Volumes of hyperbolic manifolds and mixed Tate motives

Alexander Goncharov

Department of Mathematics
Math. Institute of Technology (MIT)
Cambridge, MA 02139

USA

Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26
D-53225 Bonn

GERMANY

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Abstract

Two different constructions of an invariant of an odd dimensional hyperbolic manifold in the K-group $K_{2n-1}(\bar{\mathbb{Q}})$ are given. The volume of the manifold is equal to the value of the Borel regulator on that element. The scissor congruent groups and their relation with the algebraic K-theory of the field of complex numbers is discussed.

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1 Introduction

By hyperbolic manifold we will mean an orientable complete Riemannian manifold with constant sectional curvature -1.

1. Volumes of hyperbolic $2m$-manifolds. Let M^n be an n-dimensional hyperbolic manifold with finite volume $\mathrm{vol}(M^n)$. Suppose that $n = 2m$ is an even number. Then according to the Gauss-Bonnet theorem (see, for example, [Ch])

$$\mathrm{vol}(M^{2m}) = -c_{2m} \cdot \chi(M^{2m})$$

where $c_{2m} = 1/2\times$(volume of sphere S^{2m} of radius 1) and $\chi(M^{2m})$ is the Euler characteristic of M^{2m}. This is straightforward for compact manifolds and a bit more delicate for non-compact ones.

According to Mostow's rigidity theorem (see [Th], ch 5) volumes of hyperbolic manifolds are homotopy invariants. For even-dimensional ones this is clear from the formula above. For odd-dimensional hyperbolic manifolds volume is a much more interesting invariant. In this paper we will show that it is related to algebraic K-theory of the field of algebraic numbers. *We will denote by $\bar{\mathbb{Q}}$ the subfield of all algebraic numbers in $\mathbb{C}$.*

2. Volumes of $(2n-1)$-dimensional hyperbolic manifolds and the Borel regulator on $K_{2n-1}(\bar{\mathbb{Q}})$. Let us denote by $K_*(F)$ the Quillen K-groups of a field F. There is the Borel regulator [Bo2]

$$r_m : K_{2m-1}(\mathbb{C}) \to \mathbb{R}$$

Theorem 1.1 *Any $(2m-1)$-dimensional hyperbolic manifold of finite volume M^{2m-1} defines naturally an element $\gamma(M^{2m-1}) \in K_{2m-1}(\bar{\mathbb{Q}})\otimes\mathbb{Q}$ such that $vol(M^{2m-1}) = r_m(\gamma(M^{2m-1}))$.*

The hyperbolic volumes $\mathrm{vol}(M^{2n-1})$ form a very interesting set of numbers. If $2n-1 \geq 5$ it is discrete and, moreover, according to Wang's rigidity theorem [W] for any given $c \in \mathbb{R}$ there are only finite number of hyperbolic manifolds of volume $\leq c$. Thank to Jorgensen and Thurston we know that the volumes of hyperbolic 3-folds form a nondiscrete well-odered set of ordinal type ω^ω (see [Th]).

Theorem 1.1 together with some general conjectures in algebraic geometry suggests the highly transcendental nature of volumes of $(2n-1)$-dimensional hyperbolic manifolds. For example if the hyperbolic volumes $s_1, ..., s_k$ are algebraically dependent over $\mathbb{Q}$ then they should be linearly dependent over $\mathbb{Q}$. Of cource to check this is far beyond our abilities. The first proof of theorem 1.1 is given in chapter 3.

For an abelian group A set $A_{\mathbb{Q}} := A \otimes \mathbb{Q}$. Set $\mathbb{Q}(n) := (2\pi i)^n\mathbb{Q}$. The complex conjugation acts on $K_{2n-1}(\bar{\mathbb{Q}})$ and $\mathbb{Q}(n)$ providing the decomposition

$$K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n) = (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^+ \oplus (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^-$$

In chapters 4 and 5 we will describe a different construction of an invariant

$$\tilde{\gamma}(M^{2n-1})^- \in K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^- \tag{1}$$

which depends only on the scissor congruence class of a hyperbolic manifold. This means that if two hyperbolic manifolds can be cut on the same geodesic simplices then their invariants $\tilde{\gamma}(M^{2n-1})^-$ coincide. We will prove that theorem 1.1 remains valid if we replace the invariant $\gamma(M^{2n-1})$ by $\tilde{\gamma}(M^{2n-1})^-$. This gives a completely different and more conceptual proof this theorem.

Moreover, we will show that a formal sum of hyperbolic polyhedrons of dimension $2n-1$ whose vertices has the coordinates in $\bar{\mathbb{Q}}$, and which in addition has the Dehn invariant equal to zero, produces an element in $(K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^-$. The Borel regulator on this element is equal to the sum of volumes of the polyhedrons. In a similar situation for the spherical polyhedrons we get an element in $(K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^+$. (The definition of the Dehn invariant see in s. 2.2 below). This was known in dimension three thanks to [DS], [DPS], [S1], but it is quite surprising in higher dimensions.

Theorem 1.1 together with explicit computation of the Borel regulator for $K_3(\mathbb{C})$ (Bl1]) and $K_5(\mathbb{C})$ ([G2]) lead to much more precise results about the volumes of hyperbolic 3 and 5-manifolds. To formulate them I have to say a few words about the classical polylogarithms.

3. The classical polylogarithms. They are defined inductively as multivalued analytical functions on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$:

$$Li_n(z) := \int_0^z Li_{n-1}(w) \frac{dw}{w}, \qquad Li_1(z) = -\log(1-z)$$

The function $\log z$ has a single-valued cousin $\log|z| = \mathrm{Re} \log z$. The coresponding function for the dilogarithm was invented by D. Wigner and S. Bloch:

$$\mathcal{L}_2(z) := \mathrm{Im} Li_2(z) + \arg(1-z) \cdot \log|z| \tag{2}$$

It is continuous on $\mathbb{CP}^1$ (and, of course, real analytic on $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$). S. Bloch ([Bl1]) computed the Borel regulator for $K_3(\mathbb{C})$ using $\mathcal{L}_2(z)$.

A single-valued version of $Li_3(z)$ that was used in [G1-G2] for an explicit calculation of the Borel regulator for $K_5(\mathbb{C})$ looks as follows:

$$\mathcal{L}_3(z) := \mathrm{Re}(Li_3(z) - \log|z| \cdot Li_2(z) + \frac{1}{3} \log^2|z| \cdot Li_1(z)) \tag{3}$$

For arbitrary n there is the following function discovered by D. Zagier [Z1]:

$$\mathcal{L}_n(z) := \mathcal{R}_n \left(\sum_{k=1}^{n} \frac{B_k \cdot 2^k}{k!} Li_{n-k}(z) \cdot \log^k |z| \right) \tag{4}$$

Here $\mathcal{R}_n$ means real part for n odd and imaginary part for n even B_k are Bernoulli numbers: $\sum_{k=0}^{\infty} \frac{B_k \cdot 2^k}{k!} \cdot x^k = \frac{2x}{e^{2x}-1}$. This function is single valued on $\mathbb{CP}^1$ and coincides with functions (2) and (3) for $n = 2$ and 3.

4. Volumes of hyperbolic 3 and 5-manifolds. Recall that the wedge square $\Lambda^2\mathbb{C}^*$ of the multiplicative group $\mathbb{C}^*$ is defined as follows:

$$\Lambda^2\mathbb{C}^* := \frac{\mathbb{C}^*\otimes\mathbb{C}^*}{\{a\otimes b + b\otimes a\}} \qquad (a, b \in \mathbb{C}^*)$$

Theorem 1.2 . *Let M^3 be a hyperbolic 3-manifold of finite volume. Then there are numbers $z_i \in \bar{\mathbb{Q}}$ satisfying the condition*

$$\sum(1 - z_i) \wedge z_i = 0 \text{ in } \Lambda^2\bar{\mathbb{Q}}^* \tag{5}$$

such that

$$vol(M^3) = \sum \mathcal{L}_2(z_i)$$

This theorem follows from results of Dupont–Sah [Du1], [D-S], or Neumann-Zagier [NZ] (and was mentioned later in [Z2]).

Let X be a set. Denote by $\mathbb{Z}[X]$ the free abelian group generated by symbols $\{x\}$ where x run through all elements of X.

Let $r(x_1, x_2, x_3, x_4) = \frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)}$ be the cross-ratio of 4 distinct points on a line. Let F be a field and $R_2(F) \subset \mathbb{Z}[P^1_F \setminus \{0, 1, \infty\}]$ be the subgroup generated by the elements

$$\sum_{i=1}^{5}(-1)^i\{r(x_1, \ldots, \hat{x}_i, \ldots, x_5)\} \qquad (x_i \neq x_j \in P^1_F)$$

It is motivated by the functional equation for the Bloch-Wigner function $\sum_{i=1}^{5}(-1)^i\mathcal{L}_2(r(z_1, \ldots, \hat{z}_i, \ldots, z_5)) = 0, \quad (z_i \neq z_j \in P^1_{\mathbb{C}})$. Moreover, any functional equation for $\mathcal{L}_2(z)$ is a formal consequence of this one (see [G2], proposition 4.9). So $R_2(\mathbb{C})$ is the group of all functional equations for $\mathcal{L}_2(z)$. Now set

$$B_2(F) = \frac{\mathbb{Z}[P^1_F \setminus \{0, 1, \infty\}]}{R_2(F)}$$

Denote by $\{x\}_2$ the projection of $\{x\}$ onto $B_2(F)$.

Theorem 1.3 . *Let M^5 be a 5-dimensional hyperbolic manifold of finite volume. Then there are algebraic numbers $z_i \in \bar{\mathbb{Q}}$ satisfying the condition*

$$\sum_i \{z_i\}_2 \otimes z_i = 0 \text{ in } B_2(\bar{\mathbb{Q}})\otimes\bar{\mathbb{Q}}^*$$

such that

$$vol(M^5) = \sum_i \mathcal{L}_3(z_i)$$

Theorem 1.3 (as well as theorem 1.2) follows from theorem 1.1 and the main results of [G2].

5. Volumes of hyperbolic (2n-1)-manifolds. One can define for arbitrary n a certain subgroup $\mathcal{R}_n(F) \subset \mathbb{Z}[P^1_F]$ which for $F = \mathbb{C}$ is the subgroup of *all* functional equations for the n-logarithm $\mathcal{L}_n(z)$ (see s. 1.4 of [G2]). Set

$$\mathcal{B}_n(F) := \frac{\mathbb{Z}[P^1_F]}{\mathcal{R}_n(F)}$$

Let us define the homomorphism

$$\delta_n : \mathbb{Z}[P^1_F] \to \mathcal{B}_{n-1}(F) \otimes F^* \qquad \{x\} \mapsto \{x\}_{n-1} \otimes x$$

One can show that $\delta_n(\mathcal{R}_n(F)) = 0$ (see [G2]), so we get a homomorphism

$$\delta_n : \mathcal{B}_n(F) \to \mathcal{B}_{n-1}(F) \otimes F^*$$

Conjecture 1.4 *Let M^{2n-1} be an $(2n-1)$-dimensional hyperbolic manifold of finite volume. Then there are algebraic numbers $z_i \in \bar{\mathbb{Q}}$ satisfying the condition $(n \geq 3)$*

$$\sum_i \{z_i\}_{n-1} \otimes z_i = 0 \ in \ \mathcal{B}_{n-1}(\bar{\mathbb{Q}}) \otimes \bar{\mathbb{Q}}^*$$

such that

$$vol(M^{2n-1}) = \sum_i \mathcal{L}_n(z_i)$$

This conjecture is a consequence of a version of Zagier's conjecture [Z1] and theorem 1.1.

6. The Chern-Simons invariants of compact hyperbolic (2n-1)-manifolds. The Chern-Simons invariant $CS(M^{2n-1})$ takes the values in $S^1 = \mathbb{R}/\mathbb{Z}$. There is the Beilinson's regulator

$$r_n^{Be} : K_{2n-1}(\bar{\mathbb{Q}}) \longrightarrow \mathbb{C}/(2\pi i)^n \mathbb{Q} \tag{6}$$

Therefore $(2\pi)^n r_n^{Be}$ provides homomorphisms

$$(K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^- \longrightarrow \mathbb{R} \qquad (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^+ \longrightarrow \mathbb{R}/\mathbb{Q}$$

For an element $\gamma \in K_{2n-1}(\bar{\mathbb{Q}})$ let $\gamma^{\pm}$ be its components in $(K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^{\pm}$

Suppose that M^{2n-1} is a compact hyperbolic manifold. Then one can show that

$$CS(M^{2n-1}) = c_n r_n^{Be}(\gamma(M^{2n-1})^+)$$

where c_n is a nonzero universal constant. For 3-manifolds this was known: see [D2] if the manifold is compact, and [Y], [N], [NJ] if it has cusps. The proof of this result will be given elsewhere. Notice that according to our conventions $\mathrm{vol}(M^{2n-1}) = r_n^{Be}(\gamma(M^{2n-1})^-)$.

As far as I know the Chern-Simons invariants of noncompact hyperbolic manifolds was defined only for 3-manifolds ([Me]). So one can use the formula above as a *definition* of the $CS(M^{2n-1})$ for noncompact hyperbolic manifolds.

7. The invariants $\tilde{\gamma}(M^{2n-1})$ and $\tilde{\gamma}(M^{2n-1})^-$. There are two completely different points of view on algebraic K-theory.

i). In Quillen's definition $K_n(F)_{\mathbb{Q}}$ is the quotient of $H_n(GL(F), \mathbb{Q})$ modulo the subspace of decomposable elements. This is how elements $\gamma(M^{2m-1})$ constructed in chapter 3. The construction is straightforward for 3-dimensional compact hyperbolic manifolds: the fundamental class of such a manifold provides an element in $H_3(BSO(3,1), \mathbb{Z})$; using the local isomorphism between $SO(3,1)$ and $SL(2,\mathbb{C})$ we get a clas in $H_3(SL(2,\mathbb{C})^{\delta})$ and thus an element in $H_3(SL(\mathbb{C})^{\delta})$ via the embedding $SL_2 \hookrightarrow SL$. It is interesting that for compact hyperbolic manifolds of dimension ≥ 5 one have to use the halfspinor representation of $SO(2n-1,1)$ in order to get, starting from the fundamental class of M^{2n-1}, an interesting class in $H_{2n-1}(GL(\mathbb{C}))$. (It seems that the other fundamental representations of $SO(2n-1,1)$ lead to zero classes in $K_{2n-1}(\mathbb{C})$). For the noncompact hyperbolic manifolds $H_{2n-1}(M^{2n-1}) = 0$, and the construction of the invariant $\gamma(M^{2n-1})$ becomes rather delicate, see chapter 3.

ii). Let $gr_n^{\gamma}K_m(\mathbb{C})$ be the graded quotients of the γ-filtration on $K_m(\mathbb{C})$. According to A.A.Beilinson elements of $gr_n^{\gamma}K_{2n-1}(\mathbb{C})$ should have an interpretation as motivic extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ (here $\mathbb{Q}(n)$ is the n-th tensor power of the Tate motive $\mathbb{Q}(1)$; the latter is the inverse to the motive $H^2(\mathbb{CP}^1)$).

There is a natural map $K_{2n-1}(\bar{\mathbb{Q}}) \to gr_n^{\gamma}K_{2n-1}(\mathbb{C})$. It is expected to be an isomorphism modulo torsion (the rigidity conjecture).

In the chapter 4 (see also chapter 2 for an introduction) for any hyperbolic manifold of finite volume M^{2n-1} I will give a simple geometrical construction of a mixed Tate motif $m(M^{2n-1})$ that fits in the following exact sequence (in the category of mixed Tate motives):

$$0 \longrightarrow \mathbb{Q}(n) \longrightarrow m(M^{2n-1}) \longrightarrow \mathbb{Q}(0) \longrightarrow 0$$

Namely, I will construct an extension in the category of mixed Hodge structures that will be clearly of algebraic-geometric origin.

Moreover, it will be clear from the construction that the $\mathbb{R}$-part of the regulator map on the class of Ext^1 defined by this extension coincides with the volume of M^{2n-1}.

This extension represents an element

$$\tilde{\gamma}(M^{2n-1})^- \in (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^-$$

One should have $\tilde{\gamma}(M^{2n-1})^- = \gamma(M^{2n-1})^-$.

The element $\tilde{\gamma}(M^{2n-1})^-$ depends only on the scissor congruence class of the manifold M^{2n-1} just by the construction. Therefore the invariant $\gamma(M^{2n-1})^-$ is responsible for the scissor congruence class of the hyperbolic manifold M^{2n-1} (see s.2.1-2.2 below). Moreover, the following three conditions *should be* equivalent:

a) The volumes of hyperbolic manifolds M_1^{2n-1} and M_2^{2n-1} are coincide.

b)$\gamma(M_1^{2n-1})^- = \gamma(M_2^{2n-1})^-$.

c)M_1^{2n-1} and M_2^{2n-1} are scissor congruent.

8. The structure of the paper. In chapter 2 we define Dehn complexes, which generalize the scissor congruence groups and the classical Dehn invariant of a polyhedron, in the hyperbolic, spherical and euclidean geometries. We formulate conjectures relating the cohomology of these complexes to algebraic K-theory of $\mathbb{C}$ and (in the euclidean case) to Kahler differentials $\Omega^i_{\mathbb{R}/\mathbb{Q}}$. A part of these conjectures for noneuclidean polyhedrons whose vertices has coordinates in a number field will be proved in chapters 4 and 5. The crucial result is an interpretation of the Dehn invariant on the language of mixed Hodge structures given in chapter 4.

The chapter 2 can be considered as a continuation of the introduction: it contains some important definitions and formulations of the results and conjectures as well as ideas of some of constructions, but we avoid any technical discussion.

More generally, in chapter 4 we will construct some commutative graded Hopf algebra $S(\mathbb{C})_\bullet$ which is dual to the universal enveloping algebra of a certain remarkable negatively graded (pro)-Lie algebra. I believe that the category of finite dimensional graded modules over this Lie algebra is equivalent to the category of mixed Tate motives over $\mathbb{C}$ (compare with the Hopf algebra considered in [BGSV]). In particulary this means that the cohomology of $S(\mathbb{C})_\bullet$ should coincide with appropriate pieces of the algebraic K-theory of $\mathbb{C}$.

Chapter 5 starts from certain known to experts technical results about triangulated categories and their abelian hearts. We need them to finish the proof of theorem 2.5.

2 Dehn complexes in classical geometries and their homology

1. The scissor congruence groups . There are three classical geometry: euclidean, hyperbolic and spherical. In each of them one can consider geodesic simplices and ask the following question

Hilbert's third problem. *Suppose two geodesic simplices has the same volume. Is it possible to cut one of them on geodesic simplices and putting them together in a different way get another simplex?*

To formulate the problem in a more accurate way we need to introduce the scissor congruence groups.

Let V^n be an n-dimensional space with one of the classical geometries, i.e. V is the hyperbolic space $\mathcal{H}^n$, spherical space S^n or Euclidian space E^n

Any $n+1$ points $x_0, ..., x_n$ in the space V^n define a geodesic simplex $I(x_0, ..., x_n)$ with vertices in these points. Let us denote by $\mathcal{P}(V^n)$ the abelian group generated by symbols $\{I(x_0, ..., x_n), \alpha\}$ where α is an orientation of V^n, subject to the following relations:

a) $\{I(x_0, ..., x_n), \alpha\} = 0$ if $x_0, ..., x_n$ lie in a geodesic hyperplane.

b)$\{I(x_0, ..., x_n), \alpha\} = \{I(gx_0, ..., gx_n), g\alpha\}$ for any element g form the group of automorphisms of the corresponding geometry.

c)$\{I(x_{\sigma(0)}, ..., x_{\sigma(n)}), \beta\} = -(-1)^{|\sigma|}\{I(x_0, ..., x_n), \alpha\}$ where σ is a permutation and β is another orientation of V^n.

d) $\sum_{i=0}^{n+1}(-1)^i\{I(x_0, ..., \hat{x}_i, ..., x_{n+1}), \alpha\} = 0$ for any $n+2$ points x_i in V^n.

