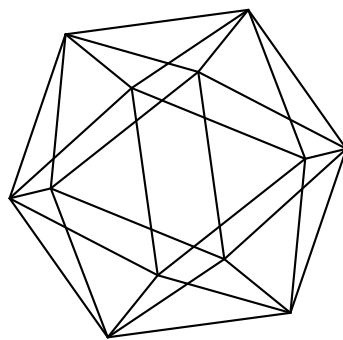


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The spherical Hall algebra of $\overline{\text{Spec}(\mathbb{Z})}$

by

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To Yuri Ivanovich Manin on his 75th birthday

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0 Introduction.

(0.1) The construction of the Hall algebra of an abelian category \mathcal{A} is known to produce interesting Hopf algebras of quantum group-theoretic nature. A condition usually imposed to ensure that the Hall algebra has a compatible comultiplication, is that \mathcal{A} is hereditary (of homological dimension 1). There are two main types of hereditary abelian categories which have been studied in this respect.

First, if Q is a *quiver*, we can form the category $\mathcal{A} = \mathcal{R}ep_{\mathbb{F}_q}(Q)$ of (finite-dimensional) representations of Q over a finite field \mathbb{F}_q . As discovered by Ringel [23], the Hall algebra of $\mathcal{R}ep_{\mathbb{F}_q}(Q)$ is related to the quantized Kac-Moody algebra whose Dynkin diagram is Q . More precisely, it contains $U_q(\mathfrak{n}_+)$, the quantization of the unipotent subalgebra on the positive root generators from the Kac-Moody root system.

Second, if X is a smooth projective *curve* over \mathbb{F}_q , we can form the category $\mathcal{A} = \mathcal{C}oh(X)$ of coherent sheaves on X . In this case the Hall algebra contains the spaces of unramified automorphic forms on the groups GL_r , $r \geq 1$ over the function field $K = \mathbb{F}_q(X)$, and the multiplication corresponds to forming Eisenstein series [12]. One can also include “orbifold curves” $G \backslash X$ where G is a finite group of automorphisms of a curve X , see [24]. The algebras obtained in this way include both quantum affine algebras [12, 24] and spherical Cherednik algebras [26].

(0.2) The goal of the present paper is to begin the study of a third, more arithmetic, type of Hall algebras. It is obtained by replacing a curve X/\mathbb{F}_q by the spectrum of the *ring of integers in a number field*, compactified at infinity by the Archimedean valuations. In this paper we consider only the basic example of $\overline{\text{Spec}(\mathbb{Z})} = \text{Spec}(\mathbb{Z}) \cup \{\infty\}$. The role of rank n vector bundles for $\overline{\text{Spec}(\mathbb{Z})}$ is played by free abelian groups L of rank n with a positive definite quadratic form in $L \otimes \mathbb{R}$, see [30, 31, 8] as well as [18, 29] for a more general point of view of Arakelov geometry. The “moduli space” of such bundles is the classical quotient of reduction theory of quadratic forms

$$\text{Bun}_n = GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R}) / O_n.$$

Functions on Bun_n are the same as automorphic forms on $GL_n(\mathbb{R})$, see [7] for a detailed study of precisely this situation.

(0.3) To describe our arithmetic analog of the Hall algebra, let $H_n = C_0^\infty(\text{Bun}_n)$ be the space of smooth functions on Bun_n with compact support. The space $H = \bigoplus_n H_n$ has a natural structure of an associative algebra, constructed in §2. From the point of view of the automorphic form theory, the multiplication in H is given by the parabolic pseudo-Eisenstein series map. If X is a curve over \mathbb{F}_q , the analogous map for unramified automorphic forms over the function field $\mathbb{F}_q(X)$ gives the multiplication in the Hall algebra of X , see [12]. So in this paper we study the space H of automorphic forms on all the $GL_n(\mathbb{R})$ as an associative algebra in its own right.

We further concentrate on the subalgebra $SH \subset H$ generated by $H_1 = C_0^\infty(\mathbb{R}_{>0})$. Extending the terminology of [27], we call SH the *spherical Hall algebra* of $\overline{\text{Spec}(\mathbb{Z})}$. From the point of view of spectral decomposition [20], SH consists of automorphic forms expressible through the Eisenstein-Selberg series [28], the simplest higher-dimensional analogs of the nonholomorphic Eisenstein-Mass series on the upper half plane. This algebra has an explicit space of generators, but relations among these generators are not directly given.

(0.4) Our first main result describes SH as a Feigin-Odesskii-type shuffle algebra, in a way similar to the results of [27] for the case of curves over a finite field. However, in our case the shuffle algebra is based not on a rational, but on a meromorphic function: the Riemann zeta function $\zeta(s)$. This function, therefore, encodes all the relations among the generators from H_1 .

Quadratic relations in SH correspond to the classical functional equation for the Eisenstein-Maass series, in a way similar to the case of function field considered in [12]. One form of writing the relations is in terms of “generating functions” (formal H -valued distributions) $\mathfrak{E}(s)$ depending on $s \in \mathbb{C}$. It has the form

$$\mathfrak{E}(s_1)\mathfrak{E}(s_2) = \frac{\zeta^*(s_1 - s_2)}{\zeta^*(s_1 - s_2 + 1)} \mathfrak{E}(s_2)\mathfrak{E}(s_1),$$

where $\zeta^*(s)$ is the full zeta function of $\overline{\text{Spec}(\mathbb{Z})}$ (the product of $\zeta(s)$ with the Gamma and exponential factors). This is discussed in §6.

Our second main result, Theorem 7.7, is that the space of the cubic relations (not following from the quadratic ones) is identified with (an appropriate completion of) the space spanned by nontrivial zeroes of $\zeta(s)$. In other

words, *the space spanned by the zeroes of $\zeta(s)$ can be realized as a certain algebraic homology space of the associative algebra H* . This is remindful of (but different from) the result of D. Zagier [33] who gave an interpretation of the zeta-space using integrals of Eisenstein-Maass series over anisotropic tori associated to real quadratic fields.

(0.5) After the first draft of this paper was written, we learned that M. Kontsevich and Y. Soibelman [16] have recently considered the algebra H as well. Their interest was in studying wall-crossing formulas in $\mathcal{B}un$, so our results practically do not intersect. We are grateful to M. Kontsevich and Y. Soibelman for explaining their work and providing us with the preliminary version of [16].

(0.6) M.K. would like to thank Universities Paris-7 and Paris-13 as well as the Max-Planck Institut für Mathematik in Bonn for hospitality and support during the work on this paper. His research was also partially supported by an NSF grant.

1 Vector bundles on $\overline{\text{Spec}(\mathbb{Z})}$.

By a *vector bundle on $\overline{\text{Spec}(\mathbb{Z})}$* we will mean a triple $E = (L, V, q)$, where V is a finite-dimensional \mathbb{R} -vector space, q is a positive definite quadratic form on V , and $L \subset V$ is a \mathbb{Z} -lattice of maximal rank. In this case, V becomes a Banach space with norm $\|v\| = \sqrt{q(v)}$.

The *rank* of E is defined as $\text{rk}(E) = \dim_{\mathbb{R}}(V) = \text{rk}_{\mathbb{Z}}(L)$. A *morphism* $f : E' = (L', V', q') \rightarrow E = (L, V, q)$ of vector bundles on $\overline{\text{Spec}(\mathbb{Z})}$ is, by definition, a linear operator $f : V' \rightarrow V$ such that, first, $f(L') \subset L$ and, second, $\|f\| \leq 1$, i.e., we have $q(f(v')) \leq q'(v')$ for each $v' \in V'$. In this way we get a category which we denote $\mathcal{B}un$. All the Hom-sets in $\mathcal{B}un$ are finite.

We denote by $\mathcal{O} = (\mathbb{Z}, \mathbb{R}, x^2)$ the *trivial bundle of rank 1*.

The *dual bundle* to E is defined as $E^\vee = (L^\vee, V^*, q^{-1})$, where q^{-1} is the inverse quadratic form on the dual space. The *tensor product* of two bundles is defined as

$$E \otimes E' = (L \otimes_{\mathbb{Z}} L', V \otimes_{\mathbb{R}} V', q \otimes q'), \quad (q \otimes q')(v \otimes v') := q(v)q'(v').$$

In particular, we have the bundle $\underline{\text{Hom}}(E, E') = E^\vee \otimes E'$. The corresponding quadratic form on $\text{Hom}_{\mathbb{R}}(V, V')$ takes $f : V \rightarrow V'$ into $\text{tr}(f^t \circ f)$, where the transpose is taken with respect to q, q' . We leave to the reader the proof of the following:

Proposition 1.1. *Let $E_i = (L_i, V_i, q_i)$, $i = 1, 2, 3$, be three vector bundles on $\overline{\text{Spec}(\mathbb{Z})}$. Then*

$$\text{Hom}_{\mathcal{B}un}(E_1, \underline{\text{Hom}}(E_2, E_3)) \subset \text{Hom}_{\mathcal{B}un}(E_1 \otimes E_2, E_3)$$

as subsets in $\text{Hom}_{\mathbb{R}}(V_1 \otimes V_2, V_3)$. \square

Note the particular case of $E_1 = \mathcal{O}$. The proposition in this case reduces to the inequality

$$\|f\| \leq \sqrt{\text{tr}(f^t \circ f)}$$

for any linear operator $f : V_2 \rightarrow V_3$. We also see why the inclusion in the proposition is not, in general, an equality. Indeed, for $E_1 = \mathcal{O}$, the Hom-set on the left consists of integer points in the domain $\text{tr}(f^t \circ f) \leq 1$, which is an ellipsoid. But the Hom-set on the right consists of integer points in the domain $\|f\| \leq 1$ which is not an ellipsoid, if $\dim(V_2), \dim(V_3) > 1$.

We also have the symmetric and exterior product functors

$$\begin{aligned} S^r(E) &= (S_{\mathbb{Z}}^r(L), S_{\mathbb{R}}^r(V), S^r(q)), & S^r(q)(v_1 \bullet \cdots \bullet v_r) &:= q(v_1) \cdots q(v_r), \\ \Lambda^r(E) &= (\Lambda_{\mathbb{Z}}^r(L), \Lambda_{\mathbb{R}}^r(V), \Lambda^r(q)), & \Lambda^r(q)(v_1 \wedge \cdots \wedge v_r) &:= \det \|B(v_i, v_j)\|. \end{aligned}$$

Here \bullet is the product in the symmetric algebra, while B is the symmetric bilinear form such that $q(v) = B(v, v)$.

Let Bun_n be the set of isomorphism classes of rank n vector bundles on $\overline{\text{Spec}(\mathbb{Z})}$. This set is a classical double quotient of the theory of automorphic forms:

$$\text{Bun}_n \overset{\sim}{\longleftarrow} GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R}) / O_n. \quad (1.2)$$

Explicitly, the double coset of $g_\infty \in GL_n(\mathbb{R})$ corresponds to the isomorphism class of the bundle $(\mathbb{Z}^n, \mathbb{R}^n, (g_\infty^t)_*^{-1}(q_{\text{st}}))$, where

$$q_{\text{st}}(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2$$

is the standard quadratic form on \mathbb{R}^n and $(g_\infty^t)_*^{-1}(q_{\text{st}})(x) = q_{\text{st}}((g_\infty^t)^{-1}(x))$ is the quadratic form corresponding to the symmetric matrix $(g_\infty^t)^{-1} \cdot g_\infty^{-1}$.

We will also need an adelic version of (1.2). Let $\mathfrak{A}^f = \prod_p^{\text{res}} \mathbb{Q}_p$ be the ring of finite adeles of the field \mathbb{Q} , let $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \subset \mathfrak{A}^f$ be the profinite completion of \mathbb{Z} , and $\mathfrak{A} = \mathbb{R} \times \mathfrak{A}^f$ be the full ring of adeles. Then $K_n := O_n \times \prod_p GL_n(\mathbb{Z}_p)$ is a maximal compact subgroup of $GL_n(\mathfrak{A})$.

Proposition 1.3. *The embedding of $GL_n(\mathbb{R})$ into $GL_n(\mathfrak{A})$ induces a bijection*

$$\text{Bun}_n \simeq GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R}) / O_n \xrightarrow{\alpha} GL_n(\mathbb{Q}) \backslash GL_n(\mathfrak{A}) / K_n.$$

Proof. The statement is of course well known. We describe the inverse map explicitly for later use. Let $g = (g_\infty, (g_p)) \in GL_n(\mathfrak{A})$, so $g_\infty \in GL_n(\mathbb{R})$ and $g_p \in GL_n(\mathbb{Q}_p)$, with $g_p \in GL_n(\mathbb{Z}_p)$ for almost all p . We associate to g a vector bundle $E_g = (L_g, V_g, q_g)$ on $\overline{\text{Spec}(\mathbb{Z})}$ by putting:

$$L_g = \mathbb{Q}^n \cap \bigcap_p g_p^t(\mathbb{Z}_p^n), \quad V_g = \mathbb{R}^n, \quad q_g = (g_\infty^t)_*^{-1}(q_{\text{st}}).$$

It is clear that $E_{\gamma g k} \simeq E_g$ for $\gamma \in GL_n(\mathbb{Q})$, $k \in K_n$, so we get a map

$$GL_n(\mathbb{Q}) \backslash GL_n(\mathfrak{A}) / K_n \xrightarrow{\beta} \text{Bun}_n.$$

By construction, $\beta\alpha = \text{Id}$; the fact that $\alpha\beta = \text{Id}$ follows since L_g is a free abelian group. \square

Example 1.4. Take $n = 1$. The set Bun_1 formed by isomorphism classes of line bundles, will be also denoted by $\text{Pic}(\overline{\text{Spec}(\mathbb{Z})})$. This set is a group under tensor multiplication. It is identified with \mathbb{R}_+^\times , the multiplicative group of positive real numbers. Explicitly, given $E = (L, V, q)$ with $\dim_{\mathbb{R}}(V) = 1$, we associate to it the number $\deg(E) = 1/\sqrt{q(l_{\min})} \in \mathbb{R}_+$, where l_{\min} is one of the two generators of L . Conversely, for $a \in \mathbb{R}_+$ we denote by $\mathcal{O}(a) = (\mathbb{Z}, \mathbb{R}, a^{-2} \cdot q_{\text{st}})$ the corresponding line bundle with $\deg(\mathcal{O}(a)) = a$. The convention, compatible with (1.2) for $n = 1$, is chosen so that for $a \gg 0$ the bundle $\mathcal{O}(a)$ has many "global sections", i.e., lattice points l such that $q(l) \leq 1$.

Example 1.5. For any n , taking the top exterior power together with the isomorphism of Example 1.4, gives a map

$$\text{Bun}_n \xrightarrow{\det} \text{Pic}(\overline{\text{Spec}(\mathbb{Z})}) \xrightarrow{\sim} \mathbb{R}_+.$$

Explicitly, $E = (L, V, q)$ is sent into $1/\text{Vol}(V/L)$, the inverse of the covolume of L with respect to the Lebesgue measure defined by q . We will denote this inverse covolume by $\text{deg}(E)$ and call it the degree of E . We denote by $\text{Bun}_{n,a}$ the set of isomorphism classes of bundles of rank n and degree a .

Consider the case $n = 2$ and take $a = 1$. In this case

$$\text{Bun}_{2,1} = SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO_2$$

is identified with the quotient $SL_2(\mathbb{Z}) \backslash \mathbb{H}$, where $\mathbb{H} \subset \mathbb{C}$ is the upper half-plane $\text{Im}(z) > 0$. More explicitly, consider the standard quadratic form on \mathbb{C} given by $q_{\text{st}}(z) = |z|^2$. Then, for $\tau \in \mathbb{H}$, the lattice $\mathbb{Z} + \mathbb{Z}\tau$ has, with respect to q_{st} , the covolume equal to $\text{Im}(\tau)$. We therefore associate to τ the bundle

$$E_\tau = (\mathbb{Z} + \mathbb{Z}\tau, \mathbb{C}, q_{\text{st}}/\text{Im}(\tau)^{1/2}) \in \text{Bun}_{2,1}.$$

Lemma 1.6. *For $\gamma \in SL_2(\mathbb{Z})$ we have $E_{\gamma(\tau)} \simeq E_\tau$, and this establishes an identification $SL_2(\mathbb{Z}) \backslash \mathbb{H} \rightarrow \text{Bun}_{2,1}$.*

Proof. It is clear that $E_\tau \simeq E_{\tau+1}$. Let us show that $E_{-1/\tau} \simeq E_\tau$. Note that

$$\text{Vol}(\mathbb{C}/L_{-1/\tau}) = \text{Im}(-1/\tau) = \text{Im}\left(\frac{-\bar{\tau}}{|\tau|^2}\right).$$

Notice also that multiplication by τ defines an isomorphism of lattices

$$L_{-1/\tau} \xrightarrow{\tau} L_\tau.$$

The determinant of the multiplication by τ being $|\tau|^2$, we conclude that this multiplication defines an isomorphism

$$(L_{-1/\tau}, \mathbb{C}, q_{\text{st}}/\text{Im}(-1/\tau)^{1/2}) \longrightarrow (L_\tau, \mathbb{C}, q_{\text{st}}/\text{Im}(\tau)^{1/2})$$

of vector bundles over $\overline{\text{Spec}(\mathbb{Z})}$. □

Let now

$$0 \rightarrow E' = (L', V', q') \xrightarrow{i} E = (L, V, q) \xrightarrow{j} E'' = (L'', V'', q'') \rightarrow 0 \quad (1.7)$$

be a sequence of vector bundles on $\overline{\text{Spec}(\mathbb{Z})}$ and their morphisms.

