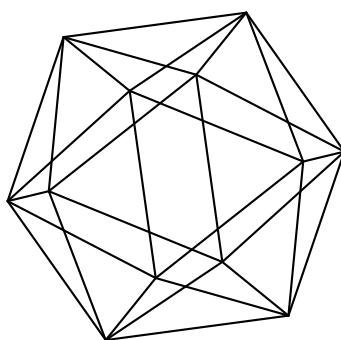# Max-Planck-Institut für Mathematik Bonn

Algebras defined by Lyndon words and Artin-Schelter regularity

by

Tatiana Gateva-Ivanova

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Tatiana Gateva-Ivanova

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

American University in Bulgaria
2700 Blagoevgrad
Bulgaria

ALGEBRAS DEFINED BY LYNDON WORDS AND ARTIN-SCHELTER REGULARITY

TATIANA GATEVA-IVANOVA

ABSTRACT. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite alphabet, and let K be a field. We study classes $\mathfrak{C}(X, W)$ of graded K-algebras $A = K\langle X\rangle/I$, generated by X and with *a fixed set of obstructions* W. Initially we do not impose restrictions on W and investigate the case when the algebras in $\mathfrak{C}(X, W)$ have polynomial growth and finite global dimension d. Next we consider classes $\mathfrak{C}(X, W)$ of algebras whose sets of obstructions W are antichains of Lyndon words. The central question is "*when a class $\mathfrak{C}(X, W)$ contains Artin-Schelter regular algebras?*" Each class $\mathfrak{C}(X, W)$ defines a Lyndon pair (N, W), which, if N is finite, determines uniquely the global dimension, $\operatorname{gl}\dim A$, and the Gelfand-Kirillov dimension, $\operatorname{GKdim} A$, for every $A \in \mathfrak{C}(X, W)$. We find a combinatorial condition in terms of (N, W), so that the class $\mathfrak{C}(X, W)$ contains the enveloping algebra $U\mathfrak{g}$, of a Lie algebra $\mathfrak{g}$. We introduce monomial Lie algebras defined by Lyndon words, and prove results on Gröbner-Shirshov bases of Lie ideals generated by Lyndon-Lie monomials. Finally we classify all two-generated Artin-Schelter regular algebras of global dimension 6 and 7 occurring as enveloping $U = U\mathfrak{g}$ of *standard monomial Lie algebras*. The classification is made in terms of their Lyndon pairs (N, W), each of which determines also the explicit relations of U.

CONTENTS

1. INTRODUCTION

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite alphabet. We denote by X^* the free monoid generated by X, and by X^+ -the free semigroup generated by X, $X^+ = X^* - \{1\}$, (the empty word is denoted by 1). Throughout the paper $K\langle X \rangle$ will denote the free associative K-algebra generated by X, where K is a field. As usual, the length of a word $w \in X^+$ is denoted by $|w|$. We shall consider the canonical grading by length on $K\langle X \rangle$, assuming each $x \in X$ has degree 1.

In the paper we shall use three distinct orderings on X^+ (denoted by distinct notation) which will be introduced gradually and used in different contexts.

The first order is the so called "divisibility order", "$\sqsubset$". This is a partial ordering on the set X^+ defined as: $a \sqsubset b$ *iff* a is a proper subword (segment) of b, i.e. $b = uav$, $|b| > |a|$, $u = 1$, or $v = 1$ is possible. In the case when $b = av, a, v \in X^+$, a is called *a proper left factor (segment) of b*. Proper right factors (segments) are defined analogously.

Let V be a set of monomials in X^+, $V \neq \emptyset$. A monomial $w \in V$ is *a minimal element of V with respect to the ordering "$\sqsubseteq$"* if it does not contain as a proper subword any monomial $u \in V$. Clearly, for any $u \in V$ there exists a unique minimal element $w \in V$, with $w \sqsubseteq u$.

A nonempty subset $W \subseteq X^+$ is called *an antichain of monomials* (or shortly *an antichain*) if any two elements $u, v \in W, u \neq v$ are not comparable with respect to "$\sqsubset$". In other words, W is an antichain of monomials if every monomial $a \in W$ is minimal with respect to "$\sqsubset$".

The following is straightforward.

Lemma 1.1. *Let V be a nonempty set of monomials in X^+. Let*

$$W = W_V = \{u \in V \mid u \text{ is minimal in } V \text{ with respect to } \sqsubset\}.$$

Then W is the unique maximal antichain of monomials contained in V. Furthermore, every $u \in V$ has a subword $w \in W$.

Let V, and W be as above. A monomial $a \in X^*$ is *V-normal (V-standard)* if a does not contain as a subword any $u \in V$. Clearly a is V-normal *iff* a is W-normal. Denote by $\mathfrak{N}(W)$ the set of W-normal words

$$\mathfrak{N}(W) = \{a \in X^* \mid a \text{ is } W\text{-normal}\}.$$

The set $\mathfrak{N}(W)$ is closed under taking subwords, Anick calls such a set *an order ideal of monomials*, or *o.i.m.*, [3].

Conversely, each nonempty order ideal of monomials $\mathfrak{N} \subset X^+$ determines uniquely an antichain of monomials $W = W(\mathfrak{N})$, so that $\mathfrak{N} = \mathfrak{N}(W)$. Indeed, the complement $V = X^* - \mathfrak{N}$ has a unique maximal antichain of monomials $W = W_V$, and the following implications are in force.

$$a \in \mathfrak{N} \Longleftrightarrow a \text{ is } W\text{- normal}; \quad v \in V \Longleftrightarrow w \sqsubseteq v, \quad \text{for some} \quad w \in W.$$

In this case W is called *the set of obstructions for $\mathfrak{N}$*, [3]. Note that W satisfies the following condition

$$W = \{w \in X^* \mid w \text{ is not in } \mathfrak{N} \text{ but any proper subword } u \text{ of } w \text{ is in } \mathfrak{N}\}.$$

The natural duality between $\mathfrak{N}$ and $W = W(\mathfrak{N})$ is discussed with more details in Section 3. We recall briefly some facts from (noncommutative) Gröbner bases

theory. By convention throughout the paper, we consider *the degree-lexicographic ordering* $\prec$ on X^*, extending the (inverse) ordering

$$(1.1) \qquad\qquad x_1 \succ x_2 \succ \cdots \succ x_{g-1} \succ x_g$$

on X. Then $(X^*, \prec)$ is a well-ordered monoid that is "$\prec$" is a total ordering on $\langle X \rangle$ compatible with the multiplication and such that every non-empty subset of $\langle X \rangle$ has a least element. In all cases when Gröbner basis of an ideal I in $K\langle X \rangle$ is considered we shall use the degree-lexicographic ordering $\prec$ on X^*, extending (1.1).

Suppose $f \in K\langle X \rangle$ is a nonzero polynomial, then its highest monomial w.r.t. $\prec$ will be denoted by $\overline{f}$. One has $f = \alpha \overline{f} + \sum_{1 \le i \le m} \alpha_i u_i$, where $\alpha, \alpha_i \in K$, $\alpha \ne 0$, $u_i \in X^*, u_i \prec \overline{f}$. Given a set $F \subseteq K\langle X \rangle$ of noncommutative polynomials, $\overline{F}$ denotes the set

$$\overline{F} = \{ \overline{f} \mid f \in F \}.$$

The free associative algebra $\mathfrak{A} = K\langle X \rangle$ is naturally graded by length,

$$\mathfrak{A} = \bigoplus_{m \ge 0} \mathfrak{A}_m, \text{ where } \mathfrak{A}_0 = K, \ \mathfrak{A}_m = \text{Span}\{ u \in X^m, m \ge 1 \}.$$

Let I be a two sided graded ideal in $K\langle X \rangle$. We shall consider graded algebras, with a minimal presentation, see Notation-convention 1.5, so without loss of generality we may assume that *I is generated by homogeneous polynomials of degree ≥ 2* , hence $I = \bigoplus_{m \ge 2} I_m$, with $I_m = I \bigcap \mathfrak{A}_m$. Then the quotient algebra $A = K\langle X \rangle / I$ is finitely generated and inherits its grading $A = \bigoplus_{m \ge 0} A_m$ from $\mathfrak{A} = K\langle X \rangle$. We shall work with the so called *normal K-basis of A*.

A monomial $u \in X^*$ is said to be *normal mod I iff* it is $\overline{I}$-normal. The set $\mathfrak{N}(I)$ of normal modulo I monomials coincides with $\mathfrak{N}(\overline{I})$.

Remark 1.2. Let $W = W(\overline{I})$ be the maximal antichain of monomials in $\overline{I}$, then u is normal (mod I) *iff* u is W-normal. Hence W is the obstruction set for $\mathfrak{N}(I)$ and there are equalities of sets

$$(1.2) \qquad \mathfrak{N}(I) = \mathfrak{N}(W) = \{ u \in X^* \mid u \text{ is normal mod } I \}.$$

In particular, the free monoid X^* splits as a disjoint union

$$(1.3) \qquad\qquad X^* = \mathfrak{N}(I) \bigsqcup \overline{I} = \mathfrak{N}(W) \bigsqcup \overline{I}.$$

The free associative algebra $K\langle X \rangle$ splits into a direct sum of K- subspaces $K\langle X \rangle \simeq \text{Span}_K \mathfrak{N} \bigoplus I$, and there are isomorphisms of vector spaces

$$A \simeq \text{Span}_K \mathfrak{N}$$

$$(1.4)$$

$$A_m \simeq \text{Span}_K \mathfrak{N}^{(m)}, \ \dim A_m = |\mathfrak{N}^{(m)}|, \ m = 0, 1, 2, 3, \cdots,$$

where $\mathfrak{N}^{(m)}$ is *the set of all normal words of length m*.

Definition 1.3. Given $A = K\langle X \rangle / I$, $\mathfrak{N} = \mathfrak{N}(I)$, and $W = W(I)$ as above we shall also refer to W as *the set of obstructions for A*, [17]. The monomial algebra $A_W := K\langle X \rangle / (W)$ is called *the monomial algebra associated to A*.

Definition 1.4. A subset $G \subseteq I$ of monic polynomials *is a Gröbner basis of I (with respect to $\prec$)* if (1) G generates I as a two-sided ideal, and (2) $\mathfrak{N}(\overline{G}) = \mathfrak{N}$. A Gröbner basis $\mathfrak{R}$ is *minimal* if the set $\mathfrak{R} - \{f\}$ is not a Gröbner basis of I, whenever $f \in \mathfrak{R}$. *Equivalently, the set $\mathfrak{R} \subseteq I$ is a minimal Gröbner basis of I*

iff $\overline{\mathfrak{R}} = W = W(\overline{I})$, *the set of obstructions.* A minimal Gröbner basis $\mathfrak{R}$ of I is *reduced* if each $f \in \mathfrak{R}$ is reduced modulo $\mathfrak{R} - \{f\}$, that is f is a linear combination of normal modulo $\mathfrak{R} - \{f\}$ monomials. Clearly, $\mathfrak{R}$ is finite if and only if the set of obstructions W is finite. In this case the algebra $A = K\langle X \rangle / (\mathfrak{R})$ is called *a standard finitely presented algebra*, or shortly *an s.f.p. algebra*.

It is known that each ideal I of $K\langle X \rangle$ has a uniquely determined reduced Gröbner basis $G = G(I)$ (with respect to $\prec$). However, G may be infinite. In [17] we initiate the study of algebraic and homological properties *of graded associative algebras for which the set of obstructions consists of Lyndon words.* Lyndon words and Lyndon-Shirshov bases are widely used in the context of Lie algebras and their enveloping algebras, and also for PI algebras (see for example the celebrated Shirshov theorem of heights, [9]). It will be interesting to explore the remarkable combinatorial properties of Lyndon words in a more general context of graded associative algebras.

Notation-Convention 1.5. We fix an enumeration on the set X of generators, $X = \{x_1, x_2, \cdots, x_n\}$, $g \geq 2$. By $\mathfrak{C}(X, W)$ we shall denote *the class of all graded K-algebras* $A = K\langle X \rangle / I$, with a finite set of generators X and a set of obstructions W of arbitrary cardinality. $\mathfrak{N} = \mathfrak{N}(W)$ will denote the set of normal words (mod W). $\mathfrak{N}$ is a K-basis for each algebra $A \in \mathfrak{C}(X, W)$. By convention we consider only *minimal presentations of A*, so

$$W \bigcap X = \emptyset, \quad \text{and therefore } X \subset \mathfrak{N}.$$

Without loss of generality we shall always assume that I *is generated by a set $\mathfrak{R}_0$ of homogeneous polynomials of degree ≥ 2* . This implies that $\mathfrak{R}$, the unique reduced Gröbner basis of I, (w.r.t. the degree-lexicographic ordering $\prec$ on X^*, extending (1.1)), is also a set of homogeneous polynomials of degree ≥ 2 . Each class $\mathfrak{C}(X, W)$ contains unique monomial algebra $A_W := K\langle X \rangle / (W)$. Clearly, all algebras $A \in \mathfrak{C}(X, W)$ share the same associated monomial algebra is A_W, and the same normal K-basis $\mathfrak{N}$. There are equalities of sets

$$\overline{\mathfrak{R}} = W \quad \mathfrak{N}(I) = \mathfrak{N}(W) = \mathfrak{N}.$$

As a consequence, all $A \in \mathfrak{C}(X, W)$ have the same Gelfand Kirillov dimension, $\mathrm{GKdim}\, A = \mathrm{GKdim}\, A_W$, and the same Hilbert series $H_A(z) = H_{A_W}$. When we work with Lyndon words we often consider a third order, "$<$" on the set X^+, namely, the pure lexicographic order which extends the ordering $x_1 < x_2 < \cdots < x_n$ on X as follows: for any $u, v \in X^+$ $u < v$ *iff* either u is a proper left factor of v ($v = ub, b \in X^+$), or $u = axb, v = ayc$, where $x < y$; $x, y \in X$, $a, b, c \in X^*$.

We study classes $\mathfrak{C}(X, W)$ such that every algebra $A \in \mathfrak{C}(X, W)$ has polynomial growth and finite global dimension. Moreover, starting from Section 3, throughout the paper we are interested in the special case when the obstructions set W consists of Lyndon words, see Definition 3.1. One of the central questions in our study is: *"When a given class $\mathfrak{C}(X, W)$ contains an Artin-Schelter regular algebra?"* .

Remark 1.6. The problem of studying classes $\mathfrak{C}(X, W)$, as above was posed first in [17] and in several talks given by the author (Bedlewo, 2013, [16], ICTP 2013). On these talks we reported the results of Theorem II, and Theorem III, (2).

A connected graded algebra A is called *Artin-Schelter regular* (or *AS regular*) if

(i) A has *finite global dimension d*, that is, each graded A-module has a free resolution of length at most d.

(ii) A has *finite Gelfand-Kirillov dimension*, meaning that the integer-valued function $i \mapsto \dim_{\mathbf{k}} A_i$ is bounded by a polynomial in i.

(iii) A is *Gorenstein*, that is, $Ext_A^i(\mathbf{k}, A) = 0$ for $i \neq d$ and $Ext_A^d(\mathbf{k}, A) \cong \mathbf{k}$.

AS regular algebras were introduced and studied first in [4, 5, 6]. Since then AS regular algebras and their geometry have intensively been studied. When $d \leq 3$ all regular algebras are classified. The problem of classification of regular algebras seems to be difficult and remains open even for regular algebras of global dimension 5, see for example [24]. The study of Artin-Schelter regular algebras, their classification, and finding new classes of such algebras is one of the basic problems in noncommutative geometry. Numerous works on this topic appeared during the last two decades, see for example [5, 6], [22, 34, 25, 11, 15, 35], and references therein.

The main results of the paper are Theorems I, II, III and IV. Theorem IV is formulated and proven in Section 9. Here we state Theorems I, II and III proven under the following hypothesis:

Hypothesis 1.7. *In Notation-convention 1.5. Suppose $X = \{x_1, x_2, \cdots, x_n\}$, and $W \subset X^+ \backslash \{X\}$ is an antichain of words of arbitrary cardinality. Denote by $\mathfrak{C}(X, W)$ the class of all graded K-algebras $A = K\langle X \rangle / I$, generated by X whose set of obstructions is W (with respect to the degree-lexicographic ordering $\prec$ on X^*, where $x_1 \succ x_2 \succ \cdots \succ x_n$). $A_W = K\langle X \rangle / (W)$ is the monomial algebra determined by W, $A_W \in \mathfrak{C}(X, W)$. Let $\mathfrak{N} = \mathfrak{N}(W)$ be the set of normal words modulo W, and let $N = \mathfrak{N} \bigcap L$ be the set of normal Lyndon words.*

Theorem I. *Assumptions as in Hypothesis 1.7. Suppose there are no d-chains on W, but there exists a $d - 1$-chain on W. Let $A = K\langle X \rangle / I$ be an arbitrary algebra in $\mathfrak{C}(X, W)$, assume that A has polynomial growth. Then the following conditions hold.*

(1) *There are equalities*

$$(1.5) \qquad \mathrm{gl\,dim}\, A = d = \mathrm{GKdim}\, A.$$

(2) *There exists a finite set M of normal words, called "atoms", in the sense of Anick, [2]: $M = \{a_1, a_2, \cdots, a_d\} \subset \mathfrak{N}$, $X \subseteq M$, such that:*

(a) *The normal K-basis $\mathfrak{N}$ of A (and A_W) can be expressed in terms of Anick's atoms as "a Poincaré-Birkhoff-Witt" basis*

$$(1.6) \qquad \mathfrak{N} = \{a_1^{k_1} a_2^{k_2} \cdots a_d^{k_d} \mid k_i \geq 0,\ 1 \leq i \leq d\}.$$

(b) *The Hilbert series of A satisfies*

$$(1.7) \qquad H_A(z) = \prod_{1 \leq i \leq d} (1 - z^{e_i})^{-1}, \quad \text{where } e_i = |a_i| \geq 1,\ 1 \leq i \leq d.$$

(c) *Any word $w \in W$ can be written as a product $w = a_j a_i, 1 \leq i < j \leq d$.*

(3) *A is standard finitely presented- the ideal I has a finite reduced Gröbner basis $\mathfrak{R}$, with*

$$(1.8) \qquad |\mathfrak{R}| = |W| \leq d(d-1)/2.$$

(4) *The following conditions are equivalent.*

(a) *$|W| = d(d-1)/2$.*

(b) *There is no atom $a \in M$ whose length is 2.*

(c) *$M = X$ and there exists a, possibly new, ordering of X:*

$$(1.9) \qquad X = \{y_1 \leftarrow y_2 \leftarrow \cdots \leftarrow y_n\}, \text{ such that } W = \{y_j y_i \mid 1 \leq i < j \leq n\}.$$

(d) $\operatorname{gl}\dim A = n$.

(e) A is a PBW algebra in the sense of Priddy (see [29, 28]), equivalently, the reduced Gröbner basis $\mathfrak{R}$ of I consists of quadratic homogeneous polynomials.

(f) $H_A(z) = \frac{1}{(1-z)^n}$.

(5) Each of the equivalent conditions in part (4) implies that A is Koszul.

The proof of the Theorem is given in Subsection 2.2

Corollary 1.8. *Given a class $\mathfrak{C}(X, W)$, where W is an arbitrary chain of monomials in X^+, suppose the monomial algebra $A_W = K\langle X\rangle/(W) \in \mathfrak{C}(X, W)$ has global dimension $\operatorname{gl}\dim A = d$ and polynomial growth. Then every algebra $A \in \mathfrak{C}(X, W)$ is standard finitely presented and $\operatorname{gl}\dim A = d = \operatorname{GKdim} A$.*

Theorem II. *Assumptions as in Hypothesis 1.7. Let $W \subset X^+$ be an antichain of arbitrary monomials.*

(1) *W is an anticain of Lyndon words if and only if the set of normal words $\mathfrak{N}$ has the shape*

(1.10) $\mathfrak{N} = \{l_1^{k_1} l_2^{k_2} \cdots l_s^{k_s} \mid l_i \in N, \ l_1 > l_2 > \cdots > l_s, \ s \geq 1, \ k_i \geq 0, \ 1 \leq i \leq s\},$

where the set of normal Lyndon words N is not necessarily finite.
  Suppose that W is an antichain of Lyndon words, so (N, W) is a Lyndon pair, see Definition 3.3. Let $A = K\langle X\rangle/I \in \mathfrak{C}(X, W)$.

(2) *A has polynomial growth of degree d if and only if N has order d, so $N = \{l_1 > l_2 > \cdots > l_d\}$. In this case the following conditions hold.*

   (a) *$\operatorname{gl}\dim A = d$;*

   (b) *A has a Poincaré-Birkhoff-Witt type K-basis*

(1.11) $\mathfrak{N} = \{l_1^{k_1} l_2^{k_2} \cdots l_d^{k_d} \mid k_i \geq 0, 1 \leq i \leq d\}$, and $H_A(t) = \prod_{1 \leq i \leq d} 1/(1 - t^{|l_i|})$.

   (c) *The algebra A is standard finitely presented, more precisely, the reduced Gröbner basis of I consists of exactly $|W|$ polynomials, where*

(1.12) $d - 1 \leq |W| \leq \frac{d(d-1)}{2}$.

(3) *The class $\mathfrak{C}(X, W)$ contains Artin-Schelter regular algebras, whenever*

$|W| = \frac{d(d-1)}{2}$ *or $|W| = d - 1$, and N is connected, see Def 5.9.*

The theorem is proven in Section 4.

Theorem III. *Suppose the class $\mathfrak{C}(X, W)$ is defined by an antichain of Lyndon words W, where the set of Lyndon atoms $N = N(W)$ has a finite order $|N| = d$.*

(1) *The following conditions are equivalent:*

   (a) *$|W| = d - 1$ and N is* a connected set of Lyndon atoms, *see Def 5.9;*

   (b) *$X = \{x < y\}$, and (up to isomorphism of monomial algebras A_W) the set N is determined uniquely:*

$$N = \{x < xy < xy^2 < \cdots < xy^{d-2} < y\};$$

(c) $X = \{x < y\}$, and (up to isomorphism of monomial algebras A_W) the set W is determined uniquely:

$$W = \{xy^i xy^{i+1} \mid 0 \leq i \leq d-3\} \bigcup \{xy^{d-1}\}.$$

In this case the class $\mathfrak{C}(X, W)$ contains the universal enveloping $U = U\mathcal{L}_{d-1}$, where $\mathcal{L}_{d-1}$ is the standard filiform algebra of dimension d and nilpotency class $d-1$. U is an AS-regular algebra with $\operatorname{gl} \dim U = d$.

(2) *The following conditions are equivalent:*
   (a) $|W| = d(d-1)/2$;
   (b) $N = X$, and $d = n$;
   (c) $W = \{x_i x_j \mid 1 \leq i < j \leq d\}$.

   In this case the class $\mathfrak{C}(X, W)$ contains an abundance of (non isomorphic) Artin-Scheler regular PBW algebras of global dimension d, each of them is presented as a skew polynomial ring with square-free binomial relations, (in the sense of [14]) and defines a solution of the Yang-Baxter equation.

The theorem is proven in subsection 7.3.

The paper is organized as follows. In Section 2 we study graded algebras with polynomial growth and finite global dimension and prove Theorem I. In Section 3 we recall the notion of Lyndon words and prove some new results. We introduce the notion of Lyndon pairs, which are central for our theory. In Section 4 we prove Theorem II, and Theorem 4.2 which is a purely combinatorial result on Lyndon pairs. In Section 5 we give some basic information on classes $\mathfrak{C}(X, W)$ containing the enveloping algebra U of a Lie algebra (possibly infinite dimensional). We show that for every such a class the set of obstruction W is an antichain of Lyndon words, and the corresponding set of Lyndon atoms $N = N(W)$ *must be connected*, see Definition 5.9. and Proposition 5.11. In Section 6 we introduce monomial Lie algebras $\mathfrak{g}$ defined by Lyndon words -these algebras and their enveloping algebras $U\mathfrak{g}$ are central for the paper. A natural problem arises: "*Classify Lyndon pairs (N, W) such that the class $\mathfrak{C}(X, W)$ contains an Artin-Scheler regular algebra occurring as an enveloping $U\mathfrak{g}$ of a concrete monomial Lie algebra $\mathfrak{g} = Lie(X)/([W])_{Lie}$*". This motivates our study of Gröbner-Shirshov bases for monomial Lie ideals. In this case we find natural combinatorial conditions in terms of the Lyndon pair (N, W). Propositions 6.12 and 6.14, give purely combinatorial conditions which in various cases help to decide easily and without computations, whether a set of bracketed monomials $[W]$ is a Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$ of the free Lie algebra $Lie(X)$. In Section 7 we present some classical Lie algebras as standard monomial Lie algebras, $Lie(X)/([W])_{Lie}$, where $[W]$ is a finite Gröbner-Shirshov basis of the ideal $J = ([W])_{Lie}$. Our first applications of the general results on GS bases of are Corollary 7.3 and Theorem 7.9 which show that the free nilpotent Lie algebra $\mathfrak{L}(m)$ of nilpotency class m and the Filiform Lie algebras $\mathcal{L}_m$ and $\mathcal{Q}_m$ are standard $\mathbb{Z}^n$ multigraded monomial Lie algebras. In Section 8 we introduce the notions of *a regular Lyndon pair*, *a standard Lyndon pair*, and *a nontrivial disconnected extension of a standard Lyndon pair*, which are essential for our classification. Important general results are Proposition 8.5, and especially Proposition 8.8. In Section 9 we prove one of the main results of the paper, Theorem IV, which gives a classification of the two-generated Artin-Scheler regular algebras of global dimensions 6 and 7, occurring as enveloping of standard monomial Lie

algebras. Our general results on Lyndon pairs from the previous sections are crucial for the proof.

