_{B(\sigma R)} {g_{(w)}}^q d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&= \Phi(R)\left(\int_{B(\sigma R)} {g_{(u)}}^q\, \chi_{\{k<u\leq h\}} d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}} \\
&= \Phi(R)\left(\int_{A(k,\sigma R)\setminus A(h,\sigma R)} {g_{(u)}}^q d\mu\right)^{\frac{1}{q}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned}$$

Hence, by Hölder inequality we have

$$\begin{aligned}
(h-k)\mu(A(h,R)) \leq \Phi(R)&\left(\int_{A(k,\sigma R)} {g_{(u)}}^p d\mu\right)^{\frac{1}{p}} \\
&\times \left(\mu(A(k,\sigma R)) - \mu(A(h,\sigma R))\right)^{\frac{1}{q}-\frac{1}{p}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned}$$

Since the functions u and $g_{(u)}$ satisfy the inequality (1), we conclude that

$$\begin{aligned}
(h-k)\mu(A(h,R)) \leq C\frac{\Phi(R)}{R}&\left(\int_{A(k,2\sigma R)} (u-k)^p d\mu\right)^{\frac{1}{p}} \\
&\times \left(\mu(A(k,\sigma R)) - \mu(A(h,\sigma R))\right)^{\frac{1}{q}-\frac{1}{p}} \mu(B(R))^{1-\frac{1}{q}}.
\end{aligned} \tag{5}$$

With $x_0 \in \partial\Omega$ fixed, let us denote for R_0, $0 < 2\sigma R \leq R_0 \leq \operatorname{diam}(X)/3$,

$$M(R, R_0) = (\operatorname*{ess\,sup}_{B(x_0,R)} u - \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta)_+.$$

Let us also define $M = M(2\sigma R, R_0)$ and

$$k_j = \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta + M(1-2^{-j}), \quad j \in \mathbb{N}.$$

Replacing now h by k_{j+1} and k by k_j in the inequality (5) we obtain

$$(k_{j+1}-k_j)\mu(A(k_{j+1},R)) \le C\frac{\Phi(R)}{R}\left(\int_{A(k_j,2\sigma R)}(u-k_j)^p d\mu\right)^{\frac{1}{p}} \times (\mu(A(k_j,\sigma R)) - \mu(A(k_{j+1},\sigma R)))^{\frac{1}{q}-\frac{1}{p}}\,\mu(B(R))^{1-\frac{1}{q}},$$

Noting that $k_{j+1}-k_j = \frac{M}{2^{j+1}}$ and denoting

$$T_j(\sigma R) = (\mu(A(k_j,\sigma R)) - \mu(A(k_{j+1},\sigma R)))^{\frac{1}{q}-\frac{1}{p}}$$

we further have

$$\begin{aligned}
&\frac{M}{2^{j+1}}\mu(A(k_{j+1},R)) \le \\
&\quad \le C\frac{\Phi(R)}{R}\left(\int_{A(k_j,2\sigma R)}(u-k_j)^p d\mu\right)^{\frac{1}{p}} T_j(\sigma R)\,\mu(B(R))^{1-\frac{1}{q}} \\
&\quad \le C\frac{\Phi(R)}{R}\,\mu(B(2\sigma R))^{\frac{1}{p}}\,(\operatorname*{ess\,sup}_{B(2\sigma R)} u - k_j)_+\,T_j(\sigma R)\,\mu(B(R))^{1-\frac{1}{q}} \\
&\quad \le C\frac{\Phi(R)}{R}\,\mu(B(R))^{1-\frac{1}{q}+\frac{1}{p}}\,\frac{M}{2^j}\,T_j(\sigma R).
\end{aligned}$$

In the last inequality we have used the doubling condition of the measure μ. Dividing both parts of the last inequality by $\frac{M}{2^{j+1}}$ and recalling the expression of $T_j(\sigma R)$ we obtain

$$\mu(A(k_{j+1},R)) \le C\frac{\Phi(R)}{R}\,\mu(B(R))^{1-\frac{1}{q}+\frac{1}{p}}\,(\mu(A(k_j,\sigma R)) - \mu(A(k_{j+1},\sigma R)))^{\frac{1}{q}-\frac{1}{p}}.$$

If $n \ge j+1$, then the set $A(k_{j+1},R)$ on the left-hand side can be replaced by $A(k_n,R)$ and the inequality remains true. We have

$$\mu(A(k_n,R))^{\frac{pq}{p-q}} \le C\left(\frac{\Phi(R)}{R}\right)^{\frac{pq}{p-q}}\mu(B(R))^{\frac{pq}{p-q}-1}\,(\mu(A(k_j,\sigma R)) - \mu(A(k_{j+1},\sigma R))).$$

Now summing up over $j = 0,1,\dots,n-1$ and using the doubling property of μ, we obtain

$$\mu(A(k_n,R))^{\frac{pq}{p-q}} \le \frac{C}{n}\left(\frac{\Phi(R)}{R}\right)^{\frac{pq}{p-q}}\mu(B(R))^{\frac{pq}{p-q}},$$

or

$$\left(\frac{\mu(A(k_n,R))}{\mu(B(R))}\right)^{\frac{pq}{p-q}} \le \frac{C}{n}\left(\frac{\Phi(R)}{R}\right)^{\frac{pq}{p-q}}. \tag{6}$$

Theorem 1.4 with k replaced by k_n and the fact that $u-k_n \le 2^{-n}M$ on $B(R)$ give

$$\begin{aligned}
\operatorname*{ess\,sup}_{B(x_0,\frac{R}{2})} u &\le k_n + C\left(\fint_{B(R)} (u-k_n)_+^p\,d\mu\right)^{\frac{1}{p}} \\
&= \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta + M(1-2^{-n}) + C\left(\frac{1}{\mu(B(R))}\int_{A(k_n,R)} (u-k_n)^p\,d\mu\right)^{\frac{1}{p}} \\
&\le \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta + M(1-2^{-n}) + \frac{CM}{2^n}\left(\frac{\mu(A(k_n,R))}{\mu(B(R))}\right)^{\frac{1}{p}}.
\end{aligned} \tag{7}$$

As the function $\frac{\Phi(R)}{R}$ is bounded for all $R\in\mathbb{R}_+$, using the estimate (6) we see that the last term on the right-hand side in (7) is at most $2^{-n-1}M$, whenever

$$n \ge n(R) = C\left(\frac{\Phi(R)}{R}\right)^{\frac{pq}{p-q}}. \tag{8}$$

Inserting the smallest integer $n\ge n(R)$ into (7) gives the following inequality

$$\operatorname*{ess\,sup}_{B(x_0,\frac{R}{2})} u \le \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta + M\left(1-\frac{1}{2^{n+1}}\right).$$

Noting that if

$$\operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta \ge \operatorname*{ess\,sup}_{B(x_0,\frac{R}{2})} u\,,$$

then

$$M(\tfrac{1}{2}R,R_0) = \big(\operatorname*{ess\,sup}_{B(x_0,\frac{R}{2})} u - \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta\big)_+ = 0\,,$$

we see that it follows from the last inequality that

$$M(\tfrac{1}{2}R,R_0) \le (1-2^{-n(R)-2})\,M(2\sigma R,R_0)\,. \tag{9}$$

Let $C>0$ and $n(R)$ be as in (8), and

$$\omega(R) = 2^{-n(2R)} = \exp\left(-C_0\left(\frac{\Phi(2R)}{2R}\right)^{\frac{pq}{p-q}}\right),$$

with $C_0 = C\log 2$.