Definition 1.8. We say that a sequence (1.7) is *short exact* (in \mathcal{Bun}), if the following hold:

- (1) The induced sequences of vector spaces and abelian groups are short exact.
- (2) The form q' is equal to $i^*(q)$, the pullback of q via i , defined by

$$(i^*q)(v') = q(i(v')), \quad v' \in V'.$$

- (3) The form q'' is equal to $j_*(q)$, the pushforward of q via j , defined by

$$(j_*q)(v'') = \min_{j(v)=v''} q(v), \quad v'' \in V''.$$

An *admissible monomorphism* (resp. *admissible epimorphism*) in \mathcal{Bun} is a morphism which can be included into a short exact sequence as i (resp. j).

Let us call a *subbundle* in E an equivalence class of admissible monomorphisms $E' \rightarrow E$ modulo isomorphisms of the source. For such a subbundle E' we have the quotient bundle $E/E' \in \mathcal{Bun}$.

Proposition 1.9. *Let $E = (L, V, q)$ be a vector bundle on $\overline{\text{Spec}(\mathbb{Z})}$. The following sets are in bijection:*

- (i) Rank r subbundles $E' \subset E$.
- (ii) Rank r primitive sublattices, i.e., subgroups $L' \subset L$ such that L/L' has no torsion.
- (iii) \mathbb{Q} -linear subspaces $W' \subset L \otimes_{\mathbb{Z}} \mathbb{Q}$ of dimension r .

Proof. The bijection between (ii) and (i) takes a primitive sublattice L' into $E' = (L', V', q')$, where $V' = L' \otimes_{\mathbb{Z}} \mathbb{R}$ and $q' = q|_{V'}$. The bijection between (iii) and (ii) takes a subspace W' into the sublattice $L' = L \cap W'$. \square

Proposition 1.10. *Let $E = (L, V, q)$ be a vector bundle on $\overline{\text{Spec}(\mathbb{Z})}$. For any $r \in \mathbb{Z}_+$ and $a \in \mathbb{R}_+$, the set of subbundles $E' \subset E$ with $\text{rk}(E') = r$ and $\text{deg}(E') \geq a$, is finite.*

Proof. Let $W = L \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider first the case $r = 1$. If $E' \subset E$ corresponds to a 1-dimensional subspace $W' \subset W$, then $\text{deg}(E') = 1/\sqrt{q(w')}$, where $w' \in W' \cap L = L'$ is one of two generators of this free abelian group of rank 1. Since the number of w' such that $q(w') \leq a$ is finite, our statement follows.

Consider now the case of arbitrary r and use the Plücker embedding of the Grassmannian $G(r, W)$ into $\mathbb{P}(\Lambda^r(W))$. If $W' \subset W$ is an r -dimensional

subspace with $L' = W' \cap L$, then $\Lambda^r(W') \subset \Lambda^r(W)$ is a 1-dimensional subspace, and $\Lambda^r(W') \cap \Lambda_{\mathbb{Z}}^r(L) = \Lambda_{\mathbb{Z}}^r(L')$ is a free abelian group of rank 1 and a primitive sublattice in $\Lambda_{\mathbb{Z}}^r(L)$. Further, $\Lambda_{\mathbb{R}}^r(V)$ is equipped with the quadratic form $\Lambda^r(q)$, and $\deg(E') = 1/\sqrt{\Lambda^r(q)(\xi')}$, where E' is the subbundle corresponding to W' and $\xi' \in \Lambda_{\mathbb{Z}}^r(L')$ is one of the two generators. We thus reduce to the case of subbundles of rank 1. \square

Let $E' = (L', V', q')$ and $E'' = (L'', V'', q'')$ be two vector bundles on $\overline{\text{Spec}(\mathbb{Z})}$. We define $\text{Ext}^1(E'', E')$ to be the set of admissible short exact sequences (1.7) modulo automorphisms of such sequences identical on E' and E'' .

Proposition 1.11. *The set $\text{Ext}^1(E'', E')$ has a natural structure of a C^∞ -manifold isomorphic to the torus $(\mathbb{R}/\mathbb{Z})^{n'n''}$, where $n' = \text{rk}(E')$ and $n'' = \text{rk}(E'')$.*

Proof. For any short exact sequence as in (1.7), the induced short exact sequence of lattices necessarily splits. Let us fix a splitting $L = L' \oplus L''$ and the induced splitting $V = V' \oplus V''$ of \mathbb{R} -vector spaces, so that i and j become the canonical embedding into and the projection from the direct sum. Let Λ be the set of positive definite quadratic forms q on V such that $i^*q = q'$ and $j_*q = q''$. By definition, Λ is a closed subset in the space of all positive definite quadratic forms on V and so has a natural topology.

The group $\text{Hom}_{\mathbb{Z}}(L'', L')$ is identified with the group of automorphisms of the split exact sequence

$$0 \rightarrow L' \xrightarrow{i} L' \oplus L'' \xrightarrow{j} L'' \rightarrow 0$$

identical on L', L'' . Therefore this group acts on Λ , and we have $\text{Ext}^1(E'', E') = \Lambda / \text{Hom}_{\mathbb{Z}}(L'', L')$.

Lemma 1.12. *The map $\Lambda \xrightarrow{\text{res}} \text{Hom}_{\mathbb{R}}(V' \otimes V'', \mathbb{R})$ which sends q into the induced pairing between the summands V' and V'' , is a homeomorphism. This map takes the action of the group $\text{Hom}_{\mathbb{Z}}(L'', L')$ on Λ into its action on $\text{Hom}_{\mathbb{R}}(V' \otimes V'', \mathbb{R})$ by translations.*

Proof: Fix a basis $e_1, \dots, e_{n'}$ of V' , orthonormal with respect to q , and a basis $v_1, \dots, v_{n''}$ of V'' . Let B', B'' be the symmetric bilinear forms on V', V''

corresponding to q', q'' , and let q be a quadratic form on V with corresponding symmetric bilinear form B . Then the condition $q \in \Lambda$ means:

$$B(e_i, e_j) = \delta_{ij} = B'(e_j, e_j),$$

$$B\left(v_p - \sum_{\mu=1}^{n'} B(v_p, e_\mu) \cdot e_\mu, v_q - \sum_{\nu=1}^{n'} B(v_q, e_\nu) \cdot e_\nu\right) = B''(v_p, v_q). \quad (1.13)$$

Indeed, the minimum in the definition of j_*q is given by the orthogonal projection to V' with respect to B , and the left hand side of the second formula above involves exactly such projections.

Denote by X the matrix $\|B(v_p, e_\mu)\|$ of size $n'' \times n'$, and let Y be the matrix $\|B(v_p, v_q)\|$ of size $n'' \times n''$. From the first condition in (1.13) we see that a quadratic form q with $i^*q = q'$ is completely determined by the datum of X and Y , while the second equation implies that $Y = B'' - X \cdot X^t$, where $B'' = \|B''(v_p, v_q)\|$. Therefore $q \in \Lambda$ is indeed completely defined by X , which is the matrix representative of $\text{res}(q)$. The action of elements of $\text{Hom}_{\mathbb{Z}}(L'', L')$ in the matrices X is the action by translation. This proves the lemma and Proposition 1.11. \square

Remark 1.14. More generally, one can consider data $\mathcal{F} = (L, V, q)$ similar to the above but where L is any finitely generated abelian group, $V = L \otimes_{\mathbb{Z}} \mathbb{R}$ and q is a positive definite quadratic form on V . They correspond to *coherent sheaves on $\overline{\text{Spec}(\mathbb{Z})}$ locally free at infinity*. We get in this way a category $\text{Coh}_{\neq \infty}(\overline{\text{Spec}(\mathbb{Z})})$, with admissible short exact sequences defined similarly to Definition 1.8. A more systematic theory should enlarge $\text{Coh}_{\neq \infty}(\overline{\text{Spec}(\mathbb{Z})})$ by allowing a meaningful concept of sheaves with torsion at ∞ . This will be done in a subsequent paper. For example, sheaves supported at ∞ can be described in terms of two positive definite quadratic forms $q \leq q'$ on one \mathbb{R} -vector space V , much in the same way as representing a finite abelian p -group as quotient of two free \mathbb{Z}_p -modules of the same rank. The role of elementary divisors is then played by the logarithms $\log \lambda_i(q : q') \in \mathbb{R}_+$, of the eigenvalues of q with respect to q' .

2 The Hall algebra.

Let

$$Y_n = GL_n(\mathbb{R})/O_n$$

be the space of quadratic forms on \mathbb{R}^n . It is a C^∞ -manifold of dimension $n(n+1)/2$. It is well known that for large N the congruence subgroup

$$GL_n(\mathbb{Z}, N) = \{\gamma \in GL_n(\mathbb{Z}) : \gamma \equiv 1 \pmod{N}\}$$

acts on Y_n freely, so $GL_n(\mathbb{Z}, N) \backslash Y_n$ is a C^∞ -manifold. The set Bun_n is the quotient of this manifold by the finite group $GL_n(\mathbb{Z}/N)$ and therefore has a structure of a C^∞ -orbifold. In particular, we can speak about C^∞ -functions on Bun_n . They are C^∞ -functions on $GL_n(\mathbb{R})$, left invariant under $GL_n(\mathbb{Z})$ and right invariant under O_n , i.e., C^∞ -automorphic forms in the classical sense. Let

$$H_n = C_0^\infty(\text{Bun}_n) = C_0^\infty(GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R})/O_n)$$

be the space of C^∞ -functions on Bun_n with compact support. Consider the direct sum

$$H = \bigoplus_{n=0}^{\infty} H_n, \quad H_0 = \mathbb{C}.$$

Let $f \in H_m, g \in H_n$. We define their *Hall product* $f * g$ to be the function $\text{Bun}_{m+n} \rightarrow \mathbb{C}$ given by the formula

$$(f * g)(E) = \sum_{E' \subset E} \deg(E')^{n/2} \deg(E/E')^{-m/2} \cdot f(E')g(E/E'), \quad (2.1)$$

where the sum is over all subbundles $E' \subset E$ of rank m .

Proposition 2.2. (a) *For every E the sum in (2.1) is actually finite, so $f * g$ is a well defined function.*

(b) *$f * g$ is again a C^∞ -function with compact support.*

(c) *The operation $f * g$ makes H into a graded associative algebra, with unit $1 \in H_0$.*

We will call the algebra H the *Hall algebra of $\overline{\text{Spec}(\mathbb{Z})}$* . In this paper we will be particularly interested in the subalgebra $SH \subset H$ generated by $H_1 = C_0^\infty(\mathbb{R}_+)$. We will call SH the *spherical Hall algebra*, adopting the terminology of [27], where a similar algebra was studied for the case of a curve over a finite field.

Remark 2.3. (a) The quantity

$$\langle E/E', E' \rangle = \deg(E')^{n/2} \deg(E/E')^{-m/2} = \sqrt{\deg \underline{\text{Hom}}(E/E', E')}$$

is the analog of the Euler form used by Ringel [23] to twist the multiplication in the Hall algebra of representations of a quiver. In our case, as well as in the case of curves over a finite field [12, 27], twisting by this form simplifies the form of commutation relations.

(b) One can get larger algebras by relaxing the condition of compact support to that of sufficiently rapid decay at infinity. More generally, there are interesting cases when f and g do not have rapid decay, but $f * g$ still makes sense as a convergent series.

Proof of Proposition 2.2: (a) Since f is with compact support, there is $A > 1$ such that $f(E') = 0$ unless $\deg(E') \in [1/A, A]$. By Proposition 1.10 all but finitely many $E' \subset E$ have $\deg(E') < A$, so that the sum in (2.1) is indeed finite.

(b) To see that $f * g$ is smooth, suppose that E_1 and E_2 are close to each other in Bun_{m+n} . Then the corresponding lattices L_1 and L_2 are identified in a canonical fashion. Therefore the sets of subbundles $E'_1 \subset E_1$ and $E'_2 \subset E_2$ of rank m , are identified, and so we have a bijection between the sets of summands in $(f * g)(E_1)$ and $(f * g)(E_2)$. Next, the number of nonzero summands in both sums is bounded by the same number by the continuity of f and g , so we can view $f * g$ as a sum of finitely many C^∞ -functions.

To see that $f * g$ has compact support, let $\Sigma_1 \subset \text{Bun}_m$ be a compact set supporting f , and Σ_2 be a compact set supporting g . For any $E_1 \in \Sigma_1$, $E_2 \in \Sigma_2$ the set of $E \in \text{Bun}_{m+n}$ that can fit into a sequence (1.7), is a compact topological space. Indeed, it is the image of a continuous map $\text{Ext}^1(E_2, E_1) \rightarrow \text{Bun}_{m+n}$, whose source is a compact torus. Let F be the total space of the fibration over $\Sigma_1 \times \Sigma_2$ with fiber over (E_1, E_2) being $\text{Ext}^1(E_2, E_1)$. Then F is compact, while the support of $f * g$ is contained in the image of F under a natural continuous map into Bun_{m+n} .

(c) To prove associativity, let $f \in H_{n_1}$, $g \in H_{n_2}$, $h \in H_{n_3}$. Then for $E \in \text{Bun}_{n_1+n_2+n_3}$ we have

$$((f * g) * h)(E) = \sum_{E_1 \subset E_2 \subset E} d_1^{\frac{n_2+n_3}{2}} d_2^{\frac{-n_1+n_3}{2}} d_3^{\frac{-n_1-n_2}{2}} \cdot f(E_1) \cdot g(E_2/E_1) \cdot h(E/E_2),$$

where E_1 runs over subbundles of E of rank $n_1 + n_2$, and E_1 runs over subbundles of E_2 of rank n_1 , and we have denoted

$$d_1 = \deg(E_1), \quad d_2 = \deg(E_2/E_1), \quad d_3 = \deg(E/E_2).$$

On the other hand

$$(f*(g*h))(E) = \sum_{\substack{E_1 \subset E \\ E'_2 \subset E/E_1}} \delta_1^{\frac{n_2+n_3}{2}} \delta_2^{\frac{-n_1+n_3}{2}} \delta_3^{\frac{-n_1-n_2}{2}} \cdot f(E_1) \cdot g(E'_2) \cdot h((E/E_1)/E'_2),$$

where we have denoted

$$\delta_1 = \deg(E_1), \quad \delta_2 = \deg(E'_2), \quad \delta_3 = \deg((E/E_1)/E'_2).$$

Let F be the set over which the first sum is extended, i.e., the set of admissible filtrations $E_1 \subset E_2 \subset E$ with $\text{rk}(E_1) = n_1$ and $\text{rk}(E_2) = n_1 + n_2$. Similarly, let F_2 be the set over which the second sum is extended, i.e., the set of pairs (E_1, E'_2) , where $E_1 \subset E$ is a subbundle of rank n_1 , and $E'_2 \subset E/E_1$ is a subbundle of rank n_2 . We have a map $\rho : F \rightarrow F'$ sending $(E_1 \subset E_2 \subset E)$ into $(E_1, E'_2 := E_2/E_1)$. The summand corresponding to any $\phi \in F$ is equal to the summand corresponding to $\rho(\phi) \in F'$. So our statement reduces to the following.