2. Graded algebras with polynomial growth and finite global dimension

2.1. **Anick's results on global dimension and growth of augmented graded algebras.** Given a finitely generated augmented graded algebra $A = K\langle X \rangle / I$ with a set of obstructions W, Anick introduces the notion of *an n-chain on W* and uses the n-chains to construct a resolution of the field K considered as an A-module. He obtains important results on augmented graded algebras with finite global dimension, see [3], and [2]. It is known that *Anick's resolution is minimal in the cases when (a) A is a monomial algebra, that is $A = K\langle X \rangle / (W)$ and (b) A is a PBW algebra*, that is the ideal I is generated by quadratic homogeneous polynomials which form a Gröbner basis w.r.t. degree-lexicographic ordering $\prec$ on X^*. We recall now some definitions and results from [2] and [3].

Definition 2.1. [2] The set of n-chains on W is defined recursively. A (-1)-chain is the monomial 1, a 0-chain is any element of X, and a 1-chain is a word in W. An (n+1)-prechain is a word $w \in X^+$, which can be factored in two different ways $w = uvq = ust$ such that $t \in W$, u is an $n-1$ chain, uv is an n-chain, and s is a proper left segment of v. An $(n+1)$-prechain is an $(n+1)$-chain if no proper left segment of it is an n-prechain. In this case the monomial q is called *the tail of the n-chain w*.

In fact the global dimension of a finitely presented monomial algebra can be effectively computed using some combinatorial properties of its defining relations, as shows the following corollary extracted from [2], Theorem 4.

Corollary 2.2. [2] *Suppose $W \subset X^+$ is an antichain of monomials. The monomial algebra $A_W = K\langle X \rangle / (W)$ has global dimension d iff there are no d-chains on W but there exists a $d-1$ chain on W.*

2.2. **Connected graded algebras with polynomial growth and finite global dimension (the general case).** In this section we shall study finitely generated graded algebras with polynomial growth and finite global dimension. We prove Theorem I assuming Hypothesis 1.7. Note that the set of obstructions W given in the hypothesis is an arbitrary antichain of monomials in X^+, no restrictions like "W consists of Lyndon words", or "W is finite" have been imposed.

Proof of Theorem I. *The algebra A and its associated monomial algebra A_W share the same set of obstructions W, hence the same K-normal basis, $\mathfrak{N} = \mathfrak{N}(W) = \mathfrak{N}(I)$, so $\mathrm{GKdim}\, A = \mathrm{GKdim}\, A_W$, and therefore A_W has polynomial growth. Moreover, by hypothesis there are no d-chains on W, but there exists a $d-1$-chain on W, hence, by Corollary 2.2 $\mathrm{gl\,dim}\, A_W < \infty$. Now a result of the author, [12], Theorem II, implies that there are equalities*

$$(2.1) \qquad \mathrm{gl\,dim}\, A = \mathrm{gl\,dim}\, A_W = d = \mathrm{GKdim}\, A = \mathrm{GKdim}\, A_W,$$

which proves part (1). It is clear, that A_W does not contain a free-subalgebra on two monomials (since A_W has polynomial growth) and therefore the monomial algebra A_W satisfies the hypothesis of [2], Theorem 5. It follows then that there exists a finite, ordered set $M = \{a_1 \leftarrow a_2 \leftarrow \cdots \leftarrow a_d\} \subset \mathfrak{N}$ consisting of atoms in the

sense of Anick, where $\leftarrow$ is a linear ordering on M, and $X \subseteq M$. M satisfies the good properties listed in [2], p. 297, which straightforwardly imply (2) (a), (b), (c). By (c) each obstruction $w \in W$ can be presented in terms of atoms as $w = a_j a_i$, where $1 \le i < j \le d$ and since there are $\binom{d}{2}$ such pairs i, j, one has $|W| \le d(d-1)/2$.

We know that the reduced Gröbner basis $\mathfrak{R}$ of I satisfies $\overline{\mathfrak{R}} = W$, where $\overline{\mathfrak{R}}$ is the set of highest monomials of the elements of $\mathfrak{R}$. Moreover, any two elements $f, g \in \mathfrak{R}, f \ne g$ satisfy $\overline{f} \ne \overline{g}$. Hence $|\mathfrak{R}| = |W| \le \binom{d}{2}$, and therefore A is standard finitely presented. This proves part (3). (4). We recall some details from Anick's results, [2]. A partial relation $\longrightarrow$ on the set of normal words, is introduced on p. 297, [2], and the set of atoms M is defined recursively as follows $M = \bigcup_{s=1}^{\infty} M_s$, where $M_1 = X$, and given M_{s-1}, M_s is defined by

$$M_s = M_{s-1} \bigcup \widetilde{M_s}, \ \ where$$
$$\widetilde{M_s} = \{ab \mid a, b \in M_{s-1}, b \leftarrow a, ab \in \mathfrak{R}, \ and \ |ab| = s\}.$$

It is proven that the set of atoms satisfy several properties, among which the following two.

E_s (Exclusivity): for $a, b \in M_s$ exactly one of the relations $a \leftarrow b, a = b$ and $a \to b$ is valid.

R_s (The relations in W): Any word $w \in W$, with $|w| \le s$ is a "would-be-atom", i.e. $w = ab$, for some $a, b \in M_{s-1}$, with $b \leftarrow a$.

By part (2) the set $X \subset M$, thus $n \le |M| = d$, and we use the restriction of the (new) linear ordering $\to$ on X to "rename" the elements of X. so that

$$X = \{y_1 \leftarrow y_2 \leftarrow \cdots \leftarrow y_n\}.$$

(4a) $\implies$ (4b). Assume $|W| = d(d-1)/2$. It follows from R_s, and $|M| = d$, that for each pair $a_j, a_i \in M$ with $j > i$ (equivalently, $a_i \leftarrow a_j$) one has $a_j a_i \in W$ (moreover, $a_j a_i \ne a_p a_q$, whenever $(j, i) \ne (p, q)$). In particular, the set

$$(2.2) \qquad \qquad \{y_j y_i \mid 1 \le i < j \le n\} \subseteq W.$$

Note that the inclusion (2.2) is in force if and only if there is an equality

$$(2.3) \qquad \qquad \widetilde{M_2} = \{ab \mid a, b \in X, b \leftarrow a, ab \in \mathfrak{R}\} = \emptyset$$

Moreover, (2.3) is equivalent to (4b). Indeed, by the recursive definition, $M_1 = X$, and $M_2 = M_1 \bigcup \widetilde{M_2}$, so there are no atoms of length 2 iff $\widetilde{M_2} = \emptyset$.

(4b) $\implies$ (4c). Assume (4b). Then $\widetilde{M_2} = \emptyset$, hence $M_2 = M_1 = X$, and (2.2) is in force. Using induction on s one verifies that $\widetilde{M_s} = \emptyset$, for all $s \ge 2$, and therefore $M = M_1 = X = \{y_1 \leftarrow y_2 \leftarrow \cdots \leftarrow y_n\}$. Moreover $W = \{y_j y_i \mid 1 \le i < j \le n\}$ which proves (4c).

(4c) $\implies$ (4d). Assume (4c). Then A_W is a quadratic monomial algebra with $\mathrm{GKdim}\, A_W = n$, and $\mathrm{gl\,dim}\, A_W = d$, hence, $d = n$, and $\mathrm{gl\,dim}\, A = \mathrm{gl\,dim}\, A_W = n$.

(4d) $\implies$ (4e). Assume $\mathrm{gl\,dim}\, A = n$. Part (1) implies $\mathrm{gl\,dim}\, A = \mathrm{gl\,dim}\, A_W = \mathrm{GKdim}\, A_W = d$, by part (2) $|M| = d$, hence $n = |M|$. Now $X \subseteq M$, and $|X| = n$, imply $M = X$. It follows that the set of obstructions W satisfies (1.9). In particular, the reduced Gröbner basis $\mathfrak{R}$ of I consists of homogeneous polynomials of degree 2, so A is a PBW algebra in the sense of Priddy, [29].

(4e) $\implies$ (4c). Assume A is a PBW algebra in the sense of Priddy. Then W consists of monomials of length 2, and A_W is a quadratic monomial algebra. By

the hypothesis of the theorem A_W has polynomial growth and finite global dimension. Hence W does not contain squares. By [15], Theorem 3.7, p 2163 there is a permutation $y_1, \cdots, y_n$ of X, such that W has the shape (1.9).

The implications $(4e) \Longrightarrow (4f)$ and $(4c) \Longrightarrow (4a)$ are clear.

$(4f) \Longrightarrow (4d)$. Suppose $H_A(z) = \frac{1}{(1-z)^n}$. Then A has polynomial growth of degree n, hence, by part (1), $\operatorname{gl} \dim A = \operatorname{GKdim} A = n$. This proves part (4).

It is known that every (quadratic) PBW algebra A in the sense of Priddy, is Koszul, see [29], and also [28], Theorem 3.1, p 84. This proves part (5). The theorem has been proved.   □

Corollary 2.3. *Let $A = K\langle X \rangle / I$ be an Artin-Shelter regular algebra of global dimension d, and with a set of obstructions W. Suppose that there are no d-chain on W. Then conditions (1) through (5) of Theorem I, 1 are satisfied. In particular, (i) there are equalities $\operatorname{GKdim} A = \operatorname{gl} \dim A = d$ and $\operatorname{gl} \dim A = \operatorname{gl} \dim A_W$; (ii) A is standard finitely presented and $|\mathfrak{R}| = |W| \leq d(d-1)/2$; (iii) Moreover, if $|W| = d(d-1)/2$, then $d = n = |X|$, and A is Koszul.*

Recall the following conjecture of Anick, [2], p.301.

Conjecture 2.4. *Suppose A is an augmented connected graded K-algebra with polynomial growth and with global dimension $d < \infty$. Then the Hilbert series of A is given by (1.7) for some positive integers $\{e_i\}$.*

Anick states in [2] that the conjecture has been verified for the cases when A is commutative, $A = U$ is an enveloping algebra of a Lie algebra, when A is a monomial algebra, and also in the case when A is is a Noetherian PI ring (an unpublished result of Lorenz). Our results verify Anick's conjecture in the case when A is an algebra whose associated monomial algebra A_W has finite global dimension.

3. Lyndon words, Lyndon atoms, and Lyndon pairs (N, W)

3.1. **Lyndon words and Lyndon atoms.** We recall first the notion of Lyndon words and some of their basic properties (we refer to [26]). Consider the *pure lexicographic ordering* "$<$" on the set X^+ which extends the ordering $x_1 < x_2 < \cdots < x_n$ on X as follows: for any $u, v \in X^+$ $u < v$ iff either u is a proper left factor of v ($v = ub, b \in X^+$), or $u = axb, v = ayc$, where $x < y$; $x, y \in X$, $a, b, c \in X^*$.

Recall that

$$L1: \quad \forall u \in X^*, \ a < b \Longleftrightarrow ua < ub.$$
$$L2: \quad \text{if } b \notin aX^+, \ \forall u, v \in X^*, \ a < b \Longrightarrow au < bv.$$

So $<$ is a linear ordering on the set X^+ compatible with the left multiplication in X^+. However, in the case when $b = au, u \neq 1$, the right multiplication does not necessarily preserve inequalities. (For example, $a < ax_1$, but $ax_2 > ax_1x_2$). Furthermore, the decreasing chain condition is not satisfied on $(X^+, <)$, for if $x, y \in X, x < y$, one has $xy > x^2y > x^3y > \cdots$.

The following simple relation between the two (fixed) orderings $\prec$ and $<$ on the free semigroup X^+ is in force.

(3.1) $u \prec v$ *iff* $|u| < |v|$, or $|u| = |v|$ and $u > v$.

For example, if $X = \{x < y\}$, one has $xxy < xyy$, but $xxy \succ xyy$.

Definition-Notation 3.1. [26] A nonperiodic word $u \in X^+$ is *a Lyndon word* if it is minimal, with respect to $<$, in its conjugacy class. In other words, $u = ab, a, b \in X^+$ implies $u < ba$. The set of Lyndon words in X^+ will be denoted by L, L_s denotes the set of all Lyndon words of length $s, s = 1, 2, 3, \cdots$.

We shall use notation, terminology and some results from [17].

Definition-Notation 3.2. [17] Given an antichain W of Lyndon words, the set of all *W-normal Lyndon words* is denoted by $N = N(W)$, we shall refer to it as *the set of Lyndon atoms corresponding to W*. By definition the set of Lyndon atoms N satisfies

$$(3.2) \qquad N = N(W) = L \bigcap \mathfrak{N}(W) \text{ where } L \text{ is the set of Lyndon words.}$$

Assume that $A = k\langle X \rangle / I$ is a graded K-algebra such that its set of obstructions W consists of Lyndon monomials, then we shall also refer to the normal Lyndon words $N = N(W)$ as *the set of Lyndon atoms for A*.

Definition 3.3. Let W be an antichain of Lyndon words, and $N = N(W) \subset L$ be the corresponding set of Lyndon atoms. Then the pair of sets (N, W) will be called *a Lyndon pair*.

The canonical duality between the notions "an antichain W of Lyndon words" and "its set of Lyndon atoms, $N = N(W)$" is discussed in the next subsection, see also [17].

Fact 3.4. (Lyndon's Theorem), [27] *11.5. Every word $w \in X^+$ can be written uniquely as a non increasing product $w = l_1 l_2 \cdots l_s$ of Lyndon words, $l_1 \geq l_2 \geq \cdots \geq l_s$.*

The following useful facts are extracted from [26], Section 5.1..

Facts 3.5.   (1) *For all $w \in L$, the equality $w = ab$, with $a, b \in X^+$, implies $a < w < b$.*
   (2) *If $a < b$ are Lyndon words then ab is a Lyndon word, so $a < ab < b$.*
   (3) *If b is the longest proper right segment of w which is a Lyndon word, then $w = ab$, where a is a Lyndon word. This is called* the (right) standard factorization *of w and denoted as $w = (a, b) = (a, b)_r$.*
   (4) *Analogously, suppose u is the longest proper left segment of w which is a Lyndon word, then $w = uv$, where v is also a Lyndon word. This determines the* left standard factorisation *of w denoted by $w = (u, v)_l$*

Notation 3.6. [17] For the monomials $a, b \in X^+$ we write $\overbrace{a, b}$ if $a = uv, b = vw$, where $u, w \in X^*$, $v, uw \in X^+$ ($b = aw$, or $a = ub$ is possible). We write $\overbrace{a, \tau, b}$ if $\tau \sqsubset ab$, and τ overlaps with a and b so that $(\overbrace{a, \tau})$ and $(\overbrace{\tau, b})$.

Lemma 3.7. [17]
   (1) *Let uv and vw be Lyndon words, $u, v \in X^+$. Then uvw is a Lyndon word.*
   (2) *If $a, b \in L$ and $\overbrace{a, b}$, then $a < b$.*
   (3) *Suppose that $a, b, w \in L$, with $\overbrace{a, w, b}$. Then $a < w < b$.*
   (4) *Let $a < b$ be Lyndon words. Then $a^k b^l$ are Lyndon words for all $k, l \geq 1$.*

(5) *If $w = (a, b)$ is the right standard factorization of w, then the (right) standard factorization of $(a^k b)$ is $(a, a^{k-1}b)$. Analogously, if $w = (a, b)_l$ is the left standard factorization of the Lyndon word w then the left standard factorization of ab^k is $(ab^k)_l = (ab^{k-1}, b)$.*

Lemma 3.8. [17] *Let $l_1 \geq l_2 \geq \cdots \geq l_s$ be Lyndon words, $s \geq 2$. If a Lyndon word u is a subword of $l_1 l_2 \cdots l_s$, then u is a subword of l_i, for some i, $1 \leq i \leq s$*

3.2. **More results on Lyndon words.** We shall prove some additional results on Lyndon words which will be used in the paper.

Lemma 3.9. *Suppose $a, w \in L, |a|, |w| \geq 2$, and a is a proper subword of w.*

(1) *If $w = (u, v)$ is the right standard factorization of w, and a is not a left segment of w, then either $a \sqsubset u$, or $a \sqsubseteq v$.*
(2) *If $w = (p, q)_l$ is the left standard factorization of w, and a is not a right segment of w, then either $a \sqsubseteq p$, or $a \sqsubset q$.*

Proof. We sketch a proof of (1). Assume the contrary. Then a overlaps with v, so that $a = a_1 a_2 \in L$, $v = a_2 v_2 \in L$, where $|a_1| \geq 1, |a_2| \geq 1$, and $a_1 a_2 v_2$ is a proper segment of w. By Lemma 3.7 (2) the product $a_1 a_2 v_2 = a_1 v$ is a Lyndon word, moreover, $a_1 v$ is a proper right segment of w with $|a_1 v| > |v|$. By assumption v is the longest Lyndon word which occurs as a right segment of w, a contradiction. The proof of part (2) is analogous. $\square$

Proposition 3.10. *Let N be a set of Lyndon words closed under taking Lyndon subwords, let $W = W(N)$ be the corresponding antichain of Lyndon words, see Remark 3.13 (2).*

(1) *If $u, v \in N, u < v$ and no Lyndon atom $a \in N$ satisfies $u < a < v$, then $uv \in W$.*
(2) *Suppose $N_0 = \{l_1 < l_2 < \cdots < l_k\} \subseteq N$ is a discrete convex in N, that is N does not contain a word u with $l_i < u < l_{i+1}$. Then $l_i l_{i+1} \in W$ for every pair $l_i, l_{i+1} \in N_0$.*
(3) *If N is a finite set of order d, then $d - 1 \leq |W|$.*

Proof. (1). By hypothesis there is no Lyndon atom a such that $u < a < v$. By Facts 3.5 $uv \in L$ and $u < uv < v$, hence $uv \in (W)$. We have to show that $uv \in W$. Assume the contrary, then (since $uv \in (W)$) some $w \in W$, is a proper subword of uv. Let $w = ab$, $a, b \in L$ be the standard factorization of w. By Remark 3.11 (1), $a, b \in N$, and by Facts 3.5 $a < ab < b$. Three cases are possible: (i) a overlaps with u and v, so that $\overbrace{u, a, v}$; (ii) b overlaps with u and v, so that $\overbrace{u, b, v}$; (iii) $u = u_1 a$, $v = bv_1$, $u_1 v_1 \neq 1$. Assume (i) is in force. Then Lemma 3.7 (3) implies $u < a < v$, which contradicts the hypothesis. Similarly, in case (ii) the relation $\overbrace{u, b, v}$ implies $u < b < v$, which is impossible. Suppose (iii) holds. Without loss of generality we may assume that a is a proper right segment of $u \in L$, so $u < a$. Now the inequalities

$$(3.3) \qquad\qquad u < a < ab < b \leq v; \quad a \in N,$$

where $a, b \in N$ contradict the hypothesis. The case when b is a proper left segment of v is analogous, which proves (1). Part (2) follows straightforwardly from (1). Clearly, (2) implies (3). $\square$

Remark 3.11. [17] Let (N, W) be a Lyndon pair. Then

(1) If u is a proper Lyndon subword of some $w \in W$ then $u \in N$. In particular, if $w \in W$ then the (right) standard factorisation $w = (u, v)$ and the left standard factorisation $w = (a, b)_l$ satisfy $a, b, u, v \in N$.

(2) If N has finite order d, and s is the maximal length of words in N then W is also finite with $|W| \leq d(d-1)/2$, and the length of each $w \in W$ is at most $2s$.

(3) Assume W is finite and let m be the maximal length of words in W. Then N is finite if and only if every word $l \in N$ has length $|l| \leq m - 1$.

3.3. **Canonical dualities on sets of monomials.** For completeness we consider the canonical dualities between (a) the notions "obtruction set", and "order ideal of monomials", see [3] and (b) the notions "antichains W of Lyndon words" and "Lyndon atoms, $N = N(W)$", [17].

Let I be an ideal in $K\langle X \rangle$, $A = K\langle X \rangle / I$. We use the notation and terminology from the previous sections, in particular (1.2), (1.3), and Definition-notation 3.2. We have seen that the set of monomials $\overline{I}$ contains uniquely determined maximal antichain of monomials $W = W(I)$ (with respect to $\sqsubset$). W satisfies (i) every element of W is minimal with respect to "$\sqsubset$", and (ii) each $a \in \overline{I}$ contains as a subword (a segment) some $v \in W$, i.e. $v \sqsubseteq a$. Denote by (W) the two-sided ideal of $K\langle X \rangle$ generated by W.

Clearly, the set of normal words $\mathfrak{N} = \mathfrak{N}(W)$ is uniquely determined by the set of obstructions W and is characterized by the properties:

N1. $X \in \mathfrak{N}$; **N2.** $u \in \mathfrak{N}, v \sqsubseteq u \Longrightarrow v \in \mathfrak{N}$; **N3.** $u \in \mathfrak{N} \Longleftrightarrow u \notin (W)$.

We have seen in the introduction that every set of monomials $\mathfrak{N} \subseteq X^*$ which satisfies conditions **N1** and **N2** determines uniquely an antichain of monomials $W \subset X^+$, so that condition **N3** holds and W is the set of obstructions for $\mathfrak{N}$.

An obstruction set $W = W(\mathfrak{N})$ is characterized by the following properties:

V1. $W \bigcap X = \emptyset$.
V2. $\forall u \in X^*, v \in W, u \sqsubset v, ; u \neq v \Longrightarrow u \in \mathfrak{N}$.
V3. $\forall a \in X^+, v \in W, v \sqsubseteq a \Longrightarrow a \notin \mathfrak{N}$.

Definition 3.12. (1) A set of monomials $\mathfrak{N} < X^+$ which satisfies conditions **N1** and **N2** is called *an order ideal of monomials* (o.i.m.), see [3].

(2) The anichain of monomials $W = W(\mathfrak{N})$ described above is called *the set of obstructions for $\mathfrak{N}$*. When W is considered as $W = W(I)$, it is also referred to as *the set of obstructions for the algebra $A = K\langle X \rangle / I$*.

Remark 3.13. (1) Suppose that $W \subset X^+$ is an antichain of Lyndon monomials, and $\mathfrak{N} = \mathfrak{N}(W)$. Then W determines uniquely a set of Lyndon atoms $N = N(W) := \mathfrak{N}(W) \bigcap L$. It satisfies

C1. $X \subseteq N$. **C2.** $\forall v \in L, \forall u \in N, v \sqsubseteq u \Longrightarrow v \in N$.
C3. $u \in N \iff u \in L$ and $u \notin (W)$.

(2) Conversely, each set N of Lyndon words satisfying conditions **C1** and **C2** determines uniquely an antichain of Lyndon monomials $W = W(N)$, such that condition **C3** holds, and N is the set of Lyndon atoms corresponding to W. In fact W is the unique maximal antichain of the complement $L - N$.

We systematize the above discussion and add some results from [17] in the following proposition.

