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SIMPLE MAPS, HURWITZ NUMBERS, AND TOPOLOGICAL RECURSION

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Abstract. We introduce the notion of fully simple maps, which are maps in which the boundaries do not touch each other, neither themselves. In contrast, maps where such a restriction is not imposed are called ordinary. We study in detail the combinatorics of fully simple maps with topology of a disk or a cylinder. We show that the generating series of simple disks is given by the functional inversion of the generating series of ordinary disks. We also obtain an elegant formula for cylinders. These relations reproduce the relation between cumulants and (higher order) free cumulants established by Collins et al [21], and implement the symplectic transformation $x \leftrightarrow y$ on the spectral curve in the context of topological recursion. We then prove that the generating series of fully simple maps are computed by the topological recursion after exchange of x and y, thus proposing a combinatorial interpretation of the property of symplectic invariance of the topological recursion.

Our proof relies on a matrix model interpretation of fully simple maps, via the formal hermitian matrix model with external field. We also deduce a universal relation between generating series of fully simple maps and of ordinary maps, which involves double monotone Hurwitz numbers. In particular, (ordinary) maps without internal faces – which are generated by the Gaussian Unitary Ensemble – and with boundary perimeters $(\lambda_1, \ldots, \lambda_n)$ are strictly monotone double Hurwitz numbers with ramifications λ above ∞ and $(2, \ldots, 2)$ above 0. Combining with a recent result of Dubrovin et al. [23], this implies an ELSV-like formula for these Hurwitz numbers.

Contents

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1. Introduction

Maps are surfaces obtained from glueing polygons, and their enumeration by combinatorial methods has been intensively studied since the pioneering works of Tutte [62]. In physics, summing over maps is a well-defined discrete replacement for the non-obviously defined path integral over all possible metrics on a given surface which underpin two-dimensional quantum gravity. The observation by t'Hooft [61] that maps are Feynman diagrams for the large rank expansion of gauge theories led Brézin-Itzykson-Parisi-Zuber [15] to the discovery that hermitian matrix integrals are generating series of maps. The rich mathematical structure of matrix models – integrability, representation theory of $U(\infty)$, Schwinger-Dyson equations, etc. – led to further insights into the enumeration of maps, *e.g.* [3, 22]. It also inspired further developments, putting the problem of counting maps into the more general context of enumerative geometry of surfaces, together with geometry on the moduli space of curves [42, 67, 47], volumes of the moduli space [54], Gromov-Witten theory [67, 14], Hurwitz theory [30, 1], etc. and unveiling a common structure of "topological recursion" [33, 28].

We say that a face f of a map is *simple* when every vertex of f is incident to atmost two edges of f. In the definition of maps, polygons may be glued along edges without restrictions, in particular faces may not be simple. This leads to singular situations, somehow at odds with the intuition of what a neat discretization of a surface should look like. This definition is the one naturally prescribed by the Feynman diagram expansion of hermitian matrix models. It is also one for which powerful combinatorial (generalized Tutte's recursion, Schaeffer bijection, etc.) and algebraic/geometric (matrix models, integrability, topological recursion, etc.) methods can be applied. Within such methods, it is possible to count maps with restrictions of a global nature (topology, number of vertices, number of polygons of degree k) and on the number of marked faces (also called boundaries) and their perimeters. Combined with probabilistic techniques, they helped in the development of a large corpus of knowledge about the geometric properties of random maps. Tutte introduced in [62] the notion of non-separable map, in which faces must all be simple. Many of the combinatorial methods aforementioned have been extended to handle non-separable maps – see *e.g.* [66, 44, 16] – but not the algebraic/geometric methods. The probabilistic aspects have not been as much studied as for (possibly separable) maps, but see for instance [5].

In the present work, we consider an intermediate problem, *i.e.* the enumeration of maps where only the boundaries are imposed to be simple. We find some remarkable combinatorial and algebraic properties in this problem. When there are several boundaries, we are led to distinguish the case of maps in which each boundary is simple from the more restrictive case of maps in which every vertex of a boundary b belongs to at most two edges of the boundaries (whether b or another one). The latter are called *fully simple maps*. We call simple the maps in which each boundary is simple, and *ordinary* the maps in which no restriction is imposed on vertices on the boundary.

Some of our arguments will apply to more general models of maps, where faces are not necessarily homeomorphic to disks. For instance, maps carrying an $O(n)$ loop model [37], an Ising model [46] or a Potts model [8], can be described as maps having faces which are either disks or cylinders [12]. We refer to [26] for more examples. The vocabulary we adopt is summarized in the following tables. Here, $(b_i)_{i=1}^n$ are the boundary faces.

boundary type	*description*
ordinary	no restriction
simple	b_i does not touch itself
fully simple	b_i does not touch $\bigcup_j b_j$

FIGURE 1

Let us mention that the random geometry of disks with simple boundaries has been studied recently in the case of quadrangulations [41], and the tree structure of the ordinary boundary of a disk has also been investigated recently [59]. It seems that a systematic enumeration of (usual or not, planar or not) maps with fully simple boundaries had not performed so far, and one of the aims of this article is a contribution to fill this gap.

type of maps	*topology of inner faces*	*matrix model* (1.1)-(6.2)
usual	disks	t_d
with loops [12]	disks and cylinders	$t_{0;d_1}$ and $t_{0;d_1,d_2}$
stuffed [7]	arbitrary	all $t_{h;d_1,\ldots,d_k}$

Figure 2

1.1. **Disks and cylinders via combinatorics.** For planar maps with one boundary (disks) or two boundaries (cylinders), we give in Section 3-4 a bijective algorithm which reconstructs ordinary maps from fully simple maps. This algorithm is not sensitive to the assumption – included in the definition of usual maps – that faces must be homeomorphic to disks. Therefore, it applies to all types of maps described in Figure 2.

We deduce two remarkable formulas for the corresponding generating series. Let F_ℓ (resp. H_ℓ) be the generating series of ordinary (resp. fully simple) disks with perimeter ℓ, and

$$W(x) = \frac{1}{x} + \sum_{\ell \geq 1} \frac{F_\ell}{x^{\ell+1}}, \qquad X(w) = \frac{1}{w} + \sum_{\ell \geq 1} H_\ell w^{\ell-1}\,.$$

Proposition 1.1. *For all types of maps in Figure 2, $X(W(x)) = x$.*

Let F_{ℓ_1,ℓ_2} (resp. H_{ℓ_1,ℓ_2}) be the generating series of ordinary (resp. fully simple) cylinders with perimeters (ℓ_1, ℓ_2), and

$$W_2^{[0]}(x_1,x_2) = \sum_{\ell_1,\ell_2 \geq 1} \frac{F_{\ell_1,\ell_2}}{x_1^{\ell_1+1}x_2^{\ell_2+1}}, \qquad X_2^{[0]}(w_1,w_2) = \sum_{\ell_1,\ell_2 \geq 1} H_{\ell_1,\ell_2}\, w_1^{\ell_1-1}w_2^{\ell_2-1}\,.$$

Proposition 1.2. *For all types of maps in Figure 2, if one sets $x_i = X(w_i)$ or equivalently $w_i = W(x_i)$,*

$$\left(W_2^{[0]}(x_1,x_2) + \frac{1}{(x_1-x_2)^2}\right)\mathrm{d}x_1\mathrm{d}x_2 = \left(X_2^{[0]}(w_1,w_2) + \frac{1}{(w_1-w_2)^2}\right)\mathrm{d}w_1\mathrm{d}w_2.$$

The identities of Propositions 1.1-1.2 are equalities of formal series in $x_i \to \infty$ and $w_i \to 0$.

1.2. **Matrix model interpretation and consequences.** It is well-known that the generating series of ordinary maps with prescribed boundary perimeters $(\ell_i)_{i=1}^n$ are computed as the moments $\langle \mathrm{Tr}\, M^{\ell_1} \cdots \mathrm{Tr}\, M^{\ell_n} \rangle$ in the formal hermitian matrix model

$$\mathrm{d}\mu(M) = \mathrm{d}M\, \exp[-N\, \mathrm{Tr}\, V(M)], \qquad V(x) = \frac{x^2}{2} - \sum_{d \geq 1} \frac{t_d x^d}{d}\,, \tag{1.1}$$

where t_d is the weight per d-gon, and the weight of a map of Euler characteristic χ is proportional to N^χ. Restricting to connected maps amounts to considering the cumulant expectation values $\kappa_n(\mathrm{Tr}\, M^{\ell_1}, \ldots, \mathrm{Tr}\, M^{\ell_n})$ instead of the moments. More generally, the measure

$$\mathrm{d}\mu(M) = \mathrm{d}M\, \exp\left(-N\,\mathrm{Tr}\,\frac{M^2}{2} + \sum_{h\geq 0}\sum_{k \geq 1}\sum_{d_1,\ldots,d_k \geq 1} \frac{N^{2-2h-k}}{k!}\,\frac{t_{h;d_1,\ldots,d_k}}{d_1\cdots d_k}\mathrm{Tr}\, M^{d_1}\cdots \mathrm{Tr}\, M^{d_k}\right) \tag{1.2}$$

generates maps with loops or stuffed maps.

We show in Section 8 that the generating series of fully simple maps with prescribed boundary perimeters $(\ell_i)_{i=1}^n$ in these models are computed as $\langle \mathcal{P}_{\gamma_1}(M) \cdots \mathcal{P}_{\gamma_n}(M) \rangle$, where γ is a permutation of $\{1, \ldots, L\}$ with n disjoint cycles $(\gamma_i)_{i=1}^n$ of respective lengths $(\ell_i)_{i=1}^n$, $L = \sum_{i=1}^n \ell_i$, and $\mathcal{P}_{\gamma_i}(M) = \prod_j M_{j,\gamma_i(j)}$. This quantity does not depend on the permutation γ, but only the lengths

$(\ell_i)_{i=1}^n$, which are encoded into a partition λ, and we write $\langle \mathcal{P}^{(\ell_1)}(M)\cdots\mathcal{P}^{(\ell_n)}(M)\rangle = \langle \mathcal{P}_\lambda(M)\rangle$. Again, the cumulants

$$\kappa_n\big(\mathcal{P}_{\gamma_1}(M),\ldots,\mathcal{P}_{\gamma_n}(M)\big) = \kappa_n\big(\mathcal{P}^{(\ell_1)}(M),\ldots,\mathcal{P}^{(\ell_n)}(M)\big)$$

generate only connected maps.

1.2.1. *Relation between ordinary and fully simple via Hurwitz theory.* $\prod_i \mathcal{P}_{\gamma_i}(M)$ is a function of M which is not invariant under $U(N)$-conjugation. Yet, as the measure μ is invariant, its expectation value must be expressible in terms of $U(N)$-invariant observables, *i.e.* as a linear combination of $\langle\prod_i \mathrm{Tr}\, M^{m_i}\rangle$. In other words, we can express the fully simple generating series in terms of the ordinary generating series. The precise formula is derived via Weingarten calculus.

Theorem 1.3. *If μ is a unitarily invariant measure on $\mathcal{H}_N$, in particular for the measures* (1.1)-(6.2) *generating any type of map in Figure 2,*

$$\langle \mathcal{P}_\lambda(M)\rangle = \sum_{\mu\vdash|\lambda|} N^{-|\mu|}\bigg(\sum_{k\geq 0}(-N)^{-k}[H_k]_{\lambda,\mu}\bigg)|C_\mu|\bigg\langle\prod_{i=1}^{\ell(\mu)}\mathrm{Tr}\,M^{\mu_i}\bigg\rangle, \tag{1.3}$$

$$\bigg\langle\prod_{i=1}^{n}\mathrm{Tr}\,M^{\mu_i}\bigg\rangle = \sum_{\lambda\vdash|\mu|} N^{|\lambda|}\bigg(\sum_{k\geq 0}N^{-k}\,[E_k]_{\mu,\lambda}\bigg)|C_\lambda|\langle\mathcal{P}_\lambda(M)\rangle\,. \tag{1.4}$$

The notation $\mu\vdash L$ means that μ is a partition of L, and $|C_\lambda|$ is the number of permutations in the conjugacy class C_λ which corresponds to the partition λ. The transition kernels $[H_k]_{\lambda,\mu}$ and $[E_k]_{\lambda,\mu}$ are universal numbers expressed via character theory of the symmetric group. Invoking the general relations [40, 56] between the enumeration of branched covers of $\mathbb{P}^1$, paths in the Cayley graph of the symmetric groups, and representation theory, we identify

- $[H_k]_{\lambda,\mu}$ with the double, weakly monotone Hurwitz numbers;
- $[E_k]_{\lambda,\mu}$ with the double, strictly monotone Hurwitz numbers.

In terms of branched covers of $\mathbb{P}^1$, λ and μ encode the ramification profiles over 0 and ∞, and k is the number of simple ramifications. Formula (1.4) is appealing as it is subtraction-free, and suggests the existence of a bijection describing ordinary maps as glueing of a fully simple map "along" a strictly monotone branched cover. We postpone such a bijective proof of (1.4) to a future work.

1.2.2. *Combinatorial interpretation of the matrix model with external field.* As a by-product, we show that the partition function of the formal hermitian matrix model with external field $A\in\mathcal{H}_N$

$$\check{Z}(A) = \int_{\mathcal{H}_N}\mathrm{d}\mu(M)\,\exp[N\mathrm{Tr}(MA)]$$

is a generating series of fully simple maps in the following sense

Proposition 1.4. *If μ is a unitarily invariant measure on $\mathcal{H}_N$ – in particular for all types of maps in Figure 2,*

$$\frac{\check{Z}(A)}{\check{Z}(0)} = \sum_\lambda \frac{\langle\mathcal{P}_\lambda(M)\rangle}{|\lambda|!}\prod_{i=1}^{\ell(\lambda)}\mathrm{Tr}\,A^{\lambda_i}\,.$$

1.2.3. *Application: an ELSV-like formula.* If we have a model in which the generating series of fully simple maps are completely known, (1.4) can be used to compute a certain set of monotone Hurwitz numbers in terms of generating series of maps. This is the case for the Gaussian Unitary Ensemble, *i.e.* $t_d = 0$ for all d in (1.1). As the matrix entries are independent

$$\kappa_{|\lambda|}\big(\mathcal{P}^{(\lambda_1)}(M),\ldots,\mathcal{P}^{(\lambda_{|\lambda|})}(M)\big)_{\mathrm{GUE}} = \prod_{i=1}^{\ell(\lambda)} \frac{\delta_{\lambda_i,2}}{N}\,.$$

Combinatorially, this formula is also straightforward: as the maps generated by the GUE have no internal faces, the only connected fully simple map is the disk of perimeter 2. Dubrovin et al. [23] recently proved a formula relating the GUE moments with all ℓ_i even, to cubic Hodge integrals. Combining their result with our (1.4) specialized to the GUE, we deduce in Section 11 an ELSV-like formula for what could be called 2-orbifold strictly monotone Hurwitz numbers.

Proposition 1.5. *Let $[E_g^\circ]_{\lambda,\mu}$ be the number of connected, strictly monotone branched covers from a curve of genus g to $\mathbb{P}^1$ with ramifications λ and μ above 0 and ∞, and simple ramifications otherwise. If $m_1,\ldots,m_n \geq 0$, set $|m| = \sum_i m_i$. We have*

$$(2|m|-1)!!\,[E_g^\circ]_{(2m_1,\ldots,2m_n),(2,\ldots,2)} = 2^g \int_{\overline{\mathcal{M}}_{g,n}} [\Delta]\cap\Lambda(-1)\Lambda(-1)\Lambda(\tfrac{1}{2})\exp\Big(\sum_{k\geq 1} a_k\kappa_k\Big)\prod_{i=1}^{n}\frac{m_i\binom{2m_i}{m_i}}{1-m_i\psi_i}\,,$$

where

$$[\Delta] = \sum_{h=0}^{g}\frac{[\Delta_h]}{2^{2h}(2h)!},\qquad \sum_{k\geq 1} a_k z^{k-1} = \operatorname*{Res}_{u\to 0}\frac{\mathrm{d}u\,e^{-u}}{u(1-z(1+u))}\,,$$

and $[\Delta_h]$ is the class of $\overline{\mathcal{M}}_{g-h,n+2h}$ included in $\overline{\mathcal{M}}_{g,n}$ by identifying pairwise the $2h$ last punctures.

1.3. **Topological recursion interpretation.**

1.3.1. *Review.* It was proved in [25, 19, 28] for maps, and [10, 12] for maps with loops, that the generating series of ordinary maps $W_n^{[g]}$ satisfies the topological recursion (hereafter, TR) formalized by Eynard and Orantin [33]. This is a universal recursion in minus the Euler characteristic $2g-2+n$, which takes as input data

- a Riemann surface $\mathcal{C}$ realized as a branched cover $p:\mathcal{C}\to\mathbb{C}$;
- an analytic function λ on $\mathcal{C}$;
- a bidifferential B with a double pole on the diagonal in $\mathcal{C}^2$.

The outcome of TR is a sequence of multidifferentials $(\omega_n^{[g]})_{g,n}$ on $\mathcal{C}^n$, including for $n=0$ a sequence of scalars $\omega_0^{[g]} = \mathcal{F}_g$. We call them "TR amplitudes". In the context of maps, $\mathcal{C}$ is the curve on which the generating series of disks can be maximally analytically continued with respect to its parameter x coupled to the boundary perimeter, and $\mathcal{C}$ has a distinguished point $[\infty]$ corresponding to $x\to\infty$.

Theorem 1.6. *The TR amplitudes for the initial data*

$$\begin{cases} p = x \\ \lambda = w = W_1^{[0]}(x) \\ B(z_1,z_2) = \Big(W_2^{[0]}(x(z_1),x(z_2)) + \frac{1}{(x(z_1)-x(z_2))^2}\Big)\mathrm{d}x(z_1)\mathrm{d}x(z_2) \end{cases} \tag{1.5}$$

compute the generating series of usual maps or maps with loops, through

$$W_n^{[g]}(x(z_1),\ldots,x(z_n)) = \frac{\omega_{g,n}(z_1,\ldots,z_n)}{\mathrm{d}x(z_1)\cdots\mathrm{d}x(z_n)},\qquad 2g-2+n>0. \tag{1.6}$$

Here z_i is a generic name for points in $\mathcal{C}$, and (1.6) *means the equality of Laurent expansion near $z_i \to [\infty]$.*

1.3.2. *Symplectic invariance.* The most remarkable, and still mysterious property of the topological recursion is its symplectic invariance

Theorem 1.7. *Assume $\mathcal{C}$ is compact and λ and p are meromorphic. Let $\check{\omega}_n^{[g]}$ be the TR amplitudes for the initial data in which the role of λ and p is exchanged and B remains the same. We have*

$$\forall g \geq 2, \qquad \check{\mathcal{F}}^{[g]} = \mathcal{F}^{[g]}$$

and this formula also holds up to an explicit corrective term for $g = 0, 1$. The TR amplitudes with $n \geq 1$ for the two initial data differ.

This theorem first appeared in [32] but some additive constants in the definition of $\mathcal{F}^{[g]}$ were overlooked, and the proof was corrected and completed in [34]. It is called symplectic invariance because the change $(p, \lambda) \to (-\lambda, p)$ preserves the symplectic form $\mathrm{d}\lambda \wedge \mathrm{d}p$ on $\mathbb{C}^2$. The invariance is believed to hold with weaker assumptions. For instance in topological strings on toric Calabi-Yau threefolds, symplectic invariance amounts to the framing independence of the closed sector, albeit involving curves of equation Polynomial$(e^p, e^\lambda) = 0$.

Propositions 1.1-1.2 tell us that swapping λ and p in the initial data (1.5) amounts to replacing the generating series of ordinary disks and cylinders with their fully simple version. We see it as the planar tip of an iceberg.

Conjecture 1.8. *For usual maps or maps with loops, let $\check{\omega}_n^{[g]}$ be the TR amplitudes for the initial data* (1.5) *after the exchange of x and w. We have*

$$X_n^{[g]}(w(z_1), \ldots, w(z_n)) = \frac{\check{\omega}_n^{[g]}(z_1, \ldots, z_n)}{\mathrm{d}w(z_1) \cdots \mathrm{d}w(z_n)}, \qquad 2g - 2 + n > 0. \tag{1.7}$$

This is an equality of formal Laurent series when $z_i \to [\infty]$.

The validity of this conjecture would give a combinatorial interpretation to the symplectic invariance, as the notions of ordinary and fully simple coincide for maps with $n = 0$ boundaries, *i.e.* $X_0^{[g]} = W_0^{[g]}$. We prove in Section 9:

Theorem 1.9. *For usual maps, Conjecture 1.8 is true.*

Our proof is however not combinatorial, and relies on the study the formal 1-hermitian matrix model with external field and the property of symplectic invariance (Theorem 1.7). It is still desirable to prove Conjecture 1.8 in a combinatorial way, as it would give an independent proof of symplectic invariance for the initial data related to maps – *i.e.* a large class of curves of genus 0. We do not solve this problem in the present article. Before obtaining a proof of Theorem 1.9, we had gathered some combinatorial evidence supporting our result. In fact, there is however no *a priori* reason for the coefficients of expansion of $\check{\omega}_n^{[g]}$ to be positive integers. Besides, for the same given perimeters, there are less fully simple maps than maps. For the initial data corresponding to quadrangulations, we have checked numerically that positivity and the expected inequalities hold for the coefficients of $\check{\omega}_1^{[1]}$ up to perimeter ≤ 14.

1.4. **Application to free probability.** The results of Theorem 1.1 for simple disks and Theorem 1.2 for fully simple cylinders coincide with the formulas found for generating series of the first and second order cumulants in [21]. We proved these formulas via combinatorics of maps, independently of [21], and also explained that they are natural in light of the topological recursion. The restriction of our Theorem 1.9 to genus 0 gives a recursive algorithm to compute the higher order free cumulants of the matrix M sampled from the large N limit of the measure (1.1). This is interesting as the relation at the level of generating series between n-th order free cumulants and cumulants is not otherwise known for $n \geq 3$ as of writing, thus imposing to work with their involved combinatorial definition via partitioned permutations.

We explain in Section 10 that a possible generalization of Conjecture 1.8 to stuffed maps – for which generating series of ordinary maps are governed by a generalization of TR – would shed light on computation of generating series of higher order cumulants in the full generality of [21]. Given the universality of the TR structure, one may also wonder if one could not formulate a universal theory of approximate higher order free cumulants, which would here be defined by taking into account the higher genus amplitudes.

Part 1. **Combinatorics**

2. Objects of study

2.1. **Maps.** Maps can intuitively be thought as graphs drawn on surfaces or discrete surfaces obtained from gluing polygons, and they receive different names in the literature: ribbon graphs, discrete surfaces, fat graphs...

Definition 2.1. A *map* M of genus g is a connected graph Γ embedded into a connected orientable surface X of genus g such that $X \setminus \Gamma$ is a disjoint union of connected components, called *faces*, each of them homeomorphic to an open disk. Each edge belongs to two faces (which may be the same) and we call *length* of a face the number of edges belonging to it.

We say that M is a map with n *boundaries* when we consider n marked faces, labeled $1, \ldots, n$, which we require to contain a marked edge, called *root*, represented by an arrow following the convention that the marked face sits on the left side of the root. A face which is not marked receives the name of *inner face.*

Two maps $M_i \subset X_i$, $i = 1, 2$, are *isomorphic* if there exists an orientation preserving homeomorphism $\varphi : X_1 \to X_2$ such that $\varphi|_{\Gamma_i}$ is a graph isomorphism between Γ_1 and Γ_2, and the restriction of φ to the marked edges is the identity.

Note that we include the connectedness condition in the definition of map. We will also work with disjoint unions of maps and we will specify we are dealing with a non-connected map whenever it is necessary. Observe that $\sum_{f \in F} \mathrm{length}(f) = 2|E|$, where the sum is taken over the set of faces F of the map and E denotes the set of edges.

We call *planar* a map of genus 0. We call *map of topology* (g, n) a map of genus g with n boundaries. For the cases $(0, 1)$ and $(0, 2)$ we use the special names: *disks* and *cylinders*, respectively.

2.1.1. *Permutational model.* The embedding of the graph into an oriented surface provides the extra information of a cyclic order of the edges incident to a vertex. More precisely, we consider half-edges, each of them incident to exactly one vertex. Let H be the set of half-edges and observe that $|H| = 2|E|$. We label the half-edges by $1, \ldots, 2|E|$ in an arbitrary way.

Every map with labeled half-edges can be encoded by a so-called *combinatorial map*, which consists of a pair of permutations (σ, α) acting on H such that all cycles of α have length 2. Given a half-edge $h \in H$, let $\sigma(h)$ be the the half-edge after h when turning around its vertex according to the orientation fixed for the underlying surface (by convention counterclockwise). On the other hand, let $\alpha(h)$ be the other half-edge of the edge to which h belongs; this information is encoded in the graph structure of the map instead.

- A cycle of σ corresponds to a vertex in the map.
- Every cycle of α corresponds to an edge of the map.

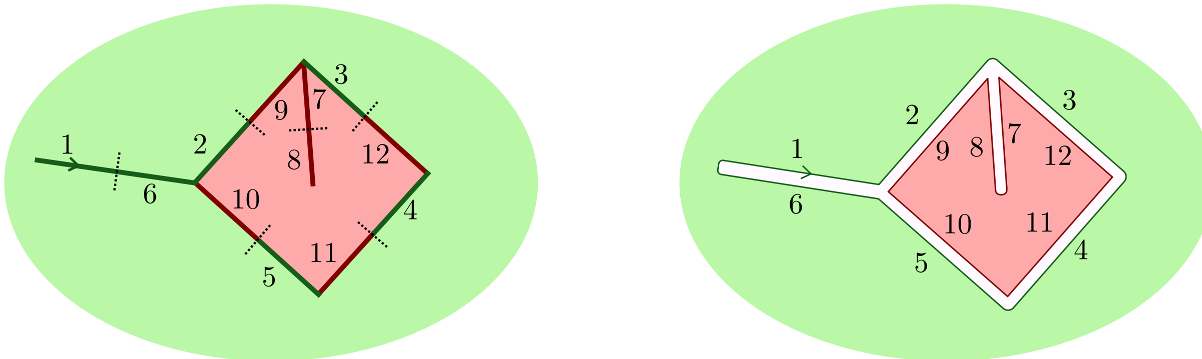


FIGURE 3. Two different ways of representing half-edges: in both cases by numbered segments, but on the left the two half-edges forming an edge are drawn consecutively, while on the right two consecutive half-edges belong to the same face and two half-edges forming an edge are drawn parallel. When maps are depicted as on the right, they are usually called ribbon graphs. We will sometimes use this representation because it can be a bit clearer, but will usually use the simpler representation on the left. On the left, the half-edges incident to a vertex are clearly the ones touching the vertex in the drawing; on the right, with our conventions, the half-edges incident to a vertex are the ones on the left viewed from the vertex in question.

- The faces may be represented by cycles of a permutation, called φ, of H.

Observe that with the convention that the face orientation is also counterclockwise, we obtain

$$\sigma \circ \alpha \circ \varphi = \mathrm{id}, \tag{2.1}$$

and hence φ can be determined by $\sigma^{-1}\alpha^{-1}$.

Rooting an edge in a face amounts to marking the associated label in the corresponding cycle of φ. Such cycles containing a root will be ordered and correspond to boundaries of the map.

Example 2.2. In Figure 3, we have

$$\begin{aligned} \sigma &= (5\ 11)(4\ 12)(3\ 9\ 7)(2\ 6\ 10), \\ \alpha &= (1\ 6)(2\ 9)(7\ 8)(3\ 12)(4\ 11)(5\ 10), \\ \varphi &= \sigma^{-1} \circ \alpha^{-1} = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12), \end{aligned}$$

where the root is the half-edge labeled 1.

The lengths of the cycles of σ and φ correspond to the degrees of vertices and faces, respectively. The Euler characteristic is given by

$$\chi(\sigma, \alpha) = |\mathcal{C}(\sigma)| - |\mathcal{C}(\alpha)| + |\mathcal{C}(\varphi)| - n,$$

where $\mathcal{C}(\cdot)$ denotes the set of cycles of a permutation and n is the number of cycles of φ containing a root.

$G = \langle \sigma, \alpha \rangle$ is the *cartographic group*. Its orbits on the set of half-edges determine the connected components of the map. If the action of G on H is transitive, the map is connected, and its genus g is given by the formula:

$$2 - 2g - n = \chi(\sigma, \alpha),$$

where n is the number of boundaries. If all orbits contain a root, the map is called *∂-connected.*

2.1.2. *Automorphisms.* Let us consider the decomposition $H = H^u \sqcup H^\partial$, where H^u is the set of half-edges belonging to unmarked faces and H^∂ is the set of half-edges belonging to boundaries.

