Max-Planck-Institut für Mathematik
Bonn

Deformation equivalence of affine ruled surfaces

by

Hubert Flenner
Shulim Kaliman
Mikhail Zaidenberg

Max-Planck-Institut für Mathematik
Preprint Series 2013 (19)

Deformation equivalence of affine ruled surfaces

Hubert Flenner
Shulim Kaliman
Mikhail Zaidenberg

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Fakultät für Mathematik
Ruhr-Universität Bochum
Geb. NA 2/72
Universitätsstr. 150
44780 Bochum
Germany

Department of Mathematics
University of Miami
Coral Gables, FL 33124
USA

Univesité Grenoble I
Institut Fourier
UMR 5582 CNRS-UJF, BP 74
38402 St. Martin d'Hères Cédex
France

MPIM 13-19

DEFORMATION EQUIVALENCE OF AFFINE RULED SURFACES

HUBERT FLENNER, SHULIM KALIMAN, AND MIKHAIL ZAIDENBERG

Abstract. A smooth family $\varphi : \mathcal{V} \to S$ of surfaces will be called *completable* if there is a logarithmic deformation $(\bar{\mathcal{V}}, \mathcal{D})$ over S so that $\mathcal{V} = \bar{\mathcal{V}} \backslash \mathcal{D}$. Two smooth surfaces V and V' are said to be deformations of each other if there is a completable flat family $\mathcal{V} \to S$ of smooth surfaces over a connected base so that V and V' are fibers over suitable points $s, s' \in S$. This relation generates an equivalence relation called *deformation equivalence*. In this paper we give a complete combinatorial description of this relation in the case of affine ruled surfaces, which by definition are surfaces that admit an affine ruling $V \to B$ over an affine base with possibly degenerate fibers. In particular we construct complete families of such affine ruled surfaces. In a few particular cases we can also deduce the existence of a coarse moduli space.

Contents

Introduction 2
1. Preliminaries on standard completions 3
2. Families of affine surfaces 7
2.1. Families of standard completions 7
2.2. Extended divisors 9
2.3. Extended divisors in families 12
3. Deformation equivalence 17
3.1. Normalized extended graph and deformation equivalence 17
3.2. A general matching principle 19
3.3. Deformation invariance of the normalized extended graph 24
4. Versal families of affine ruled surfaces 25
4.1. Complete deformation families of surfaces 25
4.2. The map into the configuration space 30
5. Moduli spaces of special Gizatullin surfaces 30
5.1. Special Gizatullin surfaces. 30
5.2. Appendix: An example 32
References 33

Acknowledgements: This research was done during a visit of the first and the second authors at the Fourier Institute, Grenoble, and of all three authors at the Max-Planck-Institute of Mathematics, Bonn. They thank these institutions for the generous support and excellent working conditions.

2010 Mathematics Subject Classification: 14R05, 14R25, 14J10, 14D22.

Key words: affine ruled surface, $\mathbb{A}^1$-fibration, moduli space, deformation equivalence.

Introduction

In classifying algebraic objects like varieties or vector bundles of a certain class one usually tries to find discrete invariants such that the objects sharing a fixed set of these invariants, form a moduli space. The model case is here the moduli space of smooth complete curves, which is known to be a $3g-3$-dimensional irreducible variety, see [DM]. Another classical case is the Hilbert scheme $\mathbb{H}_P$ parameterizing subschemes of the projective space $\mathbb{P}^n$ with fixed Hilbert polynomial P. By a classical result of Hartshorne [Ha_1] $\mathbb{H}_P$ is connected. On the other hand it is already an unsolved problem to determine the connected components of the Hilbert scheme of locally Cohen-Macaulay curves of degree d and genus g, see [Ha_3] and the references therein for partial results.

In general it may happen that for a class of varieties one cannot expect a reasonable moduli space. A typical obstruction for its existence are usually $\mathbb{P}^1$-fibrations or, in the case of open varieties, $\mathbb{A}^1$-fibrations. Instead of studying in such cases the connected components of the moduli space it is convenient to consider the relation of deformation equivalence as introduced in [Ca]. Given a deformation $f : \mathcal{X} \to S$ with all fibers in a class of varieties $\mathcal{C}$ we consider the relations $f^{-1}(s) \sim f^{-1}(t)$ if and only if $s, t \in S$ belong to the same connected component of S. This generates an equivalence relation on $\mathcal{C}$ called *deformation equivalence*.

In this paper we study this equivalence relation for the class of normal affine surfaces V that can be equipped with an *affine ruling*. By this we mean a morphism $\pi : V \to B$ onto a smooth *affine* curve with the general fiber isomorphic to $\mathbb{A}^1$. In this case there is no moduli space available except for a few special cases; see sect. 5. Our main result is that nevertheless one can characterize deformation equivalence completely. The deformations which we consider here are deformations of the open surface X which can be extended to a logarithmic deformation of a completion of X in the sense of [Ka].

The main tool for this characterization is the so called *normalized extended graph*. For smooth surfaces it can be described as follows. Given an affine ruling $\pi : V \to B$ we can extend π to a $\mathbb{P}^1$-fibration $\bar{\pi} : \bar{X} \to \bar{B}$ over the normal completion $\bar{B}$ of B. Performing suitable blowups and blowdowns along the boundary $D := \bar{X} \backslash X$ of X one can transform D or, equivalently, its weighted dual graph Γ_D into standard form, see [DG] or [FKZ_1]. The *extended divisor* D_{ext} is then the union of D with all singular fibers of $\bar{\pi}$. This extended divisor is not in general an invariant of the surface alone, but depends on the choice of the affine ruling. The components of $D_{\text{ext}} - D$ are called the *feathers*. For such feathers one has the notion of a *mother component* (see Definition 2.16 for details). A feather is not necessarily linked to its mother component. Attaching the feathers to their mother components we obtain from the dual graph Γ_{ext} of D_{ext} the *normalized extended graph* $N(\Gamma_{\text{ext}})$. With this terminology our main result is as follows.

Theorem 0.1. *Two affine ruled surfaces are deformation equivalent if and only if they share the same normalized extended graph unless D is a zigzag, i.e. a linear chain of rational curves. In the latter case two surfaces are deformation equivalent if and only if the normalized extended graphs are equal, possibly after reversing one of them (see sect. 3 for details).*

This theorem enables us to construct in section 5 a coarse moduli space for the class of special Gizatullin surface that was studied in [FKZ$_4$].

Let us mention a few special cases. First of all, the Ramanujam theorem [Ra] states that every contractible, smooth surface over $\mathbb{C}$ with a trivial fundamental group at infinity (in particular, every smooth surface over $\mathbb{C}$ homeomorphic to $\mathbb{R}^4$) is isomorphic to the affine plane $\mathbb{A}^2 = \mathbb{A}^2_{\mathbb{C}}$. Thus the deformation equivalence class for such surfaces is reduced to a single point. The normalized extended graph consists in this case just of two 0-vertices joined by an edge.

A generalization of this theorem due to Gurjar and Shastri [GS] says that every normal contractible surface with a finite fundamental group at infinity is isomorphic to $\mathbb{A}^2/G$, where G is a finite subgroup of $\mathbf{GL}_2(\mathbb{C})$ acting freely on $\mathbb{A}^2\backslash\{0\}$. Fixing G, the deformation equivalence class for such surfaces again consists of a single point.

Yet another result of this type is provided by the Danilov-Gizatullin Isomorphism Theorem [DG] (see also [CNR, FKZ$_5$]). Recall that a *Danilov-Gizatullin surface* is the complement to an ample divisor H in some Hirzebruch surface. Such a surface can be completed by the zigzag $[[0, 0, (2)_n]]$, where $n + 1 = H^2$. The Danilov-Gizatullin theorem proves that the deformation equivalence class for such a surface again consists of a single point, once we fix the length of the boundary zigzag.

The paper is organized as follows. In section 1 we recall the notion of standard completion and standard dual graph of an affine ruled surface, and in subsection 2.2 that of an extended divisor. The rest of section 2 is devoted to developing such notions in the relative case. More specifically, in Theorem 2.4 we show the existence of standard completions in completable families, and in the factorization Lemma 2.24 we shed some light onto the structure of extended divisors in the relative case. Central are here the notions of *completable families* and *resolvable families*, see Definitions 2.3 and 2.20. The normalized extended graph is introduced in section 3. Our main result Theorem 3.6 characterizes the deformation equivalence of affine ruled surfaces in terms of these graphs. The proof of this theorem starts in section 3 and is completed in the next section 4, where we construct a versal deformation space of affine ruled surfaces. In section 5 we apply our previous results in order to construct the moduli space of special Gizatullin surfaces, a subclass of the class of all Gizatullin surfaces studied in detail in our previous paper [FKZ$_4$].

All varieties in this paper are assumed to be defined over an algebraically closed field $\Bbbk$. By a *surface* we mean a connected algebraic scheme over $\Bbbk$ such that all its irreducible components are of dimension 2.

1. Preliminaries on standard completions

Let us recall the notions of a (semi-)standard zigzag and (semi-)standard graph.

1.1. Let X be a complete normal algebraic surface, and let D be an SNC (i.e. a simple normal crossing) divisor D contained in the smooth part X_{reg} of X. We say that D is a *zigzag* if all irreducible components of D are rational and the dual graph Γ_D of D is linear[1]. We abbreviate a chain of curves $C_0, C_1, \ldots, C_n$ of weights $w_0, \ldots, w_n$ by $[[w_0, \ldots, w_n]]$. We also write $[[\ldots, (w)_k, \ldots]]$ if a weight w occurs at k consecutive places.

[1]By abuse of notation, we often denote an SNC divisor and its dual graph by the same letter.

1.2. A zigzag D is called *standard* if it is one of the chains

(1) $[[(0)_i]]\,, i\le 3,$ or $[[(0)_{2i},w_2,\ldots,w_n]],$ where $i\in\{0,1\},\ n\ge 2$ and $w_j\le -2\ \forall j.$

A linear chain Γ is said to be *semi-standard* or w_1*-standard* if it is either standard or one of

(2) $[[0,w_1,w_2,\ldots,w_n]],\quad [[0,w_1,0]]\,,$ where $n\ge 1,\ w_1\in\mathbb{Z},$ and $w_j\le -2\ \forall j\ge 2\,.$

A *circular graph* is a connected graph with all vertices of degree 2. Such a weighted graph will be denoted by $((w_0,\ldots,w_n))$. A circular graph is called *standard* if it is one of

(3) $((w_1,\ldots,w_n)),\qquad ((0,0,w_1,\ldots,w_n)),\qquad ((0_l,w)),$ or $((0,0,-1,-1)),$

where $w_1,\ldots,w_n\le -2$, $n>0$, $0\le l\le 3$ and $w\le 0$.[2]

By an *inner elementary transformation* of a weighted graph we mean blowing up at an edge incident to a 0-vertex of degree 2 and blowing down the image of this vertex. By a sequence of inner elementary transformations we can successively move the pair of zeros in the standard zigzag $[[0,0,w_2,\ldots,w_n]]$ to the right:

$$[[0,0,w_2,\ldots,w_n]]\rightsquigarrow[[w_2,0,0,w_3,\ldots,w_n]]\rightsquigarrow\ldots\rightsquigarrow[[w_2,\ldots,w_n,0,0]]\,.$$

This yields the *reversion*

(4) $$D=[[0,0,w_2,\ldots,w_n]]\rightsquigarrow[[0,0,w_n,\ldots,w_2]]=:D^\vee$$

(see 1.4 in [FKZ_3]).

An *outer elementary transformation* consists in blowing up at a 0-vertex of degree ≤ 1 and blowing down the image of this vertex. A birational inner elementary transformation on a surface is rigid i.e., uniquely determined by the associated combinatorial transformation of the dual graph, whereas an outer one depends on the choice of the center of blowup.

1.3. In the sequel there is also the need to consider *NC divisors* D on a surface X. By this we mean that $D\subseteq X_{reg}$ and that the singularities of D are ordinary double points; these are given in the local ring $\mathcal{O}_{X,p}$ by an equation $xy=0$, where $x,y\in\mathfrak{m}\backslash\mathfrak{m}^2$. In particular, two different components meet in smooth points transversally, while the intersection of three different components is empty. Thus D is an SNC divisor if and only if it is an NC divisor and all its irreducible components are smooth. The dual graph Γ_D of an NC divisor D has loops which correspond to the singular points of the components. Vice versa, if the dual graph Γ_D of an NC divisor D has no loop then D is SNC. In particular, D is SNC if Γ_D is a tree.

An *NC completion* $(\bar V,D)$ of a surface V consists of a complete surface $\bar V$ and an NC divisor D on $\bar V$ such that $V=\bar V\setminus D$.

For instance, a plane nodal cubic is an NC divisor, which is not SNC. Its dual graph consists of one vertex of weight 3 and a loop.

Definition 1.4. Let Γ be the dual graph of an NC divisor D. We use the following notations.

[2]We list only those graphs for which the intersection matrix has at most one positive eigenvalue. Actually we do not use circular graphs in this paper, however without these graphs the general notion of a standard graph in 1.4 below would be incomplete.

(1) $B = B(\Gamma)$ is the set of branching points of Γ.
(2) $S = S(\Gamma)$ is the set of vertices corresponding to non-rational components of D.
(3) Following [FKZ$_1$] a connected component of $\Gamma - (B \cup S)$ is called a *segment* of Γ.
(4) A segment will be called *outer* if it contains an *extremal* (or *end*) vertex of Γ i.e., a vertex of degree 1.

Thus an outer segment is either the whole graph Γ and Γ is linear, or it is connected to exactly one vertex of $B \cup S$. The dual graph Γ of an NC divisor D on an algebraic surface $\bar{V}$ (and also D itself) will be called *(semi-)standard* if the following hold.

(i) All segments of Γ are (semi-)standard;
(ii) If a segment is outer and contains a vertex v of weight 0 then it also has an extremal vertex of weight 0. For a standard graph we require additionally that the neighbor in Γ of every extremal vertex of weight 0 is as well a zero vertex.

An NC completion $(\bar{V}, D)$ of an open surface V is called *(semi-)standard* if so is D.

These notions differ from those in [FKZ$_1$, Definition 2.13], where condition (ii) is absent.

Remark 1.5. (1) Every normal surface V has a standard NC completion $(\bar{V}, D)$. For applying [FKZ$_1$, Theorem 2.15(b)] every normal surface has an NC completion $(\bar{V}, D)$ such that Γ_D satisfies (i) in 1.4. By further elementary transformations we can achieve that also (ii) holds.