Lemma 2.4. *The map ρ is a bijection.*

Proof: An element of F has the form

$$(L_1, V_1, Q_1) \subset (L_2, V_2, q_2) \subset (L, V, q) = E,$$

where $L_1 \subset L_2 \subset L$ is a filtration by primitive sublattices, and $q_i = q|_{V_i}$, $i = 1, 2$. An image of such an element by ρ is the pair $((L_1, V_1, q_1), (L'_2, V'_2, q'_2))$, where (L_1, V_1, q_1) is as above, while $L'_2 = L_2/L_1 \subset L/L_1$, and $q'_2 = \pi'_*(q_2)$, with $\pi' : V_2 \rightarrow V_2/V_1 = V'_2$ being the canonical projection.

On the other hand, a general element of F' is a pair $((L_1, V_1, q_1), (L'_2, V'_2, q'_2))$, where (L_1, V_1, q_1) is as above, while $L'_2 \subset L/L_1$ is an arbitrary primitive sublattice of rank n_2 , and $V'_2 = L'_2 \otimes \mathbb{R}$ and q'_2 is the restriction to V'_2 of the quotient quadratic form $\pi_*(q)$ for the projection $\pi : V \rightarrow V/V_1$, i.e., $q'_2 = (i')^*(\pi_*q)$. We have therefore a Cartesian square of \mathbb{R} -vector spaces

$$\begin{array}{ccc} V_2 & \xrightarrow{i} & V \\ \pi' \downarrow & & \downarrow \pi \\ V'_2 = V_2/V_1 & \xrightarrow{i'} & V/V_1 \end{array}$$

with π, π' surjective and i, i' injective. We claim that $\pi'_* i^*(q) = i'^* \pi_*(q)$, and hence $\rho(F) = F'$. This is a particular case of the following base change property for quadratic forms.

Proposition 2.5. *Let*

$$\begin{array}{ccc} U_2 & \xrightarrow{i_1} & U \\ j' \downarrow & & \downarrow j \\ U'_2 & \xrightarrow{i_2} & U \end{array}$$

be a Cartesian square of \mathbb{R} -vector spaces, such that i_1, i_2 are injective and j, j' are surjective. Then for any positive definite quadratic form q on U we have the equality $j'_ i_1^* q = i_2^* j_* q$ of quadratic forms on U'_2 .*

Proof. Let $u'_2 \in U'_2$. Then

$$(j'_* i_1^* q)(u'_2) = \min_{u_2: j'(u_2)=u'_2} (i_1^* q)(u_2) = \min_{u_2: j'(u_2)=u'_2} q(i_1(u_2)).$$

Since the square is Cartesian, i_1 identifies $(j')^{-1}(u'_2)$ with $j^{-1}(i_2(u'_2))$, so the last minimum is equal to

$$\min_{u: j(u)=i_2(u'_2)} q(u) = (j_* q)(i_2(u'_2)).$$

□

This finishes the proof of Lemma 2.4 as well as Proposition 2.2.

Remark 2.6. One can extend the definition of the Hall algebra to the category $\text{Coh}_{\neq \infty}(\overline{\text{Spec}(\mathbb{Z})})$ defined as in Remark 1.14, using the concept of admissible exact sequences outlined there. The algebra thus obtained will be a semidirect product of H and the Hall algebra of the category of finite abelian groups, similarly to [13], §2.6.

3 The Mellin transform.

A standard tool in the theory of quantum affine algebras is the use of generating functions, i.e., passing from a collection of coefficients $(c_\alpha)_{\alpha \in \mathbb{Z}^n}$ to the Laurent series

$$F(t) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha t^\alpha, \quad t = (t_1, \dots, t_n) \in (\mathbb{C}^*)^n, \quad t^\alpha = \prod t_\nu^{\alpha_\nu}. \quad (3.1)$$

This is just the Fourier transform on the free abelian group \mathbb{Z}^n , but understood in a more pragmatic way: we do not necessarily restrict to unitary characters (they form the real torus $|t_i| = 1$) but pay attention to the domains of convergence in the space $(\mathbb{C}^*)^n$ of all characters.

A typical free abelian group to which the above is applied is, in the Hall algebra approach, $\text{Pic}(X)/\{\text{torsion}\} = \mathbb{Z}$, where X is a smooth projective curve over \mathbb{F}_q , see [12, 27]. In the present paper the corresponding role is played by the group $\text{Pic}(\overline{\text{Spec}}(\mathbb{Z})) = \mathbb{R}_+$. The Fourier transform on \mathbb{R}_+^n is known as the *Mellin transform*. We now give a summary of its properties from the same pragmatic standpoint as above.

Unitary characters of \mathbb{R}_+^n have the form

$$a = (a_1, \dots, a_n) \longmapsto a^s = \prod a_\nu^{s_\nu}, \quad s_\nu \in i\mathbb{R} \subset \mathbb{C}, \quad a_\nu^{s_\nu} = e^{s_\nu \log(a_\nu)},$$

and the Haar measure is $d^*a = \prod da_\nu/a_\nu$. Accordingly, the Mellin transform of a function (or a distribution) $f(a)$ on \mathbb{R}_+^n is the integral

$$F(s) = (\mathcal{M}f)(s) = \int_{a \in \mathbb{R}_+^n} f(a) a^s d^*a. \quad (3.2)$$

Here, a priori, $s \in i\mathbb{R}^n$, but we are interested in allowing the s_i to vary in the complex domain, i.e., in considering not necessarily unitary characters. The group isomorphism

$$\exp : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}_+^n. \quad (3.3)$$

transforms the Mellin integral into the standard Fourier integral on \mathbb{R}^n .

Example 3.4 (Paley-Wiener theorem). If $f(a)$ has compact support, then $(\mathcal{M}f)(s)$ converges for any $s \in \mathbb{C}^n$, i.e., $\mathcal{M}f$ is an entire function, analogously to the case of a Laurent series in (3.1) being a Laurent polynomial.

Recall that an entire function $F(s)$, $s \in \mathbb{C}^n$, is called a *Paley-Wiener function*, if there is a constant $B > 0$ and, for every $N > 0$ there is a $c_N > 0$ such that

$$|F(s)| \leq c_N(1 + \|s\|)^{-N} e^{B \cdot \|\operatorname{Re}(s)\|}.$$

This means, in particular, that F has a faster than polynomial decay on each vertical subspace $\{\sigma_0 + i\mathbb{R}^n, \sigma_0 \in \mathbb{R}^n\}$, while allowed to have exponential growth on any horizontal subspace. We denote by $\mathcal{PW}(\mathbb{C}^n)$ the space of Paley-Wiener functions on \mathbb{C}^n . The Paley-Wiener theorem says:

Proposition 3.5. *The Mellin transform \mathcal{M} identifies $C_0^\infty(\mathbb{R}_+^n)$ with $\mathcal{PW}(\mathbb{C}^n)$.*

Proof. The classical formulation, see, e.g., [22], Vol. II, Thm. IX.11, is for the Fourier transform of compactly supported functions on \mathbb{R}^n . The case of the Mellin transform reduces to this via exp. \square

An important point about series (3.1) is that one (meromorphic) function can have different Laurent expansions in different regions, while the region of convergence of each expansion is “logarithmically convex”, i.e., is the preimage of a convex open set $\Delta \subset \mathbb{R}^n$ under the map

$$\lambda : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n, \quad (t_i) \mapsto (\log |t_i|).$$

We now review the corresponding features of Mellin expansions. Unlike in the case of Laurent series, these features are less familiar, and a precise treatment involves L. Schwartz’s theory of Fourier transform for distributions.

For a C^∞ -manifold or orbifold M we denote by $\mathcal{D}ist(M) = C_0^\infty(M)'$ the space of distributions on M . Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions on \mathbb{R}^n , and $\mathcal{D}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^n)' \subset \mathcal{D}ist(\mathbb{R}^n)$ be the dual space of tempered distributions, see [22], Vol.I, §V.3. Recall that a C^∞ -function lies in $\mathcal{D}(\mathbb{R}^n)$ if and only if it has at most polynomial growth.

We define $\mathcal{S}(\mathbb{R}_+^n)$ and $\mathcal{D}(\mathbb{R}_+^n)$, the spaces of Schwartz functions and tempered distributions on \mathbb{R}_+^n , by means of the group isomorphism exp of (3.3). For $f \in \mathcal{D}(\mathbb{R}_+^n)$ we define $\mathcal{M}f$ to be the tempered distribution on $i\mathbb{R}^n$ given by the Fourier-Schwartz transform of $f \circ \exp$.

For a distribution $f \in \mathcal{D}ist(\mathbb{R}_+^n)$ we denote by $\operatorname{Temp}(f)$ and call the *tempering set* of f , the set of $\sigma \in \mathbb{R}^n$ such that $f(a) \cdot a^\sigma$ is a tempered distribution. It is known (see [22], Vol. II, Lemma after Th. IX. 14.1) that $\operatorname{Temp}(f)$ is a convex subset in \mathbb{R}^n . We say that f is *temperable*, if $\operatorname{Temp}(f)$

has non-empty interior. For any convex open set $\Delta \subset \mathbb{R}^n$ we denote by $U_\Delta = \{s \in \mathbb{C}^n \mid \operatorname{Re}(s) \in \Delta\}$ the corresponding tube domain.

Proposition 3.6. *Let f is a temperable distribution on \mathbb{R}_+^n , and Δ be the interior of $\operatorname{Temp}(f)$. Then $F(s) = (\mathcal{M}f)(s)$ is an analytic function in U_Δ , which has an at most polynomial growth on each vertical subspace $\sigma_0 + i\mathbb{R}^n$, $\sigma_0 \in \Delta$.*

Proof. To see holomorphy, it is enough to assume that 0 is an interior point of $T(f)$ and to show that $\mathcal{M}f$ is holomorphic in an open neighborhood of $i\mathbb{R}^n$. For a sequence of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i \in \{\pm 1\}$ let $(\mathbb{R}_+^n)_\varepsilon \subset \mathbb{R}_+^n$ be the domain given by conditions $a_i^{\varepsilon_i} > 1$, and $\mathbb{C}_\varepsilon^n \subset \mathbb{C}^n$ be given by the condition $\varepsilon_i \operatorname{Re}(s_i) < 0$. Let $\mathcal{M}_\varepsilon f$ be the partial Mellin integral of f , taken over $(\mathbb{R}_+^n)_\varepsilon$. If $s \in \mathbb{C}_\varepsilon^n$, then the function a^s decays exponentially at the infinity of $(\mathbb{R}_+^n)_\varepsilon$. Therefore, if f is a tempered distribution on \mathbb{R}_+^n (i.e., if $0 \in \operatorname{Temp}(f)$), then $\mathcal{M}_\varepsilon f$ extends to a holomorphic function in \mathbb{C}_ε^n . If, moreover, 0 is an interior point of $\operatorname{Temp}(f)$, then $\mathcal{M}_\varepsilon f$ is holomorphic in additive translates $\mathbb{C}_\varepsilon^n + \sigma$ for σ running in an open neighborhood of 0 in \mathbb{R}^n . Therefore $\mathcal{M}f = \sum_\varepsilon \mathcal{M}_\varepsilon f$ is holomorphic for $\operatorname{Re}(s)$ running in some open neighborhood of 0, as claimed.

To see that $\mathcal{M}f$ has at most polynomial growth on each $\sigma_0 + i\mathbb{R}^n$, it is again enough to treat the case $\sigma_0 = 0$. The restriction of $\mathcal{M}f$ to $i\mathbb{R}^n$ is a tempered distribution, the Fourier-Schwartz transform of $f \circ \exp$. As it is also a real analytic function, it must be of polynomial growth. \square

Next, we discuss the inverse Mellin transform, i.e., the analog of the formula which finds each coefficient c_α in (3.1) as an integral of $F(t)$ times a monomial. Formally, the inverse Mellin integral is defined by

$$f(a) = (\mathcal{N}_\Delta F)(a) = \frac{1}{(2\pi i)^n} \int_{s \in \sigma_0 + i\mathbb{R}^n} F(s) a^{-s} ds. \quad \sigma_0 \in \Delta, \quad a \in \mathbb{R}_+^n, \quad (3.7)$$

In our case, this integral should again be understood using Schwartz's theory. More precisely, we have:

Proposition 3.8. *Let $\Delta \subset \mathbb{R}^n$ be a convex open set and $F(s)$ be an analytic function in U_Δ with at most polynomial growth on each vertical subspace. Choose $\sigma_0 \in \Delta$ and define $f(a) = (\mathcal{N}_\Delta F)(a)$ as $a^{-\sigma_0}$ times the inverse Fourier transform of g as a tempered distribution on $\sigma_0 + i\mathbb{R}^n \simeq \mathbb{R}^n$ (the Fourier transform being transplanted to \mathbb{R}_+^n via \exp). Then $\mathcal{N}_\Delta F$ is independent on $\sigma_0 \in \Delta$, and is a temperable distribution on \mathbb{R}_+^n such that $\Delta \subset \operatorname{Temp}(f)$ and $\mathcal{M}f = F$.*

We will call $\mathcal{N}_\Delta(F)$ the *coefficient function* of F in U_Δ . Thus the existence of the coefficient function presupposes that F grows at most polynomially on each vertical subspace in U_Δ . As usual with the Fourier transform, the coefficient function of the product of analytic functions is the convolution (on the group \mathbb{R}_+^n) of the coefficient functions of the factors.

Proof. To show independence, it is enough to assume $0 \in \Delta$ and compare the integrals (3.7) over $i\mathbb{R}^n$ and $\sigma_0 + i\mathbb{R}^n$ for σ_0 being close to 0 in Δ . Both functions $F(s)$ and $F(s + \sigma_0)$ are tempered distributions on $i\mathbb{R}^n \simeq \mathbb{R}^n$ and so have Fourier-Schwartz transforms. Moreover, $F(s + \sigma_0)$ the sum of a Taylor series involving derivatives of $F(s)$ (evaluated on $i\mathbb{R}^n$). So the Fourier transform of $F(s + \sigma_0)$ is product of the Fourier transform of $F(s)$ and an exponential factor. This factor is accounted for by the change in a^s in the integral (3.7), showing the independence. The remaining claims follow from the inversion theorem for the Fourier-Schwartz transform. \square

Let us note the particular case $\Delta = \mathbb{R}^n$.

Corollary 3.9. *The Mellin transforms \mathcal{M} and \mathcal{N} defines mutually inverse isomorphisms between the following two spaces:*

- $\mathcal{D}(\mathbb{R}_+^n)_{\text{abs}}$, the space of absolutely tempered distributions, i.e., of distributions $f(a)$ such that $f(a)a^s$ is tempered for each $s \in \mathbb{C}^n$.
- $\mathcal{O}(\mathbb{C}^n)_{\text{pol}}$, the space of entire functions in \mathbb{C}^n with at most polynomial growth on each vertical subspace.

\square

Note that an absolutely tempered distribution has actually exponential decay at the infinity of \mathbb{R}_+^n .

For future reference we recall two elementary properties of the Mellin/Fourier transform. We denote by $\delta_c \in \mathcal{D}(\mathbb{R}_+)$ the delta function at $c \in \mathbb{R}_+$.

Proposition 3.10. (a) *Let $F(s)$ be analytic in U_Δ , with the coefficient function $f(a) = \mathcal{N}_\Delta(F)$. Then for any $\nu = 1, \dots, n$ we have*

$$\mathcal{N}_\Delta(s_\nu F(s)) = -a_\nu \frac{d}{da_\nu} f(a).$$

(b) Let Δ be an interval $(c, c') \subset \mathbb{R}$, so U_Δ is a strip in \mathbb{C} . Let $h(s)$ be analytic in U_Δ , with coefficient function $k(a)$, $a \in \mathbb{R}_+$. Consider the function of two variables

$$F(s_1, s_2) = h(s_1 - s_2), \quad (s_1, s_2) \in U_{\tilde{\Delta}} = \{c < \operatorname{Re}(s_1 - s_2) < c'\}$$

Then the coefficient function of F is found by

$$(\mathcal{N}_{\tilde{\Delta}} F)(a_1, a_2) = \delta_1(a_1 a_2) \cdot k(a_1).$$

□

Example 3.11. Let $\zeta(s)$ be the Riemann zeta function, and

$$\zeta^*(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) \tag{3.12}$$

be the zeta function of $\overline{\operatorname{Spec}(\mathbb{Z})}$. It is a meromorphic function on \mathbb{C} with simple poles at 0 and 1, satisfying $\zeta^*(s) = \zeta^*(1-s)$.