For $m = 1,2$, we divide the interval $(0,R_0)$ into two disjoint subsets as follows

$$I_m = \bigcup_{j=1}^{\infty} \left[(4\sigma)^{m-2j-1}R_0, (4\sigma)^{m-2j}R_0\right).$$

Then $I_1 \cup I_2 = (0,R_0)$, and hence for some m,

$$\int_\rho^{R_0} \omega(R)\frac{dR}{R} \le 2\int_{(\rho,R_0)\cap I_m} \omega(R)\frac{dR}{R} \tag{10}$$

For $j = 1,2,\dots$, choose $R_j \in \left[(4\sigma)^{m-2j-1}R_0, (4\sigma)^{m-2j}R_0\right)$ so that

$$\begin{aligned}\int_{(4\sigma)^{m-2j-1}R_0}^{(4\sigma)^{m-2j}R_0} \omega(R)\frac{dR}{R} &\le \frac{\omega(R_j)}{R_j}\int_{(4\sigma)^{m-2j-1}R_0}^{(4\sigma)^{m-2j}R_0} dR \\ &\le \frac{\omega(R_j)}{R_j}(4\sigma-1)(4\sigma)^{m-2j-1}R_0 \le (4\sigma-1)\omega(R_j)\,.\end{aligned} \tag{11}$$

We take some ρ, $0<\rho<R_0$. Then there exists a number $N \in \mathbb{N}$ such that $\rho \in \left[(4\sigma)^{m-2N-1}R_0, (4\sigma)^{m-2N}R_0\right)$. Summing up the inequality (11) for $j = 1,2,\dots,N$, and using the fact that $\omega(R) < 1$ for all $R > 0$, we get

$$\begin{aligned}\int_{(\rho,R_0)\cap I_m} \omega(R)\frac{dR}{R} &\le \int\limits_{((4\sigma)^{m-2j-1}R_0,R_0)\cap I_m} \omega(R)\frac{dR}{R} \\ &\le \int\limits_{((4\sigma)^{m-2j}R_0,R_0)\cap I_m} \omega(R)\frac{dR}{R} + \int\limits_{((4\sigma)^{m-2j-1}R_0,(4\sigma)^{m-2j}R_0)\cap I_m} \omega(R)\frac{dR}{R} \\ &\le (4\sigma-1)\sum_{\rho\le R_j\le R_0}\omega(R_j) + \ln(4\sigma).\end{aligned} \tag{12}$$

Now we apply the inequality (9) for R_j, $j = 1,2,\dots$, to obtain

$$\begin{aligned}M\left((4\sigma)^{m-2j-1}R_0, R_0\right) &\le M(R_j,R_0) \le \left(1-2^{-n(2R_j)-2}\right)M(4\sigma R_j,R_0) \\ &\le \left(1-\frac{\omega(R_j)}{4}\right)M(4\sigma R_j,R_0) \le \left(1-\frac{\omega(R_j)}{4}\right)M((4\sigma)^{m-2j+1}R_0,R_0).\end{aligned}$$

Iterating this estimate, we get for $0<\rho<R_0$,

$$M(\rho,R_0) \le \prod_{\rho\le R_j\le R_0}\left(1-\frac{\omega(R_j)}{4}\right)M(R_0,R_0). \tag{13}$$

Using the fact that $\log(1-t) \le -t$ for $t<1$ and noting that $\omega(R)<1$, we have

$$\begin{aligned}\prod_{\rho\le R_j\le R_0}\left(1-\frac{\omega(R_j)}{4}\right) &= \exp\log\prod_{\rho\le R_j\le R_0}\left(1-\frac{\omega(R_j)}{4}\right)\\ &= \exp\sum_{\rho\le R_j\le R_0}\log\left(1-\frac{\omega(R_j)}{4}\right)\\ &\le \exp\left(-\sum_{\rho\le R_j\le R_0}\frac{\omega(R_j)}{4}\right).\end{aligned}$$

Which gives us together with the inequality (13) the following estimate

$$M(\rho,R_0)\le \exp\left(-\frac{1}{4}\sum_{\rho\le R_j\le R_0}\omega(R_j)\right)M(R_0,R_0).$$

This inequality and the inequalities (12) and (10) imply that

$$M(\rho,R_0)\le CM(R_0,R_0)\exp\left(-\frac{1}{8(4\sigma-1)}\int_\rho^{R_0}\omega(R)\frac{dR}{R}\right),$$

or, recalling the expression of $\omega(R)$, that

$$\begin{aligned}&M(\rho,R_0)\le \qquad (14)\\ &\quad\le CM(R_0,R_0)\exp\left(-\frac{1}{8(4\sigma-1)}\int_\rho^{R_0}\exp\left(-C_0\left(\frac{\Phi(2R)}{2R}\right)^{\frac{pq}{p-q}}\right)\frac{dR}{R}\right),\end{aligned}$$

where $C=(4\sigma)^{\frac{1}{4(4\sigma-1)}}$.

The fact that the function ϑ is continuous at x_0 allows us to assume, without loss of generality, that $\vartheta(x_0)=0$.

As the function $-u$ satisfies in the pair with the function $g_{(-u)}$ Hypotheses H1-H3, in the rest of the proof it suffices to estimate $(u(x)-\vartheta(x_0))_+ = u_+(x)$ in the ball $B(x_0,\rho)$. The same estimate will hold for the function u_-.

For $R_0>0$, we have

$$M(R_0,R_0)=(\operatorname*{ess\,sup}_{B(x_0,R_0)}u-\operatorname*{ess\,sup}_{B(x_0,R_0)}\vartheta)_+\le(\operatorname*{ess\,sup}_{B(x_0,R_0)}u-\vartheta(x_0))_+=\operatorname*{ess\,sup}_{B(x_0,R_0)}u_+,$$

and from Theorem 1.4 it follows that

$$M := \operatorname*{ess\,sup}_{B(x_0,R_0)} u_+ < \infty\,.$$

For $0 < \rho < R_0$ the inequality (14) gives

$$\begin{aligned} \operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ &\le \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta_+ + M(\rho, R_0) \\ &\le \operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta_+ + \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (15) \\ &+ CM \exp\left(-\frac{1}{8(4\sigma - 1)} \int_\rho^{R_0} \exp\left(-C_0 \left(\frac{\Phi(2R)}{2R} \right)^{\frac{pq}{p-q}} \right) \frac{dR}{R} \right). \end{aligned}$$