(2) The dual graph Γ_D of the boundary divisor of a standard NC-completion (V, D) is unique up to elementary transformations as follows from [FKZ$_1$, Theorem 3.1].

(3) A Gizatullin surface is a normal affine surface V with a completion $(\bar{V}, D)$, where D is a standard zigzag. Reversing D by a sequence of inner elementary transformations performed on $(\bar{V}, D)$ we obtain a new completion $(\bar{V}^\vee, D^\vee)$, which is called the *reversed standard completion*. It is uniquely determined by $(\bar{V}, D)$.

In the presence of an affine ruling we have more precise informations.

Lemma 1.6. *Let V be a normal affine surface. Given an affine ruling $\pi : V \to B$ over a smooth affine curve B the following hold.*

(a) There is a standard SNC completion $(\bar{V}, D)$ such that π extends to a $\mathbb{P}^1$-fibration $\bar{\pi} : \bar{X} \to \bar{B}$ over the smooth completion $\bar{B}$ of B. There is a unique curve, say, C_1 in D, which is a section of $\bar{\pi}$.
(b) With $(\bar{V}, D)$ as in (a), Γ_D is a tree, and $D - C_1$ has only rational components.
(c) A completion as in (a) is unique up to elementary transformations in extremal zero vertices of Γ_D. These extremal zero vertices correspond to components of $D - C_1$ that are full fibers of $\bar{\pi}$.

Proof. (a) and (b): Let $(\bar{V}, D)$ be an SNC completion of V. Blowing up $(\bar{V}, D)$ suitably we may assume that π extends to a regular map $\bar{\pi} : \bar{V} \to \bar{B}$. There is a unique horizontal component, say, C_1 of D which is a section of $\bar{\pi}$. The surface V being affine D is connected. Every fiber of $\bar{\pi}$ is a tree of rational curves and so D is as well a tree with rational components except possibly for C_1.

Any branch $\mathcal{B}$ of D at C_1 (i.e. connected component of $D - C_1$) is contained in a fiber of $\bar{\pi}$. If the intersection form of $\mathcal{B}$ is not negatively definite, then by Zariski's lemma [FKZ$_1$, 4.3] $\mathcal{B}$ coincides with the entire fiber of $\bar{\pi}$ and so can be contracted to a single 0-curve. Thus after contracting at most linear (-1)-curves of D contained in the fibers we may suppose that every branch of Γ_D at C_1 either is minimal and negative definite or is a single extremal zero vertex. Since B is affine, extremal zero vertices do exist. Performing outer elementary transformations at such a vertex we can achieve that $(C_1)^2 = 0$. Thus conditions (i) and (ii) of Definition 1.4 are fulfilled and so Γ_D is standard.

(c) Let $(\bar V', D')$ be a second SNC completion satisfying (a). We can find a common domination of $(\bar V, D)$ and $(\bar V', D')$ by a SNC completion $(\bar V'', D'')$ of V. We may suppose that there is no (-1)-curve in $\bar V''$ that is contracted in both $\bar V$ and $\bar V'$. If $C \neq C_1$ is a curve in D, which is not a full fiber, then its proper transform in $\bar V''$ has self-intersection ≤ -2. Hence if a (-1)-curve E in $\bar V''$ is contracted in $\bar V'$, then it is necessarily the proper transform of a component of D which is a full fibers of $\bar\pi$. Thus the indeterminacy locus of the map $\bar V \dashrightarrow \bar V'$ is contained in the union of full fibers, say, $F_1 \cup \ldots \cup F_a$ of $\bar\pi$ that are contained in D. Now it is a standard fact that $(\bar V', D)$ is obtained via a sequence of elementary transformations along $F_1 \cup \ldots \cup F_a$. □

Remarks 1.7. (1) Given a normal affine surface V with a semi-standard NC completion $(\bar V, D)$, the following hold.

(a) If Γ_D contains an extremal 0-vertex, then V is affine ruled. Indeed, every extremal 0-vertex C_0 of Γ_D induces a $\mathbb{P}^1$-fibration $\bar\pi : \bar V \to \bar B$ onto a smooth complete curve $\bar B$ so that C_0 is one of the fibers, see e.g. [BHPV, Chapt. V, Proposition 4.3]. In particular $\bar\pi$ restricts to an affine ruling on V.

(b) Conversely, if V carries an affine ruling then Γ_D has extremal 0-vertices. For by Remark 1.5(1) the given semi-standard completion is obtained by elementary transformations from a standard one with extremal 0-vertices as in Lemma 1.6. Applying [FKZ_1, Corollary 3.33'], (b) follows.

(2) Recall that a normal affine surface different from $\mathbb{A}^1 \times \mathbb{A}^1_*$ [3] is Gizatullin if and only if it admits two distinct affine rulings [Gi, Du_1]. Thus the affine ruling of a non-Gizatullin surface is unique up to an isomorphism of the base. Actually the base of the canonical affine ruling $V \to B$ is equal to $B := \operatorname{Spec} \mathrm{ML}(V)$, where $\mathrm{ML}(V) \subseteq \mathcal{O}(V)$ is the Makar-Limanov invariant of V, i.e. the common kernel of nonzero locally nilpotent derivations on $\mathcal{O}(V)$. Clearly, the ruling $V \to B$ is induced by the embedding $\mathrm{ML}(V) \hookrightarrow \mathcal{O}(V)$.

(3) We note that by Remark (1) any standard completion $(\bar V, D)$ of an affine ruled surface V yields an affine ruling $\pi : V \to B$ so that $(\bar V, D)$ is a standard completion associated to π as in Lemma 1.6.

(4) In the situation of Lemma 1.6, performing elementary transformations in extremal 0-vertices one can replace a standard completion by a semi-standard one where $(C_1)^2$ is any given number. This will be useful in the sequel.

Lemma 1.8. *Given an affine ruled surface V with standard completion $(\bar V, D)$ the following hold.*

(a) If V is Gizatullin then up to reversion (4) the dual graph Γ_D of D does not depend on the choice of such a completion.

[3] As before here $\mathbb{A}^1_* = \mathbb{A}^1 \setminus \{0\}$.

(b) If V is non-Gizatullin then the divisor D and its dual graph Γ_D are unique up to elementary transformations at extremal 0-vertices of Γ_D.

Proof. (a) is shown in [DG, Theorem 2.1], see also [FKZ$_1$, Corollary 3.32].

(b) With $\pi : V \to B$ the affine ruling of V we let $(\bar{V}, D)$ be a standard completion of V as constructed in Lemma 1.6(a). By Remark 1.7(1) the dual graph $\Gamma_{D'}$ of any other standard completion $(\bar{V}', D')$ has again an extremal 0-vertex that induces a $\mathbb{P}^1$-fibration $\bar{\pi}' : \bar{V}' \to \bar{B}'$. Since V is non-Gizatullin, up to a suitable isomorphism $\bar{B} \cong \bar{B}'$ the map $\bar{\pi}'$ restricts again to π on the open part V. Thus $(\bar{V}', D')$ is also a standard completion as in Lemma 1.6(a). Now the assertion follows form part (c) of that Lemma. □

2. Families of affine surfaces

2.1. **Families of standard completions.** By a *family* (sometimes called a *flat family*) we mean a flat morphism $\varphi : \mathcal{V} \to S$ of algebraic $\Bbbk$-schemes. We call it a family of normal or affine surfaces if every fiber $\mathcal{V}(s) = \varphi^{-1}(s)$, $s \in S$, has this property.

Definition 2.1. Let $\varphi : \bar{\mathcal{V}} \to S$ be a proper family of surfaces over an algebraic $\Bbbk$-scheme S. A Cartier divisor $\mathcal{D}$ in $\mathcal{V}$ will be called a *relative NC divisor* or a *family of NC-divisors* if $\mathcal{D}$ is proper over S and for every point $p \in \mathcal{D}$ either $\varphi|\mathcal{D}$ is smooth at p, or in a suitable neighborhood of p we have $\mathcal{D} = \mathcal{D}' \cup \mathcal{D}''$, where $\mathcal{D}'$, $\mathcal{D}''$ and the scheme theoretic intersection $\mathcal{D}' \cap \mathcal{D}''$ are smooth over S.

Every fiber $\mathcal{D}(s)$, $s \in S$, of a relative NC-divisor is an NC divisor. Its dual graph $\Gamma(s)$ is locally constant in S with respect to the étale topology. We say that a relative NC divisor $\mathcal{D}$ *has constant dual graph* Γ if for every irreducible component S' of S and all $s \in S'$ the irreducible components of both $\mathcal{D}' = \mathcal{D}|S'$ and $\mathcal{D}(s)$ are in one-to-one correspondence with the vertices of Γ.

Remark 2.2. (1) A relative NC-divisor is contained in the set $\mathrm{Reg}(\mathcal{V}/S)$ of points in $\mathcal{V}$ in which φ is smooth.

(2) If $\mathrm{Sing}(\mathcal{D}/S)$ is non-empty then $\mathrm{Sing}(\mathcal{D}/S) \to S$ is an unramified covering. It may happen that the irreducible components of $\mathcal{D}$ and $\mathcal{D}(s)$ are not in one-to-one correspondence even if the base S is connected (see Example 2.27 below). In such cases the family $(\Gamma(s))_{s\in S}$ has non-trivial monodromy, and $\mathrm{Sing}(\mathcal{D}/S) \to S$ is a non-trivial covering of S.

(3) If φ is smooth and all irreducible components of $\mathcal{D}$ are smooth then a relative NC-divisor $\mathcal{D}$ amounts locally in S to a logarithmic deformation of the fiber $(\mathcal{V}(s), \mathcal{D}(s))$ in the sense of [Ka].

The following definition is central in our considerations.

Definition 2.3. A flat family $\varphi : \mathcal{V} \to S$ of normal surfaces will be called *completable* if there is a proper smooth family of surfaces $\bar{\varphi} : \bar{\mathcal{V}} \to S$ together with an open embedding $\mathcal{V} \hookrightarrow \bar{\mathcal{V}}$ such that $\mathcal{D} = \bar{\mathcal{V}} \backslash \mathcal{V} \to S$ is a relative NC divisor with constant dual graph Γ. We call such a pair $(\bar{\mathcal{V}}, \mathcal{D})$ an *NC completion* of φ. This NC completion is called *(semi-)standard* if so is Γ.

The main result in this subsection is the following relative version of the existence of standard NC completions, see Remark 1.5(a).

Theorem 2.4. *Let $\varphi : \mathcal{V} \to S$ be a family of normal surfaces admitting an NC completion $(\bar{\mathcal{V}}, \mathcal{D})$. Then locally with respect to the étale topology there exists a standard NC completion $(\bar{\mathcal{V}}, \mathcal{D})$ of φ.*

To deduce this result we need some preparations. The following well known lemma is an easy consequence of a result due to Grothendieck; in the analytic setup this is a special case of Kodaira's theorem on the stability of certain submanifolds in deformations [Ko].

Lemma 2.5. *Let $\varphi : \mathcal{V} \to S$ be a flat family of normal surfaces over S. If the regular part of a special fiber $V = \mathcal{V}(s)$ contains a (-1)-curve E then over some étale neighbourhood of s there is a unique family of (-1)-curves $\mathcal{E} \subseteq \mathcal{V}$ with $E = \mathcal{E}(s)$.*

Proof. The curve E can be considered as a point $[E]$ in the relative Hilbert scheme $\mathbb{H}_{X/S}$. Applying the smoothness criterion of Grothendieck [Gro_1, Corollaire 5.4] the morphism $\mathbb{H}_{X/S} \to S$ is étale at $[E]$. Hence on a sufficiently small open neighbourhood U of $[E]$ in $\mathbb{H}_{X/S}$ the map $U \to S$ is étale. Now the universal subspace of $X \times_S U$ yields the desired family of (-1)-curves. □

In the next result we show that in the relative case one can perform blowups and blowdowns just as in the absolute case. In order to keep the formulation short, let us say that a divisor $\mathcal{E} \subseteq \bar{\mathcal{V}}$ is a *relative (-1)-curve over S* if it is smooth over S and restricts to a (-1)-curve in every fiber.

Lemma 2.6. *Let $\varphi : \bar{\mathcal{V}} \to S$ be a proper flat family of surfaces, and let $\mathcal{D}$ be a relative NC divisor on $\bar{\mathcal{V}}$ over S with constant dual graph Γ. Then the following hold.*

(a) If $\mathcal{D}_i \subseteq \mathcal{D}$ is a relative (-1)-curve, which corresponds to a vertex of degree ≤ 2 in Γ, then contracting $\mathcal{D}_i$ fiberwise yields a proper flat family of surfaces $\varphi : \bar{\mathcal{V}}_{cont} \to S$ and a relative NC divisor $\mathcal{D}_{cont}$ on $\bar{\mathcal{V}}_{cont}$ with a constant dual graph.

(b) Assume that $\mathcal{D}_i, \mathcal{D}_j$ are irreducible components of $\mathcal{D}$ such that $\mathcal{D}_i \cap \mathcal{D}_j \cong S$ is a section of φ. Blowing up $\bar{\mathcal{V}}$ along this section yields a flat family $\varphi' : \bar{\mathcal{V}}' \to S$ together with a relative NC divisor $\mathcal{D}' = \varphi'^{-1}(\mathcal{D})$ on $\bar{\mathcal{V}}'$ with constant dual graph.

Proof. A proof of (a) can be found e.g. in [FKZ_3, Lemma 1.15]. To deduce (b) it is enough to observe that locally near $\mathcal{D}_i \cap \mathcal{D}_j$ our family is a product family. □

We frequently use the following well known fact on the local triviality of $\mathbb{P}^1$-fibrations.

Lemma 2.7. *Let $\varphi : \mathcal{X} \to S$ be a smooth morphism such that every fiber of φ is isomorphic to $\mathbb{P}^1$. Then the following hold.*

(a) φ is a locally trivial $\mathbb{P}^1$-bundle in the étale topology.

(b) If φ admits a section $\sigma : S \to \mathcal{X}$ then φ is locally trivial in the Zariski-topology.

Proof. (b) is well known. It can be shown e.g. along the same lines as Lemma 1.16 in [FKZ_3]. The assertion (a) follows from (b) since in a suitable étale neighborhood of a given point $s \in S$ the map φ has sections. □

Lemma 2.6 yields the following result.