Observe that from a combinatorial map one can recover all the information of the original map. So there is a one-to-one correspondence between maps with labeled half-edges and combinatorial maps. There is a canonical way of labeling half-edges in boundaries: assigning the first label to the root, continuing by cyclic order of the boundary and taking into account that boundaries are ordered. However, we can label half-edges of unmarked faces in many different ways. To obtain a bijective correspondence with unlabeled maps, we have to identify configurations which differ by a relabeling of H^u, i.e. (σ, α) to all $(\gamma\sigma\gamma^{-1}, \gamma\alpha\gamma^{-1})$ where γ is any permutation acting on H such that $\gamma|_{H^\partial} = \mathrm{Id}_{H^\partial}$. We call such an equivalence class *unlabeled combinatorial map* and we denote it by $[(\sigma, \alpha)]$. Note that unlabeled combinatorial maps are in bijection with the unlabeled maps we defined at the beginning of this section.

Definition 2.3. Given a combinatorial map (σ, α) acting on H, we call γ an *automorphism* if it is a permutation acting on H such that $\gamma|_{H^\partial} = \mathrm{Id}_{H^\partial}$ and

$$\sigma = \gamma\sigma\gamma^{-1}, \qquad \alpha = \gamma\alpha\gamma^{-1}.$$

Observe that for connected maps with $n \geq 1$ boundaries, the only automorphism is the identity. Note also that these special relabelings that commute with σ and α, and we call automorphisms, exist because of a symmetry of the (unlabeled) map. The symmetry factor $|\mathrm{Aut}(\sigma, \alpha)|$ of a map is its number of automorphisms.

We denote $\mathrm{Gl}(\sigma, \alpha)$ the number of elements in the class $[(\sigma, \alpha)]$ and $\mathrm{Rel}(\sigma, \alpha)$ the total number of relabelings of H^u, which is $|H^u|!$, if we consider completely arbitrary labels for the half-edges. By the orbit-stabilizer theorem, we have $\mathrm{Gl}(\sigma, \alpha) = \frac{\mathrm{Rel}(\sigma,\alpha)}{|\mathrm{Aut}(\sigma,\alpha)|}$.

We refer the interested reader to the book [48] for further details on the topic of ordinary maps, although the conventions, notations and even concepts differ a little bit.

2.2. **Simple and fully simple maps.**

Definition 2.4. We call a boundary B *simple* if no more than two edges belonging to B are incident to a vertex. We say a map is *simple* if all boundaries are simple.

To acquire an intuition about what this concept means, observe that the condition for a boundary to be simple is equivalent to not allowing edges of polygons corresponding to the boundary to be identified, except for the degenerate case of a boundary with only two edges which are identified, which is indeed considered to be a simple map.

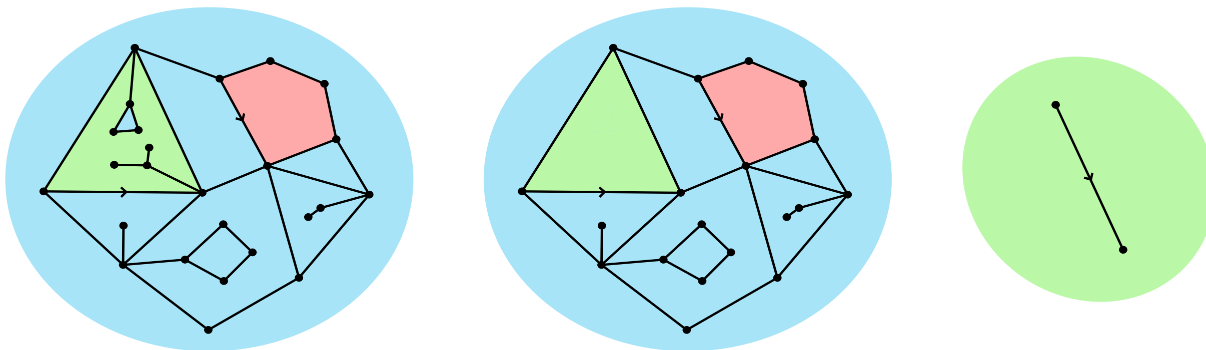
We will call *ordinary* the maps introduced in the previous section to emphasize that they are not necessarily simple.

Definition 2.5. We say a boundary B is *fully simple* if no more than two edges belonging to any boundary are incident to a vertex of B. We say a map is *fully simple* if all boundaries are fully simple.

Again one can visualize the concept of a fully simple boundary as a simple boundary which moreover does not share any vertex with any other boundary.

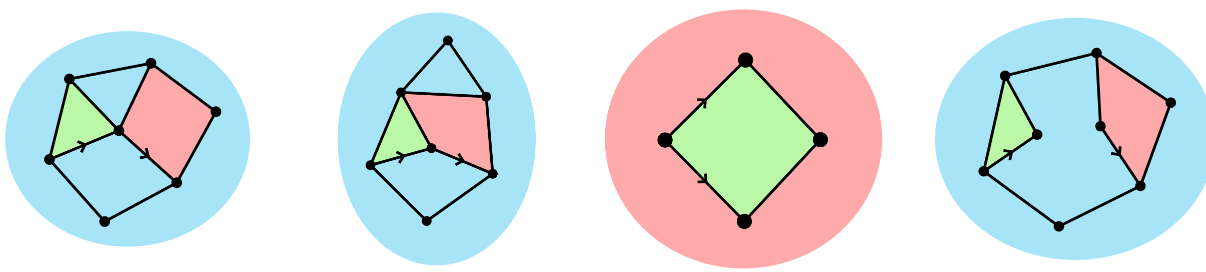
2.3. **Generating series.** We introduce now the notations and conventions for the generating series of our objects of study.

Let $\mathbb{M}_n^{[g]}$ be the set of maps of genus g with n boundaries and we denote $\mathbb{M}_n^{[g]}(v)$ the subset where we also fix the number of vertices to v.



(a) (b) (c)

FIGURE 4. (a) is an ordinary cylinder where the green boundary is non-simple and (b) is a simple cylinder. (c) is the only simple map where two edges in the boundary are identified.



(a) (b) (c) (d)

FIGURE 5. Four simple maps: (a), (b) and (c) are non-fully simple, and (d) is fully-simple. In (a) the two boundaries share a vertex, in (b) an edge, and in (c) they are completely glued to each other.

Then we define the generating series of maps of genus g and n boundaries of respective lengths $l_1, \ldots, l_n$:

$$F^{[g]}_{l_1,\ldots,l_n} := \sum_{M \in \mathbb{M}^{[g]}_n} \frac{\prod_{j\geq 1} t_j^{n_j(M)}}{|\mathrm{Aut}\, M|} \prod_{i=1}^{n} \delta_{l_i,\ell_i(M)},$$

where $n_j(M)$ denotes the number of unmarked faces of length j, and $\ell_i(M)$ denotes the length of the i-th boundary of M. For $n = 0$, we denote $F^{[g]}$ the generating series of closed maps of genus g.

We have that

$$F^{[g]}, F^{[g]}_{l_1,\ldots,l_n} \in \mathbb{Q}\,[[t_1, t_2, \ldots]],$$

that is the number of maps is finite after fixing the topology (g, n) and the number of internal faces n_j of every possible length $j \geq 1$.

Remark 2.6. Unmarked faces are often required to have length ≥ 3 and $\leq d < \infty$ in the literature. In this typical setting, it can be easily checked that the set $\mathbb{M}^{[g]}_n(v)$ is always finite, without having to fix the numbers n_j of internal faces. As a consequence, if we consider the generating series with an extra sum over the number of vertices $v \geq 1$ and a weight u^v, then it would belong to $\mathbb{Q}\,[t_3, \ldots, t_d]\,[[u]]$. We do not need this further restriction, so from now on we consider the more general setting we introduced.

We take the convention that $\mathbb{M}_1^{[0]}(1)$ contains only one map which consists of a single vertex and no edges; it is the map of genus 0 with 1 boundary of length 0, that is $F_0^{[0]} = 1$. Apart from this degenerate case, we always consider that boundaries have length ≥ 1.

Summing over all possible lengths, we define the generating series of maps of genus g and n boundaries as follows:

$$W_n^{[g]}(x_1, \ldots, x_n) := \sum_{l_1,\ldots,l_n \geq 0} \frac{F_{l_1,\ldots,l_n}^{[g]}}{x_1^{1+l_1} \cdots x_n^{1+l_n}}.$$

We have that $W_n^{[g]}(x_1, \ldots, x_n) \in \mathbb{Q}\left[\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right][[t_1, t_2, \ldots]]$ and observe that

$$F_{l_1,\ldots,l_n}^{[g]} = (-1)^n \operatorname*{Res}_{x_1 \to \infty} \cdots \operatorname*{Res}_{x_n \to \infty} x_1^{l_1} \cdots x_n^{l_n} W_n^{[g]}(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

We denote $H_{k_1,\ldots,k_n}^{[g]}$ the analogous generating series for fully simple maps of genus g and n boundaries of fixed lengths $k_1, \ldots, k_n$ and we introduce the following more convenient generating series for fully simple maps with boundaries of all possible lengths:

$$X_n^{[g]}(w_1, \ldots, w_n) := \sum_{k_1,\ldots,k_n \geq 0} H_{k_1,\ldots,k_n}^{[g]} w_1^{k_1-1} \ldots w_n^{k_n-1}.$$

Finally, we denote $G_{k_1,\ldots,k_m|l_1,\ldots,l_n}^{[g]}$ the generating series of maps with m simple boundaries of lengths $k_1, \ldots, k_m$ and n ordinary boundaries of lengths $l_1, \ldots, l_n$. We write

$$Y_{m|n}^{[g]}(w_1, \ldots, w_m \mid x_1, \ldots, x_n) = \sum_{(\mathbf{k},\mathbf{l}) \in \mathbb{N}^m \times \mathbb{N}^n} \frac{w_1^{k_1-1} \cdots w_m^{k_m-1}}{x_1^{l_1+1} \cdots x_n^{l_n+1}} G_{\mathbf{k}|\mathbf{l}}^{[g]}.$$

We use the following simplification for maps with only simple boundaries: $G_{k_1,\ldots,k_m}^{[g]}$ for $G_{k_1,\ldots,k_m|}^{[g]}$, and $Y_m^{[g]}$ for $Y_{m|}^{[g]}$.

Observe that for maps with only one boundary the concepts of simple and fully simple coincide. Therefore $G_k^{[g]} = H_k^{[g]}$ and $Y_1^{[g]} = X_1^{[g]}$.

For all the generating series introduced we allow to omit the information about the genus in the case of $g = 0$. We also use the simplification of removing the information about the number of boundaries if $n = 1$. In this way, W and X stand for $W_1^{[0]}$ and $X_1^{[0]}$.

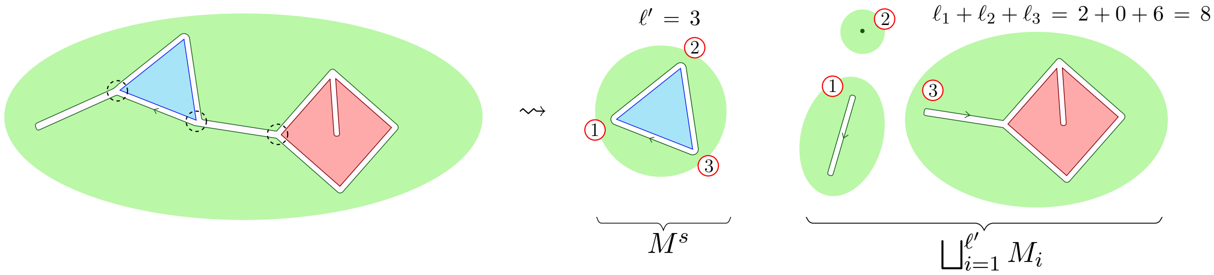
3. Simple disks from ordinary disks

We can decompose an ordinary disk M with boundary of length $\ell > 0$ into a simple disk M^s with boundary of length $1 \leq \ell' \leq \ell$ and ordinary disks of lengths $\ell_i < \ell$, using the following procedure:

Algorithm 3.1 (from ordinary to simple)**.** Set $M^s := M$. We run over all edges of M, starting at the root edge e_1 and following the cyclic order around the boundary. When we arrive at a vertex v_i from an edge e_i, we create two vertices out of it: the first remains on the connected component containing e_i, while the second one glues together the remaining connected components, giving a map M_i. We then update M^s to be the first connected component and proceed to the next edge on it. Every M_i, for $i = 1, \ldots, \ell'$, is an ordinary map consisting of

- a single vertex whenever v_i was simple, or
- a map with a boundary of positive length with the marked edge being the edge in M_i following e_i in M.

Example 3.2. Consider a non-simple map with a boundary of length 11 (non-simple vertices are circled). Applying the algorithm we obtain the simple map M^s of length 3, and 3 ordinary maps M_1, M_2, M_3.



The maps should be regarded as drawn on the sphere and the outer face is in all cases the boundary.

Using the decomposition given by the algorithm, we find that X and W are reciprocal functions:

Proposition 3.3.

$$x = X(W(x)). \tag{3.1}$$

Proof. Since the algorithm establishes a bijection, we find that

$$\forall \ell \geq 1, \qquad F_\ell = \sum_{\ell'=1}^{\ell} H_{\ell'} \sum_{\substack{\ell_1,\ldots,\ell_{\ell'} \geq 0 \\ \sum_i (\ell_i+1)=\ell}} \prod_{i=1}^{\ell'} F_{\ell_i}, \tag{3.2}$$

which implies, at the level of resolvents:

$$\begin{aligned} W(x) &= \sum_{\ell \geq 0} \frac{F_\ell}{x^{\ell+1}} = \frac{F_0}{x} + \sum_{\ell \geq 1} \frac{F_\ell}{x^{\ell+1}} = \frac{H_0}{x} + \sum_{\ell \geq 1} \frac{1}{x^{\ell+1}} \sum_{\ell'=1}^{\ell} H_{\ell'} \sum_{\substack{\ell_1,\ldots,\ell_{\ell'} \geq 0 \\ \sum_i (\ell_i+1)=\ell}} \prod_{i=1}^{\ell'} F_{\ell_i} \\ &= \frac{1}{x} \sum_{\ell' \geq 0} H_{\ell'} (W(x))^{\ell'} = \frac{W(x)}{x} X(W(x)). \end{aligned}$$

□

4. Cylinders

4.1. **Replacing an ordinary boundary by a simple boundary.** Let us consider a planar map M with one ordinary boundary of length ℓ, and one simple boundary of length k. We apply the procedure described in Algorithm 3.1 to the ordinary boundary. We have to distinguish two cases depending on the nature of M^s:

- either M^s is a planar map with one simple boundary of some length ℓ', and another simple boundary of length k (which we did not touch). Then, the rest of the pieces $M_1, \ldots, M_{\ell'}$ are planar maps with one ordinary boundary of lengths $\ell_1, \ldots, \ell_{\ell'}$.
- or M^s is a planar map with one simple boundary of some length ℓ'. And the rest consists of a disjoint union of:
  - a planar map with the simple boundary of length k which bordered M initially, and another ordinary boundary with some length ℓ_1,
  - and $\ell' - 1$ planar maps with one ordinary boundary of lengths $\ell_2, \ldots, \ell_{\ell'}$.

This decomposition is again a bijection, and hence

$$G_{k|\ell} = \sum_{\ell'=1}^{\ell} \sum_{\substack{\ell_1,\ldots,\ell_{\ell'} \geq 0 \\ \sum_i (\ell_i+1) = \ell}} \left(G_{k,\ell'} \prod_{i=1}^{\ell'} F_{\ell_i} + \ell'\, G_{\ell'} G_{k|\ell_1} \prod_{i=2}^{\ell'} F_{\ell_i} \right). \tag{4.1}$$

We deduce, at the level of resolvents, that

$$Y_{1|1}(w \mid x) = \frac{W(x)}{x} Y_2(w, W(x)) + \frac{Y_{1|1}(w \mid x)}{x} (\partial_w (w X(w)))_{w=W(x)}. \tag{4.2}$$

Isolating Y_2, we obtain:

$$Y_2(w, W(x)) = -Y_{1|1}(w \mid x)(\partial_w X(w))_{w=W(x)}. \tag{4.3}$$

4.2. **From ordinary cylinders to simple cylinders.** We consider the following operator

$$\frac{\partial}{\partial V(x)} = \sum_{k \geq 0} \frac{k}{x^{k+1}} \frac{\partial}{\partial t_k}\,, \tag{4.4}$$

which creates an ordinary boundary of length k weighted by $x^{-(k+1)}$. Therefore, we have

$$W_n^{[g]}(x_1, \ldots, x_n) = \frac{\partial}{\partial V(x_2)} \cdots \frac{\partial}{\partial V(x_n)}\, W_1^{[g]}(x_1),$$
$$Y_{1|n}^{[g]}(w \mid x_1, \ldots, x_n) = \frac{\partial}{\partial V(x_1)} \cdots \frac{\partial}{\partial V(x_n)}\, Y_1^{[g]}(w).$$

Applying $\frac{\partial}{\partial V(x_1)}$ to equation (3.1), we obtain

$$0 = (\partial_w X(w))_{w=W(x_1)} \frac{\partial}{\partial V(x_1)} W(x_1) + Y_{1|1}(W(x_1) \mid x_2)$$

and hence

$$Y_{1|1}(W(x_1) \mid x_2) = -W_2(x_1, x_2)\, (\partial_w X(w))_{w=W(x_1)}. \tag{4.5}$$

Finally, combining equations (4.3) and (4.5), we obtain the following relation between ordinary and simple cylinders:

$$Y_2(W(x_1), W(x_2)) = W_2(x_1, x_2)(\partial_w X(w))_{w=W(x_1)}(\partial_w X(w))_{w=W(x_2)}. \tag{4.6}$$

4.3. **From simple cylinders to fully simple cylinders.** We describe an algorithm which expresses a planar map M with two simple boundaries in terms of planar fully simple maps. The idea is to merge simple boundaries that touch each other. By definition, a simple boundary which is not fully simple shares at least one vertex with another boundary. By convention, whenever we refer to cyclic order in the process, we mean cyclic order of the first boundary. Let B_1 and B_2 denote the first and the second boundaries respectively.

Definition 4.1. A *pre-shared piece of length* $m > 0$ is a sequence of m consecutive edges in B_1 which are shared with B_2. We define a *pre-shared piece of length* $m = 0$ to be a vertex which both boundaries B_1 and B_2 have in common.

The first vertex sv_1 of the first edge and the second vertex sv_2 of the last edge in a pre-shared piece are called the *endpoints*. If $m = 0$, the endpoints coincide by convention with the only vertex of the pre-shared piece: $sv_1 = sv_2$.

We say that a pre-shared piece of length $m \geq 0$ is a *shared piece of length* $m \geq 0$ if the edge in B_1 that arrives to sv_1 and the edge in B_1 outgoing from sv_2 are not shared with B_2.

We define the *interior* of a shared piece to be the shared piece minus the two endpoints. The interior of a shared vertex is empty.

Before describing the decomposition algorithm, we describe a special case which corresponds to maps as in Figure 5.(c), which we will exclude. Consider a map whose two only faces are the two simple boundaries. The only possibility is that they have the same length and are completely glued to each other. We will count this kind of maps apart.

Algorithm 4.2. (From simple cylinders to fully simple disks or cylinders)

(1) Save the position of the marked edge on each boundary.

(2) Denote r the number of shared pieces. If $r = 0$, we already have a fully simple cylinder and we stop the algorithm. Otherwise, denote the shared pieces by $S_0, \ldots, S_{r-1}$. Save their lengths $m_0, \ldots, m_{r-1}$, labeled in cyclic order, and shrink their interiors so that only shared vertices remain.

Since we have removed all common edges and boundaries are simple, every shared vertex has two non-identified incident edges from B_1 and two from B_2.

(3) Create two vertices v_1, v_2 out of each shared vertex v in such a way that each v_j has exactly one incident edge from B_1 and one from B_2, which were consecutive edges for the cyclic order at v in the initial map.

In this way, we got rid of all shared pieces, and we obtain a graph drawn on the sphere formed by r connected components which are homeomorphic to a disk. We consider each connected component separately, and we glue to their boundary a face homeomorphic to a disk.

(4) For $i = 0, \ldots, r-1$, call M_i the connected component which was sharing a vertex with S_i and $S_{i+1\,(\mathrm{mod}\,r)}$. Mark the edge in M_i which belonged to the first boundary and was outgoing from sv_2^i in M_i. Then, M_i becomes a simple disk. Denote by ℓ'_i (resp. ℓ''_i) the number of edges of the boundary of M_i previously belonging to B_1 (resp. B_2). Then, the boundary of M_i has perimeter $\ell'_i + \ell''_i$.

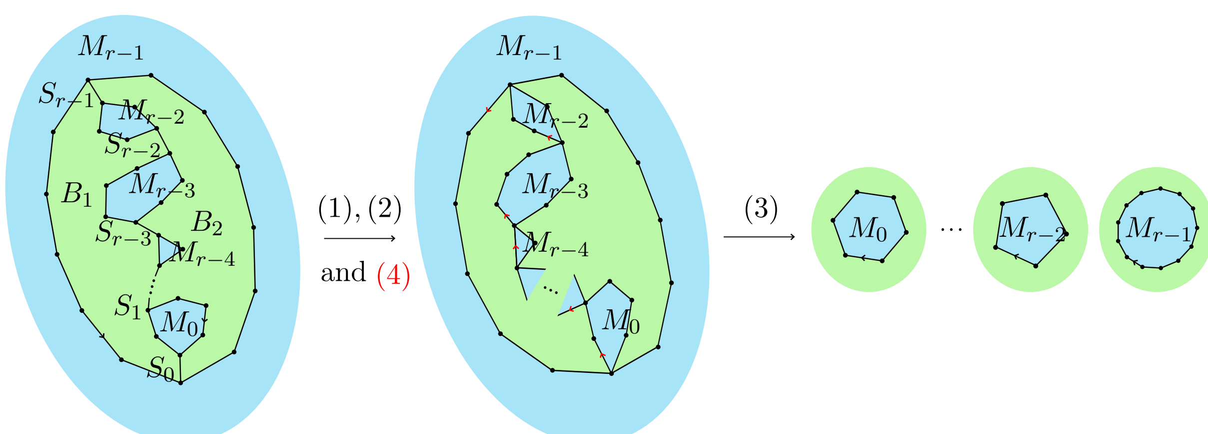


FIGURE 6. Schematic representation of Algorithm 4.2. The two green faces are the two simple, but non-fully simple, boundaries and the blue part represents the inner faces. The shared pieces $S_0, \ldots, S_{r-1}$ are drawn schematically as shared pieces of length 1. M_{r-1} is drawn as the outer face, but it does not play a special role.

Observe that by construction, $\ell'_i, \ell''_i \geq 1$ and $m_i \geq 0$. Moreover, note that the only map of length $\ell'_i + \ell''_i \geq 2$ and considered simple which is not allowed as M_i is the map with one boundary of length 2 where the two edges are identified as in Figure 4.(c), since this would correspond to a shared piece of length 1 and it would have been previously removed.

This decomposition is a bijection, since we can recover the original map from all the saved information and the obtained fully simple maps. To show this, we describe the inverse algorithm:

Algorithm 4.3. (From fully simple disks or cylinders to simple cylinders)

(1) Let r be the number of given (fully) simple discs. If $r = 0$, we already had a fully simple cylinder and the algorithms become trivial. Otherwise, observe that every M_i, for $i = 0, \ldots, r-1$, is a planar disk with two distinguished vertices v_1^i and v_2^i. The first one v_1^i is the starting vertex of the root edge e_1^i and the second one v_2^i is the ending vertex of the edge $e^i_{\ell'_i}$, where the edges are labeled according to the cyclic order of the boundary.

(2) For $i = 0, \ldots, r-1$, consider shared pieces S_i of lengths m_i.

(3) Glue sv_1^i of a shared piece S_i to $v_2^{i-1\,(\mathrm{mod}\, r)}$ in $M_{i-1\,(\mathrm{mod}\, r)}$ and sv_2^i of S_i to v_1^i in M_i.
All the marked edges in the M_i's should belong to the same simple face, which we call B_1. We call B_2 the other face, which is bordered by following the edges from v_2^i to v_1^i in every M_i, and the shared piece S_i from sv_2^i to sv_1^i, for $i = 1, \ldots, r$.

(4) Remove the r markings in B_1 and recover the roots in B_1 and B_2, which are now part of our data.
We have glued the r simple disks and shared pieces into a map with two simple (not fully simple) boundaries B_1 and B_2.

This bijection translates into the following relation between generating series of simple and fully simple cylinders:

Proposition 4.4.

$$Y_2(w_1, w_2) = X_2(w_1, w_2) + \partial_{w_1}\partial_{w_2} \ln\left(\frac{w_1 - w_2}{X(w_1) - X(w_2)}\right). \tag{4.7}$$

Proof. Let us introduce:

$$\tilde{X}(w) = X(w) - w^{-1} - w = \sum_{\ell \geq 1} \tilde{H}_\ell \, w^{\ell-1},$$

the generating series of (fully) simple disks, excluding the disk with boundary of length 0 which consists of a single vertex, and the simple disk with boundary of length 2 in which the two edges of the boundary are identified, as in Figure 4.(c).

Then, using the bijection we established, we obtain that

$$G_{L_1,L_2} = H_{L_1,L_2} + \delta_{L_1,L_2}\, L_1 + \sum_{r \geq 1} \sum_{\substack{\ell'_i,\, \ell''_i > 0,\; m_i \geq 0 \\ \sum_{i=0}^{r-1} \ell'_i + \sum_i m_i = L_1 \\ \sum_{i=0}^{r-1} \ell''_i + \sum_i m_i = L_2}} \frac{L_1 L_2}{r} \prod_{i=1}^{r} \tilde{H}_{\ell'_i + \ell''_i},$$

where the first term of the right hand side counts the case $r = 0$ in which the simple cylinders were already fully simple and the second term counts the degenerate case we excluded from the algorithm. We already observed that this degenerate case can only occur if $L_1 = L_2$ and there are L_1^2 possibilities for the two roots, but we also divide by L_1 because of the cyclic symmetry of this type of cylinders.

Summing over lengths $L_1, L_2 \geq 1$ with a weight $w_1^{L_1-1} w_2^{L_2-1}$, we get:

$$\begin{aligned} Y_2(w_1,w_2) &= X_2(w_1,w_2) - \partial_{w_1}\partial_{w_2} \ln(1-w_1w_2) \\ &\quad + \partial_{w_1}\partial_{w_2} \left(\sum_{r\geq 1} \frac{1}{r} \Big(\sum_{m\geq 0} (w_1w_2)^m \Big)^r \Big(\sum_{\ell',\ell'>0} \tilde{H}_{\ell'+\ell''} w_1^{\ell'} w_2^{\ell''} \Big)^r \right). \end{aligned}$$

Let us remark that

$$\sum_{\substack{\ell',\ell''>0 \\ \ell'+\ell''=\ell}} w_1^{\ell'} w_2^{\ell''} = \frac{w_1^{\ell+1} - w_2^{\ell+1}}{w_1 - w_2} - w_1^{\ell} - w_2^{\ell}.$$

Therefore,

$$\begin{aligned} \sum_{\ell',\ell''\geq 1} w_1^{\ell'} w_2^{\ell''} \tilde{H}_{\ell'+\ell''} &= \frac{w_1^2 \tilde{X}(w_1) - w_2^2 \tilde{X}(w_2)}{w_1 - w_2} - w_1 \tilde{X}(w_1) - w_2 \tilde{X}(w_2) \\ &= \frac{w_1 w_2 (\tilde{X}(w_1) - \tilde{X}(w_2))}{w_1 - w_2} \\ &= w_1 w_2 \frac{(X(w_1) - X(w_2))}{w_1 - w_2} - (1 - w_1 w_2). \end{aligned}$$

And finally,

$$\begin{aligned} Y_2(w_1,w_2) &= X_2(w_1,w_2) - \partial_{w_1}\partial_{w_2} \ln(1-w_1w_2) \\ &\quad - \partial_{w_1}\partial_{w_2} \ln \left[1 - \frac{1}{1-w_1w_2} \left(w_1 w_2 \frac{X(w_1) - X(w_2)}{w_1 - w_2} - (1 - w_1 w_2) \right) \right] \\ &= X_2(w_1,w_2) - \partial_{w_1}\partial_{w_2} \ln \left[-w_1 w_2 \frac{X(w_1) - X(w_2)}{w_1 - w_2} \right] \\ &= X_2(w_1,w_2) + \partial_{w_1}\partial_{w_2} \ln \left(\frac{w_1 - w_2}{X(w_1) - X(w_2)} \right). \end{aligned}$$

□

5. Combinatorial interpretation of symplectic invariance

5.1. **General result for usual maps.** In this section we remind that the generating series of ordinary maps satisfy the topological recursion and give a discrete interpretation of its important property of symplectic invariance.