As the function ϑ is Hölder continuous at x_0 and $\vartheta(x_0) = 0$, there exist some $C', \beta > 0$ such that for all sufficiently small ρ and R_0 we have

$$\operatorname*{ess\,sup}_{B(x_0,R_0)} \vartheta_+ \le C' R_0^\beta\,.$$

By the assumption (4) of the theorem, there exists (sufficiently small) $\alpha > 0$ such that

$$\int_\rho^1 \exp\left(-C_0 \left(\frac{\Phi(2R)}{2R} \right)^{\frac{pq}{p-q}} \right) \frac{dR}{R} \ge \alpha |\log \rho|\,.$$

Note also that for all $0 < R_0 < 1$,

$$\int_{R_0}^1 \exp\left(-C_0 \left(\frac{\Phi(2R)}{2R} \right)^{\frac{pq}{p-q}} \right) \frac{dR}{R} \le \int_{R_0}^1 \frac{dR}{R} = |\log R_0|\,.$$

From the inequality (15) and the last three inequalities, for sufficiently small ρ and R_0, we have

$$\operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ \le C' R_0^\beta + CM \rho^{\frac{\alpha}{8(4\sigma-1)}} R_0^{-\frac{1}{8(4\sigma-1)}}\,.$$

Choosing now $R_0 = \rho^{\alpha'}$ with $\alpha' = \frac{\alpha}{8(4\sigma-1)\beta+1}$, we obtain

$$\operatorname*{ess\,sup}_{B(x_0,\rho)} u_+ \le H \rho^\gamma\,,$$

where $H = C' + CM$ and $\gamma = \alpha'\beta$. Note that with an appropriate choice of α, $0 < \gamma \le 1$.

As the same estimate holds for the function u_-, we have

$$\operatorname*{osc}_{B(x_0,\rho)} u = \operatorname*{ess\,sup}_{B(x_0,\rho)} u - \operatorname*{ess\,inf}_{B(x_0,\rho)} u \le 2H\rho^{\gamma}\,,$$

and thus, after a redefinition on a set of measure zero, the function u is Hölder continuous at x_0. □

2 Definitions and Basic Facts on Axiomatic and Poincaré-Sobolev Spaces

In this section we recall main definitions and give a brief summary of the axiomatic theory of Sobolev spaces developed by V.M. Gol'dshtein and M. Troyanov, and we shortly present the approach to Sobolev spaces on a metric space using Poincaré inequalities, introduced in [13]. These two Sobolev-type spaces constitute the general setup of our study in the next sections. We refer the reader to the papers [6], [7] and to the paper [9] for more details on these two theories.

2.1 Preliminaries on Axiomatic Sobolev Spaces

Definition 2.1 (D-structure). *A D-structure on (X, d, μ) is an operation which associates to each function $u \in L^p_{loc}(X)$ a collection $D[u]$ of measurable functions $g : X \to \mathbb{R}_+ \cup \{\infty\}$ (called the pseudo-gradients of u). The correspondence $u \to D[u]$ is supposed to satisfy the following axioms* A1-A5*:*

Axiom A1 (Non triviality) If $u : X \to \mathbb{R}$ is non-negative and k-Lipschitz, then the function

$$g := k\chi_{\mathrm{supp}(u)} = \begin{cases} k & \text{on} \quad \mathrm{supp}(u) \\ 0 & \text{on} \quad X \setminus \mathrm{supp}(u) \end{cases}$$

belongs to $D[u]$.

Axiom A2 (Upper linearity) If $g_1 \in D[u_1]$, $g_2 \in D[u_2]$ and $g \ge |\alpha|g_1 + |\beta|g_2$ almost everywhere, then $g \in D[\alpha u_1 + \beta u_2]$.

Axiom A3 (Strong Leibnitz rule) Let $u \in L^p_{loc}(X)$. If $g \in D[u]$, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ the function

$$h(x) = (|\varphi|g(x) + \mathrm{Lip}(\varphi)|u(x)|)$$

belongs to $D[\varphi u]$.

Axiom A4 (Lattice property) Let $u := \max\{u_1, u_2\}$ and $v := \min\{u_1, u_2\}$ where $u_1, u_2 \in L^p_{loc}(X)$. If $g_1 \in D[u_1]$, $g_2 \in D[u_2]$, then

$$g := \max\{g_1, g_2\} \in D[u] \cap D[v]\,.$$

Axiom A5 (Completeness) Let $\{u_i\}$ and $\{g_i\}$ be two sequences of functions such that $g_i \in D[u_i]$ for all i. Assume that $u_i \to u$ in $L^p_{loc}(X)$ topology and $(g_i - g) \to 0$ in L^p topology, then $g \in D[u]$.

Originally, in [6] in place of Axiom A3 stated here one postulates the following

Axiom A3* (Leibnitz rule) Let $u \in L^p_{loc}(X)$. If $g \in D[u]$, then for any bounded Lipschitz function $\varphi : X \to \mathbb{R}$ the function

$$h(x) = (\sup|\varphi| g(x) + \mathrm{Lip}(\varphi)|u(x)|)$$

belongs to $D[\varphi u]$ (The absolute value of φ is replaced by $\sup|\varphi|$).

This "weaker" version of the Leibnitz rule allows the authors to include in the class of axiomatic Sobolev spaces such spaces as graphs (combinatorial Sobolev spaces). Note, however, that these "global" spaces do not satisfy certain localization properties without which it is not clear how it would be possible to achieve the regularity results of the present paper.

We define a notion of energy and the associated Sobolev space as follows:

Definition 2.2 (Energy and Sobolev space). *The p-Dirichlet energy of a function $u \in L^p_{loc}(X)$ is defined to be*

$$\mathcal{E}_p(u) = \inf\left\{\int_X g^p d\mu : g \in D[u]\right\},$$

and the p-Dirichlet space is the space $\mathcal{L}^{1,p}(X)$ of functions from $L^p_{loc}(X)$ with finite p-energy. The Sobolev space is then the space

$$W^{1,p}(X) := \mathcal{L}^{1,p}(X) \cap L^p(X).$$

Theorem 2.3. *$W^{1,p}(X)$ is a Banach space with norm*

$$\|u\|_{W^{1,p}(X)} = \left(\int_X |u|^p d\mu + \mathcal{E}_p(u)\right)^{1/p}.$$

Proof See [6].

Proposition 2.4. *Let $1 < p < \infty$. Then for any function $u \in \mathcal{L}^{1,p}(X)$, there exists a unique function $g_u \in D[u]$ such that $\int_X g_u^p d\mu = \mathcal{E}_p(u)$.*

The function g_u is called the *minimal pseudo-gradient* of u.

Proof See [6].

For an open set $\Omega \subset X$ we denote by $C_0(\Omega)$ the set of continuous functions $u : \Omega \to \mathbb{R}$ compactly supported in Ω. $\mathcal{L}_0^{1,p}(\Omega)$ is then the closure of $C_0(\Omega) \cap \mathcal{L}^{1,p}(X)$ in $\mathcal{L}^{1,p}(X)$ for the norm

$$\|u\|_{\mathcal{L}^{1,p}(\Omega,Q)} = \left(\int_Q |u|^p d\mu + \mathcal{E}_p(u)\right)^{1/p},$$

where $Q \Subset \Omega$ is a fixed relatively compact subset of positive measure.