Lemma 2.8. *Let $\varphi : \mathcal{V} \to S$ be a family of normal surfaces, which admits an NC completion $(\bar{\mathcal{V}}, \mathcal{D})$ with constant dual graph Γ. If $\Gamma' \dashrightarrow \Gamma$ is a birational transformation* *then locally in the étale topology there exists an NC completion* $(\bar{\mathcal{V}}', \mathcal{D}')$ *of* φ *such that* $\mathcal{D}'$ *has constant dual graph* Γ'.

Proof. It is sufficient to treat the case where Γ' is obtained from Γ by a single blowup or blowdown. The case of a single blowdown follows from Lemma 2.6(a), while in the case of an inner blowup part (b) of that Lemma is applicable. It remains to treat the case of an outer blowup in an irreducible component, say $\mathcal{D}_i$ of $\mathcal{D}$. Locally in the étale topology around a point $s \in S$ there are sections $S \to \mathcal{D}_i$ not meeting the other components $\mathcal{D}_j$ with $\mathcal{D}_j \neq \mathcal{D}_i$. Blowing up such a section yields the desired result. □

We are now ready to give the proof of Theorem 2.4.

Proof of Theorem 2.4. By Theorem 2.15 in [FKZ$_1$] the dual graph Γ of $(\bar{\mathcal{V}}, \mathcal{D})$ can be transformed into a standard graph by a sequence of blowdowns and blowups. Applying Lemma 2.8 the result follows. □

Next we show that for completable families $\varphi : \mathcal{V} \to S$ of affine ruled surfaces the morphism φ is affine.

Proposition 2.9. *Let* $\mathcal{V} \to S$ *be a completable family of normal surfaces admitting an NC completion* $(\bar{\mathcal{V}}, \mathcal{D})$ *with constant connected dual graph* Γ.

(a) If Γ *has an extremal 0-vertex then there exists a relative semi-ample divisor* A *on* $\bar{\mathcal{V}}$ *supported by* $\mathcal{D}$.

(b) The morphism $\mathcal{V} \to S$ *can be factorized into a proper relative simultaneous contraction* $\mathcal{V} \to \mathcal{V}'$ *(the Remmert reduction) and an affine morphism* $\mathcal{V}' \to S$.

Proof. (b) follows immediately from (a).

(a) If Γ has extremal 0-vertices then it is a tree. Letting $\mathcal{D} = \sum_{i=0}^{n} \mathcal{D}_i$, where $\mathcal{D}_0 = \mathcal{D}_{01}$ corresponds to an extremal zero vertex of Γ, we consider a $\mathbb{Q}$-divisor $A = \sum_i a_i \mathcal{D}_i$ supported by $\mathcal{D}$, where $a_i > 0\ \forall i$. We let $a_0 = 1$, and then we choose the coefficients a_i rapidly decreasing when the distance in Γ of $\mathcal{D}_i$ to $\mathcal{D}_0$ increases. Then in each fiber, $A(s) \cdot \mathcal{D}_i(s) > 0$ for $i = 0, \ldots, n$. □

Corollary 2.10. *If* $\varphi : \mathcal{V} \to S$ *is a completable family of affine ruled surfaces then* φ *is an affine morphism. In particular if the base* S *of the family is affine so is its total space* $\mathcal{V}$.

Remark 2.11. (1) Let $\varphi : \mathcal{V} \to S$ be a family of normal affine surfaces which admits an NC completion $(\bar{\mathcal{V}}, \mathcal{D})$. If some fiber of φ is an affine ruled surface then so is every fiber of φ. Indeed, the dual graph Γ of an NC completion of an affine surface V can be transformed into a standard one Γ_{std}. Due to Remark 1.7(1) the surface V is affine ruled if and only if Γ_{std} has extremal 0-vertices. Since the dual graphs $\Gamma(s)$ of the fiber over $s \in S$ and then also $\Gamma_{\text{std}}(s)$ are constant the result follows.

(2) Later on we will show that, under the assumptions of (1), locally over S the total space $\mathcal{V}$ of the family carries a relative affine ruling. One can show that then it also carries locally a relative $\mathbb{G}_a$-action.

2.2. **Extended divisors.** *In this subsection* V *denotes an affine ruled surface.* In the sequel we use the following notation.

Notation 2.12. (a) Let $(\bar{V}, D)$ be a semi-standard completion of V. Such a completion induces a $\mathbb{P}^1$-fibration $\bar{\pi} : \bar{V} \to \bar{B}$, see Remark 1.7(1). In the case that $(\bar{V}, D)$ is standard $\bar{\pi}$ was called in [FKZ$_3$] the *standard fibration* associated to $(\bar{V}, D)$. Let $\pi : V \to B$ be the induced affine ruling. We note that by Lemma 1.6 any affine ruling on V appears in this way.

(b) The singularities of $\bar{V}$ are all lying in the open part V. Thus, if $\varrho : \tilde{V} \to \bar{V}$ is the minimal resolution of singularities then D can be considered as a divisor in $\tilde{V}$ denoted by the same letter D. We call $(\tilde{V}, D)$ a *resolved semi-standard completion* of V and the induced $\mathbb{P}^1$-ruling $\tilde{\pi} := \bar{\pi} \circ \varrho : \tilde{V} \to B$ again the associated standard ruling.

(c) The horizontal section $C_1 \cong \bar{B}$ of $\bar{\pi}$ (see Lemma 1.6), considered as a curve in $\tilde{V}$, yields a horizontal section of $\tilde{\pi}$. Let $C_{0i} = \tilde{\pi}^{-1}(c_{0i})$, $i = 1, \ldots, a$, denote the full fibers of $\tilde{\pi}$ contained in D, where $c_{0i} \in \bar{B} \setminus B$ are distinct points. They correspond to the extremal 0-vertices in Γ_D and are all adjacent to C_1. The vertex C_1 is connected to further vertices, say $C_{21}, \ldots, C_{2b}$. The latter represent fiber components of the degenerate fibers, say, $\tilde{\pi}^{-1}(c_{21}), \ldots, \tilde{\pi}^{-1}(c_{2b})$ of $\tilde{\pi}$. We let in the sequel

$$C_0 = \sum_{i=1}^{a} C_{0i} \quad \text{and} \quad C_2 = \sum_{j=1}^{b} C_{2j}.$$

Since the component C_{2i} meets the section C_1 transversally, it has multiplicity 1 in the fiber $\tilde{\pi}^{-1}(c_{2i})$. Hence for every $i = 1, \ldots, b$ the rest of the fiber $\tilde{\pi}^{-1}(c_{2i}) - C_{2i}$ can be blown down to a smooth point. In this way we obtain a $\mathbb{P}^1$-ruling (i.e. a locally trivial $\mathbb{P}^1$-fibration) $\psi : X \to \bar{B}$. Thus $\tilde{V}$ is obtained from X by a sequence of blowups with centers on $C_{2i} \backslash C_1$, $i = 1, \ldots, b$, and at infinitesimally near points. To simplify notation, we keep the same letters for the curves C_{0i}, C_1, and C_{2i} on $\bar{V}, \tilde{V}$ and for their respective images in X.

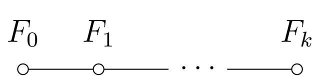
In analogy with the case of Gizatullin surfaces [FKZ$_2$], given a resolved standard completion $(\tilde{V}, D)$ of an affine ruled surface V, we associate to it the *extended divisor* D_{ext} and its *extended dual graph* Γ_{ext} of $(\tilde{V}, D)$ as follows.

Definition 2.13. The reduced divisor

$$D_{\text{ext}} = D \cup \tilde{\pi}^{-1}(\{c_{21}, \ldots, c_{2b}\})$$

is called the *extended divisor* of $(\tilde{V}, D)$ and its dual graph Γ_{ext} the *extended graph*. The connected components of $D_{\text{ext}} - D$ are called *feathers*, see [FKZ$_3$, 2.3].

As shown there each feather of D_{ext} is a linear chain of smooth rational curves on $\tilde{V}$

$\mathfrak{F}$: 

,

where the subchain $\mathfrak{R} = \mathfrak{F} - F_0 = F_1 + \ldots + F_k$ (if non-empty) contracts to a cyclic quotient singularity of V and F_0 is attached to some component C of $D - C_0 - C_1$. The curve F_0 is called the *bridge curve* of $\mathfrak{F}$. For instance, an A_k-singularity on V leads to an A_k-feather, where $\mathfrak{R}$ is a chain of (-2)-curves of length k. In particular, V can have at most cyclic quotient singularities. We note that this also follows from Miyanishi's Theorem [Mi, Lemma 1.4.4].

In the case of a smooth surface V one has $k = 0$ that is, every feather is an irreducible curve. Otherwise the dual graphs of $\mathfrak{R}$ correspond to the minimal resolutions of the singularities of V and are independent of the choice of a resolved completion. In contrast, the bridge curve F_0, its neighbor in D, and its self-intersection may depend on this choice.

We have the following uniqueness results for extended divisors.

Proposition 2.14. *(a) If the standard fibrations of two standard completions $(\bar{V}, D)$ and $(\bar{V}', D')$ of a Gizatullin surface V induce the same affine ruling $V \to \mathbb{A}^1$ then there is an isomorphism of the associated extended divisors $f : D_{\mathrm{ext}} \xrightarrow{\simeq} D'_{\mathrm{ext}}$ and of their extended dual graphs $\Gamma_f : \Gamma_{\mathrm{ext}} \xrightarrow{\simeq} \Gamma'_{\mathrm{ext}}$ sending D to D' and Γ_D to $\Gamma_{D'}$.*
(b) For a non-Gizatullin affine ruled surface V, up to an isomorphism the extended divisor D_{ext} and the extended graph Γ_{ext} do not depend on the choice of a standard completion $(\bar{V}, D)$ of V.

Proof. Performing an elementary transformation at an extremal zero vertex C_{0i} of Γ_D does not affect $D_{\mathrm{ext}} - C_{0i}$ while C_{0i} is replaced by another smooth rational curve C'_{0i}. Thus under such an operation D_{ext} and Γ_{ext} remain unchanged up to isomorphism. Now (a) and (b) follow from Lemma 1.6(c). We note that (a) is also contained in Lemma 5.12 in [FKZ$_3$]. □

2.15. In the setting of Notation 2.12, the irreducible components of $D_{\mathrm{ext}} - C_0 - C_1 - C_2$ in $\tilde{V}$ are obtained via a sequence of blowups starting from the projective ruled surface X as in 2.12,

$$\tilde{V} = X_m \to X_{m-1} \to \ldots \to X_2 \to X_1 = X \,, \tag{5}$$

with centers lying over $C_2 \backslash C_1$. Every component F of $D_{\mathrm{ext}} - C_0 - C_1 - C_2$ is created by one of the blowups $X_{k+1} \to X_k$ in (5). Since feathers do not contain (-1)-curves besides the bridge curves, the center of this blowup is necessarily a point lying on the image of D in X_k.

This justifies the following definition.

Definition 2.16. (See [FKZ$_3$, 2.3, 2.5] or [FKZ$_4$, 3.2.1].) Suppose that a component C of D is mapped onto a curve $\bar{C}$ in X_{k-1}. If under the blowup $X_k \to X_{k-1}$ the component F of D_{ext} is created by a blowup on $\bar{C}$ then C is called a *mother component* of F. If the blowup takes place in the image of the point $p_F \in C$ then p_F is called the *base point* of F.

The mother component C of a component F of a feather $\mathfrak{F}$ and its base point p_F are uniquely determined by F (see [FKZ$_3$, Lemma 2.4(a)]). Indeed, otherwise F would appear as the exceptional curve of a blowup of an intersection point $C \cap C'$ in some X_k with $C, C' \subseteq D$, so that D looses connectedness, which is impossible.

In contrast, a component $F = C$ of D can have two mother components (at most). For, if $C \subseteq D$ is the result of an inner blowup of an intersection point of two curves C' and C'' of the image of D in X_k, then the proper transforms of C' and C'' in D are both mother components of C. The curves C_{0i}, C_{2j}, and C_1 are orphans in that they have no mother component. This distinction leads to the following definition.

Definition 2.17. Consider as before a resolved completion $(\tilde{V}, D)$ of an affine ruled surface V. We say that $C \subseteq D$ is a $*$-*component* if it has two mother components in D, and a $+$-*component* otherwise. In particular, the curves C_1, C_{0i}, and C_{2i} are $+$-components.

Remark 2.18. Although the sequence (5) is not unique, in general, we can replace it by a canonical one by blowing down on each step $X_k \to X_{k-1}$ simultaneously all (-1)-curves in the fibers of the induced $\mathbb{P}^1$-fibration $\pi_k : X_k \to B$ different from the components of the curve C_2. The latter contractions are defined correctly since, on each step, no two (-1)-components in a degenerate fiber $\pi_k^{-1}(\pi_k(C_{2j}))$ different form C_{2j} are neighbors. This new sequence now only depends on the completion $(\bar{V}, D)$.

2.3. **Extended divisors in families.** Let now $\varphi : \mathcal{V} \to S$ be a family of normal affine surfaces. In order to generalize the preceding construction to the relative case, we will assume that φ is completable and admits a simultaneous resolution of singularities i.e., a proper morphism $\alpha : \mathcal{V}' \to \mathcal{V}$ such that the induced map $\varphi' : \mathcal{V}' \to S$ is smooth and $\alpha(s) : \mathcal{V}'(s) \to \mathcal{V}(s)$ is a resolution of singularities for every $s \in S$. Such a resolution will be called *minimal* if $\alpha(s) : \mathcal{V}'(s) \to \mathcal{V}(s)$ is a minimal resolution of singularities for every $s \in S$.

In many cases, a simultaneuous resolution does not exist, see Remark 2.28 for a more thorough discussion. In the case where a simultaneous resolution does exist, it can be chosen to be minimal due to the following result.

Proposition 2.19. *Let $(\bar{\mathcal{V}}, \mathcal{D})$ be an NC-completion of the family $\varphi : \mathcal{V} \to S$. Assume that $\mathcal{V}$ admits a simultaneous resolution of singularities. Then locally in the étale topology of S there is a minimal simultaneous resolution of singularities $\alpha : \tilde{\mathcal{V}} \to \bar{\mathcal{V}}$.*

Proof. Given a simultaneous resolution of singularities $\mathcal{V}' \to \mathcal{V}$ there is an NC completion $(\tilde{\mathcal{V}}, \mathcal{D})$ of $\mathcal{V}' \to S$ by this same relative NC divisor $\mathcal{D}$ so that $\tilde{\mathcal{V}} \to \bar{\mathcal{V}}$ is again a simultaneous resolution of singularities.