We consider the reparametrization

$$x(z) = \alpha + \gamma \left(z + \frac{1}{z} \right),$$

where α and γ are parameters determined in terms of the weights $u, t_3, t_4, \ldots$, and which makes $w(z) \coloneqq W_1^{[0]}(x(z))$ a rational function of z. This computation of α, γ and $W_1^{[0]}(x(z))$ is summarized later in Lemma 9.7.

We introduce the fundamental differential of the second kind:

$$B(z_1,z_2) \coloneqq \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}.$$

Up to a correction term, and written as a bidifferential form, the cylinder generating function is the fundamental differential of the second kind:

Theorem 5.1. [25]

$$W_2^{[0]}(x(z_1),x(z_2))\mathrm{d}x(z_1)\mathrm{d}x(z_2)+\frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}=B(z_1,z_2).$$

In general, we rewrite the generating functions of maps in terms of the variables z_i and as multi-differential forms in $\mathbb{CP}^1$:

$$\omega_{g,n}(z_1,\ldots,z_n)=W_n^{[g]}(x(z_1),\ldots,x(z_n))\mathrm{d}x(z_1)\cdots\mathrm{d}x(z_n)+\delta_{g,0}\delta_{n,2}\frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}.$$

The forms $\omega_{g,n}$ satisfy the so-called topological recursion with spectral curve given by (x,w) and initial data

$$(\omega_{0,1}(z_1),\omega_{0,2}(z_1,z_2))=(w(z_1)\mathrm{d}x(z_1),B(z_1,z_2)).$$

More concretely:

Theorem 5.2. [25] *For all $g\geq 0$, $n\geq 1$ with $2g-2+n>0$,*

$$\begin{aligned}\omega_{g,n}(z_1,\ldots,z_n)= \quad & \operatorname*{Res}_{z=\pm 1}K(z_1,z)\Big(\omega_{g-1,n+1}(z,1/z,z_2,\ldots,z_n)\\ & +\sum_{\substack{h=0,\ldots,g\\ I\sqcup J=\{2,\ldots,n\}}}{}^{'}\ \omega_{h,|I|+1}(z,z_I)\,\omega_{g-h,|J|+1}(1/z,z_J)\Big),\end{aligned}$$

with $\sum'$ meaning that we omit $(h,I)=(0,\emptyset)$ and $(h,J)=(g,\emptyset)$, and the following recursion kernel

$$K(z_1,z):=\frac{\frac{1}{z_1-z}-\frac{1}{z_1-1/z}}{2(w(z)-w(1/z))}\frac{\mathrm{d}z_1}{\gamma(1-z^{-2})\mathrm{d}z}.$$

For further details on these results for the generating series of ordinary maps, see the book [28].

The main property of the $\omega_{g,0}$ produced by the topological recursion is that they are symplectic invariants, i.e. they coincide for two spectral curves which are symplectically equivalent in the sense that they are given by (x,w), and (x',w') such that $\mathrm{d}x\wedge\mathrm{d}w=\mathrm{d}x'\wedge\mathrm{d}w'$.

Let us consider the simplest symplectic change of variables:

$$(x,w)\to(-w,x),$$

that is the spectral curve given by (w,x) with initial data

$$(\check{\omega}_{0,1}(z_1),\check{\omega}_{0,2}(z_1,z_2)):=(x(z)\mathrm{d}w(z),B(z_1,z_2)).$$

For $2g-2+n>0$, we call $\check{\omega}_{g,n}$ the TR amplitudes for this spectral curve.

It is natural to wonder whether the $\check{\omega}_{g,n}$ also solve some enumerative problem and, in that case, which kind of objects they are counting. We propose an answer which also offers a combinatorial interpretation of the important property of symplectic invariance:

Theorem 5.3. *The invariants $\check{\omega}_{g,n}$ enumerate fully simple maps of genus g and n boundaries in the following sense:*

$$\check{\omega}_{g,n}(z_1,\ldots,z_n)=X_n^{[g]}(w(z_1),\ldots,w(z_n))\mathrm{d}w(z_1)\cdots\mathrm{d}w(z_n)+\delta_{g,0}\delta_{n,2}\frac{\mathrm{d}w(z_1)\mathrm{d}w(z_2)}{(w(z_1)-w(z_2))^2}.$$

A full proof of Theorem 5.3, albeit non combinatorial, will be given in Section 9. Observe that if $n=0$, that is we consider maps without boundaries, we obviously have $W_0^{[g]}=X_0^{[g]}$, which agrees with symplectic invariance: $\omega_{g,0}=\check{\omega}_{g,0}$. Using our formulas relating the generating series of fully simple disks and cylinders with the ordinary ones, we obtain a combinatorial proof of the first two base cases of the theorem:

Theorem 5.4. *Theorem 5.3 is true for the two base cases* $(g,n)=(0,1)$ *and* $(0,2)$.

Proof. By Proposition 3.3, we obtain

$$\check{\omega}_{0,1}(z) \coloneqq x(z)\mathrm{d}w(z) = X(W(x(z)))\mathrm{d}w(z),$$

which is equal to $X(w(z))\mathrm{d}w(z)$ by definition. This proves the theorem for $(g,n)=(0,1)$.

For cylinders, we have, by definition: $\check{\omega}_{0,2}(z_1,z_2) = B(z_1,z_2)$. Substituting the expression for the generating series of ordinary cylinders in terms of the one for simple cylinders given by formula (4.6) in the equation for the fundamental differential of the second kind from Theorem 5.1, we obtain

$$\begin{aligned}\omega_{0,2}(z_1,z_2) = B(z_1,z_2) = &\quad W_2(x(z_1),x(z_2))\mathrm{d}x(z_1)\mathrm{d}x(z_2) + \tfrac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\\ = &\quad Y_2(w(z_1),w(z_2))\mathrm{d}w(z_1)\mathrm{d}w(z_2) + \tfrac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}.\end{aligned}$$

Finally, using Proposition 4.4, we get the theorem for $(g,n)=(0,2)$:

$$\check{\omega}_{0,2}(z_1,z_2) = X_2(w(z_1),w(z_2))\mathrm{d}w(z_1)\mathrm{d}w(z_2) + \frac{\mathrm{d}w(z_1)\mathrm{d}w(z_2)}{(w(z_1)-w(z_2))^2}.$$

$\square$

5.2. **Supporting data for quadrangulations.** Now we compare the number of fully simple, simple and ordinary maps in the case in which all the internal faces are quadrangulations, which allows us to make computations explicitly. The reasonable outcomes were initially supporting our belief that, after the symplectic transformation, the topological recursion was counting some more restrictive kind of maps. We illustrate now our results in some specific cases.

We consider maps whose internal faces are all quadrangles [62], that is $t_j = t\delta_{j,4}$. Let t be the weight per quadrangle. The spectral curve is given by

$$x(z) = c\Big(z+\frac{1}{z}\Big), \qquad w(z) = \frac{1}{cz} - \frac{tc^3}{z^3},$$

with

$$c = \sqrt{\frac{1-\sqrt{1-12t}}{6t}} = 1 + \frac{3t}{2} + \frac{63}{8}t^2 + \frac{891}{16}t^3 + \frac{57915}{128}t^4 + O(t^4).$$

The zeroes of $\mathrm{d}x$ are located at $z=\pm 1$, and the deck transformation is $\iota(z) = \frac{1}{z}$. The zeroes of $\mathrm{d}w$ are located at $z = \pm c^2\sqrt{3t}$, and the deck transformation is

$$\check{\iota}(z) = \frac{c^2 z(c^2 t + \sqrt{4tz^2 - 3c^4t^2})}{2(z^2 - tc^4)}.$$

Consider the multidifferentials $\omega_{g,n}$ and $\check{\omega}_{g,n}$ on $\mathbb{P}^1$ as at the beginning of the section but with initial data specialized for quadrangulations. We define

$$\begin{aligned}F^{[g]}_{\ell_1,\ldots,\ell_n} &= (-1)^n \operatorname*{Res}_{z_1\to\infty}(x(z_1))^{\ell_1}\cdots\operatorname*{Res}_{z_n\to\infty}(x(z_n))^{\ell_n}\left(\omega_{g,n}(z_1,\ldots,z_n) - \delta_{g,0}\delta_{n,2}\frac{\mathrm{d}x(z_1)\mathrm{d}x(z_2)}{(x(z_1)-x(z_2))^2}\right),\\ \check{F}^{[g]}_{k_1,\ldots,k_n} &= \operatorname*{Res}_{z_1\to\infty}(w(z_1))^{-k_1}\cdots\operatorname*{Res}_{z_n\to\infty}(w(z_n))^{-k_n}\left(\check{\omega}_{g,n}(z_1,\ldots,z_n) - \delta_{g,0}\delta_{n,2}\frac{\mathrm{d}w(z_1)\mathrm{d}w(z_2)}{(w(z_1)-w(z_2))^2}\right),\end{aligned}$$

and we know that $\check{F}^{[g]}_{k_1,\ldots,k_n} = H^{[g]}_{k_1,\ldots,k_n}$.

5.2.1. *Disks.* We explore two tables to compare the coefficients $[t^Q]\, F_\ell$ and $[t^Q]\, \check{F}_k$. It is remarkable that all the $[t^Q]\, \check{F}_k$ are nonnegative integers, which already suggested a priori that they may be counting some objects. Theorem 5.4 identifies $[t^Q]\, \check{F}_k$ with the number of (fully) simple disks $[t^Q]\, H_k$.

If the length of the boundary is odd, the number of disks is obviously 0.

Observe that if we consider a boundary of length $\ell = 2$, the number of ordinary disks is equal to the number of (fully) simple disks because the only two possible boundaries of length 2 are simple in genus 0. If the two vertices get identified in the non-degenerate case, either the genus is increased or an internal face of length 1 appears, which is not possible because we are counting quadrangulations.

ℓ	$Q=0$	1	2	3	4	5	6	7	8
2	1	2	9	54	378	2916	24057	208494	1876446
4	2	9	54	378	2916	24057	208494	1876446	17399772
6	5	36	270	2160	18225	160380	1459458	13646880	130489290
8	14	140	1260	11340	103950	972972	9287460	90221040	890065260

FIGURE 7. Number of ordinary disks with boundary of length ℓ and Q quadrangles.

We also remark that the $[t^Q]\, H_k$ in the second table are (much) smaller than the corresponding $[t^Q]\, F_k$, and for small number of quadrangles some of them are 0, which makes sense due to the strong geometric constraints to form maps with simple boundaries and a small number of internal faces.

k	$Q=0$	1	2	3	4	5	6	7	8
2	1	2	9	54	378	2916	24057	208494	1876446
4	0	1	10	90	810	7425	69498	663390	6444360
6	0	0	3	56	756	9072	103194	1143072	12492144
8	0	0	0	12	330	5940	89100	1211760	15540822

FIGURE 8. Number of simple disks with boundary of length k and Q quadrangles.

5.2.2. *Cylinders.* We explore now the number of cylinders imposing different constraints to the boundaries: $[t^Q]\, F_{\ell_1,\ell_2}$ and $[t^Q]\, H_{k_1,k_2}$, and also $[t^Q]\, G_{k_1|\ell_1}$ and $[t^Q]\, G_{k_1,k_2|}$ given by the formulas (4.5) and (4.6) respectively.

Since we know how to convert an unmarked quadrangle into an ordinary boundary of length 4, we can relate the outcomes for cylinders with at least one of the boundaries being ordinary of length 4 to the previous results for disks:

- $4\frac{\partial}{\partial t}F_{\ell_1} = F_{\ell_1,4} \quad \Rightarrow \quad 4Q[t^Q]\, F_{\ell_1} = [t^{Q-1}]\, F_{\ell_1,4},$
- $4\frac{\partial}{\partial t}G_{k_1} = G_{k_1|4} \quad \Rightarrow \quad 4Q[t^Q]\, G_{k_1} = [t^{Q-1}]\, G_{k_1|4}.$

If the sum of the lengths of the two boundaries is odd, the number of quadrangulations is obviously 0.

Observe that the results also satisfy the following inequalities:

$$[t^Q]\, F_{l_1,l_2} \geq [t^Q]\, G_{l_1|l_2} \geq [t^Q]\, G_{l_1,l_2} \geq [t^Q]\, H_{l_1,l_2},$$

which are compatible with the combinatorial interpretation that Theorem 5.4 offers, since we are imposing further constraints whenever we force a boundary to be simple or, even more, fully simple.

We also obtain more and more zeroes for small number of quadrangles as we impose stronger conditions on the boundaries.

(ℓ_1, ℓ_2)	$Q=0$	1	2	3	4	5	6	7	8
$(\mathbf{1},\mathbf{1})$	1	3	18	135	1134	10206	96228	938223	9382230
$(\mathbf{3},\mathbf{1})$	3	18	135	1134	10206	96228	938223	9382230	95698746
$(\mathbf{5},\mathbf{1})$	10	90	810	7560	72900	721710	7297290	75057840	782989740
$(\mathbf{7},\mathbf{1})$	35	420	4410	45360	467775	4864860	51081030	541326240	5785424190
$(\mathbf{9},\mathbf{1})$	126	1890	22680	255150	2806650	30648618	334348560	3653952120	40052936700
$(\mathbf{2},\mathbf{2})$	2	12	90	756	6804	64152	625482	6254820	63799164
$(\mathbf{4},\mathbf{2})$	8	72	648	6048	58320	577368	5837832	60046272	626391792
$(\mathbf{6},\mathbf{2})$	30	360	3780	38880	400950	4169880	43783740	463993920	4958935020
$(\mathbf{8},\mathbf{2})$	112	1680	20160	226800	2494800	27243216	297198720	3247957440	335602610400
$(\mathbf{3},\mathbf{3})$	12	108	972	9072	87480	866052	8756748	90069408	939587688
$(\mathbf{5},\mathbf{3})$	45	540	5670	58320	601425	6254820	65675610	695990880	7438402530
$(\mathbf{7},\mathbf{3})$	168	2520	30240	340200	3742200	40864824	445798080	4871936160	53403915600
$(\mathbf{9},\mathbf{3})$	630	11340	153090	1871100	21891870	250761420	2841962760	32042349360	360476430300
$(\mathbf{4},\mathbf{4})$	36	432	4536	46656	481140	5003856	52540488	556792704	5950722024
$(\mathbf{6},\mathbf{4})$	144	2160	25920	291600	3207600	35026992	382112640	4175945280	45774784800
$(\mathbf{8},\mathbf{4})$	560	10080	136080	1663200	19459440	222899040	2526189120	28482088320	320423493600

FIGURE 9. Number of ordinary cylinders with boundaries of lengths (ℓ_1, ℓ_2) and Q quadrangles: $[t^Q]\, F_{\ell_1,\ell_2}$.

Observe that forcing a boundary of length 1 or 2 to be simple does not have any effect in the planar case and therefore the corresponding rows in the first three tables coincide. However, imposing that the cylinder is fully simple is much stronger, so in the last table (Figure 12) all the entries are (much) smaller.

5.2.3. *Tori with 1 boundary.* We compute

$$\omega_{1,1}(z) = \frac{z^3(tc^4z^4 + z^2(1-5tc^4) + tc^4)}{c(z^2-1)^5(1-3tc^4)^2}\,\mathrm{d}z,$$

$$\check{\omega}_{1,1}(z) = \frac{3t^2c^9z^5[(3tc^4-2)z^4 + 3tc^4(9tc^4-1)z^2 - 27t^3c^{12}]}{(3tc^4-z^2)^5(1-3tc^4)^2}\,\mathrm{d}z.$$

We present the number of tori with 1 ordinary boundary of perimeter ℓ and Q other faces which are quadrangles, as given by Theorem 5.2.

For comparison, we present the coefficients $[t^Q]\, H_k^{[1]}$ in the same range. Again, it is remarkable that they are all nonnegative and integers; Theorem 5.3 gives a combinatorial interpretation for them. We also remark that they are always (much) smaller than the corresponding $[t^Q]\, F_\ell^{[1]}$, and

(k_1,ℓ_1)	$Q=0$	1	2	3	4	5	6	7	8
$(\mathbf{1},\mathbf{1})$	1	3	18	135	1134	10206	96228	938223	9382230
$(\mathbf{3},\mathbf{1})$	0	3	36	378	3888	40095	416988	4378374	46399392
$(\mathbf{5},\mathbf{1})$	0	0	15	315	4725	62370	773955	9287460	109306260
$(\mathbf{7},\mathbf{1})$	0	0	0	84	2520	49140	793800	11566800	158233824
$(\mathbf{9},\mathbf{1})$	0	0	0	0	495	19305	463320	8860995	148551975
$(\mathbf{2},\mathbf{2})$	2	12	90	756	6804	64152	625482	6254820	63799164
$(\mathbf{4},\mathbf{2})$	0	8	120	1440	16200	178200	1945944	21228480	231996960
$(\mathbf{6},\mathbf{2})$	0	0	42	1008	16632	235872	3095820	38864448	474701472
$(\mathbf{8},\mathbf{2})$	0	0	0	240	7920	166320	2851200	43623360	621632880
$(\mathbf{1},\mathbf{3})$	3	18	135	1134	10206	96228	938223	9382230	95698746
$(\mathbf{3},\mathbf{3})$	3	36	378	3888	40095	416988	4378374	46399392	495893502
$(\mathbf{5},\mathbf{3})$	0	15	315	4725	62370	773955	9287460	109306260	1271521800
$(\mathbf{7},\mathbf{3})$	0	0	84	2520	49140	793800	11566800	158233824	2076818940
$(\mathbf{9},\mathbf{3})$	0	0	0	495	19305	463320	8860995	148551975	2287700415
$(\mathbf{2},\mathbf{4})$	8	72	648	6048	58320	577368	5837832	60046272	626391792
$(\mathbf{4},\mathbf{4})$	4	80	1080	12960	148500	1667952	18574920	206219520	2288739240
$(\mathbf{6},\mathbf{4})$	0	24	672	12096	181440	2476656	32006016	399748608	4882643712
$(\mathbf{8},\mathbf{4})$	0	0	144	5280	118800	2138400	33929280	497306304	6911094960

FIGURE 10. Number of cylinders with the first boundary simple of length k_1 and the second boundary ordinary of length ℓ_1: $[t^Q]\, G_{k_1|\ell_1}$.

(ℓ_1,ℓ_2)	$Q=0$	1	2	3	4	5	6	7	8
$(\mathbf{1},\mathbf{1})$	1	3	18	135	1134	10206	96228	938223	9382230
$(\mathbf{1},\mathbf{3})$	0	3	36	378	3888	40095	416988	4378374	46399392
$(\mathbf{1},\mathbf{5})$	0	0	15	315	4725	62370	773955	9287460	109306260
$(\mathbf{1},\mathbf{7})$	0	0	0	84	2520	49140	793800	11566800	158233824
$(\mathbf{1},\mathbf{9})$	0	0	0	0	495	19305	463320	8860995	148551975
$(\mathbf{2},\mathbf{2})$	2	12	90	756	6804	64152	625482	6254820	63799164
$(\mathbf{2},\mathbf{4})$	0	8	120	1440	16200	178200	1945944	21228480	231996960
$(\mathbf{2},\mathbf{6})$	0	0	42	1008	16632	235872	3095820	38864448	474701472
$(\mathbf{2},\mathbf{8})$	0	0	0	240	7920	166320	2851200	43623360	621632880
$(\mathbf{3},\mathbf{3})$	3	27	252	2457	24705	253935	2653560	28089828	300480678
$(\mathbf{3},\mathbf{5})$	0	15	270	3690	45900	547560	6395760	73862280	847681200
$(\mathbf{3},\mathbf{7})$	0	0	84	2268	41076	628992	8808912	116940348	1499730876
$(\mathbf{3},\mathbf{9})$	0	0	0	495	17820	402435	7341840	118587645	1772680140
$(\mathbf{4},\mathbf{4})$	4	48	536	5952	66132	735696	8196552	91476864	1022868648
$(\mathbf{4},\mathbf{6})$	0	24	504	7728	105336	1354752	16855776	205426368	2469577896
$(\mathbf{4},\mathbf{8})$	0	0	144	4320	85200	1401120	20856960	291942144	3922233840

FIGURE 11. Number of simple cylinders with boundaries of lengths (k_1, k_2) and Q quadrangles: $[t^Q]\, G_{k_1,k_2}$.

that some of them for small number of quadrangles are 0, which indicates as before that they may be counting a subclass of ordinary maps.

Due to the strong geometric constraints to form maps with simple boundaries and few internal faces, our observations support that the $[t^Q]\, H_k^{[1]}$ are indeed counting fully simple tori with Q quadrangles.

(ℓ_1, ℓ_2)	$Q = 0$	1	2	3	4	5	6	7	8
$(\mathbf{1}, \mathbf{1})$	0	1	9	81	756	7290	72171	729729	7505784
$(\mathbf{1}, \mathbf{3})$	0	0	6	108	1458	17820	208494	2388204	27066312
$(\mathbf{1}, \mathbf{5})$	0	0	0	35	945	17010	257985	3572100	46845540
$(\mathbf{1}, \mathbf{7})$	0	0	0	0	210	7560	170100	3084480	49448070
$(\mathbf{1}, \mathbf{9})$	0	0	0	0	0	1287	57915	1563705	33011550
$(\mathbf{2}, \mathbf{2})$	0	0	6	108	1458	17820	208494	2388204	27066312
$(\mathbf{2}, \mathbf{4})$	0	0	0	40	1080	19440	294840	4082400	53537760
$(\mathbf{2}, \mathbf{6})$	0	0	0	0	252	9072	204120	3701376	59337684
$(\mathbf{2}, \mathbf{8})$	0	0	0	0	0	1584	71280	1924560	40629600
$(\mathbf{3}, \mathbf{3})$	0	0	0	48	1296	23328	353808	4898880	64245312
$(\mathbf{3}, \mathbf{5})$	0	0	0	0	315	11340	255150	4626720	74172105
$(\mathbf{3}, \mathbf{7})$	0	0	0	0	0	2016	90720	2449440	51710400
$(\mathbf{3}, \mathbf{9})$	0	0	0	0	0	0	12870	694980	21891870
$(\mathbf{4}, \mathbf{4})$	0	0	0	0	300	10800	243000	4406400	70640100
$(\mathbf{4}, \mathbf{6})$	0	0	0	0	0	2016	90720	2449440	51710400
$(\mathbf{4}, \mathbf{8})$	0	0	0	0	0	0	13200	712800	22453200

FIGURE 12. Number of fully simple cylinders with boundaries of lengths (k_1, k_2) and Q quadrangles: $[t^Q]\, H_{k_1,k_2}$.

ℓ	$Q = 0$	1	2	3	4	5	6	7	8
2	0	1	15	198	2511	31266	385398	4721004	57590271
4	1	15	198	2511	31266	385398	4721004	57590271	700465482
6	10	150	1980	25110	312660	3853980	47210040	575902710	7004654820
8	70	1190	16590	216720	2748060	34286480	423600030	5199957000	63549802260
10	420	8190	122850	1678320	21925890	279389250	3505914090	43551655560	537235675200
12	2310	51282	831600	11962566	162074682	2121490602	27174209832	343061095608	4287091638060
14	12012	300300	5261256	79891812	1126377252	15198795612	199385314128	2565902298960	32572738238040

FIGURE 13. $[t^Q]\, F_\ell^{[1]}$.

k	$Q = 0$	1	2	3	4	5	6	7	8
2	0	0	6	117	1755	23976	313227	3991275	50084487
4	0	0	0	105	2925	55215	885330	13009005	181316880
6	0	0	0	0	1260	46116	1065960	19983348	332470656
8	0	0	0	0	0	12870	585090	16073640	346928670
10	0	0	0	0	0	0	120120	6531525	208243035
12	0	0	0	0	0	0	0	1058148	66997476
14	0	0	0	0	0	0	0	0	8953560

FIGURE 14. $[t^Q]\, H_k^{[1]}$.

6. Generalization to stuffed maps

6.1. **Review of definitions.** We introduce stuffed maps as in [7], which encompass the previously studied maps since by substitution one may consider them as maps whose elementary cells are themselves maps. We will also see that all our results can be generalized to stuffed maps.

Definition 6.1. An *elementary 2-cell* of genus h and k boundaries of lengths $m_1, \ldots, m_k$ is a connected orientable surface of genus h with boundaries $B_1, \ldots, B_k$ endowed with a set $V_i \subset B_i$ of $m_i \geq 1$ vertices. The connected components of $B_i \setminus V_i$ are called *edges*. We require that each boundary has a marked edge, called the root, and by following the cyclic order, the rooting induces a labeling of the edges of the boundaries. We say that such an elementary 2-cell is *of topology* (h, k).

A *stuffed map* of genus g and n boundaries of lengths $l_1, \ldots, l_n$ is the object obtained from gluing n labeled elementary 2-cells of topology $(0, 1)$ with boundaries of lengths $l_1, \ldots, l_n$, and a finite collection of unlabeled elementary 2-cells by identifying edges of opposite orientation and with the same label in such a way that the resulting surface has genus g. The labeled cells are considered as *boundaries* of the stuffed map, and the marked edges which do not belong to the boundary are forgotten after gluing.

A map in the usual sense is a stuffed map composed only of elementary 2-cells with the topology of a disk.

We denote $\widehat{\mathbb{M}}^{[g]}_{l_1,\ldots,l_n}$ the set of stuffed maps of genus g and n boundaries of lengths $l_1, \ldots, l_n$.

To every stuffed map M we assign a Boltzmann weight as follows:

- a symmetry factor $|\mathrm{Aut}(M)|^{-1}$ as previously for maps,
- a weight $t^h_{m_1,\ldots,m_k}$ per unlabeled elementary 2-cell of genus h and k boundaries, depending symmetrically on the lengths $\mathbf{m} = (m_1, \ldots, m_k)$.

Slightly extending the notation, we add a hat to the previous symbols to denote the generating series of stuffed maps of topology (g, n):

$$\widehat{W}^{[g]}_n(x_1, \ldots, x_n) = \frac{\delta_{g,0}\delta_{n,1}}{x_1} + \sum_{l_1,\ldots,l_n \geq 1} \left(\prod_{j=1}^{n} x_j^{-(l_j+1)} \right) \left(\sum_{M \in \widehat{\mathbb{M}}^{g}_{l_1,\ldots,l_n}} \mathrm{weight}(M) \right).$$

We have that $\widehat{W}^{[g]}_n(x_1, \ldots, x_n) \in \mathbb{Q}\left[(x_j^{-1})_j\right][[(t^h_{\mathbf{m}})_{\mathbf{m},h}]]$.

A slight generalization of the permutational model for maps also works for stuffed maps. A *combinatorial stuffed map* $((\sigma, \alpha), \bigsqcup_{p=1}^{F} f_p, (h_p)_{p=1}^{F})$ consists of the following data:

- As for maps, a pair of permutations (σ, α) on the set of half-edges $H = H^u \sqcup H^\partial$, where α is a fixed-point free involution whose cycles represent the edges of the stuffed map, and $\mathcal{C}(\sigma)$ corresponds to the set of vertices. The cycles of $\varphi := (\sigma \circ \alpha)^{-1}$ are associated to the boundaries of elementary 2-cells.
- A partition $\bigsqcup_{p=1}^{F} f_p$ of $\mathcal{C}(\varphi|_{H^u})$, where every part f_p corresponds to an unlabeled elementary 2-cell with boundaries given by the cycles in f_p.
- A sequence of non-negative integers $(h_p)_{p=1}^{F}$, where every h_p is the genus of the unlabeled elementary 2-cell f_p.

F is the number of unlabeled elementary 2-cells, considered as the internal faces of the stuffed map.

To define the notion of connectedness for stuffed maps, we consider the following equivalence relation on the set of half-edges:

(i) $h \sim \sigma(h)$ and $h \sim \alpha(h)$,
(ii) if h, h' are in two cycles $c, c' \in f_p$ for some p, then $h \sim h'$.

Each equivalence class on H corresponds to a connected component of the stuffed map. We say that a stuffed map is *∂-connected* if each equivalence class has a non-empty intersection with H^∂. Observe that the notion of connectedness for maps relies on the equivalence class generated only by (i).

Since the concepts of simplicity and fully simplicity that we introduced for maps in Section 2 only refer to properties of the boundaries, they clearly extend to stuffed maps because the elementary 2-cells corresponding to boundaries in stuffed maps are imposed to be of the topology of a disk as for maps. Furthermore, all the results in Sections 3 and 4 for the generating series of maps also do not concern the inner faces, they only affect the boundaries. As a consequence, all these statements were still true in the more general setting of stuffed maps.