Definition 2.5 (Capacity). *The variational p-capacity of a pair $F \subset \Omega \subset X$ (where F is arbitrary) is defined as*

$$\operatorname{Cap}_p(F, \Omega) := \inf\{\mathcal{E}_p(u) |\ u \in \mathcal{A}_p(F, \Omega)\},$$

where the set of admissible functions is defined by

$$\mathcal{A}_p(F, \Omega) := \left\{u \in \mathcal{L}_0^{1,p}(\Omega) \,\middle|\, u \geq 1 \text{ on a neighbourhood of } F \text{ and } u \geq 0 \text{ a.e.}\right\}.$$

If $\mathcal{A}_p(F, \Omega) = \emptyset$, then we set $\operatorname{Cap}_p(F, \Omega) = \infty$. If $\Omega = X$, we simply write $\operatorname{Cap}_p(F, \Omega) = \operatorname{Cap}_p(F)$.

We now state a result about the existence and uniqueness of extremal functions for p-capacity. We first need the following definition:

Definition 2.6. (a) *A set $S \subset X$ is p-polar (or p-null) if for any pair of open relatively compact sets $\Omega_1 \subset \Omega_2 \neq X$ such that $\operatorname{dist}(\Omega_1, X \setminus \Omega_2) > 0$, we have $\operatorname{Cap}_p(S \cap \Omega_1, \Omega_2) = 0$.*

(b) *A property is said to hold p-quasi-everywhere if it holds everywhere except on a p-polar set.*

(c) *A Borel measure τ is said to be absolutely continuous with respect to p-capacity if $\tau(S) = 0$ for all p-polar subsets $S \subset X$.*

(d) *A subset F is said to be p-fat if it is a Borel subset and there exists a probability measure τ on X which is absolutely continuous with respect to p-capacity and whose support is contained in F.*

Theorem 2.7. *Let $F \subset X$ be a p-fat subset $(1 < p < \infty)$ of the space X, such that $\mathrm{Cap}_p(F) < \infty$. Then there exists a unique function $u^* \in \mathcal{L}_0^{1,p}(X)$ such that $u^* = 1$ p-quasi-everywhere on F and $\mathcal{E}_p(u^*) = \mathrm{Cap}_p(F)$. Furthermore $0 \le u^* \le 1$ for all $x \in X$.*

The function u^* is called the *capacitary function* of the set F.

Proof See [7].

Some additional properties a D-structure may satisfy are the validity of a Poincaré inequality and the locality of D-structure. We introduce these notions below.

Definition 2.8 (Poincaré inequality). *One says that a D-structure on a metric measure space X supports a weak (s,q)-Poincaré inequality, $s, q \ge 1$, if there exist two constants $\sigma \ge 1$ and $C_P > 0$ such that*

$$\left(⨍_B |u - u_B|^s d\mu\right)^{1/s} \le C_P\, r \left(⨍_{\sigma B} g^q d\mu\right)^{1/q} \tag{16}$$

for any ball $B \subset X$, any $u \in L^p_{loc}(X)$ and any $g \in D[u]$. Here r is the radius of B. Recall that

$$u_B = ⨍_B u\, d\mu = \frac{1}{\mu(B)} \int_B u d\mu.$$

By the Hölder inequality, a weak (s,q)-Poincaré inequality implies weak (s', q')-Poincaré inequalities with the same σ for all $s' \le s$ and $q' \ge q$. On the other hand, by Theorem 5.1 in [9], a weak $(1,q)$-Poincaré inequality implies a weak (s,q)-Poincaré inequality for some $s > q$ and possibly a new σ.

Definition 2.9 (Strong locality). *We say that a D-structure is strongly local if, in addition to Axioms A1-A5, the following property holds:*
Let $u_1, u_2 \in \mathcal{L}^{1,p}(X)$. If $g_1 \in D[u_1], g_2 \in D[u_2]$ and

$$g(x) = \begin{cases} g_1(x) & \text{if} \quad u_1(x) < u_2(x) \\ g_2(x) & \text{if} \quad u_1(x) > u_2(x) \\ \min\{g_1(x), g_2(x)\} & \text{if} \quad u_1(x) = u_2(x)\,, \end{cases}$$

then $g \in D[\min\{u_1, u_2\}]$.

This property enables one to "paste" two Sobolev functions along the set where they coincide. We refer the reader to [6] for other notions of locality in axiomatic Sobolev spaces.

Proposition 2.10. *Let $u \in \mathcal{L}^{1,p}(X)$ and $A \subset X$ be a relatively compact set. If the space X admits a D-structure which is strongly local, then*

$$\mathcal{E}_p(u|A) = \int_A g_u^p d\mu \, ,$$

in particular, if $u_1, u_2 \in \mathcal{L}^{1,p}(X)$ are such that $u_1 = u_2$ a.e. on A, then

$$\int_A g_{u_1}^p d\mu = \int_A g_{u_2}^p d\mu$$

Proof See [22].

Proposition 2.11. *If a D-structure on the space X is strongly local and supports a weak $(1,q)$-Poincaré inequality for some $0 < q < p$, then for every $z \in X$ and $0 < r < R < diam(X)/3$, there exists γ, $0 < \gamma < 1$, independent of R such that*

$$\frac{\mu(B(z,\frac{R}{2}))}{\mu(B(z,R))} \leq \gamma \, .$$

Proof See [22].

Theorem 2.7 gives the existence of the capacitary function of the p-Dirichlet energy in the axiomatic setting. For a p-fat subset F of X this function equals p-quasi-everywhere to 1 on F and minimizes the energy in the complement of F.

Definition 2.12 (Quasi-minimizer with boundary data). *Let $\vartheta \in L^p_{loc}(X)$. We say that a function $u \in L^p_{loc}(X)$ is a* quasi-minimizer *of the p-energy integral $\mathcal{E}_p$ on a set $\Omega \subset X$* with boundary data *ϑ if $\mu\,(\Omega \setminus supp(u - \vartheta)) = 0$ and there exists a constant $K > 0$ such that for all functions $\varphi \in \mathcal{L}^{1,p}(X)$ with $\mu(\Omega \setminus supp(\varphi)) = 0$ the inequality*

$$\int_{\varphi \neq 0} g_u^p d\mu \leq K \int_{\varphi \neq 0} g_{u+\varphi}^p d\mu$$

holds. Here, as usual, $g_{u+\varphi}$ is the minimal pseudo-gradient of $u + \varphi$. When $K = 1$, the corresponding quasi-minimizer is called the minimizer with boundary data *ϑ of the energy functional $\mathcal{E}_p$.*

When the D-structure is strongly local the capacitary function of the condenser F is a minimizer of $\mathcal{E}_p$ on the set $X \setminus F$ with boundary data 1 (see Proposition 2.19 in [22])

2.2 Poincaré-Sobolev functions

Definition 2.13 (Poincaré inequality). *Let $u \in L^1_{loc}(X)$ and $g : X \to [0, \infty]$ be Borel measurable functions. We say that the pair (u, g) satisfies a (s, q)-Poincaré inequality in $\Omega \subset X$, $s, q \geq 1$, if there exist two constants $\sigma \geq 1$ and $C_P > 0$ such that the inequality*

$$\left(\fint_B |u - u_B|^s d\mu \right)^{1/s} \leq C_P \, r \left(\fint_{\sigma B} g^q d\mu \right)^{1/q} \tag{17}$$

holds on every ball B with $\sigma B \subset \Omega$, where r is the radius of B.