The function $\Gamma(s)$ has exponential decay on each vertical line $\sigma_0 + i\mathbb{R}$, as follows from the Stirling formula. The function $\zeta(s)$ has at most polynomial growth on each vertical line, see [3], Ch.9. Therefore $\zeta^*(s)$ has exponential decay on each vertical line and therefore has a well defined coefficient function in each of the three strips of holomorphy: $\operatorname{Re}(s) > 1$, $0 < \operatorname{Re}(s) < 1$ and $\operatorname{Re}(s) < 0$. The coefficient function in $\operatorname{Re}(s) > 1$ is given by the classical formula of Riemann, see [3], §1.7:

$$(\mathcal{N}_{\operatorname{Re}(s) > 1} \zeta^*)(a) = \theta(a^2) - 1, \quad \theta(b) := \sum_{n=-\infty}^{\infty} e^{-n^2 \pi b}. \tag{3.13}$$

It can be obtained by forming the convolution (on \mathbb{R}_+) of the distribution $\sum_{n=1}^{\infty} \delta_{1/n}$, of the function $2e^{-a^2}$ and of the distribution $\delta_{1/\sqrt{\pi}}$. These three distributions are the coefficient functions for $\zeta(s) = \sum 1/n^s$, for $\Gamma(s/2) = 2 \int_0^\infty e^{-a^2} a^s d^*a$ and for $\pi^{-s/2}$ respectively. The coefficient functions in the two other strips are obtained by moving the contour past the poles of $\zeta^*(s)$ at $s = 1$ and $s = 0$ with residues $\pm 1/\sqrt{\pi}$:

$$\begin{aligned} (\mathcal{N}_{0 < \operatorname{Re}(s) < 1} \zeta^*)(a) &= \theta(a^2) - 1 - \frac{1}{a\sqrt{\pi}}, \\ (\mathcal{N}_{\operatorname{Re}(s) < 0} \zeta^*)(a) &= \theta(a^2) + 1/\sqrt{\pi} - 1 - \frac{1}{a\sqrt{\pi}}. \end{aligned}$$

4 The zeta function shuffle algebra.

We recall the formalism of shuffle algebras of Feigin-Odesskii [5], see [27] [13] for a more systematic discussion in the rational function case. We denote by \mathfrak{S}_n the symmetric group of permutations of $\{1, \dots, n\}$.

Let $\varphi(s)$ be a meromorphic function on \mathbb{C} . For any $m, n > 0$ let $Sh(m, n)$ be the set of (m, n) -shuffles, i.e., permutations $w \in \mathfrak{S}_{m+n}$ such that $w(i) < w(j)$ whenever $i < j$ and either both $i, j \in [1, m]$ or both $i, j \in [m+1, m+n]$. For any $w \in Sh(m, n)$ consider the following meromorphic function on \mathbb{C}^{m+n} :

$$\varphi_w(s_1, \dots, s_{m+n}) = \prod_{\substack{i \in [1, m] \\ j \in [m+1, m+n], \\ w(i) > w(j)}} \varphi(s_i - s_j). \quad (4.1)$$

Let $\mathcal{O}(\mathbb{C}^n) \subset \mathcal{M}er(\mathbb{C}^n)$ be the spaces of all entire and meromorphic functions on \mathbb{C}^n (defined to be equal to \mathbb{C} for $n = 0$). On the direct sum $\bigoplus_n \mathcal{M}er(\mathbb{C}^n)$ we introduce the *shuffle multiplication*

$$\mathbb{S}_{m,n} : \mathcal{M}er(\mathbb{C}^m) \otimes \mathcal{M}er(\mathbb{C}^n) \longrightarrow \mathcal{M}er(\mathbb{C}^{m+n}), \quad F \otimes F' \mapsto F \mathbb{S} F', \quad (4.2)$$

by the formula

$$\begin{aligned} & (F \mathbb{S} F')(s_1, \dots, s_{m+n}) = \\ & = \sum_{w \in Sh(m,n)} w \left(F(s_{(1)}, \dots, s_{(m)}) F'(s_{m+1}, \dots, s_{m+n}) \right) \cdot \varphi_w(s_1, \dots, s_{m+n}). \end{aligned} \quad (4.3)$$

The following is then straightforward, as in [5].

Proposition 4.4. *The shuffle multiplication \mathbb{S} makes $\bigoplus_n \mathcal{M}er(\mathbb{C}^n)$ into a graded associative algebra, with unit $1 \in \mathcal{M}er(\mathbb{C}^0)$. \square*

Assume further that the function φ satisfies the equation $\varphi(-s)\varphi(s) = 1$, and, moreover, is represented in the form

$$\varphi(s) = \lambda(s)^{-1} \lambda(-s) \quad (4.5)$$

for some meromorphic function $\lambda(s)$. For $n \geq 0$ let $\mathcal{M}er(\mathbb{C}^n)^{\mathfrak{S}_n}$ be the space of symmetric meromorphic functions on \mathbb{C}^n . On the direct sum $\bigoplus_n \mathcal{M}er(\mathbb{C}^n)^{\mathfrak{S}_n}$, we introduce the *symmetric shuffle multiplication*

$$\begin{aligned} \star_{m,n} : \mathcal{M}er(\mathbb{C}^m)^{\mathfrak{S}_m} \otimes \mathcal{M}er(\mathbb{C}^n)^{\mathfrak{S}_n} & \longrightarrow \mathcal{M}er(\mathbb{C}^{m+n})^{\mathfrak{S}_{m+n}}, \\ F \otimes F' & \mapsto F \star F', \end{aligned} \quad (4.6)$$

by the formula

$$(F \star F')(s_1, \dots, s_{m+n}) = \sum_{w \in Sh(m,n)} w \left(F(s_1, \dots, s_m) F'(s_{m+1}, \dots, s_{m+n}) \prod_{\substack{1 \leq i \leq m \\ m+1 \leq j \leq m+n}} \lambda(s_i - s_j) \right). \quad (4.7)$$

Proposition 4.8. (a) *The multiplication \star makes $\bigoplus_n Mer(\mathbb{C}^n)^{\mathfrak{S}_n}$ into a graded associative algebra with unit.*

(b) *The correspondence*

$$F(s_1, \dots, s_n) \longmapsto F(s_1, \dots, s_n) \prod_{i < j} \lambda(s_i - s_j)$$

defines an injective algebra homomorphism

$$\left(\bigoplus_n Mer(\mathbb{C}^n)^{\mathfrak{S}_n}, \star \right) \hookrightarrow \left(\bigoplus_n Mer(\mathbb{C}^n), \otimes \right).$$

(c) *Assume that $\lambda(s)$ has no poles except, possibly, a first order pole at $s = 0$. Then the graded subspace $\bigoplus_n \mathcal{O}(\mathbb{C}^n)^{\mathfrak{S}_n}$ is a subalgebra with respect to \star .*

Proof. Parts (a) and (b) are proved straightforwardly, as in [5]. For (c), let us indicate why

$$\star_{1,1} : \mathcal{O}(\mathbb{C}) \times \mathcal{O}(\mathbb{C}) \longrightarrow Mer(\mathbb{C}^2)$$

takes values in $\mathcal{O}(\mathbb{C}^2)$ (the general case is similar). Writing $\lambda(s) = cs^{-1} + h(s)$ with h entire, we have, for $f, g \in \mathcal{O}(\mathbb{C})$:

$$\begin{aligned} (f \star g)(s_1, s_2) &= \lambda(s_1 - s_2) f(s_1) g(s_2) + \lambda(s_2 - s_1) f(s_2) g(s_1) \\ &= \frac{c}{s_1 - s_2} [f(s_1) g(s_2) - f(s_2) g(s_1)] + (\text{entire}), \end{aligned}$$

and the expression in square brackets, being an entire antisymmetric function, vanishes on the diagonal $s_1 = s_2$. □

Definition 4.9. (a) We call the *shuffle algebra* associated to φ the subalgebra $\mathcal{SH}(\varphi) \subset \bigoplus_{n \geq 0} Mer(\mathbb{C}^n)$ generated by the space $\mathcal{O}(\mathbb{C}) \subset Mer(\mathbb{C}^1)$. We

call the *symmetric shuffle algebra* associated to λ the subalgebra $\mathcal{SSH}(\lambda) \subset \bigoplus_{n \geq 0} \mathcal{Mer}(\mathbb{C}^n)^{\mathfrak{S}_n}$ generated by $\mathcal{O}(\mathbb{C})$.

(b) The *Paley-Wiener shuffle algebra* $\mathcal{SH}(\varphi)_{\mathcal{PW}}$, resp. the *Paley-Wiener symmetric shuffle algebra* $\mathcal{SSH}(\lambda)_{\mathcal{PW}}$, is defined as the subalgebra in $\mathcal{SH}(\varphi)$, resp. $\mathcal{SSH}(\lambda)$, generated by the subspace $\mathcal{PW}(\mathbb{C}) \subset \mathcal{O}(\mathbb{C})$.

Thus, if φ and λ are related by (4.5), then $\mathcal{SH}(\varphi)$ is isomorphic to $\mathcal{SSH}(\lambda)$ and $\mathcal{SH}(\varphi)_{\mathcal{PW}}$ to $\mathcal{SSH}(\lambda)_{\mathcal{PW}}$. If, further, λ satisfies the condition (c) of Proposition 4.8, then $\mathcal{SSH}(\lambda)$ is a subalgebra of $\bigoplus \mathcal{O}(\mathbb{C}^n)^{\mathfrak{S}_n}$.

We now specialize $\varphi(s)$ to be the following meromorphic function:

$$\Phi(s) = \zeta^*(s)/\zeta^*(s+1). \quad (4.10)$$

It is known as the *global Harish-Chandra function* (or the *scattering matrix*) for $\overline{\text{Spec}(\mathbb{Z})}$, cf. [15, §7]. The functional equation for $\zeta(s)$ implies that $\Phi(-s)\Phi(s) = 1$. We also consider the function

$$\Lambda(s) = \zeta^*(-s)(s-1)(-s-1). \quad (4.11)$$

It has just one simple pole at $s = 0$, with $\text{res}_{s=0} \Lambda(s) = 1$, and zeroes at nontrivial zeroes of $\zeta(s)$ as well as at $s = -1$. We also have the identity

$$\Phi(s) = \Lambda(s)^{-1}\Lambda(-s). \quad (4.12)$$

Here is the main result of this paper, which will be proved in Section 5.

Theorem 4.13. *The Mellin transform $\mathcal{M} : SH_1 = C_0^\infty(\mathbb{R}_+) \xrightarrow{\sim} \mathcal{PW}(\mathbb{C})$ extends to an isomorphism of algebras $SH \rightarrow \mathcal{SH}(\Phi)_{\mathcal{PW}} \simeq \mathcal{SSH}(\Lambda)_{\mathcal{PW}}$.*

The bigger algebra $\mathcal{SSH}(\Lambda) \simeq \mathcal{SH}(\Phi)$ can be thus seen as a natural completion of SH .

5 The constant term and its Mellin transform.

The sum over shuffles appearing in the definition of the shuffle algebra turns out to match quite exactly the sum over shuffles appearing in the classical formula for the constant term of a (pseudo-)Eisenstein series, cf. [20], II.1.7. In this section we perform a detailed comparison and obtain a proof of Theorem 4.13. Our comparison can be organized into 5 steps:

- (A) Taking the constant term of an automorphic form on GL_n with respect to the Borel subgroup B_n , defines a map $CT_n : H_n \rightarrow C^\infty(\mathbb{R}_+^n)$.
- (B) We denote by \widetilde{CT}_n the twist of CT_n by the analog of the Euler form (Iwasawa Jacobian) to match the formula (2.1) for the Hall product. It is then adjoint to the Hall multiplication map

$$*_1^n = *_{1,\dots,1} : H_1^{\otimes n} \longrightarrow H_n$$

with respect to natural positive definite Hermitian scalar products on both sides. This adjointness implies that the restriction of \widetilde{CT}_n to $SH_n = \text{Im}(*_1^n)$ is an embedding $SH_n \rightarrow C^\infty(\mathbb{R}_+^n)$.

- (C) The standard principal series intertwiners for GL_n give rise to integral operators

$$M_w : C_0^\infty(\mathbb{R}_+^n) \longrightarrow C^\infty(\mathbb{R}_+^n), \quad w \in \mathfrak{S}_n,$$

whose domain of definition can be extended to include more general functions. The formula for the constant term of a pseudo-Eisenstein series then says:

$$\widetilde{CT}_{n'+n''}(f' * f'') = \sum_{w \in Sh(n', n'')} M_w(\widetilde{CT}_{n'}(f') \otimes \widetilde{CT}_{n''}(f'')), \quad f' \in H_{n'}, f'' \in H_{n''}. \quad (5.1)$$

- (D) For $f \in SH_n$ we define $Ch_n(f)$ to be the Mellin transform of $\widetilde{CT}_n(f)$. It is verified to represent a meromorphic function on \mathbb{C}^n . Taken together, the maps Ch_n define then an embedding of vector spaces $Ch : SH \rightarrow \bigoplus_n Mer(\mathbb{C}^n)$.

(E) Finally, one sees that the Mellin transform takes M_w to the operator on $\mathcal{M}er(\mathbb{C}^n)$ taking a function $F(s_1, \dots, s_n)$ to

$$(wF)(s_1, \dots, s_n) \cdot \prod_{\substack{i < j \\ w(i) > w(j)}} \Phi(s_i - s_j),$$

and so Ch takes the Hall product into the shuffle product, by comparing (5.1) with (4.3).

We now implement each step in detail.

A. The constant term. We will use both the real and the adelic interpretation of the component H_n of H :

$$H_n = C_0^\infty(GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R}) / O_n) = C_0^\infty(GL_n(\mathbb{Q}) \backslash GL_n(\mathfrak{A}) / GL_n(\widehat{O})).$$

Let $B = B_n$ be the lower triangular Borel subgroup in GL_n and U be the unipotent radical of B . For $f \in H_n$ its *constant term* is the function $\text{CT}(f)$ on \mathbb{R}_+^n defined in either interpretation by:

$$\begin{aligned} \text{CT}_n(f)(a_1, \dots, a_n) &= \int_{u \in U(\mathbb{Z}) \backslash U(\mathbb{R})} f(u \cdot \text{diag}(a_1, \dots, a_n)) du = \\ &= \int_{u_{\mathfrak{A}} \in U(\mathbb{Q}) \backslash U(\mathfrak{A})} f(u_{\mathfrak{A}} \cdot \text{diag}(a_1, \dots, a_n)) du_{\mathfrak{A}}, \quad a_i \in \mathbb{R}_+. \end{aligned} \tag{5.2}$$

Here du , resp. $du_{\mathfrak{A}}$, is the Hall measure on $U(\mathbb{R})$, resp. $U(\mathfrak{A})$, normalized so that $U(\mathbb{Z}) \backslash U(\mathbb{R})$, resp. $U(\mathbb{Q}) \backslash U(\mathfrak{A})$, has volume 1. Clearly, $\text{CT}_n(f)$ is a C^∞ -function on \mathbb{R}_+^n , bounded by $\max |f(g)|$.

Proposition 5.3. *For every $f \in H_n$ there is $c \in \mathbb{R}_+$ such that $\text{Supp}(\text{CT}_n(f))$ is contained in the domain*

$$a_1 \leq c, \quad a_1 a_2 \leq c, \quad \dots, \quad a_1 \dots a_{n-1} \leq c, \quad \frac{1}{c} \leq a_1 \dots a_n \leq c.$$

Proof. For $(a_1, \dots, a_n) \in \mathbb{R}_+^n$ and $u \in U(\mathbb{R})$ let $V(a_1, \dots, a_n; u)$ be the vector bundle on $\overline{\text{Spec}(\mathbb{Z})}$ associated to the class of $u \cdot \text{diag}(a_1, \dots, a_n)$ in the double quotient. This bundle has a canonical admissible filtration

$$V_1 \subset V_2 \subset \dots \subset V_n = V(a_1, \dots, a_n; u)$$

with $\text{rk}(V_i) = i$ and $V_i/V_{i-1} \simeq \mathcal{O}(a_i)$. But given any vector bundle V on $\text{Spec}(\mathbb{Z})$, there is $c \in \mathbb{R}_+$ such that for any admissible filtration $V_1 \subset \cdots \subset V_n = V$ with $\text{rk}(V_i) = i$, the numbers $a_i = \text{deg}(V_i/V_{i-1})$ satisfy the conditions of Proposition 5.3. This follows from Proposition 1.10, and we can clearly find a common c for bundles varying in a compact subset of Bun_n . \square

B. Twisted constant term and its adjointness. Let

$$dg = \frac{\prod_{i,j=1}^n dg_{ij}}{\det(g)^n}, \quad d^*a = \prod_{i=1}^n \frac{da_i}{a_i}$$

be the standard Haar measures on $GL_n(\mathbb{R})$ and \mathbb{R}_+^n . We introduce notation for the factors in the Iwasawa decomposition:

$$GL_n(\mathbb{R}) = U \cdot \mathbb{R}_+^n \cdot O_n, \quad g = u \cdot a \cdot k, \quad a = (a_1, \dots, a_n).$$

We write $a = a(g)$, $a_\nu = a_\nu(g)$ etc. as functions of $g \in GL_n(\mathbb{R})$. Let dk be the Haar measure on O_n of volume 1.