6.2. **Conjecture for maps carrying a loop model.** Usual maps carrying self-avoiding loop configurations are equivalent to stuffed maps for which we allow unlabeled elementary 2-cells to have the topology of a disk (usual faces) or of a cylinder (rings of faces carrying the loops). By equivalence here, we mean an equality of generating series after a suitable change of formal variables.

It is known [10, 12] that the generating series of ordinary maps with loops obey the topological recursion, with initial data $\omega_{0,1}$ and $\omega_{0,2}$ again given by the generating series of disks and cylinders. As of now, explicit expressions for $\omega_{0,1}$ and $\omega_{0,2}$ are only known for a restricted class of model with loops, *e.g.* those in which loops cross only triangle faces [37, 29, 10] maybe taking into account bending [9].

Theorems 5.1 and 5.2 apply to maps carrying a loop model. However, the analog of Theorem 5.3 is for the moment just a conjecture.

Conjecture 6.2. *After the symplectic transformation $(x, w) \to (-w, x)$ in the initial data of TR for ordinary maps with loops, the TR amplitudes enumerate fully simple maps carrying a loop model.*

A proof of this conjecture could be given along the lines of Section 9 if one could first establish that the topological recursion governs the topological expansion in the formal matrix model

$$\mathrm{d}\mu(M) = \mathrm{d}M \exp\left(N \mathrm{Tr}(MA) - N \,\mathrm{Tr}\,\frac{M^2}{2} + \sum_{h\geq 0}\sum_{k\geq 1}\sum_{d\geq 1} N\,\frac{t_d}{d}\,\mathrm{Tr}\,M^d + \sum_{d_1,d_2\geq 1}\frac{t_{d_1,d_2}}{d_1 d_2}\,\mathrm{Tr}\,M^{d_1}\,\mathrm{Tr}\,M^{d_2}\right)$$

which depends on the external hermitian matrix A.

According to our previous remark, the conjecture is true for disks and cylinders due to the validity of our Theorem 5.4.

6.3. **Vague conjecture for stuffed maps.** It was proved in [7] that the generating series of ordinary stuffed maps satisfy the so-called blobbed topological recursion, which was axiomatized in [13]. In this generalized version of the topological recursion, the invariants $\omega_n^{[g]}$ are determined by $\omega_1^{[0]}$ and $\omega_2^{[0]}$ as before, and additionally by the so-called blobs $\varphi_n^{[g]}$ for stable topologies, $2g - 2 + n > 0$. We conjecture that, after the same symplectic change of variables, and a transformation of the blobs *still to be described*, the blobbed topological recursion will enumerate fully simple stuffed maps. Again, according to our previous remark, this conjecture is true for disks and cylinders (whose expression do not involve the blobs).

Part 2. Matrix model interpretation

7. Ordinary *vs* fully simple for unitarily invariant random matrix models

We consider an arbitrary measure $\mathrm{d}\mu(M)$ on the space $\mathcal{H}(N)$ of $N \times N$ hermitian matrices which is invariant under conjugation by a unitary matrix. If $\mathcal{O}$ is a polynomial function of the entries of M, we denote $\langle \mathcal{O}(M)\rangle$ its expectation value with respect to $\mathrm{d}\mu(M)$:

$$\langle \mathcal{O}(M)\rangle = \frac{\int_{\mathcal{H}(N)} \mathrm{d}\mu(M)\, \mathcal{O}(M)}{\int_{\mathcal{H}(N)} \mathrm{d}\mu(M)}.$$

And, if $\mathcal{O}_1,\ldots,\mathcal{O}_n$ are polynomial functions of the entries of M, we denote $\kappa_n(\mathcal{O}_1(M),\ldots,\mathcal{O}_n(M))$ their cumulant with respect to the measure $\mathrm{d}\mu(M)$.

If $\gamma=(c_1\,c_2\,\ldots\,c_\ell)$ is a cycle of the symmetric group $\mathfrak{S}_N$, we denote

$$\mathcal{P}_\gamma(M) := M_{c_1,c_2}\,M_{c_2,c_3}\cdots M_{c_{\ell-1},c_\ell}\,M_{c_\ell,c_1} = \prod_{m=1}^{\ell} M_{c_m,\gamma(c_m)}.$$

We denote $l(\gamma)$ the length of the cycle γ.

We will be interested in two types of expectation values:

$$\Big\langle \prod_{i=1}^n \mathrm{Tr}\,M^{L_i}\Big\rangle \qquad \text{and} \qquad \Big\langle \prod_{i=1}^n \mathcal{P}_{\gamma_i}(M)\Big\rangle, \tag{7.1}$$

where $(L_i)_{i=1}^n$ is a sequence of nonnegative integers, and $(\gamma_i)_{i=1}^n$ is a sequence of pairwise disjoint cycles in $\mathfrak{S}_N$ with $l(\gamma_i)=L_i$ – the latter imposes N to be larger than $L=\sum_{i=1}^n l(\gamma_i)$. The first type of expectation value will be called *ordinary*, and the second one *fully simple*. The terms may be used for the "disconnected" version (7.1), or for the "connected" version obtained by taking the cumulants instead of the expectation values of the product. This terminology will be justified by their combinatorial interpretation in terms of ordinary and fully simple maps in Section 8.

Remark 7.1. The unitary invariance of μ implies its invariance under conjugation of M by a permutation matrix of size N. As a consequence, fully simple expectation values only depend on the conjugacy class of the permutation $\gamma_1\cdots\gamma_n$, thus on the partition λ encoding the lengths ℓ_i of γ_i. We can then use without ambiguity the following notations:

$$\Big\langle \textstyle\prod_{i=1}^n \mathcal{P}_{\gamma_i}(M)\Big\rangle = \big\langle \mathcal{P}_\lambda(M)\big\rangle = \Big\langle \textstyle\prod_{i=1}^n \mathcal{P}^{(\ell_i)}(M)\Big\rangle, \text{ and}$$
$$\kappa_n(\mathcal{P}_{\gamma_1}(M),\ldots,\mathcal{P}_{\gamma_n}(M)) = \kappa_n\big(\mathcal{P}^{(\ell_1)}(M),\ldots,\mathcal{P}^{(\ell_n)}(M)\big).$$

If $N<L$, we convene that these quantities are zero.

7.1. **Weingarten calculus.** If the (formal) measure on M is invariant under conjugation by a unitary matrix of size N, it should be possible to express the fully simple observables in terms of the ordinary ones – independently of the measure on M. This precise relation will be described in Theorem 7.8. We first introduce the representation theory framework which proves and explains this result.

7.1.1. *Preliminaries on symmetric functions.* The character ring of $\mathrm{GL}(N,\mathbb{C})$ – i.e. polynomial functions of the entries of M, which are invariant by conjugation – generated by $p_l(M)=\mathrm{Tr}\,M^l$ for $l\geq 0$. It is isomorphic to the ring of symmetric functions in L variables

$$\mathcal{B}_N = \mathbb{Q}[x_1,\ldots,x_N]^{\mathfrak{S}_N}$$

tensored over $\mathbb{C}$.

If λ is a partition, we denote $\mathbb{Y}_\lambda$ the corresponding Young diagram, $|\lambda|$ its number of boxes and $\ell(\lambda)$ its number of rows. By convention, we consider the empty partition $\lambda=\emptyset$ as the partition of 0. If β is a permutation of L elements, we denote $[\beta]$ its conjugacy class, and $|\beta|=|[\beta]|$ the number of elements in this conjugacy class. Associated to the permutation β, we can form a partition $\lambda=\lambda_{[\beta]}$ – by collecting the lengths of cycles in β – for which we have $|\mathcal{C}(\beta)|=\ell(\lambda_{[\beta]})$. Recall that actually the set of conjugacy classes in $\mathfrak{S}_L$ is in bijection with the set of partitions of L. We also denote $t(\beta)=t([\beta])=L-\ell(\lambda_{[\beta]})$, which can be checked to be the minimal number of transpositions in a factorization of β. We sometimes use the notation C_λ for the conjugacy class in $\mathfrak{S}_L$ described by the partition λ, and $\ell(C_\lambda)$ for $\ell(\lambda)$. We denote

$$|\mathrm{Aut}\,\lambda| := \frac{L!}{|C_\lambda|}, \qquad L = |\lambda|.$$

The power sum functions $p_{[\beta]}(M) = p_\lambda(M) := \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(M)$ with $\ell(\lambda) \leq N$ form a linear basis of the character ring of $\mathrm{GL}(N,\mathbb{C})$. Another linear basis is formed by the Schur functions $s_\lambda(M)$ with $\ell(\lambda) \leq N$, which have the following expansion in terms of power sum functions:

$$s_\lambda(M) = \frac{1}{L!} \sum_{\mu \vdash L} |C_\mu|\, \chi_\lambda(C_\mu)\, p_\mu(M), \qquad L = |\lambda|, \tag{7.2}$$

where χ_λ are the characters of $\mathfrak{S}_L$.

$\mathcal{B}_L$ are graded rings, where the grading comes from the total degree of a polynomial. We will work with the graded ring of symmetric polynomials in infinitely many variables, defined as

$$\mathcal{B} = \lim_{\infty \leftarrow L} \mathcal{B}_L$$

for the restriction morphisms $\mathcal{B}_{L+1} \to \mathcal{B}_L$ sending $p(x_1,\ldots,x_{L+1})$ to $p(x_1,\ldots,x_L,0)$. By construction, if $r \in \mathcal{B}$, it determines for any $L \geq 0$ an element $\iota_L[r] \in \mathcal{B}_L$ by setting

$$\iota_L[r](x_1,\ldots,x_L) = r(x_1,\ldots,x_L,0,0,\ldots).$$

We often abuse notation and write $r(x_1,\ldots,x_L)$ for this restriction to L variables. In fact, $\mathcal{B}$ is a free graded ring over $\mathbb{Q}$ with one generator p_k in each degree $k \geq 1$. The power sums p_λ and the Schur elements s_λ are two homogeneous linear basis for $\mathcal{B}$, abstractly related *via* (7.2). A description of the various bases for $\mathcal{B}$ and their properties in relation to representation theory can be found in [36].

Let $\mathcal{B}^{(d)}$ denote the (finite-dimensional) subspace of homogeneous elements of $\mathcal{B}$ of degree d. We later need to consider the tensor product of $\mathcal{B}$ with itself, defined by

$$\mathcal{B}\hat{\otimes}\mathcal{B} := \bigoplus_{d \geq 0} \Big(\bigoplus_{d_1+d_2=d} \mathcal{B}^{(d_1)} \otimes \mathcal{B}^{(d_2)} \Big).$$

7.1.2. *Moments of the Haar measure.* Unlike $\prod_{i=1}^n \mathrm{Tr}\, M^{L_i}$, the expression $\prod_{i=1}^n \mathcal{P}_{\gamma_i}(M)$ is not unitarily invariant. However, the unitary invariance of the measure implies that

$$\Big\langle \prod_{i=1}^n \mathcal{P}_{\gamma_i}(M) \Big\rangle = \Big\langle \int_{\mathcal{U}(N)} \mathrm{d}U \prod_{i=1}^n \mathcal{P}_{\gamma_i}(UMU^\dagger) \Big\rangle, \tag{7.3}$$

where $\mathrm{d}U$ is the Haar measure on the unitary group. Moments of the entries of a random unitary matrix distributed according to the Haar measure can be computed in terms of representation theory of the symmetric group: this is Weingarten calculus [20]. If $N \geq 1$ and $L \geq 0$ are two integers, the Weingarten function is defined as

$$G_{N,L}(\beta) := \frac{1}{L!^2} \sum_{\lambda \vdash L} \frac{\chi_\lambda(\mathrm{id})^2 \chi_\lambda(\beta)}{s_\lambda(1_N)}, \quad \text{for } \beta \in \mathfrak{S}_L.$$

Note that it only depends on the conjugacy class of β.

Theorem 7.2. [20]

$$\int_{\mathcal{U}(N)} \mathrm{d}U \Big(\prod_{l=1}^L U_{a_l,b_l} U^\dagger_{b'_l,a'_l} \Big) = \sum_{\beta,\tau \in \mathfrak{S}_L} \Big(\prod_{l=1}^L \delta_{a_l,a'_{\beta(l)}} \delta_{b_l,b'_{\tau(l)}} \Big) G_{N,L}(\beta\tau^{-1}).$$

7.1.3. *From fully simple to ordinary.* We will use this formula to compute (7.3). Let $(\gamma_i)_{i=1}^n$ be pairwise disjoint cycles:

$$\gamma_i = (j_{i,1} \quad j_{i,2} \quad \dots \quad j_{i,L_i}).$$

We denote $H^\partial = \bigsqcup_{i=1}^n \{i\} \times (\mathbb{Z}/L_i\mathbb{Z})$,

$$L = |H^\partial| = \sum_{i=1}^n L_i$$

and $\varphi^\partial \in \mathfrak{S}_{H^\partial}$ the product of the cyclic permutations sending (i,l) to $(i, l+1 \bmod L_i)$. Our notations here are motivated by the fact that when we take a certain specialization of the measure $\mathrm{d}\mu$ in Section 8, H^∂ will refer to the set of half-edges belonging to boundaries of a map and φ^∂ will be the permutation whose cycles correspond to the boundaries.

Proposition 7.3.

$$\Big\langle \prod_{i=1}^n \mathcal{P}_{\gamma_i}(M) \Big\rangle = \sum_{\mu \vdash L} \tilde{G}_{N,L}(C_\mu, \varphi^\partial)\, \langle p_\mu(M) \rangle = \sum_{\lambda \vdash L} \frac{\chi_\lambda(\varphi^\partial)\chi_\lambda(\mathrm{id})}{L!\, s_\lambda(1_N)}\, \langle s_\lambda(M) \rangle,$$

with

$$\tilde{G}_{N,L}(C,\beta) = \frac{1}{L!^2} \sum_{\lambda \vdash L} |C| \chi_\lambda(C) \chi_\lambda(\beta)\, \frac{\chi_\lambda(\mathrm{id})}{s_\lambda(1_N)}, \quad \textit{for } \beta \in \mathfrak{S}_L.$$

Proof. If M is a hermitian matrix, we denote Λ its diagonal matrix of eigenvalues – defined up to permutation. We then have

$$\int \mathrm{d}U \prod_{i=1}^n \mathcal{P}_{\gamma_i}(UMU^\dagger) = \sum_{\substack{1 \leq a_{i,l} \leq N \\ (i,l) \in H^\partial}} \int_{\mathcal{U}(N)} \mathrm{d}U \prod_{(i,l) \in H^\partial} \Lambda_{a_{i,l}} U_{j_{i,l}, a_{i,l}} U^\dagger_{a_{i,l}, j_{\varphi^\partial(i,l)}},$$

in which we can substitute Theorem 7.2. We obtain a sum over $\rho, \tau \in \mathfrak{S}_{H^\partial}$ of terms involving

$$\sum_{\substack{1 \leq a_{i,l} \leq N \\ (i,l) \in H^\partial}} \prod_{(i,l) \in H^\partial} \Lambda_{a_{i,l}} \delta_{j_{i,l}, j_{\rho(\varphi^\partial(i,l))}} \delta_{a_{i,l}, a_{\tau(i,l)}} = p_{[\tau]}(\Lambda) \prod_{(i,l) \in H^\partial} \delta_{j_{i,l}, j_{\rho(\varphi^\partial(i,l))}}.$$

As $p_{[\tau]}(\Lambda)$ is unitarily invariant, it is also equal to $p_{[\tau]}(M)$. Since we assumed the $j_{i,l}$ pairwise disjoint, this is non-zero only if $\rho = (\varphi^\partial)^{-1}$. Therefore

$$\Big\langle \prod_{i=1}^n \mathcal{P}_{\gamma_i}(M) \Big\rangle = \sum_{\tau \in \mathfrak{S}_{H^\partial}} \langle p_{[\tau]}(M) \rangle\, G_{N,L}\big((\varphi^\partial)^{-1}\tau^{-1}\big) = \sum_{\mu \vdash L} \langle p_\mu(M) \rangle\, \tilde{G}_{N,L}(C_\mu, \varphi^\partial), \tag{7.4}$$

with

$$\tilde{G}_{N,L}(C,\beta) = \sum_{\tau \in C} G_{N,L}(\beta^{-1}\tau),$$

as τ and τ^{-1} are in the same conjugacy class. To go further, we recall the Frobenius formula:

Lemma 7.4. *See e.g.* [68, Theorem 2]. *If $C_1, \dots, C_k$ are conjugacy classes of $\mathfrak{S}_L$, the number of permutations $\beta_i \in C_i$ such that $\beta_1 \circ \cdots \circ \beta_L = \mathrm{id}$ is*

$$\mathcal{N}(C_1, \dots, C_k) = \frac{1}{L!} \sum_{\lambda \vdash L} \frac{\prod_{i=1}^k |C_i| \chi_\lambda(C_i)}{\chi_\lambda(\mathrm{id})^{k-2}}.$$

Since $G_{N,L}(\beta)$ only depends on the conjugacy class of β, we compute:

$$\begin{aligned}\tilde{G}_{N,L}(C,\beta) &= \frac{1}{|\beta|}\sum_{\mu\vdash L}\mathcal{N}(C,C_\mu,[\beta])\,G_{N,L}(C_\mu)\\ &= \sum_{\mu,\lambda,\lambda'\vdash L}\left(\frac{|C|\chi_{\lambda'}(C)\,|C_\mu|\chi_{\lambda'}(C_\mu)\chi_{\lambda'}(\beta)}{L!\,\chi_{\lambda'}(\mathrm{id})}\right)\frac{\chi_\lambda(\mathrm{id})^2\chi_\lambda(C_\mu)}{s_\lambda(1_N)\,L!^2}.\end{aligned}$$

The orthogonality of characters of the symmetric group gives

$$\frac{1}{L!}\sum_{\mu\vdash L}|C_\mu|\,\chi_{\lambda'}(C_\mu)\chi_\lambda(C_\mu)=\delta_{\lambda,\lambda'}.$$

Therefore

$$\tilde{G}_{N,L}(C,\beta)=\frac{1}{L!^2}\sum_{\lambda\vdash L}|C|\chi_\lambda(C)\chi_\lambda(\beta)\,\frac{\chi_\lambda(\mathrm{id})}{s_\lambda(1_N)}.$$

The claim in terms of Schur functions is found by performing the sum over conjugacy classes C in (7.4) with the help of (7.2). ☐

7.1.4. *Dependence in N.* In Theorem 7.2, the only dependence in the matrix size N comes from the denominator. For a cell (i,j) in a Young diagram $\mathbb{Y}_\lambda$, let $\mathrm{hook}_\lambda(i,j)$ be the hook length at (i,j), where $i=1,\ldots,\ell(\lambda)$ is the row index and $j=1,\ldots,\lambda_i$ is the column index. We have the following hook-length formulas, see *e.g.* [36]

$$\chi_\lambda(\mathrm{id}) = \frac{L!}{\prod_{(i,j)\in\mathbb{Y}_\lambda}\mathrm{hook}_\lambda(i,j)}, \tag{7.5}$$

$$s_\lambda(1_N) = \prod_{(i,j)\in\mathbb{Y}_\lambda}\frac{(N+j-i)}{\mathrm{hook}_\lambda(i,j)}. \tag{7.6}$$

Therefore

$$\tilde{G}_{N,L}(C,\beta)=\frac{1}{L!}\sum_{\lambda\vdash L}\frac{|C|\chi_\lambda(C)\chi_\lambda(\beta)}{\prod_{(i,j)\in\mathbb{Y}_\lambda}(N+j-i)}.$$

The specialization of formula (7.2) gives another expression of $s_\lambda(1_N)$, and thus of $\tilde{G}_{N,L}(C,\beta)$:

$$s_\lambda(1_N)=\frac{N^L\chi_\lambda(\mathrm{id})}{L!}\left(1+\sum_{\substack{\mu\vdash L\\ C_\mu\neq[1]}}N^{-t(C_\mu)}\,\frac{|C_\mu|\chi_\lambda(C_\mu)}{\chi_\lambda(\mathrm{id})}\right),$$

where $t(C)=L-\ell(C)$. We obtain that

$$\tilde{G}_{N,L}(C,\beta)=\frac{N^{-L}}{L!}\sum_{\lambda\vdash L}\sum_{k\geq 0}(-1)^k\sum_{\substack{\mu_1,\ldots,\mu_k\vdash L\\ C_{\mu_i}\neq[1]}}\frac{|C|\chi_\lambda(C)\chi_\lambda(\beta)\prod_{i=1}^k N^{-t(C_{\mu_i})}|C_{\mu_i}|\chi_\lambda(C_{\mu_i})}{\chi_\lambda(\mathrm{id})^k}.$$

If we introduce

$$A_{L,k}^{(d)}(C,\beta)=\frac{1}{|\beta|}\sum_{\substack{\mu_1,\ldots,\mu_k\vdash L\\ \sum_i t(C_{\mu_i})=d\\ t(C_{\mu_i})>0}}\mathcal{N}(C,[\beta],C_{\mu_1},\ldots,C_{\mu_k}),$$

we can write $\tilde{G}_{N,L}(C,\beta)$ in a compact way

$$\tilde{G}_{N,L}(C,\beta) = \sum_{d\geq 0} N^{-L-d}\bigg(\sum_{k=0}^{d}(-1)^k A_{L,k}^{(d)}(C,\beta)\bigg). \tag{7.7}$$

Recall that φ^∂ has n cycles. A priori, $\tilde{G}_{N,L}(C,\varphi^\partial)\in O(N^{-L})$, but in fact there are stronger restrictions:

Lemma 7.5. *We have a large N expansion of the form:*

$$\tilde{G}_{N,L}(C,\varphi^\partial) = \sum_{g\geq 0} N^{-(L+\ell(C)-n+2g)}\tilde{G}_L^{(g)}(C,\varphi^\partial).$$

where $\tilde{G}_L^{(g)}$ does not depend on N.

Proof. The argument follows [20]. $A_{L,k}^{(d)}(C,\varphi^\partial)$ counts the number of permutations $\tau,\beta_1,\ldots,\beta_k\in\mathfrak{S}_L$ such that $\tau\in C$, $\beta_i\neq\mathrm{id}$, $\sum_{i=1}^k t(\beta_i)=d$ and

$$\tau\circ\varphi^\partial\circ\beta_1\circ\cdots\circ\beta_k=\mathrm{id}. \tag{7.8}$$

Note that $|t(\sigma)-t(\sigma')|\leq t(\sigma\sigma')\leq t(\sigma)+t(\sigma')$ and thus,

$$|\ell(C)-n| = |t(\varphi^\partial)-t(\tau)|\leq t(\beta_1\cdots\beta_k)\leq\sum_{i=1}^k t(\beta_i)=d.$$

Therefore, the coefficient of $N^{-(L+d)}$ in (7.7) is zero unless $d\geq|\ell(C)-n|$. A fortiori we must have $d\geq\ell(C)-n$. Also, computing the signature of (7.8) we must have

$$(-1)^{(L-n)+(L-\ell(C))+\sum_i t(\beta_i)}=1,$$

i.e. $n-\ell(C)+d$ is even. We get the claim by calling this even integer $2g$. $\square$

7.2. **Transition matrix via monotone Hurwitz numbers.** We dispose of a general theory relating representation theory, Hurwitz numbers and 2d Toda tau hierarchy, to which many authors contributed, and which is clearly exposed in [40]. It relies on three isomorphic descriptions of the vector space $\mathcal{B}$: as the ring of symmetric functions in infinitely many variables, the sum of centers of the group algebra of the symmetric group, and the charge 0 subspace of the Fock space *aka* semi-infinite wedge. After reviewing the aspects of this theory which are relevant for our purposes, we apply it in Section 7.2.5 to obtain a nicer form of Proposition 7.3, namely expressing the transition matrix between ordinary and fully simple observables in terms of monotone Hurwitz numbers.

7.2.1. *The center of the symmetric group algebra.* The center of the group algebra $Z(\mathbb{Q}[\mathfrak{S}_L])$ of the symmetric group $\mathfrak{S}_L$ has two interesting bases, both labelled by partitions. The most obvious one is defined by

$$\hat{C}_\lambda=\sum_{\gamma\in C_\lambda}\gamma.$$

The second one is the basis of orthogonal idempotents which can be related to the first one by

$$\hat{\Pi}_\lambda=\frac{\chi_\lambda(\mathrm{id})}{L!}\sum_{\mu\vdash L}\chi_\lambda(C_\mu)\,\hat{C}_\mu. \tag{7.9}$$

The orthogonality of the characters of $\mathfrak{S}_L$ implies that

$$\hat{\Pi}_\lambda\hat{\Pi}_\mu=\delta_{\lambda,\mu}\hat{\Pi}_\lambda.$$

The Jucys-Murphy elements of $\mathbb{Q}[\mathfrak{S}_L]$ are defined by [45, 55]

$$\hat{J}_1 = 0, \quad \hat{J}_k = \sum_{i=1}^{k-1} (i\ k), \quad k = 2, \ldots, L.$$

Their key property is that the symmetric polynomials in the elements $(\hat{J}_k)_{k=2}^{L}$ span $Z(\mathbb{Q}[\mathfrak{S}_L])$, see *e.g.* [51].

7.2.2. *Action of the center on itself and Hurwitz numbers.* Let r be a symmetric polynomial in infinitely many variables, *i.e.* $r \in \mathcal{B}$. We define

$$r(\hat{J}) \coloneqq r((\hat{J}_k)_{k=2}^{L}, 0, 0, \ldots).$$

The operator of multiplication by $r(\hat{J})$ in $\mathbb{Q}[\mathfrak{S}_L]$ acts diagonally on the basis $\hat{\Pi}_\lambda$ of idempotents, with eigenvalues equal to the evaluation on the content of the partition λ:

$$r(\hat{J})\hat{\Pi}_\lambda = r(\mathrm{cont}(\lambda), 0, 0, \ldots)\hat{\Pi}_\lambda, \qquad \mathrm{cont}(\lambda) = (j - i)_{(i,j) \in \mathbb{Y}_\lambda}.$$

A function of the form $\lambda \mapsto r(\mathrm{cont}(\lambda), 0, 0, \ldots)$ is called a *content function.* One knows (see *e.g.* [57]) that the ring of content functions is isomorphic to the ring of shifted symmetric polynomials, but we will not need this result here.

In the conjugacy class basis, the action of multiplication by $r(\hat{J})$ has a combinatorial meaning [40]:

$$r(\hat{J})\hat{C}_\mu = \sum_{\lambda \vdash L} |C_\mu|\, R_{\lambda,\mu}\, \hat{C}_\lambda, \tag{7.10}$$

where $R_{\lambda,\mu}$ enumerates paths in the Cayley graph of $\mathfrak{S}_L$ generated by transpositions, starting at an (arbitrary but) fixed permutation with cycle type λ and ending at an (arbitrary but) fixed permutation with cycle type μ. In particular, $R_{\lambda,\mu} = R_{\mu,\lambda}$. The weight of such paths is determined by the symmetric polynomial r. This is best illustrated with some examples. The coefficients $R_{\lambda,\mu}$ are generically called *double Hurwitz numbers.*

Ordinary. $p_1(\hat{J})$ is the sum of all transpositions. Therefore

$$p_1(\hat{J})^k = \sum_{\tau_1,\ldots,\tau_k} \tau_1 \cdots \tau_k$$

and thus $[P_1^k]_{\lambda,\mu}$ is the number of sequences $(\tau_1, \ldots, \tau_k, \sigma)$ such that τ_i are transpositions, $[\sigma] = \lambda$ and $\tau_1 \circ \cdots \circ \tau_k \circ \sigma$ is a given permutation with conjugacy class μ.

Strictly monotone. For the elementary symmetric polynomial e_k, we have

$$e_k(\hat{J}) = \sum_{\substack{\tau_1,\ldots,\tau_k \\ (\max \tau_i)_{i=1}^k \text{ increasing}}} \tau_1 \cdots \tau_k,$$

where $\max \tau$ for a transposition of a and b is defined as $\max(a, b)$. $[E_k]_{\lambda,\mu}$ is the number of strictly monotone k-step paths from C_λ to a given element in C_μ.

Weakly monotone. For the complete symmetric polynomials,

$$h_k(\hat{J}) = \sum_{\substack{\tau_1,\ldots,\tau_k \\ (\max \tau_i)_{i=1}^k \text{ weakly increasing}}} \tau_1 \cdots \tau_k.$$

So $[H_k]_{\lambda,\mu}$ is the number of weakly monotone k-step paths from C_λ to a given element in C_μ.

We use the word monotone Hurwitz numbers to refer to either of the two last cases.