Proposition 3.14. (1) *There exists a one-to-one correspondence between the set $\mathbb{V}$ of all antichains V in X^+ with $V \bigcap X = \emptyset$ and the set $\aleph$ consisting of all order ideals of monomials $\mathfrak{N}$, satisfying condition **N1** and **N2**. In notation as above this correspondence is defined as*

$$\begin{array}{llll} \Phi : & \mathbb{V} \longrightarrow \aleph & V \mapsto \mathfrak{N}(V) \\ \Phi^{-1} : & \aleph \longrightarrow \mathbb{V} & \mathfrak{N} \mapsto V(\mathfrak{N}). \end{array}$$

(2) *There are equalities*

$$\mathfrak{N}(V(\mathfrak{N})) = \mathfrak{N}; \quad V(\mathfrak{N}(V)) = V,$$

and each pair $(\mathfrak{N}(V), V)$ *(respectively $(\mathfrak{N}, V(\mathfrak{N}))$ obtained via this correspondence satisfies condition **N3**.*

(3) *There exists a one-to-one correspondence between the set $\mathbb{W}$ of all antichains W of Lyndon words with $X \bigcap W = \emptyset$ and the set $\mathbb{N}$ consisting of all sets N of Lyndon words satisfying conditions **C1** and **C2**. In notation as above this correspondence is defined as*

$$\begin{array}{llll} \phi : & \mathbb{W} \longrightarrow \mathbb{N} & W \mapsto N(W) \\ \phi^{-1} : & \mathbb{N} \longrightarrow \mathbb{W} & N \mapsto W(N). \end{array}$$

(4) *There are equalities*

$$N(W(N)) = N; \quad W(N(W)) = W,$$

and each pair $(N = N(W), W)$ *(respectively $(N, W = W(N))$ obtained via this correspondence satisfies condition **C3**.*

(5) [17] *Each finite antichain of Lyndon words $W \in \mathbb{W}$ determines a monomial algebra $A_W = K\langle X \rangle / (W)$ of finite global dimension, $\mathrm{gl\,dim}\, A_W \leq |W| + 1$.*

(6) [17] *If $N \in \mathbb{N}$ is a finite set of Lyndon words of order d, then the corresponding antichain $W = W(N)$ is also finite with $|W| \leq d(d-1)/2$.*

(7) *Each $N \in \mathbb{N}$ with **C1** and **C2** determines uniquely a monomial algebra $A_W = K\langle X \rangle / (W)$, with a set of defining relation $W = W(N)$ and a set of Lyndon atoms precisely N. The algebra A_W has polynomial growth of degree d iff $|N| = d$.*

Example 3.15. $X = \{x < y\}$, $A = K\langle X \rangle / (xy^2 + y^2x, \quad x^2y - yx^2)$. The algebra A is s.f.p.- the defining relations form a Gröbner basis of the ideal $I = (xy^2 + y^2x, \quad x^2y - yx^2)$. The set of obstructions $W = \{xxy, \quad xyy\}$ consists of Lyndon words. The set of Lyndon atoms is $N = \{d_1 = y > d_2 = xy > d_3 = x\}$, and (N, W) is the corresponding Lyndon pair. The normal basis of A is

$$\mathfrak{N} = \{d_1^{\alpha_1} d_2^{\alpha_2} d_3^{\alpha_3} \mid \alpha_i \geq 0\}.$$

A is a classical AS-regular algebra of $\mathrm{gl\,dim}\, A = 3$, **type A**.

4. SOME COMBINATORIAL RESULTS ON LYNDON PAIRS

In this section $X = \{x_1 < x_2 < \cdots < x_n\}$, and (N, W) is a Lyndon pair in X^+.

4.1. **Proof of Theorem II.**

Lemma 4.1. *Suppose $X = \{x_1, x_2, \cdots, x_n\}$, and let $W \subset X^+$ be an antichain of monomials of arbitrary cardinality. Let $\mathfrak{N} = \mathfrak{N}(W)$ be the set of normal words modulo W), and let $N = \mathfrak{N} \bigcap L$ be the set of normal Lyndon words. The following conditions are equivalent.*

(1) *W is an antichain of Lyndon words.*
(2) *The set $\mathfrak{N}$ of normal words has the shape given in 1.10.*

Proof. As usual, $A_W = K\langle X \rangle/(W)$ denotes the monomial algebra with defining relations $w = 0, w \in W$. Then $A_W \in \mathfrak{C}(X, W)$, and the set $\mathfrak{N}$ is a K-basis of A_W. Note that the set of normal Lyndon words $N = \mathfrak{N} \bigcap L$ satisfies conditions **C1** and **C2**, whenever W is an arbitrary antichain of monomials, but W is not necessarily a set of Lyndon words. Indeed, $X \subseteq \mathfrak{N} \bigcap L$, so N satisfies condition **C 1**. Suppose $u \in N$, and $v \in L$ is a subword of u. Then, since $u \in \mathfrak{N}$, the word v is also normal, (see **N2**) thus $v \in \mathfrak{N} \bigcap L = N$, and therefore condition **C2** also holds. Use now the canonical duality from Proposition 3.14 and consider the pair $(N, W_0 = W(N))$. Notice that if W is an arbitrary antichain of monomials, and $N = \mathfrak{N} \bigcap L$, then, in general, an inequality $W_0 = W(N) \neq W$ is possible. We claim that $W_0 = W$ holds if and only if (1.10) is in force.

$(1) \Longrightarrow (2)$. Assume that W is an antichain of Lyndon words, so $N = N(W)$ is the set of Lyndon atoms corresponding to W. Then A_W is a monomial algebra defined by Lyndon words in the sense of [17], hence, by Theorem A [17], the set of normal words $\mathfrak{N} = \mathfrak{N}(W)$ has the shape (1.10).

$(2) \Longrightarrow (1)$. Assume that the set of normal words $\mathfrak{N}$ is presented via (1.10).

We have seen that the canonical duality from Proposition 3.14 defines uniquely the pair (N, W_0), where $W_0 = W(N)$ is an antichain of Lyndon words. Then the corresponding set of Lyndon atoms, $N(W_0) = N(W(N)) = N$. Now $A_0 = K\langle X \rangle/(W_0)$ is a monomial algebra defined by Lyndon words in the sense of [17], and its set of Lyndon atoms is N. Use now the first duality in Proposition 3.14, and consider the pair $(W_0, \mathfrak{N}_0 = \mathfrak{N}(W_0))$. The set $\mathfrak{N}_0$ is the normal basis for the monomial algebra A_0. Hence by Theorem A [17] again, $\mathfrak{N}_0$ has a "PBW"-type presentation via the Lyndon atoms in N which is exactly the set given in (1.10). Therefore $\mathfrak{N}(W_0) = \mathfrak{N}(W)$, so $W_0 = W(\mathfrak{N}(W_0)) = W(\mathfrak{N}(W)) = W$. It follows that the obstruction set W is an antichain of Lyndon words. $\square$

Proof of Theorem II. *Part (1) of Theorem II is proven by Lemma 4.1.*

Suppose that W is an antichain of Lyndon words, so (N, W) is a Lyndon pair. Let $A = K\langle X \rangle/I \in \mathfrak{C}(X, W)$, let $A_W = K\langle X \rangle/(W)$ be the corresponding monomial algebra. Clearly, $A_W \in \mathfrak{C}(X, W)$.

(2). The two algebras A and A_W share the same set of obstructions W, the same set of Lyndon atoms $N = N(W)$, the same normal K-basis $\mathfrak{N}$ and the same Hilbert seies. In particular, A_W is a monomial algebra defined by Lyndon words in the sense of [17], so the results in [17] are true for A_W.

There is an obvious equality $\mathrm{GKdim}\, A = \mathrm{GKdim}\, A_W$. Therefore by [17], Theorem A, (2) each of the algebras A and A_W has a polynomial growth of degree d iff $|N| = d$. Assume A has polynomial growth of degree d. Then N has order d, say, $N = \{l_1 > l_2 > \cdots > l_d\}$. Then Theorem A, [17] implies $\mathrm{gl\,dim}\, A_W = d$. Hence there is a $d - 1$-chain on W and there is no d-chain on W. Now the algebra A

satisfies the hypothesis of Theorem I, so $\operatorname{gl\,dim} A = \operatorname{GKdim} A = d$, *which gives (2a).*

The two algebras A and A_W share the same normal K-basis $\mathfrak{N}$ and the same Hilbert series, therefore (2b) is in force. The monomial algebra A_W satisfies the hypothesis of [17], Theorem B, and therefore its set of obstructions W satisfies the inequalities (1.12), which implies (2c). Part (3) follows from Theorem III, and is given for completeness. The theorem has been proved. $\qquad\square$

4.2. **Up to isomorphism, for every** $d \geq 2$ **there exists unique Lyndon pair** (N, W)**, where** N **is connected and** $|W| = |N| - 1 = d$**.** Suppose (N, W) is a Lyndon pair in X^+, where $N = N(W)$ has finite order $|N| = d$. It follows from Theorem II that W is a finite set with

$$d - 1 \leq |W| \leq \frac{d(d-1)}{2}.$$

It follows from [17], Theorem B (2) that the upper bound $|W| = d(d-1)/2$ is attained if and only if $W = \{x_i x_j \mid 1 \leq i < j \leq d\}$, or equivalently, $d = n$ and $N = X$, see Theorem II (2). In this case the class $\mathfrak{C}(X, W)$ contains all binomial skew polynomial rings with square-free relations generated by X, in the sence of [14]. Each such a ring A is an AS-regular PBW algebra of global dimension n. For example, for $g = 8$ there are more than 2000 non-isomorphic skew-polynomial rings with binomial square-free solutions, each of which is an AS-regular and defines a set-theoretic solution of YBE. It is interesting to describe the other "extreme" case of Lyndon pairs, when the order $|W|$ attains the lower bound. Theorem 4.2 shows that *if N is connected* then the pure *numerical datum*

$$(4.1) \qquad\qquad |W| = d - 1, \quad |N| = d$$

determines uniquely (i) the order of X, $|X| = 2$, and (ii) unique Lyndon pair (N, W) in X^+ (up to isomorphisms of monomial algebras A_W). Corollary 7.8 implies that in this case the class $\mathfrak{C}(X, W)$ contains the universal enveloping $U = U\mathcal{L}_{d-1}$, where $\mathcal{L}_{d-1}$ is the standard filiform algebra of dimension d and nilpotency class $d - 1$. U is an AS-regular algebra with $\operatorname{gl\,dim} U = d$.

Theorem 4.2. *For every integer $d \geq 2$, there exists unique (up to isomorphism) Lyndon pair (N, W), where N is a connected set of finite order d and $|W| = d - 1$. Moreover, the following conditions are equivalent:*

  (1) $|W| = d - 1$ *and N is a connected set of Lyndon atoms, see Def 5.9;*
  (2) N *is a connected set of Lyndon atoms and*

$$(4.2) \qquad\qquad W = \{l_i l_{i+1} \mid 1 \leq i \leq d\}$$

  (3) *The set X has order 2, so $X = \{x < y\}$, and (up to isomorphism of monomial algebras A_W) the set of atoms $N = N(W)$ has the shape*

$$(4.3) \qquad N = \{x < xy < xyy < \cdots < xy^{d-3} < xy^{d-2} < y\}.$$

  (4)

$$W = \{xy^i xy^{i+1} \mid 0 \leq i \leq d - 3\} \bigcup \{xy^{d-1}\}.$$

Proof. (1) $\Longleftrightarrow$ (2). By Proposition 3.10 (2) there is an inclusion of sets

$$\{l_i l_{i+1} \mid 1 \leq i \leq d - 1\} \subseteq W.$$

The equivalence of (1a) and (1b) is straightforward.

(2) $\implies$ (3). Assume W has the shape given in (4.2). We shall prove (3) in several steps.

We start with an useful remark.

Remark 4.2.1. Suppose (4.2) holds, and N contains a subchain $a < b < c$. Then ac is not in W. Furthermore, if each proper Lyndon subwords $w \sqsubset ac$ is an atom, $w \in N$, one has $ac \in N$.

Step 1. $|X| = 2$. Assume the contrary, then X has at least three distinct elements, say $x < y < z$. By assumption W contains only products of Lyndon atoms which are "neighbors", so by Remark 4.2.1 $xz \in N$, and N contains the subset $x < xz < y < z$. Now by induction on k (using analogous argument) one shows that N contains an infinite subchain

$$x < xz < xz^2 < xz^3 < \cdots < x^k < \cdots < y < z,$$

which contradicts the hypothesis. By convention X has at least 2 elements, so $X = \{x < y\}$. We shall describe N.

The case $|N| = 3$ is trivial, one has $N = \{x < xy < y\}$. Assume $|N| \geq 4$.

Step 2. We prove that N does not contain simultaneously xxy and xyy. Otherwise, N contains the subchain $x < xxy < xy < xyy < y$, hence xxy and y are not neighbours, so $u = xxyy$ is not in W, and since every proper Lyndon subword of u is in N, Remark 4.2.1 implies $xxyy \in N$. Therefore N contains $x < xxy < xxyy < xy < xyy < y$. Remark 4.2.1 again implies $xxyxy \in N$, so $\{x < xxy < xxyxy < xxyy < xy < xyy < y\} \subseteq N$. Next one proves by induction on k (and using Remark 4.2.1) that N contains an infinite set of Lyndon atoms: $\{xxy(xy)^k, k \geq 1\} \subseteq N$, which contradicts the hypothesis again.

From now on, without loss of generality, we may assume $xxy \in W$. (If we assume $xyy \in W$ we shall obtain an isomorphic monomial algebra A_W). Since N is connected, it contains a Lyndon word of length 3, hence $xyy \in N$. Therefore $l_1 = x < xy = l_2 < xyy = l_3 < \cdots < l_d = y$ is a subchain in N.

Step 3. We shall prove that N satisfies (4.3). Assume the contrary, then N contains a Lyndon atom of the shape $xy^j xy^k \cdots$, $k \geq j \geq 1$. Let u_0 be *the shortest Lyndon atom of such shape*. By hypothesis N is closed under taking Lyndon subwords, and it is easy to prove that $u_0 = xy^j xy^{j+1}$, for some j. Moreover, there is no atom $v \in N$ satisfying $xy^j < v < xy^j xy^{j+1}$. Hence N contains the convex $l_1 = x < xy = l_2 < \cdots < l_{j+1} = xy^j < u_0 = xy^j xy^{j+1}$. This implies that

$$m(N) \geq |u_0| = 2j + 3.$$

We claim that $w = xy^{2j+1} \in N$. Indeed, by assumption $u_0 \in N$ is the shortest atom with $u_0 \neq xy^t$. Moreover, since $u_0 \in N, |u_0| = 2j + 3$ and N is connected, one has

(4.4) $$xy^s \in N, 1 \leq s \leq 2j + 1.$$

One has $|xy^{2j+1}| = 2j + 2$. Now consider the inequalities of Lyndon atoms:

$$l_{j+1} = xy^j < u_0 = l_{j+2} = xy^j xy^{j+1} < xy^{j+1} < y = l_d.$$

By hypothesis W satisfies (4.2), hence the product $l_{j+1}l_d = xy^j xy^{j+2}$ is not in W. Note $u_1 = l_{j+2}l_d = xy^j xy^{j+2}$, has (right) standard factorisation $u_1 = (xy^j).(xy^{j+2})$. Clearly, every Lyndon subword of u_1 which is not its left segment must be a subword of xy^{j+2} (or of xy^j), moreover, it follows from (4.2) that no

element $w \in W$ is a subword of u_1, hence by Remark 4.2.1 $u_1 = l_{j+1}l_d = xy^j xy^{j+2} \in$ N. The following sub-chain of atoms is contained in N:

$$(4.5) \qquad u_0 = l_{j+1} = xy^j xy^{j+1} < u_1 = xy^j xy^{j+2} < \cdots < xy^{j+1} \in N.$$

Similar argument implies that $u_0 = l_{j+1} = xy^j xy^{j+1} < xy^j (xy^{j+1})^2 \in N$, and using induction on k, Remark 4.2.1, and similar argument as above one proves that N contains an infinite increasing chain of Lyndon atoms:

$$(4.6) \qquad \begin{aligned} u_0 = l_{j+1} \;\; &= xy^j xy^{j+1} < xy^j (xy^{j+1})^2 < xy^j (xy^{j+1})^3 < \cdots \\ &< xy^j (xy^{j+1})^k < \cdots \in N, \; k \geq 1, \end{aligned}$$

which contradicts $|N| = d + 1$. It follows that N satisfies (4.3).

The implications $(3) \Longrightarrow (4)$ and $(3) \Longrightarrow (2)$ are clear. This proves the theorem.
□

5. Classes $\mathfrak{C}(X, W)$ containing the enveloping algebra of a Lie algebra

5.1. Finitely generated graded Lie algebras and their enveloping algebras, standard Lyndon bases. Our main references are [26], [27] [33], [21], [19].

In this subsection we set notation and recall some classical facts about finitely generated graded Lie algebras and their enveloping algebras which will be used in the sequel. As usual, $X = \{x_1, \cdots, x_n\}$, $n \geq 2$, is a finite set, K is a field of characteristic 0, $\mathfrak{L} = \mathrm{Lie}(X)$ denotes the free Lie K- algebra generated by X, and $\mathfrak{U} = \mathfrak{U}(\mathfrak{L})$ is the universal enveloping algebra of $\mathfrak{L}$. It is known that $\mathfrak{U}$ and $K\langle X \rangle$ are isomorphic as associative algebras, and we shall identify them. X^m denotes the set of all words of length $m \geq 1$ in X^*. The set of all Lyndon words in X^+ is denoted by L, and $L_m = L \bigcap X^m$ is the set of all Lyndon words of length m. The free associative algebra $\mathfrak{U} = K\langle X \rangle$ is naturally graded by length. An element $f \in \mathfrak{U}$ is called *a Lie element* (or *a Lie polynomial*) if $f \in \mathfrak{L}$. It is known that if f is a Lie element, then each homogeneous component of f is also a Lie element. A special case of *homogeneous Lie elements* are the so called $(m-1)$-*left nested Lie monomials*:

$$(5.1) \qquad f_m = [[[x_{i_1}, x_{i_2}], \cdots], x_{i_m}], \;\; x_{i_s} \in X, \; 1 \leq s \leq m.$$

Clearly, the Lie monomial f_m may also be considered as an element of the free associative algebra $\mathfrak{U} = K\langle X \rangle$, via the well-known equality $[a, b] = ab - ba$, so $f_m \in \mathfrak{U}_m$. The free Lie algebra $\mathfrak{L} = \mathrm{Lie}(X)$ is also naturally graded by length,
$$(5.2)$$
$$\begin{aligned} \mathfrak{L} \;\; &= \bigoplus_{m \geq 0} \mathfrak{L}_m, \;\; \text{where} \;\; \mathfrak{L}_0 = K, \;\; \mathfrak{L}_1 = KX, \\ \mathfrak{L}_m \;\; &= \mathrm{Span}_K \{ f_m = [[\cdots [x_{i_1}, x_{i_2}], \cdots], x_{i_m}] \mid x_{i_s} \in X, 1 \leq s \leq m \}, \; m \geq 2. \end{aligned}$$

Throughout the paper *the right standard factorisation* $w = (u, v)$ will be called simply "*the standard factorisation of* w". Recall that v is the longest proper right segment of w which is a Lyndon word, in this case u is also a Lyndon word.

Definition 5.1. [26] Let $l \in L$. Consider the unique *(right) standard factorization* denoted $l = (u, v)_r$. The (right) standard bracketing $[l] = [l]_r$ on the set L of all Lyndon words is defined inductively as follows.

$$[x] = x, \forall x \in X, \quad [l]_r := [[u]_r, [v]_r], \text{ where } l \in L - X, \; l = (u, v)_r.$$

The left standard bracketing $[w]_l$ is defined analogously, but using the left standard factorization.

Definition-Convention 5.2. The notation $[a]$ stands for the right standard bracketing, and $[a]_l$ will be used for the left standard bracketing of a. The Lie element $[u]$ is called also *a Lyndon-Lie monomial (corresponding to u)*. Given a nonempty set V of Lyndon words, $V \subseteq L$, we denote by $[V]$ the set of standard bracketings of all elements of V:

$$(5.3) \qquad\qquad [V] = \{[v] \mid v \in V\},$$

and refer to it as *"the bracketing of V"*.

Remark 5.3.   (1) For each Lyndon word u, the standard bracketing $[u]$ considered as an element of the algebra $\mathfrak{U} = K\langle X \rangle$ is a (noncommutative) polynomial which is a linear combination

$$(5.4) \qquad\qquad [u] = u + \sum c_a a,$$

where each nonzero coefficient c_a is an integer, all (distinct) monomials $a \in X^+$ in this presentation satisfy $a \prec u, |a| = |u|$, in particular $\overline{[u]} = u$, see [26], Lemma 5.3.2, or [33], Theorem 5.1.

(2) It is not difficult to see that considered as "commutative terms" all monomials occurring in the right-hand side of (5.4) are equal, that is the monomials a and u have *the same multi-degree* as u, (see Definition-Notation 6.3).

(3) The elements

$$[L] = \{[u] \mid u \in L\}$$

form a basis of $\mathrm{Lie}(X)$, see [26], Theorem 5.3.1, or [33], Theorem 4.9.

In particular, for each $m \geq 1$ the set $[L_m]$ of *Lyndon Lie monomials* of degree m,

$$[L_m] = \{[u] \mid u \in L_m\}$$

is a K-basis of the graded component $\mathfrak{L}_m$ (considered as a vector space).

(4) On the other side, for each $m \geq 2$ the graded component $\mathfrak{L}_m$ is spanned by all left-nested Lie monomials $[[[x_{i_1}, x_{i_2}], \cdots], x_{i_m}]$, $x_{i_j} \in X$, hence there are equalities of sets

$$(5.5) \quad \underset{K}{\mathrm{Span}}\{[u] \mid u \in L_m\} = \mathfrak{L}_m = \underset{K}{\mathrm{Span}}\{f_m = [[\cdots[[x_{i_1}, x_{i_2}], \cdots], x_{i_{m-1}}], x_{i_m}]\}.$$

The works [21] and [17] are central for this subsection. These two works use different terminology for equivalent notions and, for convenience, we adapt the terminology from [21] to our equivalent terminology given in [17], and typical for the theory of non-commutative Gröbner bases. The equivalence is given in Remark 5.5. The notion of *"an atom"* was introduced by Anick, [2] in a more general context, the special properties of "the atoms" in the sense of Anick are listed in [2], Theorem 5. It is easy to see that if (N, W) is a Lyndon pair, then the set of Lyndon words N has the properties analogous to the properties of a set of atoms, in the sense of Anick. In fact N coincides with the set of atoms for the monomial algebra A_W, see also our comments in [17]. Recall that, by convention, all Gröbner bases of (associative) ideals I in $K\langle X \rangle$ and all Lyndon-Shyrshov bases of Lie ideals J in $\mathrm{Lie}(X)$ are considered with respect to "$\prec$" -the degree- lexicographic well-ordering on X^*.

Notation-Convention 5.4. From now on J will denote a Lie ideal in $\mathfrak{L} = \mathrm{Lie}(X)$, and I will denote the two-sided ideal in $K\langle X \rangle$ generated by J, where J is considered

as a set of associative polynomials, $I = (J_{ass})$. Let

$$(5.6) \qquad \mathfrak{g} = \mathrm{Lie}(X)/J, \quad U = U\mathfrak{g} = K\langle X\rangle/I.$$

It is known that U is the enveloping algebra of the Lie algebra $\mathfrak{g}$, see for example [21], p. 1823. $W = W(I)$ denotes the corresponding set of obstructions for the associative algebra $U = U\mathfrak{g}$. As usual, $\mathfrak{N} = \mathfrak{N}(I) = \mathfrak{N}(W)$ denotes the set of normal words modulo I (w.r.t. $\prec$). $N = N(I) = N(W)$, is the set of all Lyndon words which are normal mod I, we may also write $N = N(J)$. By definition $N = \mathfrak{N} \bigcap L$, where L is the set of Lyndon words. We shall prove that the obstruction set W is *an antichain of Lyndon words*, see Proposition 5.6 (1), so, by convention, we call $N = N(I) = N(W)$ *the set of Lyndon atoms for U*. In this case (N, W) is *a Lyndon pair*.

The remark given below provides an useful interpretation of the terminology of [21] in terms of our equivalent notions introduced in this paper and in [17]. Note that our terminology is introduced in more general settings not necessarily in the context in Lie algebras, and independently. We use results from [17] and [21], see Theorem (2.1), Corollaries (2.5) and (2.8) to deduce straightforwardly the following.

Remark 5.5. In notation and assumption as above, $\mathfrak{g} = \mathrm{Lie}(X)/J$, $U = U\mathfrak{g} = K\langle X\rangle/I$, and W is the set of obstructions for U. (1) A word $a \in X^*$ is *standard* modulo I in the sense of [21] *iff* a is *normal* modulo I, that is no $f \in I$ satisfies $\overline{f} \sqsubseteq a$ (equivalently, no $w \sqsubseteq a$, for some $w \in W$). Moreover, the set $\mathfrak{N}(I)$ of normal (mod I) monomials coincides with the set of "*standard mod I monomials*" in the sense of [21]. (2) The set N of Lyndon atoms, where $N = \mathfrak{N}(I) \bigcap L$ coincides with the set of "*Lie-standard Lyndon words*", in the sense of [21]. Moreover, the set $[N] \subset \mathfrak{g}$ is a K-basis of $\mathfrak{g}$. (3) The enveloping algebra $U = U\mathfrak{g}$ has a K-basis of "a Poincaré-Birkhoff-Witt" type expressed in terms of its Lyndon atoms via (1.10). This basis coincides with the set of normal words $\mathfrak{N}$ (modulo I).