The automorphism groups are: the group $O(n,1)$ for the hyperbolic case, $O(n+1)$ for the spherical geometry and the semidirect product of $O(n)$ and translation group E^n for the Euclidian geometry

Lemma 2.1 $\mathcal{P}(S^{2n}) = 0.$

Proof. Points $-x_0, x_0, x_1, ..., x_{2n-1}$ belong to a geodesic hyperplane in S^{2n}. Therefore $\{I(-x_0, x_0, x_1, ..., x_{2n-1}), \alpha\} = 0$.

Using this fact we see that the additivity axiom d) for $(-x_0, x_0, , x_1, ..., x_{2n})$ together with a) implies that $(I(x_0, x_1, ..., x_{2n}), \alpha) = (I(-x_0, x_1, ..., x_{2n}), \alpha)$ and hence $(I(x_0, x_1, ..., x_{2n}), \alpha) = (I(-x_0, -x_1, ..., -x_{2n}), \alpha)$. The relation b) for the antipodal involution is $(I(x_0, x_1, ..., x_{2n}), \alpha) = (I(-x_0, -x_1, ..., -x_{2n}), \beta)$. Applying c) we get lemma 2.1.

The volume of a geodesic simplex provide homomorphisms

$$vol_{\mathcal{H}} : \mathcal{P}(\mathcal{H}^n) \to \mathbb{R}, \qquad vol_S : \mathcal{P}(S^{2n-1}) \to \mathbb{R}/\mathbb{Z}, \qquad vol_E : \mathcal{P}(E^n) \to \mathbb{R}$$

It is not hard to see (and in the euclidean case was known to Euclid) that the volume provides isomorphisms

$$\mathcal{P}(S^1) = \mathbb{R}/\mathbb{Z}, \qquad \mathcal{P}(\mathcal{H}^1) = \mathcal{P}(\mathcal{H}^2) = \mathbb{R}, \qquad \mathcal{P}(E^1) = \mathcal{P}(E^2) = \mathbb{R}$$

A more accurate formulation of the Hilbert's third problem is the following question: is it true that the volume homomorphism is injective in dimension three?

The negative answer to this problem was given by Max Dehn (1898). He discovered that in dimension ≥ 3 the volume *does not* separate the elements of the scissor congruence group, because a new phenomena appears:

2. The Dehn invariant. This is a homomorphism

$$D_n^V : \mathcal{P}(V^n) \longrightarrow \bigoplus_{i=1}^{n-2} \mathcal{P}(V^i) \otimes \mathcal{P}(S^{n-i-1}) \qquad (7)$$

Here, as usual, V^n can be $\mathcal{H}^n$, S^n or E^n.

It is defined as follows. Let (I, α) be a generator of $\mathcal{P}(V^n)$. For each i-dimensional edge A of a (geodesic) simplex I consider the corresponding geodesic i-plane. This plane inherits the same type of geometry as V^n (i.e. it is a hyperbolic plane in hyperbolic geometry etc.) and vertices of the edge A define a geodesic simplex in it. The numeration of its vertices is induced by the one of I. Choose an orientation α_A of this i-plane. We get an element of $\mathcal{P}(V^i)$ that will be denoted as I_A.

Now look at a $(n-i)$-plane A' orthogonal to A and intersecting it at a certain point e. This is a Euclidian plane in all geometries. The intersection of the sphere in A' centered in e with $i+1$-dimensional edges of I containing A defines a spherical simplex. Its vertices has a natural numeration induced by the numeration of the vertices if I. Finally, there is an orientation $\alpha_{A'}$ of A' such that the orientation of $\mathcal{H}^n$ defined by α_A and $\alpha_{A'}$ coincides with the orientation α. So we get an element $I_{A'} \in \mathcal{P}(S^{n-i})$. By definition

$$D_n^H(I, \alpha) := \sum_A I_A \otimes I_{A'}$$

where the sum is over all edges of the simplex I of dimension i, $0 < i < n$

Useful general references on this subject are [Sah 1] and [C]. In particular according to S.H.Sah $\oplus\mathcal{P}(S^n)$ is a Hopf algebra with spherical Dehn invariant D_n^S as a comultiplication; hyperbolic (respectively euclidean) Dehn invariant (7) provides $\mathcal{P}(\mathcal{H}^n)$ (respectively $\mathcal{P}(E^n)$) with a structure of a comudule over it. Notice that our groups $\mathcal{P}(S^n)$ are different (smaller) from the ones defined in [Sah]. For those groups , for example, one has $\mathcal{P}(S^{2n}) = \mathcal{P}(S^{2n-1})$.

3. Scissor congrunce class of a hyperbolic manifold. Any hyperbolic manifold of finite volume can be cut into a finite number of geodesic simplices (see, for example, chapter 2 below). If the manifold is compact these simplices have vertices in $\mathcal{H}^n$. Therefore one can consider the sum of corresponding elements in $\mathcal{P}(\mathcal{H}^n)$. If the manifold is noncompact, the vertices of these simplices might be in the absolute $\partial\mathcal{H}^n$. However one can introduce the scissor congruent group $\mathcal{P}(\bar{\mathcal{H}}^n)$ generated by simplices with vertices at $\bar{\mathcal{H}}^n = \mathcal{H}^n \cup \partial\mathcal{H}^n$. It turns out that

Proposition 2.2 *([Sah 2],) The natural inclusion*

$$\mathcal{P}(\mathcal{H}^n) \hookrightarrow \mathcal{P}(\bar{\mathcal{H}}^n) \tag{8}$$

is an isomorphism.

The sum of the elements of $\mathcal{P}(\mathcal{H}^n)$ corresponding to these simplices clearly does not depend on the cutting. Therefore *any* hyperbolic manifold produces an element $s(V^n) \in \mathcal{P}(\mathcal{H}^n)$

Proposition 2.3 *The Dehn invariant of $s(M^n)$ is equal to zero.*

Proof. This is quite clear for compact manifolds. Namely,consider a k-dimensional edge A of the triangulation of V^n on geodesic simplices. Let I^i be the set of all simplices containing the edge A. Each simplex I^i defines an element $I^i_{A'} \in \mathcal{P}(S^{n-k})$: the "inner angle" at the edge A (see above). It follows from the very definition that I_A will appear in formula for D_n^H with factor $\sum_i I^i_{A'}$. But $\sum_i I^i_{A'} = [S^{n-k}] = 0$ in $\mathcal{P}(S^{n-k})$ because M^n is a manifold without boundary.

For noncompact manifolds proposition 2.3 is less obvious and we have to proceed as follows. One can define the (extended) Dehn invariant

$$D^H_{2n-1} : \mathcal{P}(\bar{\mathcal{H}}^{2n-1}) \longrightarrow \bigoplus_{i=1}^{n-1} \mathcal{P}(\bar{\mathcal{H}}^{2i-1}) \otimes \mathcal{P}(S^{2(n-i)-1}) \tag{9}$$

providing that the homomorphism (8) commutes with the Dehn invariant. Namely, the definition of the components of (9) corresponding to edges of dimension bigger then 1 is *verbatim* the same. For 1-dimensional edges let us use the following Thurston's regularization procedure. Let $I(x_1, ..., x_{2n})$ be a simplex in $\mathcal{P}(\bar{\mathcal{H}}^{2n-1})$ .For each infinite vertex x_i delete a small horoball centered at x_i. Denote by $x_i(j)$ the intersection point of the edge $x_i x_j$ with the horosphere. If x_k is a point inside hyperbolic space, set $x_k(j) = x_k$. Then the component of (9) corresponding to the edge $x_i x_j$ is the tensor product of the 1-dimensional hyperbolic simplex $x_i(j)x_j(i)$ and the spherical $2n-1$ simplex that appears in the usual definition of Dehn invariant (" inner angle" at the edge $x_i x_j$, see above). The expression we get does not depend on the choice of horoballs because of the following reason. Consider a simplex in Euclidian space $\mathbb{R}^{2n-2}$. Each its vertex v defines a spherical simplex and hence an element of $\mathcal{P}(S^{2n-2})$ ("inner angle" at v). Then the sum of the elements corresponding to all vertices of the simplex is zero (this generalises the fact that the sum of angles of an euclidean triangle is π and true only for simplices in even-dimensional spaces).

After this the proof for noncompact hyperbolic manifolds is the same as for compact ones.

4. Kernel of the Dehn invariant in non-euclidean geometry and $K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}$.

Conjecture 2.4 *There are canonical injective homomorphisms*

$$KerD^H_{2n-1} \otimes \mathbb{Q} \hookrightarrow (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^-$$

$$KerD^S_{2n-1} \otimes \mathbb{Q} \hookrightarrow (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^+$$

such that the following diagrams are commutative:

$$\begin{tikzcd}
KerD^H_{2n-1} \otimes \mathbb{Q} \arrow[rr] \arrow[dr, "vol"'] & & (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^-, \arrow[dl, "r_n"] \\
& \mathbb{R} &
\end{tikzcd}$$

$$\begin{tikzcd}
KerD^S_{2n-1} \otimes \mathbb{Q} \arrow[rr] \arrow[dr, "vol"'] & & (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^+, \arrow[dl, "r_n^{Be}"] \\
& \mathbb{R}/\mathbb{Z} &
\end{tikzcd}$$

Let $\mathcal{P}(\mathcal{H}^{2n-1}, \bar{\mathbb{Q}})$ (respectively $\mathcal{P}(S^{(2n-1)}, \bar{\mathbb{Q}})$ be the subgroup of the hyperbolic (resp. spherical) scissor congruence group generated by the simplices defined over $\bar{\mathbb{Q}}$ (i.e. their vertices have coordinates in $\bar{\mathbb{Q}}$.

Theorem 2.5 *There are the following commutative diagrams*

$$\begin{tikzcd} KerD^H_{2n-1}|_{\mathcal{P}(\mathcal{H}^{2n-1},\bar{\mathbb{Q}})} \otimes \mathbb{Q} \arrow[rr] \arrow[dr, "vol"'] & & (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^-, \arrow[dl, "r_n"] \\ & \mathbb{R} & \end{tikzcd}$$

$$\begin{tikzcd} KerD^S_{2n-1}|_{\mathcal{P}(S^{2n-1}),\bar{\mathbb{Q}})} \otimes \mathbb{Q} \arrow[rr] \arrow[dr, "vol"'] & & (K_{2n-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}(n))^+, \arrow[dl, "r_n^{Be}"] \\ & \mathbb{R}/\mathbb{Z} & \end{tikzcd}$$

The proof of this theorem is based on the following ideas:

1. A $(2n-1)$-dimensional non-euclidean geodesic simplex M defines a mixed Tate motive $\gamma(M)$ with an additional data: n-framing. See s. 2.6 below and all the details in chapter 4.

2. The Dehn invariant also has a motivic interpretation and moreover each element $[M] \in KerD^H_{2n-1}|_{\mathcal{P}(\mathcal{H}^{2n-1},\bar{\mathbb{Q}})} \otimes \mathbb{Q}$ defines an exact sequence in the category of mixed Tate motives:

$$0 \longrightarrow \mathbb{Q}(n) \longrightarrow \gamma([M]) \longrightarrow \mathbb{Q}(0) \longrightarrow 0$$

i.e. an element of $Ext^1_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}}, \mathbb{Q}(n)_{\mathcal{M}})$. See chapter 4.

3. We already have an abelian category of mixed Tate motives over a number field. This follows from the existence of a certain triangulated (not abelian!) category of mixed motives over a field F constructed by Levine [L] and Voevodsky [V] in which the Ext-groups isomorphic to appropriate parts of the K-theory, and some formal arguments that use the Borel theorem, see the details in chapter 5.

The following conjecture, I guess, express a general expectation about the kernel of euclidean Dehn invariant. It tells us that the Dehn invariant and the volume homomorphism separate all elements of the euclidean scissor congruence group. Some speculations about its generalization and an "explanation" see below in s. 2.7.

Conjecture 2.6 $KerD^E = \mathbb{R}$

5. Dehn complexes in non euclidean geometry and algebraic K-theory of $\mathbb{C}$. The next natural question is what can we say about the cokernel of the Dehn invariant. First of all it turns out that the Dehn invariant in higher (bigger then 3) dimensions is only the beginning of a certain complex, which I will call the Dehn complex. Namely, consider the following complexes. (Here, again, V means one of the three geometries)

$$\begin{aligned}\mathcal{P}^\bullet_V(n) \quad &: \mathcal{P}(V^{2n-1}) \xrightarrow{d_v} \oplus_{i_1+i_2=2n-1}\mathcal{P}(V^{i_1}) \otimes \mathcal{P}(S^{i_2}-1) \xrightarrow{d_v} \ldots \\ &\xrightarrow{d_v} \oplus_{i_1+\ldots+i_k=2n-1}\mathcal{P}(V^{i_1}) \otimes \mathcal{P}(S^{i_2-1}) \otimes \ldots \otimes \mathcal{P}(S^{i_k-1}) \xrightarrow{d_v} \ldots \end{aligned} \quad (10)$$

Here the first group is placed in degree 1 and the differentials have degree +1;

$$d_v = D^V \otimes id \otimes \ldots \otimes id - id \otimes D^S \otimes id \otimes \ldots \otimes id + \ldots \pm \otimes id \otimes \ldots \otimes id \otimes D^S \quad (11)$$

Remark. The spherical Dehn invariant provides $\oplus\mathcal{P}(S^{2j-1})$ with a structure of a coalgebra. Further, the Dehn invariant provides $\oplus\mathcal{P}(V^{2i-1})$ with a structure of a comodules over this coalgebra. The complexes above are just the cobar complexes computing the degree n part of the cohomology of the coalgebra $\oplus\mathcal{P}(S^{2n-1})$ with coefficients in comodule $\oplus\mathcal{P}(\mathcal{H}^{2n-1})$.

Conjecture 2.7 . *There are canonical homomorphisms*

$$H^i(\mathcal{P}^\bullet_{\mathcal{H}}(n)) \longrightarrow (gr^\gamma_n K_{2n-i}(\mathbb{C}) \otimes \mathbb{Q}(n))^- \quad (12)$$

$$H^i(\mathcal{P}^\bullet_S(n)) \longrightarrow (gr^\gamma_n K_{2n-i}(\mathbb{C}) \otimes \mathbb{Q}(n))^+ \quad (13)$$

The following conjecture is due to D. Ramakrishnan and generalizes Milnor's conjecture about the values of the Lobachevsky function (see 7.1.2 in [R])

Conjecture 2.8 *The Beilinson regulator is injective modulo torsion.*

Conjectures (2.4), (2.8) would imply that in hyperbolic and spherical geometry the

The Extended Hilbert Third Problem. *Do the Dehn invariant and the volume separate all points of scissor congruence groups?*

should have an affirmative answer.

Problem *Is it true that the homomorphisms (12), (13) are isomorphisms modulo torsion?*

For $n = 2$ the answer is yes. This follows from the results of J.Dupont and S.H.Sah(see [D], [DS1], [DPS], [Sah3]).Unfortunately their methods use essentially classical isomorphisms between simple Lie groups in low dimensions (like the local isomorphism between $SO(3,1)_0$ and $SL_2(\mathbb{C})$) and a lot of arguments "ad hoc" that one does not see how to generalize to higher dimensions, even hypothetically. Their approach deal with the homology of orthogonal groups and the relation with algebraic K-theory was quite mysterious.

In the next section and in the chapter 4 below I will give a motivic interpretation of Dehn complexes. After this algebraic K-theory shows up immediately thanks to Beilinson's conjectures about the category of mixed Tate motives. In particularly this approach clarifies why the dilogarithm appears in the classical computations of volumes of geodesic 3-simplices (see, for example, [Co] and [M2]).

6. Geodesic simplices in non-Euclidian geometry and mixed Tate motives. In the chapters 4 and 5 I will deduce conjectures (2.4) and (2.7) from standard conjectures about mixed Tate motives. The key idea is that any geodesic simplex M in spherical or hyperbolic geometry define certain mixed Tate motive. To get its Hodge realization in the case , say, hyperbolic geometry one has to proceed as follows.

In the Klein model the Lobachevsky space $\mathcal{H}^m$ is realized as the interior of a ball in $\mathbb{R}^m$ and the distance $\rho(P_1, P_2)$ is defined as $|\log r(Q_1, Q_2, P_1, P_2)|$ where Q_1, Q_2 are the intersection points of the line P_1P_2 with the absolute: the sphere Q, and Q_1, P_1, P_2, Q_2 is the order of the points on the line.

Then geodesics are straight lines and so a geodesic simplices are just the usual ones inside Q.

Let us complexify and compactify this picture. We will get $\mathbb{CP}^m$ together with a quadric Q corresponding to the absolute and a collection of hyperplanes $M = (M_1, \ldots, M_{m+1})$ corresponding to faces of a geodesic simplex.

For any nondegenerate quadric $Q \subset \mathbb{CP}^m$ and a simplex $M = (M_1, \ldots, M_{m+1})$ in generic position with respect to Q let us denote by $H^m(\mathbb{CP}^m \setminus Q, M)$ the m-th cohomology group of $\mathbb{CP}^m \setminus Q, \mathbb{Q})$ modulo $Q \setminus (Q \cap M)$. This space has canonical mixed Hodge structure, which we denote as $h(Q, M)$. The weights are $0, 2, \ldots, 2 \cdot \left[\frac{m-1}{2}\right]$ and the corresponding graded quotients of weight $2j$ are isomorphic to direct sum of the Tate structures $\mathbb{Q}(-j)$. Such mixed Hodge structures are called mixed Hodge-Tate structures. In particularly a geodesic simplex produces a Hodge-Tate structure.

In complete analogy with this a quadric $Q \in \mathbb{CP}^m$ without points in $\mathbf{RP}^m$ defines an elliptic geometry in $\mathbf{RP}^m$; the geodesic simplices are just the usual ones.

Now let us suppose that $m = 2n - 1$. There is a canonical, up to a sign, meromorphic $2n - 1$ - form ω_Q in $\mathbb{CP}^{2n-1}\backslash Q$ with polar singularity along Q. If we choose coordinates $x_1, x_2, ..., x_{2n}$ in $\mathbb{C}^{2n}$ and an equation $\tilde{Q} = \sum_{i,j} q_{ij}x_i x_j = 0$ of the quadric Q, then

$$w_Q := 1/(2\pi i)^n \cdot \sqrt{\det \tilde{Q}} \cdot \frac{\sum_{i=1}^{2n}(-1)^i x_i dx_i \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{2n}}{\tilde{Q}^n} \tag{14}$$

It does not depend on the choice of the coordinates x_i and the equation $\tilde{Q}$.

Let Δ_M be a relative cycle representing a generator of $H_{2n-1}(\mathbb{CP}^{2n-1}, M)$. Set

$$v(Q, M) = \int_{\Delta_M} \omega_Q \tag{15}$$

If a pair (Q, M) corresponds to a geodesic simplex S in $\mathcal{H}^{2n-1}$ then there is a natural choice for the relative cycle Δ_M and $v(Q, M) = \mathrm{vol}(S)$. So the volume of an odd-dimensional geodesic simplex is the maximal period of the corresponding mixed Hodge structure.

Mixed Tate motives over $\mathbb{C}$ should be objects of certain abelian $\mathbb{Q}$-category $\mathcal{M}$. There should be a realization functor from $\mathcal{M}$ to the category of mixed Hodge-Tate structures $\mathcal{H}_T$. According to general philosophy mixed Tate motives can be considered as those of mixed Hodge-Tate structures that can be realised in cohomology of algebraic varieties. Therefore the mixed Hodge-Tate structure $h(Q, M)$ constructed above definitely corresponds to a certain mixed Tate motive.