Definition 2.14 (Poincaré-Sobolev functions). *A function $u \in L^1_{loc}(X)$ for which there exists $0 \leq g \in L^q(X)$ such that the pair (u, g) satisfies a $(1, q)$-Poincaré inequality in X is called a Poincaré-Sobolev function. We denote by $PW^{1,q}(X)$ the set of all Poincaré-Sobolev functions.*

The Poincaré inequality (17) is the only relationship between the functions u and g. Working in this setting P. Hajłasz and P. Koskela developed in [9] quite a rich theory of these Sobolev-type functions on metric spaces.

The pairs (u, g) may satisfy some additional properties crucial for our purposes. These are the truncation property and the p-De Giorgi condition which we next consider.

Given a function v and $\infty > t_2 > t_1 > 0$, we set

$$v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}.$$

Definition 2.15 (Truncation property). *Let the pair (u, g) satisfies a $(1, q)$-Poincaré inequality in Ω. Assume that for every $b \in \mathbb{R}$, $\infty > t_2 > t_1 > 0$, and $\varepsilon \in \{-1, 1\}$, the pair $(v_{t_1}^{t_2}, g\chi_{\{t_1 < v \leq t_2\}})$, where $v = \varepsilon(u - b)$, satisfies the $(1, q)$-Poincaré inequality in Ω (with fixed constants C_P, σ). Then we say that the pair (u, g) has the truncation property.*

The truncation property for Poincaré-Sobolev functions is the notion similar to the one of the strong locality in axiomatic Sobolev spaces. These notions reflect some localization properties of the Sobolev spaces under consideration. Note that in the Euclidean space $\mathbb{R}^n$ both conditions mean that the gradient of a function, which is constant on some set, equals zero a.e. on that set.

As we will see in the next section, the quasi-minimizers of the axiomatic p-Dirichlet energy satisfy the De Giorgi condition (Hypothesis H1). Note that this is also the case for the quasi-minimizers in the upper gradients Sobolev spaces (see [12]). For the class of Poincaré-Sobolev functions the possible notion of energy is not consistent, in particular it is not clear how it would be possible to prove the existence of corresponding minimizers, since in this case the corresponding Sobolev space is not a Banach, but only a quasi-Banach space. The De Giorgi condition however is still legitimate for the Poincaré-Sobolev functions. Thus, as it seems that there exists an intimate connection between extremal functions and the functions satisfying the De Giorgi condition, in the case of Poincaré-Sobolev functions, the functions whose regularity we establish are those which satisfy the following property:

Definition 2.16 (p-De Giorgi condition). *We say that a Poincaré-Sobolev function u (satisfying a $(1,q)$-Poincaré inequality with some function g) enjoys the p-De Giorgi condition on the set Ω if for all $k \in \mathbb{R}$, $z \in X$ and $0 < \rho < R \leq diam(X)/3$, the following inequality*

$$\int_{A(k,\rho)} g^p d\mu \leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u-k)^p d\mu, \tag{18}$$

holds, provided $\mu(\Omega \setminus A(k,R)) = 0$, where $A(k,r) = B(z,r) \cap \{x \colon u(x) > k\}$, $p \in \mathbb{R}$, $p > q$.

3 Boundary Regularity of Quasi-minimizers in Axiomatic Sobolev Spaces

Assume that the metric measure space (X,d,μ) is equipped with a D-structure and that F is a p-fat subset of X. In this section we show that under some additional conditions on the D-structure on X, a quasi-minimizer u^* of the energy functional $\mathcal{E}_p$ on the set $X \setminus F$ with a boundary data function ϑ is Hölder continuous at a boundary point of the set F.

For this we adapt the notations of Chapter 1 on the regularity at the boundary in an abstract setting and prove that the quasi-minimizer u^* satisfies Hypotheses H1-H3.

Namely, we set $\Omega = X \setminus F$ and we suppose that the quasi-minimizer u^* coincides a.e. with the function ϑ on the set $X \setminus \Omega = F$ and that ϑ is Hölder continuous at a boundary point x_0 of Ω.

For the proof of Proposition 3.3 we need the following

Lemma 3.1. *Let $f(r)$ be a nonnegative function defined on the interval $[R_1, R_2]$, where $R_1 \geq 1$. Suppose that for all $R_1 \leq r_1 < r_2 \leq R_2$,*

$$f(r_1) \leq \theta f(r_2) + \frac{A}{(r_2 - r_1)^\alpha} + B\,,$$

where $A, B \geq 0$, $\alpha > 0$ and $0 \leq \theta < 1$. Then there exists $C > 0$ depending only on α and θ such that for all $R_1 \leq r_1 < r_2 \leq R_2$,

$$f(r_1) \leq C\left(\frac{A}{(r_2 - r_1)^\alpha} + B\right).$$

Proof See, e.g., Lemma 5.1 in [5].

The main result of this section is the following

Theorem 3.2. *Assume that the D-structure on X is strongly local and that it supports for some q, $q < p$, a weak $(1, q)$-Poincaré inequality. If, in addition, the following condition is satisfied*

$$\liminf_{\rho \to 0} \frac{1}{|\log \rho|} \int_\rho^1 \exp\left(-C\left(\frac{R^{-q}\mu(B(x_0, R))}{Cap_q\left(B(x_0, \frac{1}{2}R) \setminus \Omega, B(x_0, R)\right)}\right)^{\frac{p}{p-q}}\right)\frac{dR}{R} > 0\,, \tag{19}$$

for some constant $C > 0$, then the quasi-minimizer u^* is Hölder continuous at x_0.

This theorem follows from Theorem 1.5 and the following

Proposition 3.3. *If the D-structure on X is strongly local and it supports a weak $(1, q)$-Poincaré inequality, then the quasi-minimizer u^* and its minimal pseudo-gradient g_{u^*}, as well as $-u^*$ and g_{u^*}, satisfy at x_0 Hypotheses* H1-H3. *In this case the function Φ of Hypothesis* H3 *can be chosen to be*

$$\Phi(\Omega, x_0, R) = C\frac{\mu(B(x_0, R))^{\frac{1}{q}}}{\mathrm{Cap}_q\left(B(x_0, \frac{1}{2}R) \setminus \Omega, B(x_0, R)\right)^{\frac{1}{q}}}\,,$$

with some $C > 0$.

Proof Hypothesis H1: For the point $x_0 \in \partial\Omega$ and $0 < \rho < R \leq \mathrm{diam}(X)/3$, let η be a $\frac{1}{(R-\rho)}$–Lipschitz cutoff function so that $0 \leq \eta \leq 1$, $\eta = 1$ on $B(x_0, \rho)$ and the support of η is contained in $B(x_0, R)$.
Set

$$v = -\eta \max\{u^* - k, 0\},$$

where $k \geq k^* := \operatorname{ess\,sup}_{B(x_0,R)} \vartheta$. Then

$$u^* + v = (1-\eta)\,(u^* - k) + k$$

on $A(k,R) = \{x \in B(x_0,R) :\ u^*(x) > k\}$.