To make this resolution minimal let us fix a point $s_0 \in S$. If $\tilde{\mathcal{V}}(s_0) \to \bar{\mathcal{V}}(s_0)$ is not a minimal resolution then there is a (-1)-curve E in the fiber $\tilde{\mathcal{V}}(s_0)$. By Lemma 2.5 near[4] s_0 there is a unique family of (-1)-curves $\mathcal{E} \to S$ in $\tilde{\mathcal{V}}$ with $\mathcal{E}(s_0) = E$. Since $\mathcal{V}$ is a family of affine surfaces, the fibers of $\mathcal{E}$ are contracted in $\mathcal{V}$ and then also in $\bar{\mathcal{V}}$. According to Lemma 2.6(a) contracting $\mathcal{E}$ fiberwise leads to a flat family $\tilde{\mathcal{V}}'$ over S. Replacing $\tilde{\mathcal{V}}$ by $\tilde{\mathcal{V}}'$ and repeating the argument, after a finite number of steps we arrive at a simultaneous resolution near s_0 denoted again $\tilde{\mathcal{V}} \to \bar{\mathcal{V}}$, which is minimal in the fiber over s_0.

We claim that near s_0, the map $\tilde{\mathcal{V}}(s) \to \bar{\mathcal{V}}(s)$ is a minimal resolution for every $s \in S$. Indeed, consider the relative Hilbert scheme $\mathbb{H}_{\tilde{\mathcal{V}}/S}$. Every of its irreducible components is proper over S (see [Gro$_2$]), and the (-1)-curves in $\mathbb{H}_{\tilde{\mathcal{V}}/S}$ form a constructible subset, say, A. Then also the image of A in S is constructible.

Hence, if in every neighborhood of s_0 there are points for which the resolution $\tilde{\mathcal{V}}(s) \to \bar{\mathcal{V}}(s)$ is not minimal, then there is a smooth curve T, a morphism $\gamma : T \to S$, and a point $t_0 \in T$ with $\gamma(t_0) = s_0$ together with a T-flat family of curves $\mathcal{C} \subseteq \tilde{\mathcal{V}} \times_S T$ such

[4]In the étale topology.

that over $t \neq t_0$ the fiber $C_t := \mathcal{C}(t)$ is a (-1)-curve in $\tilde{\mathcal{V}}(\gamma(t))$. Over t_0 we obtain a curve $C = \mathcal{C}(t_0)$ in $\tilde{V} = \tilde{\mathcal{V}}(s_0)$.

We claim that C is contracted in $\bar{V}$. To show this, let $p : \mathcal{C} \to \bar{\mathcal{V}}$ be the induced map. As $\mathcal{C}$ is flat over T, the complement $\mathcal{C}\backslash C$ is dense in $\mathcal{C}$ and so by continuity $p(\mathcal{C}\backslash C)$ is dense in $p(\mathcal{C})$. Since for $t \neq t_0$ the curve C_t is contracted in $\bar{\mathcal{V}}$ to a point, the image $p(\mathcal{C}\backslash C)$ and then also $p(\mathcal{C})$ are curves in $\bar{\mathcal{V}}$. Thus C is contracted to the point in $p(\mathcal{C})$ lying over s_0 in $\bar{V}$, proving the claim.

Because of $C.D = C_t.\mathcal{D}(t) = 0$ for $t \neq t_0$ the curve C does not meet D and so is contained in $\tilde{V}\backslash D$. In particular, since $\tilde{V} \to \bar{V}$ is a minimal resolution of singularities every component C_i of C is smooth and $C_i^2 \leq -2$.

Letting $C = \sum n_i C_i$, $n_i > 0$, and K_t be the canonical divisor on $\mathcal{V}'(\gamma(t))$, we get

$$-1 = C_t.K_t = C.K = \sum n_i C_i.K\,,$$

where $K = K_{t_0}$. Hence $C_i.K \leq -1$ for some i. This is only possible if $C_i^2 \geq -1$, which contradicts the minimality of resolution in the fiber over s_0. Now the proof is completed. □

We need the following definition.

Definition 2.20. Let $\varphi : \mathcal{V} \to S$ be a flat family of normal surfaces, $\varphi' : \tilde{\mathcal{V}} \to S$ a flat proper family of smooth surfaces and $\mathcal{D} \subseteq \tilde{\mathcal{V}}$ a divisor. We call $(\tilde{\mathcal{V}}, \mathcal{D})$ a *resolved completion* of $\varphi : \mathcal{V} \to S$ if the following conditions are satisfied.

(a) $\varphi' : \mathcal{V}' := \tilde{\mathcal{V}}\backslash\mathcal{D} \to S$ is a minimal simultaneous resolution of singularities of $\varphi : \mathcal{V} \to S$;

(b) $(\tilde{\mathcal{V}}, \mathcal{D})$ is a NC completion of $\varphi' : \mathcal{V}' \to S$ as in Definition 2.3; in particular it has a constant dual graph Γ.

We call the family $(\tilde{\mathcal{V}}, \mathcal{D})$ *standard or semi-standard* if so is Γ.

In the next Proposition we show that one can organize the standard morphisms of Notation 2.12 into a family.

Proposition 2.21. *Let $\varphi : \mathcal{V} \to S$ be a family of affine ruled surfaces, which admits a resolved semi-standard completion $(\tilde{\mathcal{V}}, \mathcal{D})$ with associated map $\tilde{\varphi} : \tilde{\mathcal{V}} \to S$. Then there exists a factorization of $\tilde{\varphi}$ as*

$$\tilde{\mathcal{V}} \xrightarrow{\tilde{\Pi}} \mathcal{B} \longrightarrow S\,,$$

where $\mathcal{B} \to S$ is a family of smooth complete curves and $\tilde{\Pi}$ induces in every fiber over $s \in S$ the standard morphism $\tilde{\pi}_s$ from Notation 2.12.

Proof. The extremal zero vertices of Γ correspond to families of curves $\mathcal{C}_{0i} \subseteq \mathcal{D}$, $i = 1, \ldots, a$. Their common neighbor $\mathcal{C}_1 \subseteq \mathcal{D}$ is a family of curves over S. Given a point $s \in S$ we let C_s and $\tilde{V}_s$ be the fibers of $\mathcal{C} = \mathcal{C}_{01}$ and $\tilde{\mathcal{V}}$ over s, respectively. We consider a semi-standard $\mathbb{P}^1$-fibration $\tilde{\pi}_s : \tilde{V}_s \to \bar{B}_s$ on $\tilde{V}_s$ as in 2.12. The curve C_s is the full fiber $\tilde{\pi}^{-1}(c_s)$ of $\tilde{\pi}_s$ over some point $c_s \in \bar{B}_s$. The direct image sheaf $R^1\tilde{\pi}_{s*}(\mathcal{O}_{\tilde{V}_s}(mC_s))$ vanishes for $m \geq 0$, while $\tilde{\pi}_{s*}(\mathcal{O}_{\tilde{V}_s}(mC_s)) \cong \mathcal{O}_{\bar{B}_s}(mc_s)$. Applying the Leray spectral sequence we obtain that $H^1(\tilde{V}_s, \mathcal{O}_{\tilde{V}_s}(mC_s)) \cong H^1(\bar{B}_s, \mathcal{O}_{\bar{B}_s}(mc_s)) = 0$ for $m \gg 0$ and all $s\in S$. Hence $R^1\tilde\varphi_*(\mathcal{O}_{\tilde{\mathcal{V}}}(m\mathcal{C}))$ vanishes and is in particular locally free. By [Ha$_2$, Chapt. III, Theorem 12.11] the map

$$\tilde\varphi_*(\mathcal{O}_{\tilde{\mathcal{V}}}(m\mathcal{C}))_s\to H^0(\tilde V_s,\mathcal{O}_{\tilde V_s}(mC_s))$$

is surjective for all $s\in S$. Shrinking S we may assume that there are sections

$$\beta_0,\dots,\beta_N\in H^0(S,\tilde\varphi_*(\mathcal{O}_{\tilde{\mathcal{V}}}(m\mathcal{C})))$$

whose images in $H^0(\tilde V_s,\mathcal{O}_{\tilde V_s}(mC))$ generate the latter vector space for every $s\in S$. Consider now the morphism

$$f=((\beta_0:\ldots:\beta_N),\tilde\varphi):\tilde{\mathcal{V}}\to\mathbb{P}^N\times S\,.$$

By construction its restriction $f_s:\tilde V_s\to\mathbb{P}^N$ to the fiber over s factors through the standard morphism

$$\tilde V_s\xrightarrow{\tilde\pi_s}\bar B_s\xrightarrow{\gamma_s}\mathbb{P}^N\,.$$

Here γ_s denotes the map given by the linear system $|m\cdot c_s|$ on B_s.

We claim that the sheaf $\mathcal{O}_{\mathcal{B}}=f_*(\mathcal{O}_{\tilde{\mathcal{V}}})$ is flat over S. As this is a local problem, we may suppose in the proof of this claim that $S=\operatorname{Spec}A$ is affine. Using the vanishing of $H^1(\tilde V_s,\mathcal{O}_{\tilde V_s})$ for all $s\in S$ by [Ha$_2$, Chapt. III, Proposition 2.10] the left exact functor

$$M\mapsto T(M)=H^0(\tilde{\mathcal{V}},\mathcal{O}_{\tilde{\mathcal{V}}}\otimes_A M)$$

on finite A-modules is also right exact. Applying Proposition 12.5 in [Ha$_2$, Chapt. III] the natural map $T(A)\otimes_A M\to T(M)$ is an isomorphism. Thus the functor $M\mapsto T(A)\otimes_A M$ is exact and so $T(A)$ is a flat A-module. Consequently

$$\mathcal{O}_{\mathcal{B}}:=f_*(\mathcal{O}_{\tilde{\mathcal{V}}})$$

is a S-flat sheaf on $\mathbb{P}^N\times S$, proving the claim.

The sheaf $\mathcal{O}_{\mathcal{B}}$ gives rise to a flat family of smooth curves $\mathcal{B}\to S$ with fibers $\bar B_s$. By construction the morphism $\tilde\Pi=f:\tilde{\mathcal{V}}\to\mathcal{B}$ induces over each point $s\in S$ the standard fibration from Notation 2.12, proving the Lemma. $\square$

Definition 2.22. Let $\varphi:\mathcal{V}\to S$ be as before a family of affine ruled surfaces with a resolved semi-standard completion $(\tilde{\mathcal{V}},\mathcal{D})$. The morphism $\tilde\Pi:\tilde{\mathcal{V}}\to\mathcal{B}$ onto a family of curves $\mathcal{B}\to S$ constructed in the proof of Proposition 2.21 by means of the linear system $|m\mathcal{C}|$, $m\gg 0$, will be called the *standard morphism* associated to $(\bar{\mathcal{V}},\mathcal{D})$. This is a $\mathbb{P}^1$-fibration over $\mathcal{B}$ which induces in each fiber over $s\in S$ the standard $\mathbb{P}^1$-fibration.

2.23. The vertex of Γ which corresponds to $\mathcal{C}_1$ has neighbors given by $\mathcal{C}_{01},\dots,\mathcal{C}_{0a}$ and further ones given (locally in S) by smooth families of rational curves $\mathcal{C}_{21},\dots,\mathcal{C}_{2b}\subseteq\tilde{\mathcal{V}}$ over S. The families $\mathcal{C}_{01},\dots,\mathcal{C}_{0a}$ arise as preimages under $\tilde\Pi$ of sections, say, $\gamma_{0i}:S\to\mathcal{B}$, $i=1,\dots,a$, while $\mathcal{C}_{2j}$ are projected in $\mathcal{B}$ to sections, say, $\gamma_{2j}:S\to\mathcal{B}$, $j=1,\dots,b$. Moreover $\mathcal{C}_1\cong\mathcal{B}$ is a section of the standard morphism $\tilde\Pi$. On the curve $\bar B_s=\mathcal{B}(s)$ the sections γ_{0i} and γ_{2j} yield the points $c_{0i}=\gamma_{0i}(s)$ and $c_{2j}=\gamma_{2j}(s)$ (cf. Definition 2.12).

The following Factorization Lemma will be useful in the sequel.

Lemma 2.24. (Factorization Lemma) *Let $\mathcal{V} \to S$ be a family of affine ruled surfaces, which admits a resolved standard completion $(\tilde{\mathcal{V}}, \mathcal{D})$ (see Definition 2.20). Then locally in the étale topology of S there is a factorization of the associated standard morphism $\tilde{\Pi} : \tilde{\mathcal{V}} \to \tilde{B}$ as*

$$\tilde{\mathcal{V}} = \mathcal{X}_m \to \mathcal{X}_{m-1} \to \ldots \to \mathcal{X}_1 \to \mathcal{B}\,, \tag{6}$$

where

(a) every morphism $\mathcal{X}_i \to \mathcal{X}_{i-1}$ is a blowup of a section $\gamma_i : S \to \mathcal{X}_{i-1}$ contained in the image of $\mathcal{D}$ in $\mathcal{X}_{i-1}$ with the exceptional divisor $\mathcal{E}_i \subseteq \mathcal{X}_i$ being a relative (-1)-curve over S,

(b) $\mathcal{B} \to S$ is a smooth family of complete curves over S, and

(c) $\mathcal{X}_1 \to \mathcal{B}$ is a locally trivial $\mathbb{P}^1$-bundle.

Proof. On the fiber over some point s we consider the standard fibration

$$\tilde{\pi}_s =: \tilde{\Pi}|\tilde{V}_s : \tilde{V}_s = \tilde{\mathcal{V}}(s) \to \bar{B}_s\,.$$

We let $C_{0i,s} = \mathcal{C}_{0i}(s)$, $C_{1,s} = \mathcal{C}_1(s)$, and $C_{2j,s} = \mathcal{C}_{2j}(s)$ be the respective fibers over s so that $C_{2j,s}$ is contained in a fiber $\tilde{\pi}_s^{-1}(c_{2j,s})$ over some point $c_{2j,s}$ of $\bar{B}_s$. Blowing down the divisors $\tilde{\pi}_s^{-1}(c_{2j,s}) - C_{2j,s}$, $j = 1, \ldots, b$, we arrive at a locally trivial $\mathbb{P}^1$-bundle, say, $\Psi : X \to \bar{B}$ such that $\bar{V}$ is obtained by a sequence of blowups of X as in (5) with centers on $\bigcup_{j=1}^b \Psi^{-1}(c_{2j}) \backslash C_1$ and infinitesimally near points.[5]

In particular, in the last blowup $X_m = \tilde{V} \to X_{m-1}$ in (5) there is a (-1)-curve, say, $E_m \subseteq \tilde{V}$ which is blown down in X_{m-1}. By Lemma 2.5(a) near s_0 there is a family of (-1)-curves $\mathcal{E}_m \subseteq \tilde{\mathcal{V}}$ inducing E_m over s_0. Applying Lemma 2.6(a) we can blow down $\mathcal{E}_m$ and obtain a morphism $\mathcal{X}_m = \tilde{\mathcal{V}} \to \mathcal{X}_{m-1}$ inducing $X_m \to X_{m-1}$ in the fiber over s_0. Repeating this procedure we arrive at a factorization (6).