However Ext-groups between even simplest mixed Tate motives $\mathbb{Q}(n)_{\mathcal{M}}$ in the category $\mathcal{M}$ should be quite different from those of their Hodge counterparts $\mathbb{Q}(n)_{\mathcal{H}_T}$. For example, one should have ([B1])

$$Ext^1_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}}, \mathbb{Q}(n)_{\mathcal{M}}) \otimes \mathbb{Q} = gr_n^{\gamma} K_{2n-1}(\mathbb{C}) \otimes \mathbb{Q} \tag{16}$$

while in the category of mixed Hodge-Tate structures one have

$$Ext^1_{\mathcal{H}_T}(\mathbb{Q}(0)_{\mathcal{H}_T}, \mathbb{Q}(n)_{\mathcal{H}_T}) = \mathbb{C}/(2\pi i)^n\mathbb{Q} \tag{17}$$

and the realization functor provides a homomorphism

$$Ext^1_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}}, \mathbb{Q}(n)_{\mathcal{M}}) \longrightarrow Ext^1_{\mathcal{H}_T}(\mathbb{Q}(0)_{\mathcal{H}_T}, \mathbb{Q}(n)_{\mathcal{H}_T}) \tag{18}$$

that should coincide with the Beilinson's regulator (6) after identifications (16) (17).

The mixed Hodge-Tate structure $h(Q, M)$ has the following additional data (n -framing): the form ω_Q defines a vector $[w_Q] \in gr^W_{2n}h(Q, M)$ and a generator of $H_{2n-1}(\mathbb{C}P^{2n-1}, M)$ can be viewed as a functional in $gr^W_0 h(Q, M)$. (Compare with [BGSV]).

n-framed mixed Hodge-Tate structures form an abelian group $\mathcal{H}_n$ (see [BGSV], or s.4.2). There is a homomorphism of a scissor congruence group in a $(2n-1)$- dimensional non-Euclidian space to $\mathcal{H}_n$ defined on generators as described above.

The next important idea is that the Dehn invariant has a natural interpretation in the language of mixed Hodge-Tate structures (theorem 4.8). Moreover, any element in the kernel of the Dehn invariant corresponds to an element of $Ext^1\mathcal{H}_T(\mathbb{Q}(0)_{\mathcal{H}_T}, \mathbb{Q}(n)_{\mathcal{H}_T})$. It is clearly of motivic origin and hence gives us an element of (16). Therefore we get a map predicted by conjecture 2.4 in the hyperbolic case.

7. Some speculations about euclidean Dehn complexes and mixed Tate motives over the dual numbers. Let $\Omega^i_{\mathbb{R}/\mathbb{Q}}$ be the Kahler differential i-forms of the field $\mathbb{R}$

Conjecture 2.9 $H^i(\mathcal{P}^\bullet_E(2n-1)) \otimes \mathbb{Q} = \Omega^{i-1}_{\mathbb{R}/\mathbb{Q}}$

For $n = 2$ this is the beautyful theorem of Sydler, later reproved by Jessen and Thorup and Dupont-Sah [DS2].

Notice that $\Omega^0_{\mathbb{R}/\mathbb{Q}} = \mathbb{R}$, so for $i = 1$ and arbitrary n we get the conjecture (2.6).

Here is a "motivic interpretation" of conjecture (2.9). One could think of Euclidian geometry as of degeneration of hyperbolic geometry. Namely, consider the Cayley model of the hyperbolic geometry inside of the sphere Q_ε given by the equation $\varepsilon(x_1^2 + ...x_n^2) = x_0^2$ in homogeneous coordinates. Then in the limit $\varepsilon \longrightarrow 0$ we will get the Euclidean geometry in $\mathbb{R}^n$. Now let ε be a formal variable with $\varepsilon^2 = 0$. Then one should imagine that $H(Q_\varepsilon, M) := H^n(\mathbb{P}^n \backslash Q_\varepsilon, M)$ is a variation (or better to say a deformation because it should split at $\varepsilon = 0$) of mixed Tate motives over $Spec\mathbb{R}[\varepsilon]$. Notice that right now there is no even a hypothetical definition of the category of mixed motives or mixed Hodge structures over a scheme with nilpotents. The Euclidian Dehn invariant should have a natural interpretation in the language of deformations of mixed Hodge structures over $Spec\mathbb{R}[\varepsilon]$ similar to the one for non euclidean Dehn invariant given in the chapter 4.

Let $D_T(\varepsilon)$ be the hypothetical category of mixed Tate motives over $Spec\mathbb{R}[\varepsilon]$ which split over $Spec\mathbb{R}$. It is natural to conjecture that

$$Ext^i_{D_T(\varepsilon)}(\mathbb{Q}(0),\mathbb{Q}(n)) = Ker\Big(gr^{\gamma}_n K_{2n-i}(\mathbb{R}[\varepsilon]) \longrightarrow gr^{\gamma}_n K_{2n-i}(\mathbb{R})\Big)\otimes\mathbb{Q}$$

One can compute the right hand side using the comparision with the cyclic homology ([FT], [Goo]) and get the following theorem (I am indebted to B. Tsygan for scketching me the proof):

Theorem 2.10 $Ker\Big(gr^{\gamma}_n K_{2n-i}(\mathbb{R}[\varepsilon]) \longrightarrow gr^{\gamma}_n K_{2n-i}(\mathbb{R})\Big)\otimes\mathbb{Q} = \Omega^{i-1}_{\mathbb{R}/\mathbb{Q}}$

Therefore one should get a canonical map $H^i\mathcal{P}^{\bullet}_E(2n-1) \longrightarrow \Omega^{i-1}_{\mathbb{R}/\mathbb{Q}}$. I hope that it is an isomorphism.

3 Volumes of hyperbolic n-manifolds and continuous cohomology of $SO(n,1)$

Any oriented hyperbolic manifold M^n, $\dim M^n = n$, can be represented as a quotient $M^n = \Gamma\setminus\mathcal{H}^n$ where Γ is a torsion free subgroup of $SO(n,1)$.

1.Let us denote by BG the classifying space of a group G. Set $G^n := \underbrace{G\times\cdots\times G}_{ntimes}$. There is Milnor's simplicial model for BG: $BG_{\bullet} = EG_{\bullet}/G$ where

$$EG_{\bullet}:\ G \leftleftarrows G^2 \,\substack{\leftarrow\\[-2pt]\leftarrow\\[-2pt]\leftarrow}\, G^3 \,\substack{\leftarrow\\[-2pt]\leftarrow\\[-2pt]\leftarrow\\[-2pt]\leftarrow}\,\ldots$$

The inclusion $j:\Gamma\hookrightarrow SO(n,1)$ induces a map

$$j_*: B\Gamma \to BSO(n,1)^{\delta}$$

(Here G^{δ} is a Lie group considered as a discrete group). Notice that $\Gamma\setminus\mathcal{H}^n = B\Gamma$.

2. Homological interpretation of the volume of a compact hyperbolic manifold. Now let us suppose that M^n is *compact*. Then $H_n(M^n) = \mathbb{Z}$. Let us denote by b_n the generator of $H_n(M^n)$. Then

$$j_*(b_n)\in H_n(BSO(n,1)^{\delta}) = H_n(SO(n,1)^{\delta}) \tag{19}$$

Further, let G be a Lie group, $C^n(G)$: the space of continuous functions on G^n. There is a differential

$$d: C^n(G)\to C^{n+1}(G), df(g_0,\ldots,g_n) = \sum_{i=0}^{n}(-1)^i f(g_0,\ldots,\hat{g}_i,\ldots,g_n)$$

Let $C^n(G)^G$ is the subspace of functions invariant under the left diagonal action of G on G^n. Then by definition

$$H_c^*(G,R) := \mathcal{H}^{*+1}(\dots \xrightarrow{d} C^n(G)^G \xrightarrow{d} C^{n+1}(G)^G \xrightarrow{d} \dots) \quad (20)$$

are the continuous cohomology of the Lie group G.

Let us denote by $I(g_0x, \dots, g_nx)$ the geodesic simplex in the hyperbolic space $\mathcal{H}^n$ with vertices at points $g_0x, \dots, g_nx$, where $g_i \in SO(n,1)$ and x is a given point in $\mathcal{H}^n$. Then

$$(I(g_0x, \dots, g_nx)) \quad (21)$$

is a continuous n-cocycle of $SO(n,1)$ because it

a) is invariant under the left diagonal action of $SO(n,1)$

b) satisfies the cocycle condition

$$\sum_{i=0}^{n+1}(-1)^i \mathrm{vol}(I(g_0x, \dots, \widehat{g_ix}, \dots, g_{n+1}x)) = 0$$

Let

$$v_n \in H_c^n(SO(n,1), R) \subset H^n(SO(n,1)^\delta, R) \quad (22)$$

be the cohomology class of this cocycle.

Theorem 3.1. *Let M^n be a compact hyperbolic manifold. Then*

$$\mathrm{vol}(M^n) = \langle v_n, j_*(b_n) \rangle$$

Proof. There is a triangulation

$$M^n = \bigcup_k I_k$$

of M^k in geodesic simplices I_k. To obtain it choose N generic points $y_1, \dots, y_N$ in M^n and consider the corresponding Dirichlet domains

$$D(y_j) := \{x \in M^n | \rho(x, y_\alpha) \le \rho(x, y_\beta) \text{ for any } \beta \ne \alpha\} \quad (23)$$

Then the dual triangulation is the desired one. (The vertices of the dual simplices are the points y_α, two vertices y_{α_1} and y_{α_2} are connected by the edge if domains $D(y_{\alpha_1})$ and $D(y_{\alpha_2})$ have common codimension 1 face and so on.)

Let $\pi : \mathcal{H}^n \to \Gamma \backslash \mathcal{H}^n$. The group Γ acts on $\mathcal{H}^n$ freely. Therefore $\pi^{-1}(v_\alpha)$ is a principal homogeneous space of Γ. Let us choose elements $g_\alpha \in SO(n,1)$ such that $g_\alpha x \in \pi^{-1}(v_\alpha)$. (and so $\pi^{-1}(v_\alpha) = \Gamma \cdot g_\alpha x$). Let $I(g_0^{(k)}x, \dots, g_n^{(k)}x)$ be a geodesic simplex in $\mathcal{H}^n$ that projects onto I_k. One can choose elements $g_i^{(k)}$ in such a way that $g_i^{(k)} \in \Gamma \cdot g_\alpha x$ for some α. Let us do this and consider the following n-chain in $BSO(n,1)^\delta$.

$$\sum_k (g_0^{(k)}, \dots, g_n^{(k)}) \tag{24}$$

Lemma 3.2.
a) *The boundary of this n-chain is zero.*
b) *Its homology class in $H_n(BSO(n,1)^\delta)$ coincides with $j_*(b_n)$.*

Proof. a) Let $I(h_0^{(1)}x, \dots, h_{n-1}^{(1)}x)$ and $I(h_0^{(2)}x, \dots, h_{n-1}^{(2)}x)$ be codimension 1 faces of geodesic simplices $I(g_0^{(k)}x, \dots, g_n^{(k)}x)$ that project to the same face in $\Gamma \setminus \mathcal{H}^n$. Then (thanks to the special choice of elements $g_i^{(k)}$) there is an element $\gamma \in \Gamma$ such that $\gamma \cdot h_i^{(1)} = h_i^{(2)}$ for all i. Further, it is clear that these faces will appear with opposite signs. The statement a) is proved. The proof of statement b) follows from the definitions.

The value of the cocycle (21) on the cycle (24) is equal to $\mathrm{vol}(\Gamma \setminus \mathcal{H}^n)$ just by definition. So theorem 3.1 follows immediately from Lemma 3.2.

3. The strategy for noncompact hyperbolic manifolds. Now let $M^n = \Gamma \setminus \mathcal{H}^n$ be a noncompact hyperbolic manifold with finite volume. We would like to prove an analog of theorem 3.1. The first problem is that $H_n(M^n) = 0$. Let H be a subgroup of a group G. Set

$$H_*(G, H; \mathbb{Q}) := H_*(\mathrm{Cone}(BH_\bullet \to BG_\bullet), \mathbb{Q})$$

We will show that M^n does produce an element

$$\tilde{c}(M_n) \in H_n(SO(n,1)^\delta, T_n(s)^\delta; \mathbb{Q}) \tag{25}$$

where $T_n(s)$ is a subgroup of $SO(n,1)$ consisting of transformations preserving $s \in \partial\mathcal{H}^n$ and acting as translations on horospheres based at s. (If s is the point at infinity in the upper half-space realization of $\mathcal{H}^n$, then $T_n(s)$ is the group of translations in the hyperplane $x_n = c$ where x_n is the vertical coordinate).

Further, let H be a Lie subgroup of a Lie group G. The inclusion $i : H \hookrightarrow G$ induces a homomorphism of complexes $C^*(G)^G \longrightarrow C^*(H)^H$. Set

$$H_c^*(G, H; \mathbb{R}) := H^* \mathrm{Cone}(C^*(G)^G \xrightarrow{i^*} C^*(H)^H)[-1]$$

By definition an n-cocycle in $\mathrm{Cone}(C^*(G)^G \xrightarrow{i^*} C^*(H)^H)[-1]$ is a pair (f, h) where f is a continuous n-cocycle of G, h is an $(n-1)$-cochain on H and

$$i^* f = dh$$

In other words cocycles in $\mathrm{Cone}(i^*)$ are just those cocycles of G whose restriction to H is cohomologous to zero. Choose a point $x \in \mathcal{H}^n$. Then $(h_i \in T_n(s))$

$$\tilde{v}_n(x) := (\mathrm{vol}I(g_0x, \ldots, g_nx), \quad \mathrm{vol}I(s, h_0x, \ldots, h_{n-1}x)) \tag{26}$$

is a cocycle representing certain cohomology class

$$\tilde{v}_n \in H_c^n(SO(n,1), T_n(s))$$

Indeed,

$$\mathrm{vol}I(h_0x, \ldots, h_nx) = \sum_{i=0}^{n} (-1)^i \mathrm{vol}I(s, h_0x, \ldots, \widehat{h_ix}, \ldots, h_nx)$$

We will prove that $\mathrm{vol}(M^n) = \langle \tilde{v}_n, \tilde{c}(M^n) \rangle$.

To produce the class $\tilde{c}_n(M^n)$ we have to recall some basic facts about noncompact hyperbolic manifolds (see §5 of Thurston's book [Th])

4. A decomposition of hyperbolic manifolds Let $x \in \mathcal{H}^n$. Denote by $\Gamma_\varepsilon(x)$ the subgroup generated by all elements of Γ which move x to a distance $\leq \varepsilon$.

Theorem 3.3 (Kazhdan-Margoulis [KM]). *There is an $\varepsilon > 0$ such that for every discrete group Γ of isometries of $\mathcal{H}^n$ and for every $x \in \mathcal{H}^n$, $\Gamma_\varepsilon(x)$ has an abelian subgroup of finite index.*

Let $s \in \partial\mathcal{H}^n$. A part of $\mathcal{H}^n$ located inside an horosphere centered at s is called horoball. Iqn the upper half space realization of $\mathcal{H}^n$ horoball centered at infinity is defined as $\{x_n \geq c\}$. Let us denote by $E_n(s)$ the stabilizer in $SO(n,1)$ of the point s. Let $\Delta_n(s)$ be a discrete subgroup of $E_n(s)$ that has a subgroup of finite index isomorphic to $\mathbb{Z}^{n-1}$ and acting by translation on horospheres centered at s. Suppose also that $\Delta_n(s)$ acts freely in $\mathcal{H}^n$. Then the quotient of a horoball centered at s is called cusp.

Denote by $\ell(v)$ the length of the shortest geodesic loop based at v.

Theorem 3.4. (Decomposition Theorem). *Let M^n be an orientable hyperbolic manifold of finite volume. Then there is $\delta > 0$ such that $\ell^{-1}[0,\delta] \subset M^n$ consists of finitely many components and each of these components is isometric to a cusp.*

This theorem follows from theorem 3.3.

Recall that $M^n = \Gamma \setminus \mathcal{H}^n$. Let us denote by $C(\Gamma)$ the subset of $\partial\mathcal{H}^n$ consisting of the points c such that

a) the isotropy subgroup of c in Γ is nontrivial

b) the isotropy subgroup of any geodesic ending at c in Γ is trivial

Remark 3.5. A nontrivial element γ of Γ preserving a point $c \in \partial\mathcal{H}^n$ and a geodesic ending at c can not stabilize points on this geodesic because by the assumption each point of $\mathcal{H}^n$ has trivial stabilizer in Γ. So the element γ move the geodesic along itself. Therefore the point c does not correspond to a cusp of $\Gamma \setminus \mathcal{H}^n$. It follows from the Decomposition theorem that $C(\Gamma)$ consists of a finite number of Γ-orbits. (Each of them corresponds to a cusp of $\Gamma \setminus \mathcal{H}^n$). Let us choose $\{c_i\} \in C(\Gamma)$ such that

$$C(\Gamma) = \bigcup_i \Gamma c_i \tag{27}$$

The manifold M^n is compact if and only if $C(\Gamma) = \emptyset$.

5. A triangulation of a noncompact hyperbolic manifold by ideal geodesic simplices. We would like to define a Dirichlet decomposition of $\mathcal{H}^n$ corresponding to the set $C(\Gamma) \in \partial\mathcal{H}^n$, and then take the dual triangulation. The problem is that the distance $\rho(x,c)$ from a point $x \in \mathcal{H}^n$ to $c \in C(\Gamma)$ is infinite. To define a regularized distance $\tilde{\rho}(x,c)$ let us choose a horosphere $h(c)$ for each $c \in C(\Gamma)$ in such a way that $h(\gamma \cdot c) = \gamma \cdot h(c)$ for any $\gamma \in \Gamma$. Then by definition $\tilde{\rho}(x,c)$ is the distance from x to the horosphere $h(c)$. It is negative if x is inside of $h(c)$. Now the Dirichlet domains are defined as usual:

$$D(c) := \{x \in \mathcal{H}^n | \tilde{\rho}(x,c) \leq \tilde{\rho}(x,c') \text{ for any } c' \in C(\Gamma),\ c' \neq c\} \tag{28}$$

They are polyhedrons with finite number of faces.

For a generic choice of horospheres $h(c_i)$ the dual polyhedrons are simplices with vertices at points of $C(\Gamma)$. This triangulation is Γ-invariant.

6. Theorem 3.6. *Any hyperbolic manifold of finite volume M^n defines canonically a class*

$$\tilde{c}(M^n) \in H_n(SO(n,1)^\delta, T_n(v)^\delta; \mathbb{Q})$$

such that

$$\langle \tilde{v}_n, \tilde{c}(M^n)\rangle = \mathrm{vol}(M^n)$$

Proof. Let $\widetilde{\partial\mathcal{H}^n}$ be the set of all pairs (v,ℓ) where $v \in \partial\mathcal{H}^n$ and ℓ is a geodesic ended at v.