7.2.3. *Topological interpretation of Hurwitz numbers.* Hurwitz numbers enumerate branched coverings of $\mathbb{S}^2$ with various constraints. Let $Y = \{y_1, \ldots, y_k\}$ be an ordered finite set of points in the topological sphere $\mathbb{S}^2$. There is a one- to-one correspondence between the set of topological branched coverings of the sphere $\mathbb{S}^2$ with ramification locus Y and with sheets labeled from 1 to L, and the set of representations of $\pi_1(\mathbb{S}^2 \setminus Y)$ into $\mathfrak{S}_L$. Actually, loops γ_j around every special point y_j in Y give a presentation of the fundamental group:

$$\pi_1(\mathbb{S}^2 \setminus Y) = \Big\{\gamma_1, \ldots, \gamma_k \quad \Big| \quad \prod_{j=1}^{k} \gamma_j = \mathrm{id}\Big\}.$$

and hence a representation of $\pi_1(\mathbb{S}^2 \setminus Y)$ into $\mathfrak{S}_L$ is a sequence of permutations $(\beta_i)_{j=1}^k$ such that

$$\beta_1 \circ \cdots \circ \beta_k = \mathrm{id}. \tag{7.11}$$

Starting from a representation of $\pi_1(\mathbb{S}^2 \setminus Y)$ into $\mathfrak{S}_L$, we can construct a branched covering of the sphere

$$\Big(\bigsqcup_{i=1}^{L} \mathbb{S}^2\Big)/\sim \ \overset{\pi}{\longrightarrow}\ \mathbb{S}^2 \tag{7.12}$$

as follows. For every $y_j \in Y$, we denote $y_j^{(i)} \in \pi^{-1}(y_j)$ its preimage in the i-th copy of $\mathbb{S}^2$. In the source of (7.12), the equivalence relation identifies the points $y_j^{(i)}$ and $y_j^{(\beta_j(i))}$ for every $y_j \in Y$. In this way, we get $|\pi^{-1}(y_j)| = |\mathcal{C}(\beta_j)|$. Conversely, if we are given a branched covering, the monodromy representation indeed gives a representation of $\pi_1(\mathbb{S}^2 \setminus Y)$ into $\mathfrak{S}_L$. The total space of the covering is connected if and only if $\beta_1, \ldots, \beta_k$ act transtively on $[\![1, L]\!]$.

Let λ_j be the partition of L describing the conjugacy class of β_j, that is also the ramification profile over y_j. The Euler characteristic of the total space is given by the Riemann-Hurwitz formula:

$$\chi = 2L - \sum_{j=1}^{k} t(\lambda_j).$$

In particular, if there are $r = k-2$ transpositions and two conjugacy classes described by partitions λ, μ of lengths n and m, we have

$$\chi = n + m - r.$$

This special case corresponds to the one described previously: double Hurwitz numbers, which count the number of ways $\mathrm{id} \in \mathfrak{S}_L$ may be factorized into a product of $k-2$ transpositions and two permutations of cycle types given by λ and μ, and equivalently the number of L-sheeted branched coverings of the sphere with ramification profile over two points given by λ, μ and $k-2$ other simple ramifications. Various constraints can be put on the factorization (7.11), which can in turn be interpreted as properties of the branched coverings.

7.2.4. *Hypergeometric tau-functions.* We consider Frobenius' characteristic map

$$\Phi : \bigoplus_{L \geq 0} Z(\mathbb{Q}[\mathfrak{S}_L]) \longrightarrow \mathcal{B}$$

defined by

$$\Phi(\hat{C}_\lambda) = \frac{p_\lambda}{|\mathrm{Aut}\,\lambda|} = \frac{|C_\lambda|\, p_\lambda}{L!}, \qquad |\lambda| = L.$$

This map is linear and it is a graded isomorphism – namely it sends $Z(\mathbb{Q}[\mathfrak{S}_L])$ to $\mathcal{B}^{(L)}$. This definition together with the formula (7.2) and the formula for change of basis (7.9) imply that

$$\Phi(\hat{\Pi}_\lambda) = \frac{\chi_\lambda(\mathrm{id})}{L!}\, s_\lambda.$$

The action of $Z(\mathbb{Q}[\mathfrak{S}_L])$ on itself by multiplication, can then be assembled into an action of $\mathcal{B}$ on itself. Concretely, if $r \in \mathcal{B}$ this action is given by

$$r(\hat{J}) := \Phi \circ \bigg(\bigoplus_{L \geq 0} r\big((\hat{J}_k)_{k=2}^{L}, 0, 0, \ldots\big)\bigg) \circ \Phi^{-1}.$$

Definition 7.6. A *hypergeometric tau-function* is an element of $\mathcal{B}\hat{\otimes}\mathcal{B}$ of the form $\sum_\lambda A_\lambda\, s_\lambda \otimes s_\lambda$ for some scalar-valued $\lambda \mapsto A_\lambda$ function which is a content function.

Remark 7.7. A *2d Toda tau-function* is an element of $\mathcal{B}\hat{\otimes}\mathcal{B}$ which satisfies the Hirota bilinear equations – these are the analog of Plücker relations in the Sato Grassmannian. It is known that, if A is a content function, $\sum_\lambda A_\lambda s_\lambda \otimes s_\lambda$ is a 2d Toda tau-function [17, 58]. We adopt here the name "hypergeometric" coined by Orlov and Harnad for those particular 2d Toda tau-functions. Let us mention there exist 2d Toda tau-functions which are diagonal in the Schur basis but with coefficients which are not content functions.

We can identify $\mathcal{B}\hat{\otimes}\mathcal{B}$ with the ring of symmetric functions in two infinite sets of variables $\mathbf{z} = (z_1, z_2, \ldots)$ and $\widetilde{\mathbf{z}} = (\widetilde{z}_1, \widetilde{z}_2, \ldots)$. There is a trivial hypergeometric tau-function:

$$\mathcal{T}_\emptyset := \exp\Big(\sum_{k \geq 1} \frac{1}{k}\, p_k(\mathbf{z})\, p_k(\widetilde{\mathbf{z}})\Big) = \prod_{i,j=1}^{\infty} \frac{1}{1 - z_i \widetilde{z}_j} = \sum_\lambda s_\lambda(\mathbf{z})\, s_\lambda(\widetilde{\mathbf{z}}) = \sum_\mu \frac{1}{|\mathrm{Aut}\,\mu|}\, p_\mu(\mathbf{z})\, p_\mu(\widetilde{\mathbf{z}}),$$

where the two last sums are over all partitions and for the equality in the middle we have used Cauchy-Littlewood formula [49, Chapter 1].

$r \in \mathcal{B}$ acts on the set of hypergeometric tau-functions by action on the first factor via $r(\hat{J}) \otimes \mathrm{Id}$. More concretely, the action on $\mathcal{T}_\emptyset$ reads

$$\mathcal{T}_r = \sum_\lambda r(\mathrm{cont}\,\lambda)\, s_\lambda \otimes s_\lambda = \sum_{L \geq 0} \sum_{|\lambda| = |\mu| = L} \frac{|C_\lambda||C_\mu|}{L!}\, R_{\lambda,\mu}\, p_\lambda \otimes p_\mu\,. \tag{7.13}$$

7.2.5. *Main result.* We prove that the transition matrix from ordinary to fully simple expectation values is given by double, weakly monotone Hurwitz numbers (with signs), while the transition matrix from fully simple to ordinary is given by the double, strictly monotone Hurwitz numbers.

Theorem 7.8. *With respect to any $\mathcal{U}_N$-invariant measure on the space $\mathcal{H}(N)$ of $N \times N$ hermitian matrices, we obtain*

$$\begin{aligned} \big\langle \mathcal{P}_\lambda(M) \big\rangle &= \sum_{\mu \vdash |\lambda|} N^{-|\mu|} \Big(\sum_{k \geq 0} (-N)^{-k} [H_k]_{\lambda,\mu}\Big) |C_\mu| \big\langle p_\mu(M) \big\rangle, \\ \big\langle p_\mu(M) \big\rangle &= \sum_{\lambda \vdash |\mu|} N^{|\lambda|} \Big(\sum_{k \geq 0} N^{-k} [E_k]_{\mu,\lambda}\Big) |C_\lambda| \big\langle \mathcal{P}_\lambda(M) \big\rangle, \end{aligned}$$

where E_k (resp. H_k) are the Hurwitz numbers related to the elementary symmetric (resp. complete symmetric) polynomials.

Proof. We introduce an auxiliary diagonal matrix $\widetilde{M}$ and deduce from Proposition 7.3 and Equation (7.2) that

$$\frac{1}{L!}\sum_{\lambda\vdash L}|C_\lambda|\,p_\lambda(\widetilde{M})\big\langle\mathcal{P}_\lambda(M)\big\rangle=\sum_{\mu\vdash L}\frac{\chi_\mu(\mathrm{id})}{L!\,s_\mu(1_N)}\,s_\mu(\widetilde{M})\,\big\langle s_\mu(M)\big\rangle. \tag{7.14}$$

The formulas (7.5)-(7.6) show that $\frac{\chi_\mu(\mathrm{id})}{L!\,s_\mu(1_N)}$ is a content function coming from the complete symmetric polynomials

$$\frac{\chi_\mu(\mathrm{id})}{L!\,s_\mu(1_N)}=\prod_{(i,j)\in\mathbb{Y}_\mu}\frac{1}{N+\mathrm{cont}(i,j)}=N^{-|\mu|}\sum_{k\geq 0}(-N)^{-k}\,h_k(\mathrm{cont}\,\mu). \tag{7.15}$$

We denote r_N the corresponding element of $\mathcal{B}[[N^{-1}]]$. The identity (7.14) then translates into:

$$\frac{1}{L!}\sum_{\lambda\vdash L}|C_\lambda|\,p_\lambda(\widetilde{M})\,\big\langle\mathcal{P}_\lambda(M)\big\rangle=\sum_{\mu\vdash L}r_N(\mathrm{cont}\;\mu)\,s_\mu(\widetilde{M})\big\langle s_\mu(M)\big\rangle. \tag{7.16}$$

To interpret these expressions as r_N acting on $\mathcal{T}_\emptyset$ as in (7.13), we remind that $\mathcal{T}$ in $\mathcal{B}\hat{\otimes}\mathcal{B}$ can be seen as a function of two sets of infinitely many variables Λ and $\widetilde{\Lambda}$. Moreover, we consider $\mathcal{T}$ evaluated at two matrices M and $\widetilde{M}$ of size N, by substituting Λ (resp. $\widetilde{\Lambda}$) by the set of N eigenvalues of M (resp. $\widetilde{M}$) completed by infinitely many zeros. We then write $\mathcal{T}(M,\widetilde{M})$ to stress that we have a function of two matrices. In this way, we identify the summation of (7.16) over $L\geq 0$ with $\langle\mathcal{T}_{r_N}(M,\widetilde{M})\rangle$, where the expectation value is taken with respect to any unitarily-invariant measure on M – while $\widetilde{M}$ is a matrix-valued parameter.

Now comparing with (7.13), we find that

$$\big\langle\mathcal{P}_\lambda(M)\big\rangle=\sum_{\mu\vdash|\lambda|}(R_N)_{\lambda,\mu}\,|C_\mu|\big\langle p_\mu(M)\big\rangle, \tag{7.17}$$

which yields the first formula we wanted to prove. To obtain the second formula, we observe that

$$s_N(\mathrm{cont}\;\lambda)=\prod_{(i,j)\in\mathbb{Y}_\lambda}(N+\mathrm{cont}(i,j))=N^{|\lambda|}\sum_{k\geq 0}N^{-k}\,e_k(\mathrm{cont}\;\lambda)$$

defines an element $s_N\in\mathcal{B}[N,N^{-1}]]$ which is inverse to r_N in $\mathcal{B}[N,N^{-1}]]$. We denote $[S_N]_{\lambda,\mu}$ the Hurwitz numbers it determines via (7.10). If we act by $s_N(\hat{J})$ on $\mathcal{T}_{r_N}$, we get back the trivial tau-function:

$$\langle\mathcal{T}_\emptyset(\widetilde{M},M)\rangle=\sum_\mu\frac{p_\mu(\widetilde{M})}{|\mathrm{Aut}\,\mu|}\,\langle p_\mu(M)\rangle. \tag{7.18}$$

On the other hand, representing this action in the power sum basis using (7.13) and (7.16) yields

$$\langle\mathcal{T}_\emptyset(\widetilde{M},M)\rangle=\sum_{L\geq 0}\sum_{|\lambda|=|\mu|=L}\frac{|C_\lambda|}{|\mathrm{Aut}\,\mu|}\,[S_N]_{\mu,\lambda}\,p_\mu(\widetilde{M})\langle\mathcal{P}_\lambda(M)\rangle. \tag{7.19}$$

Finally, we can identify the coefficients of (7.18) and (7.19) to obtain the desired formula. □

Remark 7.9. Our definition of Hurwitz numbers, given by (7.10), differs from other conventions in the literature by some automorphism factors. For instance, double Hurwitz numbers are defined and related to ours as follows in [2]:

$$\frac{|C_\lambda||C_\mu|}{L!}[H_k]_{\lambda,\mu}=\frac{1}{L!}[\hat{C}_{(1^L)}]\,\hat{C}_\lambda\,\hat{C}_\mu\,h_k(\hat{J}), \tag{7.20}$$

where $|\lambda| = |\mu| = L$ and (1^L) denotes the partition $\sigma = (1, \ldots, 1)$ with $\ell(\sigma) = L$.

With our proof, we obtain some intermediate formulas that will be useful later. However, for the derivation of Theorem 7.8 it is not crucial to write our generating series in the form of τ-functions. Using Proposition 7.3, (7.15) and the following expression for Hurwitz numbers derived from the definition (7.20):

$$[H_k]_{\lambda,\mu} = \frac{1}{L!} \sum_{\nu \vdash L} \chi_\nu(C_\lambda)\chi_\nu(C_\mu) h_k(\mathrm{cont}\,\nu), \tag{7.21}$$

Theorem 7.8 is straightforward. Apart from the reason already mentioned, we also consider it is interesting to illustrate the relation with the world of τ-functions.

Finally, we would like to remark that, instead of our formula (7.15), we could have used [56, Theorem 2.1 and 2.2], but our formula is more elementary.

7.3. **Relation with the matrix model with external field.** The Itzykson-Zuber integral [43] is a function of an integer N and two matrices A and B of size N defined by

$$\mathcal{I}_N(A, B) := \int_{U(N)} \mathrm{d}U \, \exp\big[N \, \mathrm{Tr}(AUBU^\dagger)\big], \tag{7.22}$$

where $\mathrm{d}U$ is the Haar measure on $U(N)$ normalized to have mass 1. It admits a well-known expansion in terms of characters of the unitary group, *i.e.* Schur functions:

Theorem 7.10. [6, Eq. (4.6)]

$$\mathcal{I}_N(A, B) = \sum_\lambda \frac{N^{|\lambda|}\chi_\lambda(\mathrm{id})}{L!\, s_\lambda(1_N)}\, s_\lambda(A)\, s_\lambda(B).$$

For any unitarily invariant measure μ on $\mathcal{H}_N$, we define

$$\check{Z}(A) = \int_{\mathcal{H}_N} \mathrm{d}\mu(M) e^{N\,\mathrm{Tr}(AM)}.$$

We also introduce the factor which allows us to go from (ordered) tuples $(\ell_1, \ldots, \ell_n)$ with $L := \sum_{i=1}^n \ell_i$ to (unordered) partitions of L:

$$g_\lambda := \frac{\ell(\lambda)!}{\prod_{i=1}^L m_i(\lambda)!} = \frac{\ell(\lambda)!\, |C_\lambda| \prod_{i=1}^{\ell(\lambda)} \lambda_i}{|\lambda|!}, \tag{7.23}$$

where $L = \sum_{i=1}^L i\, m_i(\lambda) = \sum_{i=1}^n \lambda_i$.

Corollary 7.11. *We denote $\langle \cdot \rangle$ the expectation value and $\kappa_n(\cdot)$ the n-th order cumulant with respect to any unitarily invariant measure μ on $M \in \mathcal{H}_N$. We have the formulas*

$$\begin{aligned} \frac{\check{Z}(A)}{\check{Z}(0)} &= 1 + \sum_{\lambda \neq \emptyset} \frac{|C_\lambda|}{|\lambda|!} N^{|\lambda|} p_\lambda(A) \big\langle \mathcal{P}_\lambda(M) \big\rangle = \big\langle \mathcal{I}_N(A, M) \big\rangle, \\ \ln\Big(\frac{\check{Z}(A)}{\check{Z}(0)}\Big) &= \sum_{n \geq 1} \sum_{\ell_1, \ldots, \ell_n \geq 1} N^L \, \kappa_n\big(\mathcal{P}^{(\ell_1)}(M), \ldots, \mathcal{P}^{(\ell_n)}(M)\big) \prod_{i=1}^n \frac{p_{\ell_i}(A)}{\ell_i}, \end{aligned}$$

where we recall that the corresponding expectation value is zero whenever $|\lambda|$ or $L := \sum_i \ell_i$ exceeds N.

Proof. Comparing Theorem 7.10 with (7.16) gives the first line. If we replace the sum over partitions by the sum over tuples of positive integers multiplying by $\frac{1}{g_\lambda}$, we find

$$\begin{aligned}\frac{\check{Z}(A)}{\check{Z}(0)} &= 1+\sum_{n\geq 1}\frac{1}{n!}\sum_{\ell_1,\ldots,\ell_n\geq 1} N^L\left\langle \mathcal{P}^{(\ell_1)}(M)\cdots\mathcal{P}^{(\ell_n)}(M)\right\rangle\prod_{i=1}^{n}\frac{p_{\ell_i}(A)}{\ell_i}\\ &= \left\langle \exp\left(\sum_{i\geq 1}\frac{N^{\ell_i}p_{\ell_i}(A)}{\ell_i}\mathcal{P}^{(\ell_i)}(M)\right)\right\rangle.\end{aligned}$$

Taking the logarithm gives precisely the cumulant generating series as in the second formula. □

In other words, the fully simple observables for the matrix model μ are naturally encoded in the corresponding matrix model with an external source A. Compared to Theorem 7.8, this result is in agreement with the combinatorial interpretation of the Itzykson-Zuber integral in terms of double monotone Hurwitz numbers [38].

8. Combinatorial interpretation in terms of ordinary and fully simple maps

The relation between ordinary and fully simple observables through monotone Hurwitz numbers is universal in the sense that it does not depend on the unitarily invariant measure considered. This section is devoted to the relation between matrix models and the enumeration of maps for a specific unitarily invariant measure. This relation is well-known for ordinary maps, but we recall it here to later give the derivation for fully simple maps. This specialization is motivated by our study of the general ordinary and fully simple observables.

We introduce the Gaussian probability measure on the space $\mathcal{H}(N)$ of $N\times N$ hermitian matrices:

$$\mathrm{d}\mu_0(M)=\frac{\mathrm{d}M}{Z_0}\,e^{-N\mathrm{Tr}\,\frac{M^2}{2}},\qquad Z_0=\int_{\mathcal{H}_N}\mathrm{d}M\,e^{-N\mathrm{Tr}\,\frac{M^2}{2}},$$

and the generating series:

$$\begin{aligned}\tilde{T}_{h,k}(w_1,\ldots,w_k) &= \sum_{m_1,\ldots,m_k\geq 1}\frac{t^h_{m_1,\ldots,m_k}}{m_1\cdots m_k}\prod_{i=1}^{k}w_i^{m_i},\\ T_{h,k}(w_1,\ldots,w_k) &= \tilde{T}_{h,k}(w_1,\ldots,w_k)-\delta_{h,0}\delta_{k,1}\frac{w_1^2}{2}.\end{aligned}$$

We consider the formal measure

$$\mathrm{d}\mu(M)=\frac{\mathrm{d}M}{Z_0}\,\exp\left(\sum_{h\geq 0,k\geq 1}\frac{N^{2-2h-k}}{k!}\mathrm{Tr}\,T_{h,k}\left(M_k^{(1)},\ldots,M_k^{(k)}\right)\right),\tag{8.1}$$

where $M_k^{(i)}:=\bigotimes_{j=1}^{i-1}I_N\otimes M\otimes\bigotimes_{j=i+1}^{k}I_N$, in the sense that the expectation value of any polynomial function of M with respect to this measure is defined as a formal series in the t's.

For a combinatorial map (σ,α), we consider here a special structure for the set of half-edges $H=H^u\sqcup H^\partial$ which will be convenient for our derivation:

$$H^\partial=\bigsqcup_{i=1}^{n}\{i\}\times(\mathbb{Z}/L_i\mathbb{Z}),\qquad H^u=\bigsqcup_{m=1}^{r}\{m\}\times(\mathbb{Z}/k_m\mathbb{Z}).$$

The permutation $\varphi:=(\sigma\circ\alpha)^{-1}$ acting on H, whose cycles correspond to faces of the map, is hence given by $\varphi((i,l))=(i,l+1)$. With this special structure of the set of half-edges, counting the number of relabelings of H^u amounts to choosing an order of the unmarked faces and a root for each of them. Therefore

$$\mathrm{Rel}(\sigma,\alpha)=r!\,\prod_{m=1}^{r}k_m.$$

8.1. **Ordinary usual maps.** Consider first the case where $T_{h,k}=0$ for $(h,k)\neq(0,1)$, *i.e.*

$$\mathrm{d}\mu(M)=\frac{\mathrm{d}M}{Z_{\mathrm{GUE}}}\,\exp\left\{N\,\mathrm{Tr}\Big(-\frac{M^2}{2}+\sum_{k\geq 1}\frac{t_k\,M^k}{k}\Big)\right\},\qquad Z=\int_{\mathcal{H}(N)}\mathrm{d}\mu(M). \tag{8.2}$$

We denote $\langle\cdot\rangle_{\mathrm{GUE}}$ the expectation value with respect to the Gaussian measure $\mathrm{d}\mu_0$. The matrix elements have covariance:

$$\langle M_{a,b}M_{c,d}\rangle_{\mathrm{GUE}}=\frac{1}{N}\delta_{a,d}\delta_{b,c}. \tag{8.3}$$

Let $(L_i)_{i=1}^n$ be a sequence of nonnegative integers, and $L=\sum_{i=1}^n L_i$. The expectation values with respect to $\mathrm{d}\mu$ are computed, as formal series in $(t_k)_k$:

$$\Big\langle\prod_{i=1}^{n}\mathrm{Tr}\,M^{L_i}\Big\rangle=\frac{1}{Z}\sum_{r\geq 0}\sum_{k_1,\ldots,k_r\geq 1}\frac{N^r}{r!}\prod_{m=1}^{r}\frac{t_{k_m}}{k_m}\sum_{\substack{1\leq j_h\leq N,\\ h\in H}}\Big\langle\prod_{h\in H}M_{j_h,j_{\varphi(h)}}\Big\rangle_{\mathrm{GUE}}. \tag{8.4}$$

With the help of Wick's theorem for the Gaussian measure and (8.3), we obtain:

$$\Big\langle\prod_{h\in H}M_{j_h,j_{\varphi(h)}}\Big\rangle_{\mathrm{GUE}}=N^{-\frac{|H|}{2}}\sum_{\alpha\in\mathfrak{I}_H}\prod_{h\in H}\delta_{j_h,j_{\alpha(\varphi(h))}}.$$

where $\mathfrak{I}_H\subset\mathfrak{G}_H$ is the set of all fixed-point free involutions, i.e. all pairwise matchings, on H.

We observe that the product on the right hand side is 1 if $h\mapsto j_h$ is constant over the cycles of $\alpha\circ\varphi$, otherwise it is 0. Therefore,

$$\sum_{\substack{1\leq j_h\leq N,\\ h\in H}}\Big\langle\prod_{h\in H}M_{j_h,j_{\varphi(h)}}\Big\rangle_{\mathrm{GUE}}=N^{-\frac{|H|}{2}}\sum_{\alpha\in\mathfrak{I}_H}N^{|\mathcal{C}(\alpha\circ\varphi)|}.$$

To recognize (8.4) as a sum over combinatorial maps, we let $\alpha\in\mathfrak{I}_H$ correspond to the edges of maps whose faces are given by cycles of φ, and $\sigma:=(\alpha\circ\varphi)^{-1}$, whose cycles will correspond to the vertices. Observe that $2|\mathcal{C}(\alpha)|=|H|$.

$$\Big\langle\prod_{i=1}^{n}\mathrm{Tr}\,M^{L_i}\Big\rangle=\frac{1}{Z}\sum_{(\sigma,\alpha)}\frac{N^{|\mathcal{C}(\sigma)|-|\mathcal{C}(\alpha)|+|\mathcal{C}(\varphi)|-n}}{\mathrm{Rel}\,(\sigma,\alpha)}\prod_{f\in\mathcal{C}(\varphi|_{H^u})}t_{\ell(f)},$$

where the sum is taken over non-connected combinatorial maps (σ,α) with n boundaries of lengths $L_1,\ldots,L_n$.

To transform this sum over combinatorial maps into a generating series for (unlabeled) maps, we have to multiply by the number of combinatorial maps which give rise to the same unlabeled combinatorial map $[(\sigma,\alpha)]$, which is $\mathrm{Gl}(\sigma,\alpha)=\frac{\mathrm{Rel}(\sigma,\alpha)}{|\mathrm{Aut}(\sigma,\alpha)|}$, as we explained in section 2.1.2.

A similar computation can be done separately for Z, and we find it is the generating series of maps with empty boundary. The contribution of connected components without boundaries factorizes in the numerator and, consequently,

$$\Big\langle\prod_{i=1}^{n}\mathrm{Tr}\,M^{L_i}\Big\rangle=\sum_{\substack{\partial\text{-connected } M=[(\sigma,\alpha)]\\ \text{with }\partial\text{ lengths }(L_i)_{i=1}^n}}\frac{N^{\chi(\sigma,\alpha)}}{|\mathrm{Aut}(\sigma,\alpha)|}\prod_{f\in\mathcal{C}(\varphi|_{H^u})}t_{\ell(f)}.$$

We remark that the power of N sorts maps by their Euler characteristic. Finally, a standard argument shows that taking the logarithm for closed maps or the cumulant expectation values for maps with boundaries, we obtain the generating series of connected maps:

Proposition 8.1. [28]

$$\ln Z = \sum_{g\geq 0} N^{2-2g} F^{[g]}, \qquad \kappa_n(\mathrm{Tr}\, M^{L_1}, \ldots, \mathrm{Tr}\, M^{L_n}) = \sum_{g\geq 0} N^{2-2g-n} F^{[g]}_{L_1,\ldots,L_n}.$$

This kind of results appeared first for planar maps in [15], but for a modern and general exposition see also [28].

8.2. **Ordinary stuffed maps.** For the general formal measure

$$\begin{aligned} \mathrm{d}\mu(M) &= \frac{\mathrm{d}M\, e^{-N\,\mathrm{Tr}\frac{M^2}{2}}}{Z_{\mathrm{GUE}}} \exp\left(\sum_{h\geq 0, k\geq 1} \frac{1}{k!} N^{2-2h-k}\, \mathrm{Tr}\, \tilde{T}_{h,k}\left(M_k^{(1)}, \ldots, M_k^{(k)}\right)\right) \\ &= \frac{\mathrm{d}M\, e^{-N\,\mathrm{Tr}\frac{M^2}{2}}}{Z_{\mathrm{GUE}}} \exp\left(\sum_{h\geq 0, k\geq 1} \frac{1}{k!} N^{2-2h-k} \sum_{m_1,\ldots,m_k\geq 1} t^h_{m_1,\ldots,m_k} \prod_{i=1}^{k} \frac{\mathrm{Tr}\, M^{m_i}}{m_i}\right), \end{aligned}$$

the expectation values are generating series of stuffed maps. We denote here Z_S, $\langle\cdot\rangle_S$ and $\kappa_n(\cdot)_S$ the partition function, the expectation value and the n-th order cumulant expectation values with respect to this general measure to distinguish these more general expressions from the previous ones.