It remains to prove that $\mathcal{X}_1 \to \mathcal{B}$ is a locally trivial $\mathbb{P}^1$-fibration. Let us show first that the morphism $\mathcal{X}_1 \to \mathcal{B}$ is flat. Indeed, using in every step of our construction Lemma 2.6(a) $\mathcal{X}_i \to S$ is a flat morphism. In particular $\mathcal{X}_1 \to S$ is flat. Since for every $s \in S$ also $\mathcal{X}_1(s) \to \mathcal{B}(s)$ is flat, the flatness of $\mathcal{X}_1 \to \mathcal{B}$ follows from [Ei, Corollary 6.9].

Now the fact that $\mathcal{X}_1 \to \mathcal{B}$ is a locally trivial $\mathbb{P}^1$-fibration is a consequence of Lemma 2.7. For $\mathcal{C}_1 \to \mathcal{B}$ is an isomorphism and so the inclusion $\mathcal{C}_1 \hookrightarrow \mathcal{X}_1$ yields a section of $\mathcal{X}_1 \to \mathcal{B}$. □

By construction all blowups in the sequence (6) take place over disjoint sections $\gamma_{21}(S), \ldots, \gamma_{2b}(S)$ of $\mathcal{C}_2 \setminus \mathcal{C}_1 = \bigcup_{j=1}^b \mathcal{C}_{2j} \backslash \mathcal{C}_1$ over S in $\mathcal{X}_1$. In analogy with the absolute case we introduce the extended divisor of $(\bar{\mathcal{V}}, \mathcal{D})$ as

$$\mathcal{D}_{\text{ext}} = \mathcal{D} \cup \tilde{\Pi}^{-1}\left(\gamma_{21}(S) \cup \ldots \cup \gamma_{2b}(S)\right) \tag{7}$$

(see Definition 2.13) . We emphasize that this is not, in general, a relative SNC divisor in the sense of Definition 2.1 (see Example 2.27 below). However, we have the following result.

Corollary 2.25. *Under the assumptions of Lemma 2.24 the extended divisor $\mathcal{D}_{\text{ext}}$ is flat over S, and each fiber $\mathcal{D}_{\text{ext}}(s)$ over $s \in S$ is just the extended divisor of $(\tilde{V}_s, D_s)$.*

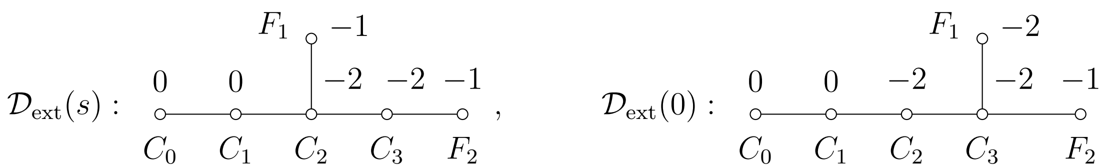
[5]As in Notation 2.12 we use the same letters for C_{0i}, C_1 and for their images in X.

Proof. Locally $\mathcal{D}_{\text{ext}}$ is the set of zeros of a non-zero divisor on $\tilde{\mathcal{V}}$ that is also a non-zero divisor in each fiber. Thus the first part follows e.g. from [Ei, Corollary 6.9]. The second part follows from the fact that blowing up a section of $\tilde{\mathcal{V}} \to S$ commutes with taking the fiber over s. □

Remark 2.26. In analogy with the absolute case we call a component $\mathcal{A}$ of the extended divisor $\mathcal{D}_{\text{ext}}$ from (7) a *boundary component* if it is in $\tilde{\mathcal{V}}$ a component of $\mathcal{D}$, and otherwise a *feather component*. Since the dual graph of $\mathcal{D}(s)$ is constant, $\mathcal{A}$ is a boundary or a feather component if and only if its fiber $\mathcal{A}(s)$ over some point $s \in S$ is. However the neighbouring components of $\mathcal{A}(s)$ may change in nearby fibers. We call this phenomenon "jumping" of feathers.

Let us give an example where this actually happens.

Example 2.27. Letting V be the Danilov-Gizatullin surface with dual zigzag $\Gamma = [[0, 0, -2, -2]]$ we consider the trivial family $\mathcal{V} = V \times S$ over $S = \mathbb{A}^1$. We construct a family of completions $(\bar{\mathcal{V}}, \mathcal{D})$ over S with constant dual zigzag Γ, where the dual graph $\mathcal{D}_{\text{ext}}(s)$ of the extended divisor is not constant on S. Actually we will see that the dual graphs of $\mathcal{D}_{\text{ext}}(s)$, $s \neq 0$, and $\mathcal{D}_{\text{ext}}(0)$ are as follows:

(8) 

The construction starts from the quadric $X_1 = Q = \mathbb{P}^1 \times \mathbb{P}^1$ with the $\mathbb{P}^1$-fibration given by the first projection $X_1 \to B = \mathbb{P}^1$ and with the curves

$$C_0 = \{\infty\} \times \mathbb{P}^1, \qquad C_1 = \mathbb{P}^1 \times \{\infty\}, \quad \text{and} \quad C_2 = \{0\} \times \mathbb{P}^1 .$$

Blowing up X_1 at the point $(0, 0)$ creates a feather F_1 on the blown up surface X_2. Letting $\mathcal{X}_2 = X_2 \times \mathbb{A}^1$, $\mathcal{C}_i = C_i \times \mathbb{A}^1$, $i = 0, 1, 2$, and $\mathcal{F}_1 = F_1 \times \mathbb{A}^1$, we blow up $\mathcal{X}_2$ along a "diagonal" section $\gamma_3 : S \to \mathcal{C}_2$, which meets $\mathcal{F}_1$ over $s = 0$ only. By this blowup in every fiber over s the boundary component C_3 is created. Blowing up a section $\gamma_4(S) \subseteq \mathcal{C}_3$, which does not meet $\mathcal{C}_2 \cup \mathcal{F}_1$, we obtain a second feather $\mathcal{F}_2$ of $\mathcal{D}_{\text{ext}}$ on the new threefold $\bar{\mathcal{V}} = \mathcal{X}_4$. Thus the feather $\mathcal{F}_1(s)$, $s \neq 0$, jumps from C_2 to C_3 over the point $s = 0$ of the base as indicated by the dual graphs.

We end this section with a remark on the existence of a resolved completion in our setup.

Remark 2.28. (1) Let V be a normal affine ruled surface with a standard completion $(\bar{V}, D)$ and associated standard fibration $\bar{V} \to \bar{B}$. Then every deformation $\mathcal{V} \to S$ of V over an Artinian germ S can be extended to a deformation $(\bar{\mathcal{V}}, \mathcal{D}) \to S$ of the completion $(\bar{V}, D)$. Indeed, the infinitesimal deformations and obstructions of $(\bar{V}, D)$ are given by

$$T^1 = H^1(\bar{V}, R\mathcal{H}om_{\bar{V}}(\Omega^1_{\bar{V}}\langle D\rangle, \mathcal{O}_{\bar{V}})) \quad \text{and} \quad T^2 = H^2(\bar{V}, R\mathcal{H}om_{\bar{V}}(\Omega^1_{\bar{V}}\langle D\rangle, \mathcal{O}_{\bar{V}}))$$

respectively, see [Se]. Consider the triangle in the derived category

$$0 \to \Theta_{\bar{V}}\langle D\rangle \to R\mathcal{H}om_{\bar{V}}(\Omega^1_{\bar{V}}\langle D\rangle, \mathcal{O}_{\bar{V}}) \to \mathcal{G}^\bullet \to 0 . \qquad (9)$$

The first and the second cohomology of $\mathcal{G}^\bullet$ are just

$$T^1_{\rm loc} = {\rm Ext}^1_V(\Omega^1_V, \mathcal{O}_V)) \quad \text{and} \quad T^2_{\rm loc} = {\rm Ext}^2_V(\Omega^1_V, \mathcal{O}_V))\,,$$

which control the deformations and obstructions of the affine part, respectively (cf. [Se]). Using the long exact cohomology sequence of (9) we get an exact sequence

$$T^1 \to T^1_{\rm loc} \to H^2(\bar V, \Theta_{\bar V}\langle D\rangle) \to T^2 \to T^2_{\rm loc}\,.$$

The term in the middle is dual to ${\rm Hom}_{\bar V}(\Theta_{\bar V}\langle D\rangle, \omega_{\bar V})$ and so vanishes as the restriction of an element of the latter group to a generic fiber of $\bar V \to \bar B$ vanishes.

Thus by standard deformation theory (see[Se]) the functor assigning to a deformation of $(\bar V, D)$ the deformations of the open part is formally smooth.

(2) Consider, for instance, a surface V which has a cyclic quotient singularity with several components in the versal deformation. Its versal family $\mathcal{V} \to S$ admits at least formally a simultaneous completion by (1) but does not admit a simultaneous resolution. See [KS] for examples of cyclic quotients for which the versal deformation space is not irreducible and does not coincide with the Artin component.

3. Deformation equivalence

3.1. **Normalized extended graph and deformation equivalence.** The purpose of this and the next sections is to characterize in combinatorial terms the deformation equivalence for affine ruled surfaces, which we introduce as follows.

Definition 3.1. We say that two normal surfaces V and V' are *deformations of each other* if there exists a flat family of surfaces $\varphi : \mathcal{V} \to S$ over a connected base S with the following properties.

(a) φ admits a resolved completion (see Definition 2.20), and
(b) $V \cong \mathcal{V}(s)$ and $V' \cong \mathcal{V}(s')$ for some points $s, s' \in S$.

This generates an equivalence relation called *deformation equivalence*: $V \sim V'$ if and only if there exists a chain $V = V_1, \dots, V_n = V'$ such that V_i and V_{i+1} are deformations of each other for every $i = 1, \dots, n-1$.

In the sequel V will be a normal affine ruled surface. Let $(\bar V, D)$ be a standard completion of V, and let $\tilde V \to \bar V$ be the minimal resolution of singularities. The extended divisor $D_{\rm ext}$ as in Definition 2.13 and the extended graph $\Gamma_{\rm ext} = \Gamma(D_{\rm ext})$ depend in general on the choice of a completion $(\bar V, D)$. We associate to $\Gamma_{\rm ext}$ now another graph $N(\Gamma_{\rm ext})$ called the *normalized extended graph* which will turn out to be independent of the choice of completion and thus is an invariant of the affine surface.

Definition 3.2. Given a component C of Γ we let δ_C denote the number of feather components F of $\Gamma_{\rm ext}$ with mother component C (see Definitions 2.13 and 2.16). The *normalized extended graph* $\Delta = N(\Gamma_{\rm ext})$ of $(\bar V, D)$ is the weighted graph obtained from $\Gamma = \Gamma_D$ by attaching to every component C of Γ exactly δ_C extremal (-1)-vertices called the *feathers of* $N(\Gamma_{\rm ext})$.

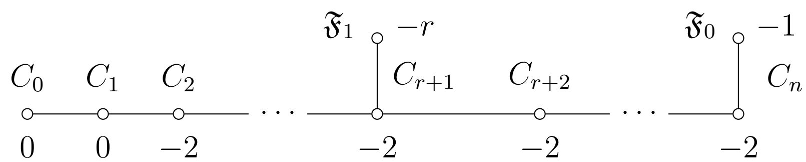
Thus $\Gamma_{\rm ext}$ and $N(\Gamma_{\rm ext})$ contain both Γ as a distinguished subgraph and have the same number of feather components, and even the same number of them with a given mother component. Furthermore, every feather of $N(\Gamma_{\rm ext})$ consists of a single extremal (-1)-vertex, and these vertices are in one-to-one correspondence with the feather components of Γ_{ext}. For instance, for the family $(\mathcal{X}(s), \mathcal{D}(s))$ as in Example 2.27 we have $\delta_{C_2} = \delta_{C_3} = 1$ at any point $s \in S$, so the normalized graph Δ is in both cases the dual graph on the left in (8).

Remark 3.3. The normalized extended graph $N(\Gamma_{\mathrm{ext}})$ can be uniquely recovered from the extended graph Γ_{ext} via the simultaneous contraction procedure described in 2.15.

In the case of a Gizatullin surface V with a standard completion $(\bar{V}, D)$ the boundary zigzag Γ can be reversed by moving the two zeros to the other end as in (4). Since Γ is contained in the extended divisor Γ_{ext} and also in $\Delta = N(\Gamma_{\mathrm{ext}})$, we can perform the same operation in Γ_{ext} and in Δ. In this way we obtain a new normalized extended graph $\Delta^\vee = N(\Gamma_{\mathrm{ext}}^\vee)$ called the *reversion* of Δ.

Example 3.4. Recall that every Danilov-Gizatullin surface $V = V_n$ (see the Introduction) can be realized as the complement of an ample section σ in a Hirzebruch surface Σ_d with self-intersection $\sigma^2 = n > d$. By a theorem of Danilov-Gizatullin [DG, Theorem 5.8.1][6] the isomorphism class of V_n depends only on n and not on the choice of σ or of the concrete Hirzebruch surface Σ_d. According to Example 1.22 in [FKZ$_3$], for every r in the range $1, \ldots, n-1$ the surface V_n admits a standard completion $(\bar{V}_{n,r}, D)$ with extended graph

(10) $\Gamma_{\mathrm{ext},r}$:



,

where the bottom line Γ_D corresponds to the boundary zigzag D, the feather $\mathfrak{F}_1$ consists of a single $(-r)$-component F_1, and $\mathfrak{F}_0$ of a single (-1)-component (for $r+1 = n$ both feathers $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are attached to C_n). For every $r = 1, \ldots, n$ the mother component of F_1 is C_2. Thus all these completions have the same normalized extended graph $N(\Gamma_{\mathrm{ext},r}) = \Gamma_{\mathrm{ext},1}$. In particular, $\Gamma_{\mathrm{ext},1}$ is as well the normalized extended graph of the reversed standard completion $(\bar{V}_{n,r}^\vee, D^\vee)$ with the extended graph $\Gamma_{\mathrm{ext},n-r+2}$.