The Haar measure dg on $GL_n(\mathbb{R})$ has, in Iwasawa coordinates, the well known form

$$dg = \delta(a) du \cdot dk \cdot d^*a, \quad (5.4)$$

where the *Iwasawa Jacobian* $\delta(a)$ is defined by

$$\delta(a) = \delta_n(a) = \prod_{1 \leq i < j \leq n} \frac{a_j}{a_i} = \prod_{i=1}^n a_i^{-n+2i-1}. \quad (5.5)$$

See, e.g., [32], §4.1, Exercise 20 for upper-triangular matrices. We also write $\delta_n(g) = \delta_n(a(g))$ for $g \in GL_n(\mathbb{R})$.

Let us make H_n and $C_0^\infty(\mathbb{R}_+^n) \supset H_1^{\otimes n}$ into pre-Hilbert spaces via the positive definite Hermitian scalar products

$$(f_1, f_2)_H = \int_{GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R})} f_1(g) \overline{f_2(g)} dg, \quad (\varphi_1, \varphi_2) = \frac{1}{2^n} \int_{\mathbb{R}_+^n} \varphi_1(a) \overline{\varphi_2(a)} d^*a.$$

More generally, in each case the scalar product makes sense whenever only one of the arguments has compact support. Define the *twisted constant term* of $f \in H_n$ to be the function

$$\widetilde{\text{CT}}_n(f)(a_1, \dots, a_n) = \text{CT}(f)(a_1, \dots, a_n) \cdot \delta(a)^{1/2}. \quad (5.6)$$

Proposition 5.7. *The map $\widetilde{\text{CT}}_n : H_n \rightarrow C^\infty(\mathbb{R}_+^n)$ is adjoint to $*_{1^n} : H_1^{\otimes n} \rightarrow H_n$, i.e., we have*

$$(*_{1^n}(\varphi), f)_H = (\varphi, \widetilde{\text{CT}}_n(f)), \quad \varphi \in H_1^{\otimes n}, f \in H_n.$$

Proof. This is standard, we provide details for convenience of the reader. For $\varphi \in C_0^\infty(\mathbb{R}_+^n)$ we define a function $\tilde{\varphi}$ on $U \backslash GL_n(\mathbb{R}) / O_n$ by

$$\tilde{\varphi}(g) = \varphi(a_1(g), \dots, a_n(g)) \cdot \delta(g)^{-1/2}.$$

Translating the (iterated) formula (2.1) for the Hall product, into group-theoretical terms, we have

$$(*_{1^n}(\varphi))(g) = \sum_{\gamma \in B_n(\mathbb{Z}) \backslash GL_n(\mathbb{Z})} \tilde{\varphi}(\gamma g)$$

(a pseudo-Eisenstein series). The adjointness then follows from the expression of dg in terms of the Iwasawa factorization:

$$\begin{aligned} (*_{1^n}(\varphi), f)_H &= \int_{g \in GL_n(\mathbb{Z}) \backslash GL_n(\mathbb{R})} \overline{f(g)} \sum_{\gamma \in B(\mathbb{Z}) \backslash GL_n(\mathbb{Z})} \tilde{\varphi}(\gamma g) dg \\ &= \int_{x \in B(\mathbb{Z}) \backslash GL_n(\mathbb{R})} \overline{f(x)} \tilde{\varphi}(x) dx \stackrel{\text{def}}{=} \int_{x \in B(\mathbb{Z}) \backslash GL_n(\mathbb{R})} \overline{f(x)} \varphi(x) \delta(x)^{-1/2} dx \\ &= \frac{1}{2^n} \int_{y \in U(\mathbb{Z}) \backslash GL_n(\mathbb{R})} \overline{f(y)} \varphi(y) \delta(y)^{-1/2} dy \\ &= \frac{1}{2^n} \int_{z \in U(\mathbb{R}) \backslash GL_n(\mathbb{R})} \int_{u \in U(\mathbb{Z}) \backslash U(\mathbb{R})} \overline{f(uz)} \varphi(z) \delta(z)^{-1/2} du dz \\ &\stackrel{(5.4)}{=} \frac{1}{2^n} \int_{a \in \mathbb{R}_+^n} \overline{\text{CT}_n(f)(a)} \varphi(a) \delta(a)^{+1/2} d^*a = (\varphi, \widetilde{\text{CT}}_n(f)). \end{aligned}$$

□

Corollary 5.8. *The map $\widetilde{\text{CT}}_n : SH_n \rightarrow C_0^\infty(\mathbb{R}_+^n)$ is injective.*

Proof. By definition of SH as the subalgebra generated by H_1 , a non-zero element $f \in SH_n$ has the form $f = *_{1^n}(\varphi)$ for some $\varphi \in H_1^{\otimes n}$. We can regard φ as an element of $C_0^\infty(\mathbb{R}_+^n)$. To prove that $\widetilde{\text{CT}}_n(*_{1^n}(\varphi)) \neq 0$, we notice that by adjointness and by the positivity of the scalar product on H , we have

$$(\varphi, \widetilde{\text{CT}}_n(*_{1^n}(\varphi))) = (*_{1^n}(\varphi), *_{1^n}(\varphi))_H = (f, f)_H > 0.$$

□

C. The principal series intertwiners. We use the intertwiners in their adelic form, as this form accounts for the appearance of the factors involving the Riemann zeta in the function $\Phi(s)$ defining the shuffle algebra, see (4.10).

Let A_n be the diagonal subgroup in GL_n . We have the identification

$$\mathbb{R}_+^n = U(\mathfrak{A})A_n(\mathbb{Q}) \backslash GL_n(\mathfrak{A}) / K_n, \quad K_n = O_n \prod_p GL_n(\mathbb{Z}_p).$$

For $w \in \mathfrak{S}_n$ let $U_w = U \cap (w^{-1}Uw)$. Using the above identification, we define the operator

$$M_w : C_0^\infty(\mathbb{R}_+^n) \longrightarrow C^\infty(\mathbb{R}_+^n), \quad (M_w \varphi)(g) = \int_{u \in (U(\mathfrak{A}) \cap U_w(\mathfrak{A})) \backslash U(\mathfrak{A})} \varphi(wug) du,$$

cf. [20], II.1.6. More generally, $M_w(\varphi)$ can be defined if, for any g , the function $u \mapsto \varphi(wug)$ on the domain of integration has sufficiently fast decay (for example, has compact support). Here is an example, to be used later.

We consider the following domain in \mathbb{C}^n :

$$\mathbb{C}_>^n = \{s = (s_1, \dots, s_n) : s_\nu - s_{\nu+1} > 1, \nu = 1, \dots, n\}, \quad (5.9)$$

where we put $s_{n+1} = 0$. For $w \in \mathfrak{S}_n$ put

$$\Phi_w(s) = \prod_{\substack{1 \leq i < j \leq n \\ w(i) > w(j)}} \Phi(s_i - s_j). \quad (5.10)$$

Proposition 5.11. *If $s = (s_1, \dots, s_n) \in \mathbb{C}_>^n$, then applying M_w to the function $a \mapsto a^s$ gives a convergent integral, and it is found as follows:*

$$M_w(a^s) = a^{w(s)} \Phi_w(s).$$

Proof. This is a version of the classical Gindikin-Karpelevich formula. More precisely, the value of the adelic intertwiner is found as the Euler product of the values of similarly defined local intertwiners (involving the integration over the p -adic or real group). Each local integral is found by Gindikin-Karpelevich to contribute the factor

$$\prod_{\substack{1 \leq i < j \leq n \\ w(i) > w(j)}} \frac{\zeta_p(s_i - s_j)}{\zeta_p(s_i - s_j + 1)},$$

where ζ_p is the p th Euler factor of the Riemann zeta, or the Gamma factor for $p = \infty$. \square

For $\varphi' \in C^\infty(\mathbb{R}_+^{n'})$ and $\varphi'' \in C^\infty(\mathbb{R}_+^{n''})$ we define $\varphi' \otimes \varphi'' \in C^\infty(\mathbb{R}_+^{n'+n''})$ by

$$(\varphi' \otimes \varphi'')(a_1, \dots, a_{n'+n''}) = \varphi'(a_1, \dots, a_{n'})\varphi''(a_{n'+1}, \dots, a_{n'+n''}). \quad (5.12)$$

We will use similar notation in other situations without special explanation.

Having now defined all the ingredients of the equality (5.1), we explain how it is proved. This is again a standard argument, using the Bruhat decomposition of a Grassmannian into cells labelled by shuffles, cf. [20], II.1.7 for the case of any parabolic subgroup in any reductive group.

To give some details in our particular case, let $n = n' + n''$ and $P_{n',n''} \subset GL_n$ be the parabolic (block-lower-triangular) subgroup corresponding to (n', n'') . We denote $U_{n',n''}$ its unipotent radical and $A_{n',n''} = GL_{n'} \times GL_{n''}$ the Levi subgroup. Then the Iwasawa decomposition implies that

$$\begin{aligned} (GL_{n'}(\mathbb{Q}) \backslash GL_{n'}(\mathfrak{A}) / K_{n'}) \times (GL_{n''}(\mathbb{Q}) \backslash GL_{n''}(\mathfrak{A}) / K_{n''}) &\xrightarrow{\sim} \\ \xrightarrow{\sim} (U_{n',n''}(\mathfrak{A}) A_{n',n''}(\mathbb{Q})) \backslash GL_n(\mathfrak{A}) / K_n. &\end{aligned} \quad (5.13)$$

Given $f' \in H_{n'}$, $f'' \in H_{n''}$, let f be the function on the right hand side of (5.13) corresponding to the function

$$(g', g'') \mapsto |\det(g')|^{n''/2} \cdot |\det(g'')|^{-n'/2} \cdot f'(g') f''(g'')$$

on the left hand side. Here $|a|$ is the adelic norm of a . The Hall product $f' * f''$ is then given by the parabolic pseudo-Eisenstein series

$$(f' * f'')(g) = \sum_{\gamma \in P_{n',n''}(\mathbb{Q}) \backslash GL_n(\mathbb{Q})} f(\gamma g).$$

Now, writing

$$\widetilde{CT}_n(f' * f'')(g) = \int_{u \in U(\mathbb{Q}) \backslash U(\mathfrak{A})} \sum_{\gamma \in P_{n',n''}(\mathbb{Q}) \backslash GL_n(\mathbb{Q})} f(\gamma u g) \delta_n(g)^{1/2} du,$$

we notice that the Grassmannian $\text{Gr}(n', \mathbb{Q}^n) = P_{n',n''}(\mathbb{Q}) \backslash GL_n(\mathbb{Q})$ splits, under the right $U(\mathbb{Q})$ -action, into $\binom{n}{n'}$ orbits (Schubert cells)

$$\Sigma_w = P_{n',n''}(\mathbb{Q}) \backslash wU(\mathbb{Q}), \quad w \in Sh(n', n'').$$

Notice that for $w \in Sh(n', n'')$ we have $U_w = U \cap w^{-1}P_{n', n''}w$. This means that we can write each $\gamma \in \Sigma_w$ uniquely in the form $\gamma = P_{n', n''}(\mathbb{Q}) \cdot w \cdot v$ for $v \in U_w(\mathbb{Q}) \setminus U(\mathbb{Q})$ and so

$$\begin{aligned} \widetilde{CT}_n(f' * f'')(g) &= \sum_{w \in Sh(n', n'')} \int_{u \in U(\mathbb{Q}) \setminus U(\mathfrak{A})} \sum_{v \in U_w(\mathbb{Q}) \setminus U(\mathbb{Q})} f(wvug) \delta_n(g)^{1/2} du \\ &= \sum_{w \in Sh(n', n'')} \int_{\tilde{u} \in U_w(\mathbb{Q}) \setminus U(\mathfrak{A})} f(w\tilde{u}g) \delta_n(g)^{1/2} d\tilde{u}, \end{aligned}$$

and we identify the integral over \tilde{u} corresponding to w , with $M_w(\widetilde{CT}_{n'}(f') \otimes \widetilde{CT}_{n''}(f''))$. Note that this argument shows, in particular, that M_w is indeed applicable in this case as the domain of integration reduces to a compact one (since all we did was re-partition the integral for $\widetilde{CT}_n(f' * f'')(g)$, which was over a compact domain to begin with). We leave the rest to the reader.

Let us note a version of the above statement for the constant term of the n -tuple Hall product. The proof is similar.

Proposition 5.14. *Let $\varphi_1, \dots, \varphi_n \in C_0^\infty(\mathbb{R}_+)$ and $\varphi = \varphi_1 \otimes \dots \otimes \varphi_n \in C_0^\infty(\mathbb{R}_+^n)$. Then*

$$\widetilde{CT}_n(*_{1^n}(\varphi)) = \sum_{w \in \mathfrak{S}_n} M_w(\varphi). \quad \square$$

D. The Mellin transform of the constant term. For $f \in H_n$ we set $Ch_n(f) = \mathcal{M}(\widetilde{CT}_n(f))$.

Proposition 5.15. *The Mellin integral for $Ch_n(f)$ converges to an analytic function in the region $\mathbb{C}_>^n$.*

Proof. The Mellin transform of $\widetilde{CT}_n(f)(a) = \delta_n(a)^{1/2} CT_n(a)$ differs from $\mathcal{M}(CT_n(a))$ by a shift of variables, and our statement is equivalent to saying that $\mathcal{M}(CT_n(a))$ converges for

$$\operatorname{Re}(s_1 - s_2) > 0, \operatorname{Re}(s_2 - s_3) > 0, \dots, \operatorname{Re}(s_{n-1} - s_n) > 0, \operatorname{Re}(s_n) > 0.$$

To see this, note that by Proposition 5.3 and of boundedness of $CT_n(f)$, the integral is bounded by

$$\begin{aligned} \text{const} \int_{a_1=0}^c \int_{a_1 a_2=0}^c \dots \int_{a_1 \dots a_n=0}^c & a_1^{s_1 - s_2} (a_1 a_2)^{s_2 - s_3} \dots (a_1 \dots a_n)^{s_n} \times \\ & \times d^* a_1 d^*(a_1 a_2) \dots d^*(a_1 \dots a_n). \end{aligned}$$

Since $\int_0^c a^s d^*a$ converges for $\operatorname{Re}(s) > 0$, the claim follows. \square

Proposition 5.16. *For any $f \in SH_n$, the function $Ch_n(f)$ extends to a meromorphic function on \mathbb{C}^n .*

Before giving the proof, we recall the properties of a classical type of Eisenstein series due to Selberg [28].

For any $s \in \mathbb{C}$ we denote by $\mathfrak{E}(s)$ the following function on Bun_1 :

$$\mathfrak{E}(s) : E \longmapsto \deg(E)^s = \exp(s \cdot \ln(\deg(E))). \quad (5.17)$$

The (formal) Hall product

$$\mathfrak{E}(s_1) * \cdots * \mathfrak{E}(s_n) = {}_{*1^n}(a_1^{s_1} \cdots a_n^{s_n}) \quad (5.18)$$

is a series of functions on Bun_n , known as the (primitive) *Eisenstein-Selberg series*, see [28] and [11] §8.3.