Let $\widetilde{C(\Gamma)} \subset \widetilde{\partial\mathcal{H}^n}$ be the set of all pairs (c,ℓ) where $c \in C(\Gamma)$ and ℓ is an edge of one of geodesic simplices from the constructed triangulation of $\mathcal{H}^n$. Notice that the group Γ acts freely on $\widetilde{C(\Gamma)}$ Choose geodesics ℓ_{ij} such that $(c_i, \ell_{ij}) \subset \widetilde{C(\Gamma)}$ and

$$\widetilde{C(\Gamma)} = \bigcup_{i,j} \Gamma \cdot (c_i, \ell_{ij}) \tag{29}$$

Take a point $(v_0, \ell_0) \in \widetilde{\partial\mathcal{H}^n}$ such that $v_0 \notin C(\Gamma)$. There exist elements $g_{ij} \in SO(n,1)$ with properties

$$a)(c_i, \ell_{ij}) = g_{ij}\cdot(v_0,\ell_0) \qquad b)g_{ij}^{-1}g_{ik} \in T_n(v_0) \tag{30}$$

(Recall that $T_n(v_0)$ is the subgroup of translations). Therefore any $(c,\ell) \in \widetilde{C(\Gamma)}$ can be written as

$$(c,\ell) = g_{(c,\ell)}\cdot(v_0,\ell_0), \tag{31}$$

$$g_{(c,\ell)} := \gamma_{(c,\ell)}\cdot g_{ij}, \tag{32}$$

where $\gamma_{(c,\ell)} \in \Gamma$ is defined from the condition $(c,\ell) = \gamma_{(c,\ell)}\cdot(c_i,\ell_{ij})$

Now let us choose a representation S_α in each class of Γ-equivalence of simplices from the constructed triangulation of $\mathcal{H}^n$. This is a finite set $\{S_\alpha\}$ and $\{\pi(S_\alpha)\}$ is a triangulation of $\Gamma \setminus \mathcal{H}^n$. (Recall that $\pi : \mathcal{H}^n \to \Gamma\setminus\mathcal{H}^n$).

Let $F(\{S_\alpha\}) \subset \widetilde{C(\Gamma)}$ be the set of all pairs (v,ℓ) where v is a vertex of a simplex S_α and ℓ is its edge. Each simplex S_α defines $n(n+1)$ elements of the set $F(\{S_\alpha\})$. They correspond to vertices of the "truncated simplex" $\tilde{S}_\alpha$.

Let us subdivide polyhedrons $\tilde{S}_\alpha$ on simplices $\tilde{S}_{\alpha,\beta}$. Denote by $f^0_{\alpha,\beta},\ldots,f^n_{\alpha,\beta}$ the vertices of $\tilde{S}_{\alpha,\beta}$. According to (31) there are uniquely defined elements $g^i_{\alpha,\beta}\in SO(n,1)$ such that

$$f^i_{\alpha\beta}=g^i_{\alpha\beta}\cdot(v_0,\ell_0)\qquad(0\le i\le n)\tag{33}$$

Set

$$\tilde{c}'(M^n):=\sum_{\alpha,\beta}(g^0_{\alpha,\beta},\ldots,g^n_{\alpha,\beta})\in BSO(n,1)(n)\tag{34}$$

The boundary of chain (34) is computed as follows. Each vertex $v_a(S_\alpha)$ of simplex S_α gives us n elements

$$(v_a(S_\alpha),\ell^1_a(S_\alpha)),\ldots,(v_a(S_\alpha),\ell^n_a(S_\alpha))$$

in $F(\{S_\alpha\})$. So there are $h^b_a\in SO(n,1)$, chosen according to (31)-(32) such that

$$(v_a(S_\alpha),\ell^b_a(S_\alpha))=h^b_a(S_\alpha)\cdot(v_0,\ell_0)\tag{35}$$

($\{h^b_a(S_\alpha)\}=\{g^i_{\alpha,\beta}\}$ of course). So each vertex $v_a(S_\alpha)$ produces a chain

$$(h^1_a(S_\alpha),\ldots,h^n_a(S_\alpha))\in BSO(n,1)_{(n)}\tag{36}$$

It follows from the definitions that

$$\partial\tilde{c}'_n(M^n)=\sum_{v_a(S_\alpha)}(h^1_a(S_\alpha),\ldots,h^n_a(S_\alpha))\tag{37}$$

where summation is over all vertices of simplices S_α.

Lemma 3.7. *For every c_i (see (27))*

$$\sum_{v_a(S_\alpha)\in\{\Gamma\cdot c_i\}}(h^1_a(S_\alpha),\ldots,h^n_a(S_\alpha))\in BSO(n,1)_{n-1}\tag{38}$$

is a cycle.

Proof. Clear.

Lemma 3.8. *The cycle (38) is the image of a cycle $\tilde{c}''_{(i)}(M^n)$ in $BG^{(i)}(v_0)_{(n-1)}$ where $G^{(i)}(v_0)$ is a semidirect product of a finite group and the group of translations $T(v_0)$.*

Proof. Look at definitions (30). Each element $h_a^b(S_\alpha)$ is equal to

$$\gamma \cdot g_{ik} \cdot (v_0, \ell_0) = \gamma \cdot g_{ij} \cdot t \cdot (v_0, \ell_0) = g_{ij} \cdot (g_{ij}^{-1} \cdot \gamma \cdot g_{ij}) \cdot t \cdot (v_0, \ell_0)$$

where $\gamma \in \Delta_n(c_i)$ (the stabilizer of c_i in Γ) and $t \in T(v_0)$. Notice that $g_{ij}^{-1} T(c_i) g_{ij} = T(v_0)$ and $\Delta_n(c_i)$ is a semidirect product of a finite group and a subgroup in $T(c_i)$. Lemma 3.8 is proved.

It follows from Lemmas 3.7 and 3.8 that

$$c_E(M^n) := \left(\sum_i \tilde{c}''_{(i)}(M^n), \tilde{c}'(M^n) \right) \tag{39}$$

is a cycle in

$$\mathrm{Cone}(BG_n^{(i)}(v_0)_\bullet \to BSO(n,1)_\bullet)$$

Proposition 3.9. $\langle \tilde{v}_n, c_E(M^n) \rangle = \mathrm{vol}(M^n)$.

Proof. The cocycle

$$v_n(x) = (\mathrm{vol} I(g_o x, \ldots, g_n x), \mathrm{vol} I(v_o, h_0 x, \ldots, h_{n-1} x)) \tag{40}$$

represents the cohomology class $\tilde{v}_n \in H_c^n(SO(n,1), E_n(v_0))$. The cocycles $v_n(x)$ for different points $x \in \mathcal{H}^n$ are (canonically) cohomologous. Now let us move point x in (40) to the boundary point v_0. Then the limit of value of the component $\mathrm{vol}(I(g_0 x, \ldots, g_n x))$ on the chain $\tilde{c}'(M^n)$ (see (34)) exists and is equal to $\sum_\alpha \mathrm{vol}(S_\alpha) = \mathrm{vol}(M^n)$. From the other hand the limit of the second component $\mathrm{vol} I(v_0, h_0 x, \ldots, h_{n-1} x)$ of cocycle $v_n(x)$ on the chain $\sum_i c''_{(i)}(M^n)$ is zero. Proposition 3.9 is proved.

To prove theorem 3.6 it remains to show that there is a cycle $c_T(M^n)$ in

$$\mathrm{Cone}(BT_n(v_0)_\bullet \to BSO(n,1)_\bullet)$$

homologous to $N \cdot c_E(M^n)$ for certain integer N. This is a consequence of the following

Lemma 3.10. *The homomorphism*

$$j_* : H_*(T_n(v_0), \mathbb{Q}) \to H_*(G^i(v_0), \mathbb{Q})$$

induced by the inclusion $j : T_n(v_0) \hookrightarrow G^{(i)}(v_0)$ *is a map onto.*

Proof. Set $A := G^{(i)}(v_0)/T_n(v_0)$. There is Hochshild-Serre spectral sequence

$$E_{pq}^2 = H_p(A, H_q(T_n(v_0), \mathbb{Q})) \Rightarrow H_{p+q}(G^{(i)}(v_0), \mathbb{Q})$$

Further, for a finite group A and an A-module V, $H_i(A, V \otimes \mathbb{Q}) = 0$ for $i > 0$. Therefore

$$H_*(G^{(i)}(v_0), \mathbb{Q}) = H_*(T_n(v_0), \mathbb{Q})_A.$$

Lemma 3.10 and hence theorem 3.6 is proved.

7. $\tilde{c}(M^{2n-1})$ **and** $H_{2n-1}(GL_N(\bar{\mathbb{Q}}), \mathbb{Q})$. It is usefull to keep in mind the following trivial lemma

Lemma 3.11 *Let $\tilde{G} \to G$ be a surjective homomorphism of groups with finite kernel. Then it induces isomorphism on homology with rational coefficients.*

Let $N = 2^{n-1}$ and $s_+ : Spin(2n-1, 1) \to GL_N(\mathbb{C})$ be the halfspinor representation. Set $\tilde{T}_{2n-1}(v) := s_+(T_{2n-1}(v))$. We will produce a basis in this representation and a 1-dimensional subgroup $H \subset diag(GL_N)$ isomorphic to $\mathbb{G}_m$ and diagonal in this basis such that the following 3 conditions hold.

1. $s_+(Lie(T_{2n-1}(v)))$ is an abelian Lie algebra contained in the upper triangular Lie algebra.

2. It is normalized by a subgroup H.

3. The eigenvalues of an element $t \in H(\mathbb{Q}) = \mathbb{Q}^*$ acting on $s_+(Lie(T_{2n-1}(v)))$ by conjugation are equal to t.

For this let me remind the construction of the halfspinor representation. The spinor representation of the complex Lie algebra $o(V_{2n})$, where V_{2n} is a $2n$-dimensional complex vector space, can be described as follows. Let us choose a decomposition $V_{2n} = U \oplus W$ where U and W are (maximal) isotropic planes. Then $o(V_{2n})$ is isomorphic to a Lie subalgebra of the Lie algebra of all superdifferential operators of order ≤ 2 acting on the vector space $\Lambda^* W$ (which is the space of regular functions on the odd variety W). To be precise, $o(V_{2n})$ is the Lie subalgebra of all superdifferential operators of total degree 0 or 2.

Notice that

$$\Lambda^* W = S^- \oplus S^+$$

where $S^- = \Lambda^{odd} W$, $S^+ = \Lambda^{even} W$. Each subspace is preserved by the action of $o(V_{2n})$. The corresponding representations are the half spinor representations s_+ and s_-.

Let me describe the spinor representation in coordinates. Choose a basis $e_1, ..., e_n; f_1, ..., f_n$ in V_{2n} such that $W =< e_1, ..., e_n >; U =< f_1, ..., f_n >$ and

$(e_i, f_j) = \delta_{ij}$. Then an element $X \in o(2n)$ looks in this basis as follows:

$$\begin{matrix} A & B \\ C & -A^t \end{matrix}$$

where $B^t = -B$ and $C^t = -C$. One has

$$s : X \longmapsto -\frac{1}{2}trA + \sum_{1 \leq i,j \leq n} a_{ij}\xi_i \partial_{\xi_j} + \sum_{1 < i < j < n} \left(b_{ij}\xi_i \wedge \xi_j + c_{ij}\partial_{\xi_i} \wedge \partial_{\xi_j} \right)$$

Let $\mathcal{N} \subset s(o(2n))$ be the Lie subalgebra acting in the halfspinor representation by the following operators:

$$\sum_{1 < j < n} a_{1j}\xi_1 \partial_{\xi_j} + \sum_{1 < j < n} b_{1j}\xi_1 \wedge \xi_j$$

We will denote by N the corresponding subgroup of the group GL_N. The group $\tilde{T}_{2n-1}(v)$ is conjugated to a subgroup of N.

Let us embed the group $\mathbb{G}_m$ to GL_N as follows: $t \in \mathbb{G}_m$ multiplies a vector $\xi_{i_1} \wedge \ldots \wedge \xi_{i_k}$ where $i_1 < \ldots < i_k$ by t if $i_1 = 1$ and by 1 otherwise. This is the subgroup H. Let P be the semidirect product of the groups H and N.

Let us denote by $\hat{c}(M^{2n-1})$ the image of $\tilde{c}(M^{2n-1})$ under the composition of natural homomorphisms

$$H_{2n-1}(SO(2n-1,1), T_{2n-1}(v); \mathbb{Q}) \xrightarrow{s_+} H_{2n-1}(GL_N(\mathbb{C}), \tilde{T}_{2n-1}(v); \mathbb{Q}) \longrightarrow$$
$$H_{2n-1}(GL_N(\mathbb{C}), P(\bar{\mathbb{Q}}); \mathbb{Q})$$

The second arrow is provided by a conjugation in $GL_N(\bar{\mathbb{Q}})$. Let $P(\bar{\mathbb{Q}})$ be the group of $\bar{\mathbb{Q}}$-points of P.

Lemma 3.12 a) *Let M^{2n-1} be a compact hyperbolic manifold. Then (see 19)*

$$j_*(b_{2n-1}) \in H_n(SO(2n-1,1)(\bar{\mathbb{Q}}), \mathbb{Z})$$

b)Let M^{2n-1} be a noncompact hyperbolic manifold. Then

$$\hat{c}(M^{2n-1}) \in H_{2n-1}(GL_N(\bar{\mathbb{Q}}), P(\bar{\mathbb{Q}}); \mathbb{Q})$$

Proof According to Weil's rigidity, see 6.6, 6.7, 7.13 in [Ra], the subgroup $\Gamma \in SO(2n-1,1)$ is conjugate to a subgroup whose entries are algebraic numbers.

There is the usual exact sequence of cone:

$$\ldots \to H_{2n-1}(P(\bar{\mathbb{Q}});\mathbb{Q}) \longrightarrow H_{2n-1}(GL_N(\bar{\mathbb{Q}});\mathbb{Q}) \xrightarrow{e_*} H_{2n-1}(GL_N(\bar{\mathbb{Q}}),P(\bar{\mathbb{Q}});\mathbb{Q})$$

$$\longrightarrow H_{2n-2}(P(\bar{\mathbb{Q}});\mathbb{Q}) \longrightarrow H_{2n-2}(GL_N(\bar{\mathbb{Q}});\mathbb{Q}) \tag{41}$$

Theorem 3.13. *There exists an element* $c(M^{2n-1}) \in H_{2n-1}(GL_N(\bar{\mathbb{Q}});\mathbb{Q})$ *such that* $\hat{c}(M^{2n-1}) = e_* c(M^{2n-1})$.

Proof. It follows from

Proposition 3.14. $i_* : H_*(P(\bar{\mathbb{Q}});\mathbb{Q}) \to H_*(GL_N(\bar{\mathbb{Q}});\mathbb{Q})$ *is injective.*

Proof. Recall that $P = H \cdot N$ where H is the diagonal part of P and N is the unipotent one. Notice that $H(\bar{\mathbb{Q}}) = \bar{\mathbb{Q}}^*$.

Lemma 3.15. *The natural map* $H_*(H(\bar{\mathbb{Q}});\mathbb{Q}) \to H_*(P(\bar{\mathbb{Q}});\mathbb{Q})$ *is an isomorphism.*

Proof. There is the Serre-Hochshild spectral sequence

$$H_p(H, H_q(N,\mathbb{Q})) \Rightarrow H_{p+q}(P;\mathbb{Q}).$$

For a number field F the group $N(F)$ is isomorphic to the additive group of a finite dimensional $\mathbb{Q}$-vector space $V(F)$. So $H_q(N(F),\mathbb{Q}) \simeq \Lambda^q_{\mathbb{Q}} V(F)$. Further, any integer $n \in \mathbb{Q}^* = H(\mathbb{Q})$ acts on $H_q(N,\mathbb{Q})$ by multiplication on n^q. Therefore $H_*(H(\bar{\mathbb{Q}}), H_q(N(\bar{\mathbb{Q}});\mathbb{Q}))$ is anihilated by multiplication on $n^q - 1$.

This is a particular case of the following general fact (Proposition 5.4 in [M]): let G be a group and V is a G-module. Then the action of any element $g \in G$ on G by conjugation and on V via the G-module structure induces the identity map on $H_*(G,V)$. Lemma 3.15 is proved.

To complete the proof of the proposition 3.14 consider the homomorphism $det : GL_N(\bar{\mathbb{Q}}) \longrightarrow \bar{\mathbb{Q}}^*$. Notice that $\bar{\mathbb{Q}}^* = H(\bar{\mathbb{Q}}) \to GL_N(\bar{\mathbb{Q}}) \stackrel{det}{\to} \bar{\mathbb{Q}}^*$ is given by $x \longmapsto x^k$ for a certain positive integer k. Therefore its composition with $H_*(H(\bar{\mathbb{Q}});\mathbb{Q}) \longrightarrow H_*(GL_N(\bar{\mathbb{Q}});\mathbb{Q}) \longrightarrow H_*(\bar{\mathbb{Q}}^*;\mathbb{Q})$ is injective. The proposition is proved.

8. The spinor representation and the Pfaffian. The supertrace Str in the spinor representation $S = S^+ \oplus S^-$ is defined as follows:

$$Str := Tr|S_+ - Tr|S_- \qquad \text{i.e.} \quad Str(A) := Trs_+(A) - Trs_-(A)$$

There is a remarkable invariant polynomial of degree n on the Lie algebra $o(2n)$: the Pfaffian. Its restriction to the Cartan subalgebra $(t_1, ..., t_n)$, $e_i \longmapsto t_i e_i, f_i \longmapsto -t_i f_i$ is given by formula $Pf(t_1, ..., t_n) = t_1 \cdot ... \cdot t_n$. For a skewsymmetric $2n \times 2n$ matrix A one has $Pf(A)^2 = (-1)^n det(A)$.

Proposition 3.16 *The invariant polynomial* $A \longmapsto (-1)^{n(n-1)/2} n! \times Pf(A)$ *on* $o(2n)$ *is equal to the restriction of the invariant polynomial* $A \longmapsto Str(A^n)$ *in the spinor representation.*

Proof. Let us compute the restriction of the invariant polynomial $Str(A^n)$ in the spinor representation to the Cartan subalgebra $e_i \longmapsto t_i e_i, f_i \longmapsto -t_i f_i$

We will get

$$Str\Big(\sum_{i=1}^{n} t_i \xi_i \partial_{\xi_i} - \frac{1}{2}(t_1 + ... + t_n)\Big)^n = (-1)^{n(n-1)/2} n! \cdot t_1 \cdot ... \cdot t_n$$

Indeed, it is easy to see that the supertrace of the superdifferential operator $\xi_{i_1} \wedge ... \wedge \xi_{i_m} \partial_{\xi_1} \wedge ... \wedge \partial_{\xi_{i_m}}$ is 0 if $m < n$ and $(-1)^{n(n-1)/2} n!$ if $m = n$.

Consider the involution σ of the Lie algebra $o(2n, C)$ which corresponds to the involution interchanging the "horns" of the Dynkin diagram, whose involutive subalgebra is $o(2n-1, C)$.

Lemma 3.17 *The involution* σ *of the Lie algebra* $o(2n, C)$ *interchanges the halfspinor representations.*

Proof. The halfspinor representation is the fundamental representation corresponding to a vertex of a "horn".

9. The Lobachevsky class $v_{2n-1} \in H_c^{2n-1}(SO(2n-1, 1), \mathbb{R})$ can be obtained by restriction of the Borel class in the halfspinor representation. Recall that for a Lie group G with maximal compact subgroup K one has the Van Est isomorphism

$$H_c^*(G, \mathbb{R}) = H^*(LieG, LieK)$$

(On the right we have the relative Lie algebra cohomology). From the other hand

$$H^*(LieG, LieK) = H^*(LieG \otimes_{\mathbf{R}} \mathbb{C}, LieK \otimes_{\mathbf{R}} \mathbb{C})$$

Let us suppose that $K = G^\sigma$ where σ is an involution of the group G, (i.e. G, K, σ) is an involutive pair). Let $G_{\mathbf{C}}$ (resp $K_{\mathbf{C}}$) be the complex Lie group corresponding to $LieG \otimes_{\mathbf{R}} \mathbb{C}$ (resp $LieK \otimes_{\mathbf{R}} \mathbb{C}$). Then $K_{\mathbf{C}}$ is the fixed point set of an involution $\sigma_{\mathbf{C}}$ of the complex Lie group $G_{\mathbf{C}}$. Finally, let G_u be the maximal compact subgroup of $G_{\mathbb{C}}$. Then there is an involution σ_u of G_u such that $K = G_u^{\sigma_u}$. In this situation on has the isomorphism

$$H_c^*(G, \mathbb{R}) = H^*(LieG, LieK) = H_{top}^*(G_u/K, \mathbb{R})$$

which is functorial with respect to the maps of the symmetric pairs.