In the next result we show that $N(\Gamma_{\mathrm{ext}})$ is essentially independent of the choice of the completion $(\bar{V}, D)$ of an affine ruled surface V and thus is an invariant of V.

Theorem 3.5. *Let V be a normal affine ruled surface.*

(a) If V is not Gizatullin then up to an isomorphism its normalized extended graph $N(\Gamma_{\mathrm{ext}})$ is independent of the choice of a resolved standard completion $(\tilde{V}, D)$ of V.[7]

(b) If V is Gizatullin then its normalized extended graph $N(\Gamma_{\mathrm{ext}})$ is uniquely determined by V up to reversion.

Proof. Assertion (a) follows from Corollary 2.14(b). Indeed, due to this corollary for a non-Gizatullin affine ruled surface V the extended graph Γ_{ext}, and hence also the normalized extended graph $N(\Gamma_{\mathrm{ext}})$, does not depend on the choice of a standard completion.

[6] See also [CNR, FKZ$_5$].

[7] Hence this graph $N(\Gamma_{\mathrm{ext}})$ is a combinatorial invariant of the affine ruled surface V.

In the smooth case the proof of (b) is a consequence of [FKZ$_3$, Corollary 3.4.3]. For the normal case we provide a proof in Theorem 3.11 below. □

Now we come to the main theorem of our paper. In terms of the normalized extended graphs deformation equivalence can be characterized as follows.

Theorem 3.6. *Two affine ruled surfaces V and V' with resolved standard completions $(\bar V, D)$ and $\bar V', D')$ are deformation equivalent if and only if the following two conditions hold:*

(i) The associated normalized extended graphs are isomorphic or, in the case of Gizatullin surfaces, are isomorphic up to reversion.
(ii) The horizontal curves $C_1 \subseteq D$ and $C_1' \subseteq D'$ have the same genus.

We establish the 'only if'-part in subsection 3.3, and the 'if'-part in Corollary 4.8 in section 4.

3.2. **A general matching principle.** Part (b) of Theorem 3.5 follows immediately from Theorem 3.11(b) below. To formulate this theorem we introduce the following notation.

Notation 3.7. Let C be a component of $D - C_0 - C_1$. Then C_1 is sitting on a unique branch $\Gamma_{D'}$ of the tree Γ_D at the vertex C; note that D' contains a mother component of C. We let $C \cap D' = \{\infty_C\}$. For a +-component C (see Definition 2.17) we let $C^* = C\backslash\{\infty_C\} = C\backslash D'$.

A $*$-component C has two mother components, say, $C' \subseteq D'$ and $C'' \subseteq D - D'$. For such a component C we let $C^* = C\backslash(D' \cup D'') = C \setminus \{\infty_C, 0_C\}$, where $D'' \neq D'$ is the branch of D at C containing C'' and $\{0_C\} = C \cap D''$.

Next we define the *configuration invariant* of V. In the particular case of smooth Gizatullin surfaces, this invariant was introduced in [FKZ$_4$, §3.2].

Definition 3.8. (cf. [FKZ$_4$, (3.2.3)]) Given a component C of $D - C_0 - C_1$ we consider the cycle on C

$$Q_C = \sum_F p_F\,,$$

where F runs over all feather components with mother component C, and $p_F \in C$ is the base point of F (see Definition 2.16). From the fact that every feather has just one mother component it follows that the cycle Q_C is contained in C^*. Letting $\delta_C = \deg Q_C$ be the number of feathers with mother component C, we consider the Hilbert scheme $\mathbb{H}_{\delta_C}(C^*)$ of subschemes of length δ_C in C^*, or, equivalently, of effective zero cycles of degree δ_C on C^*. This space is the quotient of the Cartesian power $(C^*)^{\delta_C}$ modulo the symmetric group $\mathcal{S}_{\delta_C}$. Let $\mathrm{Aut}(C^*, \infty_C)$ denote the group of automorphisms of the curve $C \cong \mathbb{P}^1$ fixing the point $\infty_C = C \cap D'$ and, if C is a $*$-component, also the point $0_C = C \cap D''$ (see 3.7). This group acts diagonally on $(C^*)^{\delta_C}$ commuting with the $\mathcal{S}_{\delta_C}$-action. Hence it also acts on the Hilbert scheme $\mathbb{H}_{\delta_C}(C^*)$. Let us consider the quotient

$$\mathfrak{C}_{\delta_C}(C^*) = \mathbb{H}_{\delta_C}(C^*)/\mathrm{Aut}(C^*, \infty_C)\,.$$

The cycle Q_C defines a point denoted by the same letter in the space $\mathfrak{C}_{\delta_C}(C^*)$. Letting

$$\mathfrak{C}(\bar V, D) = \prod_{C\subseteq D-C_0-C_1} \mathfrak{C}_{\delta_C}(C^*)$$

we obtain a point

$$Q(\bar V, D) = \{Q_C\}_{C\subseteq D-C_0-C_1} \in \mathfrak{C}(\bar V, D)$$

called the *configuration invariant* of V.

Remark 3.9. Performing in $(\tilde V, D)$ elementary transformations with centers at the components C_{0i} of C_0 we neither change $\tilde\Pi$ nor the extended divisor (except for the self-intersection index C_1^2) and thus leave the data δ_C and $Q(\bar V, D)$ invariant. Hence we can define these invariants also for any semi-standard completion $(\bar V, D)$ of V by sending the latter via elementary transformations with centers on C_0 into a standard completion.

In order to show that the configuration invariant is independent of the choice of a standard completion we need the following definition.

Definition 3.10. Let V be an affine ruled surface. Given two standard completions $(\bar V, D)$ and $(\bar V', D')$ of V, we consider the birational map $f : \bar V \to \bar V'$, which extends the identity map of V. We distinguish between the following two cases.

(1) If V is non-Gizatullin then according to Proposition 2.14(b) the extended divisor of a standard completion is uniquely determined. In other words, f induces a canonical isomorphism of extended divisors D_{ext} and D'_{ext}. The component in D' corresponding under this isomorphism to the component $C \subseteq D$ will be denoted C^f.

(2) Assume further that V is a Gizatullin surface, and let $D = C_0 \cup \ldots \cup C_n$ and $D' = C'_0 \cup \ldots \cup C'_n$ be the standard zigzags of the corresponding completions. Then $(\bar V', D')$ is symmetrically linked either to the completion $(\bar V, D)$, or to its reversion $(\bar V^\vee, D^\vee)$, see [FKZ$_4$, 2.2.1-2.2.2].[8] In the first case we set $C_i^f = C'_i$ and in the second one $C_i^f = C'_{i^\vee}$, where

$$i^\vee := n - i + 2\,.$$

If $(\bar V', D')$ is symmetrically linked to both completions $(\bar V, D)$ and $(\bar V^\vee, D^\vee)$, then we define C_i^f to be either C'_i or $C'_{i^\vee}$.

The following theorem in the particular case of smooth Gizatullin surfaces is proven in [FKZ$_4$, Proposition 3.3.1].

Theorem 3.11. (General Matching Principle) *Let V be an affine ruled surface with two standard completions $(\bar V, D)$ and $(\bar V', D')$, and let C be a component of D with the corresponding component C^f of D'. Then the following hold.*

(a) C is a $*$-component if and only if C^f is;

(b) there is a canonical isomorphism $C \cong C^f$ mapping C^* to $(C^f)^*$ and ∞_C to ∞_{C^f}. Under this isomorphism the cycle Q_C on C is mapped onto the cycle Q_{C^f} on C^f;

(c) the induced isomorphism $\mathfrak{C}(\bar V, D) \cong \mathfrak{C}(\bar V', D')$ sends the configuration invariant $Q(\bar V, D)$ to $Q(\bar V', D')$.

[8]The latter means that there is a sequence of blowups and blowdowns transforming $(\bar V', D')$ into one of $(\bar V, D)$ or $(\bar V^\vee, D^\vee)$ and inducing an isomorphism of the corresponding dual graphs, which can be written symmetrically as (γ, γ^{-1}), where $\Gamma_{D'} \to \Gamma'$ is a birational transformation.

Proof. Clearly (c) is a consequence of (a) and (b). If V is not a Gizatullin surface, then the extended divisors $D_{\rm ext}$ and $D'_{\rm ext}$ are isomorphic, see Proposition 2.14(b). Hence (a) follows in this case. Since the cycles Q_C can be read off from this extended divisor, also (b) follows.

In the case of a Gizatullin surface V, the birational map $(\bar{V}, D) \dashrightarrow (\bar{V}', D')$ induced by id_V can be uniquely decomposed into a sequence

$$D = Z_1 \rightsquigarrow Z_2 \rightsquigarrow \ldots \rightsquigarrow Z_n = D'\,,$$

where each Z_i is a semi-standard zigzag and each step is either

(i) the reversion of a standard zigzag, or

(ii) an elementary transformation at an extremal 0-vertex,

see [DG, Theorem 1] or [FKZ$_4$, Proposition 2.3.3] for the existence part. A transformation of type (ii) does not alter the extended divisor except for the weight C_1^2, so it preserves the configuration invariant. Hence we are done in this case as before. It remains to deduce (a) and (b) in the case, where $(\bar{V}', D')$ is the reversion of $(\bar{V}, D)$. This is the content of the following proposition. □

Proposition 3.12. *Let V be a Gizatullin surface with standard completion $(\bar{V}, D)$ and standard zigzag $D = C_0 \cup \ldots \cup C_n$. Let further $(\bar{V}^\vee, D^\vee)$ be the reversed completion with the reversed standard zigzag $D^\vee = C_0^\vee \cup \ldots \cup C_n^\vee$. Then the following hold.*

(a) C_i is a $*$-component if and only if $C_{i^\vee}^\vee$ is;

(b) there is a natural isomorphism $C_i \cong C_{i^\vee}^\vee$ mapping C^* to $(C^\vee)^*$ and ∞_C to $\infty_{C_{i^\vee}^\vee}$. Under this isomorphism the cycle Q_{C_i} is mapped onto the cycle $Q_{C_{i^\vee}^\vee}$.

The proof is given in 3.19 below. Let us recall first the following notation.

Notation 3.13. Given a Gizatullin surface V with boundary zigzag $D = C_0 + C_1 + \ldots + C_n$, for $t \in \{1, \ldots, n\}$ we let $D_{\rm ext}^{\geq t}$ denote the branch of $D_{\rm ext}$ at C_{t-1} containing C_t. Moreover we let $D^{\geq t} = D \cap D_{\rm ext}^{\geq t}$ and $D_{\rm ext}^{>t} = D_{\rm ext}^{\geq t} - C_t$ (see [FKZ$_4$, 3.2.1]).

The proof of Proposition 3.12 is similar to the proof for smooth Gizatullin surfaces given in [FKZ$_4$, Prop. 3.3.1]. As in *loc.cit.* the main tool is the *correspondence fibration*. Let us recall this notion from [FKZ$_4$, 3.3.2-3.3.3].

Definition 3.14. Using inner elementary transformations we can move the pair of zeros in the zigzag $D = [[0, 0, w_2, \ldots, w_n]]$ several places to the right. In this way we obtain a new resolved completion (W, D_W) of V with boundary zigzag

$$D_W = [[w_2, \ldots, w_{t-1}, 0, 0, w_t, \ldots, w_n]]$$

for some $t \in \{2, \ldots, n+1\}$. For $t = 2$, $D_W = D = [[0, 0, w_2, \ldots, w_n]]$ is the original zigzag, while for $t = n+1$, $D_W = D^\vee = [[w_2, \ldots, w_n, 0, 0]]$ is the reversed one. The new zigzag D_W can also be written as

$$D_W = C_n^\vee \cup \ldots \cup C_{t^\vee}^\vee \cup C_{t-1} \cup C_t \cup \ldots \cup C_n = D^{\geq t-1} \cup D^{\vee \geq t^\vee}\,,$$

where for all $t-1 \leq i \leq n$ and $t^\vee \leq j \leq n$ we identify $C_i \subseteq \tilde{V}$ and $C_j^\vee \subseteq \tilde{V}^\vee$ with their proper transforms in W. In W we have $C_{t-1}^2 = C_{t^\vee}^{\vee 2} = 0$. Likewise in [FKZ$_4$, 3.3.3] the map

$$\psi : W \to \mathbb{P}^1$$

defined by the linear system $|C_{t-1}|$ on W will be called the *correspondence fibration* for the pair of curves $(C_t, C^\vee_{t^\vee})$. The components C_t and $C^\vee_{t^\vee}$ represent sections of ψ. Their projections to the base $\mathbb{P}^1$ yield isomorphisms $C_t \cong C^\vee_{t^\vee} \cong \mathbb{P}^1$; in what follows we identify points of all three curves under these isomorphisms.

Since the feathers of D_{ext} and $D^\vee_{\text{ext}}$ are not contained in the boundary zigzags they are not contracted in W. We denote their proper transforms in W by the same letters. It will be clear from the context where they are considered. We observe that every such feather $\mathfrak{F}$ can be written as $\mathfrak{F} = F + \mathfrak{R}$, where F is a unique component of $\mathfrak{F}$ with $F \cdot D_W \geq 1$ and $\mathfrak{R}$ is the exceptional divisor of the minimal resolution in W of a cyclic quotient singularity of V with $\mathfrak{R} \cdot D_W = 0$.

The following lemma is proven in [FKZ$_4$, Lemma 3.3.4] for smooth Gizatullin surfaces; the proof in the general case is similar and can be left to the reader.

Lemma 3.15. *Under the assumptions of Proposition 3.12, with the notation as in 3.14 the following hold.*

(a) The divisor $D^{\geq t+1}_{\text{ext}}$ is contained in some fiber $\psi^{-1}(q)$, $q \in \mathbb{P}^1$. Similarly, $D^{\vee \geq t^\vee+1}_{\text{ext}}$ is contained in some fiber $\psi^{-1}(q^\vee)$. The points q and $q^\vee$ are uniquely determined unless $D^{\geq t+1}_{\text{ext}}$ and $D^{\vee \geq t^\vee+1}_{\text{ext}}$ are empty, respectively.