A generalization of the technique reviewed in § 8.1 shows that

$$\left\langle \mathrm{Tr}\, M^{L_1} \cdots \mathrm{Tr}\, M^{L_n} \right\rangle_S = \sum_{\substack{\partial\text{-connected stuffed map } M \\ \text{with } \partial \text{ lengths } (L_i)_{i=1}^n}} \frac{N^{\chi(M)}}{|\mathrm{Aut}(M)|} \prod_{p=1}^{F} t^{h_p}_{(\ell(c))_{c\in f_p}}.$$

As in the previous subsection, taking the logarithm or the cumulant expectation values in absence or presence of boundaries respectively, we obtain the generating series of connected stuffed maps:

Proposition 8.2. [7]

$$\ln Z_S = \sum_{g\geq 0} N^{2-2g} \widehat{F}^{[g]}, \qquad \kappa_n(\mathrm{Tr}\, M^{L_1}, \ldots, \mathrm{Tr}\, M^{L_n})_S = \sum_{g\geq 0} N^{2-2g-n} \widehat{F}^{[g]}_{L_1,\ldots,L_n}.$$

8.3. **Fully simple maps.** Let $(\gamma_i)_{i=1}^n$ be pairwise disjoint cycles:

$$\gamma_i = (j_{i,1} \to j_{i,2} \to \cdots \to j_{i,L_i}) \tag{8.5}$$

and $L = \sum_{i=1}^n L_i$. We want to compute $\left\langle \prod_{i=1}^n \mathcal{P}_{\gamma_i}(M) \right\rangle$ and the idea is that only fully simple maps will make a non-zero contribution, so we will be able to express it as a generating series for fully simple maps. Let us describe this expression for the measure (8.2) in terms of maps. Repeating the steps of § 8.1, we obtain

$$\begin{aligned} \left\langle \prod_{i=1}^{n} \mathcal{P}_{\gamma_i}(M) \right\rangle &= \frac{1}{Z} \sum_{r\geq 0} \sum_{k_1,\ldots,k_r\geq 1} \frac{N^r}{r!} \prod_{m=1}^{r} \frac{t_{k_m}}{k_m} \sum_{\substack{1\leq j_h \leq N, \\ h\in H^u}} \left\langle \prod_{h\in H} M_{j_h, j_{\varphi(h)}} \right\rangle_0 \\ &= \frac{1}{Z} \sum_{r\geq 0} \sum_{k_1,\ldots,k_r\geq 1} \frac{N^r}{r!} \prod_{m=1}^{r} \frac{t_{k_m}}{k_m} N^{\frac{-|H|}{2}} \sum_{\substack{1\leq j_h \leq N, \\ h\in H^u}} \sum_{\alpha\in \mathfrak{I}_H} \prod_{h\in H} \delta_{j_h, j_{\alpha(\varphi(h))}}. \end{aligned}$$

We consider as before the permutation $\sigma := (\alpha \circ \varphi)^{-1}$ which will correspond to the vertices of the maps. The difference with (8.4) lies in the summation over indices j_h between 1 and N only for $h \in H^u$, while j_h for $h = (i,l) \in H^\partial$ is prescribed by (8.5). As $j_{i,l}$ are pairwise distinct, the only non-zero contributions to the sum will come from maps for which $(i,l) \in H^\partial$ belong to pairwise distinct cycles of σ and this is the characterization for fully simple maps in the permutational model setting. The function $h \mapsto j_h$ must be constant along the cycles of σ, and its value for every $h \in H^\partial$ is prescribed by (8.5). So, the number of independent indices of summation among $(j_h)_{h \in H^u}$ is

$$|\mathcal{C}(\sigma)| - L.$$

Thus,

$$\begin{aligned}
\Big\langle \prod_{i=1}^{n} \mathcal{P}_{\gamma_i}(M) \Big\rangle &= \frac{1}{Z} \sum_{r \geq 0} \sum_{k_1,\ldots,k_r \geq 1} \frac{N^r}{r!} \prod_{m=1}^{r} \frac{t_{k_m}}{k_m} N^{-\frac{|H|}{2}} \sum_{\alpha \in \mathfrak{I}_H} N^{|\mathcal{C}(\alpha\circ\varphi)|-L}. \\
&= \sum_{\substack{\partial\text{-connected fully} \\ \text{simple } M=[(\sigma,\alpha)] \\ \text{with } \partial \text{ lengths } (L_i)_{i=1}^n}} \frac{N^{\chi(\sigma,\alpha)-L}}{|\mathrm{Aut}(\sigma,\alpha)|} \prod_{f \in \mathcal{C}(\varphi|_{H^u})} t_{\ell(f)}.
\end{aligned}$$

The generalization to the measure (8.5) is straightforward and gives rise to generating series for fully simple stuffed maps:

$$\Big\langle \prod_{i=1}^{n} \mathcal{P}_{\gamma_i}(M) \Big\rangle_S = \sum_{\substack{M \\ \text{fully simple stuffed map} \\ \partial \text{ perimeters } (L_i)_{i=1}^n \\ \partial-\text{connected}}} \frac{N^{\chi(M)-L}}{|\mathrm{Aut}(M)|} \prod_{p=1}^{F} t^{h_p}_{(\ell(c))_{c \in f_p}}.$$

As before, the cumulant expectation values give the generating series of connected fully simple maps and stuffed maps for the more general measure:

Proposition 8.3.

$$\begin{aligned}
\kappa_n(\mathcal{P}_{\gamma_1}(M), \ldots, \mathcal{P}_{\gamma_n}(M)) &= \sum_{g \geq 0} N^{2-2g-n-L} H^{[g]}_{L_1,\ldots,L_n}, \\
\kappa_n(\mathcal{P}_{\gamma_1}(M), \ldots, \mathcal{P}_{\gamma_n}(M))_S &= \sum_{g \geq 0} N^{2-2g-n-L} \widehat{H}^{[g]}_{L_1,\ldots,L_n}.
\end{aligned}$$

8.4. **Generating series.** We define $f_\lambda(\mathbf{w}) := \prod_{i=1}^{\ell(\lambda)} w_i^{\lambda_i}$ and $f_\lambda(\mathbf{x}^{-1}) := \prod_{i=1}^{\ell(\lambda)} x_i^{-\lambda_i}$. We recall that g_λ in (7.23) was the factor to pass from tuples $(L_1, \ldots, L_n)$ with $L := \sum_{i=1}^{n} L_i$ to partitions of L.

We shall rephrase Theorem 7.8 in terms of the generating series of maps and in the topological expansions, which exist for maps by construction. If $\mathbf{L} = (L_i)_{i=1}^n$ is a sequence of integers, we denote $[\mathbf{L}]$ the corresponding partition. The following relation

$$\sum_{L_1,\ldots,L_n \geq 1} \big\langle \mathcal{P}_{[\mathbf{L}]}(M) \big\rangle f_{[\mathbf{L}]}(\mathbf{w}) = \sum_{\ell(\lambda)=n} g_\lambda \big\langle \mathcal{P}_\lambda(M) \big\rangle f_\lambda(\mathbf{w})$$

motivates the introduction of the generating series of disconnected fully simple maps and their topological expansion classifying disconnected maps by their Euler characteristic:

$$\overline{X}_n(w_1, \ldots, w_n) = \sum_{\ell(\lambda)=n} g_\lambda \big\langle \mathcal{P}_\lambda(M) \big\rangle f_\lambda(\mathbf{w}) \prod_{i=1}^{n} w_i^{-1} = \sum_{\chi} N^{\chi-n} \, \overline{X}_n^{[\chi]} \left(\frac{\mathbf{w}}{N} \right),$$

and their ordinary counterpart:

$$\overline{W}_m(x_1,\ldots,x_m) = \sum_{\ell(\mu)=m} g_\mu \langle p_\mu(M)\rangle\, f_\mu(\mathbf{x}^{-1}) \prod_{i=1}^{m} x_i^{-1} = \sum_{\chi} N^{\chi}\, \overline{W}_m^{[\chi]}(\mathbf{x}).$$

We remark that the coefficients $\overline{X}_n^{[\chi]}$ and $\overline{W}_m^{[\chi]}$ of these topological expansions do not depend on N, as happened for the connected generating series $X_n^{[g]}$ and $W_m^{[g]}$. This will also be true for the topological expansions of the generating series of Hurwitz numbers that will be introduced later in this section.

Their connected versions are given by the inclusion-exclusion principle:

$$\begin{aligned} X_n(\mathbf{w}) &= \sum_{J_1\sqcup\cdots\sqcup J_s=[\![1,n]\!]} (s-1)!(-1)^{s-1} \prod_{i=1}^{s} \overline{X}_{|J_i|}(\mathbf{w}_{J_i}) \\ &= \sum_{g\geq 0} N^{2-2g-2n}\, X_n^{[g]}\Big(\frac{\mathbf{w}}{N}\Big), \\ W_m(\mathbf{x}) &= \sum_{J_1\sqcup\cdots\sqcup J_s=[\![1,m]\!]} (s-1)!(-1)^{s-1} \prod_{i=1}^{s} \overline{W}_{|J_i|}(\mathbf{x}_{J_i}) \\ &= \sum_{g\geq 0} N^{2-2g-m}\, W_m^{[g]}(\mathbf{x}). \end{aligned}$$

We introduce now the generating series of double Hurwitz numbers in their disconnected version:

$$\begin{aligned} \overline{\mathcal{H}}_{n;m}^{(\chi_0)}(\mathbf{w};\mathbf{x}) &= \sum_{\substack{|\lambda|=|\mu| \\ \ell(\lambda)=n \\ \ell(\mu)=m}} (-1)^{n+m-\chi_0} \frac{g_\mu}{g_\lambda}\, |C_\lambda|\, [H_{n+m-\chi_0}]_{\lambda,\mu}\, f_\lambda(\mathbf{w})\, f_\mu(\mathbf{x}), \\ \overline{\mathcal{E}}_{m;n}^{(\chi_0)}(\mathbf{x};\mathbf{w}) &= \sum_{\substack{|\lambda|=|\mu| \\ \ell(\lambda)=n \\ \ell(\mu)=m}} \frac{g_\lambda}{g_\mu}\, |C_\mu|\, [E_{n+m-\chi_0}]_{\mu,\lambda}\, f_\mu(\mathbf{x}^{-1})\, f_\lambda(\mathbf{w}^{-1}), \end{aligned}$$

and with all topologies:

$$\begin{aligned} \overline{\mathcal{H}}_{n;m}(\mathbf{w};\mathbf{x}) &= \sum_{\chi_0} N^{\chi_0-(n+m)}\, \overline{\mathcal{H}}_{n;m}^{[\chi_0]}(\mathbf{w}/N;\mathbf{x}), \\ \overline{\mathcal{E}}_{m;n}(\mathbf{x};\mathbf{w}) &= \sum_{\chi_0} N^{\chi_0-(n+m)}\, \overline{\mathcal{E}}_{m;n}^{[\chi_0]}(\mathbf{x};\mathbf{w}/N). \end{aligned}$$

Their connected versions are

$$\mathcal{H}_{n;m}(\mathbf{w};\mathbf{x}) = \sum_{\substack{\sqcup_{i=1}^{s} J_i=[\![1,m]\!] \\ \sqcup_{j=1}^{s} K_i=[\![1,n]\!]}} (s-1)!(-1)^{s-1} \prod_{i=1}^{s} \overline{\mathcal{H}}_{|K_i|,|J_i|}(\mathbf{w}_{K_i};\mathbf{x}_{J_i}),$$

$$\mathcal{E}_{m;n}(\mathbf{x};\mathbf{w}) = \sum_{\substack{\sqcup_{i=1}^{s} J_i=[\![1,m]\!] \\ \sqcup_{j=1}^{s} K_i=[\![1,n]\!]}} (s-1)!(-1)^{s-1} \prod_{i=1}^{s} \overline{\mathcal{E}}_{|J_i|,|K_i|}(\mathbf{x}_{J_i};\mathbf{w}_{K_i}), \tag{8.6}$$

and can be decomposed as

$$\begin{aligned}\mathcal{H}_{n;m}(\mathbf{w};\mathbf{x}) &= \sum_{g\geq 0} N^{2-2g-(n+m)}\,\mathcal{H}^{[g]}_{n;m}(\mathbf{w}/N;\mathbf{x}),\\ \mathcal{E}_{m;n}(\mathbf{x};\mathbf{w}) &= \sum_{g\geq 0} N^{2-2g-(n+m)}\,\mathcal{E}^{[g]}_{m;n}(\mathbf{x};\mathbf{w}/N).\end{aligned}$$

From Theorem 7.8, we obtain the following expressions relating the generating series we have introduced:

Corollary 8.4. *We have the following relations between disconnected generating series:*

$$\begin{aligned}\overline{X}^{[\chi']}_n(\mathbf{w}) &= \prod_{i=1}^n w_i^{-1}\sum_{m\geq 1}\sum_{\chi_0+\chi=\chi'+n+m}\oint \overline{\mathcal{H}}^{[\chi_0]}_{n;m}(\mathbf{w};\mathbf{x})\,\overline{W}^{[\chi]}_m(\mathbf{x})\prod_{j=1}^m\frac{\mathrm{d}x_j}{2\mathrm{i}\pi},\\ \overline{W}^{[\chi']}_m(\mathbf{x}) &= \prod_{i=1}^m x_i^{-1}\sum_{n\geq 1}\sum_{\chi_0+\chi=\chi'+n+m}\oint \overline{\mathcal{E}}^{[\chi_0]}_{m;n}(\mathbf{w};\mathbf{x})\,\overline{X}^{[\chi]}_n(\mathbf{w})\prod_{j=1}^n\frac{\mathrm{d}w_j}{2\mathrm{i}\pi}.\end{aligned}$$

Corollary 8.5. *We have the following relations between connected generating series:*

$$X^{[g]}_n(\mathbf{w}) = \prod_{i=1}^n w_i^{-1}\sum_{s,m\geq 1}\sum_{\substack{m_1,\ldots,m_s\geq 1\\ \sum_{j=1}^s m_j=m}}\sum_{\substack{g_0,\ldots,g_s\geq 0\\ (m-s)+\sum_{j=0}^s g_j=g}}\frac{m!}{\prod_{j=1}^s m_j!}$$

$$\oint \mathcal{H}^{[g_0]}_{n;m}(\mathbf{w};\mathbf{x})\prod_{j=1}^s W^{[g_j]}_{m_j}(x_{1+\sum_{i=1}^{j-1}m_i},\ldots,x_{\sum_{i=1}^j m_i})\prod_{j=1}^m\frac{\mathrm{d}x_j}{2\mathrm{i}\pi},$$

$$W^{[g]}_m(\mathbf{x}) = \prod_{i=1}^m x_i^{-1}\sum_{r,n\geq 1}\sum_{\substack{n_1,\ldots,n_r\geq 1\\ \sum_{j=1}^r n_j=n}}\sum_{\substack{g_0,\ldots,g_r\geq 0\\ (n-r)+\sum_{j=0}^r g_j=g}}\frac{n!}{\prod_{j=1}^r n_j!}$$

$$\oint \mathcal{E}^{[g_0]}_{m;n}(\mathbf{x};\mathbf{w})\prod_{j=1}^r X^{[g_j]}_{n_j}(w_{1+\sum_{i=1}^{j-1}n_i},\ldots,w_{\sum_{i=1}^j n_i})\prod_{j=1}^n\frac{\mathrm{d}w_j}{2\mathrm{i}\pi}.$$

The integer $n-r$ (or $m-s$) is called the *genus defect.* Note that if $g=0$, we must have $g_j=0$ for all j and $r=n$ ($s=m$), thus $n_j=1$ ($m_j=1$) for all j. Therefore,

$$\begin{aligned}X^{[0]}_n(\mathbf{w}) &= \prod_{i=1}^n w_i^{-1}\sum_{m\geq 1}\oint \mathcal{H}^{(0)}_{n;m}(\mathbf{w};x_1,\ldots,x_m)\prod_{j=1}^m\frac{W(x_j)\,\mathrm{d}x_j}{2\mathrm{i}\pi},\\ W^{[0]}_m(\mathbf{x}) &= \prod_{i=1}^m x_i^{-1}\sum_{n\geq 1}\oint \mathcal{E}^{(0)}_{m;n}(\mathbf{x};w_1,\ldots,w_n)\prod_{j=1}^n\frac{X(w_j)\mathrm{d}w_j}{2\mathrm{i}\pi}.\end{aligned}$$

9. Proof of Theorem 5.3 (usual maps)

In this section we show

Theorem 9.1. *Conjecture 1.8 is true for usual maps.*

The starting point of our proof[1] is the representation of the generating series of connected fully simple maps as the free energies $\ln\check{Z}(A)$ of the 1-hermitian matrix model with external

[1]The idea of this argument first appeared the derivation of Bouchard-Marino conjecture proposed in [11]. In this article, a generating series of simple Hurwitz numbers were represented in terms of a matrix model with external field for a complicated V, albeit it was later pointed out by D. Zvonkine that this representation was

field [50]. This model is considered here to be valued in formal series. The topological expansion of its correlators satisfies Eynard-Orantin topological recursion, for a well-characterized spectral curve $(\mathcal{S}_A, x, y)$ [35]. On the other hand, the generating series $X_n^{[g]}$ we are after are encoded into the n-th order Taylor expansion of $\ln \check{Z}(A)$ around $A = 0$. Using the symplectic invariance and the properties of topological recursion under deformations of the spectral curve [33], we relate these n-th order Taylor coefficients to TR amplitudes of the topological recursion applied to the curve $(\mathcal{S}_0, y, x)$. As the matrix model $\check{Z}(0)$ generates usual maps, the spectral curve $\mathcal{S}_0$ must be the initial data mentioned in Theorem 1.6. Unfortunately, our proof is not combinatorial and relies on the symplectic invariance itself.

Prior to applying the result of [35], we detail the definition of the topological expansion of $\check{Z}(A)$ and the computation of the spectral curve from Schwinger-Dyson equations in the realm of formal series. These aspects are well-known to physicists.

9.1. **Defining the topological expansion.** Let $N \geq \nu \geq 1$ and $d \geq 3$ be integers, $a_1, \ldots, a_\nu, t_3, \ldots, t_d$ formal parameters. We consider the measure on the space of hermitian matrices of size N

$$\mathrm{d}\mu_A(M) = \mathrm{d}M\, \exp\big[N\,\mathrm{Tr}(MA - V(M))\big], \qquad \check{Z}(A) = \int \mathrm{d}\mu_A(M)\,, \tag{9.1}$$

with

$$V(x) = \frac{x^2}{2} - \sum_{m=3}^{d} \frac{t_m}{m}\, x^m, \qquad A = \mathrm{diag}(a_1, \ldots, a_\nu, 0, \ldots, 0)\,.$$

It is a measure valued in the ring

$$\check{\mathcal{R}} := \mathbb{C}[[a_1, \ldots, a_\nu, t_3, \ldots, t_d]]\,.$$

Expectation values for this measure are denoted $\langle \cdots \rangle_A$ but one should not forget they depend on A. We denote $\langle \cdots \rangle_{\mathrm{GUE}}$ the expectation value with respect to the Gaussian measure only – *i.e.* setting t and A to 0 in (9.1).

Expanding the measure in its formal parameters, we obtain

$$\hat{\mathcal{Z}}(A) := \frac{\check{Z}(A)}{\check{Z}(0)} = \sum_{f,s \geq 0} \frac{N^{f+s}}{f!s!} \sum_{\substack{m_1, \ldots, m_f \geq 3 \\ 1 \leq i_1, \ldots, i_s \leq \nu}} \Big\langle \prod_{j=1}^{r} \mathrm{Tr}\, M^{m_i} \prod_{k=1}^{s} M_{i_k, i_k} \Big\rangle_{\mathrm{GUE}} \prod_{j=1}^{f} \frac{t_{m_j}}{m_j} \prod_{k=1}^{s} a_{i_k}\,. \tag{9.2}$$

We use Wick theorem to compute the expectation values for the Gaussian measure, and find that

$$\Big\langle \prod_{j=1}^{f} \mathrm{Tr}\, M^{m_i} \prod_{k=1}^{s} M_{i_k, i_k} \Big\rangle_{\mathrm{GUE}}$$

is a Laurent polynomial in N. Therefore, $\hat{\mathcal{Z}}(A)$ defines an element of

$$\mathcal{R} = \mathbb{C}[N^{-1}, N][[a_1, \ldots, a_\nu, t_3, \ldots, t_d]].$$

More precisely, it is an element of the subring of $\mathcal{R}^{\mathfrak{S}_\nu} \subset \mathcal{R}$ consisting of the elements invariants under permutations of $a_1, \ldots, a_\nu$. This element is a generating series of usual maps with a certain number f_m faces of degree m, s faces of degree 1 with labels $i_1, \ldots, i_s$, e edges and v vertices. The weight of such a map is

$$N^{v-e+f-s} \prod_{m \geq 3} t_m^{f_m} \prod_{k=1}^{s} a_{i_k}\,.$$

ill-defined even in the realm of formal series. This issue is not relevant here as we start with a well-defined matrix model in formal series, and are careful to justify all steps by legal operations within formal series.

Lemma 9.2. *Let* $\mathcal{R}_- = \mathbb{C}[N^{-1}][[t_3,\ldots,t_d,a_1,\ldots,a_\nu]]$*. For any element* X *of* $\mathcal{R}$*, there is a unique decomposition*

$$X = \sum_{\chi\in\mathbb{Z}} N^{\chi}\, X^{\{\chi\}}, \qquad X^{\{\chi\}} \in \mathcal{R}_- \tag{9.3}$$

as an equality in $\mathcal{R}$*. The same is true if* $\mathcal{R}$ *is replaced by* $\mathcal{R}^{\mathfrak{S}_\nu}$ *or a ring of formal series over* $\mathcal{R}$*. This is called the* topological expansion *of* X*.*

Proof. $X^{\{\chi\}}$ just collects in X the monomials proportional to $N^{\chi-s}\prod_{m\geq 3} t_m^{f_m}\prod_{i=1}^{\nu} a_i^{\ell_i}$ such that $s=\sum_{i=1}^{\nu}\ell_i$, and divide them by N^{χ}. □

Remark 9.3. Not all series of the form of the right-hand side of (9.3) give a well-defined element of $\tilde{\mathcal{R}}$. Indeed, the definition of $\tilde{\mathcal{R}}$ implies that for given f_m and ℓ_i, there exists a finite χ_0 such that the monomial $\prod_{m\geq 3} t_m^{f_m}\prod_{i=1}^{\nu} a_i^{\ell_i}$ does not appear in $X^{\{\chi\}}$ for $\chi > \chi_0$.

Coming back to (9.2), we see that $\hat{\mathcal{Z}}^{\{\chi\}}(A)$ give by construction the generating series of maps with Euler characteristic χ. Taking the logarithm, we have the topological expansion

$$\hat{F}(A) = \ln \hat{\mathcal{Z}}(A) = \sum_{g\geq 0} N^{2-2g}\,\hat{F}_g(A), \qquad F^{[g]}(A) := F^{\{2-2g\}}(A) \in \mathcal{R}_-^{\mathfrak{S}_\nu}\,, \tag{9.4}$$

where $F^{[g]}(A)$ generates certain maps of genus g. The same arguments show the topological expansion of the correlators

$$\hat{W}_n(x_1,\ldots,x_n;A) = \kappa_{n;A}\Big(\mathrm{Tr}\,\frac{1}{x_1-M},\ldots,\mathrm{Tr}\,\frac{1}{x_n-M}\Big) = \sum_{g\geq 0} N^{2-2g}\,\hat{W}_n^{[g]}(x_1,\ldots,x_n)$$

gives generating series

$$\hat{W}_n^{[g]}(x_1,\ldots,x_n;A) := \hat{W}_n^{\{2-2g-n\}}(x_1,\ldots,x_n;A) \in \mathcal{R}_-^{\mathfrak{S}_\nu}[[x_1^{-1},\ldots,x_n^{-1}]]$$

of certain maps of genus g with n boundaries.

Remark 9.4. We stress that the topological expansion is not naively a large N expansion, and that the genus g part of F and W_n here depends on N. This brings some complications which were absent for the formal 1-hermitian matrix model. As the number ν of non-zero entries in A is arbitrary, once we work in $\mathcal{R}$ or $\mathcal{R}_-$, we can set $\nu = N$. We will do so in the remaining of the proof, as it simplifies the exposition.

9.2. **A new generating series.** Corollary 7.11 tells us that

$$\hat{F}_g(A) = \sum_{n\geq 1}\sum_{\ell_1,\ldots,\ell_n\geq 1} \frac{p_{\ell_1}(A)}{\ell_1}\cdots\frac{p_{\ell_n}(A)}{\ell_n}\kappa_n^{[g]}\big(\mathcal{P}^{(\ell_1)}(M),\ldots,\mathcal{P}^{(\ell_n)}(M)\big).$$

9.3. **The spectral curve.** We shall make use of the first Schwinger-Dyson equation of this model, which is proved in [35] in the more general context of the chain of matrices.

Lemma 9.5. *We have the identity in* $\mathcal{R}[[x^{-1},y^{-1}]]$

$$\begin{aligned} 0 \;=\; & \kappa_{2;A}\Big(\mathrm{Tr}\,\frac{1}{x-M},\mathrm{Tr}\,\frac{1}{x-M}\,\frac{1}{y-A}\Big) + (W_1(x;A) - NV'(x) + Ny)\Big\langle \mathrm{Tr}\,\frac{1}{x-M}\,\frac{1}{y-A}\Big\rangle_A \\ & -NW_1(x;A) + \Big\langle \mathrm{Tr}\,\frac{V'(x)-V'(M)}{x-M}\,\frac{1}{y-A}\Big\rangle_A . \end{aligned}$$

Proof. This is obtained from the relation

$$0=\sum_{i,j=1}^{N}\sum\int \mathrm{d}M\,\partial_{M_{i,j}}\bigg(\Big(\frac{1}{x-M}\,\frac{1}{y-A}\Big)_{j,i}\exp\big[N\,\mathrm{Tr}(MA-V(M))\big]\bigg).$$

□

We now look at the first term in the topological expansion of this equation, also called planar limit. Let us introduce simplified notations

$$W^{(i)}(x;A)=\Big\langle\Big[\frac{1}{x-M}\Big]_{i,i}\Big\rangle_A^{[0]},\qquad W(x;A)=W_1^{[0]}(x;A)=\sum_{i=1}^{N}W^{(i)}(x;A)\,.$$

Proposition 9.6. *We have*

$$W(x;A)^2-[V'(x)W(x;A)]_-+\sum_{i=1}^{N}\tfrac{a_i}{N}W^{(i)}(x;A)=0\,, \tag{9.5}$$

and for any $i\in\{1,\ldots,N\}$, we have

$$W^{(i)}(x;A)\big(W(x;A)+a_i\big)-\tfrac{1}{N}[V'(x)W^{(i)}(x;A)]_-=0\,, \tag{9.6}$$

where $[\cdots]_-$ takes the negative part of the Laurent expansion when $x\to\infty$.

Proof. In the planar limit of (9.5), U_2 disappears

$$0=(W(x;A)-V'(x)+y)\Big\langle\mathrm{Tr}\,\frac{1}{x-M}\frac{1}{y-A}\Big\rangle_A-W(x;A)+\frac{1}{N}\Big\langle\mathrm{Tr}\,\frac{V'(x)-V'(M)}{x-M}\,\frac{1}{y-A}\Big\rangle_A^{[0]}.$$

The right-hand side rational function of y, with simple poles at $y\to a_i$ and $y\to\infty$. Identifying the coefficient of these poles gives an equivalent set of equations. At $y\to\infty$ we get a trivial relation. At $y\to a_i$ we get

$$(W(x;A)-V'(x)+\tfrac{a_i}{N})W^{(i)}(x;A)+\Big\langle\Big[\frac{V'(x)-V'(M)}{x-M}\Big]_{i,i}\Big\rangle_A^{[0]}=0\,, \tag{9.7}$$

in which we recognize (9.6). Summing this relation over $i\in\{1,\ldots,N\}$, we obtain (9.5). □

For $A=0$, we recognize in (9.5) the planar limit of the first Schwinger-Dyson equation of the hermitian matrix model

$$W(x;0)^2-[V'(x)W(x;0)]_-=0\,, \tag{9.8}$$

which is equivalent to Tutte's equation for the generating series of disks. Its solution is well-known, see *e.g.* [28].