For example when $G = Spin(2n-1,1)$ one has $K = Spin(2n-1)$, $G_u = Spin(2n)$ and

$$H_c^*(Spin(2n-1,1), \mathbb{R}) = H_c^*(SO(2n-1,1), \mathbb{R}) = H_{top}^*(S^{2n-1}, \mathbb{R})$$

So there is just one (up to a scalar) nontrivial continuous cohomology class of the Lie group $SO(2n-1,1)$. Such a class v_{2n-1} was produced in s.2 above. We will call it the Lobachevsky class.

Similarly

$$H_c^*(GL_N(\mathbb{C}), \mathbb{R}) = H_{top}^*(U(N) \times U(N)/U(N)_{diag}, \mathbb{R}) = \Lambda^*(b_1, b_3, ..., b_{2N+1})$$

where $b_{2i-1} \in H_c^{2i-1}(GL_N(\mathbb{C}), \mathbb{R})$ are the continuous cohomology classes corresponding to the primitive generators B_{2i-1} of $H_{top}^*(U(N), \mathbb{Z})$. Namely there is canonical projection $p : U(N) \longrightarrow S^{2N-1} \subset \mathbb{C}^N$ and $B_{2N-1} := p^*[S^{2N-1}]$. Further, the restriction map $H^{2i-1}(U(N), \mathbb{Z}) \longrightarrow H^{2i-1}(U(i), \mathbb{Z})$ sends B_{2i-1} to the described above generator. I will call them the Borel classes. (Borel [Bo2] used a different normalisation of these classes).

Theorem 3.18. *Let $s_+ : Spin(2n-1,1) \to Aut(S_+)$ be the halfspinor representation. Then the restriction of the Borel class b_{2n-1} is proportional to the Lobachevsky class: $s_+^* b_{2n-1} = c \cdot v_{2n-1}$, $c \neq 0$.*

Proof. We will need the following 3 symmetric pairs (Lie group, an involutive Lie subgroup)

$$(Spin(2n-1,1), Spin(2n-1)), \quad (Spin(2n, \mathbb{C}), Spin(2n-1, \mathbb{C})),$$

$$(Spin(2n), Spin(2n-1))$$

We start with the half spinor representation providing us with a morphism of symmetric pairs

$$(Spin(2n-1,1), Spin(2n-1)) \longrightarrow (GL_{2^{n-1}}(\mathbb{C}), U(2^{n-1}))$$

Then we complexify it, getting a morphism of symmetric pairs

$$(Spin(2n, \mathbb{C}), Spin(2n-1, \mathbb{C})) \longrightarrow (GL_{2^{n-1}}(\mathbb{C}) \times GL_{2^{n-1}}(\mathbb{C}), GL_{2^{n-1}}(\mathbb{C}))$$

Finally restricting it to the maximal compact subgroups we get a morphism of symmetric pairs

$$a : (Spin(2n), Spin(2n-1)) \longrightarrow (U(2^{n-1}) \times U(2^{n-1}), U(2^{n-1})_{diag})$$

where $U(2^{n-1})_{diag}$ is the diagnal subgroup. The last map is $a : u \longmapsto (u, \sigma(u))$ where σ is the involution whose involutive subgroup is $Spin(2n-1)$.

The projections of $a(Spin(2n))$ to the first and second factors are half-spinor and antyhalfspinor representations. Indeed, the involution σ of the Lie algebra $o(2n, C)$ whose involutive subalgebra is $o(2n-1, C)$ interchanges the halfspinor representations (lemma 3.17).

So we get a commutative diagram

$$\begin{array}{ccc} Spin(2n) & \xrightarrow{a} & U(N) \times U(N) \\ \downarrow p_1 & & \downarrow p_2 \\ \frac{Spin(2n)}{Spin(2n-1)} & \xrightarrow{a'} & \frac{U(N)\times U(N)}{U(N)_{diag}} \end{array} \tag{42}$$

Under the map $p_2 : (u_1, u_2) \to u_1 u_2^{-1}$ the primitive generator $B_{2n-1} \in H^{2n-1}_{top}(U(N))$ goes to $p_2^* B_{2n-1} = B_{2n-1} \otimes 1 - 1 \otimes B_{2n-1}$

The key topological statement is that $a'(B_{2n}) \neq 0$. It is a corollary of proposition 3.16. To see this let us remind that there is the Euler class

$$E_{2n-1} := \pi^*([S^{2n-1}]) \in H^*_{top}(SO(2n))$$

where $\pi : SO(2n) \to S^{2n-1}$ is the natural projection. There are the corresponding classes

$$e_{2n-1} \in H^*(o(2n, \mathbb{C})) \qquad \text{and} \quad \bar{e}_{2n} \in H^*_{top}(BSO(2n))$$

One can identify $H^*_{top}(BSO(2n))$ with the ring of invariant polynomials $H^*(o(2n))^{o(2n)}$ on the Lie algebra $o(2n)$. Then the Euler class corresponds to (a non zero multiple of) the Pfaffian. The constant c_n can be calculated as follows. Set $E_{2n-1} = \alpha_n[Pf], B_{2n-1} = \beta_n[TrA^n]$ where $[Pf]$ and $[TrA^n]$ are the topological cohomology classes of S^{2n-1} and $U(N)$ coresponding to the invariant symmetric polynomials given by the Pfaffian and TrA^n. Then $c_n = (-1)^{n(n-1)/2} n! \cdot \beta_n/\alpha_n$.

Theorem 3.18 is proved.

Theorem 1.1 follows immediately from theorems 3.18, 3.1 (in the compact case), 3.6 (in the noncompact case) and 3.13 and lemma 3.12. First of all the image of the element $c(M^{2n-1}) \in H_{2n-1}(GL_N(\bar{\mathbb{Q}});\mathbb{Q})$ in $K_{2n-1}(\bar{\mathbb{Q}})$ gives the invariant of M^{2n-1} we are looking for. It is well defined because $H_{2n-1}(P(\bar{\mathbb{Q}});\mathbb{Q}) = \bar{\mathbb{Q}}^*$ projects to zero in $K_{2n-1}(\bar{\mathbb{Q}})$ (notice that $2n-1 > 1$). Further, we have an embedding

$$\varphi : \Big(SO(2n-1,1)(\bar{\mathbb{Q}}), T_n(v)(\bar{\mathbb{Q}})\Big) \hookrightarrow \Big(GL_N(\bar{\mathbb{Q}}), P(\bar{\mathbb{Q}})\Big)$$

The restriction of the cohomology class b_{2n-1} to $P(\bar{\mathbb{Q}})$ is zero, so it provides a relative class $\tilde{b}_{2n-1}$. Similarly one has the relative Lobachevsky class $\tilde{v}_{2n-1}$. It follows from theorem 3.18 that $\varphi^*(\tilde{b}_{2n-1}) = \tilde{v}_{2n-1}$. According to theorems 3.1, 3.18 and 3.13

$$\begin{aligned} vol(M^{2n-1}) \quad &= \quad < \tilde{v}_{2n-1}, \tilde{c}(M^{2n-1} > \quad = \quad < \tilde{b}_{2n-1}, \varphi_*\tilde{c}(M^{2n-1}) >= \\ & \qquad\qquad < b_{2n-1}, \varphi_* c(M^{2n-1}) > \end{aligned}$$

Theorem 1.3 now follows from theorem 1.1 and main results of [G1] or [G2] where the Borel regulator on $K_5(F)$ where F is a number field was computed via the trilogarithm. See s. 3 and 5 in [G2].

Theorem 1.2 follows, for example, from theorem 1.1 and results of s.2 in [G1].

Remark. The halfspinor representations seems to be the only ones among the fundamental representations of $SO(2n-1,1)$ such that the restriction of the Borel class to $SO(2n-1,1)$ is not zero. This means that only the halfspinor representations lead to a nontrivial invariant $c(M^{2n-1}) \in H_{2n-1}(GL_N(\bar{\mathbb{Q}});\mathbb{Q})$.

10. A plan of the second approach to the main results Let us say that a geodesic simplex is defined over $\bar{\mathbb{Q}}$ if it is equivalent under the motion group to a geodesic simplex which has vertices with coordinates in $\bar{\mathbb{Q}}$.

Proposition 3.19 *Any hyperbolic manifold can be decomposed on geodesic simplices defined over $\bar{\mathbb{Q}}$.*

Corollary 3.20 *The scissor congruence class of a hyperbolic manifold belongs to $\mathcal{P}(\mathcal{H};\bar{\mathbb{Q}})$.*

Proof. By Weil's rigidity theorem ([Ra]) $\pi_1(M)$ is congugated to a subgroup whose entries are algebraic numbers. If M is compact let us take sufficiently many points whose coordinates are algebraic numbers inside of the fundamental domain. Consider the Dirichlet decomposition corresponding to these points and their orbits under the action of the group $\pi_1(M)$. The intersection of $n+1$ geodesic hyperplane defined over $\bar{\mathbb{Q}}$ is a defined over $\bar{\mathbb{Q}}$ point. So the vertices of this decomposition defined over $\bar{\mathbb{Q}}$.

If M has cusps then their coordinates are also defined over $\bar{\mathbb{Q}}$, and the Dirichlet decomposition with respect to chosen points and the cusps gives the desired decomposition of M on geodesic simplices defined over $\bar{\mathbb{Q}}$ (see also [EP]).

After this applying results of the next chapter and chapter 5 we get another, more conceptual proof of the main theorem.

4 Noneuclidean polyhedrons and mixed Tate motives

1. The mixed Hodge structure corresponding to a non-Euclidian geodesic simplex. Let $Q \subset \mathbb{C}P^{2n-1}$ be a nondegenerate quadric and $M = (M_1, \ldots, M_{2n})$ be a simplex in generic position with respect to Q. Recall that $h(Q, M)$ is a notation for the mixed Hodge-Tate structure in $H^{2n-1}(\mathbb{CP}^{2n-1} \setminus Q, M; \mathbb{Q})$ and the weights are from 0 to $2n$. There are canonical isomorphisms

$$\mathbb{Q}(0) = H^{2n-1}(\mathbb{CP}^{2n-1}, M; \mathbb{Q}) \longrightarrow W_0 H^{2n-1}(\mathbb{CP}^{2n-1} \setminus Q, M; \mathbb{Q}) \qquad (43)$$

$$gr^W_{2n} H^{2n-1}(\mathbb{CP}^{2n-1} \setminus Q, M; \mathbb{Q}) \longrightarrow H^{2n-1}(\mathbb{CP}^{2n-1} \setminus Q; \mathbb{Q}) = \mathbb{Q}(n) \qquad (44)$$

The cohomology class $[w_Q]$ of (defined up to a sign) meromorphic $2(n-1)$ - form

$$w_Q := 1/(2\pi i)^n \cdot \sqrt{\det \tilde{Q}} \cdot \frac{\sum_{i=1}^{2n} (-1)^i x_i dx_i \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_{2n}}{\tilde{Q}^n} \qquad (45)$$

has the following geometrical interpretation. The quadric Q has 2 families of maximal isotopic subspaces. Let us denote by α_Q and β_Q the corresponding cohomology classes in $H^{2n-2}(Q)$.

The exact sequence

$$\ldots \to H^{2n-1}(\mathbb{CP}^{2n-1} \setminus Q) \longrightarrow H^{2n}_Q(\mathbb{CP}^n) = H^{2n}(Q)(-1) \longrightarrow H^{2n}(\mathbb{CP}^n) \to \ldots$$

shows that $\alpha_Q - \beta_Q \in \mathrm{Im}\delta$. In fact $\alpha_Q - \beta_Q = \pm\delta([w_Q])$. So to choose a sign in (45) one has to choose one of the families of maximal isotropic subspaces in Q.

Further, an orientation of the simplex M (provided by the numeration of hyperplanes M_i) corresponds to a generator of $H_{2n-1}(\mathbb{CP}^{2n-1}, M)$. Therefore if we choose a sign of w_Q and a simplex M then the mixed Hodge structure $h(Q,M)$ has the following n-frame: in $gr^W_{2n}h(Q,M)$ a vector is distinguished: $[w_Q]$, and in $gr^W_0 h(Q,M)$ a functional is distinguished: a generator of $H_{2n-1}(\mathbb{C}P^{2n-1}, M)$.

3. Hodge-Tate structures and the Hopf algebra $\mathcal{H}_\bullet$. For the convenience of the reader I will just reproduce some definitions from [BGSV]. Let us call a Hodge-Tate structure a mixed $\mathbb{Q}$ Hodge structure with weight factors isomorphic to $\mathbb{Q}(k)$. Say that H is n-framed Hodge-Tate structure if it is supplied with a nonzero vector in $gr^W_{2n}H$ and a nonzero functional in $gr^W_0 H$. Consider the coarsest equivalence relation on the set of all n-framed Hodge-Tate structures for which $H_1 \sim H_2$ if there is a map $H_1 \to H_2$ compatible with frames. For example any n-framed Hodge-Tate structure is equivalent to a one H with $W_{-2}H = 0$, $W_{2n}H = H$. Let $\mathcal{H}_n$ be the set of equivalence classes.

One may introduce on $\mathcal{H}_n$ the structure of an abelian group as follows. For $[H_1], [H_2] \in \mathcal{H}_n$ a vector in $gr^W_{2n}(H_1 \oplus H_2)$ will be the one whose components are distinguished vectors in $gr^W_{2n}H_1$ and $gr^W_{2n}H_2$.

The frames in $gr^W_0 H_i$ define the maps $gr^W_0(H_1 \oplus H_2) \to \mathbb{Z}(0) \oplus \mathbb{Z}(0)$. Its composition with the sum map $\mathbb{Z}(0) \oplus \mathbb{Z}(0) \to \mathbb{Z}(0)$ will be a distinguished functional in $gr^W_0 H$.

Let $-[H]$ be the class of H in which the frame in $gr^W_{2n}H$ is multiplied by -1. (We will get the same class multiplying by -1 the functional in $gr^W_0 H$).

The tensor product of mixed Hodge structures induces the commutative multiplication

$$\mu : \mathcal{H}_k \otimes \mathcal{H}_\ell \to \mathcal{H}_{k+\ell}$$

Let us define the comultiplication

$$\nu = \bigoplus_{k+\ell=n} \nu_{k\ell} : \mathcal{H}_n \to \bigoplus_{k+\ell=n} \mathcal{H}_k \otimes \mathcal{H}_\ell$$

Let $[H] \subset \mathcal{H}_n$, and (v_n, f_0) is a framing for $H : v_n \in gr^W_{2n}H$, $f_0 \in (gr^W_0 H)^*$. Let $R \subset gr^W_{2k}H$ be the lattice and $R^* \subset (gr^W_{2k}H)^*$ is the dual one. Define homomorphisms

$$\varphi : R \to \mathcal{H}_k, \qquad \psi : R^* \to \mathcal{H}_{n-k}$$

Namely, for $x \in R$ (resp. $y \in R^*$), $\varphi(x)$ (resp. $\psi(y)$) is the class of the mixed Hodge structure H with framing (x, f_0) (resp. (v_n, y)).

Let $\{e_j\}$, $\{e^j\}$ be dual bases in R, R^*. Then

$$\nu_{k,n-k}([H]) := \sum_j \varphi(e_j) \otimes \psi(e^j)$$

It is easy to see that

$$\nu(a \cdot b) = \nu(a) \cdot \nu(b)$$

Where $a \cdot b = \mu(a, b)$. Therefore the abelian group $\mathcal{H}_\bullet := \oplus \mathcal{H}_n$, where $\mathcal{H}_0 := \mathbb{Z}$, has a structure of graded Hopf algebra with the commutative multiplication μ and the comultiplication ν.

The following result is the main reason to consider the Hopf algebra $\mathcal{H}_\bullet$. More detailes can be found in [BMS].

Theorem 4.1. *The category of mixed $\mathbb{Q}$-Hodge-Tate structures is canonically equivalent to the category of finite-dimensional graded $\mathcal{H}_\bullet$-comodules.*

Namely, the equivalence assigns to a Hodge structure H the graded comodule $M(H)$, $M(H)_n = gr^W_{2n}(H)$ with $\mathcal{H}_\bullet$- action $M(H) \otimes M(H)^* \longrightarrow \mathcal{H}_\bullet$ given by the formula $x_m \otimes y_n \longrightarrow$ class of mixed Hodge structure H framed by x_m, y_n.

3. The Hopf algebra $S(\mathbb{C})_\bullet$. Let (Q, α) be a nondegenerate quadric $Q \in \mathbb{CP}^{2n-1}$ together with a choice of one of the families of maximal isotropic subspaces on Q denoted by α.

Let us denote by $S_{2n}(\mathbb{C})$ the abelian group generated by generic pairs $[(Q, \alpha); M]$ subject to the following relations:

R0) If $M_0 \cap \ldots \cap M_{2n-1} \neq \emptyset$ then $[(Q, \alpha); M] = 0$

R1) (*Projective invariance*). For any projective transformation $g \in PGL_{2n}(\mathbb{C})$

$$[(Q, \alpha); M] = [(gQ, g\alpha); gM]$$

R2) (*Skew symmetry*) a) For any permutation σ of the set {0, ... ,2n-1} one has

$$[(Q, \alpha); M] = (-1)^{|\sigma|}[(Q, \alpha); M_\sigma]$$

where $M_\sigma = (M_{\sigma(0)}, ..., (M_{\sigma(2n-1)})$

b) Let β be another family of maximal isotropic subspaces on the quadric $Q \in \mathbb{CP}^{2n-1}$. Then

$$[(Q, \alpha); M] = -[(Q, \beta); M]$$

R3) (*Additivity*). Let $M_0, ..., M_{2n}$ be hyperplanes such that $Q \cap M_I$ is a nondegenerate quadric for any subset $I \in \{0, ..., 2n\}$. Set $M^{(j)} := (M_0, ..., \hat{M}_j, ..., M_{2n})$. Then

$$\sum_{j=0}^{2n+1} (-1)^j [(Q, \alpha); M^{(j)}] = 0$$

For example, $S_2(\mathbb{C})$ is generated by 4-tuples of points (q_0, q_1, m_0, m_1) on $\mathbb{CP}^1$. Here by definition $\alpha = q_0$. The cross-ratio defines an isomorphism $r : S_2(\mathbb{C}) \to \mathbb{C}^*$.

Remark 4.2. The ordering of points and relation (R2a) is necessary to have this isomorphism. Moreover if we have it, the orientation of the quadric Q together with relation (R2a) in all dimensions is needed in order to define comultiplication Δ.

Recall that there is n-framed Hodge-Tate structure $h[(Q,\alpha); M]$ related to $H^{2n-1}(CP^{2n-1}\backslash Q, M; \mathbb{Q})$ and a choice of a family of maximal isotropic subspaces α on Q.

Proposition 4.3 *There is a homomorphism of abelian groups* $h : S_{2n}(\mathbb{C}) \to \mathcal{H}_n$ *defined on generators as* $h : [(Q,\alpha); M] \longrightarrow h[(Q,\alpha); M]$

Proof. We have to show that h maps relations to zero. For R1) and R2) this is clear from the definitions. If $\cap M_i \neq \emptyset$ then $gr_0^W h[(Q,\alpha); M] = 0$ and so this n-framed Hodge-Tate structure is equivalent to zero.