5.2. **Basic information on classes $\mathfrak{C}(X, W)$ containing the enveloping algebra U of a Lie algebra.**

Proposition 5.6. *Suppose $\mathfrak{g}$ is a Lie algebra, $\mathfrak{g} = Lie(X)/J$, $U = U\mathfrak{g} = K\langle X\rangle/I$ is its enveloping algebra. Let W be the set of obstructions for U, as usual, $\mathfrak{N} = \mathfrak{N}(I)$ denotes the set of normal monomial, and $N = N(W) = \mathfrak{N} \bigcap L$ is the set of normal Lyndon words. The following conditions hold.*

(1) *W is an antichain of Lyndon words, so (N, W) is a Lyndon pair, and $A_W = K\langle X\rangle/(W)$ is a monomial algebra defined by Lyndon words, in the sense of [17].*

(2) *Assume furthermore, that J is generated by homogeneous Lie elements, so the enveloping algebra U is canonically graded. Then the algebras U, and A_W are in the class $\mathfrak{C}(X, W)$. The following conditions are equivalent.*
    (a) *U is an Artin-Schelter regular algebra.*
    (b) *U has polynomial growth.*
    (c) *The Lie algebra $\mathfrak{g}$ is finite dimensional.*
    (d) *The set of Lyndon atoms N is finite.*
    Each of these equivalent conditions implies that

$$\mathrm{gl}\dim(U) = \mathrm{GKdim}(U) = \dim_K \mathfrak{g} = |N| = d.$$

> *Moreover, U is a standard finitely presented algebra and*

$$(5.7) \qquad d - 1 \leq |W| \leq d(d-1)/2, \quad where \ \ d = |N|.$$

Proof. (1). The algebra U has a K-basis of "a Poincaré-Birkhoff-Witt" type expressed in terms of its Lyndon atoms via (1.10), see Remark 5.5. Hence by Lemma 4.1 the set of obstruction W is an antichain of Lyndon words. Clearly, then (N, W) is a Lyndon pair. Moreover, $A_W = K\langle X \rangle/(W)$ is a monomial algebra defined by Lyndon words and all results of [17] are applicable.

(2). The algebras A_W and U belong to $\mathfrak{C}(X, W)$, they have the same K-basis, $\mathfrak{N} = \mathfrak{N}(W)$, and the same set of Lyndon atoms $N = N(W) = \mathfrak{N} \bigcap L$, in particular, $\mathrm{GKdim}\, U\mathfrak{g} = \mathrm{GKdim}\, A_W$. The equivalence below follows from [17], Theorem A (2).

$$(5.8) \qquad \mathrm{GKdim}(A_W) = d \Longleftrightarrow |N| = d.$$

Moreover, [17], Theorem B (2) gives the following relations

$$(5.9) \qquad \mathrm{GKdim}(A_W) = d \Longrightarrow \mathrm{gl\,dim}(A_W) = d \quad d - 1 \leq |W| \leq d(d-1)/2.$$

Now [21], Theorem (2.1) implies straightforwardly that the set $[N] = \{[u] \mid u \in N\}$ is a K-basis of the Lie algebra $\mathfrak{g}$, there is an isomorphism of vector spaces $\mathfrak{g} \cong \mathrm{Span}_K[N]$. Thus $\mathfrak{g}$ is finite dimensional, if and only if the set of atoms N is finite, in this case $\dim \mathfrak{g} = |N|$. Assume now that U has polynomial growth of degree d, so

$$\mathrm{GKdim}(U) = d = \mathrm{GKdim}(A_W).$$

It follows from (5.8) that $|N| = d$, hence (2b)$\Longleftrightarrow$ (2d), and (2b) $\Longleftrightarrow$ (2c). By (5.9) $\mathrm{gl\,dim}(A_W) = d$, and by [12], Theorem II, $\mathrm{gl\,dim}(U) = \mathrm{gl\,dim}(A_W) = d$. The obstruction set W is finite (by (5.9) again) therefore U is standard finitely presented.

Suppose now U is Artin-Schelter regular, then by definition it has polynomial growth of degree, say d, so (2a)$\Longrightarrow$ (2b), and therefore the remaining conditions (c) and (d) are also in force. The discussion above implies $\mathrm{gl\,dim}\, U = d = \mathrm{GKdim}\, U = \dim \mathfrak{g} = |N|$.

Conversely, assume that $\mathfrak{g}$ is finite dimensional. It is known by the experts that the enveloping algebra $U = U\mathfrak{g}$ of a finite dimensional positively graded Lie algebra $\mathfrak{g}$ is Artin-Schelter regular. A reference to the original result is difficult to find, and we refer to the proof provided by Fløystad and Vatne, see [11], Theorem 2.1. $\quad\square$

We have seen that the enveloping algebra $U\mathfrak{g}$ of any Lie algebra $\mathfrak{g}$ generated by X has a set of obstructions W, consisting of Lyndon words, in general, W may be infinite. Moreover, if $\mathfrak{g}$ has finite dimension $\dim \mathfrak{g} = d$, then the enveloping algebra $U = U\mathfrak{g}$ is a standard finitely presented Artin-Schelter regular algebra of global dimension d. The number $r = |W|$ of its relations satisfies the inequalities (5.7). The following question arises naturally.

Question 5.7. *Let (N, W) be a Lyndon pair, where N is a finite set. Under what combinatorial conditions is W the set of obstructions for the enveloping $U\mathfrak{g}$ of some Lie algebra $\mathfrak{g}$?*

Proposition 5.11 gives a necessary condition in terms of $N = N(W)$, the set of Lyndon atoms. In section 6 we introduce and study *the monomial Lie algebras* $\mathfrak{g} = Lie(X)/([W])$. In this case the enveloping algebra $U\mathfrak{g}$ belongs to $\mathfrak{C}(X, W)$ if and only if $[W]$ is a Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$ of $Lie(X)$.

5.3. **If the class** $\mathfrak{C}(X, W)$ **contains the enveloping algebra** U **of a Lie algebra then the set** $N = N(W)$ **is connected.**

Remark 5.8. Due to the duality $W = W(N)$, $N = N(W(N))$, see Proposition 3.14, each class of associative graded algebras $\mathfrak{C}(X, W)$, with obstruction set W consisting of Lyndon words is also uniquely determined by the set of Lyndon atoms $N = N(W)$. Conversely, each set N of Lyndon words closed under taking Lyndon subwords, with $X \subseteq N$ determines uniquely a class $\mathfrak{C}(X, W)$, where $W = W(N)$, so that (N, W) is a Lyndon pair. It is interesting to classify all Lyndon pairs (N, W) such that the monomial algebra A_W has polynomial growth and $\operatorname{gl\,dim} A_W = d$, where $d \geq 1$ is fixed, we use our result $\operatorname{gl\,dim} A_W = \operatorname{GKdim} A_W = |N| = d$ (in this case W is always finite). In fact it is more convenient (i) to classify the sets of Lyndon atoms N of a fixed finite order $|N| = d$, N must satisfy **C1, C2**, and only then (ii) to write down the corresponding set of obstructions $W = W(N)$, so that (N, W) is a Lyndon pair. In particular, one has

$$\mathfrak{C}(X, W) = \mathfrak{C}(X, N), \text{ where } (W, N) \text{ is a Lyndon pair.}$$

Till the end of the paper we shall consider only the special case when *the ideal* $I = (J_{ass})$ *of* $K\langle X \rangle$ *is graded by length, respectively, multi-graded,* thus the enveloping algebra $U = K\langle X \rangle / I$ is also graded, respectively, multi-graded. Moreover, U has a set of obstructions W consisting of Lyndon words, and $U \in \mathfrak{C}(X, W)$.

Note that every Lyndon word u of length $|u| \geq 3$ in a finite alphabet X, contains a Lyndon subword of length 2. Moreover, if N is a finite set of Lyndon monomials in $X = \{x, y\}$ with **C1, C2**, and N contains a word of length ≥ 4, then $N \bigcap L_3 \neq \emptyset$.

Definition 5.9. Suppose N is a set of Lyndon words (possibly infinite) with conditions **C1** and **C2**. (i.e. $X \subseteq N$, and N is closed under taking Lyndon subwords).

(1) We say that N is *connected* if

$$(5.10) \qquad [N \bigcap L_m \neq \emptyset, \ m \geq 2] \quad \text{implies} \quad [N \bigcap L_s \neq \emptyset, \ \forall s, 1 \leq s \leq m].$$

In particular, if $N = \{l_1 < l_2 < l_3 \cdots < l_d\}$ is a finite set, then N is *connected iff* (5.10) is in force for $m = \max\{|l_i| \mid 1 \leq i \leq d\}$, and all $s, 1 \leq s \leq m$.

(2) Suppose N is not connected, that is $N \bigcap L_j = \emptyset$, $N \bigcap L_{j+1} \neq \emptyset$ for some j. Then *the connected component of* N, denoted by N_{con} is defined as

$$(5.11) \qquad \begin{array}{l} N_{con} := \bigcup_{1 \leq s \leq k} (N \bigcap L_s), \text{ where } N \bigcap L_s \neq \emptyset, \forall s, 1 \leq s \leq k, \\ \text{and } N \bigcap L_{k+1} = \emptyset. \end{array}$$

For completeness we set $N_{con} = N$, whenever N is connected. Note that N_{con} is the maximal *connected subset* of N, which satisfies **C1** and **C2**.

One can find explicit examples in the last section, where we have classified the Lyndon pairs (N, W) in the alphabet $X = \{x < y\}$, with $|N| = 7$, see Subsec. 9.3. Each of the first 12 pairs (N, W), (7.4.1) through (7.6.12), has connected set of atoms N. Each of the remaining pairs has a disconnected set of Lyndon atoms, but we have identified explicitly the connected components N_{con}. We illustrate this with the Lyndon pair (N, W) given in (7.7.15). N is disconnected: $N = \{x < xy < xy^2 < xy^2xy^3 < xy^3 < xy^4 < y\}$. One has $N \bigcap L_7 = \{u = xy^2xy^3\}$, and $N \bigcap L_6 = \emptyset$. In this case N has a connected component $N_{con} = N \setminus \{xy^2xy^3\} = N(\mathcal{L}_5)$, the set of atoms corresponding to the Filiform Lie algebra $\mathcal{L}_5$ of dimension 6 and nilpotency class of degree 5

Suppose N is finite. Clearly, the property "N *is connected*" is recognizable. Moreover, this is a necessary condition so that "$\mathfrak{C}(X, W) = \mathfrak{C}(X, N)$ contains the enveloping algebra $U = U\mathfrak{g}$ of some Lie algebra $\mathfrak{g}$", as shows Proposition 5.11.

Definition 5.10. Recall that $\mathfrak{g}$ is *a nilpotent Lie algebra of nilpotency class* m if m is the minimal value for which $\mathfrak{g}^{m+1} = 0$, where

$$\mathfrak{g}^1 = \mathfrak{g}, \text{ and } \mathfrak{g}^t = [\mathfrak{g}^{t-1}, \mathfrak{g}], \ t \geq 2$$

is the lower central series of $\mathfrak{g}$. In this case we shall also write $\mathfrak{g} \in \mathfrak{N}_m$.

Proposition 5.11. *Suppose (N, W) is a Lyndon pair, such that the class $\mathfrak{C}(X, W)$ contains the enveloping algebra $U = U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$. Then the following conditions hold.*

(1) *If $N \bigcap L_s = \emptyset$, for some $s \geq 2$ then $N \bigcap L_{s+k} = \emptyset$, for all $k \geq 1$. In particular, N is finite. Moreover, if s is the minimal positive integer with this property, then $\mathfrak{g}$ is a nilpotent Lie algebra of nilpotency class $s - 1$.*

(2) *If N is an infinite set, then $N \bigcap L_m \neq \emptyset$ for all $m \geq 1$, so N is connected.*

(3) *Suppose N is a finite set, $|N| = d$ and let $m = \max\{|u| \mid u \in N\}$. Then the Lie algebra $\mathfrak{g}$ is nilpotent of nilpotency class $m \leq d - |X| + 1$.*

(4) *The set N of Lyndon atoms is connected, and $[N]$ is a K-basis for $\mathfrak{g}$.*

Proof. It follows from the hypothesis that there exists a Lie algebra $\mathfrak{g} = Lie(X)/J$, where J is a (graded) Lie ideal whose enveloping $U = U\mathfrak{g} = K\langle X\rangle/I$, has set of obstructions W (I is the associative two-sided ideal generated by J). In fact U determines uniquely the Lyndon pair (N, W), given in the hypothesis.

(1) Assume $N \bigcap L_s = \emptyset$, for some integer $s \geq 2$. It follows from the equalities (5.5) that there is an inclusion of subspaces

$$\mathfrak{L}_s = \operatorname*{Span}_K \{[[[[x_{i_1}, x_{i_2}], x_{i_3}] \cdots], x_{i_s}]\} \subset J.$$

This implies that for $q \geq s$ every nested Lie monomial $w_q = [[[[x_{i_1}, x_{i_2}], \cdots x_{i_s}], \cdots], x_{i_q}] \in J$, i.e. $\mathfrak{g}$ is a nilpotent Lie algebra of nilpotency class $\leq s - 1$. Suppose that s is the minimal integer with $N_s = \emptyset$. Then $\mathfrak{g}^{s-1} \neq 0$, and $\mathfrak{g}^s = [[\mathfrak{g}^{s-1}, \mathfrak{g}] = 0$, hence $\mathfrak{g}$ is a nilpotent Lie algebra of nilpotency class $s - 1$, see Definition 5.10.

Part (1) implies straightforwardly part (2).

(4). Suppose that N is a finite set. It follows from Definition 5.9 and part (1) that N is connected. By part (2) N is connected, whenever it is an infinite set.

(3) Assume that N has finite order $|N| = d$ and $m = \max\{|u| \mid u \in N\}$. Without loss of generality we may assume $m \geq 2$. The set N is connected, hence $N_s = N \bigcap L_s \neq \emptyset, 1 \leq s \leq m$. The equality $N = \bigcup_{1 \leq s \leq m} N_s$ implies

$$d = |N| = |X| + \sum_{2 \leq s \leq m} |N_s| \geq |X| + m - 1,$$

therefore $m \leq d + 1 - |X|$. It follows from part (1) that $\mathfrak{g}$ is a nilpotent Lie algebra of nilpotency class $m \leq d - |X| + 1$. $\square$

Corollary 5.12. *Let (N, W) be a Lyndon pair, where N is finite. Suppose N is not a connected set of Lyndon words. Then (i) the class $\mathfrak{C}(X, W)$ does not contain the enveloping algebra $U = U\mathfrak{g}$ of any Lie algebra $\mathfrak{g}$. (ii) In particular, $[W]$ is not a Lyndon-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$ of $Lie(X)$.*

6. MONOMIAL LIE ALGEBRAS DEFINED BY LYNDON WORDS

As usual, $X = \{x_1, \cdots, x_n\}$ is a finite set and W denotes an antichain of Lyndon words in X^+. Our goal is to find explicitly (in terms of generators and relations) classes $\mathfrak{C}(X, W)$ containing enveloping algebras $U\mathfrak{g}$ of prescribed global dimension. As a first step in this direction we introduce and study the *monomial Lie algebras defined by Lyndon words*.

6.1. **Basic definitions and first properties of Monomial Lie algebras.**

Definition 6.1. We say that $\mathfrak{g} = \mathrm{Lie}(X)/J$ is *a monomial Lie algebra defined by Lyndon words*, or shortly, *a monomial Lie algebra*, if J is a Lie ideal generated by the Lie elements $[W] = \{[w] \mid w \in W\}$, where W is an antichain of Lyndon words. If moreover, the set of Lie monomials $[W]$ is a Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$ (see Definition 6.7) we shall refer to $\mathfrak{g}$ as *a standard monomial Lie algebra* and denote it by $\mathfrak{g}_W$.

Definition 6.7 is one of the several equivalent definitions of a Gröbner-Shirshov basis of a Lie ideal in $\mathrm{Lie}(X)$ (or shortly a GS basis). Proposition 6.12 gives several equivalent conditions each of which can be used for a definition. For now we shall use the fact that $[W]$ is a Gröbner-Shirshov basis of the Lie ideal J *iff* the set $[W]_{ass}$ considered as "associative" elements of $K\langle X\rangle$ is a *Gröbner basis of the associative ideal* $I = (J_{ass})$, w.r.t the ordering $\prec$ on X^*, see Proposition 6.12. Our first examples are classical.

Example 6.2.   (1) The free-nilpotent Lie algebra $\mathfrak{L}(m+1)$, $m \geq 2$, generated by X, is a standard monomial Lie algebra defined by Lyndon words, see Corollary 7.3.

(2) The standard Filiform Lie algebra $\mathcal{L}_m$, of dimension $m+1$ and nilpotency class m, see Subsec 7.4.

Each monomial Lie algebra defined by Lyndon words, $\mathfrak{g} = Lie(X)/([W])_{Lie}$ is graded by length, and is also positively $\mathbb{Z}^n$-graded. Moreover, the two sided ideal $I = (J_{ass})$ of $K\langle X\rangle$ is generated by homogeneous elements and is also $\mathbb{Z}^n$-graded, so the enveloping algebra $U = U\mathfrak{g} = K\langle X\rangle/I$ is (non-negatively) $\mathbb{Z}^n$-graded.

Definition-Notation 6.3. A monomial $w \in X^*$, has (*a non-negative*)*multi-degree* $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$, $\alpha_i \geq 0$, if w, considered as a commutative term, can be written as $w = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. In this case we write $\deg(w) = \alpha$. In particular, the unit $1 \in X^*$ has multi-degree $\mathbf{0} = (0, \cdots, 0)$, and $\deg(x_1) = (1, 0, \cdots, 0)$, $\deg(x_2) = (0, 1, \cdots, 0), \cdots, \deg(x_n) = (0, 0, \cdots, 1)$. For each $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{Z}^n, \alpha_i \geq 0$, denote by X_α the set $X_\alpha = \{w \in \langle X\rangle \mid \deg(w) = \alpha\}$. Then the free monoid X^* is naturally $\mathbb{Z}^n$-graded:

$$X^* = \bigsqcup_{\alpha \in \mathbb{Z}^n, \alpha \geq \mathbf{0}} X_\alpha, \quad X_{\mathbf{0}} = \{1\},\ X_\alpha.X_\beta \subseteq X_{\alpha+\beta}.$$

The set of Lyndon words L inherits this grading canonically, it splits as a disjoint union

$$L = \bigsqcup_{\alpha \in \mathbb{Z}^{\times n}, \alpha > \mathbf{0}} L_\alpha,\ \text{where}\ L_\alpha = L\bigcap X_\alpha.$$

The free associative algebra generated by X is also canonically $\mathbb{Z}^{\times n}$-graded:

$$U = K\langle X\rangle = \bigoplus_{\alpha \in \mathbb{Z}^{\times n}} U_\alpha,\ \text{where}\ U_\alpha = \mathrm{Span}\, X_\alpha,\ \alpha \in \mathbb{Z}^{\times n}.$$

For example, if $X = \{x < y\}$, then

$$L_{(3,3)} = \{xxxyyy < xxyxyy < xxyyxy\} = \{xxxyyy \succ xxyxyy \succ xxyyxy\}$$

Recall that a graded Lie algebra is a Lie algebra endowed with a gradation compatible with the Lie bracket. The free Lie algebra is naturally $\mathbb{Z}^n$-graded:

$$\mathfrak{L} = \text{Lie}(X) = \bigoplus_{\alpha \in \mathbb{Z}^{\times n}} \mathfrak{L}_\alpha, \ \ \mathfrak{L}_\alpha = \text{Span}\{[u] \mid u \in L_\alpha\}, \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n, \ \alpha_i \geq 0\}$$

If $w \in L$ has multi-degree $\deg(w) = \alpha$, then the standard (right) bracketing $[w]$, considered as an associative polynomial in $\mathfrak{U}$ is a linear combination of words $u \in X_\alpha, u \preceq w$, so $[w] \in \mathfrak{L}_\alpha$, and its highest monomial w.r.t. $\prec$ is precisely $\overline{[w]} = w$. Sometimes we shall also write

$$L(w) := L_\alpha, \ \ L(\preceq w) := \{u \in L(w), u \preceq w\}, \ \ L(\prec w) := \{u \in L(w), u \prec w\}.$$

For example, $w = xxxyyy \in L_{(3,3)}$, $L(\prec w) = \{xxyxyy \succ xxyyxy\}$. Let W be an antichain of Lyndon monomials in X^+. Then the Lie ideal $J = ([W])_{Lie}$ is multi-graded, $J = \bigoplus_\alpha J_\alpha$, where $J_\alpha = J \bigcap \mathfrak{L}_\alpha, \alpha \in \mathbb{Z}^n, \alpha \geq \mathbf{0}$.

Remark 6.4. Recall that if $\mathcal{I} = (W)$ is a monomial ideal in $K\langle X \rangle$, where W is an arbitrary antichain of monomials ($W \bigcap X = \emptyset$), then $\mathcal{I}$ is X^+-graded. If $f = \sum c_a a \in \mathcal{I}$, with $a \in X^+, c_a \in K^\times$ then every monomial a in this presentation is also contained in $\mathcal{I}$, or equivalently, $w \sqsubseteq a$ for some $w \in W$. In contrast, a monomial Lie ideal $J = ([W])_{Lie}$ generated by Lyndon Lie monomials may contain a Lie element $g \in J$, where $g = \sum c_a[a], \ a \in L, \ c_a \in K^\times$, such that $[a]$ is not in J for some a. See the following example. Moreover, the reduced Gröbner-Shirshov Lie basis of a monomial Lie ideal $J = ([W])$ in $\text{Lie}(X)$ may contain a Lie element $f \in J$, which is not a Lie monomial, as shows Example 8.4

Example 6.5. Let $W = \{xxy\}$, $J = ([xxy])_{Lie}$. Then, clearly, $g = [[[xxy], y], y] \in J$. Consider the equalities:

$$\begin{aligned} f = & \ [[xxy], y] = [x, [xyy]] \in J, \\ g = & \ [f, y] = [[[xxy], y], y] = [[x, [xyy]], y] = [[xy], [xyy]] + [x, [xyyy]] \\ = & \ [xyxyy] + [xxyyy] \in J. \end{aligned}$$

So $[xyxyy] + [xxyyy] = g \in J$, but the Lie monomials $[xxyyy]$ and $[xyxyy]$ are not in J. Note that

$$\begin{aligned} xxyyy = & \ \overline{[xxyyy]} \succ \overline{[xyxyy]} = xyxyy \\ & \text{hence } \overline{g} = \overline{[[[xxy], y], y]} = xxyyy, w = xxy \sqsubset \overline{g}. \end{aligned}$$

6.2. Gröbner-Shirshov bases of monomial Lie ideals, preliminaries.
In this subsection we present in terms of Lyndon words some notions and results from Shirshov's theory based on Shirshov's *regular words*. Recently some authors use the so called "Lyndon-Shirshov words", which are analogous to Lyndon words (in the classical sense), but are defined differently and the so called "Lyndon-Shirshov words" can not be used directly (without certain modification), for our theory based on the original notion of Lyndon word. By convention, throughout the paper we use the original definition of Lyndon words, following [26] and [21].

Our goal is to simplify the general procedure for finding Gröbner-Shirshov bases in the special case of $\mathbb{Z}^n$-graded monomial Lie ideals $J = ([W])_{Lie}$ in $\text{Lie}(X)$. For the purpose we combine some known facts about Gröbner-Shirshov bases with results proven in [3], [21], [17] and especially with our combinatorial results in

this paper. As a first step we yield Proposition 6.12, which can be used in some special cases to decide *straightforwardly* (avoiding computations) whether $[W]$ is the Gröbner-Shirshov basis of the monomial Lie ideal $J = ([W])_{Lie}$. We next prove Propositions 6.14, and 8.5 which (in various cases) can be and will be used to find straightforwardly the reduced Gröbner-Shirshov basis of the Lie ideals $J = ([W])_{Lie}$ (and avoid the need of heavy computations prescribed by the standard procedure). As an application, in Section 9 we classify all 2-generated AS-regular algebras U of global dimension 6 and 7 which occur as enveloping algebras $U = U\mathfrak{g}$, of standard Lie monomial algebras $\mathfrak{g} = \mathrm{Lie}(x, y)/([W])$, (in this case $[W]$ is a Gröbner-Shirshov Lie basis).

Recall that if $f \in K\langle X\rangle$ is a nonzero polynomial, then its highest monomial with respect to the degree-lexicographic order $\prec$ on X^* (with $x_1 \succ x_2 \succ \cdots \succ x_n$) is denoted by $\overline{f}$. One has $f = \alpha\overline{f} + \sum_{1 \leq i \leq m} \alpha_i u_i$, where $\alpha, \alpha_i \in K$, $\alpha \neq 0$, $u_i \in X^*, u_i \prec \overline{f}$.

Let $u \in L$ be a Lyndon word of length ≥ 2. We know that the right bracketing $[u] = [u]_r$ and the left bracketing $[u]_l$ are Lie elements which (considered as associative elements) are monic polynomials with equal highest terms

$$\overline{[u]} = u = \overline{([u]_l)}.$$

More generally, consider a new bracketing ("rebracketing") $[u]_*$ of u, that is a Lie monomial, obtained by writing Lie brackets in some way. It is called *a regular bracketing*, or *a regular Lie monomial* of u if it is a monic (associative) polynomial with highest monomial $\overline{([u]_*)} = u$.