Note that as $u^* = \vartheta := 1$ a.e. on $B(x_0,R) \setminus \Omega$, $v = 0$ a.e. on $B(x_0,R) \setminus \Omega$ and $u^* = \vartheta \leq k$ a.e. on the set $X \setminus \Omega$. Moreover, $v = 0$ outside $B(x_0,R)$ by the definition of η. Therefore, $\mu(\Omega \setminus \operatorname{supp}(v)) = 0$, and by the energy quasi-minimizing property of u^*, we have

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \leq \int_{v\neq 0} g_{u^*}^p d\mu \leq K \int_{v\neq 0} g_{u^*+v}^p d\mu \leq K \int_{A(k,R)} g_{u^*+v}^p d\mu\,,$$

where K is the constant in the definition the quasi-minimizer u^*.

From the strong locality of the D-structure it follows (see Proposition 2.10) that

$$\int_{A(k,R)} g_{u^*+v}^p d\mu = \int_{A(k,R)} g_{(1-\eta)(u^*-k)+k}^p d\mu\,,$$

Note that $\frac{1}{(R-\rho)} \in D[1-\eta]$. Axioms A1, A2 and A3 imply

$$(u^* - k)\frac{1}{(R-\rho)} + (1-\eta) g_{u^*} \in D[(1-\eta)(u^*-k)+k].$$

From this and the last two inequalities we obtain

$$\begin{aligned}\int_{A(k,\rho)} g_{u^*}^p d\mu &\leq K \int_{A(k,R)} ((u^*-k)^p \frac{1}{(R-\rho)^p} + (1-\eta)^p g_{u^*}^p) d\mu \\ &\leq \frac{C}{(R-\rho)^p} \int_{A(k,R)} (u^*-k)^p d\mu + C \int_{A(k,R)\setminus A(k,\rho)} g_{u^*}^p d\mu,\end{aligned}$$

where $C = K\,2^{p-1}$. Here we used the fact that $1-\eta = 0$ on $A(k,\rho)$. Adding the term $C\int_{A(k,\rho)} g_{u^*}^p d\mu$ to the left and right hand sides of the inequality above, we see that

$$(1+C)\int_{A(k,\rho)} g_{u^*}^p d\mu \leq C\int_{A(k,R)} g_{u^*}^p d\mu + \frac{C}{(R-\rho)^p}\int_{A(k,R)} (u^*-k)^p d\mu,$$

or

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \leq \frac{C}{1+C}\int_{A(k,R)} g_{u^*}^p d\mu + \frac{C}{1+C}\frac{1}{(R-\rho)^p}\int_{A(k,R)} (u^*-k)^p d\mu.$$

Hence, if $\rho < r \le R$, then

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \le \frac{C}{1+C}\int_{A(k,r)} g_{u^*}^p d\mu + \frac{C}{1+C}\frac{1}{(r-\rho)^p}\int_{A(k,R)} (u^*-k)^p d\mu.$$

From the last inequality and Lemma 3.1 we conclude that there is a constant $C > 0$ depending on p and K only, so that

$$\int_{A(k,\rho)} g_{u^*}^p d\mu \le \frac{C}{(R-\rho)^p}\int_{A(k,R)} (u^*-k)^p d\mu$$

and hence the pair u^* and g_{u^*} satisfies Hypothesis H1.

Hypothesis H2: For the proof of Hypothesis H2 we do not use the fact that u^* is a quasi-minimizer of the Dirichlet energy and, thus, the proof does not depend on the region where u^* is minimal. In particular, the proof is the same whether we consider a boundary or an interior point of Ω (see also Remark 1.2). We refer, therefore, the reader to [22] for the corresponding proof in the interior case.

Hypothesis H3: The function Φ of Hypothesis H3 does depend on the domain of minimization Ω. Hence, the previous remark on the proof of Hypothesis H2 for the boundary case is not suitable for Hypothesis H3.

As h and k in the definition of the function w are such that $h > k > k^* = \operatorname{ess\,sup}_{B(x_0,R)} \vartheta$, we have that $w = 0$ a.e. on $B(x_0,R)\setminus\Omega$. In fact, it is not difficult to check that $w = u_k^h = (u-k)_+ - (u-h)_+$ and as $u = \vartheta$ a.e. on the complement of Ω, in particular on $B(x_0,R)\setminus\Omega$, we have $(u-k)_+ = (u-h)_+ = 0$ a.e. on this set.

Let

$$\bar{w} = \left(⨍_{B(x_0,R)} w^q d\mu \right)^{\frac{1}{q}}$$

and η be a $\frac{2}{R}$-Lipschitz function vanishing outside $B(x_0,R)$ such that $0 \le \eta \le 1$ and $\eta = 1$ on $B(x_0,\frac{1}{2}R)$. Then the function $f = \eta(1-\frac{w}{\bar{w}})_+$ is admissible for the capacity $\operatorname{Cap}_q\big(B(x_0,\frac{1}{2}R)\setminus\Omega, B(x_0,R)\big)$. From Axioms A1,A2 and A3 and from the strong locality it follows that

$\frac{1}{\bar w} g_w \chi_{\{\frac{w}{\bar w}<1\}} + \frac{2}{R}(1-\frac{w}{\bar w})_+ \in D[w]$. Hence

$$\begin{aligned}\operatorname{Cap}_q\big(B(\tfrac12 R)\setminus\Omega, B(R)\big) &\le \int_{B(R)} g_f^q d\mu\\ &\le \int_{B(R)} \Big(\tfrac{1}{\bar w} g_w \chi_{\{\frac{w}{\bar w}<1\}} + \tfrac{2}{R}(1-\tfrac{w}{\bar w})_+\Big)^q d\mu\\ &\le \frac{2^{q-1}}{\bar w^q}\int_{B(R)} g_w^q d\mu + \frac{2^{q-1}2^q}{R^q \bar w^q}\int_{B(R)} |w-\bar w|^q d\mu\,.\end{aligned}$$

Denoting the ball $B(R)$ by B, by the Minkowski inequality we have

$$\left(\int_B |w-\bar w|^q d\mu\right)^{\frac1q} \le \left(\int_B |w-w_B|^q d\mu\right)^{\frac1q} + |\bar w - w_B|\mu(B)^{\frac1q}\,.$$

Using a weak (q,q)-Poincaré inequality and the doubling condition of μ, we estimate the second term of the last inequality as follows

$$\begin{aligned}|\bar w - w_B|\mu(B)^{\frac1q} &= \big|\|w_B\|_{L^q(B)} - \|w\|_{L^q(B)}\big|\\ &\le \|w-w_B\|_{L^q(B)} \le CR\left(\int_{\sigma B} g_w^q d\mu\right)^{\frac1q},\end{aligned}$$

where $C>0$. The last three inequalities, a weak (q,q)-Poincaré inequality and the doubling condition now give

$$\operatorname{Cap}_q\big(\tfrac12 B\setminus\Omega, B\big) \le \frac{C}{\bar w^q}\int_{\sigma B} g_w^q d\mu\,,$$

or

$$\bar w^q = \fint_B w^q d\mu \le \frac{C}{\operatorname{Cap}_q\big(\frac12 B\setminus\Omega, B\big)}\int_{\sigma B} g_w^q d\mu\,.$$