Let us prove that

$$\sum_{j=0}^{2n} (-1)^j h[(Q,\alpha), M^{(j)}] = 0 \tag{46}$$

One has the following map

$$H^{2n-1}(\mathbb{CP}^{2n-1}\backslash Q, \cup_{i=0}^{2n} M_i; \mathbb{Q}) \xrightarrow{\oplus r_j} \bigoplus H^{2n-1}(\mathbb{CP}^{2n-1}\backslash Q, M^{(j)}; \mathbb{Q}) \tag{47}$$

Consider in the left mixed Hodge structure the n-framing $(\omega_Q, 0)$ Then it is easy to see that $\oplus r_j$ induces map of this Hodge-Tate structure just to the left-hand side of (46) Therefore (47) holds. Proposition (4.3) is proved.

Set $S_0(\mathbb{C}) = \mathbb{Z}$ and $S(\mathbb{C})_\bullet := \oplus_{n=0}^{\infty} S_{2n}(\mathbb{C})$

Our next goal is to define on $S(\mathbb{C})_\bullet$ a structure of graded commutative Hopf algebra.

The product of elements $[(Q,\alpha); M] \in S_{2n}(\mathbb{C})$ and $[(Q',\alpha'); M'] \in S_{2n'}(\mathbb{C})$ is defined as follows. Let V be a vector space of dimension $2n$. Choose a quadratic equation $\tilde{Q}$ of the quadric Q and a maximal isotropic subspace $\tilde{\alpha}$ from the family α . Denote by $\tilde{M}_i$ the codimension 1 subspace projected to M_i. There is an analogous data in a vector space V' related to $[(Q',\alpha'); M']$. Then in $V \oplus V'$ there are the quadratic form $\tilde{Q} \oplus \tilde{Q}'$, the coordinate simplex consisting of subspaces $\tilde{M}_i \oplus V'$ and $V \oplus \tilde{M}_i'$ and the maximal isotropic subspace $\tilde{\alpha} \oplus \tilde{\alpha}'$. The projectivisation of this data defines a pair $[(Q * Q'; \alpha * \alpha'), M * M']$ .Its projective equivalence class does not depend on the choice of $\tilde{Q}, \tilde{Q}'$ and $\tilde{\alpha}, \tilde{\alpha}'$. By definition the product

$$[(Q, \alpha); M] * [(Q', \alpha'); M'] \in S_{2(n+n')}(\mathbb{C})$$

is the class of this pair.

Proposition 4.4. *The product is well defined.*

Proof. Clear from the definitions. For example to check

$$(\sum_{j=0}^{2n}(-1)^j[(Q, \alpha); M^j]) * [(Q', \alpha'); M'] = 0$$

consider $2(n+n')+1$ hyperplanes in $P(V')$ corresponding to $\{M_i \oplus V'\} \bigcup \{V \oplus M'_{i'}\}$ where $0 \le i \le 2n$; and $0 \le i' \le 2n-1$. Then on the level of generators the left side of (4.6) is just the additivity relation written for $(Q * Q'; \alpha * \alpha')$ and these hyperplanes.

Now let us define the comultiplication

$$\Delta : S_{2n}(\mathbb{C}) \longrightarrow \bigoplus_{k=0}^{n} S_{2k}(\mathbb{C}) \otimes S_{2(n-k)}(\mathbb{C})$$

Let $[(Q, \alpha); M]$ be a generator of $S_{2n}(\mathbb{C})$. If $\cap M_i \neq \emptyset$ then $\Delta[(Q, \alpha); M] = 0$ by definition. Now let us suppose that $\cap M_i = \emptyset$. Choose a data $(\tilde{Q}, \tilde{\alpha}); \{\tilde{M}_i\}$ in a vector space V corresponding to $[(Q, \alpha); M]$. Let $\tilde{Q}_I$ be the restriction of the quadratic form $\tilde{Q}$ to the subspace $\tilde{M}_I$. Set $\tilde{\alpha}_I := \tilde{\alpha} \cap M_I$

Let $\bar{I} := \{\bar{i}_1 < \ldots < \bar{i}_{2(n-k)}\}$ be the complement to I in $\{0, ..., 2n-1\}$. Then there is the family of hyperplanes $M_{\bar{i}_1} \cap M_I, \ldots, M_{\bar{i}_{2(n-k)}} \cap M_I$ in M_I that will be reffered to as $M_{\{\bar{I}\}}$.

Further, let $\tilde{Q}^I$ (resp. $\tilde{\alpha}^I$) be the quadratic form (resp. maximal isotropic subspace) induced in V/M_I by $\tilde{Q}$ (resp. $\tilde{\alpha}$). The hyperplanes M_i where $i \in I$ induce the family M^I of hyperplanes in $P(V/\tilde{M}_I$. Set $\Delta = \oplus_{k=0}^{n} \Delta_{2(n-k),2k}$ and

$$\Delta_{2(n-k),2k}([Q, \alpha]; M) := \sum_{|I|=2k} ([(\tilde{Q}^I, \tilde{\alpha}^I); \tilde{M}^I]) \otimes [(\tilde{Q}_I, \tilde{\alpha}_I); \tilde{M}_{\bar{I}}) \in S_{2k}(\mathbb{C}) \otimes S_{2(n-k)}(\mathbb{C})$$

There is also a duality $i : S_{2n}(\mathbb{C}) \to S_{2n}(\mathbb{C})$ defined as follows. A non-degenerate quadric $Q \in \mathbb{CP}^m$ provides an isomorphism $i_Q : \mathbb{CP}^m \to \hat{\mathbb{C}}^m$.

Namely, let $x \in \mathbb{CP}^m$. Consider the set of lines through x tangent to Q. The tangency locus is a section of Q by a hyperplane $i_Q(x)$. Let $v_j = \cap_{i \neq j} M_i$ be the vertices of simplex M. Set $\hat{M} := (i_Q(v_0), ..., i_q(v_m))$. Then

$$i : [(Q, \alpha); M] \longrightarrow [(Q, \beta); \hat{M}]$$

Theorem 4.5 a) *The comultiplication Δ is well defined.*

b) The multiplication $*$ and the comultiplication Δ provide the structure of graded commutative Hopf algebra on $S(\mathbb{C})$.

c)The duality i is an antipode, i.e. $i(a * b) = i(a) * i(b); \Delta(i(a)) = t(i \otimes i)\Delta(a)$, where t interchange factors in the tensor product.

Proof. a) We have to check that $\Delta_{2k, 2(n-k)}$ takes relations R0)-R3) to zero. This is obvious for R1)-R2) and true by definition for R0). It remains to check R3).

It is convenient to extend the definition of Δ to generators $[(Q, \alpha); M]$ with $\bigcap M_i \neq \emptyset$ using (4.8) where the summation is over all subsets I such that $|I| = 2k$, $\text{codim} M_I = 2k$ and $M_{I'} \neq M_I$ for any $I' \supset I$ with $I' \neq I$. It is obvious that then $\Delta[(Q, \alpha); M] = 0$ because the second factor in (4.8) will be always 0 thanks to R0).

Now let $M_0, ..., M_{2n}$ be hyperplanes in $\mathbb{CP}^{2n-1}$ such that $Q \bigcap M_J$ is a nondegenerate quadric for any subset J.

We have to prove that

$$\Delta_{2k,2(n-k)}(\sum_{j=0}^{2n} (-1)^j [(Q, \alpha); M^{(j)}]) = 0$$

Compute the left side of (4.9) using definition (4.8).The formula we get is a sum of contributions corresponding to codimension $2k$ edges of the configuration $(M_0, ...M_{2n})$. Consider one of the contributions related to a certain edge E. Let I be the set of all indices i such that $E \subset M_i$. If $|I| \geq 2k + 2$ then all simplices $M^{(j)}$ are degenerate and so R3) is provided by R0). Suppose that $I = 2k + 1$. Then in the part of the formula under consideration the second factor in all summands is the same, and the sum of the first factors is 0 thanks to relation R3). Similary if $I = 2k$ the first factor is the same, and the sum of the second ones is 0 according to R3)

b)We have to prove that $\Delta(a * b) = \Delta(a) * \Delta(b)$ This is an immediate consequence of definitions and the following

Lemma 4.6. *Let $V = V_1 \oplus V_2, \tilde{Q} = \tilde{Q}_1 \oplus \tilde{Q}_2, \tilde{M} = \tilde{M}_1 \oplus \tilde{M}_2$. Suppose that $dimV_i$ is odd for $i = 1, 2$. Then the element of $S(C)_\bullet$ corresponding to $([Q, \alpha]; M)$ is equal to zero modulo 2-torsion.*

Proof. The transformation $g = (-id_{V_1}, id_{V_2})$ has determinant -1 and so interchange connected components of the manifold of maximal isotropic subspaces of the quadric Q and the simplex M. Therefore

$$([Q, \alpha]; M) = ([gQ, g\alpha]; gM) = ([Q, \beta]; M) = -([Q, \alpha]; M)$$

Lemma 4.6 is proved.

The coassociativity of Δ also follows immediately from the definitions and lemma 4.4.

c) Clear from the definitions.

Remark 4.7 (compare with remark 4.2). If we omit from the definition of group $S_{2n}(\mathbb{C})$ orientation of the quadric Q and hence relation (R2a) we would missed lemma 4.5 and therefore would get $\Delta(a * b) \neq \Delta(a) * \Delta(b)$

The comultiplication Δ can be defined in terms of projective geometry. Namely Q_I and $M_{\{I\}}$ are the quadric and the simplex in M_I. To define the second factor in the wright-hand side of (4.8) consider the projective space formed by hyperplanes containing M_I. The tangency condition with Q defines a quadric in this projective space. Hyperplanes $M_i, i \in I$ can be considered as vertices of a simplex in it. Finally an orientation of the quadric Q provides orientations of both quadrics we get.

Accoding to proposition 4.3 we have the morphism of abelian groups

$$h : S(C)_\bullet \to \mathcal{H}_\bullet$$

Theorem 4.8. *h is a homomorphism of Hopf algebras.*

Proof. Let us prove that h commute with comultiplication. We need the following elementary facts about the cohomology of quadrics.

a) If $Q \in \mathbb{CP}^m$ is a nonsingular quadric then $H^i(\mathbb{CP}^m \backslash Q; \mathbb{Q}) = 0$ if m is even, $i > 0$ or m is odd, $i \neq 2n-1$ and $H^{2n-1}(\mathbb{CP}^{2n-1} \backslash Q; \mathbb{Q}) = \mathbb{Q}(-n)$

b) If Q_0 is a singular quadric in $\mathbb{C}^m$ then $H^i(\mathbb{C}^m, Q_0; \mathbb{Q}) = 0$ for all i

Let us compute

$$H^*(\mathbb{CP}^{2n-1} \backslash Q, M \backslash (M \cap Q); \mathbb{Q}) \tag{48}$$

Notice that (Q, M) is a normal crossing divisor. Recall that for a subset $I \in \{0, ..., 2n-1\}$ we set $Q_I := M_I \cap Q$. Consider the corresponding simplicial scheme

$$\mathbb{CP}^{2n-1}\backslash Q \leftleftarrows \bigcup_{|I|=1} M_I\backslash Q_I \leftleftarrows \cdots \leftleftarrows \bigcup_{|I|=2n-1} M_I\backslash Q_I$$

It produces the spectral sequence with

$$E_1^{p,q} = H^p(M_I\backslash Q_I; \mathbb{Q})$$

where $|I| = q$. This spectral sequence degenerates at E_1 (because of the weight considerations). The filtration it induces on (48) coincides with the weight filtration. Therefore

$$gr^W_{2(n-k)} H^{2n-1}(\mathbb{CP}^{2n-1}\backslash Q, M; \mathbb{Q}) = \bigoplus_{|I|=2k} H^{2(n-k)-1}(M_I\backslash Q_I; \mathbb{Q})$$

Moreover, for each subset I with $|I| = 2k$ there is an obvious injective morphism of mixed Hodge structures

$$i_I : h(Q_I, M_I) \to h(Q, M)$$

Recall that there are canonical up to a sign vectors $[\omega_{Q_I}] \in gr^W_{2(n-k)} h(Q_I, M_I)$. If we choose signes, their images form a basis in $gr^W_{2(n-k)} h(Q, M)$. Further, there are morphisms of mixed Hodge structures

$$H^{2n-1}(\mathbb{CP}^{2n-1}\backslash Q, M; \mathbb{Q}) \longrightarrow H^{2n-1}(\mathbb{CP}^{2n-1}\backslash Q, \cup_{i\in I} M_i; \mathbb{Q})$$

Proposition 4.9. *One has canonical isomorphism of mixed Hodge structures*

$$H^{2n-1}(\mathbb{CP}^{2n-1}\backslash Q, \cap_{i\in I} M_i; \mathbb{Q})(n-k) = h(M^I, Q^I)$$

Proof. By Poincare duality

$$H^{2n-1}(\mathbf{C}P^{2n-1}\backslash Q, \cup_{i\in I} M_i; \mathbb{Q}) = H^{2n-1}(\mathbb{CP}^{2n-1}\backslash \cup_{i\in I} M_i, Q; \mathbb{Q})^*(-(2n-1))$$

Notice that

$$H^{2n-1}(\mathbb{CP}^{2n-1}\backslash Q, \cup_{i\in I} M_i; \mathbb{Q}) = R^{2n-1} f_*(j_{M*} j_{Q!} \mathbb{Q})$$

where

$$\mathbb{CP}^{2n-1}\backslash(\cup_{i\in I}M_i \cup Q) \overset{j_Q}{\hookrightarrow} \mathbb{CP}^{2n-1}\backslash(\cap_{i\in I}M_i) \overset{j_M}{\hookrightarrow} \mathbb{CP}^{2n-1}\backslash M_I$$

and f is a composition

$$\mathbb{CP}^{2n-1}\backslash M_I \xrightarrow{f_1} P(V/\tilde{M}_I)\backslash M^I \xrightarrow{f_2} * \tag{49}$$

Let us compute $R^{2n-1}f_{1*}(j_{M*}j_{Q!}\mathbb{Q})$. A point x of the plane $P(V/\tilde{M}_I)$ corresponds to a plane H_x of dimension $\dim M_I + 1$ containing M_I. Set $h_x := H_x\backslash M_I$. The fiber

$$i_x^* R^i f_{1*}(j_{M*}j_{Q!}\mathbb{Q})$$

is isomorphic to $H^i(h_x, h_x \cap Q)$. Recall that x belongs to the quadric if and only if $h_x \cap Q$ is singular. So the fiber (49) at such points is zero according to b). The fiber (49) at points $x \in P(V/\tilde{M}_I)\backslash \mathbb{Q}^I \cap M^{\{I\}}$ is $Q(-k)$ for $i = 2k$ and zero for other i. Therefore

$$Rf_{1*}(j_{M*}j_{Q!}\mathbb{Q}) = j_{M^I*}j_{Q^I!}\mathbb{Q}(-k)[-2k]$$

and applying again Poincare duality we get proposition 4.9.

8. An analog of Schläfli formula. To write it I need some facts about geometry of even-dimensional smooth quadric Q. There are exactly 2 different families α and β of maximal isotropic subspaces on Q (they are of middle dimension). For each isotropic subspace γ of dimension one less there are just 2 maximal isotropic subspaces $\alpha_{(\gamma)}$ and $\beta_{(\gamma)}$ containing it. They belong to families α and β respectively. Let H be a generic codimension 2 hyperplane. Choose an isotropic subspace $\gamma \subset H \cap Q$. Then there are just two hyperplanes H_γ^α and H_γ^β containing H and tangent to quadric Q (here $\alpha_{(\gamma)} \subset H_\gamma^\alpha$).

Let us return to our data (Q, M).Set $M_{ij} = M_i \cap M_j$. Choose a maximal isotropic subspace $\gamma_{ij} \subset M_{ij} \cap Q$. Let $H_{\gamma_{ij}}^\alpha$ and $H_{\gamma_{ij}}^\beta$ be the corresponding hyperplanes tangent to Q and containing M_{ij}. The hyperplanes containing M_{ij} form a projective line. Let $r(M_i, M_j, H_{\gamma_{ij}}^\alpha, H_{\gamma_{ij}}^\beta)$ be the cross - ratio of 4 points on it corresponding to the hyperplanes

Recall that $v_\alpha(Q, M) := \int_{\Delta_M}\omega_Q$ is the period integral related to the pair (Q, M) where the sighn of the form ω_Q corresponds to the family α of isotropic planes on Q. Set $Q_{ij} := Q \cap M_{ij}$. The hyperplanes $M_k, k \neq i, j$, cut a simplex M^{ij} in M_{ij}.

Theorem 4.10

$$dv_\alpha(Q,M)=\sum_{i<j} v_{\gamma_{ij}}(Q_{ij},M^{ij})\cdot d\log r(M_i,M_j,H^\alpha_{\gamma_{ij}},H^\beta_{\gamma_{ij}})$$

Notice that if γ'_{ij} is a maximal isotropic subspace from a different family then both the cross-ratio and period v change sign.

Using this formula one can easyly prove that the integrals $v_\alpha(Q,M)$ can be expressed by the hyperlogarithms of order n. Using the fact that hyperlogarithms of order 3 can be expressed by the classical trilogarithm (see [G4]) we come to the expression of integrals $v_\alpha(Q,M)$ by means of classical trilogarithms and products of dilogarithm and logarithms. See [K1] for another method of proving this result.

9.The category of mixed Tate motives over C. According to [B1], [BD] it should be a tensor $\mathbb{Q}$-category with a fixed invertible object $\mathbb{Q}(1)_{\mathcal{M}}$ such that any simple object is isomorphic to $\mathbb{Q}(n)_{\mathcal{M}}:=\mathbb{Q}(1)^{\otimes m}_{\mathcal{M}}$ $\quad m\in\mathbb{Z}$ and these objects are mutually nonisomorphic.

$$Ext^1_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}},\mathbb{Q}(m)_{\mathcal{M}})=0 \quad \text{for} \quad m\le 0. \tag{50}$$

$$Ext^i_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}},\mathbb{Q}(n)_{\mathcal{M}})=gr^\gamma_n K_{2n-i}(\mathbb{C})\otimes\mathbb{Q} \tag{51}$$

Furhter, any object in this category carries a canonical increasing finite filtration (weight filtration), whose graded quotient of degree 2n is isomorphic to a direct sum of $\mathbb{Q}(-n)_{\mathcal{M}}$'s. Finally,

$$Ext^i_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}},\mathbb{Q}(n)_{\mathcal{M}})=gr^\gamma_n K_{2n-i}(\mathbb{C})\otimes\mathbb{Q} \tag{52}$$

In complete analogy with the case of Hodge-Tate structures one can define n-framed mixed Tate motives. Their equivalence classes form an abelian group denoted $\mathcal{A}_n$ and $\mathcal{A}_\bullet:=\oplus_{n\ge 0}\mathcal{A}_n$ is equipped with a structure of a commutative graded Hopf algebra. The category of mixed Tate motives is canonically equivalent to the category of finite-dimensional graded $\mathcal{A}_\bullet$-modules. This equivalence sends $\mathbb{Q}(n)_{\mathcal{M}}$ to the trivial one-dimensional $\mathcal{A}_\bullet$-comodule siting in degree n and one has

$$Ext^i_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}},\mathbb{Q}(n)_{\mathcal{M}})=H^i_{(n)}(\mathcal{A}_\bullet,\mathbb{Q}) \tag{53}$$

Here the group in the right side is the degree n part of the graded vector space $H^i(\mathcal{A}_\bullet,\mathbb{Q})$. Therefore according to axiom (52) one should have

$$H^i_{(n)}(\mathcal{A}_\bullet,\mathbb{Q})=gr^\gamma_n K_{2n-i}(\mathbb{C})\otimes\mathbb{Q} \tag{54}$$

Each generator of $S_{2n}(\mathbb{C})$ defines an n-framed Hodge-Tate structure realised in cohomology of a (simplicial) algebraic variety and therefore an n-framed mixed Tate motive. Similar arguments provides us with canonical homomorphism of graded Hopf algebras

$$S(\mathbb{C})_\bullet \longrightarrow \mathcal{A}_\bullet \tag{55}$$

Therefore should be canonical homomorphisms

$$H^i_{(n)}(S(\mathbb{C})_\bullet, \mathbb{Q}) \longrightarrow gr^\gamma_n K_{2n-i}(\mathbb{C}) \otimes \mathbb{Q} \tag{56}$$

One can compute the left side as the cohomology of degree n part of the cobar complex $S^\bullet(n)$ for the graded Hopf algebra $S(\mathbb{C})_\bullet$:

$$S_n(\mathbb{C}) \longrightarrow \oplus_{i_1+i_2=n} S_{i_1}(\mathbb{C}) \otimes S_{i_2}(\mathbb{C}) \longrightarrow \oplus_{i_1+i_2+i_3=n} S_{i_1}(\mathbb{C})_{i_1} \otimes S_{i_2}(\mathbb{C}) \otimes S_{i_3}(\mathbb{C}) \longrightarrow \ldots \tag{57}$$

Here the left group is placed at degree 1 and the coboudary is of degree $+1$. Therefore we arrive to the following

Conjecture 4.11 . a) *There is canonical homomorphism*

$$H^i(S^\bullet(n)) \otimes \mathbb{Q} \longrightarrow gr^\gamma_n K_{2n-i}(\mathbb{C}) \otimes \mathbb{Q} \tag{58}$$

b) *It is injective for* $i = 1$

Problem 4.12 *Wherether it is true that homomorphism (58) is an isomorphism?*

Notice that (56) is an isomorphism if and only if (55) is an isomorphism. I believe that (55) is at least injective; this provides the part b) of conjecture 4.11.