Suppose $\tau, u \in L$, and u is a proper subword of $\tau, \tau = aub, ab \in X^+$. As usual, $[u]$ denotes the (right) standard bracketing of u. It follows from Shirshov's results that there exists a special regular bracketing denoted $(\tau)_u$ of τ which respects the bracketing $[u]$ and (considered as associative polynomial) has highest monomial $\overline{(\tau)_u} = \tau$. We shall refer to $(\tau)_u$ as *a regular bracketing of τ respecting $[u]$* and denote it also by $(a[u]b) := (\tau)_u$. Clearly, if u is a proper right segment: $\tau = au$, one has $(a[u]) = [au]$. Suppose $\tau \in L_\alpha$, then its standard bracketing $[\tau] \in \mathrm{Span}[L_\alpha]$, and the regular bracketing $(a[u]b)$ is a Lie element satisfying

$$\text{(6.1)} \qquad \begin{aligned} \overline{(a[u]b)} &= aub = \tau, \\ (a[u]b) &= [\tau] + \Sigma_{v \prec \tau} c_v[v] \in \mathrm{Span}[L_\alpha], \ c_v \in K, \ \forall v. \end{aligned}$$

Moreover, the regular bracketing $(a[u]b)$ is an element of the Lie ideal $([u])_{Lie}$ of $\mathrm{Lie}(X)$, generated by $[u]$.

Lemma 6.6. *Let ω be a Lyndon word, $\omega \in L_\alpha$. Let $\omega = uv$, where u, v are Lyndon words. If $[u]_*$, and $[v]_*$ are regular bracketings (regular Lie monomial) of u and v, then $([u]v)_* := [[u]_*, [v]_*]$ is a regular bracketing of $\omega = uv$ respecting $[u]_*$ (and $[v]_*$).*

Proof. It follows from the hypothesis that $u < uv = \omega < v < vu$. Recall that for any two elements of the free associative algebra, $f, g \in K\langle X\rangle$ one has

$$\text{(6.2)} \qquad\qquad\qquad \overline{f.g} = (\overline{f})(\overline{g}).$$

The following relations are considered in the free associative algebra:

$$[[u]_*, [v]_*] \quad = ([u]_*).([v]_*) - ([v]_*).([u]_*)$$

$$\overline{([u]_*).([v]_*)} \quad = (\overline{([u]_*)}).(\overline{([v]_*)})$$

$$= u.v < v.u = (\overline{([v]_*)})(\overline{([u]_*)}) = \overline{([v]_*).[u]}$$

Recall that for monomials of the same length in X^+ we have $a < b$ *iff* $a \succ b$, so $\overline{[u].([v]_*)} = u.v \succ v.u = \overline{([v]_*).[u]}$, and therefore

(6.3) $$\overline{[[u]_*, [v]_*]} = uv,$$

as desired. Clearly this bracketing of uv respects both $[u]_*$ and $[v]_*$. $\qquad\square$

Definition 6.7. Let W be an antichain of Lyndon words. The set of Lie monomials $[W]$ is *a Gröbner-Shirshov basis of the Lie ideal* $J = ([W])_{Lie}$ if every nonzero Lie element $f \in J$, can be written as

$$f = c[u] + \sum_{v \prec u} c_v[v] \in \mathrm{Span}[L],$$

where $u = \overline{f}, c \in K^\times$, and $u = awb$, for some $w \in W, a, b \in X^*$. Note that in this case $[W]$ is *the reduced Gröbner-Shirshov basis* of J, and $[W]_{ass}$ is *the reduced Gröbner basis* of the associative ideal $I = (J_{ass})$ in $K\langle X\rangle$.

Definition 6.8. Let W be an antichain of Lyndon words, $W \subset L \setminus X$.

(1) [2, 3] *A 2-prechain* is a word $\omega = hbt$, where $u = hb \in W$, $v = bt \in W$, and $h, t \in X^+$. A 2-prechain ω is a *2-chain* if no proper left segment of ω is a 2-prechain. Note that every 2-prechain ω is a Lyndon word, see for example [17], Proposition 6.4.

(2) Suppose $\omega = hbt$ is *a 2-prechain*, where $u = hb \in W$, $v = bt \in W$, and $h, t \in X^+$. The difference of Lie elements

$$(u, v)_\omega = ([u]t) - (h[v]) = ([u]t) - [\omega]$$

is called *a Lie composition of overlap*, or also *a Lie composition of intersection* (in the sense of Shirshov).

(3) Suppose $u, \omega \in L$, with $\omega = aub$. *The composition of inclusion* (in the sense of Shirshov), corresponding to the pair u, ω, is defined as

$$(\omega, u)_\omega := [(a[u]b) - \omega].$$

Note that by assumption W is *an antichain* of Lyndon words, that is no $u \in W$ is a proper sub-word of some $v \in W$. Thus compositions of inclusion, in the sense of Shirshov, will not occur in the initial step of the procedure for finding a Gröbner-Shirshov basis of the Lie ideal $([W])_{Lie}$. However, in the process of resolving a composition of overlap, compositions of inclusion may also occur.

Remark 6.9. (1) The Lie element $(u, v)_\omega = ([u]t) - [\omega]$ is in the ideal J. Indeed, $u, v \in W$, thus $([u]t) \in J$ and $(h[v]) = [hv] = [\omega] \in J$.

(2) One has $\overline{([u]t)} = \omega = \overline{([\omega]}$, hence, either $(u, v)_\omega = 0$, or

$$0 \neq (u, v)_\omega, \text{ and } \overline{(u, v)_\omega} \prec \omega.$$

(3) Suppose a two-prechain $\omega \in L_\alpha$, (notation as above). Then each of the Lie elements $([u]t)$, and $(h[v]) = [\omega]$ is in the graded component $\mathfrak{L}_\alpha = \text{Span}[L_\alpha]$. Therefore $(u,v)_\omega = ([u]t) - [\omega]$ is a homogenous Lie element of multi-degree α, or $(u,v)_\omega = 0$. More precisely, $(u,v)_\omega \in \text{Span}[L_\alpha(\prec \omega)]$. In particular, if ω is the minimal element of L_α (w.r.t. $\prec$) then, clearly, $(u,v)_\omega = ([u]t) - [\omega] = 0$, and the composition is solvable

Definition 6.10. Suppose ω is a 2-prechain of multi-degree α, so $\omega \in L_\alpha$. The composition of overlap $(u,v)_\omega$ is *solvable mod* $[W]$ if either $(u,v)_\omega = 0$, or there exist an explicit presentation

$$(6.4) \qquad (u,v)_\omega = c_0(a_0[w_0]b_0) + c_1(a_1[w_1]b_1) + \cdots + c_j(a_j[w_j]b_j),$$

where (i) $a_j w_j b_j \prec \cdots \prec a_1 w_1 b_1 \prec a_0 w_0 b_0 \prec \omega$ are words in L_α; (ii) $w_i \in W$, and $(a_i[w_i]b_i)$ is a regular bracketing respecting the right standard bracketing $[w_i]$, $\forall i,\ 0 \le i \le j$.

One can find effectively whether a presentation of the form (6.4) exists. If not, there exists a unique presentation

$$(u,v)_\omega = c_0[v_0] + \cdots + c_s[v_s],$$

where $c_0 \ne 0$, $\omega \succ v_0$, and $v_0 \succ \cdots \succ v_s$ are Lyndon atoms in $N \bigcap L_\alpha$. In this case W is not the obstruction set of the associative ideal $I = (J_{ass})$, neither is N the set of Lyndon atoms for U.

The next corollary follows from Anik's results, see [3], 2.*Diamond Lemma revisited* (pp 646-647), part (ii) follows from [21].

Corollary 6.11. *If all compositions on 2-chains are solvable, then the compositions on 2-prechains are also solvable, so (i) $[W]_{ass}$ is a Gröbner basis of the associative ideal $I = ([W]_{ass})$ in $K\langle X \rangle$; (ii) $[W]$ is a Gröbner-Shirshov basis of the Lie ideal $([W])_{Lie}$.*

The following statement is a consequence from well-known results on Gröbner bases, Corollary 6.11, and our assumption that W is an antichain of Lyndon words.

Proposition 6.12. *Let (N, W) be a Lyndon pair, let $J = ([W])_{Lie}$ be the Lie ideal of $Lie(X)/J$, generated by $[W]$, let I be the two sided ideal $I = ([W]_{ass})$ in the free associative algebra $K\langle X \rangle$. Suppose $\mathfrak{g} = Lie(X)/J$, so $U = U\mathfrak{g} = K\langle X \rangle/I$ is the enveloping algebra of $\mathfrak{g}$. The following conditions are equivalent.*

(1) *The set of Lie monomials $[W]$ is a Gröbner-Shirshov basis of the Lie ideal J, see Definition 6.7.*
(2) *For every 2-chain $\omega = abc, u = ab \in W, v = bc \in W$, the Lie composition of overlap $(u,v)_\omega$ is solvable mod $[W]$.*
(3) *The set*
$$[N] = \{[u] \mid u \in N\}$$
is a K-basis of $\mathfrak{g}$.
(4) *The set $[W]_{ass}$ (considered as a set of "associative" elements of $K\langle X \rangle$) is a Gröbner basis of the (associative) ideal $I = ([W]_{ass})$ (w.r.t. the ordering $\prec$ on X^*).*
(5) *The enveloping algebra $U\mathfrak{g} = K\langle X \rangle/I$ is a graded algebra whose set of obstructions is W, that is every $f \in I$ satisfies $\overline{f} = awb$, for some $w \in W, a, b \in X^*$.*

(6) *The set of monomials*

$$\mathfrak{N} = \{l_1 l_2 \cdots l_s \mid s \geq 1,\ l_1 \geq l_2 \geq \cdots \geq l_s,\ l_i \in N,\ 1 \leq i \leq s\} \bigcup \{1\}$$

is a normal K- basis of $U\mathfrak{g}$.

In this case N is a connected set of Lyndon atoms.

Definition-Notation 6.13. Let (N, W) be a Lyndon pair. We introduce the following numerical invariants.

$$(6.5) \qquad \begin{aligned} m &= m(N) := \max\{|u| \mid u \in N\} \\ c &= c(W) := \min\{|\omega| \mid \omega \text{ is a two-chain on } W = W(N)\}. \end{aligned}$$

The following Proposition gives *some purely combinatorial conditions* sufficient for $[W]$ to be a Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$.

Proposition 6.14. *In notation and assumptions of Proposition 6.12. Let (N, W) be a Lyndon pair.*

(1) *Suppose $u = ab, v = bc \in W$, and $\omega = abc$ is a 2-chain on W of multi-degree $\deg \omega = \alpha$, or equivalently, $\omega \in L_\alpha$. If $(L_\alpha(\prec \omega)) \bigcap N = \emptyset$, then the composition of overlap $(u, v)_\omega$ is solvable mod $[W]$. Clearly, this is so, whenever $m(N) < |\omega|$.*

(2) *Suppose $u \in W$, and $\omega = aub \in L_\alpha$. The composition of inclusion corresponding to the pair ω, u, $(\omega, u)_\omega := [\omega] - (a[u]b)$, is solvable mod $[W]$, whenever $(L_\alpha(\prec \omega)) \bigcap N = \emptyset$. In particular, this is so if $m(N) < |\omega|$.*

(3) *Suppose $|N| = d < \infty$, N is connected and some of the following conditions is in force:*

    (a) *$m(N) < c(W)$ that is each 2-chain ω on W has length $|\omega| > |u|, \forall u \in N$;*

    (b) *Every 2-chain ω on W is a minimal word in its $\mathbb{Z}^n$-component, L_α, that is $L_\alpha(\prec \omega) = \emptyset$.*

    (c) *Every $u \in N$, belongs to some $\mathbb{Z}^n$-component, L_α, which does not contain any 2-chain ω.*

    Then $[W]$ is a Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$ in $Lie(X)$, the Lie algebra $\mathfrak{g} = \mathfrak{g}_W$ is a standard monomial Lie algebra with $\dim \mathfrak{g} = d$ and a K-basis $[N]$. In this case the enveloping algebra $U = U\mathfrak{g}$ belongs to the class $\mathfrak{C}(X, W) = \mathfrak{C}(X, N)$ and is an Artin-Schelter regular algebra of $\mathrm{gl\,dim}\, U = d$.

7. Some classical monomial Lie algebra defined by Lyndon words

7.1. The free nilpotent Lie algebra $\mathfrak{L}(m)$ of nilpotency class m is a standard monomial Lie algebra defined by Lyndon words.

Notation 7.1. Denote by J_{m+1} the Lie ideal of $\mathrm{Lie}(X)$ *generated* by all m-(left) nested Lie monomials f_{m+1} (length $m + 1$), see (7.1).

The definition of J_{m+1} and (5.5) imply
$$(7.1)$$
$$\begin{aligned} J_{m+1} &\supseteq \mathrm{Span}\{f_{m+1} = [[[[x_{i_1}, x_{i_2}], \cdots], x_{i_m}], x_{i_{m+1}}] \mid x_{i_s} \in X,\ 1 \leq s \leq m+1\} \\ &= \mathrm{Span}[L_{m+1}]. \end{aligned}$$

Moreover,

$$(7.2) \qquad\qquad\qquad J_m \supseteq J_{m+1} \supseteq \cdots, \quad m \geq 2.$$

Remark 7.2. Suppose that J is a Lie ideal in $\mathrm{Lie}(X)$ and $\mathfrak{g} = \mathrm{Lie}(X)/J$ is a graded *nilpotent Lie algebra of nilpotency class m.* Then all m-left nested Lie monomials f_{m+1} of degree $\geq m+1$ are zero in $\mathfrak{g}$.

The Lie algebra $\mathfrak{L}(m) = \mathrm{Lie}(X)/J_{m+1}$ is the *free nilpotent Lie algebra of degree m,* see [19].

Recall that L_s denotes the set of all Lyndon words of length $s, s = 1, 2, 3, \cdots$. Denote by $L(m)$, $m \geq 2$, the set of all Lyndon words of length $\leq m$. Clearly, $L(m)$ is closed under taking Lyndon subword, so **C1** and **C2** are in force. Let $W(m)$ be (the unique) antichain of Lyndon words whose set of Lyndon atoms is $L(m)$, so

(7.3) $\begin{aligned} L(m) \quad &= \{u \in L \mid |u| \leq m\} = \bigcup_{1 \leq s \leq m} L_s, \\ W(m) \quad &:= W(L(m)), \text{ and } (L(m), W(m)) \text{ is a Lyndon pair} \end{aligned}$

Consider the Lie algebra

(7.4) $\mathfrak{F}_m = \mathrm{Lie}(X)/J(m), \quad J(m) := ([W(m)])_{Lie},$

where $J(m)$ denotes the Lie ideal of $\mathrm{Lie}(X)$ generated by the set of Lyndon-Lie monomials

$$[W(m)] = \{[w] \mid w \in W(m)\}.$$

$$W(m) \bigcap L_s = \emptyset, \ 1 \leq s \leq m, \quad W(m) \supset L_{m+1}$$

Corollary 7.3. *In notation as above the following conditions hold.*
   (1) *The set $[W(m)]$ is a Lyndon-Shirshov Lie basis of the Lie ideal $J(m)$, so $\mathfrak{F}_m$ is a standard monomial Lie algebra, see Definition 6.1*
   (2) *The Lie ideal J_{m+1} generated by all m-nested Lie monomials (Notation 7.1) coincides with the ideal $J(m)$, generated by the Lyndon-Lie monomials $[W(m)]$, $J_{m+1} = J(m)$.*
   (3) *The free nilpotent Lie algebra $\mathfrak{L}(m)$ of nilpotency class m is isomorphic to the standard monomial Lie algebra $\mathfrak{F}_m$, given in (7.4).*
   (4) *Moreover, the enveloping algebra $U\mathfrak{L}(m)$ is standard finitely presented as*

$$U\mathfrak{L}(m) \cong K\langle X\rangle/([W(m)]_{ass}) = U\mathfrak{F}_m,$$

   where the finite set $[W(m)]_{ass}$ is an associative Gröbner basis for the ideal $I = ([W(m)]_{ass})$ of $K\langle X\rangle$.
   (5) *The class $\mathfrak{C}(X, W(m))$ contains an AS-regular algebra of global dimension $d = |L(m)|$, namely the enveloping algebra $U(m) = U\mathfrak{L}(m)$.*

Proof. (1) Every 2-chain ω on $W(m)$ has length $|\omega| \geq m+2$, and the corresponding composition is solvable, since there are no atoms of length $\geq m+1$. Hence the set $[W(m)]$ is a Lyndon-Shirshov basis of the Lie ideal $J(m) = ([W(m)])_{Lie}$. (2) Clearly, every $w \in W(m)$ has length $q = |w| \geq m+1$, hence $[w] \in [L_q] \subset J_q \subseteq J_{m+1}$ (by (7.1) and (7.2)). Since the set of generators $\{[w] \mid w \in W(m)\}$ for the Lie ideal $J(m)$ is contained in J_{m+1}, one has

$$J(m) \subseteq J_{m+1}.$$

But $L_{m+1} \subset W(m)$, and (5.5) gives

$$Span_K\{[u] \mid u \in L_{m+1}\} = \underset{K}{\mathrm{Span}}\{[[\cdots[[x_{i_1}, x_{i_2}], \cdots], x_{i_m}], x_{i_{m+1}}]\},$$

so $J_{m+1} \subseteq J(m)$, and therefore $J(m) = J_{m+1}$. (3) The equality of the two ideals implies straightforwardly

$$\mathfrak{F}_m = \mathrm{Lie}(X)/J(m) \cong \mathrm{Lie}(X)/J_{m+1} = \mathfrak{L}(m),$$

so $\mathfrak{F}_m$ is isomporphic to the free nilpotent Lie algebra of of degree m (or nilpotency class m) see Remark 7.2. (3) implies (4). (5) follows straightforwardly from Proposition 5.6. $\qquad\square$

Remark 7.4. The free nilpotent algebra $\mathfrak{L}(m)$ with g generators $x_1, \cdots, x_n$ and nilpotency class $\mathfrak{N}_m$, has a K-basis

$$\{[w] \mid w \in \bigcup_{1 \le s \le m} L_s\},$$

and dimension

$$\dim \mathfrak{L}(m) = |L(m)| = \sum_{1 \le i \le m} M(g, i),$$

where $|X| = g$, $M(g, i)$ is the necklace polynomials, or (Moreau's) necklace-counting function

$$g^n = \sum_{d \mid n} dM(\alpha, d).$$

One can also use the Witt's formula

$$|L_n| = \frac{1}{n} \sum_{d \mid n} \mu(d)(\alpha)^{n/d}, \text{ where } \mu \text{ is the Möbius function.}$$

The numbers of binary Lyndon words of each length, starting with length one, form the integer sequence

$$2, 1, 2, 3, 6, 9, 18, 30, 56, 99, 186, 335, \cdots.$$

Corollary 7.5. *Let J be a Lie ideal in $Lie(X)$, $\mathfrak{g} = Lie(X)/J$, $U = U\mathfrak{g} = K\langle X \rangle / I$ (in usual notation), and (N, W) is the corresponding Lyndon pair for the enveloping algebra U. Suppose the set of Lyndon atoms $N = N(W)$ is finite, and $m > 2$ is the minimal integer, such that N contains atoms of length m, but there are no atoms of length $m + 1$. Then all m-nested Lie monomials f_{m+1} belong to J, so $J_{m+1} \subseteq J$, and therefore the algebra $\mathfrak{g} = Lie(X)/J$ is nilpotent of degree m. In particular, $m = \max\{|u| \mid u \in N\}$.*

Remark 7.6. Let $\mathfrak{g} = \mathrm{Lie}(X)/J$, where J is a Lie ideal in $\mathrm{Lie}(X)$, and $U = U\mathfrak{g} = K\langle X \rangle / I$ be its enveloping algebra. Suppose (N, W) is the Lyndon pair for the enveloping algebra U, so W is the set of obstructions for U. In general, this datum does not determine uniquely the Gröbner -Shirshov basis of the Lie ideal J. It is possible that $[W]$ is a Gröbner-Shirshov basis of the Lie ideal $T = ([W])_{Lie}$, and W is the obstructions set of J, but $J \ne T$. However in some "extreme cases", see Corollaries 7.7 and 7.11 the set of obstructions determines uniquely the defining relations of U, and the ideal I.

Corollary 7.7. *Let J be a Lie ideal in $Lie(X)$, $\mathfrak{g} = Lie(X)/J$, $U = U\mathfrak{g} = K\langle X \rangle / I$ (in usual notation), and suppose the Lyndon pair for the enveloping algebra U is exactly $(N, W) = (L(m), W(m))$, 7.3. Then $J = ([W(m)])_{Lie}$*

7.2. **The $\mathbb{Z}^2$-graded Filiform Lie algebras, basic facts.** Filiform Lie algebras have been actively studied and numerous works on their algebraic and geometric properties can be found in the literature. As an application of our approach (and for completeness) we shall show that each of the classical Filiform algebras $\mathcal{L}_n$ and $\mathcal{Q}_n$ has a canonical presentation as a naturally $\mathbb{Z}^2$-graded standard monomial Lie algebra defined by Lyndon words. Similar presentation is given for the infinite Filiform Lie algebras $\mathfrak{M}_0$, but, of course, its set of defining Lyndon-Lie monomial relations is infinite.

Recall that *a finite dimensional filiform Lie algebra* is a nilpotent Lie algebra $\mathfrak{F}$ whose nil index is maximal, that is $\mathfrak{F}$ has nil index n, (nilpotency class n), where $\dim \mathfrak{F} = n + 1$. It is well-known that, *up to isomorphism*, there are two families of filiform Lie algebras of rank 2.

$$\{\mathcal{L}_n : \ n \geq 2\} \ \text{and} \ \{\mathcal{Q}_n : \ n \geq 5, \ n \ \text{odd}\}.$$

Both algebras $\mathcal{L}_n$ and $\mathcal{Q}_n$ have dimension $n + 1$. Each of them has a basis $X_0, X_1, \cdots, X_n$, their nonzero brackets of basis elements are given below.

$$(7.5) \qquad \mathcal{L}_n : \qquad [X_0, X_j] = X_{j+1}, \ \text{for} \ 1 \leq j \leq n - 1.$$

$$(7.6) \qquad \begin{aligned} \mathcal{Q}_n : \quad & [X_0, X_j] = X_{j+1}, \ \text{for} \ 1 \leq j \leq n - 1 \\ & [X_j, X_{n-j}] = (-1)^i X_n, \ \text{for} \ 1 \leq j \leq n - 1 \\ & n \geq 3, \ n \ \text{is odd}. \end{aligned}$$

Infinite Filiform Lie algebra $\mathfrak{M}_0$, [10], is presented via generators $e_i, i \geq 1$, and relations $[e_1, e_i] = e_{i+1}, \forall i \geq 2$. This algebra was first studied in [32], where was shown that its enveloping U is an integral domain with a subexponential growth, not bounded by any polynomial.