In the proof of Hypothesis H3 for the functions u^* and g_{u^*} in the interior of Ω, it was shown in [22] that the function $g_{(w)} := g_{u^*}\,\chi_{\{k<u^*\le h\}}$ belongs to the set $D[w]$ of the pseudo-gradients of w. Therefore, from the last inequality we have

$$\int_B w^q d\mu \le \frac{C\mu(B)}{\operatorname{Cap}_q\big(\frac12 B\setminus\Omega, B\big)}\int_{\sigma B} g_{(w)}^q d\mu\,,$$

and, thus, as a function Φ of Hypothesis H3 we can take the function

$$\Phi(\Omega, x_0, R) = C\frac{\mu(B(x_0,R))^{\frac1q}}{\operatorname{Cap}_q\big(B(x_0,\frac12 R)\setminus\Omega, B(x_0,R)\big)^{\frac1q}}\,,$$

with some $C > 0$.

Note that since by Axiom A2, $D[-u] = D[u]$ for any function $u \in L^p_{loc}(X)$, the function $-u^*$ is also a quasi-minimizer of the energy functional $\mathcal{E}_p$. Moreover the function $-u^*$ coincides with the function $-\vartheta = -1$ q.e. outside the set $\Omega = X \setminus F$. Therefore, the reasoning similar to the one for the pair (u^*, g_{u^*}) shows that Hypotheses H1-H3 are also true for the function $-u^*$ and the pseudo-gradient g_{u^*}. □

Remark 3.4. *The condition* (19) *in Theorem* 3.2 *is an analog, for our case, of the classical Wiener criterion for continuity at a boundary point of a domain in* $\mathbb{R}^n$. *Note, however, that the capacity which appears in many Wiener criteria for solutions of various classes of elliptic equations is, in fact, the Sobolev capacity, not the variational capacity which we have in* (19). *But, due to Lemma* 3.6 *below, the variational capacity* Cap_q *of the criterion* (19) *could be replaced by the Sobolev capacity* C_q. *In this case, the Sobolev capacity* C_q *is described by the following*

Definition 3.5 (Sobolev capacity). *The* Sobolev q-capacity *of a set* $F \subset X$ *is defined by*

$$C_q(F) := \inf\left\{ \|u\|^q_{W^{1,q}(X)} \mid u \in W^{1,q}(X), u \geq 1 \text{ near } F \text{ and } u \geq 0 \text{ a.e.}\right\}.$$

Lemma 3.6. *Under the hypotheses of Theorem* 3.2, *for a set* $E \subset \frac{1}{2}B = B(x_0, \frac{1}{2}R)$, *we have*

$$\mathrm{Cap}_q(E, B) \geq \frac{C_q(E)}{C(1 + R^q)}$$

and

$$\mathrm{Cap}_q(E, B) \geq \frac{\mu(E)}{CR^q},$$

for some constant $C > 0$.

Proof Let v be a function admissible for the variational capacity $\mathrm{Cap}_q(E, B)$. Then

$$\left(\fint_{2B} v^q d\mu\right)^{\frac{1}{q}} \leq \left(\fint_{2B} |v - v_{2B}|^q d\mu\right)^{\frac{1}{q}} + |v_{2B}|.$$

By the Hölder inequality and the fact that $v \in \mathcal{L}^{1,q}_0(B)$, we have

$$|v_{2B}| \leq \fint_{2B} v\chi_B d\mu \leq \left(\fint_{2B} v^q d\mu\right)^{\frac{1}{q}} \left(\frac{\mu(B)}{\mu(2B)}\right)^{1-\frac{1}{q}} \leq \left(\fint_{2B} v^q d\mu\right)^{\frac{1}{q}} \gamma_0,$$

for some γ_0, $0 < \gamma_0 < 1$ (see Proposition 2.11).

The last two inequalities, a weak (q,q)-Poincaré inequality, the fact that $v \in \mathcal{L}_0^{1,q}(B)$ and the strong locality of the D-structure give

$$\int_B v^q d\mu \leq CR^q \int_B g_v^q d\mu \,,$$

for some constant $C > 0$.

As v is admissible for the capacity $\mathrm{Cap}_q(E, B)$, we further have

$$\mu(E) \leq \int_B v^q d\mu \leq CR^q \int_B g_v^q d\mu$$

and

$$\mathrm{C}_q(E) \leq \int_X v^q d\mu + \int_X g_v^q d\mu \leq C(1 + R^q) \int_B g_v^q d\mu \,.$$

Taking the infimum over all admissible v completes the proof. □

Corollary 3.7. *The condition* (19) *is also satisfied if the complement of* Ω *has a corkscrew at* x_0*, i.e. if the set* $B(x_0, R) \setminus \Omega$ *contains a ball with the radius* CR*, for some* $C > 0$*, or, more generally, if*

$$\mu(B(x_0, R) \setminus \Omega) \geq C\mu(B(x_0, R)) \,.$$

4 Regularity of Extremal Poincaré-Sobolev Functions

The condition (19) for the Hölder continuity at a boundary point x_0 of Theorem 3.2 is expressed in terms of the capacities of certain sets related to this point. As it was already underlined in Section 2.2 the notion of a capacity is, in a sense, meaningless in the class of the Poincaré-Sobolev functions. Nevertheless, it is still possible to consider in this space the problem of regularity at a boundary point of a set. In this case we change the capacities in the criterion of regularity by the measures of appropriate sets (cf. Lemma 3.6 and Corollary 3.7).

Let Ω be a subset of X, $x_0 \in \partial\Omega$, a boundary point of Ω, be fixed and a function $\vartheta \in L^p(X)$ be given. Suppose that the function ϑ is Hölder continuous at the point x_0.

We will also assume that any pair (u, g), $u \in L^1_{loc}(X)$, $g \in L^q(X)$, satisfying a $(1, q)$-Poincaré inequality in X has the truncation property.

We have then the following

Theorem 4.1. *Let $u, -u \in PW^{1,q}(X)$ satisfy the p-De Giorgi condition in Ω $(p > q)$. If $u = \vartheta$ a.e. on $X \setminus \Omega$ and the following condition*

$$\liminf_{\rho\to 0} \frac{1}{|\log \rho|} \int_\rho^1 \exp\left(-C\left(\frac{\mu(B(x_0,R))}{\mu(B(x_0,R)\setminus\Omega)}\right)^{\frac{(q-1)p}{p-q}}\right)\frac{dR}{R} > 0\,,$$

holds for some constant $C > 0$, then the function u is Hölder continuous at x_0.

This theorem follows from Theorem 1.5 and the following

Proposition 4.2. *If the pairs (u, g) and $(-u, g)$ satisfy the requirements of Theorem* 4.1*, then they satisfy Hypotheses* H1-H3*. In this case the function Φ of Hypothesis* H3 *can be chosen to be*

$$\Phi(\Omega, x_0, R) = CR\left(\frac{\mu(B(x_0,R))}{\mu(B(x_0,R)\setminus\Omega)}\right)^{1-\frac{1}{q}}\,,$$

with some $C > 0$.