Theorem 4.13 *Conjecture 4.11 valid for* $n = 2$

Let me deduce conjectures 2.4 and 2.7 from conjecture 4.11. Let us represent $\mathbb{CP}^n$ as a complexification of $\mathbf{RP}^n$. Then the complex conjugation acts on $\mathbb{CP}^n$. Set

Theorem 4.14. a) *There are canonical homomorphisms*

$$\mathcal{P}(\mathcal{H}^{2n-1}) \xrightarrow{\psi_{\mathcal{H}}(n)} (S_{2n}(\mathbb{C}) \otimes \mathbb{Z}(n))^- \tag{59}$$

$$\mathcal{P}(S^{2n-1}) \xrightarrow{\psi_S(n)} (S_{2n}(\mathbb{C}) \otimes \mathbb{Z}(n))^+ \tag{60}$$

b) They transform Dehn invariant just to the comultiplication, providing the following commutative diagram in hyperbolic case

$$\begin{array}{ccc} \mathcal{P}(\mathcal{H}^{2n-1}) & \xrightarrow{D^H_{2n-1}} & \oplus\mathcal{P}(\mathcal{H}^{2k-1})\otimes\mathcal{P}(S^{2(n-k)-1}) \\ \downarrow \psi_{\mathcal{H}}(n) & & \downarrow \psi_{\mathcal{H}}(k)\otimes\psi_S(n-k) \\ S_{2n}(\mathbb{C}) & \xrightarrow{\Delta} & \oplus S_{2k}(\mathbb{C})\otimes S_{2(n-k)}(\mathbb{C}) \end{array}$$

and a similar diagram in the spherical case.

c) *The homomorphisms (59) , (60) provide the following homomorphisms from the hyperbolical and spherical Dehn complexes* (see (10)

$$\psi^\bullet_{\mathcal{H}}(n) : \mathcal{P}^\bullet_{\mathcal{H}}(n) \longrightarrow (S^\bullet(n)\otimes\mathbb{Z}(n))^- \tag{61}$$

$$\psi^\bullet_S(n) : \mathcal{P}^\bullet_S(n) \longrightarrow (S^\bullet(n)\otimes\mathbb{Z}(n))^+ \tag{62}$$

Proof. Follows from the definition of Δ and theorem 4.5a) The action of the complex conjugation on the image of scissor congruence groups computed easyly looking on the frames. Notice that homomorphisms $\psi_{\mathcal{H}}(n)$ and $\psi_S(n)$ are injective but certainly not isomorphisms unless for $\psi_{\mathcal{H}}(n)$

Combining this homomorphisms with conjectures 4.11a) and 4.11b) we get conjecture 2.7 and conjecture 2.4 respectively.

10. The structure of the complexes $S^\bullet(n)$ for $n \leq 3$.

Conjecture 4.15. *There exists canonical homomorphisms of complexes*

$$\begin{array}{ccc} S_2(\mathbb{C}) & \longrightarrow & \mathbb{C}^*\otimes\mathbb{C}^* \\ \downarrow s_2 & & \downarrow \\ B_2(\mathbb{C}) & \longrightarrow & \Lambda^2\mathbb{C}^* \end{array}$$

and

$$\begin{array}{ccccc} S_3(\mathbb{C}) & \longrightarrow & \mathbb{C}^*\otimes S_2(\mathbb{C})\oplus S_2(\mathbb{C})\otimes\mathbb{C}^* & \longrightarrow & \mathbb{C}^*\otimes\mathbb{C}^*\otimes\mathbb{C}^* \\ \downarrow s_3 & & \downarrow id\otimes s_2 + s_2\otimes id & & \downarrow \\ B_3(\mathbb{C}) & \longrightarrow & B_2(\mathbb{C})\otimes\mathbb{C}^* & \longrightarrow & \Lambda^3\mathbb{C}^* \end{array}$$

which induce isomorphisms on homology modulo torsion.

Here the group $B_2(\mathbb{C})$ was introduced in 1.4 and the group $B_3(\mathbb{C})$ is an explicit version of the group $\mathcal{B}_3(\mathbb{C})$ from s.1.5, see [G2]. The differentials in the B-complexes are defined by the formulas $\{x\}_2 \longmapsto (1-x) \wedge x$ and $\{x\}_3 \longmapsto \{x\}_2 \otimes x$, $\{x\}_2 \otimes y \longmapsto (1-x) \wedge x \wedge y$. A similar results, of cource, should be valid for any algebraicly closed field.

The most interesting problem here is a construction of an *explicit* homomorphisms $S_3(\mathbb{C}) \longrightarrow B_3(\mathbb{C})$ and $S_2(\mathbb{C}) \longrightarrow B_2(\mathbb{C})$ which commutes with the differentials. I believe that these homomorphisms should have a very beautyfull geometrical description. For example, the homomorphism $S_2(\mathbb{C}) \longrightarrow B_2(\mathbb{C})$ is easy to define for simplices whose vertices are on the quadric Q. Namely, $Q = \mathbb{C}P^1 \times \mathbb{C}P^1$, so such a simplex is defined by a 4-tuple $[(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)]$ of points of $P^1 \times P^1$. Its image in $B_2(\mathbb{C})$ should be $\{r(x_1, x_2, x_3, x_4)\}_2 - \{r(x_1, x_2, x_3, x_4)\}_2$.

The homomorphisms s_n $(n = 2, 3)$ will lead to explicit formulas for volumes of noneuclidean simplices via classical di and trilogarithms (compare with the work of R.Kellerhals [K1]). More precisely, the composition

$$\mathcal{P}(\mathcal{H}^{2n-1}) \longrightarrow S_n(\mathbb{C}) \xrightarrow{s_n} B_n(\mathbb{C}) \xrightarrow{\mathcal{L}_n} \mathbb{R}$$

should coinside with the volume homomorphism. The kernel of the maps s_2, s_3 should be equal to $S_1(\mathbb{C}) * S_1(\mathbb{C})$ and $S_1(\mathbb{C}) * S_2(\mathbb{C})$.

11. A construction of an element in $Ext^1_{\mathcal{M}}(\mathbb{Q}(0)_{\mathcal{M}}, \mathbb{Q}(n)_{\mathcal{M}})$ corresponding to a hyperbolic $(2n-1)$- manifold . Recall that a hyperbolic manifold M^{2n-1} produces an element $s(M^{2n-1}) \in \mathcal{P}(\mathcal{H}^{2n-1})$ that is obtained by a decomposition of the manifold on geodesic simplices (see s.2.1).

Proposition 2.3 provides $s(M^{2n-1}) \in KerD^{H}_{2n-1}$ and so thanks to theorem 4.14b) its image $\psi_n(s(M^{2n-1}))$ under homomorphism (61) belongs to KerΔ.

Therefore according to theorem 4.8 the comultiplication ν of the corresponding n-framed Hodge-Tate structure $h \circ \psi_{\mathcal{H}}(n)(s(M^{2n-1}))$ is also 0.

By the definition of the graded Hopf algebra $\mathcal{H}_\bullet$ one has $h \circ \psi_{\mathcal{H}}(n)(s(M^{2n-1})) \in \mathcal{H}_n$. Look at the cobar complex for the Hopf algebra $\mathcal{H}_\bullet$:

$$\mathcal{H}_\bullet \xrightarrow{\nu} \mathcal{H}_\bullet \otimes \mathcal{H}_\bullet \longrightarrow \ldots \tag{63}$$

The fact that the comultiplication ν of $h \circ \psi_{\mathcal{H}}(n)(s(M^{2n-1}))$ is zero just means that $h \circ \psi_{\mathcal{H}}(n)(s(M^{2n-1}))$ is a 1-cocycle in this complex.

Therefore according to theorem 4.1 the cohomology class of this 1-cocycle provides us an element of $Ext^1_{\mathcal{H}_T}(\mathbb{Q}(0)_{\mathcal{H}_T}, \mathbb{Q}(n)_{\mathcal{H}_T})$ that is by construction of algebraic-geometrical origin. So we have constructed a motivic extension promised in s. 1.5.

12. Problem 2.7 has positive answer for $n = 2$.

Theorem 4.16([D],[DS1],[DPS]) *The sequences*

$$0 \longrightarrow H_3(SL_2(\mathbb{C}))^- \longrightarrow \mathcal{P}(\mathcal{H}^3) \xrightarrow{D_3^H} \mathbb{R} \otimes S^1 \longrightarrow H_2(SL_2(\mathbb{C}))^- \longrightarrow 0 \quad (64)$$

$$0 \longrightarrow H_3(SU(2)) \longrightarrow \mathcal{P}(S^3) \xrightarrow{D_3^S} S^1 \otimes S^1 \longrightarrow H_2(SU(2)) \longrightarrow 0 \quad (65)$$

are exact.

According to [Su 2] and [Sah 3] $H_3SL_2(\mathbb{C}) = K_3^{ind}(\mathbb{C})$ modulo torsion. Further, $H_2SL_2(\mathbb{C}) = K_2(\mathbb{C})$, $gr_2^{\gamma}K_2(\mathbb{C}) = K_2(\mathbb{C})$ and $gr_2^{\gamma}K_3(\mathbb{C}) = K_3^{ind}(\mathbb{C})$. Therefore the complex

$$\mathcal{P}(\mathcal{H}^3) \xrightarrow{D_3^H} \mathbb{R} \otimes S^1 \quad (66)$$

computes $K_3^{ind}(\mathbb{C})^-$ modulo torsion and $K_2(\mathbb{C})^-$.

Moreover, the complex (66) is just the "-" part of the following Bloch complex [D]:

$$0 \longrightarrow B_2(\mathbb{C}) \xrightarrow{\delta} \Lambda^2\mathbb{C}^* \longrightarrow 0 \qquad \delta\{x\}_2 := (1-x) \wedge x \quad (67)$$

(By Matsumoto theorem Coker$\delta = K_2(\mathbb{C})$ and according to [Su 2] and [Sah 3] Ker$\delta = K_3^{ind}(\mathbb{C})$ modulo torsion.)

The construction of the map of complexes is provided by the isomorphisms $\mathcal{P}(\mathcal{H}^3) = \mathcal{P}(\partial\mathcal{H}^3)$ and $\partial\mathcal{H}^3 = \mathbb{CP}^1$; the coincidence of differentials is not obvious and follows from calculations.

Finally, in [DPS] was given a rather involved construction (using the Hopf map $S^3 \longrightarrow S^2$) of a homomorphism of the spherical complex

$$\mathcal{P}(S^3) \xrightarrow{D_3^S} S^1 \otimes S^1 \quad (68)$$

to the "+" part of (67). It is not an isomorphism but induces isomorphisms on cohomologies ([WS], [DPS]).

13. A symplectic approach to the Hopf algebra $S(\mathbb{C})_\bullet$. Let W_{2n} be a symplectic vector space decomposed to a direct sum of Lagrangian subspaces $W_{2n} = E \oplus F$. Suppose also that coordinate hyperplanes $E_1, ..., E_n$ in E

are given. The symplectic structure provides an isomorphism $F = E^*$ and therefore there are dual hyperplanes $F_1, ..., F_n$ in F.

The subgroup of $Sp(W_{2n})$ preserving this data is an n-dimensional torus T_n (a maximal Cartan subgroup).

Let L_n^0 be the manifold of all Lagrangian subspaces $H \in W_{2n}$ in generic position with respect to coordinate hyperplanes $E_i \oplus F, E \oplus F_j$. Each of them can be considered as a graph of a map $h : E \to F$ and hence defines a bilinear form $h \in E^* \otimes F^*$. The condition that H is isotropic just means that h is symmetric bilinear form.

Lemma 4.17. *Points of L_n^0/T_n are in 1-1 correspondence with configurations (i.e. projective equivalence classes) of pairs $(Q; M)$ where Q is a nondegenerate quadric and M is a simplex in generic position with respect to Q in $P(E)$.*

Proof. The coordinate hyperplanes $E_1, ..., E_n$ define a simplex in $P(E)$. The torus T_n is the subgroup of all transformations in $GL(E)$ preserving this simplex. Lemma 4.17 follows from these remarks.

5 Proof of the theorem 2.5

1. Some results on the t-structure on a triangulated category. Recall that a t-structure on a triangulated category $\mathcal{D}$ is a pair of subcategories $\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}$ satisfying the following conditions:

1)$Hom_{\mathcal{D}}(X, Y) = 0$ for all objects $X \in Ob\mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$.

2)For any $X \in Ob\mathcal{D}$ there exists an exact triangle

$$X_{\leq 0} \longrightarrow X \longrightarrow X_{\geq 1} \longrightarrow X_{\leq 0}[1]$$

with $X_{\leq 0} \in \mathcal{D}^{\leq 0}$ and $X_{\geq 1} \in \mathcal{D}^{\geq 1}$.

3)$\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\leq 0}$

Here $\mathcal{D}^{\leq a} := \mathcal{D}^{\leq 0}[-a]$, $\mathcal{D}^{\geq a} := \mathcal{D}^{\geq 0}[-a]$, $\mathcal{D}^{[a,b]} := \mathcal{D}^{\leq b} \cap \mathcal{D}^{\geq a}$ and $\mathcal{D}^a := \mathcal{D}^{[a,a]}$.

The exact triangle in (2) is defined uniquely up to an isomorphism and depends functorially on X. The subcategory $\mathcal{D}^0$ is called the heart of a t-structure. It is an abelian category ([BBD]).

Let $\mathcal{A}$ and $\mathcal{B}$ be two sets of isomorphism classes of objects in $\mathcal{D}$. Denote by $\mathcal{A} * \mathcal{B}$ the set of all objects X in $Ob\mathcal{D}$ which can be included into an exact triangle $A \longrightarrow X \longrightarrow B \longrightarrow A[1]$ with $A \in \mathcal{A}, B \in \mathcal{B}$.

Lemma 5.1 $(\mathcal{A} * \mathcal{B}) * \mathcal{C} = \mathcal{A} * (\mathcal{B} * \mathcal{C})$

Proof. Follows from the octahedron lemma, see [BBD].

Theorem 5.2 *a)Let $\mathcal{D}$ be a triangulated $\mathbb{Q}$-category and $\mathcal{Q}$ be a full semisimple subcategory generating $\mathcal{D}$ as a triangulated category. Suppose that for any two objects $Q_1, Q_2 \in Ob\mathcal{Q}$ one has*

$$Hom_{\mathcal{D}}^{-i}(Q_1, Q_2) = 0 \qquad i > 0 \tag{69}$$

Then there is canonical t-structure on $\mathcal{D}$ with the abelian heart $\mathcal{M} = \cup \mathcal{Q} * \mathcal{Q} * \ldots * \mathcal{Q}$.

b) If in addition $Hom_{\mathcal{D}}^{i}(Q_1, Q_2) = 0$ for $i \geq 2$, then the tensor category $\mathcal{D}$ is equivalent to the derived category of $\mathcal{M}$.

Theorem 5.3 *Let $\mathcal{D}$ be a triangulated tensor $\mathbb{Q}$-category and $\mathcal{Q}$ be a full semisimple subcategory generated by non isomorphic objects $\mathbb{Q}(m)$, $m \in \mathbb{Z}$ such that $\mathbb{Q}(1)$ is invertible, $\mathbb{Q}(m) = \mathbb{Q}(1)^{\otimes m}$, and*

$$Hom_{\mathcal{D}}^{i}(\mathbb{Q}(m), \mathbb{Q}(n)) = 0 \quad \text{if} \quad m > n, \qquad Hom_{\mathcal{D}}(\mathbb{Q}(0), \mathbb{Q}(0)) = \mathbb{Q}$$

Then the abelian heart $\mathcal{M}$ from the theorem (5.2) is a tensor category. It is equivalent to the tensor category of finite dimensional representations of a certain free negatively graded (pro)-Lie algebra.

Let $\mathcal{D}$ be a triangulated category and $\mathcal{M} \subset \mathcal{D}$ be a full subcategory.

Theorem 5.4 *$\mathcal{M}$ is a heart for the unique bounded t-structure on $\mathcal{D}$ if and only if*

i) $\mathcal{M}$ generates $\mathcal{D}$ as a triangulated category

ii) $\mathcal{M}$ is closed with respect to extensions

iii) $Hom_{\mathcal{D}}(X, Y[i]) = 0$ for any $X, Y \in \mathcal{M}$, $i < 0$

iv) $\mathcal{M} * \mathcal{M}[1] \subset \mathcal{M}[1] * \mathcal{M}$

For the proof of this theorem see [BBD] or [P]. Let me scetch the construction of the t-structure on $\mathcal{D}$. We will use the following

Lemma 5.5 *Let*

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{f} X[1]$$

be an exact triangle. Then $f = 0$ if and only if $Y = X \oplus Z$

Proof. Let us show that $f=0$ implies $Y=X\oplus Z$. The composition of the identity morphism $Z\xrightarrow{id}Z$ with f is zero, so one has a morphism $g:Z\longrightarrow Y$ making the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "a"] & Y \arrow[r] & Z \arrow[r, "f"] & X[1] \\ & & Z \arrow[ul, "g"] \arrow[u, "id"] \arrow[ur, "=0"'] & \end{tikzcd}$$

So there is a morphism $X\oplus Z\longrightarrow X$. The universality property of this morphism follows immediately from $f=0$.

Let $\mathcal{D}^{[a,b]}$ be the minimal full subcategory of $\mathcal{D}$ containing $\mathcal{M}[-i]$ for $a\leq i\leq b$ and closed under extensions. The subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$ are defined similarly and satisfy 1) thanks to iii). Notice that $Hom_{\mathcal{D}}(\mathcal{M},\mathcal{M}[-i])=0$ implies $\mathcal{M}*\mathcal{M}[i+1]=\mathcal{M}\oplus\mathcal{M}[i+1]$ for $i>0$ by lemma 5.5. So $\mathcal{M}*\mathcal{M}[n]\subset\mathcal{M}[n]*\mathcal{M}$ for any $n>0$. Using ii) and lemma 5.1 we get $\mathcal{D}^{[a,b]}=\mathcal{M}[-a]*\mathcal{M}[-a-1]*...*\mathcal{M}[-b]$. This proves 2). It remains to show $\mathcal{D}^{\leq 0}\cap\mathcal{D}^{\geq 1}\subset\mathcal{D}^{[a,b]}$. If $X=Y_c*Y_{c+1}*...*Y_d$ where $Y_i\in\mathcal{M}[-i]\in\mathcal{D}^{\geq a}$ then $Hom(Y_c,X)=0$ for $c<a$ and so we get from the exact triangle

$$Y_c\longrightarrow X\longrightarrow Y_{c+1}*...*Y_d\longrightarrow Y_c[1]$$

that $Y_{c+1}*...*Y_d=X\oplus Y_c[1]$. Now $Hom_{\mathcal{D}}(Y_c[1],Y_{c+1}*...*Y_d)=0$ by iii) and so $Y_c=0$.