Proposition 5.19. (a) *The series (5.18) converges for $s = (s_1, \dots, s_n) \in \mathbb{C}_{>}^n$, to a C^∞ -function on Bun_n .*

(b) *For any $g \in \operatorname{Bun}_n$ the function $(\mathfrak{E}(s_1) * \cdots * \mathfrak{E}(s_n))(g)$ extends to a meromorphic function in the s_i , with position and order of poles independent on g .*

(c) *The twisted constant term of $(\mathfrak{E}(s_1) * \cdots * \mathfrak{E}(s_n))(g)$ as a function on g is given by*

$$\widetilde{\operatorname{CT}}_n(\mathfrak{E}(s_1) * \cdots * \mathfrak{E}(s_n))(a_1, \dots, a_n) = \sum_{w \in \mathfrak{S}_n} a_1^{s_{w(1)}} \cdots a_n^{s_{w(n)}} \prod_{\substack{i < j \\ w(i) > w(j)}} \Phi(s_i - s_j).$$

Proof. For (a), see, e.g., [11], §8.5, Remark, and take into account the Ringel twist in the definition of $*$ which translates the shifts by $1/2$ into shifts by 1 . See also [7], Proposition 10.4.3 for a slightly weaker statement.

For (b), see [11], §8.6-7.

Finally, (c) follows by the formula (5.1) applied to the function a^s , $s \in \mathbb{C}_{>}^n$ (the application is legal because of the decay conditions) and then using Proposition 5.11. \square

Proof of Proposition 5.16: It is enough to assume that $f = f_1 * \cdots * f_n$, where $f_\nu \in H_1 = C_0^\infty(\mathbb{R}_+)$. Let $F_\nu = \mathcal{M}(f_\nu) \in \mathcal{PW}(\mathbb{C})$ be the Mellin transform of f_ν . Then $f_\nu = \mathcal{N}(F_\nu)$, and the inverse Mellin integral (understood as in Proposition 3.8) can be taken along any vertical line $\operatorname{Re}(s) = \sigma_\nu$.

Let us now choose $\sigma_1, \dots, \sigma_n$ such that $\sigma_{\nu+1} - \sigma_\nu > 1$ for each $\nu = 1, \dots, n-1$ and $\sigma_n > 1$. The equalities $\mathcal{N}(F_\nu) = f_\nu$ then imply that

$$f(g) = \frac{1}{(2\pi i)^n} \int_{\operatorname{Re}(s_\nu) = \sigma_\nu} F_1(s_1) \cdots F_n(s_n) (\mathfrak{E}(-s_1) * \cdots * \mathfrak{E}(-s_n))(g) ds_1 \cdots ds_n.$$

Substituting the formula for the twisted constant term of $(\mathfrak{E}(-s_1) * \cdots * \mathfrak{E}(-s_n))(g)$ from Proposition 5.19(c) into the integral for $f(g)$, we represent $\widetilde{\text{CT}}_n(f)$ as the inverse Mellin transform of the function

$$F(s_1, \dots, s_n) = \sum_{w \in \mathfrak{S}_n} F_1(s_{w(1)}) \cdots F_n(s_{w(n)}) \prod_{\substack{i < j \\ w(i) > w(j)}} \Phi(s_j - s_i),$$

which is analytic in the region $\operatorname{Re}(s_{\nu+1}) - \operatorname{Re}(s_\nu) > 1$. Further, if we take $\sigma_1, \dots, \sigma_n$ such that $\sigma_{\nu+1} - \sigma_\nu > 1$, $\sigma_n > 1$, then F is bounded on the vertical subspace $\operatorname{Re}(s_\nu) = \sigma_\nu$. Indeed, each F_i , being a Paley-Wiener function, decays exponentially at the imaginary infinity. On the other hand, the lemma below shows that $\Phi(s)$ is bounded on vertical lines $\operatorname{Re}(s) = \sigma_0 > 1$. Therefore we can apply the Mellin inversion (Proposition 3.8) to F and obtain that $Ch_n(f) = \mathcal{M}(\widetilde{\text{CT}}_n(f)) = F(s_1, \dots, s_n)$ and so it is meromorphic.

Lemma 5.20. *For every $\sigma_0 > 1$, the function $\Phi(\sigma_0 + it)$ is bounded, as a function of $t \in \mathbb{R}$, and decays as $|t| \rightarrow \infty$.*

Proof. Indeed, for $s = \sigma_0 + it$, $\sigma_0 > 1$ we have

$$\zeta(s)/\zeta(s+1) = \sum_{n=1}^{\infty} \varphi(n) n^{-s-1},$$

where $\varphi(n) = |(\mathbb{Z}/n)^\times|$ is the Euler function. This is bounded by

$$\sum n \cdot n^{-\sigma_0-1} = \zeta(\sigma_0).$$

Further, $\Gamma(\frac{s}{2})/\Gamma(\frac{s+1}{2})$ decays at infinity as $s^{-1/2}$, as it follows from the Stirling formula. \square

E. Intertwiners and the constant term. We now study the action of the intertwiners M_w on the Mellin transform of the constant term.

Proposition 5.21. For $\varphi \in C_0^\infty(\mathbb{R}_+^n)$ and any $w \in \mathfrak{S}_n$ we have

$$\mathcal{M}(M_w(\varphi))(s) = \mathcal{M}(\varphi)(w(s)) \cdot \Phi_w(s).$$

Proof. Write φ as the inverse Mellin integral of a Paley-Wiener function F over any vertical subspace $\sigma + i\mathbb{R}^n$ inside $\mathbb{C}_>^n$, and apply Proposition 5.11. \square

At this point, we can finish the proof of Theorem 4.13. It remains only to prove that Ch is a homomorphism of algebras, i.e., that

$$Ch_n(f' * f'') = Ch_{n'}(f') \mathbb{S} Ch_{n''}(f''), \quad n = n' + n'' \quad (5.22)$$

for any $f' \in SH_{n'}$ and $f'' \in SH_{n''}$. Using the formula (5.1) for the left hand side and the definition of the shuffle product \mathbb{S} for the right hand side, we write this as an equality of two sums over shuffles

$$\begin{aligned} & \sum_{w \in Sh_{n', n''}} \mathcal{M}(M_w(\widetilde{CT}_{n'}(f') \otimes \widetilde{CT}_{n''}(f'')))(s) = \\ & = \sum_{w \in Sh_{n', n''}} \mathcal{M}(\widetilde{CT}_{n'}(f') \otimes \widetilde{CT}_{n''}(f''))(w(s)) \cdot \Phi_w(s). \end{aligned} \quad (5.23)$$

As f', f'' belong to the subalgebra SH , we can write them as

$$f' = *_{1^{n'}}(\varphi'), \quad f'' = *_{1^{n''}}(\varphi'')$$

for some $\varphi' \in C_0^\infty(\mathbb{R}_+^{n'})$, $\varphi'' \in C_0^\infty(\mathbb{R}_+^{n''})$. By Proposition 5.14, we have

$$\widetilde{CT}_{n'}(f') = \sum_{w' \in \mathfrak{S}_{n'}} M_{w'}(\varphi'),$$

and similarly for $\widetilde{CT}_{n''}(f'')$. Substituting this to the LHS of the putative equality (5.23), we find that it is equal to

$$\sum_{w \in \mathfrak{S}_n} \mathcal{M}(M_w(\varphi)) \stackrel{5.21}{=} \sum_{w \in \mathfrak{S}_n} \mathcal{M}(\varphi)(w(s)) \cdot \Phi_w(s), \quad \varphi = \varphi' \otimes \varphi''. \quad (5.24)$$

On the other hand, writing $s \in \mathbb{C}^n$ as (s', s'') with $s' \in \mathbb{C}^{n'}$, $s'' \in \mathbb{C}^{n''}$, we have

$$\mathcal{M}(\widetilde{CT}_{n'}(f') \otimes \widetilde{CT}_{n''}(f''))(s) = \mathcal{M}(\widetilde{CT}_{n'}(f'))(s') \cdot \mathcal{M}(\widetilde{CT}_{n''}(f''))(s''),$$

and so the summand in the RHS of (5.23) corresponding to $w \in Sh_{n',n''}$, is equal by Proposition 5.14, to $\Phi_w(s)$ times

$$\begin{aligned} & \sum_{\substack{w' \in \mathfrak{S}_{n'} \\ w'' \in \mathfrak{S}_{n''}}} \mathcal{M}(\varphi')(w'(s')) \cdot \mathcal{M}(\varphi'')(w''(s'')) \cdot \Phi_{w'}(s') \Phi_{w''}(s'') = \\ & = \sum_{\substack{w' \in \mathfrak{S}_{n'} \\ w'' \in \mathfrak{S}_{n''}}} \mathcal{M}_{w' \times w''}(\varphi)((w' \times w'')(s)) \cdot \Phi_{w' \times w''}(s), \end{aligned}$$

and further summation over w gives the same result as (5.24). \square

6 Quadratic relations and Eisenstein series.

Let

$$S = \bigoplus_{n=0}^{\infty} S_n, \quad S_0 = \mathbb{C},$$

be a graded associative algebra over \mathbb{C} . The space of degree n relations among elements of degree 1 is then

$$R_n = \text{Ker}\{S_1^{\otimes n} \longrightarrow S_n\} \subset S_1^{\otimes n}. \quad (6.1)$$

Here we are interested in quadratic relations ($n = 2$) for the algebra SH generated by $SH_1 = H_1 = C_0^\infty(\mathbb{R}_+)$. Because of the analytic nature of elements of H it is not reasonable to look for relations inside the algebraic tensor product $H_1 \otimes H_1$ and we consider a completion of it, namely the space

$$H_1 \widehat{\otimes} H_1 := \mathcal{D}(\mathbb{R}_+^2)_{\text{abs}}$$

of absolutely tempered distributions on \mathbb{R}_+^2 , see Corollary 3.9.

Proposition 6.2. *If $f \in H_1 \widehat{\otimes} H_1$, then the series*

$$\widehat{*}_{1,1}(f)(E) = \sum_{E' \subset E} \text{deg}(E')^{1/2} \text{deg}(E/E')^{-1/2} f(\text{deg}(E'), \text{deg}(E/E')), \quad E \in \text{Bun}_2,$$

converges absolutely, defining a distribution $\widehat{}_{1,1}(f)$ on Bun_2 . The resulting linear map $\widehat{*}_{1,1} : H_1 \widehat{\otimes} H_1 \rightarrow \mathcal{D}(\text{Bun}_2)$ extends the Hall multiplication $*_{1,1} : H_1 \otimes H_1 \rightarrow H_2$.*

Proof. The points $(\alpha, \beta) = (\deg(E'), \deg(E/E'))$ lie on the hyperbola $\alpha\beta = \deg(E)$. An absolutely tempered distribution decays exponentially at the infinity of \mathbb{R}_+^2 , in particular at the infinity of any such hyperbola. Now the number of subbundles in $E = (L, V, q)$ of given degree $\alpha = 1/a$ is one half the number of primitive vectors in L of norm a . This number of all lattice vectors of norm a grows linearly with a , so exponential decay of f ensures the convergence. \square

Remark 6.3. It is possible that one can extend H to a bigger algebra, consisting of some analogs of absolutely tempered distributions on the Bun_n , which have sufficient decay at the infinity. Note that the concept of a tempered distribution on a semisimple Lie group was introduced by Harish-Chandra [10].

We will therefore understand quadratic relations in SH in a wider sense, as elements of the space

$$\widehat{R}_2 = \text{Ker}(\widehat{*}_{1,1}) \subset H_1 \widehat{\otimes} H_1. \quad (6.4)$$

Let also \mathcal{R}_2 be the space of entire functions $F \in \mathcal{O}(\mathbb{C}^2)_{\text{pol}}$ such that

$$F(s_1, s_2) + \Phi(s_1 - s_2)F(s_2, s_1) = 0. \quad (6.5)$$

Proposition 6.6. *The Mellin transform identifies \widehat{R}_2 with \mathcal{R}_2 .*

Proof. This follows from an instance of Eq. (5.22) for $m = n = 1$ but applied to absolutely tempered distributions instead of functions with compact support. The proof in the new case is the same, given the decay (to define the Hall product) and the analyticity of the Mellin transform. \square

Note that \mathcal{R}_2 is a module over the ring $\mathcal{O}(\mathbb{C}^2)_{\text{pol}}^{\text{sym}}$ of symmetric entire functions of polynomial growth on vertical planes.

Example 6.7. Let $P(s) = s(s-1)(s+1)$. Then the function

$$F_{1,1}(s_1, s_2) = P(s_1 - s_2)\zeta^*(s_1 - s_2)$$

belongs to \mathcal{R}_2 . Further, for any $\lambda_1, \lambda_2 \in \mathbb{R}_+$ the function

$$F_{\lambda_1, \lambda_2}(s_1, s_2) = (\lambda_1^{s_1} \lambda_2^{s_2} + \lambda_1^{s_2} \lambda_2^{s_1}) F_{1,1}(s_1, s_2)$$

again lies in \mathcal{R}_2 by the remark above. Let

$$\nabla_a = P\left(a\frac{d}{da}\right) = a^3\frac{d^3}{da^3} - a^2\frac{d^2}{da^2}.$$

The inverse Mellin transform of $F_{1,1}$ is, in virtue of Proposition 3.10 and the Riemann formula (3.13), equal to

$$\Psi_{1,1}(a_1, a_2) = \delta_1(a_1 a_2) \cdot \nabla_{a_1} \theta(a_1^2) \in \widehat{R}_2,$$

and the inverse Mellin transform of F_{λ_1, λ_2} is the distribution

$$\Psi_{\lambda_1, \lambda_2}(a_1, a_2) = \Psi_{1,1}(a_1/\lambda_1, a_2/\lambda_2) + \Psi_{1,1}(a_1/\lambda_2, a_2/\lambda_1) \in \widehat{R}_2.$$

This gives a 2-parameter family of quadratic relations in SH .

Remark 6.8. This 2-parameter family of relations is analogous to the family of relations

$$[\mathcal{O}(m+1)] * [\mathcal{O}(n)] - q[\mathcal{O}(n)] * [\mathcal{O}(m+1)] = q[\mathcal{O}(m)] * [\mathcal{O}(n+1)] - [\mathcal{O}(n+1)] * [\mathcal{O}(m)]$$

in the Hall algebra of the category of vector bundles on $\mathbb{P}_{\mathbb{F}_q}^1$, see [12] §5.2 or [2], Lemma 16.

We now explain the relation of the above quadratic relations with the functional equation for *Eisenstein-Maas series*

$$\mathbf{E}(\tau, s) = \frac{1}{2} \sum_{(m,n)=1} \frac{\text{Im}(\tau)^s}{|m + n\tau|^{2s}}, \quad \tau \in \mathbb{H}, \text{Re}(s) > 1,$$

see [7], §3.1. It is classical that $\mathbf{E}(\tau, s)$ extends to a function meromorphic in the entire s -plane and satisfying the functional equation

$$\mathbf{E}(\tau, s) = \frac{\zeta^*(2s-1)}{\zeta^*(2s)} \mathbf{E}(\tau, 1-s).$$

Further, the poles of $\mathbf{E}(\tau, s)$ are all among the poles of the ratio of the ζ^* -functions, in particular, they do not depend on τ .

On the other hand, recall (5.17) the function

$$\mathfrak{E}(t) : \text{Bun}_1 \rightarrow \mathbb{C}, \quad E \mapsto \deg(E)^t.$$

Here $t \in \mathbb{C}$ is a fixed complex number. This function does not lie in $H_1 = C_0^\infty(\mathbb{R})$. Nevertheless, the correspondence $t \mapsto \mathfrak{E}(t)$ can be seen as a kind of H -valued distribution (“operator field”) on \mathbb{C} (or, rather, on $i\mathbb{R} \subset \mathbb{C}$). That is, for any Paley-Wiener function $G(t)$ we have a well defined element

$$\int_{i\mathbb{R}} \mathfrak{E}(t)G(t)dt \in H_1.$$

This simply the function $E \mapsto f(\deg(E)^{-1})$, where $f = \mathcal{N}(G) \in C_0^\infty(\mathbb{R}_+)$.