(b) A fiber $\psi^{-1}(p)$ contains at most one component C not belonging to $D^{\geq t}_{\text{ext}} \cup D^{\vee \geq t^\vee}_{\text{ext}}$. Such a component C meets both $D^{\geq t}$ and $D^{\vee \geq t^\vee}$.

The next lemma is crucial in the proof of Proposition 3.11; see [FKZ$_4$, 3.3.6 and 3.3.9] for the case of a smooth Gizatullin surface. The proof in the general case is far more involved.

Lemma 3.16. *Given a point $q \in \mathbb{P}^1$, we let $F_0, \ldots, F_k$ denote the feather components of the extended divisor D_{ext} with mother component C_t contained in the fiber $\psi^{-1}(q)$. Assuming that there exists at least one such component, i.e. $k \geq 0$, the following hold.*

(a) With a suitable enumeration, the components $F_0, \ldots, F_k$ form a chain in $D^{\geq t}_{\text{ext}}$ contained in some feather $\mathfrak{F}$ of $D^{\geq t}_{\text{ext}}$.

(b) Let F_0 have the smallest distance to C_t in the chain above. Then the fiber $\psi^{-1}(q)$ contains a further component F_{k+1} such that $F_1, \ldots, F_{k+1}$ are all components of $D^\vee_{\text{ext}}$ with mother component $C^\vee_{t^\vee}$ contained in the fiber $\psi^{-1}(q)$. The components $F_1, \ldots, F_{k+1}$ form a chain in some feather $\mathfrak{F}^\vee$ of $D^{\vee \geq t^\vee}_{\text{ext}}$.

Proof. All base points p_{F_i} are equal since by assumption the curves F_i are contained in the same fiber over q. Let us denote this common base point by $p \in C_t$.

To deduce (a) we perform contractions in $D^{\geq t}_{\text{ext}}$ until one of the components $F_0, \ldots, F_k$, say F_0, meets the first time the component C_t. On the contracted surface, say, W', the images of $F_0, \ldots, F_k$ necessarily form a chain $[[-1, (-2)_k]]$. Any curve in W between F_0 and F_j has also mother component C_t and the same base point p whence $F_0, \ldots, F_k$ form as well a chain in D_{ext}. Being connected this chain has to be contained in some feather $\mathfrak{F}$.

(b) After renumbering we may suppose that any two consecutive curves in the chain $F_0, \ldots, F_k$ meet. The dual graph of $\mathfrak{F}_0 = F_0 + \ldots + F_k$ in W' is $[[-1, (-2)_k]]$, where W' is as in (a). Since under the map $W \to W'$ only components of the fiber $\psi^{-1}(q)$ are contracted, the $\mathbb{P}^1$-fibration $\psi : W \to \mathbb{P}^1$ factors through a $\mathbb{P}^1$-fibration $\psi' : W' \to \mathbb{P}^1$.

The component F_0 meets C_t in W' and so it disconnects the rest of the fiber $\psi'^{-1}(q)$ from C_t. Hence this fiber cannot contain any component of the zigzag $D^{\geq t}$.

Being contractible in W' to a smooth point $p \in C_t$ the chain $\mathcal{F}_0$ cannot exhaust the full fiber $\psi'^{-1}(q)$. After contracting $\mathfrak{F}_0$ in W' the section C_t still meets the resulting fiber in one point transversally. Hence there is an extra component F_{k+1} of the fiber $\psi'^{-1}(q)$ with $F_{k+1} \cdot \mathfrak{F}_0 = F_{k+1} \cdot F_k = 1$. Clearly, F_{k+1} has multiplicity 1 in the fiber. Therefore the divisor $\psi'^{-1}(q) - F_k - F_{k+1}$ can be contracted on W' to a smooth point. Thus we arrive at a smooth surface W'' still fibered over $\mathbb{P}^1$ with two (-1)-curves F_k and F_{k+1} in the fiber over q.

The fiber component F_{k+1} when considered as a curve on W, cannot belong to $D^{\vee \geq t^\vee}$ since otherwise the feather $\mathfrak{F}$ containing $\mathfrak{F}_0$ would meet the boundary twice. If on W the fiber $\psi^{-1}(q)$ contains an extra component C as in Lemma 3.15(b), then C must be contracted on W''. Indeed, C has to meet both $D^{\geq t}$ and $D^{\vee \geq t^\vee}$, whereas F_{k+1}, F_k when considered in W belong neither to $D^{\vee \geq t^\vee}$ nor to $D^{\geq t}$.

If F_{k+1} were a component of $D^{\geq t}_{\text{ext}}$ on W then it would also be a feather component with mother component C_t and base point p, contradicting the maximality of the collection $\{F_0, \ldots, F_k\}$.

According to Lemma 3.15(b) F_{k+1} is a component of $D^{\vee \geq t^\vee}_{\text{ext}}$ with mother component $C^\vee_{t^\vee}$ and base point $p^\vee \in C^\vee_{t^\vee}$ over q; this can be seen from the fiber structure on the surface W''. Thus F_{k+1} must be a component of a feather, say, $\mathfrak{F}^\vee$ of $D^{\vee \geq t^\vee}_{\text{ext}}$.

Assume that $k > 0$. Then the feather $\mathfrak{F}$ containing $\mathfrak{F}_0$ can be written as $\mathfrak{F} = F + \mathfrak{R}$, where F is the bridge curve of $\mathfrak{F}$ and $\mathfrak{R}$ contracts to a singular point x on V. The image of F_{k+1} on V, and then also that of $\mathfrak{F}^\vee$, contains x. It follows that $\mathfrak{F}^\vee = F^\vee + \mathfrak{R}$ with $F^\vee$ being the bridge curve of $\mathfrak{F}^\vee$. In particular, the chain $\mathfrak{F}_0 - F_0 + F_{k+1} = F_1 + \ldots + F_{k+1}$ is contained in $\mathfrak{F}^\vee$.

On the surface W' we can contract all components of the fiber $\psi'^{-1}(q)$ except for $F_0, \ldots, F_{k+1}$. The remaining fiber $F_0 + \ldots + F_k + F_{k+1}$ has then dual graph $[[-1, (-2)_k, -1]]$. The subchain $F_1 + \ldots + F_{k+1}$ of this fiber can be contracted on the resulting surface to the point $p^\vee \in C^\vee_{t^\vee}$. Hence this is a subchain of a maximal chain of feather components of $D^{\vee \geq t^\vee}_{\text{ext}}$ with the same mother component $C^\vee_{t^\vee}$ and the same base point $p^\vee$ as F_{k+1}.

If there were a further component $F^\vee$ of $D^{\vee \geq t^\vee}_{\text{ext}}$ with the same mother component and the same base point as F_{k+1}, then interchanging the roles of $\bar{V}$ and $\bar{V}^\vee$ the above reasoning would give at least $k+2$ feather components in $D^{\geq t}_{\text{ext}}$ with mother component C_t and base point $q = p_{F_0}$, contradicting our assumption. Now the proof is completed. □

Definition 3.17. Following [FKZ$_4$, 3.3.7] a pair of feathers $(\mathfrak{F}, \mathfrak{F}^\vee)$ as in Lemma 3.16 will be called a *matching pair*.

The next lemma is shown in [FKZ$_4$, Lemma 3.3.10] for a smooth Gizatullin surface; the proof applies as well to an arbitrary Gizatullin surface.

Lemma 3.18. *C_t is a $*$-component if and only if $C^\vee_{t^\vee}$ is. Furthermore, in the latter case the base points q and $q^\vee$ as in Lemma 3.15(a) coincide.*

Now we are ready to deduce Proposition 3.12.

3.19. *Proof of Proposition 3.12.* (a) is just Lemma 3.18. By Lemma 3.16(b), the identification $C_t \cong C^\vee_{t^\vee}$ given by the correspondence fibration ψ provides a one-to-one correspondence between the set of feathers of D_{ext} with mother component C_t and base point $p \in C_t$ and the set of feathers of $D^\vee_{\text{ext}}$ with mother component $C^\vee_{t^\vee}$ and base point $p^\vee \in C^\vee_{t^\vee}$. Moreover, the points

$$\{\infty_{C_t}\} = C_t \cap C_{t-1} \quad \text{and} \quad C^\vee_{t^\vee} \cap C_{t-1} = C^\vee_{t^\vee} \cap C^\vee_{t^\vee-1} = \{\infty_{C^\vee_{t^\vee}}\}$$

are identified under ψ and as well the points p and $p^\vee$ are. This leads to the equality $0_{C_t} = 0_{C^\vee_{t^\vee}}$ and shows (b). □

3.3. **Deformation invariance of the normalized extended graph.** The 'only if' part of Theorem 3.6 is an immediate consequence of the following proposition.

Proposition 3.20. *If a family $\pi : \mathcal{V} \to S$ of affine ruled surfaces over a connected base S admits a resolved standard completion $(\tilde{\mathcal{V}}, \mathcal{D})$, then the normalized extended graph $N(\Gamma_{\text{ext}}(s))$ of $(\bar{\mathcal{V}}(s), \bar{D}(s))$ does not depend on $s \in S$.*

Proof. The proof is based on the Factorization Lemma 2.24. Given a point $s_0 \in S$ the fiber $(\tilde{V}, D) = (\tilde{\mathcal{V}}(s_0), \mathcal{D}(s_0))$ of the pair $(\tilde{\mathcal{V}}, \mathcal{D})$ over s_0 is a resolved standard completion of V with dual graph Γ independent of s. According to Lemma 2.24, shrinking S to a suitable étale neighborhood of s_0 we can decompose $\tilde{\mathcal{V}} \to S$ into a sequence

$$\tilde{\mathcal{V}} = \mathcal{X}_m \to \mathcal{X}_{m-1} \to \ldots \to \mathcal{X}_1 \to \mathcal{B} \to S\,, \tag{11}$$

where $\mathcal{B} \to S$ is a smooth family of curves, $\mathcal{X}_1 \to \mathcal{B}$ is a locally trivial $\mathbb{P}^1$-fibration, and $\mathcal{X}_i \to \mathcal{X}_{i-1}$ is a blowup of a section $\gamma_i : S \to \mathcal{X}_{i-1}$ with exceptional set $\mathcal{E}_i \subseteq \mathcal{X}_i$, $i = 2, \ldots, m$. Restricting (11) to the fiber over s_0 we obtain a decomposition

$$\tilde{V} = X_m \to X_{m-1} \to \ldots \to X_1 \to \bar{B}\,. \tag{12}$$

Let now F be a component of a feather $\mathfrak{F}$ of $D_{\text{ext}}(s_0)$ created in the blowup $X_i \to X_{i-1}$ with center on the mother component $C \subseteq D$ of F. Then $\gamma_i(s_0) \in \mathcal{D}$ is a smooth point of $\mathcal{D}$ and so $\gamma_i(S) \subseteq \mathcal{C}$, where $\mathcal{C}$ is a component of $\mathcal{D}$ near s_0 with $\mathcal{C}(s_0) = C$. For $s \in S$ we let $F(s)$ be the proper transform of $\mathcal{E}_i(s)$ in $\tilde{\mathcal{V}}(s)$ so that $F(s_0) = F$. As follows from the definition of a constant dual graph of a family of completions (see Definition 2.1) $F(s)$ cannot be a component of $\mathcal{D}(s)$. Hence this is a feather component in $\tilde{\mathcal{V}}(s)$ with mother component $\mathcal{C}(s)$. Since every feather of $\tilde{\mathcal{V}}(s)$ appears in this way and the total number of components of the extended graph $D_{\text{ext}}(s)$ stays constant, the number δ_C of feather components in $\tilde{\mathcal{V}}(s)$ with mother component $\mathcal{C}(s)$ does not depend on $s \in S$. Hence the normalized extended graph $N(\Gamma_{\text{ext}}(s))$ is indeed independent of $s \in S$. □

The above proof provides in fact a little piece more of information.

Corollary 3.21. *The identification $N(\Gamma_{\text{ext}}(s_0)) \cong N(\Gamma_{\text{ext}}(s))$ as in the proof of Proposition 3.20 is independent of the factorization (11). Thus locally in S with respect to the étale topology there is a* natural *identification $N(\Gamma_{\text{ext}}(s_0)) \cong N(\Gamma_{\text{ext}}(s))$ for $s \in S$.*

Proof. Given another factorization

$$\tilde{\mathcal{V}} = \mathcal{X}_m \to \mathcal{X}'_{m-1} \to \ldots \to \mathcal{X}'_1 \to \mathcal{B} \to S\,, \tag{13}$$

in the first step $\mathcal{X}_m \to \mathcal{X}_{m-1}$ and $\mathcal{X}_m \to \mathcal{X}'_{m-1}$ two families of (-1)-curves $\mathcal{E}$ and $\mathcal{E}'$, respectively, are contracted. These relative (-1)-curves are either equal, or they are disjoint in all fibers by the uniqueness part in Lemma 2.5. In the latter case it is possible to contract $\mathcal{E}$ and $\mathcal{E}'$ simultaneously. In this way we obtain a new family $\tilde{\mathcal{V}}_1$ with again two different factorizations. Using induction, these two factorizations provide the same identification of extended divisors. □

The monodromy of the base points of the feather components can be non-trivial even in a family with constant dual graph. To exclude this possibility we give the following definition.

Definition 3.22. Let $\mathcal{V} \to S$ be a family of affine ruled surfaces with resolved completion $(\tilde{\mathcal{V}}, \mathcal{D})$ and associated dual graph Γ. We call $(\tilde{\mathcal{V}}, \mathcal{D})$ a *family of constant type* (Δ, Γ) if there is a family of isomorphisms $\Delta \cong \Gamma_{\text{ext}}(s)$ such that for every point $s_0 \in S$ the isomorphism $N(\Gamma_{\text{ext}}(s)) \cong \Delta \cong N(\Gamma_{\text{ext}}(s_0))$ is the natural identification of Corollary 3.21 for s near s_0.

4. Versal families of affine ruled surfaces

4.1. **Complete deformation families of surfaces.** In this subsection we construct a sufficiently big family of affine ruled surfaces admitting a resolved completion with a given normalized extended graph Δ. This family is complete in the sense that every other such family can be obtained, at least locally, from this one by a base change. In particular, every individual affine ruled surface admitting a resolved completion with normalized extended graph Δ appears as a member of our family.