Note that if $\mathfrak{g}$ is a graded Lie algebra generated by generators of degree 1, $\mathfrak{g}_1 = Span X$, then the data $\dim \mathfrak{g} = n + 1$; $\mathfrak{g} \in \mathfrak{N}_n$ implies

$$(7.7) \qquad (i) \ \ 2 = \dim \mathfrak{g}_1 = |X|; \quad (ii) \ \ \dim \mathfrak{g}_k = 1, \ 2 \leq k \leq n.$$

At the same time we have a result on "the minimality of the order $|W|$ of W independently of the context of Lie algebras. Recall that if (N, W) is a Lyndon pair in X, with $|N| = d$, Theorem I implies that W is a finite set with $d - 1 \leq |W| \leq \frac{d(d-1)}{2}$. Theorem 4.2 shows that *if N is connected* then the pure *numerical datum*

$$(7.8) \qquad \qquad |N| = d, \quad |W| = d - 1$$

determines uniquely (up to isomorphisms of monomial algebras A_W): (i) the order of X, $|X| = 2$, and (ii) the Lyndon pair (N, W) in X^+ with $|W| = d - 1$:

$$(7.9) \qquad \begin{aligned} N \ \ &= \{x < xy < xyy < \cdots < xy^{d-3} < xy^{d-2} < y\}. \\ W \ \ &= \{xy^i xy^{i+1} \mid 0 \leq i \leq d - 3\} \bigcup \{xy^{d-1}\}. \end{aligned}$$

Corollary 7.8. *Let $X = \{x < y\}$, let $d \geq 2$ be an integer, and let (N, W) be the Lyndon pair (7.9). Then*

(1) *The set of Lie monomials $[W]$ is a Lyndon-Shirshov Lie basis of the ideal $J = ([W])_{Lie}$, so $[N]$ is a K-basis of $\mathfrak{g}$, and W is the set of obstructions of the enveloping algebra $U\mathfrak{g}$.*

(2) *The 2-generated monomial Lie algebra $\mathfrak{g} = Lie(X)/J$ is isomorphic to the standard filiform Lie algebra $\mathcal{L}_{d-1}$ of dimension d and nilpotency class $d-1$.*

(3) *In particular, the class $\mathfrak{C}(X, W)$ contains the enveloping $U = U\mathfrak{g}$ which is an AS-regular algebra of global dimension d.*

Proof. Each two-chain ω on W has multi-degree $\deg(\omega) = \alpha = (k, j)$, where $k \geq 2$. But $L_\alpha \bigcap N = \emptyset$, hence by Proposition 6.14 part (3c) $[W]$ is a Lyndon-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$. This implies that $[N]$ is a K basis of $\mathfrak{g}$, and W is the set of obstructions of the enveloping algebra $U = K\langle X\rangle/([W])_{ass}$. Clearly, $\mathfrak{g} \cong \mathcal{L}_d$, the standard filiform Lie algebra of dimension $d + 1$, and nilpotency class d. $\qquad\square$

7.3. **Proof of Theorem III.**

Proof of Theorem III. *(1). Theorem 4.2 proves the equivalence of conditions (1.a), (1.b), (1.c). In this case the Lyndon pair (N, W) is given by (7.9), hence the hypothesis of Corollary 7.8 is satisfied. This implies that the set of Lie monomials $[W]$ is a Lyndon-Shirshov Lie basis of the ideal $J = ([W])_{Lie}$ in $Lie(X)$ so $\mathfrak{g} = Lie(X)/([W])_{Lie} = \mathfrak{g}_W$ is isomorphic to the standard filiform Lie algebra $\mathcal{L}_{d-1}$ of dimension d and nilpotency class $d - 1$. In particular, the class $\mathfrak{C}(X, W)$ contains the enveloping $U = U\mathcal{L}_{d-1}$ which is an Artin-Schelter regular algebra of global dimension d. This proves (1)*

(2). The equivalences of conditions (2.a), (2.b) and (2.c) follow from [17], Theorem B (2). In this case $n = d$ and $W = \{x_i x_j \mid 1 \leq i < j \leq n\}$. The class $\mathfrak{C}$ contains all binomial skew polynomial rings introduced and studied by the author, see [14]. Each of them is an Artin-Schelter regular PBW algebra of global dimension n, see [18], and also [15], and defines in a natural way a square-free solution of the Yang-Baxter equation. When $|X| = 8$ more that 2400 (non-isomorphic) square-free solutions were found by Schedler via a computer program.

7.4. **The $\mathbb{Z}^2$-graded Filiform Lie algebras presented as standard monomial Lie algebras defined by Lyndon words.** In this subsection, as an application of our approach (and for completeness) we show that each of the classical Filiform algebras $\mathcal{L}_n$ and $\mathcal{Q}_n$ has a canonical presentation as a naturally $\mathbb{Z}^2$-graded standard monomial Lie algebra defined by Lyndon words.

Theorem 7.9. *Let (N, W) be a Lyndon pair, in the alphabet $X = \{x < y\}$. Let $J = ([W])_{Lie}$, and I be the corresponding ideals, and $\mathfrak{g} = Lie(X)/J$, $U = U\mathfrak{g} = K\langle X\rangle/I$. Assume that N is connected, with order $|N| \geq 4$, and*

$$(7.10) \qquad |N \bigcap L_m| \leq 1, \forall m \geq 2.$$

Without loss of generality we may assume that $xyy \in N$.
The following conditions hold.

(1) *N is infinite if and only if*

$$(7.11) \qquad N = \{xy^k \mid k \geq 0\} \bigcup \{y\}, \quad \text{or equivalently,} \quad W = \{xy^k xy^{k+1} \mid k \geq 0\}.$$

   $[W]$ is a Gröbner-Shirshov basis of J, so $\mathfrak{g} = \mathfrak{g}_W$ is a standard monomial Lie algebra with a K-basis $[N]$, and $\mathfrak{g} \simeq \mathfrak{M}_0$, the infinite-dimensional Filiform algebra, see [10].
(2) *N is a finite set of order d if and only if one of the conditions (a), or (b) is in force.*

(a)

(7.12)
$$N = \{xy^k \mid 0 \le k \le d-2\} \bigcup \{y\} = N(\mathcal{L}_{d-1})$$
$$W = \{xy^k xy^{k+1} \mid 0 \le k \le d-2\} \bigcup \{xy^{d-1}\}.$$

> In this case $\mathfrak{g} = \mathfrak{g}_W$ is a standard monomial algebra with a K-basis $[N]$, moreover, $\mathfrak{g} \cong \mathcal{L}_{d-1}$, the standard filiform Lie algebra of dimension d and nilpotency class $d-1$.

(b) *One has $d = 2m \ge 6$ and $(N, W) = (N_{2m}, W_{2m})$, where*

$$N_{2m} \quad = \{xy^j \mid 0 \le j \le 2m-3\} \bigcup \{xy^{m-2}xy^{m-1}\} \bigcup \{y\} = N(\mathcal{Q}_{2m})$$

(7.13) $W_{2m} = \{xy^{2m-2}, (xy^{m-2})(xy^{m-2}xy^{m-1}), (xy^{m-2}xy^{m-1})(xy^{m-1})\}$

$$\bigcup \{xy^k xy^{k+1} \mid 0 \le k \le 2m-4, \text{ and } k \ne m-2\}.$$

> Again, $\mathfrak{g} = \mathfrak{g}_W$ is a standard monomial algebra with a K-basis $[N]$, $\mathfrak{g} \cong \mathcal{Q}_n$, the second type Filiform Lie algebra of dimension $d = 2m \ge 6$ and nilpotency class $d-1$.

Lemma 7.10. *In notation and assumptions as in the theorem, let $d = 2m \ge 6$. Let (N, W) be a Lyndon pair, where $N = \{x = l_1 < l_2 < \cdots < l_d = y\}$ satisfies:*

(7.14) $$|N \bigcap L_k| = 1, \; \forall k, 2 \le k \le d.$$

Suppose $xxy \in W$. The following are equivalent.

(1) *$[W]$ is a Lyndon-Shirshov Lie basis of the ideal J, and N contains an atom different from xy^k, for $k \ge 1$.*

(2)

(7.15) $$N = N_{2m} = \{xy^k \mid 0 \le k \le 2m-3\} \bigcup \{y\} \bigcup \{xy^{m-2}xy^{m-1}\},$$

and $[N]$ is a K-basis of $\mathfrak{g}$.

(3) *The Lie algebra $\mathfrak{g}$ is generated by the set $X = \{x < y\}$, $W = W_{2m}$ and its Lyndon-Lie monomial relations $[W]$ form a Gröbner-Shirshov basis, where*

(7.16)
$$W = W_{2m} = \{xy^k xy^{k+1} \mid k \ne m-2, 0 \le k \le 2m-4\} \bigcup \{xy^{2m-2}, xy^{m-2}xy^m\})$$
$$\bigcup \{(xy^{m-2})(xy^{m-2}xy^{m-1}), (xy^{m-2}xy^{m-1})(xy^{m-1})\}.$$

Proof. Without loss of generality we have assume $xyy \in N$, hence $xxy \, W$. It is clear that $(N_{2m}, W_{2m},)$ given in (7.15) and (7.16), respectively, is a Lyndon pair. The implication $(2) \implies (1)$ is obvious.

We shall prove that for $W = W_{2m}$, the bracketing $[W]$ is a is a Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$, we use Proposition 6.14. Indeed, there are exactly three-types of two-chains ω on W. (a) $\omega \in L_\alpha$, where $\alpha = (2, k), k \ge 2m-2$. Hence $|\omega| \ge 2m > max\{|l| \mid l \in N\} = 2m-1$, and $N \bigcap L_\alpha = \emptyset$; (b) $\omega \in L_\alpha$ where $\alpha = (3, s), s \ge 3$, then, clearly, $N \bigcap L_\alpha = \emptyset$; (c) $\omega \in L_\alpha$ where $\alpha = (4, s), s \ge 2m-1$, hence $N \bigcap L_\alpha = \emptyset$ again. It follows from Proposition 6.14 part (1) that in each of the cases (a), (b), and (c) the corresponding composition of overlap is solvable, therefore $[W]$ is a Gröbner-Shirshov basis of the Lie ideal J. Now the equivalence $(2) \iff (3)$ is clear.

$(1) \implies (2)$. By the hypothesis of (1) $[W]$ is a Gröbner-Shirshov basis of the ideal J. Moreover N contains an atom of multi-degree (k, s), $k \ge 2, s \ge 1$ Since $xxy \in W$, and N is closed under taking Lyndon subwords, it follows that N contains

a subword of the shape $xy^k xy^m \cdots$, with $k, m \geq 1$. Let u be the shortest word in N with this property. Clearly, $u = xy^s xy^{s+1} \in N$, for some $s \geq 1$. Moreover, N contains a convex subchain

$$(7.17) \qquad l_1 = x < l_2 = xy < \cdots l_s = xy^{s-1} < l_{s+1} = xy^{s-1} xy^s.$$

In particular $xy^{s-1} xy^s \in N$, with $|xy^{s-1} xy^s| = 2s + 1$, and since N contains no other atom of length $2s + 1$, one has

$$(7.18) \qquad xy^{2s} \in W, \quad xy^{2s+t} \text{ is not in } N, \forall t \geq 0.$$

We claim that $u = l_{s+1} = xy^{s-1} xy^s$ is the longest Lyndon atom in N, and $s = m$. Note first that $|u| = |l_{s+1}| = |xy^{s-1} xy^s| = 2s + 1$, and our assumption (7.14) and (7.18) give (a) all elements $l \in N$ with $2 \leq |l| < 2s + 1$ have the shape $l = xy^k, 1 \leq k \leq 2s - 1$. This implies an inclusions of sets

$$(7.19) \qquad \{xy^k xy^{k+1} \mid 0 \leq k \leq s - 2\} \bigcup \{xy^{2s}\} \subset W.$$

We shall prove that $N \bigcap L_{2s+2} = \emptyset$. Indeed, xy^{2s+1} is not in N, and there are no atoms $l \in N$ of length $|l| = 2s + 2$ of multi-degree $\alpha = (3, k)$, hence it will be enough to prove that

$$(7.20) \qquad w = xy^{s-1} xy^{s+1} \in W.$$

It follows from (7.19) that $a = xxy, b = xy^{2s} \in W$, hence $\omega = xxy^{2s} = (xxy)y^{2s-1} = x(xy^{2s})$ is a two-chain on W, $\omega \in \mathcal{L}(2, 2s)$. So there is a composition of overlap:

$$(xxy, xy^{2s})_\omega = ([xxy]y^{2s-1}) - (x[xy^{2s}]) = [xxy^{2s}]_l - [xxy^{2s}] \in J.$$

which must be solvable, since by assumption $[W]$ is a GS-Lie basis. Detailed computation implies

$$[xxy^{2s}]_l - [xxy^{2s}] \equiv c[xy^{s-1} xy^{s+1}] \text{ (modulo } J),$$

for some integer $c \neq 0$. Hence $[xy^{s-1} xy^{s+1}] \in J$, and therefore $v = xy^{s-1} xy^{s+1}$ is not an atom. All proper Lyndon subwords of v are atoms, indeed these are

$$xy^{s-1} xy^s \in N, xy^k \in N, 0 \leq k \leq s + 1, y \in N,$$

therefore $l = xy^{s-1} xy^{s+1} \in W$.

This together with the information we already have (i) $xy^{2s}, xy^{s-1} xy^{s+1} \in W$ and $xy^{s-1} xy^s$ is the shortest Lyndon word in which x occurs more than once imply that that there is no Lyndon atom $d \in N$ of length $|d| = 2s + 2$. By assumption $[W]$ is a GS-Lie basis, so by Lemma 5.11 N is connected, thus $N \bigcap L_t = \emptyset, t \geq 2s + 2$.

It follows that $s = m$, and N satisfies(7.15), so $N = N_{2m}$, and therefore $|N| = \dim \mathfrak{g} = 2m$. Clearly, $\mathfrak{g}$ is a filiform Lie algebra of nilpotency class $2m - 1$, and its defining Lyndon Lie monomial relations are $[W_{2m}]$. This proves $(1) \Longrightarrow (2)$. $\qquad \square$

Proof of Theorem. *7.9. Without loss of generality we have assumed that $xyy \in N$, hence $xxy \in W$. Two cases are possible: (i) all atoms in N have the shape $xy^k, k \geq 1$, or (ii) N contains an atom different from xy^k, for $k \geq 1$. Assume (i) is in force. Then every atom $u \in N$ of length $|u| \geq 2$ has the shape $u_s = xy^s$, where $1 \leq s$, and exactly one of the cases (i.1), or (i.2) given below is in force. (i.1) N contains all words $xy^s, s \geq 0$, then $|N \bigcap L_s| = 1, \forall s \geq 2$. Thus N and W satisfy (7.11). In this case $[W]$ is an (infinite) Gröbner-Shirshov basis of the Lie ideal J (this can be proved recursively), $[N]$ is a K-basis of $\mathfrak{g}$ and $\mathfrak{g} \cong \mathfrak{M}_0$, the*

infinite Filiform Lie algebra. The case (i.2) is explicitly described by Corollary 7.8. Assume (ii) is in force, then Lemma 7.10 completes the proof of the theorem .   □

Corollary 7.11. *Let J be a $\mathbb{Z}^n$ (non-negatively) graded Lie ideal in $Lie(X)$, $\mathfrak{g} = Lie(X)/J$, $U = U\mathfrak{g} = K\langle X\rangle/I$ (in usual notation), and suppose the Lyndon pair for the enveloping algebra U satisfies one of the conditions (a) $N, W) = (N(\mathcal{L}_d), W(\mathcal{L}_d))$, is the pair corresponding to the Filiform Lie algebra $\mathcal{L}_d$ of nilpotency class d and dimension the $d+1$, see (7.12); or (b) $(N, W) = (N_{2m}, W_{2m})$, is the Lyndon pair corresponding to Fillifotm Lie algebra of second type, $\mathcal{Q}_d$, see (7.15), and (7.16). Then the set of obstructions determines uniquely the Gröbner-Shirshov basis of J and the defining relations of U. More precisely, in case (a) $J = [W(\mathcal{L}_d)]_{Lie}$, $\mathfrak{g} = \mathcal{L}_d$, and U is the enveloping algebra of the Filiform Lie algebra $\mathcal{L}_d$, $\mathrm{gl\,dim}\,U = d + 1$. and in the case (b) $J = [W(\mathcal{Q}_d)]_{Lie}$, $\mathfrak{g} = \mathcal{Q}_d$, and U is the enveloping algebra of the Filiform Lie algebra $\mathcal{Q}_d$, $\mathrm{gl\,dim}\,U = d+1$.*

Proof. Note that in each of the cases (a) and (b) every obstruction $w \in W$ is minimal (w.r.t. $\prec$) in its $\mathbb{Z}^n$-component, L_α, where $\deg(w) = \alpha$. Hence the only monic homogeneous Lie element $f \in \mathrm{Lie}(X)$ with multi-degree α and with highest monomial $\overline{f} = w$ is $f = [w]$. Therefore in the special cases when one of the conditions (a) or (b) holds, the obstruction set W determines uniquely the set of defining relations $\mathfrak{R}$ for U, one has $\mathfrak{R} = [W]$.   □

8. More on Gröbner-Shirshov bases of monomial Lie ideals. Applications

8.1. More on Gröbner-Shirshov bases of monomial Lie ideals. We have a natural problem.

Problem 8.1. *Given X and a Lyndon pair (N, W), where $|N| = d$. Decide whether $[W]$ is a Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$ in $\mathrm{Lie}(X)$.*

Proposition 6.12 (2) implies an effective method (Procedure) to check whether $[W]$ is a Gröbner-Shirshov basis of the Lie ideal $J = ([W])$. Moreover, Proposition 6.14 can be and will be used to simplify the procedure by omitting the unnecessary computation of compositions corresponding to "long" 2-chains.

Definition 8.2.   (1) We say that a Lyndon pair (N, W) is *regular* if (i) N is finite, and (ii) the class $\mathfrak{C}(X, W)$ contains the enveloping $U = U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$. In this case (i) U is an s.f.p. AS-regular algebra of global dimension $d = |N|$, and (ii) the set $[N]$ is a K-basis of $\mathfrak{g}$.
   (2) We say that the Lyndon pair (N, W) is *standard*, if $[W]$ is a Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$ in $\mathrm{Lie}(X)$. Every standard Lyndon pair (W, N) with a finite set N is regular, since the class $\mathfrak{C}(X, W)$ contains the enveloping algebra U of the standard monomial Lie algebra $\mathfrak{g}_W = \mathrm{Lie}(X)/J$.

We are interested in Lyndon pairs (N, W), where N is a finite set of order d.

Remark 8.3. Suppose (N, W) is a Lyndon pair, and $J = ([W])_{Lie}$ is the corresponding monomial Lie ideal in $\mathrm{Lie}(X)$, $\mathfrak{g} = \mathrm{Lie}(X)/J$, $U = U\mathfrak{g}$.
   (1) In general, the reduced Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$ may contain a Lie element $f \in J$, which is not a Lie monomial, as shows Example 8.4.

(2) If N is a finite set of order d, then the monomial Lie ideal $J = ([W])_{Lie}$ has a finite reduced Gröbner-Shirshov basis and determines uniquely the corresponding Lyndon pair $(\widetilde{N}, \widetilde{W})$, where $\widetilde{W}$ is the set of obstructions for the enveloping algebra $U = U\mathfrak{g}$. One has $\widetilde{N} \subseteq N$. Moreover, the Lie algebra $\mathfrak{g} = \mathrm{Lie}(X)/J$ is finite dimensional with a K-basis $[\widetilde{N}]$ and $\widetilde{d} = \dim \mathfrak{g} = |\widetilde{N}| \leq d$. The enveloping $U = U\mathfrak{g}$ is in the class $\mathfrak{C}(X, \widetilde{W}) = \mathfrak{C}(X, \widetilde{N})$, U an AS-regular algebra of $\mathrm{gl}\dim U = \widetilde{d}$. The set $[\widetilde{W}]$ is not necessarily a Gröbner-Shirshov basis of the Lie ideal $\widetilde{J} = ([\widetilde{W}])_{Lie}$, see Example 8.4.

Example 8.4. Let $X = \{x < y\}$ and let $(N, W = W(N))$ be the Lyndon pair, determined by the set of Lyndon atoms N given below, $\mathfrak{g} = \mathrm{Lie}(X)/([W])$,notation as usual :

$$N = \{x < xxy < xxyxy < xxyxyy < xxyxyy < xxyxyyy < xxyy$$
$$< xxyyxy < xxyyxyy < xxyyy < xxyyyxy < xy < xyxyxyy$$
$$< xyxyy < xyxyyy < xyy < xyyy < y\}$$

$|N| = 18, \quad m(N) = 7, \quad N$ is connected.

$W = \{xxxy, xyyy\}\bigcup\{\text{appropriate Lyndon words of length } \geq 7\}.$

$\omega = (xxxy)yyy = xx(xyyy)$ is the unique 2-chain of length $|\omega| = 7$.

Every 2-chain ρ on W, $\rho \neq \omega$ has length $|\rho| > 7$.

By Proposition 6.14, if a 2-chain ρ on W has length $|\rho| > m(N) = 7$, then the corresponding composition is solvable. Thus there is only one composition of overlap which needs computation, namely the unique composition corresponding to the two-chain ω of length ≤ 7. One has

(8.1)
$$(xxxy, xyyyy)_\omega = [[[[xxxy], y]y]y] - [xxxyyy]$$
$$= [xxyxyyy] - 3[xxyyxyy] - 4[xxyyyxy] + 3[xyxyxyy] = f$$

The Lie element f is in J, it is presented as a linear combination of Lyndon Lie monomials, each of which is in $[N]$ (i.e. normal modulo $[W]$). Its highest monomial (w.r.t. $\prec$) is $\overline{f} = \overline{[xxyxyyy]} = xxyxyyy$ of length 7. The set $\Gamma_1 = \{f\}\bigcup[W]$ generates J. Moreover, every composition of overlap is solvable, all compositions of inclusion correspond to certain monomials of length ≥ 8, and therefore are solvable. The unique composition of overlap corresponding to a two-chain of order 7 is now solvable, so Γ_1 is a GS-Lie basis of J. Denote

$$W_0 = W \setminus \{u \mid u \in W, |u| \geq 8, \overline{f} \sqsubseteq u\}$$

Then the set

$$\Gamma = \{f\}\bigcup[W_0]$$

is the reduced GS Lie basis of the Lie ideal J. Note that $[\widetilde{W}] = \{[xxyxyy]\}\bigcup[W_0]$ is not a GS Lie basis of J. Moreover the Lyndon pair $(\widetilde{N}, \widetilde{W})$, where

$$\widetilde{N} = N \setminus \{xxyxyyy\},$$
$$\widetilde{W} = \{xxyxyy\}\bigcup(W \setminus \{u \mid u \in W, |u| \geq 8, \overline{f} = xxyxyyy \sqsubseteq u\})$$

corresponds to the enveloping algebra $U = U\mathfrak{g}, \mathfrak{g} = \mathrm{Lie}/([W])_{Lie} = \mathrm{Lie}/(\Gamma)_{Lie}$.

Using the terminology from Definition 8.2, the Lyndon pair $(\widetilde{N}, \widetilde{W})$ is regular, i.e. $\widetilde{W}$ is the set of obstructions of the enveloping $U = U\mathfrak{g}$ (equivalently, $[\widetilde{N}]$ is a K-basis of $\mathfrak{g}$). However, this Lyndon pair is not standard since the set $[\widetilde{W}]$ is not a GS-Lie basis of the Lie ideal $([\widetilde{W}])_{Lie}$.

Recall that given a Lyndon pair (N, W), the connected component N_{con} of N is defined in Definition 5.9 (2).

Proposition 8.5. *Let (N, W) be a Lyndon pair, $N = \{l_1 < l_2 < l_3 \cdots < l_d\}$. Let $J = ([W])_{Lie}$ be the Lie ideal in $Lie(X)$, generated by $[W]$, $\mathfrak{g} = Lie(X)/J$, and let I be the corresponding associative ideal in $K\langle X \rangle$. Denote by $(\widetilde{N}, \widetilde{W})$ the Lyndon pair corresponding to the enveloping algebra $U = U\mathfrak{g} = K\langle X \rangle/I$. Let N_{con} be the connected component of N, say $N_{con} = \{l_1 = u_1 < u_2 < \cdots < u_s\} \subseteq N$. Then the following conditions hold.*

1. $\widetilde{N} = N$ *if and only if $[W]$ is the reduced Gröbner-Shirshov basis of J. In this case N is connected.*
2. *In the general case, one has $\widetilde{N} \subseteq N_{con} \subseteq N$.*
3. $\widetilde{W}$ *is a finite antichain of Lyndon words, so the monomial Lie ideal $J = ([W])_{Lie}$ has a finite reduced Gröbner-Shirshov basis $\mathfrak{R}$, and the enveloping $U = K\langle X \rangle/I$ is standard finitely presented.*
4. *Assume that N is disconnected, then*

$$\widetilde{d} = |\widetilde{N}| \leq |N_{con}| = s < |N| = d$$

   and U is an s.f.p. Artin-Schelter regular algebra of global dimension $\widetilde{d} < d$.
5. *The Lie algebra $\mathfrak{g}$ has nilpotency class $m(\widetilde{N}) \leq m(N_{con})$.*

Proof. Note first that if $[W]$ is a Gröbner-Shirshov basis of the Lie ideal J, then it is a reduced Gröbner-Shirshov basis.

(1). Assume $\widetilde{N} = N$, then $\widetilde{W} = W$, and W is the set of obstructions of the ideal I, hence $[W]$ (considered as a set of associative polynomials) is the reduced Gröbner basis of the ideal I, and therefore $[W]$ is the reduced Gröbner-Shirshov basis of the Lie ideal J. Conversely, if $[W]$ is the reduced Gröbner-Shirshov basis of the Lie ideal J then $[W]$ (considered as a set of associative polynomials) is the reduced Gröbner basis of the (associative) ideal I, hence W is the set of obstructions for U, and $\widetilde{W} = W$, hence $\widetilde{N} = N$.

(2) Suppose $a \in \widetilde{N}$ is an atom for U, then a is normal modulo I (equivalently, a does not contain as a subword the highest term of any element $g \in I$). In particular, for every $w \in W$ we have $[w]_{ass} \in I$, and $\overline{[w]_{ass}} = w$, hence $w \sqsubseteq a$ is impossible, so $a \in N$. This proves $\widetilde{N} \subseteq N$, but since $\widetilde{N}$ is connected, one has $\widetilde{N} \subseteq N_{con} \subseteq N$.