Proof Hypothesis H1: For the point $x_0 \in \partial\Omega$ and $0 < \rho < R \le \mathrm{diam(X)}/3$, choose $k \ge k^* := \operatorname{ess\,sup}_{B(x_0,R)} \vartheta$. Then for all $k \ge k^*$ we have

$$\mu(\Omega \setminus A(k,R)) = 0,$$

where $A(k,R) = B(x_0,R) \cap \{u > k\}$. Note that $u = \vartheta$ a.e. on $X \setminus \Omega$. Thus, we can apply the p-De Giorgi condition (18) for the functions u and g, which gives Hypothesis H1 for the pair (u, g).

Hypothesis H2: See [22] and the proof of Hypothesis H2 in Theorem 3.2.

Hypothesis H3: As $u = \vartheta$ a.e. on the complement of Ω, for $h > k > k^* = \operatorname{ess\,sup}_{B(x_0,R)} \vartheta$, we have $w = 0$ a.e. on the set $B(x_0,R) \setminus \Omega$.

By the truncation property the pair $(w, g_{(w)})$ satisfies a $(1, q)$-inequality and, thus, a (q, q)-Poincaré inequality. Therefore, we have

$$\begin{aligned}\left(\fint_{B(R)} w^q d\mu\right)^{\frac{1}{q}} &\le \left(\fint_{B(R)} |w - w_{B(R)}|^q d\mu\right)^{\frac{1}{q}} + |w_{B(R)}| \\ &\le C_P R\left(\fint_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}} + |w_{B(R)}|.\end{aligned}$$

By the Hölder inequality we obtain

$$
\begin{aligned}
|w_{B(R)}| &= \frac{1}{\mu(B(R))}\int_{B(R)} w d\mu = \frac{1}{\mu(B(R))}\int_{B(R)} w\,\chi_{\{w>0\}} d\mu \\
&\leq \left(\fint_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \left(\frac{\mu(B\cap\Omega)}{\mu(B)}\right)^{1-\frac{1}{q}}.
\end{aligned}
$$

These two inequalities and the doubling property of the measure μ now give

$$
\left(1-\left(\frac{\mu(B\cap\Omega)}{\mu(B)}\right)^{1-\frac{1}{q}}\right)\left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \leq CR\left(\int_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}},
$$

or

$$
\left(\int_{B(R)} w^q d\mu\right)^{\frac{1}{q}} \leq CR\left(\frac{\mu(B)}{\mu(B\setminus\Omega)}\right)^{1-\frac{1}{q}}\left(\int_{B(\sigma R)} g_{(w)}^q d\mu\right)^{\frac{1}{q}},
$$

for some $C>0$. Hence, Hypothesis H3 is satisfied for the pair (u,g) with the function

$$
\Phi(\Omega, x_0, R) = CR\left(\frac{\mu(B(x_0,R))}{\mu(B(x_0,R)\setminus\Omega)}\right)^{1-\frac{1}{q}}.
$$

□

Remark 4.3. *In the proof of Hypothesis* H2 *in* [22] *the following condition on the measure μ is used. For every $z\in X$ and $0<R\leq diam(X)/3$, there exists γ, $0<\gamma<1$, such that*

$$
\frac{\mu(B(z,\frac{R}{2}))}{B(z,R)} \leq \gamma\,.
$$

Note that in the axiomatic setting this condition is proved in Proposition 2.11*. In the case of Poincaré-Sobolev spaces this condition, therefore, must be assumed for the results of this section to hold.*

References

[1] J. Björn *Boundary continuity for quasiminimizers on metric spaces.* Illinois J. Math. **46** (2002), no. 2, 383–403.

[2] E. De Giorgi, *Sulla differenziabilità e l'analicità delle estremali degli integrali multipli regolati.* Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. **3** (1957), 25–43.

[3] R. Gariepy and W. P. Ziemer, *A regularity condition at the boundary for solutions of quasilinear elliptic equations.* Arch. Rational Mech. Anal. **67** (1977), 25–39.

[4] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems.* Annals of Math. Studies No. 105, Princeton Univ. Press, Princeton (1983).

[5] M. Giaquinta, *Introduction to Regularity Theory for Nonlinear Elliptic Systems.* Birkhäuser Verlag, Basel (1993).

[6] V. M. Gol'dshtein and M. Troyanov, *Axiomatic Theory of Sobolev Spaces.* Exposition. Math. **19** (2001), 289–336.

[7] V. M. Gol'dshtein and M. Troyanov, *Capacities on Metric Spaces.* Integr. Equ. Oper. Theory **44** (2002), no. 2, 212–242.

[8] P. Hajłasz, *Sobolev spaces on an arbitrary metric space.* Potential Anal. **5** (1996), no. 4, 403–415.

[9] P. Hajłasz and P. Koskela, *Sobolev met Poincaré.* Memoirs Amer. Math. Soc. **688** (2000).

[10] J. Heinonen, *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York (2001).

[11] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry.* Acta Math. **181** (1998), 1–61.

[12] J. Kinnunen and N. Shanmugalingam, *Regularity of quasi-minimizers on metric spaces.* Manuscripta Math. **105** (2001), 401–423.

[13] P.Koskela and P.MacManus, *Quasiconformal mappings and Sobolev classes.* Studia Math. **131** (1998), 1-17.

[14] W. Littman, G. Stampacchia and H.F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients*. Ann. Sc. Norm. Sup. Pisa **17** (1963), 45–79.

[15] J. Malý and W.P. Ziemer, *Fine Regularity of Solutions of Elliptic Partial Differential Equations.* American Mathematical Society, Providence, RI (1997).

[16] V. G. Maz'ya *Regularity at the boundary of solutions of elliptic equations and conformal mapping.* Soviet Math. Dokl. **4** (1963), 1547–1551.

[17] V. G. Maz'ya *On the continuity at a boundary point of solutions of quasi-linear elliptic equations.* Vestnik Leningrad Univ. Math. **3** (1976), 225–242.

[18] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations.* Comm. Pure Appl. Math. **13** (1960), 457–468.

[19] J. Nash, *Contunuity of solutions of parabolic and elliptic equations.* Amer. J. Math. **80** (1958), 931–954.

[20] J. Serrin, *Local behavior of solutions of quasilinear equations.* Acta Math. **III** (1964), 247–302.

[21] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Rev. Mat. Iberoamericana **16** (2000), 243–279.

[22] S. Timoshin, *Regularity on metric spaces: Hölder continuity of extremal functions in axiomatic and Poincaré-Sobolov spaces.* Preprint of the Max-Planck-Institut für Mathematik Bonn, MPIM 2007-115.

[23] N. Wiener *The Dirichlet problem.* J. Math. Phys. **3** (1924), 127–146.

[24] W. P. Ziemer *Bounary regularity for quasiminima.* Arch. Rational Mech. Anal. **92** (1986), 371–382.