Proposition 5.19(a) implies that for $\operatorname{Re}(t_1 - t_2) > 0$ the Hall product $\mathfrak{E}(t_1) * \mathfrak{E}(t_2)$ defined as a formal series, converges to a real analytic function on Bun_2 . This function essentially reduces to the series $\mathbf{E}(\tau, s)$ above. Indeed, let E_τ be the bundle of rank 2 and degree 1 corresponding to τ as in Example 1.5. Rank 1 subbundles $E' = E'_{m,n}$ in E_τ are parametrized by pairs $(m, n) \in \mathbb{Z}^2$ of coprime integers, taken modulo simultaneous change of sign. Explicitly, the primitive sublattice $L'_{m,n}$ of $E'_{m,n}$ is spanned by $m + n\tau$, and we have

$$\deg(E'_{m,n}) = \frac{\operatorname{Im}(\tau)^{1/2}}{|m + n\tau|}, \quad \deg(E_\tau/E'_{m,n}) = \frac{|m + n\tau|}{\operatorname{Im}(\tau)^{1/2}}.$$

Therefore

$$(\mathfrak{E}(t_1) * \mathfrak{E}(t_2))(E_\tau) = \mathbf{E}(\tau, (t_1 - t_2 + 1)/2). \quad (6.9)$$

This means that the product $\mathfrak{E}(t_1) * \mathfrak{E}(t_2)$ extends to a meromorphic function of t_1, t_2 (with values in the space of functions on Bun_2) and we can write a formula looking like “quadratic commutation relations” in H :

$$\mathfrak{E}(t_1) * \mathfrak{E}(t_2) - \Phi(t_1 - t_2) \mathfrak{E}(t_2) * \mathfrak{E}(t_1) = 0. \quad (6.10)$$

The two summands in (6.10) are given by series converging in different regions, having no points in common, and the relations should be understood via analytic continuation. This way of understanding commutation relations is quite standard in the theory of vertex operators [6]. In our situation it is modified as follows.

In order to translate the relations (6.10) into actual elements of \widehat{R}_2 , we can rewrite them in the form “free of denominators”

$$*_{1,1} \{ P(t_1 - t_2) \cdot \zeta^*(t_1 - t_2 + 1) \cdot a_1^{t_1} a_2^{t_2} - P(t_1 - t_2) \cdot \zeta^*(t_1 - t_2) \cdot a_1^{t_2} a_2^{t_1} \} = 0. \quad (6.11)$$

Here we write $\mathfrak{E}(t_1) \otimes \mathfrak{E}(t_2)$ as the function $(a_1, a_2) \mapsto a_1^{t_1} a_2^{t_2}$ on $\text{Bun}_1 \times \text{Bun}_1 = \mathbb{R}_+^2$. We then “compare coefficients” in both sides of this equality at any $\lambda_1^{t_1} \lambda_2^{t_2}$, $\lambda_\nu \in \mathbb{R}_+$, by multiplying with $\lambda_1^{-t_1} \lambda_2^{-t_2}$ and integrating (performing the inverse Fourier-Schwartz transform) along any vertical 2-plane, which we can choose separately for each summand. This gives a family of distributions $\Psi_{\lambda_1, \lambda_2}(a_1, a_2) \in \widehat{R}_2$ which is the same as in Example 6.7.

We can thus say that quadratic relations such as (6.10) are built into the very definition of the shuffle algebra.

7 Wheels, cubic relations, and zeta roots.

A. Wheels. Let $\lambda(s)$ be a meromorphic function on \mathbb{C} with a simple pole at $s = 0$ and no other singularities. In this section we sketch a general approach to higher order relations in the symmetric shuffle algebra $\mathcal{SSH}(\lambda)$ and illustrate it on the case of cubic relations in the shuffle algebra completion of the spherical Hall algebra SH , which corresponds to

$$\lambda(s) = \Lambda(s) = \zeta^*(-s)(s-1)(-s-1).$$

Our approach is based on studying the following additive patterns of roots of λ which were introduced in [4] and used in the case when λ is rational.

Definition 7.1. A *wheel* of length n for λ is a sequence (s_1, \dots, s_n) of distinct complex numbers such that

$$\lambda(s_2 - s_1) = 0, \lambda(s_3 - s_2) = 0, \dots, \lambda(s_n - s_{n-1}) = 0, \lambda(s_1 - s_n) = 0.$$

Wheels (s_1, \dots, s_n) and $(s_1 + c, \dots, s_n + c)$ for $c \in \mathbb{C}$, will be called *equivalent*.

In other words, equivalence classes of wheels are the same as ordered sequences

$$(z_1, \dots, z_n) \in (\mathbb{C}^*)^n, \quad \lambda(z_i) = 0, \quad \sum_{i=1}^n z_i = 0, \quad \sum_{i=p}^q z_i \neq 0, \quad (p, q) \neq (1, n).$$

Example 7.2. All wheels for $\Lambda(s)$ have length 3 or more. The sequences corresponding to wheels of length 3 have, up to permutation, the form

$$(z_1, z_2, z_3) = (\rho, 1 - \rho, -1),$$

where ρ runs over nontrivial zeroes of $\zeta(s)$. Indeed, zeroes of Λ are of the form $s = \rho$ together with one more zero $s = -1$. So there are no pairs of them summing up to 0 and the only triples summing to up 0 are as stated.

B. Relations and bar-complexes. Let S be a graded associative algebra as in §6. A systematic way of approaching relations in S is via the bar-complexes

$$B_n^\bullet = B_n^\bullet(S) = \left\{ S_1^{\otimes n} \rightarrow \cdots \rightarrow \bigoplus_{i+j+k=n} S_i \otimes S_j \otimes S_k \rightarrow \bigoplus_{i+j=n} S_i \otimes S_j \rightarrow S_n \right\}.$$

Here i, j, k, \dots run over positive integers. The grading is such that $S_1^{\otimes n}$ is in degree $(-n)$, while S_n is in degree (-1) . The differential is given by

$$d(s_1 \otimes \cdots \otimes s_p) = \sum_{i=1}^{p-1} (-1)^{i-1} s_1 \otimes \cdots \otimes s_{i-1} \otimes s_i s_{i+1} \otimes s_{i+2} \otimes \cdots \otimes s_p,$$

so that the condition $d^2 = 0$ follows from the associativity of S . It is well known that

$$H^{-i}(B_n^\bullet(S)) = \text{Tor}_i^S(\mathbb{C}, \mathbb{C})_n,$$

the part of the Tor-group which has degree n w.r.t. the grading induced from that on S . In particular, the rightmost cohomology has the meaning of the space of generators in degree n , and the previous one is interpreted as the space of relations which have degree n with respect to the grading on the generators (which, a priori, can be present in any degree).

As in (6.1), let R_n be the space of degree n relations among generators in degree 1. For instance, quadratic relations are found as $R_2 = H^{-2}(B_2^\bullet)$. The next case of cubic relations corresponds to the complex

$$B_3^\bullet = \{ S_1 \otimes S_1 \otimes S_1, \xrightarrow{d-3} (S_2 \otimes S_1) \oplus (S_1 \otimes S_2) \xrightarrow{d-2} S_3 \}.$$

We treat this case directly. Denote

$$R_{12} = R_2 \otimes S_1, \quad R_{23} = S_1 \otimes R_2 \quad \subset \quad S_1 \otimes S_1 \otimes S_1.$$

We have then an inclusion $R_{12} + R_{23} \subset R_3$ of subspaces in $S_1^{\otimes 3}$. The left hand side of this inclusion is, by definition, the space of those cubic relations which follow algebraically from the quadratic ones. Thus the quotient

$$R_3^{\text{new}} = R_3 / (R_{12} + R_{23})$$

can be seen as the space of “new”, essentially cubic, relations.

Proposition 7.3. *Assume that the multiplication map $S_1 \otimes S_1 \rightarrow S_2$ is surjective. Then R_3^{new} is identified with $H^{-2}(B_3^\bullet)$, the middle cohomology space of B_3^\bullet .*

Proof. Denote for short

$$V = S_1^{\otimes 3}, \quad A = R_{12}, \quad B = R_{23}, \quad C = R_3,$$

so that $A, B \subset C \subset V$. Under our assumption, the complex B_3^\bullet can be written as

$$V \xrightarrow{\delta_{-3}} (V/A) \oplus (V/B) \xrightarrow{\delta_{-2}} V/C,$$

with δ_{-3} being the difference of the two projections, and δ_{-2} being the sum of the two projections. It is a general fact that in such a situation the middle cohomology is identified with $C/(A+B)$. Explicitly, if $(v+A, w+B) \in \text{Ker}(\delta_{-2})$, then $v+w \in C$. The image of $v+w$ in $C/(A+B)$ depends only on the class of $(v+A, w+B)$ in $\text{Ker}(\delta_{-2})/\text{Im}(\delta_{-3})$. We leave the rest to the reader. \square

C. Localization of the bar-complexes. We now apply the above to the two graded algebras

$$\mathcal{SSH}(\lambda) \subset \mathcal{S} := \left(\bigoplus_n \mathcal{O}(\mathbb{C}^n)^{\mathfrak{S}_n}, \star \right).$$

By definition, these algebras coincide in degrees 0 and 1, and $\mathcal{SSH}(\lambda)$ is the subalgebra in \mathcal{S} generated by the degree 1 part which is $\mathcal{S}_1 = \mathcal{O}(\mathbb{C})$. Accordingly, the space of relations of any degree n among degree 1 generators in \mathcal{S} and $\mathcal{SSH}(\lambda)$ are the same. As in §6, we will look at relations as elements of the completed tensor product. That is, for any two Stein manifolds M and N we write

$$\mathcal{O}(M) \widehat{\otimes} \mathcal{O}(N) := \mathcal{O}(M \times N)$$

and understand $\mathcal{S}_1^{\widehat{\otimes} n} = \mathcal{O}(\mathbb{C}^n)$ accordingly. The version of the bar-complex of \mathcal{S} using $\widehat{\otimes}$, has the form

$$\mathbf{B}_n^\bullet = \left\{ \mathcal{O}(\mathbb{C}^n) \rightarrow \cdots \rightarrow \bigoplus_{i+j+k=n} \mathcal{O}(\mathbb{C}^n)^{\mathfrak{S}_i \times \mathfrak{S}_j \times \mathfrak{S}_k} \rightarrow \bigoplus_{i+j=n} \mathcal{O}(\mathbb{C}^n)^{\mathfrak{S}_i \times \mathfrak{S}_j} \rightarrow \mathcal{O}(\mathbb{C}^n)^{\mathfrak{S}_n} \right\}$$

Notice that each term of this complex is a module over the ring $\mathcal{O}(\mathbb{C}^n)^{\mathfrak{S}_n}$ of symmetric entire functions, and the differentials, coming from multiplication

in \mathcal{S} , are $\mathcal{O}(\mathbb{C}^n)^{\mathfrak{S}_n}$ -linear. This means that \mathbf{B}_n^\bullet is the complex of global sections of a complex of vector bundles \mathcal{B}_n^\bullet on the Stein manifold $\text{Sym}^n(\mathbb{C})$. Explicitly, for $i_1 + \dots + i_p = n$ we denote by

$$\pi_{i_1, \dots, i_p} : \text{Sym}^{i_1}(\mathbb{C}) \times \dots \times \text{Sym}^{i_p}(\mathbb{C}) \longrightarrow \text{Sym}^n(\mathbb{C})$$

the symmetrization map (a finite flat morphism). Then

$$\mathcal{B}_n^{-p} = \bigoplus_{i_1 + \dots + i_p = n} (\pi_{i_1, \dots, i_p})_* \mathcal{O}_{\prod \text{Sym}^{i_\nu}(\mathbb{C})}, \quad (7.4)$$

in particular, \mathcal{B}_n^\bullet is a complex of holomorphic vector bundles on $\text{Sym}^n(\mathbb{C})$. This allows us to approach the cohomology of \mathbf{B}_n^\bullet (and, in particular, relations in \mathcal{S}) in a more geometric way, by studying the cohomology of the fibers

$$\mathcal{B}_{n,T}^\bullet = \mathcal{B}_n^\bullet \otimes_{\mathcal{O}_{\text{Sym}^n(\mathbb{C})}} \mathcal{O}_T$$

of the complex \mathcal{B}_n^\bullet over various points $T \in \text{Sym}^n(\mathbb{C})$. Now, our main technical result is as follows.

Theorem 7.5. *Let $T = \{s_1^0, \dots, s_n^0\} \in \text{Sym}^n(\mathbb{C})$ be an unordered collection of distinct points. Suppose that no subset of T (in any order) is a wheel. Then $\mathcal{B}_{n,T}^\bullet$ is exact everywhere except the leftmost term, where the cohomology is 1-dimensional.*

Recall that similar exactness of all the bar-complexes $B_n^\bullet(S)$ for a graded algebra S means that S is quadratic Koszul. The wheels represent therefore local obstructions to Koszulity for \mathcal{S} .

D. Cubic relations in SH and zeta roots. Before giving the proof of Theorem 7.5, let us explain how to apply it to the case of cubic relations for $\lambda = \Lambda$. Let ρ be a nontrivial zero of $\zeta(s)$. Denote by $W_\rho \subset \text{Sym}^3(\mathbb{C})$ the subset of points $\{s_1, s_2, s_3\}$ such that, after some renumbering of the s_i we have $s_2 - s_1 = \rho$, $s_3 - s_2 = 1 - \rho$ (such a renumbering is then unique). Let W be the union of the W_ρ over all nontrivial zeroes ρ of $\zeta(s)$. The following is then straightforward.

Proposition 7.6. (a) *Each W_ρ is a complex submanifold in $\text{Sym}^3(\mathbb{C})$, isomorphic to \mathbb{C} , the symmetric function $s_1 + s_2 + s_3$ establishing an isomorphism.*
 (b) *For $\rho \neq \rho'$ we have $W_\rho \cap W_{\rho'} = \emptyset$.*
 (c) *A point $\{s_1, s_2, s_3\} \in \text{Sym}^3(\mathbb{C})$ lies in W , if and only if it is a wheel (in some numbering). \square*

Theorem 7.7. *Let $\lambda(s) = \Lambda(s)$.*

(a) *The multiplication map $\mathcal{S}_1 \widehat{\otimes} \mathcal{S}_1 \rightarrow \mathcal{S}_2$ is surjective, so, by Proposition 7.3, the space*

$$H^{-2}(\mathbf{B}_3^\bullet) = H^0(\mathrm{Sym}^3(\mathbb{C}), \underline{H}^{-2}(\mathbf{B}_3^\bullet))$$

is identified with the space of new cubic relations in \mathcal{S} as well as in in $\mathcal{SSH}(\Lambda)$.

(b) *The support of the coherent sheaf $\underline{H}^{-2}(\mathbf{B}_3^\bullet)$ is equal to $W = \bigsqcup W_\rho$. If ρ is a simple root of $\zeta(s)$, then $\underline{H}^{-2}(\mathbf{B}_3^\bullet) \simeq \mathcal{O}_{W_\rho}$ in a neighborhood of W_ρ .*

Remark 7.8. From the point of view of this section, a cubic relation in \mathcal{SSH} is an entire function $F(s_1, s_2, s_3) \in \mathcal{O}(\mathbb{C}^3) = \mathcal{S}_1^{\widehat{\otimes} 3}$ mapped to the zero element of \mathcal{S}_3 by the symmetric shuffle multiplication. On the other hand, from the more immediate point of view of §6, a cubic relation in the spherical Hall algebra SH is a distribution $f(a_1, a_2, a_3)$ on $\mathbb{R}_+^3 = (\mathrm{Bun}_1)^3$, mapped to the zero distribution on Bun_3 by the Hall multiplication. The relation between f and F is that of the Mellin transform. Note that whenever $f(a_1, a_2, a_3)$ is a relation, then so is the rescaling $f(\alpha a_1, \alpha a_2, \alpha a_3)$ for any $\alpha \in \mathbb{R}_+$. Taking a weighted average of such rescalings, i.e., a convolution

$$\int_0^\infty f(\alpha a_1, \alpha a_2, \alpha a_3) \varphi(\alpha) d^* \alpha$$

corresponds, on the Mellin transform side, to multiplying $F(s_1, s_2, s_3)$ by a function of the form $\psi(s_1 + s_2 + s_3)$. Since $s_1 + s_2 + s_3$ is a global coordinate on each W_ρ , Theorem 7.7 admits the following striking interpretation: *the space of new cubic relations in SH modulo rescaling is identified with the space spanned by nontrivial zeroes of $\zeta(s)$.*

This fact is also true (with a similar proof) for the Hall algebras corresponding to arbitrary compactified arithmetic curves (= spectra of rings of integers in number fields) as well as (with an easier, more algebraic proof) for Hall algebras of smooth projective curves X/\mathbb{F}_q . Note that for $X = \mathbb{P}^1$ there are no new cubic relations [12, 2], while for X elliptic, new cubic relations were found in [25]. Our results show that presence of cubic relations is a general phenomenon, holding for all curves X/\mathbb{F}_q of genus ≥ 1 .

We will give a detailed proof of Theorem 7.5 and a sketch of proof of Theorem 7.7, which will be taken up and generalized in a subsequent paper.