Lemma 9.7. *The equation* (9.8) *determines completely $W(x;0)$. Let $\mathcal{R}^0=\mathbb{Q}[[t_3,\ldots,t_d]]$. There exist unique $\alpha,\gamma\in\mathcal{R}^0$ such that*

$$x(\zeta;0)=\alpha+\gamma(\zeta+\zeta^{-1}),\qquad w(\zeta;0)=[V'(x(\zeta))]_+$$

satisfy $W_1^{[0]}(x(\zeta);0)=V'(x(\zeta))-w(\zeta;0)$. Here, $[\cdots]_+$ takes the polynomial part in ζ. More precisely, α and γ are determined by the conditions

$$[\zeta^0]\,V'(x(\zeta))=0,\qquad [\zeta^{-1}]\,V'(x(\zeta))=\gamma^{-1}\,.$$

and we have $\gamma=1+O(\mathbf{t})$. □

Our next goal is to prove:

Lemma 9.8. *The equations* (9.5)-(9.6) *determine uniquely $W^{(i)}(x;A)$ for all $i\in\{1,\ldots,p\}$.*

Proof. We introduce two gradings in the ring $\mathcal{R}_-[[x_1^{-1}]]$: the first one denoted $\deg_A$ assigns a degree 1 to the variables a_i, and 0 to the other generators; the second one $\deg_t$ assigns a degree 1 to the variables t_j and 0 to the other generators. We denote $W^{(i)}_\alpha$ and W_α the homogeneous part of $W^{(i)}$ and W with $\deg_A = \alpha$. We remark that $W_0^{(i)}(x;A) = W^{(i)}(x;0)$ is independent of A and i and thus

$$W_0^{(i)}(x;A) = \frac{W(x;0)}{N}.$$

Besides, we observe that

$$\forall \alpha \geq 1, \quad \forall i \in \{1,\ldots,N\}, \qquad W_\alpha^{(i)}(x;A) = O(x^{-2}). \tag{9.9}$$

We proceed by induction on $\deg_A$. We already know that the $\deg_A = 0$ part of (9.5)-(9.6) has a unique solution given by Lemma 9.7. Let $\alpha \geq 1$, and assume (9.5)-(9.6) determine uniquely $W^{(i)}_{\alpha'}$ for $\alpha' < \alpha$. Decomposing (9.5) in homogeneous degree $\alpha \geq 1$, we find

$$\mathcal{K}[W_\alpha(-;A)](x) + \sum_{\substack{0<\alpha_1,\alpha_2<\alpha \\ \alpha_1+\alpha_2=\alpha}} W_{\alpha_1}(x;A)W_{\alpha_2}(x;A) + \sum_{i=1}^N \frac{a_i}{N} W^{(i)}_{\alpha-1}(x;A) = 0, \tag{9.10}$$

where $\mathcal{K}$ is the linear operator

$$\mathcal{K}[f](x) = 2W(x;0)f(x) - [V'(x)f(x)]_-.$$

Let us write

$$V(x) = \frac{x^2}{2} + \delta V(x), \qquad \deg_t(\delta V) = 1.$$

When $f(x) = O(x^{-2})$, we have

$$\mathcal{K}[f](x) = (2W(x;0) - x)f(x) - [(\delta V)'(x)f(x)]_-.$$

Under this assumption, the equation

$$\mathcal{K}[f](x) = g(x) \tag{9.11}$$

in $\mathcal{R}_-[x^{-1}]$ has a unique solution, which we denote $f(x) = \mathcal{K}^{-1}[g](x)$. Indeed, $(2W(x;0) - x)$ is invertible and (9.11) determine recursively the $\deg_t$ homogeneous components of f recursively in terms of those of g. Taking into account (9.9), we can apply this remark to (9.10) and find that

$$W_\alpha(x;A) = -\mathcal{K}^{-1}\Bigg[\sum_{\substack{0<\alpha_1,\alpha_2<\alpha \\ \alpha_1+\alpha_2=\alpha}} W_{\alpha_1}(-;A)W_{\alpha_2}(-;A) + \sum_{i=1}^N \frac{a_i}{N} W^{(i)}_{\alpha-1}(-;A)\Bigg](x)$$

is determined. We turn to the $\deg_A = \alpha$ part of (9.6)

$$\mathcal{K}[W_\alpha^{(i)}(-;A)](x) + \tfrac{1}{N}\big(W(x;0)W_\alpha(x;A) + \frac{a_i}{N}W^{(i)}_{\alpha-1}(x;A)\big) + \sum_{\substack{0<\alpha_1,\alpha_2<\alpha \\ \alpha_1+\alpha_2=\alpha}} W^{(i)}_{\alpha_1}(x;A)W^{(i)}_{\alpha_2}(x;A) = 0.$$

Hence

$$W^{(i)}(x;A) = -\mathcal{K}^{-1}\Bigg[\tfrac{1}{N}\big(W(-;0)W_\alpha(-;A) + \frac{a_i}{N}W^{(i)}_{\alpha-1}(-;A)\big) + \sum_{\substack{0<\alpha_1,\alpha_2<\alpha \\ \alpha_1+\alpha_2=\alpha}} W^{(i)}_{\alpha_1}(-;A)W^{(i)}_{\alpha_2}(-;A)\Bigg](x)$$

is determined as well. We conclude by induction. □

Lemma 9.9. *There exists a unique polynomial $w(z;A)\in\mathcal{R}_-[z]$ and $b_k=b_k(A)\in\mathcal{R}_-$ for $k\in\{1,\ldots,N\}$, such that*

$$x(z;A)=z+\frac{1}{N}\sum_{i=1}^{N}\frac{1}{w'(b_i;A)(z-b_i)}\,, \tag{9.12}$$

together with

$$w(z;A)=V'(x(z))+O(z^{-1}),\qquad w(b_k;A)=a_k\,. \tag{9.13}$$

Besides, this unique data is such that

$$w(z;A)=V'(x(z))+z^{-1}+O(z^{-2}). \tag{9.14}$$

Proof. Let $\beta_i(A),b_i(A)$ be elements of $\mathcal{R}_-$, so far undetermined. We assume $\beta_i(A)$ invertible, and $\beta:=\beta_i(0)$ and $b:=b_i(0)$ independent of i. We define

$$x(z;A)=z+\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\beta_i(A)(z-b_i(A))} \tag{9.15}$$

and introduce the polynomial $w(z;A)\in\mathcal{R}_-[z]$ such that

$$V'(x(z;A))=w(z;A)+O(z^{-2}),\qquad z\to\infty \tag{9.16}$$

Equivalently

$$w(z;A)=\oint\frac{\mathrm{d}\tilde{z}}{2\mathrm{i}\pi}\,\frac{V'(x(\tilde{z};A))}{\tilde{z}-z},$$

where the contour is close enough to $\tilde{z}=\infty$. We are going to prove that the system of equations

$$\forall i\in\{1,\ldots,N\},\qquad \left\{\begin{array}{lcl} w(b_i(A);A) & = & a_i \\ w'(b_i(A);A) & = & \beta_i(A)\end{array}\right. \tag{9.17}$$

has unique solutions $b_i(A)$ and $\beta_i(A)$ in $\mathcal{R}_-$, which we will adopt to define (9.15)-(9.16).

For $A=0$, our definition gives

$$x(z;0)=z+\frac{1}{\beta(z-b)}.$$

Let $\alpha,\gamma\in\mathcal{R}^0$ be as in Lemma 9.7. We recall that $\gamma=1+O(t)$, hence γ is invertible in $\mathcal{R}^0$. By making the change of variable $z=\gamma\zeta+\alpha$ and choosing

$$\beta_i(A)=\gamma^{-2}+O(A),\qquad b_i(A)=\alpha+O(A),$$

we find by comparing with Lemma 9.7 that

$$W(x(z);0)=V'(x(z))-w(z;0),\qquad w(b;0)=0,\qquad w'(b;0)=\beta,$$

in terms of the functions introduced in (9.15)-(9.16).

Next, we introduce a grading in $\mathcal{R}_-$ by assigning degree 1 to each a_i, and 0 to all other generators. We write $x_d(z;A)$, $w_d(z;A)$, $\beta_{i,d}(A)$ and $b_{i,d}(A)$ for the degree d component of the corresponding quantities. Fix $d\geq 1$, and assume we have already determined all these quantities in degree $d'<d$. Let us examine the degree d part of the system (9.17), and isolate the pieces involving $\beta_{i,d}(A)$ and $b_{i,d}(A)$. We find for all $i\in\{1,\ldots,N\}$

$$\left\{\begin{array}{lcl} w'(\gamma^{-2};0)b_{i,d}(A)+\frac{c_2}{N}\Big(\sum_{i=1}^{N}\frac{\beta_{i,d}(A)}{\gamma^{-4}}\Big)+\frac{c_3}{N}\Big(\sum_{i=1}^{N}\frac{b_{i,d}(A)}{\gamma^{-2}}\Big) & = & Y_{i,d}(A), \\ w''(\gamma^{-2};0)b_{i,d}(A)+\frac{c_3}{N}\Big(\sum_{i=1}^{N}\frac{\beta_{i,d}(A)}{\gamma^{-4}}\Big)+\frac{c_4}{N}\Big(\sum_{i=1}^{N}\frac{b_{i,d}(A)}{\gamma^{-2}}\Big) & = & \tilde{Y}_{i,d}(A),\end{array}\right. \tag{9.18}$$

where

$$c_k = \oint \frac{\mathrm{d}\tilde{z}}{2\mathrm{i}\pi}\,\frac{V''(x(\tilde{z};0))}{(\tilde{z}-\alpha)^k}$$

and $Y_{i,d}$ and $\tilde{Y}_{i,d}$ are polynomials in the t_ks, a_is, and $\beta_{j,d'}(A)$ and $b_{j,d'}(A)$ with $d' < d$. In this formula we used that $\beta_{j,0} = \gamma^{-2}$ and $b_{j,0} = \alpha$ for all j. For instance, in degree 1

$$Y_{i,1}(A) = a_i, \qquad \tilde{Y}_{i,1}(A) = 0.$$

As $V''(x) = 1 + O(t)$, we deduce by moving the contour to surround $\tilde{z} = \alpha$ that

$$\forall k \geq 2, \qquad c_k = O(t).$$

Therefore, the system (9.18) takes the matrix form

$$\left[\begin{pmatrix} \gamma^{-2}\,\mathrm{Id}_N & 0 \\ 0 & -\mathrm{Id}_N \end{pmatrix} + N^{-1}O(t)\right]\begin{pmatrix} b_{\bullet,d} \\ \beta_{\bullet,d} \end{pmatrix} = \begin{pmatrix} Y_{\bullet,d}(A) \\ \tilde{Y}_{\bullet,d}(A) \end{pmatrix}.$$

The matrix in the left-hand side is invertible in $\mathcal{R}_-$, hence $\beta_{i,d}(A)$ and $b_{i,d}(A)$ are uniquely determined. By induction, we conclude to the existence of unique $\beta_i(A)$ and $b_i(A)$ in $\mathcal{R}_-$ satisfying (9.17).

Let $c_\infty \in \mathcal{R}_-$ be such that

$$V'(x(z;A)) = w(z;A) + c_\infty\, z^{-1} + O(z^{-2}). \tag{9.19}$$

We claim that $c_\infty = 1$. Indeed, the second set of equations in (9.17) imply

$$\sum_{i=1}^{N} \operatorname*{Res}_{z\to b_i} x(z;A)\mathrm{d}w(z;A) = 1,$$

while as $x(z;A) = O(z)$ when $z \to \infty$, we obtain by moving the contour to ∞ and using (9.19)

$$\sum_{i=1}^{N} \operatorname*{Res}_{z\to b_i} x(z;A)\mathrm{d}w(z;A) = -\operatorname*{Res}_{z\to\infty} x(z;A)\mathrm{d}w(z;A) = c_\infty - \operatorname*{Res}_{x\to\infty} x\,\mathrm{d}V' = c_\infty.$$

The last equality holds because V' is a polynomial in x. □

Lemma 9.10. *There exists a unique polynomial $P_i(\xi;A) \in \mathcal{R}_-[\xi]$ of degree $d-2$ with leading coefficient $\frac{t_d}{N}$, such that*

$$V'(x(z;A)) - w(z;A) = \sum_{i=1}^{N} \frac{P_i(x(z);A)}{w(z;A) - a_i}. \tag{9.20}$$

Proof. We have

$$x(z;A) = \frac{S_{N+1}(z)}{T_N(z)}, \qquad w(z;A) = U_{d-1}(z),$$

where S, T, U are polynomials in z with coefficients in $\mathcal{R}_-$, of degree indicated by the subscript, and T_N is monic. Therefore, the resultant

$$Q(X,Y) = \mathrm{res}_z\big[XT_N(z) - S_{N+1}(z), T_N(z)(Y - U_{d-1}(z))\big]$$

is a polynomial with coefficients in $\mathcal{R}_-$ and of degree $N+d-1$ in X, and $N+1$ in Y, which gives a polynomial relation

$$Q(x(z;A), w(z;A)) = 0.$$

In fact, we can argue that the degree in X smaller, as follows. We study the slopes of the Newton polygon of Q associated to $x \to \infty$ or $w \to \infty$. We have $w \to \infty$ if and only if $z \to \infty$, and in this case $x \to \infty$ and we have

$$w \sim -t_d x^{d-1}. \tag{9.21}$$

Besides, the only other situation where $x \to \infty$ is when $w \to a_i$ for some $i \in \{1, \ldots, N\}$, and in this case we read from (9.12) that

$$(w - a_i)x \ \sim \ 1. \tag{9.22}$$

This determines slopes which must be in the Newton polygon of Q. A closer look to the determinant defining the resultant shows, using $T_N(0) = (-1)^N \prod_{i=1}^N a_i$ and that $XT_N(z) - S_{N+1}(z) = -z^{N+1} + O(z^N)$, that the top degree term of $Q(X,Y)$ in the variable Y is $Y^{N+1} \prod_{i=1} a_i^N$. In particular, the coefficient of $X^i Y^{N+1}$ which could a priori exist, vanish for $i \in \{1, \ldots, d-1+N\}$. The existence of the previous slopes then forces the Newton polygon to be included in the shaded region of Figure 15. In particular, Q must be irreducible. And the precise behaviors (9.21)-(9.22) leads to a decomposition

$$Q(X,Y) = c\Big(Y^{N+1} + t_d X^{d-1} \prod_{i=1}^{N} (Y - a_i)\Big) + \tilde{Q}(X,Y), \qquad c := \prod_{i=1}^{N} a_i^N, \tag{9.23}$$

for some polynomial $\tilde{Q}(X,Y)$ with coefficients in $\mathcal{R}_-$ such that

$$\deg_X \tilde{Q} \leq d-2, \qquad \deg_Y \leq N. \tag{9.24}$$

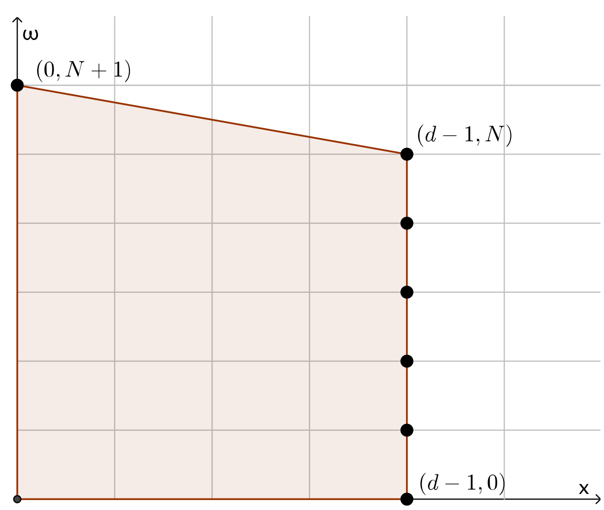


FIGURE 15. The Newton polygon of $Q(x,w)$. The coefficients identified in (9.23) correspond to the nodes in the picture.

Now, let us examine

$$L(z) = \big(V'(x(z;A)) - w(z;A)\big)\prod_{i=1}^{N}(w(z;A) - a_i).$$

It can also be written

$$L(z) = -w(z;A)^{N+1} - t_d x(z;A)^{d-1}\prod_{i=1}^{N}(w(z;A) - a_i) + \tilde{L}(x(z;A), w(z;A)),$$

where $\tilde{L}(X,Y)$ is a polynomial satisfying the same degree bound as (9.24). Using the relation (9.23), we can eliminate the first terms and get

$$L(z) = \hat{L}(x(z;A), w(z;A)),$$

for some polynomial $\hat{L}(X,Y)$ with coefficients in the localization $\mathcal{R}_-^{(c)}$ of $\mathcal{R}_-$ at c, and with $\deg_X \hat{L} \leq d-2$ and $\deg_Y \hat{L} \leq N$. We deduce the existence of polynomials $(P_i(X))_{i=1}^N$ and $P_\infty(X)$ of degree $\leq d-2$ and with coefficients in $\mathcal{R}_-^{(c)}$, such that

$$V'(x(z;A)) - w(z;A) = P_\infty(x(z;A)) + \sum_{i=1}^{N}\frac{P_i(x(z;A))}{w(z;A) - a_i},$$

after partial fraction decomposition. According to (9.13), the left-hand side behaves is $O(z^{-1})$ when $z \to \infty$. As $w(z;A) = -t_d z^{d-1}$ and $P_i(x(z;A)) = O(z^{d-2})$, this is also true for the sum over i. Any non-zero monomial in P_∞ would disagree with this behavior at $z \to \infty$. Hence $P_\infty = 0$, and we have prove the existence result for polynomials $P_i(x(z;A))$ with coefficients in $\mathcal{R}_-^{(c)}$.

Now, we prove uniqueness. By construction, $w(z;A) - a_i$ is a polynomial of degree $d-1$ in z with coefficients in the local ring $\mathcal{R}_-$, and $b_i(A)$ is a root. We remark that

$$w(z;A) = -t_d z^{d-1} + z + \tilde{w}(z;A),$$

where $\tilde{w}(z;A) = O(A,t)$ is a polynomial of degree $\leq (d-2)$. Therefore, $w(z;A) - a_i$ has $(d-2)$-other roots, counted with multiplicity, which belong to the local ring $\widehat{\mathcal{R}}_-$ obtained from $\mathcal{R}_-$ by adjunction of a finite set $\mathfrak{b}$ consisting of $t_d^{-1/(d-2)}$ and all its Galois conjugates. Besides, in $\widehat{\mathcal{R}}_-$ those $(d-2)$ roots are pairwise distinct.

By construction, $V'(x(z);A) - w(z;A)$ is a rational function of z, with poles at $z = b_i$ and $z = \infty$. As the denominator of the i-th term has $(d-2)$ roots $\mathfrak{b}_i$ which are not poles of $\mathcal{W}(z;A)$ as set of root, we deduce that $P_i(x(z);A)$ must have roots at $\mathfrak{b}_i$, hence

$$P_i(\xi;A) = \tfrac{t_d}{N}\prod_{\rho(A)\in\mathfrak{b}_i(A)}(\xi - x(\rho;A)).$$

Therefore, (9.20) determines uniquely the polynomial $P_i(\xi;A)$. Note that the coefficients of this polynomial belong to the localization of $\mathcal{R}_-^{(t_d)}$ of $\mathcal{R}_-$ at t_d, and not only to the localization of $\widehat{\mathcal{R}}_-$ at t_d, as the product runs over Galois orbits. By comparison, the P_i constructed in the existence part had coefficients in $\mathcal{R}_-^{(c)}$. We deduce that P_i has coefficients in $\mathcal{R}_-^{(c)} \cap \mathcal{R}_-(t_d) = \mathcal{R}_-$. □

Corollary 9.11. *We have*

$$W(x(z);A) = V'(x(z);A) - w(z;A), \qquad W^{(i)}(x(z);A) = \frac{P_i(x(z);A)}{w(z;A) - a_i}.$$

Proof. As $x = z + O(z^{-1})$, we can perform a Lagrange inversion and define unique elements $\mathcal{W}(x;A)$ and $\mathcal{W}^{(i)}(x;A)$ in $\mathcal{R}_-[[x^{-1}]]$ such that

$$\mathcal{W}(x(z;A);A) = V'(x(z;A)) - w(z;A), \qquad \mathcal{W}^{(i)}(x(z;A);A) = \frac{P_i(x(z;A);A)}{w(z;A) - a_i}.$$

By construction, we have

$$\sum_{i=1}^{N} \mathcal{W}^{(i)}(x;A) = \mathcal{W}(x;A), \tag{9.25}$$

and

$$\forall i \in \{1,\ldots,N\}, \qquad \mathcal{W}^{(i)}(x;A)(\mathcal{W}(x;A) - V'(x) + a_i) + P_i(x) = 0.$$

Therefore, $\mathcal{W}^{(i)}$ and $\mathcal{W}$ satisfy (9.5)-(9.6). Note that (9.14) ensures that

$$\mathcal{W}(x) = \tfrac{1}{x} + O(\tfrac{1}{x^2}), \qquad x \to \infty,$$

and that $V'(x) = -t_d x^{d-1}$ also implies

$$\mathcal{W}^{(i)}(x) = \tfrac{1}{Nx} + O(\tfrac{1}{x^2}), \qquad x \to \infty.$$

Since the solution of these equations is unique according to Lemma 9.8, we conclude that $W = \mathcal{W}$ and $W^{(i)} = \mathcal{W}^{(i)}$. ☐

9.4. **Topological recursion and symplectic invariance.** Eynard and Prats-Ferrer have analyzed in [35] the (topological expansion) of tower of Schwinger-Dyson equations which results from variation of the potential V, and involve the n-point correlators. Their final result is

Theorem 9.12. *Let $\omega_{g,n}^A$ be the TR amplitudes for the initial data*

$$\begin{cases} \mathcal{C} = \mathbb{P}^1 \\ p(z) = x(z;A) \\ \lambda(z) = -w(z;A) \\ B(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}. \end{cases}$$

For any $g,n \geq 0$, the equality

$$W_{g,n}^A(p(z_1),\ldots,p(z_n)) \prod_{i=1}^{n} \mathrm{d}p(z_i) = \omega_{g,n}^A(z_1,\ldots,z_n) - \delta_{g,0}\delta_{n,2}\frac{\mathrm{d}p(z_1)\mathrm{d}p(z_2)}{(p(z_1)-p(z_2))^2} + \delta_{g,0}\delta_{n,1}\,\mathrm{d}V(p(z_1))$$

holds in Laurent expansion near $z_i \to \infty$.

As the initial data is a compact curve with two meromorphic functions λ and p, we can use the symplectic invariance stated from Theorem 1.7.

Corollary 9.13. *The $\hat{F}_g(A)$ of (9.4) are the TR free energies for the initial data*

$$\begin{cases} \mathcal{C} = \mathbb{P}^1 \\ p(z) = w(z;A) \\ \lambda(z) = x(z;A) \\ B(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}. \end{cases}$$

☐

9.5. **Deformations of the spectral curve.**

Lemma 9.14. *We have, for any $i\in\{1,\ldots,N\}$*

$$\Omega_i(z;A)=\partial_{a_i}x(z;A)w'(z;A)-\partial_{a_i}w(z;A)x'(z;A)=\frac{1}{w(b_i;A)}\,\frac{1}{(z-b_i)^2}, \tag{9.26}$$

where the derivative with respect to a_i is taken at z fixed, and $'$ denotes the derivative with respect to the variable z.

Proof. From the form of $x(z;A)$ and $w(z;A)$, we know that $\Omega_i(z,A)$ is a rational function of z, with atmost double poles at $z\to b_k$ for $k\in\{1,\ldots,N\}$, and maybe a pole at ∞. We are going to identify $\Omega_i(z,A)$ from its singular behavior at these poles. It is easier to start by computing

$$\tilde{\Omega}_j(z;A):=\partial_{b_j}x(z;A)w'(z;A)-\partial_{b_j}w(z;A)x'(z;A)$$

and then use the relation

$$\Omega_i(z;A)=\sum_{j=1}^{N}\frac{\partial b_j}{\partial a_i}\,\tilde{\Omega}_j(z;A)\,. \tag{9.27}$$

We start by examining $z\to\infty$. From the equation

$$w(z;A)=V'(x(z;A))+\tfrac{1}{z}+O(\tfrac{1}{z^2})\,,$$

we deduce

$$\begin{aligned} w'(z;A) &= x'(z;A)V''(x(z;A))-\tfrac{1}{z^2}+O(\tfrac{1}{z^3})\,,\\ \partial_{b_j}w(z;A) &= \partial_{b_j}x(z;A)V''(x(z;A))+O(\tfrac{1}{z^2})\,, \end{aligned}$$

and from the form of x

$$\partial_{b_j}x(z;A)=O(1),\qquad x'(z;A)=O(1)\,.$$

This implies

$$\begin{aligned} \tilde{\Omega}_i(z;A) &= \partial_{b_j}x(z;A)\big(x'(z;A)V''(x(z;A))+O(\tfrac{1}{z^2})\big)-\big(\partial_{b_j}x(z;A)V''(x(z;A)+O(\tfrac{1}{z^2})\big)x'(z;A)\\ &= O(\tfrac{1}{z^2})\,, \end{aligned}$$

and therefore $\tilde{\Omega}_i(z;A)$ has no pole at ∞.

Next, we examine $z\to b_k$. We have

$$\begin{aligned} \partial_{b_j}x(z;A) &= \frac{\delta_{j,k}}{N}\bigg(-\frac{w''(b_k;A)}{(w'(b_k;A))^2}\,\frac{1}{z-b_k}+\frac{1}{w'(b_k;A)}\,\frac{1}{(z-b_k)^2}\bigg)-\frac{1}{N}\,\frac{\partial_{b_j}w'(b_k;A)}{(w'(b_k;A))^2}\,\frac{1}{z-b_k}+O(1)\,,\\ w'(z;A) &= w'(b_k;A)+w''(b_k;A)(z-b_k)+O(z-b_k)^2\,,\\ \partial_{b_j}w(z;A) &= \partial_{b_j}w(b_k;A)+\partial_{b_j}w'(b_k;A)(z-b_k)+O(z-b_k)^2\,,\\ x'(z) &= -\frac{1}{w'(b_k;A)}\,\frac{1}{(z-b_k)^2}+O(1)\,. \end{aligned}$$

Hence, we obtain after simplification

$$\tilde{\Omega}_j(z;A)=\frac{1}{N}\bigg(\delta_{j,k}+\frac{\partial_{b_j}w'(b_k;A)}{w'(b_k;A)}\bigg)\,\frac{1}{(z-b_k)^2}+O(1)\,.$$

So, the only singularities of $\tilde{\Omega}_j(z;A)$ are double pole without residues at b_k, and we get

$$\tilde{\Omega}_j(z;A)=\frac{1}{N}\sum_{k=1}^{N}\bigg(\delta_{j,k}+\frac{\partial_{b_j}w'(b_k;A)}{w'(b_k;A)}\bigg)\,.$$

We finally return to $\Omega_j(z;A)$. Differentiating the relation $w(b_k;A) = a_k$ with respect to a_i we get

$$\partial_{a_i} w(b_k;A) + \partial_{a_i} b_k w'(b_k;A) = \delta_{i,k}\,.$$

We insert it in (9.27) and find a simplification

$$\Omega_i(z;A) = \frac{1}{N}\,\frac{1}{w'(b_i;A)}\,\frac{1}{(z-b_i)^2}\,.$$

□

A general property of the TR amplitudes, which we only state here in the genus 0 case, is the following:

Theorem 9.15. [31] *Let* $(\mathbb{P}^1, p, \lambda, B = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2})$ *be a smooth family of initial data for TR, depending on a parameter* $t \in \mathbb{C}$, *and* $\omega_{g,n}^t$ *its TR amplitudes. Assume there exists a current* $\gamma \subset \mathbb{P}^1$ *whose support does not contain the zeroes of* $\mathrm{d}p$, *such that*

$$\partial_t\lambda(z)\mathrm{d}y(z) - \partial_t\lambda(z)\mathrm{d}x(z) = \int_\gamma B(z,\cdot).$$

where the derivatives are taken with z *fixed. Then, for any* $g \geq 0$ *and* $n, m \geq 0$

$$\partial_t^m \omega_{g,n}(z_1,\ldots,z_n)|_{t=t_0} = \int_{\gamma^m} \omega_{g,n+m}^{t_0}(z_1,\ldots,z_n,\cdots).$$

where the derivatives are taken at z_i *fixed, and the last* m *variables in the integrand are integrated over* γ^m. *In particular, for* $n = 0$, *we obtain* $\hat{F}_g(A)$.

If we specialize to $n = 0$ and the deformation (9.26), we find

Corollary 9.16. *We have for any* $n \geq 1$

$$\partial_{a_1}\cdots\partial_{a_n}\hat{F}^{[g]}(A) = \frac{\breve{\omega}_n^{[g]}(z_1,\ldots,z_n)}{\mathrm{d}w(z_1)\cdots\mathrm{d}w(z_n)},$$

where the z_i *are points in* $\mathcal{C}$ *defined by* $w(z_i;0) = a_i$.