(3) Without loss of generality we may assume that $[W]$ is not a Gröbner-Shirshov basis of the Lie ideal J. It follows that there exist a 2-chain $\omega = hbt$ where $u = hb \in W$, $v = bt \in W$, and $h, t \in X^+$, such that the Lie composition of overlap

$$(u, v)_\omega = ([u]t) - (h[v]) = ([u]t) - [\omega]$$

is not solvable. Let ω be the minimal 2-chain with this property. Recall that the composition $(u, v)_\omega \in J$. Since it is not solvable, we can present it (modulo J) as a linear combination of Lie monomials,

(8.2) $\quad \begin{aligned} &(u, v)_\omega \equiv \alpha[w] + \sum_{1 \leq i \leq m} \alpha_i[w_i] = f \pmod{J}, \text{ where} \\ &\overline{f} = w \succ w_i, \ w, w_i \in N, \ \alpha, \alpha_i \in K, \alpha \neq 0, \ 1 \leq i \leq m. \end{aligned}$

This implies that the monomial w is not a Lyndon atom for U, and $\widetilde{N} \subseteq N \setminus \{w\}$. In fact the Lyndon pair $(\widetilde{N}, \widetilde{W})$ is uniquely determined by the ideal I, where $\widetilde{W}$ is the set of obstructions for I, equivalently it is determined by the Lie ideal J. We

have shown that $\widetilde{N}$ is finite, with $|\widetilde{N}| = \widetilde{d} < d = |N|$, therefore, by Theorem B, $\widetilde{W}$ is also finite and satisfies

$$\widetilde{W} \leq \frac{\widetilde{d}(\widetilde{d} - 1)}{2} < \frac{d(d - 1)}{2}.$$

It follows that U is standard finitely presented, and the reduced Gröbner-Shirshov basis of the Lie ideal J is also finite.

Note that the Lie element f found in (8.2) is in the reduced Gröbner-Shirshov basis of J. One can effectively find the the reduced Gröbner-Shirshov basis of J, applying a standard procedure for constructing a Gröbner-Shirshov basis of J, but this is not in the scope of this paper.

(4). The set of Lyndon atoms $\widetilde{N}$ is connected and $\widetilde{N} \subseteq N$ implies $\widetilde{N} \subseteq N_{con} \subseteq N$. Part (5) is clear. $\square$

Corollary 8.6. *Suppose (N, W) is a standard Lyndon pair with $|N| = d$. Then the universal enveloping algebra $U = U\mathfrak{g} = K\langle X \rangle / I$ is an AS-regular algebra of global dimension $\gl \dim U = d = |N| = \mathrm{GKdim}\, U$. Clearly, then the class $\mathfrak{C}(X, W)$ contains AS-regular algebras of global dimension $d = |N|$.*

8.2. **Nontrivial disconnected extensions of standard Lyndon pairs.** In this subsection X has arbitrary finite order $n \geq 2$. In the cases when Filiform algebras occur, $X = \{x < y\}$, as usual.

Definition 8.7. Let $(N^\star, W^\star)$ be a standard Lyndon pair (that is $[W^\star]$ is a Gröbner-Shirshov basis of the Lie ideal $J = ([W^\star])_{Lie}$ in $\mathrm{Lie}(X)$). A Lyndon pair (N, W) is called *a nontrivial disconnected extension* of $(N^\star, W^\star)$ if N is disconnected and $N^\star = N_{con} \subsetneq N$.

Proposition 8.8. *Suppose (N, W) is a nontrivial disconnected extension of the standard Lyndon pair $(N^\star, W^\star)$, where $|N^\star| = |N_{con}| = d$. In notation as usual, $J = ([W])_{Lie}$ is the corresponding Lie ideal in $\mathrm{Lie}(X)$, $\mathfrak{g} = \mathrm{Lie}(X)/J$, and $U = U\mathfrak{g}$ is the corresponding enveloping algebra. The following conditions hold.*

(1) *$[N^\star]$ is a K-basis of $\mathfrak{g}$.*
(2) *The enveloping algebra U is in the class $\mathfrak{C}(X, W^\star) = \mathfrak{C}(X, N^\star)$. U is an s.f.p. Artin-Schelter regular algebra with $\gl \dim U = d < |N|$.*
(3) *Furthermore, assume that $(N^\star, W^\star)$ satisfies one of the following conditions: (a) $N^\star = N(\mathcal{L}_d)$; (b) $N^\star = N(\mathcal{Q}_d)$; (c) $N^\star = L(m)$, (the set of all Lyndon words of length $\leq m$ Then*
   (a) *$[W^\star]$ is the Gröbner-Shirshov basis of the Lie ideal J.*
   (b) *The monomial Lie algebra $\mathfrak{g} = \mathrm{Lie}(X)/([W])_{Lie}$ coincides with the standard monomial algebra $\mathfrak{g}_{W^\star} = \mathrm{Lie}(X)/([W^\star])_{Lie}$*
   (c) *The enveloping algebra $U = U\mathfrak{g} = K\langle X \rangle / ([W]) = K\langle X \rangle / I$ has defining relations $[W^\star]_{ass}$ which form a (finite) Gröbner basis of the corresponding ideal I in $K\langle X \rangle$.*

Proof. (1) As in Proposition 8.5 denote by $(\widetilde{N}, \widetilde{W})$ the Lyndon pair corresponding to the enveloping algebra $U = U\mathfrak{g} = K\langle X \rangle / I$. It will be enough to prove $(\widetilde{N}, \widetilde{W}) = (N^\star, W^\star)$. By Proposition 8.5, and by the hypothesis one has

(8.3) $$\widetilde{N} \subseteq N_{con} = N^\star.$$

We shall prove that $N^\star \subseteq \widetilde{N}$, that is every Lyndon word $a \in N^\star$ is normal modulo J. We claim that a word $a \in N^\star$ can not be affected in the process of resolving

a composition of overlap corresponding to some 2-chain on W. Indeed, consider
the set of two-chains on W. Denote by $m^\star = m(N^\star)$ the maximum of all lengths
of words in $N^\star$. Note that if a word $w \in W$ has length $|w| \leq m^\star$, then $w \in W^\star$.
Suppose ω is a two-chain on W. Two cases are possible (i) $\omega = ut = hv, u, v \in W$,
has length $|\omega| \leq m$, then ω is composed out of the Lyndon words $u, v \in W$, of
lengths $|u| < m^\star, |v| < m^\star$. It follows hat $u, v \in W^\star$, so ω is a two-chain on
$W^\star$. Then the composition of overlap induced by ω does not affect any of the
elements $a \in N^\star$, since $(N^\star, W^\star)$ is a standard Lyndon pair. (ii) $\omega = ut = hv$
has length $|\omega| \geq m + 1$. Then the composition of overlap, induced by ω is either
0, or it is a linear combination of elements $[b_i]$, with the same multi-degree as ω,
$b_i \in L(\omega), b_i \prec \omega$. But $|\omega| \geq m + 1$, hence $L(\omega) \bigcap N^\star = \emptyset$. It follows that the
composition of overlap induced by ω can not affect any of the Lyndon words in
$N^\star$. We have shown that every Lyndon word in $N^\star$ is normal modulo the ideal
$([W])_{Lie} = J$. Therefore $N^\star \subseteq \widetilde{N}$, which together with (8.3) implies the desired
equality $N^\star = \widetilde{N}$. This implies, that there is an equality of Lyndon pairs

$$(8.4) \qquad\qquad (\widetilde{N}, \widetilde{W}) = (N^\star, W^\star).$$

Hence $[N^\star]$ is a K-basis of the Lie algebra $\mathfrak{g}$. Part (2) is straightforward. In
particular $U \in \mathfrak{C}(X, W^\star) = \mathfrak{C}(X, N^\star)$

However, as we have already discussed, in general, the set of obstructions $W^\star$
of U does not determine the Gröbner-Shirshov basis of the ideal $J = ([W])_{Lie}$, its
elements are not necessarily Lyndon Lie monomials.

Part (3) follows from Corollary 7.7, and Corollary 7.11. We see that under the
restrictive hypothesis of part (3) the set of obstructions $W^\star$ determines explicitly
the the Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$, and the relations of
the AS regular algebra U. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

8.3. The Fibonacci monomial algebras F_n, and the Lie algebras $\mathfrak{g}_n = \mathbf{Lie}(X)/([W(F_n)])$.
In this subsection $X = \{x < y\}$. In [17], section 7, we define
the sequence of *Fibonacci-Lyndon words* $\{f_n(x, y)\}$ recursively, as follows: $f_0 :=
x, f_1 := y$ and, then for $n \geq 1$

$$(8.5) \qquad f_{2n} = f_{2n-2} f_{2n-1}, \quad f_{2n+1} = f_{2n} f_{2n-1} = f_{2n-2} f_{2n-1} f_{2n-1}.$$

This give the sequence

0	1	2	3	4	5	6
x	y	xy	xyy	$xyxyy$	$xyxyyxyy$	$xyxyyxyxyyxyy$.

Note that if we let a be $f_2(x, y) = xy$ and b be $f_3(x, y) = xyy$, then the Fibonacci-
Lyndon word $f_m(a, b) = f_{m+2}(x, y)$.

Facts 8.9. [17] *The following holds:*

    a. *The word $f_n(x, y)$ is a Lyndon word and its length is the n'th Fibonacci
       number.*

    b. *For the lexicographic order we have*

$$f_0 < f_2 < \cdots < f_{2n} < \cdots < f_{2n+1} < \cdots < f_3 < f_1.$$

*Let $W(F_\infty)$ consist of all Lyndon words $f_{2n-2} f_{2n}$ and $f_{2n+1} f_{2n-1}$, where $n \geq 1$. A
Lyndon word w in x and y is not in the ideal $(W(F_\infty))$ of $K\langle X \rangle$ if and only if w
is a Fibonacci-Lyndon word.*

Let $W(F_n)$ be the antichain of all minimal elements in $W(F_\infty) \cup \{f_n(x, y)\}$, with respect to the divisibility order $\sqsubset$. (This is a finite set, since f_n is a factor of the Fibonacci-Lyndon words later in the sequence.) The corresponding monomial algebra

$$F_n = K\langle X \rangle/(W(F_n))$$

is called the Fibonacci-Lyndon monomial algebra, [17] (or shortly, the Fibonacci algebra).

The set of Lyndon atoms with respect to the ideal $(W(F_n))$ in $K\langle X \rangle$ is $N(F_n) = \{f_0, \ldots, f_{n-1}\}$. We shall call the corresponding Lyndon pair $(N(F_n), W(F_n))$ the nth Fibonacci Lyndon pair. The monomial algebra F_n has Hilbert series $H_{F_n} = \prod_{i=0}^{n-1} (1 - t^{|f_i|})^{-1}$, where f_i is the i'th Fibonacci number. Clearly, the global dimension and the Gelfand-Kirillov dimension of F_n are both n. Proposition 8.1., [17] states that the Fibonacci-Lyndon monomial algebra F_6 does not deform to a bigraded Artin-Schelter regular algebra.

Corollary 8.10. [17] *The class $\mathfrak{C}(X, W(F_6))$ does not contain AS-regular algebras.*

The next corollary is a particular case of Proposition 8.8, part (3), and shows that each Fibonacci pair $(N(F_n), W(F_n))$, with $n \geq 4$ induces different defining relations, $[W(F_n)]$, for the corresponding monomial Lie algebra $\mathfrak{g}_n = Lie(X)/([W(F_n)])_{Lie}$, but the result is isomorphic to the same Lie algebra: $\mathfrak{g}_n = Lie(X)/([W(F_n)])_{Lie} \cong \mathcal{L}_3$, the 4-dimensional Filiform Lie algebra!

Corollary 8.11. *Let $X = \{x < y\}$. Let $(N(F_n), W(F_n))$ be the nth Fibonacci Lyndon pair, $n \geq 4$. Denote by $J(F_n)$ the Lie ideal generated by $[W(F_n)]$ in $Lie(X)$, $\mathfrak{g}_n = Lie(X)/J(F_n)$, I_n denotes the two sided ideal $I_n = ([W(F_n)]_{ass})$ in $K\langle X \rangle$, $U_n = U\mathfrak{g}_n = K\langle X \rangle/I_n$ is the enveloping algebra of $\mathfrak{g}_n$. Then*

   (1) *The class $\mathfrak{C}(X, W(F_n)) = \mathfrak{C}(X, N(F_n))$ contains an AS-regular algebra occurring as an enveloping algebra U of some Lie algebra $\mathfrak{g}$ if and only if $2 \leq n \leq 4$.*
   (2) *$N(F_4) = \{x < xy < xyy < y\} = N(\mathcal{L}_3)$, $\mathfrak{g}_4 = Lie(X)/J_{(F_4))} \cong \mathcal{L}_3$, the standard Filiform Lie algebra of dimension 4.*
   (3) *Let $n \geq 5$, then the connected component of $N(F_n)$ is exactly $N(\mathcal{L}_3)$. There are equalities $J(F_n) = J(F_4)$, $I_n = I_4$, $\forall n \geq 4$, and*

   $$\mathfrak{g}_n = Lie(X)/J(F_n) \cong \mathcal{L}_3, \quad U_n = U_4 \cong U\mathcal{L}_3, \, \forall n \geq 4.$$

   In particular, $U_n \in \mathfrak{C}(X, W(F_4))$, $\mathrm{gl\,dim}\, U_n = 4$ for all $n \geq 4$.

9. Two-generated AS-regular algebras of global dimensions 6 and 7, occurring as enveloping of standard monomial Lie algebras

In this section we prove one of the main results of the paper, Theorem IV.

9.1. More properties of Lyndon pairs.

Remark 9.1. Suppose (N, W) is a Lyndon pair in the alphabet $X = \{x_1, \cdots, x_n\}$, $N = \{l_1 < l_2 < l_3 \cdots < l_{d+1}\}$, let $m = \max\{|l| \mid l \in N\}$. Then $l_1 = x_1$, $l_{d+1} = x_n$, and the following conditions hold

   (1) Assume $u \in N \setminus X$ is maximal w.r.t. $\sqsubseteq$, that is u is not a proper subword of any $l \in N$ (this is always so if $|u| = m$). Then the set $N_0 = N \setminus \{u\}$ satisfies conditions **C1** and **C2**. Let $W_0 = W(N_0)$ be the corresponding antichain so

(N_0, W_0) is a Lyndon pair, $(|N_0| = d)$. Let $A_0 = A_{W_0}$, be the corresponding monomial algebra. Then every algebra $A \in \mathfrak{C}(X, W_0) = \mathfrak{C}(X, N_0)$ satisfies GKdim $A = d = \operatorname{gl} \dim A$.

(2) Conversely, let (N_0, W_0) be a Lyndon pair, with $|N_0| = d$. Suppose that $u, v \in N_0, u < v$ are such that $w = uv \in W_0$, or, equivalently, $w = uv$ is not in N_0, but every Lyndon sub-word of uv belongs to N_0. Then the set $N = N_0 \bigcup \{w\}$ satisfies conditions **C1** and **C2**. Let $W = W(N)$ be the corresponding antichain of Lyndon words, so that (N, W) is a Lyndon pair, one has $|N| = d + 1$, and every algebra $A \in \mathfrak{C}(X, W) = \mathfrak{C}(X, N)$ satisfies GKdim $A = d + 1 = \operatorname{gl} \dim A$.

Problem 9.2. **Given** a full list $\mathfrak{P}_d$ of all Lyndon pairs (N, W) with $|N| = d$, (up to isomorphism of monomial algebras A_W). **Find** the list $\mathfrak{P}_{d+1}$ all Lyndon pairs (N, W) with $|N| = d + 1$, (up to isomorphism of monomial algebras A_W).

Remark 9.1 gives a base for the strategy, conditions (2) and (1) can be and will be used for inductive classification of all Lyndon pairs (N, W) in X^+, with a fixed order $|N| = d + 1$. The interested reader may find a procedure for an effective solution of Problem 9.2 in [20].

Till the end of the section we assume $X = \{x < y\}$. Applying directly our strategy, we have found the sets $\mathfrak{P}_6$ and $\mathfrak{P}_7$ independently of [20], and without using computer programs. In Subsections 9.2, 9.3 we give lists of Lyndon pairs (N, W) in X^+, where $X = \{x < y\}$ and $|N| = 6, 7$. The enumeration (6.m.j), or (7.m.j) stands for a Lyndon pair (N_j, W_j), where $m = m(N_j)$ is the maximal length of all word $l \in N_j$. In each case when N is not connected we identify the connected component $N_{con} =: N^\star$, and we prove that (in these cases) $(N^\star, W^\star)$ is a standard Lyndon pair, corresponding to an explicitly given standard monomial Lie algebra $\mathfrak{g}_{W^\star}$. Note that, in general, for large values of d this is not true.

9.2. A complete list of Lyndon pairs (N, W) where $|N| = 6$.

$$
\begin{array}{lll}
(6.4.1) & N_1 & = \{x < x^2y < xy < xy^2 < xy^3 < y\}, \\
        & W_1 & = \{x^3y,\ x^2y^2\ x^2yxy, xyxy^2,\ xy^2xy^3\ xy^4\} \\
(6.4.2) & N_2 & = \{x < x^2y < x^2y^2 < xy < xy^2 < y\}, \\
        & W_2 & = \{x^3y,\ xy^3\ x^2yxy\ xyxy^2,\ x^2yx^2y^2,\ x^2y^2xy\} \\
(6.5.3) & N_3 & = \{x < xy < xy^2 < xy^3 < xy^4 < y\} = N(\mathcal{L}_5) \\
        & W_3 & = \{xy^ixy^{i+1},\ 0 \le i \le 3,\ xy^5\} \\
(6.5.4) & N_4 & = \{x < xy < xyxy^2 < xy^2 < xy^3 < y\} = N(\mathcal{Q}_5) \\
        & W_4 & = \{x^2y,\ xyxyxy^2,\ xyxy^2xy^2,\ xy^2xy^3,\ xy^4\} \\
(6.5.5) & N_5 & = \{x < xxy < xy < xyxy^2 < xyy < y\}, \qquad N^\star = N(\mathfrak{L}(3)) \\
(6.7.6) & N_6 & = \{x < xy < xy^2 < xy^2xy^3 < xy^3 < y\}, \qquad N^\star = N(\mathcal{L}_4)) \\
(6.7.7) & N_7 & = \{x < xy < xyxyxy^2 < xyxy^2x < xy^2 < y\}, \qquad N^\star = N(\mathcal{L}_3)) \\
(6.8.8) & N(F_6) & = \{x, y, xy, xyy, xyxyy, xyxyyxyy\}, \qquad\qquad N^\star = N(\mathcal{L}_3))
\end{array}
$$

$N^\star := N_{con}$ is the connected component of N.

9.3. A complete list of Lyndon pairs (N, W) where $|N| = 7$.

(7.4.1) $N_1 = \{x < x^3y < x^2y < x^2y^2 < xy < xy^2 < y\}$,
           $W_1 = \{x^4y,\ x^3y^2,\ x^3yx^2y,\ x^3yxy\ x^2yxy,\ xyxy^2,\ x^2y^2xy,\ xy^3\}$

(7.4.2) $N_2 = \{x < x^3y < x^2y < xy < xy^2 < xy^3 < y\}$,
           $W_2 = \{x^4y,\ x^2y^2\ x^2y^2,\ x^3yx^2y,\ x^2yxy,\ xyxy^2,\ xy^2xy^3,\ xy^4\}$

(7.4.3) $N_3 = \{x < x^2y < x^2y^2 < xy < xy^2 < xy^3 < y\}$,
           $W_3 = \{x^3y,\ x^2y^3,\ x^2yxy,\ xyxy^2,\ x^2y^2xy,\ x^2yx^2y^2,\ xy^2xy^3,\ xy^4\}$

(7.5.4) $N_4 = \{x < xy < xyxy^2 < xy^2 < xy^3 < xy^4 < y\}$,
           $W_4 = \{x^2y,\ xyx^yxy^2,\ x^yxy^2xy^2,\ x^yxy^3,\ xy^2xy^3,\ xy^3xy^4\ xy^5\}$

(7.5.5) $N_5 = \{x < x^2y < xy < xy^2 < xy^3 < xy^4 < y\}$,
           $W_5 = \{x^3y,\ x^2y^2,\ x^2yxy,\ xyxy^2,\ xy^2xy^3,\ xy^3xy^4,\ xy^4\}$

(7.5.6) $N_6 = \{x < x^2y < xy < xyxyy < xy^2 < xy^3 < y\}$,
           $W_6 = \{x^3y,\ x^2y^2,\ xxyxy\ xyxyxyy,\ xyxy^3\ xyxy^2xy^2\ xy^2xy^3,\ xy^4\}$

(7.5.7) $N_7 = \{x < x^2y < x^2yxy < x^2y^2 < xy < xy^2 < y\}$,
           $W_7 = \{x^3y, x^2yx^2yxy, x^2yxyxy, x^2yxyx^2y^2, x^2yx^2y^2, xyxy^2,$
               $x^2y^2xy, xy^3\}$

(7.5.8) $N_8 = \{x < x^2y < x^2y^2 < xy < xyxy^2 < xy^2 < y\}$,
           $W_8 = \{x^3y\ x^2yxy\ x^2yx^2y^2\ x^2y^2xy\ xyxyxy^2\ xyxy^2xy^2\ xy^3\}$,

(7.6.9) $N_9 = \{x < xy < xy^2 < xy^3 < xy^4 < xy^5 < y\}$,
           $W_9 = \{xy^ixy^{i+1}, 0 \le i \le 4\} \bigcup \{xy^6\}$

(7.5.10) $N = \{x < x^2y < x^2yxy < xy < xy^2 < xy^3 < y\}$,
           $W = \{x^3y,\ x^2y^2,\ x^2yx^2yxy,\ x^2yxyxy,\ xyx^2,\ xy^2xy^3,\ xy^4\}$

(7.5.11) $N = \{x < x^3y < x^2y < xy < xyxy^2 < xy^2 < y\}$,
           $W = \{x^4y,\ x^2y^2,\ xy^3,\ \cdots\}$

(7.6.12) $N = \{x < xy < xyxy^2 < xyxy^3 < xy^2 < xy^3 < y\}$,
           $W = \{xxy\ xy^4\ \cdots\}$

(7.5.13) $N = \{x < x^2y < x^2yxy < xy < xyxy^2 < xy^2 < y\}$, $N^\star = N(\mathfrak{L}(3))$

(7.6.14) $N = \{x < x^2y < x^2y^2 < x^2y^2xy < xy < xy^2 < y\}$, $N^\star = N_2$
           $N^\star = \{x < x^2y < x^2y^2 < xy < xy^2 < y\} = N_2$,   see (6.4.2)

(7.7.15) $N = \{x < xy < xy^2 < xy^2xy^3 < xy^3 < xy^4 < y\}, m = 7$, $N^\star = N(\mathcal{L}_5)$

(7.7.16) $N = \{x < xy < xyxy^2 < xy^2 < (xy^2)(xy^3) < xy^3 < y\}$, $N^\star = N(\mathcal{Q}_5)$

(7.7.17) $N = \{x < x^2y < xy < xy^2 < (xy^2)(xy^3) < xy^3 < y\}$, $N^\star = N_1$
           $N^\star = \{x < x^2y < xy < xy^2 < xy^3 < y\} = N_1$,   see (6.4.1)

(7.7.18) $N = \{x < xy < (xy)(xyxy^2) < xyxy^2 < xy^2 < xy^3 < y\}$, $N^\star = N(\mathcal{Q}_5)$

(7.7.19) $N = \{x < x^2y < (x^2y)(x^2y^2) < x^2y^2 < xy < xy^2 < y\}$, $N^\star = N_2$
           $N^\star = \{x < x^2y < x^2y^2 < xy < xy^2 < y\} = N_2$,   see (6.4.2)

(7.7.20) $N = \{x < x^2y < xy < xy(xyxy^2) < xyxy^2 < xy^2 < y\}$, $N^\star = N(\mathfrak{L}(3))$

(7.8.21) $N = \{x < x^2y < xy < xyxy^2 < (xyxy^2)(xy^2) < xy^2 < y\}$, $N^\star = N(\mathfrak{L}(3))$

(7.8.22) $N = \{x < xy < xyxy^2 < (xyxy^2)(xy^2) < xy^2 < xy^3 < y\}$, $N^\star = N(\mathcal{Q}_5)$

(7.8.23) $N = \{x < xy < (xy)(xyxy^2) < xyxy^2 < (xyxy^2)(xy^2) < xy^2 < y\}$, $N^\star = N(\mathcal{L}_3)$

(7.9.24) $N = \{x < xy < xy^2 < xy^3 < (xy^3)(xy^4) < xy^4 < y\}$, $N^\star = N(\mathcal{L}_5)$

(7.9.25) $N = \{x < xy < (xy)(xyxyxy^2) < (xy)(xyxy^2) < xyxy^2 < xy^2 < y\}$, $N^\star = N(\mathcal{L}_3)$

(7.10.26) $N = \{x < xy < xy^2 < (xy^2)(xy^2xy^3) < xy^2xy^3 < xy^3 < y\}$, $N^\star = N(\mathcal{L}_4)$

(7.11.27) $N = \{x < xy < xy^2 < xy^2xy^3 < (xy^2xy^3)(xy^3) < xy^3 < y\}$, $N^\star = N(\mathcal{L}_4)$

(7.11.28) $N = \{x, xy, xyxy^2, (xyxy^2)(xy^2), (xyxy^2xy^2)(xy^2), xy^2 < y\}$, $N^\star = N(\mathcal{L}_3)$

(7.12.29) $N = \{x, xy, xy(xyxy^2), (xyxyxy^2)(xyxy^2), xyxy^2, xy^2 < y\}$, $N^\star = N(\mathcal{L}_3)$

(7.13.30) $N = N(F_7) = \{x, y, xy, xyy, xyxyy, xyxyyxyy, (xyxyy)(xyxyyxyy)\}$, $N^\star = N(\mathcal{L}_3)$.