E. Permutohedra and the proof of Theorem 7.5. Our approach, similar to that of [1, 19], uses the *permutohedron*, which is the convex polytope

$$P_n = \text{Conv}(\mathfrak{S}_n \cdot (1, 2, \dots, n)) \subset \mathbb{R}^n$$

of dimension $(n - 1)$. Thus vertices of P_n are the $n!$ vectors (i_1, \dots, i_n) for all the permutations. It is well known that faces of P_n are in bijection with sequences (I_1, \dots, I_p) of subsets of $\{1, \dots, n\}$ which form a disjoint decomposition. We denote $[I_1, \dots, I_p]$ the case corresponding to (I_1, \dots, I_p) . Subfaces of $[I_1, \dots, I_p]$ correspond to sequences obtained by *refining* (I_1, \dots, I_p) , i.e., by replacing each I_ν , in its turn, by a sequence $(J_{\nu,1}, \dots, J_{\nu,q_\nu})$ of subsets of I_ν forming a disjoint decomposition. Thus, as a polytope,

$$[I_1, \dots, I_p] \simeq P_{|I_1|} \times \dots \times P_{|I_p|}, \quad \dim[I_1, \dots, I_p] = n - p.$$

Let $C^\bullet(P_n)$ be the cochain complex of P_n with complex coefficients. The basis of $C^m(P_n)$ is formed by the $\mathbf{1}_F$, the characteristic functions of the m -dimensional faces. We choose an orientation for each face. Then

$$d(\mathbf{1}_F) = \sum_{F' \supset F} \varepsilon_{FF'} \cdot \mathbf{1}_{F'}.$$

Here the sum is over $(m + 1)$ -dimensional faces F' containing F , and $\varepsilon_{FF'} = \pm 1$ is the sign factor read from the orientations of F and F' .

On the other hand, (7.4) gives a natural basis of $\mathcal{B}_{n,T}^{n-1-m}$ labeled by the disjoint union of the preimages

$$\pi_{i_1, \dots, i_p}^{-1}(\{s_1^0, \dots, s_n^0\}), \quad i_1 + \dots + i_p = n.$$

For a subset $I \subset \{1, \dots, n\}$ let $T_I = \{s_i^0 | i \in I\} \subset T$. Elements of each $\pi_{i_1, \dots, i_p}^{-1}(\{s_1^0, \dots, s_n^0\})$ are precisely the

$$(T_{I_1}, \dots, T_{I_p}) \in \text{Sym}^{i_1}(\mathbb{C}) \times \dots \times \text{Sym}^{i_p}(\mathbb{C})$$

for all sequences of subsets (I_1, \dots, I_p) , forming a disjoint decomposition of $\{1, \dots, n\}$. Denoting by e_{I_1, \dots, I_p} the corresponding basis vector in $\mathcal{B}_{n,T}^{n-1-m}$, we get an isomorphism of graded vector spaces

$$\mathcal{B}_{n,T}^\bullet \xrightarrow{\sim} C^\bullet(P_n)[n], \quad e_{I_1, \dots, I_p} \mapsto \mathbf{1}_{[I_1, \dots, I_p]}. \quad (7.9)$$

To see the differential in $\mathcal{B}_{n,T}^\bullet$ from this point of view, consider the matrix

$$\mathfrak{L} = \|\lambda_{ij}\|, \quad \lambda_{ij} = \lambda(s_i^0 - s_j^0), \quad 1 \leq i, j \leq n, \quad i \neq j.$$

Let $F \subset F'$ be a codimension 1 embedding of faces of P_n . That is, $F' = [I_1, \dots, I_p]$ and F is a minimal refinement of F' , i.e., is obtained by replacing some I_ν by (I', I'') where I', I'' are nonempty sets forming a disjoint decomposition of I_ν . We put

$$\lambda_{FF'} = \prod_{\substack{i' \in I' \\ i'' \in I''}} \lambda_{i'i''}.$$

It is immediately so see that the $\lambda_{FF'}$ satisfy the multiplicativity property for any pair of composable codimension 1 embeddings:

$$\lambda_{FF'} \lambda_{F'F''} = \lambda_{FF''}, \quad F \subset F' \subset F''.$$

This implies that by putting

$$d_{\mathfrak{L}}(\mathbf{1}_F) = \sum_{F' \supset F} \lambda_{FF'} \cdot \varepsilon_{FF'} \cdot \mathbf{1}_{F'},$$

we obtain a differential $d_{\mathfrak{L}}$ in $C^\bullet(P_n, \mathbb{C})$ with square 0. This is a certain perturbation of the cochain differential for P_n . We then see easily:

Proposition 7.10. *The isomorphism (7.9) defines an isomorphism of complexes*

$$B_{n,T}^\bullet \longrightarrow (C^\bullet(P_n), d_{\mathfrak{L}})[n].$$

□

Note that the perturbed differential $d_{\mathfrak{L}}$ can be written for any system $\mathfrak{L} = \|\lambda_{ij}\|_{i \neq j}$ of complex numbers. Conceptually, \mathfrak{L} is a \mathbb{C} -valued function on the root system of type A_{n-1} . We simply refer to \mathfrak{L} as a matrix.

By a *wheel* for \mathfrak{L} we mean a sequence of i_1, \dots, i_m of indices such that

$$\lambda_{i_1, i_2} = \lambda_{i_2, i_3} = \dots = \lambda_{i_{p-1}, i_p} = \lambda_{i_p, i_1} = 0.$$

Theorem 7.5 is now a consequence of the following result.

Proposition 7.11. *Let $\mathfrak{L} = \|\lambda_{ij}\|_{i \neq j}$ be an n by n matrix without wheels. Then $(C^\bullet(P_n), d_{\mathfrak{L}})$ is exact outside of the leftmost term, where the cohomology (kernel) is 1-dimensional.*

Proof: For a face $F = [I_1, \dots, I_p]$ of P_n we put

$$\lambda_F = \prod_{\mu < \nu} \prod_{\substack{i \in I_\mu \\ j \in I_\nu}} \lambda_{ij}. \quad (7.12)$$

Then for an embedding $F \subset F'$ of codimension 1 we have

$$\lambda_F = \lambda_{F'} \cdot \lambda_{FF'}.$$

This means that we have a morphism of complexes

$$\Psi : (C^\bullet(P), d_\Psi) \longrightarrow (C^\bullet(P), d), \quad \Psi(\mathbf{1}_F) = \lambda_F \cdot \mathbf{1}_F,$$

where d is the usual cochain differential. As P_n is a convex polytope, $(C^\bullet(P), d)$ is exact outside the leftmost term, with $H^0 = \mathbb{C}$. We now analyze the kernel and cokernel of Ψ . For a face F of P_n as before we call the *depth* of P the number of factors in (7.12) which are zero. In other words, we put

$$Z = \{(i, j) : i \neq j, \lambda_{ij} = 0\} \subset \{1, \dots, n\}^2. \quad (7.13)$$

Then the depth of F is the number

$$\text{dpt}(F) = \#\{(i, j) \in Z : \exists \mu < \nu : i \in I_\mu, j \in I_\nu\}. \quad (7.14)$$

Note that if F is a subface of F' , then $\text{dpt}(F) \geq \text{dpt}(F')$. Therefore we have a descending chain of polyhedral subcomplexes

$$P^{(r)} = \bigcup_{\text{dpt}(F) \geq r} F \subset P_n, \quad r \geq 0.$$

Lemma 7.15. (a) *The complex $\text{Coker}(\Psi)$ is isomorphic to the relative cochain complex $C^\bullet(P_n, P^{(1)})$.*

(b) *The complex $\text{Ker}(\Psi)$ has a filtration with quotients isomorphic to the relative cochain complexes $C^\bullet(P^{(r)}, P^{(r+1)})$, $r \geq 1$.*

Proof: The matrix of Ψ is diagonal in the chosen bases, and $\text{Im}(\Psi) \subset C^\bullet(P_n)$ is spanned by the $\mathbf{1}_F$, $F \in P^{(1)}$, which shows (a). As for (b), for each $r \geq 0$ we have the cochain subcomplex $C^\bullet(P_n)^{\geq r} \subset C^\bullet(P)$ spanned by $\mathbf{1}_F$ with $\text{dpt}(F) \geq r$, with $C^\bullet(P_n)^{\geq 1} = \text{Ker}(\Psi)$. The quotient $C^\bullet(P_n)^{\geq r} / C^\bullet(P_n)^{\geq r+1}$ is identified with $C^\bullet(P^{(r)}, P^{(r+1)})$ in a way similar to (a). \square

Note that the weights of faces of P_n and the polyhedral subcomplexes $P^{(r)}$ are defined entirely in terms of the subset Z in (7.13) which can be, a priori, arbitrary. Now, absence of wheels in \mathfrak{L} (or, what is the same, in Z) means that after an appropriate renumbering of $\{1, \dots, n\}$, any $(i, j) \in Z$ satisfies $i < j$. Such renumbering does not change the combinatorial type of any of the $P^{(r)}$. Proposition 7.11 is therefore a consequence of the following purely combinatorial fact.

Proposition 7.16. *Let $Z \subset \{(i, j) \mid 1 \leq i < j \leq n\}$ be any subset of positive roots for A_{n-1} . Then each polyhedral complex $P^{(r)}$ is either empty or contractible.*

Proof. For a permutation $\sigma \in \mathfrak{S}_n$ let

$$O(\sigma) = \{(i < j) \mid \sigma(i) < \sigma(j)\}$$

be the set of order preserving pairs of σ . Thus the weak Bruhat order on \mathfrak{S}_n is given by

$$\sigma \leq \tau \quad \text{iff} \quad O(\tau) \subseteq O(\sigma).$$

Now, for a face $F \subset P_n$ we have

$$\text{dpt}(F) = \min_{[\sigma] \in \text{Vert}(F)} |O(\sigma) \cap Z|. \quad (7.17)$$

Indeed, for $F = [\sigma]$ a vertex this is precisely the definition (7.14), while for $F = [I_1, \dots, I_p]$ the minimum in the RHS of (7.17) is achieved for σ arranging each I_ν in the decreasing order and is equal to $\text{dpt}(F)$.

Let $D = |Z|$. Then for $r > d$ we have $P^{(r)} = \emptyset$, while for $r \leq d$ we have that $P^{(r)}$ contains at least the vertex $[e]$ corresponding to the unit permutation. Further, by (7.17), the set $\text{Vert}(P^{(r)}) \subset \mathfrak{S}_n$ is a “left order ideal” with respect to the weak Bruhat order: with each τ , it contains all $\sigma \leq \tau$. This implies that $P^{(r)}$ contracts onto $[e]$. \square

This finishes the proof of Theorem 7.5.

F. Proof of Theorem 7.7 (sketch). (a) It is enough to prove that the map of the fibers $\mathcal{B}_{2,T}^{-2} \rightarrow \mathcal{B}_{2,T}^{-1}$ over any $T = \{s_1^0, s_2^0\} \in \text{Sym}^2(\mathbb{C})$ is surjective. If $s_1^0 \neq s_2^0$, it follows from Theorem 7.5, as there are no wheels of length 2. Assume now that $s_1^0 = s_2^0 = s^0$. The fiber of $p_{1,1*}\mathcal{O}_{\mathbb{C}^2}$ at $\{s^0, s^0\}$ is then

$\mathcal{O}(\mathbb{C})/\mathfrak{m}_{s^0}^2$, the space of first jets of sections of $\mathcal{O}_{\mathbb{C}}$ at s^0 . Since $\Lambda(s)$ has a first order pole at 0 with residue 1, for any analytic function $f(s_1, s_2)$ we have

$$\lim_{s_1, s_2 \rightarrow s^0} (\widehat{\star}_{1,1} F)(s_1, s_2) = \frac{1}{2} \frac{d}{dt} \Big|_{t=0} F(s^0 + t, s^0 - t).$$

This implies that the subspace $\mathfrak{m}_{s^0}/\mathfrak{m}_{s^0}^2$ of jets vanishing at s , will map surjectively onto the fiber of $\mathcal{O}_{\text{Sym}^2(\mathbb{C})}$ at $\{s^0, s^0\}$.

(b) For $T = \{s_1, s_2, s_3\} \in \text{Sym}^3(\mathbb{C})$ let \mathbb{C}_T be the skyscraper sheaf at T . We have a spectral sequence

$$E_2^{ij} = \text{Tor}_i^{\text{Sym}^3(\mathbb{C})}(\underline{H}^j(\mathcal{B}_3^\bullet), \mathbb{C}_T) \implies H^{j-i}(\mathcal{B}_{3,T}^\bullet). \quad (7.18)$$

We analyze it backwards, using the information about the abutment to say something about E_2 and then about the $\underline{H}^j(\mathcal{B}_3^\bullet)$. Some parts of this analysis involve straightforward computations which we omit, highlighting the conceptual points only.

First, let $\Delta \subset \text{Sym}^3(\mathbb{C})$ be the locus of T such that $s_i = s_j$ for some $i \neq j$. Note that $W \cap \Delta = \emptyset$. Theorem 7.5 implies that for $T \notin W \cup \Delta$ the abutment of (7.18) is zero for $j - i > -3$ and this implies that both \underline{H}^{-2} and \underline{H}^{-1} of \mathcal{B}_3^\bullet are zero outside $W \cup \Delta$.

Next, $\mathcal{B}_3^{-1} = \mathcal{O}_{\text{Sym}^3(\mathbb{C})}$, so $d_{-2}(\mathcal{B}_3^{-1})$ is a sheaf of ideals there and therefore $\underline{H}^{-1}(\mathcal{B}_3^\bullet)$ is the structure sheaf of an analytic subspace $\mathcal{W} \subset \text{Sym}^3(\mathbb{C})$. By the above the support of \mathcal{W} is contained in $W \cup \Delta$.

Next, we analyze (7.18) in the case when $T \in W$. The permutohedron P_3 is a hexagon, so for $T \notin \Delta$ the complex $\mathcal{B}_{3,T}^\bullet$ is, by Proposition 7.10, the perturbed cochain complex of this hexagon corresponding to the matrix $\mathfrak{L} = \|\lambda_{ij}\| = \|\Lambda(s_i - s_j)\|$. If $T \in W$, then, after renumbering, we have $\lambda_{12} = \lambda_{23} = \lambda_{31} = 0$, while other $\lambda_{ij} \neq 0$. From this it is an elementary computation to find the dimensions of the cohomology spaces of $\mathcal{B}_{3,T}^\bullet$ to be

$$h^{-3} = 3, \quad h^{-2} = 3, \quad h^{-1} = 1.$$

This, shows that \mathcal{W} contains W . Further, let ρ be a nontrivial zero of $\zeta(s)$ of multiplicity ν and \mathcal{W}_ρ be the part of \mathcal{W} supported on W_ρ . We can then analyze the last map in complex \mathcal{B}_3^\bullet near $T = \{\rho + c, 1 - \rho + c, -1 + c\} \in W_\rho$ directly, using the family of perturbed differentials $d_{\mathfrak{L}} : C^1(P_3) \rightarrow C^2(P_3)$

with $\mathfrak{L} = \|\Lambda(s_i - s_j)\|$ depending on $\{s_1, s_2, s_3\}$ near W_ρ . This is again an elementary computation which yields that \mathcal{W}_ρ is isomorphic to the ν th infinitesimal neighborhood of W_ρ in an embedded surface. In particular, if ρ is a simple root, then $\mathcal{W}_\rho = W_\rho$ as an analytic subspace.

This means that $\mathcal{W} = \underline{H}^{-1}(\mathcal{B}_3^\bullet)$ is given locally in $\text{Sym}^3(\mathbb{C}) - \Delta$ by two equations and so $\dim \text{Tor}_1^{\text{Sym}^3(\mathbb{C})}(\underline{H}^{-1}(\mathcal{B}_3^\bullet), \mathbb{C}_T) = 2$ for any $T \in W$. From the equality $h^{-2}(\mathcal{B}_{3,T}^\bullet) = 3$ and the spectral sequence (7.18) we then conclude that $\dim(\underline{H}^{-2}(\mathcal{B}_3^\bullet) \otimes \mathbb{C}_T) = 1$, and so $W \subset \text{supp}(\underline{H}^{-2}(\mathcal{B}_3^\bullet))$. The statement that $\underline{H}^{-2}(\mathcal{B}_3^\bullet) = \mathcal{O}_{W_\rho}$ near W_ρ for a simple root ρ , uses an additional local calculation which we omit. We also omit the analysis of the case $T \in \Delta$ which shows that the support of $\underline{H}^{-2}(\mathcal{B}_3^\bullet)$ does not meet Δ . \square

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