Proof. Ω_i represents the infinitesimal deformation of the TR initial data $(p = x(z;A), \lambda = w(z;A))$, and an equivalent form of (9.26) is

$$\Omega_i(z;A)\mathrm{d}z = \operatorname*{Res}_{\zeta\to b_i} \frac{B(z,\zeta)}{w(\zeta;A) - w(b_i;A)}.$$

Therefore,

$$\begin{aligned}
\partial_{a_1}\cdots\partial_{a_n}\hat{F}^{[g]}(A) &= \operatorname*{Res}_{\zeta_1\to b_1}\cdots\operatorname*{Res}_{\zeta_n\to b_n} \frac{\breve{\omega}_n^{[g]}(\zeta_1,\ldots,\zeta_n)}{\prod_{i=1}^n (w(\zeta_i;A) - w(b_i;A))} \\
&= \frac{\breve{\omega}_n^{[g]}(b_1,\ldots,b_n)}{\mathrm{d}w(b_1;A)\cdots\mathrm{d}w(b_n;A)}, \qquad (9.28)
\end{aligned}$$

where the differential acts on the first variable of w. □

9.6. **Conclusion.** Let $\mathcal{R}_-^0 = \mathbb{Q}[[t_3,\ldots,t_d]]$. We have $\mathcal{R}_-^{\mathfrak{S}_N} \cong \mathcal{R}_-^0[p_1,p_2,\ldots]$ in terms of the power sums of a_i, this defines a graded ring by assigning degree 1 to each generator p_ℓ. As an element of $\mathcal{R}_-^{\mathfrak{S}_N}$, the free energy decomposes as

$$\hat{F}^{[g]} = \sum_{n\geq 1}\sum_{\ell_1,\ldots,\ell_n} \hat{F}^{[g]}_{n;\ell_1,\ldots,\ell_n}\prod_{i=1}^n \frac{p_{\ell_i}(A)}{\ell_i}, \qquad \hat{F}_{g,n}[\ell_1,\ldots,\ell_n]\in\mathcal{R}_-^0.$$

Corollary 9.17. *For $2g-2+n>0$, we have the equality in $\mathcal{R}_-$*

$$\check{\omega}_{g,n}^{A=0}(b_1,\ldots,b_n) = \sum_{\ell_1,\ldots,\ell_n\geq 1}\hat{F}^{[g]}_{n;\ell_1,\ldots,\ell_n}\prod_{i=1}^n a_i^{\ell_i-1} \qquad \text{with} \quad w(b_i;0)=a_i.$$

Proof. If X is a graded ring, we denote $[X]_m$ its degree m subspace. Let $n\geq 1$ be an integer. We denote $\pi_{\mathcal{R},n}$ the projection from $\mathcal{R}_-^{\mathfrak{S}_N}$ to its degree n subspace. We introduce the ring $\mathcal{Q}_-^{(n)} = \mathcal{R}_-^0[[a_1,\ldots,a_n]][[p_1,p_2,\ldots]]$. We make it a graded ring by assigning degree 1 to each generator p_ℓ. We denote $\pi_{\mathcal{Q},0}$ the projection from $\mathcal{Q}_-^{(n)}$ to its degree 0 subspace. We can define a linear map $\Upsilon^{(n)} : \mathcal{R}_-^{\mathfrak{S}_N}\to\mathcal{Q}_-^{(n)}$ by

$$\Upsilon^{(n)}(p_\lambda) = \sum_{\substack{\mu\sqcup\mu'=\lambda \\ \ell(\mu)=n}} \mu_1\cdots\mu_n \mathrm{M}_\mu(a_1,\ldots,a_n)\,p_{\mu'}, \qquad \mathrm{M}_\mu(a_1,\ldots,a_n) = \sum_{\sigma\in\mathfrak{S}_n}\prod_{i=1}^n a_{\sigma(i)}^{\mu_i}$$

and extending by linearity. In other words

$$\Upsilon^{(n)}(f) = a_1\partial_{a_1}\cdots a_n\partial_{a_n} f.$$

$\Upsilon^{(n)}$ is a homogeneous map of degree $-n$, in particular it sends p_λ with $\ell(\lambda)<n$ to zero. Besides, from the degree n part to the degree 0 part it induces an isomorphism, which is just the change of basis from power sums to (unnormalized) symmetric monomials, and we have

$$\Upsilon^{(n)}\circ\pi_{\mathcal{R},n} = \pi_{\mathcal{Q},0}\circ\Upsilon^{(n)}. \tag{9.29}$$

We would like to access

$$\Upsilon^{(n)}\circ\pi_{\mathcal{R},n}(\hat{F}_g) = \sum_{\ell_1,\ldots,\ell_n\geq 1}\hat{F}^{[g]}_{n;\ell_1,\ldots,\ell_n}\prod_{i=1}^n a_i^{\ell_i}, \tag{9.30}$$

and from Corollary 9.16, we know since $w(b_i;A)=a_i$ that

$$\Upsilon^{(n)}(\hat{F}_g) = \check{\omega}_{g,n}(b_1,\ldots,b_n;A)\prod_{i=1}^n\frac{w(b_i;A)}{\mathrm{d}w(b_i;A)}.$$

According to (9.29) the degree 0 part of the right-hand side computes the quantity in (9.30). The spectral curve $x(z;A)$ and $y(z;A)$ is symmetric in $a_1,\ldots,a_N$. Therefore when we compute the right-hand side with TR, we obtain a formal series in $p_\ell(A)$, whose term of degree 0 is

$$\check{\omega}_{g,n}^{A=0}(b_1,\ldots,b_n)\prod_{i=1}^n\frac{w(b_i;0)}{\mathrm{d}w(b_i;0)}, \qquad \text{with} \quad w(b_i;0)=a_i.$$

□

Part 3. Applications

10. Relation with free probability

We explain how the results of Part 1 fit in the context of free probability, and a possible application of our Theorem 9.1 and more general conjectures of Section.......

10.1. Review of higher order free probabilities. Voiculescu [63, 64] introduced the notion of freeness to capture simultaneously a property of algebraic and probabilistic independence of two subsets $\mathcal{A}_1, \mathcal{A}_2$ of a non-commutative probability space $(\mathcal{A}, \varphi)$. If $\mathcal{A}_1, \mathcal{A}_2$ are free, moments of any element in the subalgebra generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ can be computed solely in terms of moments of elements in $\mathcal{A}_1$, and moments of elements in $\mathcal{A}_2$. The precise relation is elegantly expressed via free moments. If $a \in \mathcal{A}$, the free moments $(\kappa_\ell(a))_{d\geq 0}$ are determined by the moments $(\varphi(a^\ell))_{\ell\geq 0}$ and tailored such that

$$\forall \ell \geq 0, \qquad K_\ell(a+b) = K_\ell(a) + K_\ell(b).$$

The combinatorics behind the definition of free moment is compactly handled at the level of generating series by the R-transform. Departing from the usual normalizations in free probability, we introduce

$$\mathcal{W}(x) = \sum_{\ell \geq 0} \frac{\varphi(a^\ell)}{x^{\ell+1}} = \varphi\Big(\frac{1}{x-a}\Big), \qquad \mathcal{X}(w) = \sum_{\ell\geq 0} K_\ell(a)\, w^{\ell-1}.$$

For comparison, Voiculescu R-transform is $\mathcal{R}(w) = \mathcal{X}(w) - w^{-1}$. Free moments are computed in terms of moments via functional inversion

Theorem 10.1. [63]

$$\mathcal{X}(\mathcal{W}(x)) = x.$$

The combinatorics behind the R-transform was later related to non-crossing partitions [60].

This theory has been generalized to second order cumulants by Mingo and Speicher [53], and Collins, Mingo, Speicher and Śniady for higher order cumulants [21]. They introduce a notion of higher order non-commutative probability space $\mathcal{A}$. Beyond a linear trace $\varphi = \varphi_1$, it is equipped with multilinear traces $\varphi_n : \mathcal{A}^n \to \mathbb{C}$ satisfying properties similar to those of n-th order cumulants. Then, they define n-th order free cumulants $K_{\ell_1,\ldots,\ell_n}(a_1,\ldots,a_n)$ via the combinatorics of non-crossing partitions on n circles. Roughly speaking, a family of pairwise disjoint subsets is free if for any $2 \leq m \leq n$, the m-th order free cumulants of any m-uple of elements in those subsets vanish whenever at least two of these elements belong to two different subsets. In particular, second order freeness amounts to $\varphi_2(a,b) = 0$ for any elements a, b belonging to two different subsets, and gives

$$K_{\ell_1,\ell_2}(a+b, a+b) = K_{\ell_1,\ell_2}(a,a) + K_{\ell_1,\ell_2}(b,b).$$

The relation between second order cumulants and free cumulants is elucidated in [21, Theorem 2.12], written here with adapted notations.

Theorem 10.2. *Let*

$$\begin{aligned} \mathcal{W}_2(x_1,x_2) &= \sum_{\ell_1,\ell_2\geq 0} \frac{\varphi_2(a^{\ell_1}, a^{\ell_2})}{x_1^{\ell_1+1} x_2^{\ell_2+1}} = \varphi_2\Big(\frac{1}{x_1-a}, \frac{1}{x_2-a}\Big), \\ \mathcal{X}_2(w_1,w_2) &= \sum_{\ell_1,\ell_2\geq 0} K_{\ell_1,\ell_2}(a,a)\, w_1^{\ell_1-1} w_2^{\ell_2-1}. \end{aligned}$$

We have

$$\Big(\mathcal{W}_2(x_1,x_2) + \frac{1}{(x_1-x_2)^2}\Big)\mathrm{d}x_1\mathrm{d}x_2 = \Big(\mathcal{X}_2(w_1,w_2) + \frac{1}{(w_1-w_2)^2}\Big)\mathrm{d}w_1\mathrm{d}w_2,$$

where $w_i = \mathcal{W}(x_i)$ *or equivalently* $x_i = \mathcal{X}(w_i)$.

For $n \geq 3$, it is not easy to characterize n-th order freeness in terms of φ_n, and the computation of via generating series has not yet been devised.

This theory is partly driven by its application to asymptotics of unitarily invariant random matrices. Indeed, if M and $\tilde{M}$ are two independent hermitian random matrices of size N, and the distribution of M is unitarily invariant, M and $\tilde{M}$ determine in the large N limit (higher order) free elements of a (higher order) non-commutative probability space when this limit exist. A precise statement can be found in [64] at first order, [52] at second order, and [21] in what regards higher order. Therefore, the large N limit of the cumulants $\kappa_n(\mathrm{Tr}\, P_1(M, \widetilde{M}), \ldots, \mathrm{Tr} P_n(M, \widetilde{M}))$ for any non-commutative polynomials in two variables $(P_i)_{i=1}^n$ can be computed solely in terms of the cumulants of M, and the cumulants of $\widetilde{M}$. Doing so explicitly for $P_i = (M + \widetilde{M})^{\ell_i}$ motivates Theorems 10.1-10.2 and their potential generalizations to higher order.

10.2. **Comments.** Comparing with our Propositions 3.3-4.4, we see that the relation between fully simple and ordinary generating series in the case of disks and cylinders, coincides the relation between generating series of free cumulants and cumulants for first order and second order. This is also true with more boundaries Indeed, if M is a unitarily invariant random hermitian matrix of size N such that the limit

$$\lim_{N \to \infty} N^{n-2} \kappa_n(\mathrm{Tr}\, M^{\ell_1} \cdots \mathrm{Tr}\, M^{\ell_n})$$

exists for $n \geq 1$, this large N limit defines a higher order non-commutative probability space generated by a single element m, and the free cumulants of m are represented as

$$K_{\ell_1,\ldots,\ell_n}(\mathrm{m}, \ldots, \mathrm{m}) = \lim_{N \to \infty} N^{n-2+\sum_i \ell_i} \kappa_n\big(\mathcal{P}_{\gamma_1}(M), \ldots, \mathcal{P}_{\gamma_n}(M)\big),$$

where γ_i are n pairwise disjoint cycles in $\{1, \ldots, N\}$ of respective lengths ℓ_i, see [21, Theorem 4.4] and the asymptotic analysis which follows. For the matrix ensembles (8.5), the comparison with Proposition 8.3 identifies $K_{\ell_1,\ldots,\ell_n}(\mathrm{m}, \ldots, \mathrm{m})$ with the generating series of genus 0 fully simple maps with boundaries of perimeters $\ell_1, \ldots, \ell_n$.

In this article, we have proved the formulas for $\mathcal{X}_1$ and $\mathcal{X}_2$ via combinatorics of maps – instead of non-crossing partitions – independently of [21]. Although Weingarten calculus and the HCIZ integral is also used in [21] to relate higher order free cumulants to cumulants, we have explained in Section 7.2.5 that the relation is naturally expressed in terms of monotone Hurwitz numbers. As Hurwitz theory develops rapidly, this fact may give insight into the structure of higher order cumulants generating series.

Our Proposition 5.3 applies to matrix ensembles in which $\ln\big(\frac{\mathrm{d}\mu(M)}{\mathrm{d}M}\big)$ is linear in the trace of powers of M, *i.e.* with $T_{h,k} = 0$ for $(h,k) \neq (0,1)$, and it is Conjecture 6.2 if we turn on $T_{0,2}$ as well. These are the models governed by the topological recursion, and whose combinatorics is captured by usual maps, or maps carrying a loop model. We therefore have a computational tool for the free cumulants in these models, via the topological recursion: going free amounts to performing the symplectic transformation $x \mapsto y$. More general unitarily invariant ensembles are rather governed by the blobbed topological recursion of [7] and related to stuffed maps. Concretely, the initial data for the blobbed topological recursion is a spectral curve as in (1.5), supplemented with blobs $(\varphi_{g,n})_{2g-2+n>0}$ which play the role of extra initial data intervening in topology (g,n) and beyond. For matrix ensembles of the form (8.5), there exist specific values for the blobs, such that the blobbed topological recursion computes the large N expansion of the correlators. It would be interesting to know if the $x \leftrightarrow y$ can be supplemented with transformation of the blobs, such that the blobbed topological recursion for the transformed initial data computes the free cumulants. Restricting to genus 0, it would give a computational scheme to handle free cumulants of any order – in full generality – via the blobbed topological recursion. The higher genus theory should capture finite size corrections to freeness, and the universality of the topological recursion suggests that it may be possible to formulate a universal theory of approximate freeness, for which unitarily invariant matrix ensembles would provide examples.

11. An ELSV-like formula for monotone Hurwitz numbers

11.1. **GUE and monotone Hurwitz numbers.** The Gaussian Unitary Ensemble is the probability measure on the space of hermitian matrices of size N

$$\begin{aligned} \mathrm{d}\mu(M) &= 2^{-\frac{N}{2}}\big(\tfrac{N}{\pi}\big)^{\frac{N(N+1)}{2}}\,\mathrm{d}M\,\exp(-N\,\mathrm{Tr}\,\tfrac{M^2}{2}) \\ &= 2^{-\frac{N}{2}}\big(\tfrac{N}{\pi}\big)^{\frac{N(N+1)}{2}}\prod_{i=1}^{N}\mathrm{d}M_{i,i}\,e^{-NM_{i,i}^2/2}\prod_{i<j}\mathrm{dRe}\,M_{i,j}\,e^{-N(\mathrm{Re}M_{i,j})^2}\prod_{i<j}\mathrm{dIm}\,M_{i,j}\,e^{-N(\mathrm{Im}\,M_{i,j})^2}. \end{aligned}$$

We recall that the cumulants of the GUE have a topological expansion

$$\kappa_n(\mathrm{Tr}\,M^{\mu_1},\ldots,\mathrm{Tr}\,M^{\mu_n}) = \sum_{g\geq 0} N^{2-2g-n}\,\kappa_n^{[g]}(\mathrm{Tr}\,M^{\mu_1},\ldots,\mathrm{Tr}\,M^{\mu_n}).$$

For any fixed positive integers μ_i, this sum is finite. The numbers $\kappa_n^{[g]}$ count genus g maps whose only faces are the n marked faces, and are sometimes called generalized Catalan numbers. They have been extensively studied by various methods [65, 42, 39, 18, 4], but we are able to make a seemingly new observation, relating them to monotone Hurwitz numbers of a certain kind.

Proposition 11.1. *For any $g\geq 0$ and $n\geq 1$,*

$$\kappa_n^{[g]}(\mathrm{Tr}\,M^{\mu_1},\ldots,\mathrm{Tr}\,M^{\mu_n}) = [E_g^{\circ}]_{\mu,(2,\ldots,2)}|C_{(2,\ldots,2)}|,$$

where $[E_g^{\circ}]_{\mu,\lambda}$ are the connected double weakly monotone Hurwitz numbers of genus g.

Proof. As the entries of M are independent and gaussian, we can easily evaluate

$$\langle\mathcal{P}_\lambda(M)\rangle = \prod_{i=1}^{n}\frac{\delta_{\lambda_i,2}}{N},\qquad \kappa_n(\mathcal{P}_{\lambda_1}(M),\ldots,\mathcal{P}_{\lambda_n}(M)) = \frac{\delta_{n,1}\delta_{\lambda_1,2}}{N}.$$

From Theorem 7.8 we deduce

$$\Big\langle\prod_{i=1}^{n}\mathrm{Tr}\,M^{\mu_i}\Big\rangle = \sum_{\lambda\vdash|\mu|} N^{\frac{|\mu|}{2}}\Big(\sum_{k\geq 0}N^{-k}[E_k]_{\mu,(2,\ldots,2)}\Big)|C_{(2,\ldots,2)}|,.$$

where

$$|C_{(2,\ldots,2)}| = \frac{|\lambda|!}{2^{\ell(\lambda)}\ell(\lambda)!} = (2\ell(\lambda)-1)!! = (|\mu|-1)!!$$

since here $|\mu| = 2\ell(\lambda)$. We then use inclusion-exclusion formula to compute the cumulants. This replaces monotone Hurwitz numbers by their connected counterpart. The genus g contribution to the n-order cumulant is the coefficient of N^{2-2g-n}. It remains to check that the coefficient of N^{2-2g-n} on the Hurwitz side also selects the genus g Hurwitz numbers. This is guaranteed by the Riemann-Hurwitz formula, which says that the covers counted in $[E_g^{\circ}]$ have $k = 2g-2+\ell(\mu)+\frac{|\mu|}{2}$ and here $n = \ell(\mu)$. □

This specialization of our result recovers a particular case of [2, Prop. 4.8] which says that the enumeration of hypermaps is equivalent to the strictly monotone orbiforld Hurwitz problem. This suggests it is natural to investigate if our results generalize to the more general setting of hypermaps.

It is well-known that the GUE correlation functions are computed by the topological recursion for the spectral curve,

$$\mathcal{C}=\mathbb{P}^1, \qquad p(z)=z+\frac{1}{z}, \qquad \lambda(z)=\frac{1}{z}, \qquad B(z_1,z_2)=\frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}\,. \tag{11.1}$$

Therefore, Proposition 11.1 gives a new proof that the 2-orbifold strictly monotone Hurwitz numbers are computed by the topological recursion, a fact already known as a special case of more general results, see *e.g.* [24, 1].

11.2. **GUE and Hodge integrals.** Dubrovin, Liu, Yang and Zhang [23] recently discovered a relation between Hodge integrals and the even GUE moments. For $2g-2+n>0$, the Hodge bundle $\mathbb{E}$ is the holomorphic vector bundle over Deligne-Mumford compactification of the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ whose fiber above a curve with punctures $(\mathcal{C},p_1,\ldots,p_n)$ is the g-dimensional space of holomorphic 1-forms on $\mathcal{C}$. We denote

$$\Lambda(t)=\sum_{j=0}^{g}\mathrm{ch}_j(\mathbb{E})\,t^h$$

its Chern character. Let ψ_i be the first Chern class of the line bundle $T^*_{p_i}\mathcal{C}$, and introduce the formal series

$$Z_{\mathrm{Hodge}}(t;\hbar)=\exp\bigg(\sum_{2g-2+n>0}\frac{\hbar^{2g-2+n}}{n!}\sum_{i_1,\ldots,i_n\geq 0}\int_{\overline{\mathcal{M}}_{g,n}}\Lambda(-1)\Lambda(-1)\Lambda(\tfrac{1}{2})\prod_{i=1}^{n}\psi_i^{d_i}t_{d_i}\bigg).$$

On the GUE side, the cumulants have a topological expansion

$$\kappa_n(\mathrm{Tr}\,M^{\ell_1},\ldots,\mathrm{Tr}\,M^{\ell_n})=\sum_{g\geq 0}N^{2-2g-n}\,\kappa_n^{[g]}(\mathrm{Tr}\,M^{\ell_1},\ldots,\mathrm{Tr}\,M^{\ell_n}),$$

where $\kappa_n^{[g]}$ are independent of N, and the sum is always finite. We introduce the formal series

$$Z_{\mathrm{even}}(s;N)=\frac{e^{-A(s;N)}\prod_{j=1}^{N-1}j!}{2^N\pi^{\frac{N(N+1)}{2}}}\int_{\mathcal{H}_N}\mathrm{d}M\,\exp\Big[N\,\mathrm{Tr}\Big(-\tfrac{M^2}{2}+\sum_{j\geq 1}s_j\,\mathrm{Tr}\,M^{2j}\Big)\Big], \tag{11.2}$$

where we choose

$$\begin{aligned} A(s;N) &= \frac{\ln N}{12}-\zeta'(-1) \\ &\quad +N^2\Big[-\frac{3}{4}+\sum_{j\geq 1}\frac{1}{j+1}\binom{2j}{j}s_j+\frac{1}{2}\sum_{j_1,j_2\geq 1}\frac{j_1j_2}{j_1+j_2}\binom{2j_1}{j_1}\binom{2j_2}{j_2}s_{j_1}s_{j_2}\Big]. \end{aligned}$$

The normalization factor in (11.2) is related to the volume of $U(N)$ and the factor $e^{-A(s;N)}$ cancels the non-decaying terms in its large N asymptotics, as well as the contributions of $\kappa_1^{[0]}$ and $\kappa_2^{[0]}$. The large N asymptotics of the outcome reads

$$\begin{aligned} Z_{\mathrm{even}}(s;N) &= \exp\bigg(\sum_{g\geq 2}\frac{N^{2-2g}B_{2g}}{4g(g-1)} \\ &\quad +\sum_{2g-2+n>0}\frac{N^{2g-2+n}}{n!}\sum_{\ell_1,\ldots,\ell_n\geq 0}\kappa_n^{[g]}(\mathrm{Tr}\,M^{2\ell_1},\ldots,\mathrm{Tr}\,M^{2\ell_n})\prod_{i=1}^{n}s_{\ell_i}\bigg), \end{aligned}$$

and we consider it as an element of $N^2\mathbb{Q}[N^{-1}][[s_1,s_2,\ldots]]$.

Theorem 11.2. [23] *With the change of variable*

$$T_{i,\pm}(s;N) = \sum_{k\geq 1} k^{i+1}\binom{2k}{k} s_k - \delta_{i\geq 2} \pm \frac{\delta_{i,0}}{2N},$$

we have the identity of formal series

$$Z_{\mathrm{even}}(s;N) = Z_{\mathrm{Hodge}}(T_+(s;N),\sqrt{2}N^{-1})Z_{\mathrm{Hodge}}(T_-(s;N),\sqrt{2}N^{-1}).$$

We can extract from this result an explicit formula for the even GUE moments, which gives an ELSV-like formula for the monotone Hurwitz numbers with even ramification above ∞, and $(2,\ldots,2)$ ramification above 0.

Corollary 11.3. *For $g\geq 0$ and $n\geq 1$ such that $2g-2+n>0$ and $m_1,\ldots,m_n\geq 0$, we have*

$$\begin{aligned}|C_{(2,\ldots,2)}|[E_g^{\circ}]_{(2m_1,\ldots,2m_n),(2,\ldots,2)} &= \kappa_n^{[g]}(\mathrm{Tr}\,M^{2m_1},\ldots,\mathrm{Tr}\,M^{2m_n}) \\ &= 2^g\int_{\overline{\mathcal{M}}_{g,n}} [\Delta]\,\cap\,\Lambda(-1)\Lambda(-1)\Lambda(\tfrac{1}{2})\exp\Big(\sum_{d\geq 1} a_d\kappa_d\Big)\prod_{i=1}^{n}\frac{m_i\binom{2m_i}{m_i}}{1-m_i\psi_i}.\end{aligned}$$

κ_d are the pushforwards of ψ_{n+1}^{d+1} via the morphism forgetting the last puncture, and the numbers a_d are determined by

$$\sum_{d\geq 1} a_d z^{d-1} = \operatorname*{Res}_{u\to 0}\frac{\mathrm{d}u}{u}\,\frac{e^{-u}}{u-z(1+u)}. \tag{11.3}$$

Denoting $[\Delta_h]$ the class of the boundary strata $\overline{\mathcal{M}}_{g-h,n+2h}\subset\overline{\mathcal{M}}_{g,n}$ which comes from the pairwise glueing of the last $2h$ punctures, we have introduced

$$[\Delta] = \sum_{h\geq 0}\frac{[\Delta_h]}{2^{2h}(2h)!}.$$

Proof. Identifying the coefficient of $\frac{N^{2-2g-n}}{n!}\,s_{m_1}\cdots s_{m_n}$ in Theorem 11.2 yields

$$\begin{aligned}&\kappa_n^{[g]}(\mathrm{Tr}\,M^{2m_1},\ldots,\mathrm{Tr}\,M^{2m_n}) \\ &= \sum_{h=0}^{\lfloor\frac{g}{2}\rfloor}\sum_{\ell\geq 0}\frac{2^{g-2h}}{(2h)!\ell!}\int_{\overline{\mathcal{M}}_{g-h,n+\ell+2h}}\Lambda(-1)\Lambda(-1)\Lambda(\tfrac{1}{2})\prod_{i=1}^{n}\frac{m_i\binom{2m_i}{m_i}}{1-m_i\psi_i}\prod_{i=n+1}^{n+\ell}\frac{-\psi_i^2}{1-\psi_i}.\end{aligned} \tag{11.4}$$

This sum is actually finite as the degree of the class to integrate goes beyond the dimension of the moduli space. We can get rid of the ℓ factors of ψ-classes by using the pushforward relation

$$(\pi_\ell)_*\Big(X\prod_{i=1}^{\ell}\psi_i^{d_i+1}\Big) = \sum_{\sigma\in\mathfrak{S}_\ell}\prod_{\gamma\in\mathcal{C}(\sigma)}\kappa_{\sum_{i\in\gamma}d_i}, \tag{11.5}$$

where $\pi_\ell:\overline{\mathcal{M}}_{g',k+\ell}\to\overline{\mathcal{M}}_{g',k}$ is morphism forgetting the last ℓ punctures, and X is the pullback via π_ℓ of an arbitrary class on $\overline{\mathcal{M}}_{g',k}$. In general, if we introduce formal variables $\hat{a}_1,\hat{a}_2,\ldots$, we deduce from (11.5) the relation

$$\sum_{\ell\geq 0}\frac{1}{\ell!}\sum_{d_1,\ldots,d_\ell\geq 1}(\pi_\ell)_*\Big(X\prod_{i=k+1}^{k+\ell}\psi_i^{d_i+1}\Big)\prod_{i=1}^{\ell}\hat{a}_{d_j} = X\exp\Big(\sum_{d\geq 1}a_d\kappa_d\Big),$$

where

$$1+\sum_{d\geq 1}a_d v^d = \exp\Big(\sum_{d\geq 1}\hat{a}_d v^d\Big).$$

To simplify (11.4) we should apply this relation with $\hat{a}_d = -1$ for all $d \geq 1$. Therefore

$$a_d = \operatorname*{Res}_{v \to 0} \frac{\mathrm{d}v}{v^{d+1}} \exp\Big(-\frac{v}{1-v}\Big).$$

The change of variable $v = \frac{u}{1-u}$ brings it to

$$a_d = \operatorname*{Res}_{u \to 0} \frac{\mathrm{d}u\,(1+u)^{d-1}e^{-u}}{u^{d+1}},$$

whose generating series is indeed (11.3). Consequently

$$\begin{aligned} & \kappa_n^{[g]}(\mathrm{Tr}\,M^{2m_1}, \cdots, \mathrm{Tr}\,M^{2m_n}) \\ = \quad & \sum_{h=0}^{\lfloor \frac{g}{2} \rfloor} \frac{2^{g-2h}}{(2h)!} \int_{\overline{\mathcal{M}}_{g-h,n+2h}} \Lambda(-1)\Lambda(-1)\Lambda(\tfrac{1}{2}) \exp\Big(\sum_{d \geq 1} a_d \kappa_d\Big) \prod_{i=1}^{n} \frac{m_i \binom{2m_i}{m_i}}{1 - m_i \psi_i}. \end{aligned}$$

The claim is a compact rewriting of this formula, using the pushforward via the inclusions $\iota_h : \overline{\mathcal{M}}_{g-h,n+2h} \to \overline{\mathcal{M}}_{g,n}$. ☐

The fact that the correlators of the GUE satisfy the topological recursion implies according to the general theory in [27] a representation in terms of integrals over the moduli space of curves. Computing explicitly this representation is complicated by the presence of two ramification points in the spectral curve (11.1), and we did not manage to simplify it even by restricting to the even GUE moments as in Theorem 11.2. It is therefore unclear to us whether the ELSV-type formula that can in principle be derived from [27] is equivalent to our Corollary 11.3.

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