The enumeration (7.m.j) indicates that $m = m(N_j) := \max\{|l| \mid l \in N_j\}$. The connected component of N is $N^\star := N_{con}$, $\mathfrak{L}(3)$ denotes the free nilpotent algebra of nilpotency class 3, and $\mathcal{L}_d$ is the Filiform Lie algebra of dimension $d + 1$.

Theorem IV. *Let $X = \{x < y\}$, and (N, W) be the Lyndon pairs in X^+ given in subsections 9.2 and 9.3. For each such pair $J = ([W])_{Lie}$ is the Lie ideal of $Lie(X)$ generated by $[W]$, $\mathfrak{g} = Lie(X)/J$, I denotes the two sided ideal $I = ([W]_{ass})$ in $K\langle X \rangle$, so $U = U\mathfrak{g} = K\langle X \rangle/I$ is the enveloping algebra of $\mathfrak{g}$.*

    I *Suppose $|N| = 6$. Then*
        (a) *There are eight Lyndon pairs $(N_i, W_i), 1 \leq j \leq 8$, up to isomorphism of monomial algebras A_W, see the list 9.2. Suppose (N_i, W_i) is from this list.*
        (b) *The algebra $U_i = K\langle X \rangle/([W_i]_{ass})$ is an Artin-Schelter regular algebra of global dimension 6 if and only if $1 \leq i \leq 4$. In this case $U_i \in \mathfrak{C}(X, W_i)$.*
        (c) *The class $\mathfrak{C}(X, N(F_6)) = \mathfrak{C}(X, W_8)$ does not contain any Artin-Schelter regular algebra.*
        (d) *The class $\mathfrak{C}(X, W_i)$, $5 \leq i \leq 7$ does not contain AS regular algebras U presented as an enveloping $U = U\mathfrak{g}$ of a Lie algebra.*
        (e) *For $5 \leq i \leq 8$ the algebra U_i is an AS regular algebra with $\operatorname{gl\,dim} U_i \leq 5$.*
    II *Suppose $|N| = 7$. Then*
        (a) *There are 30 Lyndon pairs $(N_i, W_i), 1 \leq i \leq 30$, up to isomorphism of monomial algebras A_W, see the list 9.3. Suppose (N_i, W_i) is a pair from this list.*
        (b) *The algebra $U_i = K\langle X \rangle/([W_i]_{ass})$ is an Artin-Schelter regular algebra of global dimension 7 if and only if $1 \leq i \leq 9$. In this case $U_i \in \mathfrak{C}(X, W_i)$.*
        (c) *Each class $\mathfrak{C}(X, W_i)$, $12 \leq i \leq 30$, does not contain any AS regular algebra U presented as an enveloping $U = U\mathfrak{g}$ of a Lie algebra.*
        (d) *For $10 \leq i \leq 30$, U_i is a finitely presented AS regular algebra of global dimension $\operatorname{gl\,dim} U_i \leq 6$. These algebras are identified in Corollary 9.4, and Remark 9.3.*

Proof. I. (b) Consider the Lyndon pairs $(N_j, W_j), 1 \leq i \leq 4$ given in subsection 9.2. We claim that each of the sets $[W_j]$ is a Gröbner-Shirshov basis of the Lie ideal J_j. Note that for $j = 1, 2$ the number $m_j = \max\{|u| \mid u \in N_j\}$ is $m_j = 4$, and the minimal length of a 2-chain is $c_j = 6 > m_j$. Indeed, $\omega = (xyxyy)y = xy(xyyy)$ is the only 2-chain on W_j with length $|\omega| = 6$. All remaining 2-chains have lengths > 6. It follows from Proposition 6.14 (3.a), that all compositions of overlap are solvable, hence $[W_j]$ is a Gröbner-Shirshov basis of the Lie ideal $J_j = ([W_i])_{Lie}$. For $i = 3, 4$ we use the results from subsections 7.2 and 7.4. The Lyndon pair (N_3, W_3) determines $\mathfrak{g}_3 \simeq \mathcal{L}_5$, the standard 6-dimensional Filiform Lie algebra of nilpotency class 5, while the pair (N_4, W_4) determines $\mathfrak{g}_4 \simeq \mathcal{Q}_6$, the second type Filiform algebra of dimension 6. In particular, each of the sets $[W_i], i = 3, 4$ is a Gröbner-Shirshov basis of the Lie ideal J_i, and the set $[N_i]$ forms a K-basis of $\mathfrak{g}_i$, $i = 3, 4$. It follows that $U_i = U\mathfrak{g}_i \in \mathfrak{C}(X, W_i)$ for all $1 \leq i \leq 4$. Moreover, U_i is an Artin-Schelter regular algebra of $\operatorname{gl\,dim} U_i = 6$. Part (d) implies that for $5 \leq i \leq 8$, $[W]_i$ is not a Grobner-Shirshov basis, and $\operatorname{gl\,dim} U_i \leq 5$. This completes the proof of (b).

(c) By Corollary 8.10 the class $\mathfrak{C}(X, W(F_6))$ does not contain AS-regular algebras.

(d) Each of the sets N_i, $5 \leq i \leq 8$ is not connected, hence by Corollary 5.12, the class $\mathfrak{C}(X, W_i)$ does not contain the enveloping algebra $U = U\mathfrak{g}$ of any Lie algebra $\mathfrak{g}$. In particular, $[W_i]$ is not a Lyndon-Shirshov basis of the Lie ideal $J_i = ([W_i])_{Lie}$ of $Lie(X)$. For $i, 5 \leq i \leq 8$ (N_i, W_i) is a nontrivial disconnected extension of its connected components $N_i^\star = (N_i)_{con}$, which is identified explicitly in the list. We use Proposition 8.8 to identify the Lie algebra $\mathfrak{g}_i = Lie(X)/[W_i])_{Lie}$, $5 \leq i \leq 8$, and $U\mathfrak{g}_i$. In the case (6.5.5) the connected component $N_5^\star = L(3)$ -the set of all Lyndon words of length ≤ 3. Proposition 8.8 implies that $W_5^\star = W(N_5^\star) = W(L(3))$, satisfies $([W_5^\star])_{Lie} = ([W_5])_{Lie} = J_5$. The set $[W_5^\star]$ is a Gröbner-Shirshov basis of the Lie ideal J_5. Moreover, $\mathfrak{g}_5 = Lie(X)/J_5 \cong \mathfrak{L}(3)$, the free nilpotent Lie algebra of nilpotency class 3. Its enveloping algebra $U_5 \cong U\mathfrak{L}(3) \in \mathfrak{C}(X, W(L_3))$ is an Artin-Schelter regular algebra of $\mathrm{gl\,dim}\, U_5 = 5$. In case (6.5.6) $N_6^\star = N(\mathcal{L}_4)$, so (N_6, W_6) is a nontrivial disconnected extension of the standard Lyndon pair $(N(\mathcal{L}_4), W(\mathcal{L}_4))$, corresponding to the Filiform Lie algebra $\mathcal{L}_4$. Proposition 8.8 again implies that $J_6 = ([W(\mathcal{L}_4))])_{Lie}$, hence $\mathfrak{g}_6 \cong \mathcal{L}_4$, and $U_6 = U\mathfrak{g}_6 \cong U\mathcal{L}_4$, $U_6 \in \mathfrak{C}(X, W(\mathcal{L}_4))$ is an Artin-Schelter regular algebra of $\mathrm{gl\,dim}\, U_6 = 5$. In each of the cases (6.7.7) and (6.8.8) $\mathfrak{g}_i \cong \mathcal{L}_3$, so $U = U\mathfrak{g}_i \in \mathfrak{C}(X, W(\mathcal{L}_3))$ is an AS regular algebra with $\mathrm{gl\,dim}\, U_i = 4$, $i = 7, 8$. Part I of the theorem has been proved.

II. (b) We shall prove that for $1 \leq i \leq 9$ the Lie algebra $[W]_i$ is a Gröbner-Shirshov basis of the Lie ideal J_i. We split this in several cases.

(b.1). The cases (7.4.i), $1 \leq i \leq 3$. For each of the Lyndon pairs (N_i, W_i), the maximal length of a Lyndon atom is $m = 4$, at the same time the minimal length of a two-chain on W_i is 6, so Proposition 6.14 implies that all compositions of overlap are solvable, and $[W_i]$ is a GS-basis of the Lie ideal $J_i = ([W_i])_{Lie}$. Therefore $U_i \in \mathfrak{C}(X, W_i)$ is an AS regular algebra of $\mathrm{gl\,dim}\, U_i = 7$

(b.2) The Lyndon pairs (N_4, W_4), (N_7, W_7), (N_8, W_8) are similar. For $i = 4, 7, 8$, the maximal length of Lyndon atoms in N_i is $m_i = 5$, and the minimal length of a two-chain on W_i is $c_i = 6$. Proposition 6.14 implies again that $[W_i]$ is a Gröbner-Shirshov basis of the ideal J_i, and $U_i \in \mathfrak{C}(X, W_i)$ is an AS regular algebra of $\mathrm{gl\,dim}\, U_i = 7$.

(b.3) The Lyndon pair (N_9, W_9) corresponds to the standard filiform Lie algebra $\mathcal{L}_7$, in particular $[W_9]$ is a Gröbner-Shirshov basis of the Lie ideal J_9.

(b.4) It remains to consider only two cases: (7.5.5) and (7.5.6.) We have

$$(7.5.5) \quad \begin{aligned} &N_5 = \{x < x^2y < xy < xy^2 < xy^3 < xy^4 < y\},\ m(N_5) = 5 \\ &W_5 = \{x^3y,\ x^2y^2,\ xxyxy,\ xyxy^2,\ xy^2xy^3,\ xy^3xy^4,\ xy^4\} \end{aligned}$$

$$(7.5.6) \quad \begin{aligned} &N_6 = \{x < xxy < xy < xyxyy < xy^2 < xy^3 < y\},\ m(N_6) = 5 \\ &W_6 = \{x^3y,\ x^2y^2,\ xxyxy\ xyxyxyy,\ xyxy^3\ xyxy^2xy^2\ xy^2xy^3,\ xy^4\} \end{aligned}$$

For $i = 5, 6$ there is unique 2-chain of length 5, $\omega = (xxxy)y = x(xxyy)$, on W_i. The 2-chain ω implies the following composition of overlap

$$(9.1) \quad \begin{aligned} (xxxy, xxyy)_\omega \quad &= [[xxxy], y] - [x, [xxyy]] = [[x, xxy], y] - [x, [xxyy]] \\ &= [[xy, xxy] + [x, [xxy, y]]] - [x, [xxyy]] \\ &= -[xxy, xy] + [[xxxyy] - [xxxyy]] \\ &= -[xxy, xy] = -[xxyxy] \in Span[W_i],\ i = 5, 6. \end{aligned}$$

Hence this composition is solvable. All remaining 2-chains on W_i have length $\geq 6 > m(N_i)$, i = 5,6, hence the corresponding compositions are solavble. It follows that $[W_i]$ is a GS basis of the Lie ideal J_i.

The cases (7.x.i), where $13 \leq i \leq 30$. Then each of the pairs (N, W) is a nontrivial disconnected extension of a standard Lyndon pair $(N^\star, W^\star)$, therefore by Corollary 5.12, $[W_i]$ is not a Lyndon-Shirshov basis of the Lie ideal $J_i = ([W_i])_{Lie}$ of Lie(X). Moreover, the class $\mathfrak{C}(X, W_i)$, $13 \leq i \leq 30$ does not contain the enveloping algebra $U = U\mathfrak{g}$ of any Lie algebra $\mathfrak{g}$. Explicit details for the Lyndon-Shirshov basis of the Lie ideals J_i, the corresponding monomial algebras $\mathfrak{g}$ and their enveloping U in these cases are given in Corollary 9.4.

It remains to study the cases (7.5.10), (7.5.11), (7.6.12), where the Lyndon pair (N, W) has connected set N, but $[W]$ is not a Gröbner-Shirshov basis of the LIe ideal J. Detailed information for each of these cases is given in Remark 9.3. $\square$

Remark 9.3. In notation and assumptions of Theorem IV.

    (1) The Lyndon pair (N_{10}, W_{10}) given in (7.5.10) defines a Lie algebra $\mathfrak{g}$ whose enveloping $U = U\mathfrak{g}$ has Lyndon atoms $\widetilde{N}$ and relations $[\widetilde{W}]$, where

$$(9.2) \qquad \begin{aligned} \widetilde{N} &= \{x < xxy < xy < xy^2 < xy^3 < y\} \\ \widetilde{W} &= \{x^3y,\ x^2y^2,\ x^2xy,\ xyx^2,\ xy^2xy^3,\ xy^4\}. \end{aligned}$$

        This is exactly the Lyndon pair (N_1, W_1) given by (6.4.1). Moreover, the GS- basis of the Lie ideal J_{10} is $[W_1]$, so $\mathfrak{g} = \mathfrak{g}_{W_1}$ has basis $[N_1]$, and is 6 dimensional, $U_{10} \in \mathfrak{C}(X, W_1)$ is an AS regular algebra with gl dim $U_{10} = 6$.

    (2) The Lyndon pair (N_{11}, W_{11}) given in (7.5.11) defines a Lie algebra $\mathfrak{g}$ whose enveloping $U = U\mathfrak{g}$ has Lyndon atoms $\widetilde{N}$ and relations $[\widetilde{W}]$, where

$$\begin{aligned} \widetilde{N} &= \{x < x^3y < x^2y < xy < xy^2 < y\} \\ \widetilde{W} &= \{x^4y,\ x^2y^2,\ x^3yx^2y,\ x^2yxy,\ xyx^2,\ xy^3\}. \end{aligned}$$

        The monomial algebra $A_{\widetilde{W}}$ is isomorphic to A_{W_1}, corresponding to case (6.4.1). $U \in \mathfrak{C}(X, \widetilde{W}$ is an AS-regular algebra of gl dim $U = 6$.

    (3) For the Lyndon pair (N_{12}, W_{12}) given in (7.6.12) one has $\mathfrak{g} = \text{Lie}/([W_{12}]) \simeq \mathcal{Q}_5$, the second type filiform Lie algebra of dimension 6, $U = U\mathcal{Q}_5 \in \mathfrak{C}(X, W(\mathcal{Q}_5))$ is an AS regular algebra with gl dim $U = 6$.

Proof. We give a proof of (1). Consider the pair (N_{10}, W_{10}), given in (7.5.10). For simplicity we set $N = N_{10}, W = W_{10}, J = J_{10}, I = I_{10}$. The two-chain $\omega = xxxyy = (xxxy)y = x(xxyy)$ on W gives a composition of overlap, $(xxxy, xxyy)_\omega$, as in (9.1), so the same computations imply that $(xxxy, xxyy)_\omega = -[xxyxy] \in J$. By assumption $xxyxy \in N$, thus $[xxyxy] \in [N] \bigcap J$, and therefore the composition $(xxxy, xxyy)_\omega$ is not solvable. It follows that $[W_{10}]$ is not a Gröbner Shirshov basis of the Lie ideal J_{10}. Consider $N_0 = N \setminus \{xxyxy\}$. It is easy to show that the (actual) Lyndon pair corresponding to the ideal I_{10} is the pair $(\widetilde{N}, \widetilde{W})$ given in (9.2). But this is exactly the Lyndon pair (6.4.1) (N_1, W_1). Hence the the Gröbner-Shirshov basis of J_{10} is $[\widetilde{W}] = [W_1]$, see (6.4.1), $\mathfrak{g} = \mathfrak{g}_{W^1}$ is 6 dimensional and U_{10} is an AS regular algebra of global dimension 6, $U_{10} \in \mathfrak{C}(X, \widetilde{W})$.

Parts (2) and (3) are verified by similar arguments. In the case (2), (7.5.11) one shows that $(xxyy, xyyy)_\omega = [xy, xyy] \in J_{11} \bigcap N_{11}$; and in the case (3) (7.6.12) the composition $(xxy, xyyyy)_\omega = [xyxyyy] \in J_{12} \bigcap N_{12}$. Analogous argument as in (1) completes the proof. $\square$

Corollary 9.4. *Notation and assumptions as in Theorem IV, part II. Let (N, W) be a Lyndon pair listed in subsection 9.3, denote its connected component by $N^\star = N_{con}$, $(N^\star, W^\star)$ is the corresponding Lyndon pair.*

(1) *In each of the cases (7.8.23), (7.9.25), (7.11.28), (7.12.29), (7.13.30)*

$$N^\star = N(\mathcal{L}_3) = \{x < xy < xyy < y\}, \quad W^\star = W(\mathcal{L}_3) = \{xxy, xyxy^2, xy^3\}.$$
$$J = ([W])_{Lie} = ([W(\mathcal{L}_3)])_{Lie}, \quad \mathfrak{g} = Lie(X)/J \simeq \mathcal{L}_3,$$
$$U = U\mathfrak{g} \in \mathfrak{C}(X, W(\mathcal{L}_3)), \text{ is an AS regular algebra with } \mathrm{gl}\dim U = 4.$$

(2) *In each of the cases (7.7.20), (7.8.21)*

$$N^\star = \{x < x^2y < xy < xy^2 < y\} = L(3) = N(\mathfrak{L}(3)), \ W^\star = W(3) = W(\mathfrak{L}(3)).$$
$$J = ([W])_{Lie} = ([W(3)])_{Lie} \simeq J_3, \quad \mathfrak{g} = Lie(X)/J \simeq \mathfrak{L}(3)$$
$$U = U\mathfrak{g} \in \mathfrak{C}(X, W(3)), \text{ is an AS regular algebra with } \mathrm{gl}\dim U = 5.$$

(3) *In each of the cases (7.10.26), (7.11.27)*

$$N^\star = N(\mathcal{L}_4) = \{x < xy < xyy < xyyy < y\},$$
$$W^\star = W(\mathcal{L}_4) = \{xy^k xy^{k+1} \mid 0 \le k \le 2\} \bigcup \{xy^4\}.$$
$$J = ([W])_{Lie} = ([W(\mathcal{L}_4)])_{Lie}, \quad \mathfrak{g} = Lie(X)/J \simeq \mathcal{L}_4$$
$$U = U\mathfrak{g} \in \mathfrak{C}(X, W(\mathcal{L}_4)), \text{ is an AS regular algebra with } \mathrm{gl}\dim U = 5.$$

(4) *In each of the cases (7.7.15), (7.9.24)*

$$N^\star = N(\mathcal{L}_5) = \{x < xy < xyy < xyyy < xyyyy < y\},$$
$$W^\star = W(\mathcal{L}_5) = \{xy^k xy^{k+1} \mid 0 \le k \le 3\} \bigcup \{xy^5\}.$$
$$J = ([W])_{Lie} = ([W(\mathcal{L}_5)])_{Lie}, \quad \mathfrak{g} = Lie(X)/J \simeq \mathcal{L}_5$$
$$U = U\mathfrak{g} \in \mathfrak{C}(X, W(\mathcal{L}_5)), \text{ is an AS regular algebra with } \mathrm{gl}\dim U = 6.$$

(5) *In each of the cases (7.7.16), (7.7.18), (7.8.22),*

$$N^\star = N(\mathcal{Q}_5) = \{x < xy < xyxyy < xyy < xyyy < y\},$$
$$W^\star = W(\mathcal{Q}_5) = \{xxy, (xy)^2 xyy, xy(xyy)^2, xy^2 xy^3, xy^4\}.$$
$$J = ([W])_{Lie} = ([W(\mathcal{Q}_5)])_{Lie}, \quad \mathfrak{g} = Lie(X)/J \simeq \mathcal{Q}_5$$
$$U = U\mathfrak{g} \in \mathfrak{C}(X, W(\mathcal{L}_4)), \text{ is an AS regular algebra with } \mathrm{gl}\dim U = 6.$$

(6) *Each of the cases (7.6.14), (7.7.19) "degenerates" to (6.4.2), more precisely*

$$N^\star = \{x < x^2y < x^2y^2 < xy < xy^2 < y\} = N_2, \text{ see case (6.4.2)}$$
$$W^\star = \{x^3y, xy^3 \, x^2yxy \, xyxy^2, x^2yx^2y^2, x^2y^2xy\} = W_2, \text{ see case (6.4.2)}$$
$$J = ([W])_{Lie} = ([W_2])_{Lie}, \quad \text{has a G-S basis the set } [W_2]$$
$$\mathfrak{g} = Lie(X)/J \simeq Lie(X)/([W_2])_{Lie} \quad \text{has a K-basis } [N^\star]$$
$$U = U\mathfrak{g} \in \mathfrak{C}(X, W_2), \text{ is an AS regular algebra with } \mathrm{gl}\dim U = 6.$$

(7) *The case (7.7.17) "degenerates" to (6.4.1), more precisely*

$$N^\star = \{x < x^2y < xy < xy^2 < xy^3 < y\} = N_1, \text{ see (6.4.1)}$$
$$W^\star = \{x^3y, x^2y^2 \, x^2yxy, xyxy^2, xy^2xy^3 \, xy^4\} = W_1, \text{ see (6.4.1)}$$
$$J = ([W])_{Lie} = ([W_1])_{Lie}, \quad \text{has a G-S basis the set } [W_1]$$
$$\mathfrak{g} = Lie(X)/J \simeq Lie(X)/([W_1])_{Lie} \quad \text{has a K-basis } [N^\star]$$
$$U = U\mathfrak{g} \in \mathfrak{C}(X, W_1), \text{ is an AS regular algebra with } \mathrm{gl}\dim U = 6.$$

Proof. Recall that by Proposition 8.8 a nontrivial disconnected extension (N, W) of a standard Lyndon pair $(N^\star, W^\star)$, (here $|N^\star| = |N_{con}| < |N|$) does not "contribute" new classes of Lie algebras, whenever $N^\star$ is one of the following (a) $N^\star = N(\mathcal{L}_d)$; (b) $N^\star = N(\mathcal{Q}_d)$; (c) $N^\star = L(m)$, the set of all Lyndon words of length $\le m$. Clearly, parts (1) through (5) follow straightforwardly from Proposition 8.8. To

prove parts (6) and (7) one uses first Proposition 8.8 to deduce that the correct obstruction set for $U\mathfrak{g}$ is exactly $W^\star$. Note that every word $w \in W^\star$ is minimal (w.r.t. $\prec$) in its multi-degree component L_α, where $\deg(w) = \alpha$, therefore $[w]$ is the unique Lie element in $K[L_\alpha]$, whose highest monomial is w. This implies that the monomial Lie algebra $\mathfrak{g} = \mathrm{Lie}(X)/([W])_{Lie}$ is isomorphic to the standard monomial Lie algebra $\mathfrak{g}_{W^\star} = \mathrm{Lie}(X)/([W^\star])_{Lie}$ occurring in the list 9.2. $\qquad\square$

Remark 9.5. We have seen that for $d \le 7$ each non-regular Lyndon pair (N, W) with $|N| = d$ "degenerates" to a standard Lyndon pair $(N^\star, W^\star)$ with $|N^\star| < d$, moreover, in these cases the monomial Lie algebra $\mathfrak{g} = \mathrm{Lie}(X)/([W])_{Lie}$ is isomorphic to some (already known) standard monomial Lie algebra $\mathfrak{g}_{W^\star} = \mathrm{Lie}(X)/([W^\star])_{Lie}$ with a K-basis $[N^\star] \subsetneq [N]$, and the enveloping $U\mathfrak{g} \in \mathfrak{C}(X, W^\star)$ is AS-regular with $\mathrm{gl\,dim}\, U < d$. However, Example 8.4 shows that when d is large enough, (in this case $d = 18$) a Lyndon pair (N, W), with $|N| = d$ may define a monomial Lie algebra $\mathfrak{g} = \mathrm{Lie}(X)/([W])_{Lie}$, such that the reduced Gröbner-Shirshov basis of the Lie ideal $J = ([W])_{Lie}$ contains Lie elements which are not Lyndon Lie monomials. This way we can obtain Artin-Schelter regular algebras $U = U\mathfrak{g}$ with obstruction set $\widetilde{W} \neq W$, set of atoms $\widetilde{N} \subsetneq N$ ($\mathrm{gl\,dim}\, U < d$) and explicitly given more sophisticated relations which are non-trivial linear combinations of Lyndon-Lie monomials.

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American University in Bulgaria, 2700, Blagoevgrad, Bulgaria, and, Max-Plank Institute for Mathematics, Bonn, Germany

E-mail address: `tatyana@aubg.edu`