Proof of theorem5.2. a). Let us use theorem 5.4. The properties i)-ii) are obvious. iii) is easy to prove using (69), so we have to check only iv). Using the associativity of $*$ operation we see that one has to prove only that $Q_1*Q_2[1]\subset Q_2[1]*Q_1$ for any two simple objects in $\mathcal{Q}$. One has

$$Q_1\longrightarrow X\longrightarrow Q_2[1]\xrightarrow{f}Q_1[1]$$

There are only two possibilities for f:

1. f is an isomorphism; then $X=0$.
2. $f=0$; then $X=Q_1\oplus Q_2[1]$ by lemma 5.5.

Part b).

Proposition 5.6 *a) Let $\mathcal{D}$ be a triangulated category and $\mathcal{M}$ be the heart of a t-structure on $\mathcal{D}$. Suppose that one has*

$$Hom^i_{\mathcal{D}}(X,Y)=0 \quad \textit{for any} \quad X,Y\in Ob\mathcal{M}, \quad i>1 \tag{70}$$

Then the category $D^b(\mathcal{M})$ is equivalent to the category $\mathcal{D}$.

b) If we suppose in addition that the triangulated category $\mathcal{D}$ is a subcategory of a derived category $D^b(\mathcal{N})$ for some abelian category $\mathcal{N}$, then $D^b(\mathcal{M})$ is equivalent to $\mathcal{D}$ as a triangulated category.

Proof. Let $\tilde{\mathcal{D}}$ be the full subcategory of $\mathcal{D}$ whose objects are direct sums $\oplus A_i[-i]$ where $A_i \in \mathcal{D}^{-i}$, i.e. $H^j_{\mathcal{M}}(A_i) = 0$ for $j \neq -i$. There is a canonical functor $i : \tilde{\mathcal{D}} \hookrightarrow \mathcal{D}$. Then (70) is a necessary and sufficient condition for i to be an equivalence of triangulated categories. Indeed, let us show that every object Y in $\mathcal{D}$ is isomorphic to $i(X)$ for some X. We may suppose $Y \in \mathcal{D}^{[0,n]}$ and will use induction by n. Consider the exact triangle

$$H^0_{\mathcal{M}}(Y) \xrightarrow{f} Y \longrightarrow C \longrightarrow H^0_{\mathcal{M}}(Y)[1]$$

provided by the canonical morphism $f : H^0_{\mathcal{M}}(Y) \longrightarrow Y$. Then $C \in \mathcal{D}^{[1,n]}$ and $H^0_{\mathcal{M}}(Y)[1] \in \mathcal{D}^{-1}$, so it is easy to see that the morphism $C \longrightarrow H^0_{\mathcal{M}}(Y)[1]$ must equal to 0 because $Hom^{\geq 2}$ are zero. Therefore $Y = H^0_{\mathcal{M}}(Y) \oplus C$ by the lemma above.

Let us show that $D^b(\mathcal{M})$ has cohomological dimension one, i.e.

$$Hom^i_{D^b(\mathcal{M})}(X, Y) = 0 \quad \text{for any} \quad X, Y \in Ob\mathcal{M}, \quad i > 1 \tag{71}$$

Notice that for any $X, Y \in \mathcal{M}$ one has

$$Hom^i_{D^b(\mathcal{M})}(X, Y) = Ext^i_Y(X, Y)$$

where $Ext^i_Y(X, Y)$ is the Yoneda Ext-groups in $\mathcal{M}$. Also $Hom^1_{\mathcal{D}}(X, Y) = Ext^1_Y(X, Y)$.

One has $Ext^2_Y(X, Y) \subset Ext^2_{\mathcal{D}}(X, Y)$. Therefore $Ext^2_{\mathcal{D}}(X, Y) = 0$ implies $Ext^2_Y(X, Y) = 0$. Any element in Ext^n_Y can be represented as a product of certain elements from Ext^1_Y and Ext^{n-1}_Y. So if $Ext^2_Y(X, Y) = 0$ for any 2 objects X, Y in $\mathcal{M}$, then $Ext^i_Y(X, Y) = 0$ for all $i \geq 2$. So we have an equivalence of the categories. Part b) is proved using [BBD]. The proposition is proved.

The proof of the theorem (5.3) is rather standard. One shows that $\mathcal{M}$ is a mixed Tate category (see [BD] or [G1] for the definitions) thanks to the conditions imposed on Hom's between $\mathbb{Q}(i)$'s; then the Tannakian formalism leads to the theorem (5.3) (see again [BD] or [G1]).

2. An application: the abelian category of mixed Tate motives over a number field. I will work with the category $\mathcal{DM}_F$ of triangulated mixed motives over a field F from [V]. An object in $\mathcal{DM}_F$ is a "complex" of regular (but not necessarily projective) varieties $X_1 \longrightarrow X_2 \longrightarrow ... \longrightarrow X_n$ where the morphisms are given by finite correspondences and the composition of any two successive morphisms is zero.

A pair $(\mathbb{P}^n \backslash Q, L)$ where Q is a nondegenerate quadric and L is a simplex provides an object $m(Q, L)$ in $\mathcal{DM}_F$. Namely, for an subset set $I = \{j_1 < ... < j_{n-i}\}$ of $\{0, 1, ..., n\}$ let $L(I) := L_{j_1} \cap L_{j_2} \cap ... \cap L_{j_{n-i}}$ be the corresponding i-dimensional face of L and $L^Q(I) := L(I) \backslash (Q \cap L(I))$. Let $L^Q(i) := \cup_{|I|=n-i} L^Q(I)$. Then $m(Q, L) := Hom(\tilde{m}(Q, L), \mathbb{Q})$ Here Hom is the inner Hom in the category $\mathcal{DM}_F$ and

$$\tilde{m}(Q, L) := L^Q(0) \longrightarrow L^Q(1) \longrightarrow ... \longrightarrow L^Q(n-1) \longrightarrow \mathbb{P}^n \backslash Q$$

where the first group is sitting in degree 0 and the differentials decrease the degree and given by the usual rules in the simplicial resolution.

There are the objects $\mathbb{Q}(n) \in \mathcal{DM}_F$ satisfying almost all the needed properties including the relation with K-theory:

$$Ext^i_{\mathcal{DM}_F}(\mathbb{Q}(0), \mathbb{Q}(n)) = gr^\gamma_n K_{2n-i}(F) \otimes \mathbb{Q} \tag{72}$$

The formula above follows from the results of Bloch, Suslin and Voevodsky; the key step is the relation of Higher Chow groups and algebraic K-theory proved by Bloch (see [Bl3] and the moving lemma in [Bl4]). For the relation between the Higher Chow groups and motivic cohomology (i.e. the left hand side in (72)) see [V], Proposition 4.2.9 (Higher Chow groups = Borel-Moore motivic homology) and [V], Proposition 4.3.7 (duality for smooth varieties).

The only serious problem is the Beilinson-Soulé vanishing conjecture which should guarantee that the negative Ext's are zero. However if F is a number field the vanishing conjecture follows from the results of Borel and Beilinson ([B2], [Bo2]). So $Hom^{-i}_{\mathcal{DM}}(\mathbb{Q}, \mathbb{Q}(n)) = 0$ for all $i, n > 0$. The category $\mathcal{DM}_F$ is a subcategory of the derived category of "sheaves with transfers", see [V], so we may apply part b) of the proposition 5.6. Therefore we can apply the "category machine" from s. 5.1 and get the abelian category $\mathcal{M}_T(F)$ of mixed Tate motives over a number field F with *all* the needed properties. In particulary we have the Hopf algebra $\mathcal{A}_\bullet(F)$ provided by the Tannakian formalism.

For any embedding $\sigma : F \hookrightarrow \mathbb{C}$ there is the realisation functor H_σ from the abelian category of mixed motives over a number field F to the abelian category of mixed Hodge Tate structures:

$$H_\sigma : \mathcal{M}_T(F) \longrightarrow \mathcal{H}_T$$

It follows from the Borel theorem (injectivity of the regulator map on $K_{2n-1}(F)\otimes \mathbb{Q}$ for number fields) that $\oplus_\sigma H_\sigma$ induces an injective map on

$$\oplus_\sigma H_\sigma : Ext^1_{\mathcal{M}_T(F)}(\mathbb{Q}(0),\mathbb{Q}(n)) \hookrightarrow \oplus_\sigma Ext^1_{\mathcal{H}_T(F)}(\mathbb{Q}(0),\mathbb{Q}(n)) \qquad (73)$$

3. Proof of the theorem (2.5): the final step. The Hodge realisation provides a diagram

$$\begin{array}{ccc} S(F)_\bullet & \stackrel{\Delta}{\longrightarrow} & S(F)_\bullet \otimes S(F)_\bullet \\ \downarrow & & \downarrow \\ \oplus_\sigma \mathcal{H}_\bullet & \stackrel{\nu}{\longrightarrow} & \oplus_\sigma \mathcal{H}_\bullet \otimes \mathcal{H}_\bullet \end{array}$$

Here the vertical arrows are embeddings because (73) is injective. This diagram is commutative thanks to the main results of the chapter 4 (especially theorem 4.8).

Further, one has the diagram

$$\begin{array}{ccc} S_\bullet(F) & \stackrel{\Delta}{\longrightarrow} & S_\bullet(F) \otimes S_\bullet(F) \\ \downarrow & & \downarrow \\ \mathcal{A}_\bullet(F) & \stackrel{\Delta}{\longrightarrow} & \mathcal{A}_\bullet(F) \otimes \mathcal{A}_\bullet(F) \\ \downarrow & & \downarrow \\ \oplus_\sigma \mathcal{H}_\bullet & \stackrel{\nu}{\longrightarrow} & \oplus_\sigma \mathcal{H}_\bullet \otimes \mathcal{H}_\bullet \end{array}$$

where the composition of vertical arrows coincides with the corresponding vertical arrow in the previous diagram. (Abusing notations we denoted the comultiplication in the two different Hopf algebras by the same letter Δ). This, together with injectivity of vertical arrows in the bottom square of the second diagram implies the commutativity of the upper square of that diagram. Theorem (2.5) follows from the commutativity of the upper square in the second diagram. Indeed, the kernel of the middle horisontal arrow coincides with $K_{2n-1}(F)\otimes\mathbb{Q}$ and the Beilinson regulator comes from the bottom square of that diagram.

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REFERENCES

[Ao] Aomoto K.: *Analitic structure of Schlafli function*, Nagoya Math. J. 68 (1977), 1-16

[B1] Beilinson A.A.: *Height pairings between algebraic cycles*, Lecture Notes in Math. N. 1289, (1987), p. 1–26.

[B2] Beilinson A.A.:*Higher regulators and special values of L-functions* VINITI, 24 (1984), 181-238. English translation: J.Soviet Math., 30 (1985), 2036-2070.

[BBD] Beilinson A.A., Bernstein J., Deligne P.: *Faisceaux pervers* Asterisque 100, 1982.

[BD] Beilinson A.A., Deligne P.: *Motivic polylogarithms and Zagier's conjecture.* Preprint to appear

[BMS] Beilinson A.A., Macpherson R.D. Schechtman V.V: *Notes on motivic cohomology.* Duke Math. J., 1987 vol 55 p. 679-710.

[BGSV] Beilinson A.A., Goncharov A.B., Schechtman V.V, Varchenko A.N.: *Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of a pair of triangles in the plane* . The Grothendieck Feschtrift, Birkhoser, vol , 1990, p 131-172.

[Bl1] Bloch S.: *Higher regulators, algebraic K- theory and zeta functions of elliptic curves*, Lect. Notes U.C. Irvine, 1977.

[Bl2] Bloch S.: *Application of the dilogarithm function in algebraic K-theory and algebraic geometry.* Proc. Int. Symp. Alg. geom., Kyoto (1977), 1-14.

[Bl3] Bloch S.: *Algebraic cycles and higher K-theory* Advances in Math. 61, n 3, (1985), 65-79.

[Bl4] Bloch S.: *Moving lemma for Higher Chow groups* Preprint (1994).

[Bo] Böhm J.: *Inhaltsmessung in* $\mathbb{R}^5$ *constanter Krümmung* , Arch.Math. 11 (1960), 298-309.

[Bo1] Borel A.: *Cohomologie des espaces fibrés principaux*, Ann. Math. 57 (1953), 115–207.

[Bo2] Borel A.: *Cohomologie de* SL_n *et valeurs de fonctions zêta aux points entiers.* Annali Scuola Normale Superiore Pisa 4 (1977), 613–636.

[Ch] Chern S.-S.: *A simple intrinsic proof of the Gauss-Bonnet formula for closed Riemannian manifolds*, Ann. of Math., vol 45, 1944, N 4.

[Co] Coxeter H.S.M. *The function of Schlafli and Lobachevsky*, Quart.J.Math. (Oxford) 6 (1935), 13-29

[Du1] Dupont J.: *Algebra of polytops and homology of flag complexes* Osaka J. Math. 19 (1982), 599-641

[Du2] Dupont J.: *The dilogarithm as a characteristic class for a flat bundles*, J. Pure and Appl. Algebra, 44 (1987), 137–164.

[DPS] Dupont J., Parry W., Sah H.: *Homology of classical groups made discrete.II.*H_2, H_3*, and relations with scissors congruences* J. Pure and Appl. Algebra, 113, 1988, p.215-260.

[DS1] Dupont J., Sah H.: *Scissors congruences II*, J. Pure and Appl. Algebra, v. 25, 1982, p. 159–195.

[DS2] Dupont J., Sah H.: *Homology of Euclidean groups of motion made discrete and Euclidean scissor congruences*, Acta Math. 164 (1990) 1-24.

[EP] Epstein D.B.A., Penner .R. *Euclidean decomposition of non-compact hyperbolic manifolds* J.Diff. Geom. 27 (1988), 67-80.

[FV] Feigin B, Tsygan B.: *Additive K-theory* Lect. Notes in Math. 1289.

[FV] Friedlander E., Voevodsky V.: *Bivariant cycle homology* Preprint 1994.

[G1] Goncharov A.B.: *Polylogarithms and motivic Galois groups* Proseedings of AMS Reasearch Summer Conference "Motives", Symposium in Pure Mathematics, vol 55, part 2, 43-97.

[G2] Goncharov A.B.: *Geometry of configurations, polylogarithms and motivic cohomology* Advances in Mathematics, vol. 144, N2, 1995. 197-318.

[G3] Goncharov A.B.: *Chow polylogarithm and regulators* Mathematical Research Letters, 1995, vol 2, N 1 95-112.

[G4] Goncharov A.B.: *Hyperlogarithms, multiple zeta numbers and mixed tate motives.* Preprint MSRI 1993.

[Goo] Goodwillie Annals of Math. 24 (1986) 344-399.

[Gu] Guichardet A.: *Cohomologie des groupes topologiques et des algèbres de Lie*, Nathan (1980).

[K1] Kellerhals R.: *On the volume of hyperbolic 5-orthoschemes and the trilogarithm* , Preprint MPI/91-68, (Bonn).

[K2] Kellerhals R.: *The Dilogarithm and the volume of Hyperbolic Polytops* , Chapter 14 in [Le].

[Le] *Structural Properties of Polylogarithms*, edited by L.Lewin, AMS series of Mathematical surveys and monographs, vol 37, 1991.

[Lei] Gerhardt C.I. (ed).*G.W.Leibniz Mathematisch Schriften* III/1, 336-339 Georg Olms Verlag, Hildesheim. New York 1971.

[Me] Meyerhoff R.: *Density of the Chern-Simons invariants for hyperbolic 3-manifolds*, London Mathematical Sosiety Lecture Notes, vol 112, 1986, 210–226.

[M1] Milnor J.: *Computation of volume*, Chapter 7 in [Th].

[M2] Milnor J.: *Algebraic K-theory and quadratic forms*, Invent. Math. 9 (1970), 318–340.

[MM] Milnor J., Moore J.: *On the structure of Hopf algebras*, Ann. of Math. (2) 81 (1965) 211-264.

[Mu] Müller P.: *Über Simplexinhalte in nichteuklidischen Räumen*, Dissertation Universität Bonn, 1954

[N] Neumann W.: *Combinatorics of triangulations and the Chern-Simons invariant for hyperbolic manifolds* in Topology 90, Proc. of Res. Semesterin Low Dimensional topology at Ohio State (Walter de Gruyter Verlag, Berlin - New York 1992, 243-272

[N-Z] Neumann W., Zagier D.: *Volumes of hyperbolic three- manifolds*, Topology 24 (1985), 307-331.

[N] Neumann W. Yang J.: *Invariants from triangulations of hyperbolic 3-manifolds* Electronic Res. Announcements, (1995) vol. 1 Issue 2, 72-79

[P] Positselsky L.: *Some remarks on triangulated categories.* Preprint 1995.

[Ra] Raghunatan M.S. :*Discrete subgroups of Lie groups*, Springer Verlag, 1972.

[R] Ramakrishnan D.: *Regulators, algebraic cycles and values of L-functions*, Contemporary Math., vol 83, 183–310.

[Sah1] Sah H.: *Hilbert's third problem: scissor congruences* Research notes in mathematics 33), Pitman Publishing Ltd., San-Francisco-London-Melbourne, 1979

[Sah2] Sah H.: *Scissor congruences, I: The Gauss-Bonnet map*, Math. Scand. 49 (1981), 181-210.

[Sah3] Sah H.: *Homology of classical groups made discrete. III* J.pure appl. algebra 56 (1989) 269-312.

[SW] Sah H., Wagoner J.B: *Second homology of Lie groups made discrete.* Comm. Algebra 5(1977), 611-642.

[Sch] Schläfli: *On the multiple integral* $\int^n dxdy...dz$, *whose limits are* $p_1 = a_1x + b_1y + ... + h_1z > 0, p_2 > 0, ..., p_n > 0, and\ x^2 + y^2 + ... + z^2 < 1$, in Gesammelte Mathematische Abhandlungen II, pp. 219-270, Verlag Birkhauser, Basel, 1953.

[S1] Suslin A.A.: *Algebraic K-theory of fields.* Proceedings of the International Congress of Mathematicians, Berkeley, California, USA, 1986, 222–243.

[S2] Suslin A.A.: *K_3 of a field and Bloch's group*, Proceedings of the Steklov Institute of Mathematics 1991, Issue 4.

[Th] Thurston W.:*Geometry and topology of 3-manifolds*, Princeton Univercity, 1978.

[V] Voevodsky V.: *Triangulated category of motives over a field* Preprint 1994.

[Y] Yoshida : *η-invariants of hyperbolic 3-manifolds*, Inventiones Math., 1986.

[Z1] Zagier D.: *Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields.* Arithmetic Algebraic Geometry (G.v.d.Geer, F.Oort, J.Steenbrink, eds.), Prog. Math., Vol 89, Birkhauser, Boston, 1991, pp. 391–430. Proceedings of the Texel conference on Arithmetical Algebraic Geometry, 1990.

[Z2] Zagier D.: *Hyperbolic manifolds and special values of Dedekind zeta functions*, Inventiones Math. 83 (1986), 285–301.

[Z3] Zagier D.: *The remarkable Dilogarithm* J.Math. and Phys. Soc. 22 (1988), 131-145.

[W] Wang H.C.: *Topics in totally discontinuous groups*, in Symmetric spaces, Boothby-Weiss, Editors, New-York, 1972.