4.1. We let Γ be a semi-standard tree with an extremal 0-vertex and a fixed embedding $\Gamma \hookrightarrow \Delta$ into a normalized extended weighted tree. All extremal 0-vertices say, $v_{01}, \ldots, v_{0a}$ of Γ are joined to the same vertex v_1, which is uniquely determined. The subgraph Δ^1 of Δ consisting of v_1 and all its neighbors in Δ will be called the *one-skeleton* of Δ. We consider it as a weighted graph by assigning to v_1 the same weight as in Δ, while all other vertices get the weight zero. Thus Δ^1 consists of v_1, v_{0i}, $1 \le i \le a$, and the remaining neighbors say, $v_{21}, \ldots, v_{2b}$ of v_1. We assume that Δ is obtained from its 1-skeleton by a sequence of blow ups

$$\Delta = \Delta^m \to \Delta^{m-1} \to \ldots \to \Delta^1 \,. \tag{14}$$

We suppose also that the following conditions are satisfied.

(a) If in a blowup $\Delta^i \to \Delta^{i-1}$ a vertex v of Γ is created then in the subsequent blowups we first create all the feathers of v, and only after that the next vertices of Γ. In other words, if v and w are vertices of Γ created in the blowups $\Delta^i \to \Delta^{i-1}$ and $\Delta^j \to \Delta^{j-1}$, respectively, where $j > i$, then the vertices created in the further blowups $\Delta^k \to \Delta^{k-1}$, $k > j$, are not feathers of v.

(b) There is a genus g assigned to v_1, and v_1 has weight $-2g$.

The construction of a versal family of surfaces associated to Δ proceeds in several steps as follows.

4.2. We let $\mathcal{M}$ denote the moduli space of marked curves (C, p) of genus g with $p \in C$ and with a level l structure, where $l \ge 3$. The latter means that we fix a symplectic basis of the group $H^1(C;\mathbb{Z}/l\mathbb{Z})$. It is known that $\mathcal{M}$ is irreducible, quasi-projective [DM] and smooth [Po]. According to Theorem 10.9 in [Po, Lect. 10] there exists a universal family of curves $\mathcal{Z}\to\mathcal{M}$, which constitutes a smooth projective morphism.

Step 0: By [Gro$_2$, Théorème 3.2] the relative Picard functor $\mathrm{Pic}_{\mathcal{Z}/\mathcal{M}}$ is representable over $\mathcal{M}$. In view of the fact that there is a section provided by the marking, the representability of the Picard functor means that there exists a scheme $\mathcal{P}$ locally of finite type over $\mathcal{M}$ and a universal line bundle $\mathcal{L}$ over $\mathcal{Z}\times_{\mathcal{M}}\mathcal{P}$, see [Gro$_2$, Corollaire 2.4]. Letting $\mathcal{P}'\subseteq\mathcal{P}$ be the connected component corresponding to the line bundles of degree $-2g$ and $\mathcal{B}^0=\mathcal{Z}\times_{\mathcal{M}}\mathcal{P}'$, there are morphisms[9]

$$\mathcal{X}_1^0=\mathbb{P}_{\mathcal{B}^0}(\mathcal{O}_{\mathcal{B}^0}\oplus\mathcal{L})\to\mathcal{B}^0\to S^0=\mathcal{P}'\,, \tag{15}$$

where the first one is a locally trivial $\mathbb{P}^1$-fibration and $\mathcal{B}^0\to S^0$ is a complete family of marked curves of genus g with a level l structure over S^0. The composition $\mathcal{X}_1^0\to S^0$ is a complete family of projective ruled surfaces over S^0. The marking gives rise to a section $\sigma: S^0\to\mathcal{B}^0$. Taking the fiber product yields a family of rational curves $\mathcal{C}_{01}^0=\mathcal{X}_1^0\times_\sigma S^0$ in $\mathcal{X}_1^0$ over S^0. The projection $\mathcal{O}_{\mathcal{B}^0}\oplus\mathcal{L}\to\mathcal{O}_{\mathcal{B}^0}$ defines a section $\mathcal{C}_1^0\cong\mathcal{B}^0$, where

$$\mathcal{C}_1^0=\mathbb{P}_{\mathcal{B}^0}(\mathcal{O}_{\mathcal{B}^0})\hookrightarrow\mathcal{X}_1^0=\mathbb{P}_{\mathcal{B}^0}(\mathcal{O}_{\mathcal{B}^0}\oplus\mathcal{L}).$$

Thus the fiber $X_1=\mathcal{X}_1^0(t)$ over a point $t\in S^0$ contains the curves

$$C_{01}=\mathcal{C}_{01}^0(t)\quad\text{and}\quad C_1=\mathcal{C}_1^0(t)\,.$$

Clearly X_1 is a $\mathbb{P}^1$-bundle over the genus g curve $B=\mathcal{B}^0(t)$, the curve C_{01} is a full fiber of $X_1\to B$ and $C_1\cong B$ is a section. By construction $\mathcal{O}_{C_1}(C_1)\cong\mathcal{L}|B$ so C_1 has in X_1 self-intersection $C_1^2=\deg(\mathcal{L}|B)=-2g$.

Step 1: Let now

$$(\mathcal{B}^0)^{a+b-1}=\mathcal{B}^0\times_{S^0}\times\ldots\times_{S^0}\mathcal{B}^0$$

be the $(a+b-1)$-fold fiber product and $S^1\subseteq(\mathcal{B}^0)^{a+b-1}$ be the subset in the fiber over $s\in S^0$ consisting of all points

$$(p_{02},\ldots,p_{0a},p_{21},\ldots,p_{2b})$$

with pairwise distinct coordinates different from $p_{01}=\sigma(s)$. By base change $S^1\to S^0$ we obtain from (15) morphisms

$$\mathcal{X}_1^1\to\mathcal{B}^1\to S^1\,. \tag{16}$$

On $\mathcal{X}_1^1$ we have again the families of curves $\mathcal{C}_{01}^1=\mathcal{C}_{01}^0\times_{S^0}S^1$ (corresponding to the section σ) and $\mathcal{C}_1^1=\mathcal{C}_1^0\times_{S^0}S^1$. With $\pi_i: S^1\to\mathcal{B}^0$ being the ith projection the morphisms

$$\pi_i\times\mathrm{id}: S^1\to\mathcal{B}^1\subseteq\mathcal{B}^0\times_{S^0}S^1,\quad i=1,\ldots,a+b-1,$$

yield sections $\sigma_{02},\ldots,\sigma_{0a}$ and $\sigma_{21},\ldots,\sigma_{2b}$. The preimages of $\sigma_{ij}(S^1)\subseteq\mathcal{B}^1$ under $\mathcal{X}^1\to\mathcal{B}^1$ give rise to families of curves $\mathcal{C}_{02}^1,\ldots,\mathcal{C}_{0a}^1$ and $\mathcal{C}_{21}^1,\ldots,\mathcal{C}_{2b}^1$ in $\mathcal{X}^1$.

Further steps: We construct in 4.3-4.4 below a sequence of morphisms

$$S^m\to S^{m-1}\to\ldots\to S^1\,, \tag{17}$$

which corresponds to the sequence (14). Furthermore, for each $i=1,\ldots,m$ we define

[9] By abuse of notation, we denote the restriction $\mathcal{L}|\mathcal{B}^0$ still by $\mathcal{L}$.

(a) a sequence of morphisms

$$\mathcal{X}_i^i \to \mathcal{X}_{i-1}^i \to \ldots \to \mathcal{X}_1^i \to \mathcal{B}^i \to S^i, \tag{18}$$

where $\mathcal{B}^i = \mathcal{B}^{i-1} \times_{S^{i-1}} S^i$, and

(b) families of curves over S^i

$$\mathcal{E}_j^i \subseteq \mathcal{X}^i,\ j \le i, \quad \mathcal{C}_{0\alpha}^i \subseteq \mathcal{X}^i\,,\ 1 \le \alpha \le a, \quad \mathcal{C}_1^i \subseteq \mathcal{X}^i \text{ and } \mathcal{C}_{2\beta}^i \subseteq \mathcal{X}^i\,,\ 1 \le \beta \le b,$$

corresponding to the vertices of the 1-skeleton of Δ^i in (14),

with the following properties.

(i) Except for the morphism $\mathcal{X}_i^i \to \mathcal{X}_{i-1}^i$, (18) is obtained from the corresponding sequence of morphisms in level $i-1$ by base change $S^i \to S^{i-1}$.

(ii) $\mathcal{E}_i^i$ is a family of (-1)-curves obtained by blowing up a section of $\mathcal{X}_{i-1}^{i-1} \to S^i$. Moreover we have $\mathcal{E}_j^i = \mathcal{E}_j^j \times_{S^j} S^i$ for $j < i$ and similarly

$$\mathcal{C}_{0\alpha}^i = \mathcal{C}_{0\alpha}^1 \times_{S^1} S^i\,, \quad \mathcal{C}_1^i = \mathcal{C}_1^1 \times_{S^1} S^i \text{ and } \mathcal{C}_{2\beta}^i = \mathcal{C}_{2\beta}^1 \times_{S^1} S^i\,.$$

(iii) The divisor $\mathcal{D}_{\text{ext}}^i$ on $\mathcal{X}^i$ formed of the families of curves $\mathcal{E}_j^i$, $\mathcal{C}_{0\alpha}^i$, $\mathcal{C}_1^i$ and $\mathcal{C}_{2\beta}^i$, restricts in each fiber over S^i to an SNC divisor with normalized dual graph Δ^i as in (14).

(iv) The divisor, say, $\mathcal{D}^i$ formed of families of curves as in (ii) that correspond to vertices in Γ, represents a family of SNC divisors with dual graph $\Gamma \cap \Delta^i$.

4.3. Suppose that the sequences (17) and (18) are already constructed up to a step $i < m$. Then at the step $i+1$ we construct these data as follows.

(A) Suppose that in the blowup $\Delta^{i+1} \to \Delta^i$ in (14) a feather v is created by an outer blowup of, say, $v' \in \Gamma \cap \Delta^i$. Let $\mathcal{C}'$ be the corresponding family of curves on $\mathcal{X}^i$ so that $\mathcal{C}' = \mathcal{E}_j^i$ for some $j \le i$, or $\mathcal{C}'$ is one of the families of curves $\mathcal{C}_{2,\beta}^i$ for some β. Letting

$$S^{i+1} = \mathcal{C}' \backslash \mathcal{D}' \quad \text{with} \quad \mathcal{D}' = \mathcal{D}^i - \mathcal{C}',$$

we define $\mathcal{X}_{i+1}^{i+1}$ to be the blowup of $\mathcal{X}_i^{i+1} = \mathcal{X}_i^i \times_{S^i} S^{i+1}$ along S^{i+1}, which we consider as a subscheme of $\mathcal{X}_i^{i+1}$ via the diagonal embedding into $\mathcal{X}_i^i \times_{S^i} S^{i+1}$. Now we let $\mathcal{E}_{i+1}^{i+1}$ be the resulting family of (-1)-curves on $\mathcal{X}_{i+1}^{i+1}$, while the remaining curves in (b) are defined as in (ii).

(B) Suppose next that in the blowup $\Delta^{i+1} \to \Delta^i$ in (14) a vertex v of Γ is created. Then we proceed as follows according to whether v has one or two mother components.

(B1) If v has one mother component, then it is created by an outer blowup of, say, $v' \in \Gamma \cap \Delta^i$. Let $\mathcal{C}'$ be the corresponding component of $\mathcal{D}^i$. We let $S^{i+1} = \mathcal{C}' \backslash \mathcal{D}'$, where $\mathcal{D}' = \mathcal{D}^i - \mathcal{C}'$, and define now $\mathcal{X}_{i+1}^{i+1}$ and $\mathcal{E}_{i+1}^{i+1}$ similarly as in (A).

(B2) If v has two mother components, then it is created by an inner blowup of an edge connecting two vertices $v', v'' \in \Gamma \cap \Delta^i$. These vertices correspond to families of curves $\mathcal{C}'$ and $\mathcal{C}''$ of $\mathcal{D}^i$, respectively. In this case we let $S^{i+1} = S^i$, and we let $\mathcal{X}_{i+1}^{i+1}$ be the blowup of $\mathcal{X}^i$ along the section $\mathcal{C}' \cap \mathcal{C}''$.

4.4. On the last step m we arrive at a family of surfaces $\mathcal{X}_j = \mathcal{X}_j^m$ over $S = S^m$ with morphisms

$$\mathcal{X}_m \to \mathcal{X}_{m-1} \to \ldots \to \mathcal{X}_1 \to \mathcal{B} = \mathcal{B}^m \to S\,. \tag{19}$$

Likewise, omitting the upper index m we obtain families of curves $\mathcal{C}_{0\alpha}$, $\mathcal{C}_1$, $\mathcal{C}_{2\beta}$ and $\mathcal{E}_j$. These yield a relative SNC divisor $\mathcal{D}$ over S with dual graph Γ, and in every fiber of

$\mathcal{X}_m \to S$ over $s \in S$ also an extended divisor $\mathcal{D}_{\rm ext}(s)$. The normalized dual graph of $\mathcal{D}_{\rm ext}(s)$ is Δ for each point $s \in S$, while the dual graph of $\mathcal{D}_{\rm ext}(s)$ might depend on s.[10] We arrive in this way at a family of smooth quasi-projective surfaces $\mathcal{X}_m \backslash \mathcal{D}$ over S. In a final step using Proposition 2.9, we can contract all feather components in $\mathcal{X}_m \backslash \mathcal{D}$ that are complete curves in $\mathcal{X}_m \backslash \mathcal{D}$. Thus we obtain a flat family of normal affine surfaces $\mathcal{V} \to S$ along with a resolved $(-2g)$-standard completion $(\tilde{\mathcal{V}}, \mathcal{D}) = (\mathcal{X}_m, \mathcal{D})$.

Definition 4.5. Given a normalized extended tree Δ together with a subtree Γ as in 4.1 we call

– $S = S_{\Delta,\Gamma}$ as in 4.4 the *presentation space of normal affine surfaces of type* (Δ, Γ);
– $\mathcal{V} = \mathcal{V}_{\Delta,\Gamma} \to S$ the *universal family over* S *of normal surfaces of type* (Δ, Γ);
– $(\tilde{\mathcal{V}}, \mathcal{D}) = (\tilde{\mathcal{V}}_{\Delta,\Gamma}, \mathcal{D}_{\Delta,\Gamma})$ the *universal resolved* $(-2g)$*-standard completion of* $\mathcal{V} \to S$ *of type* (Δ, Γ).

Remark 4.6. 1. The construction of the presentation space $S_{\Delta,\Gamma}$ and the universal families $(\tilde{\mathcal{V}}_{\Delta,\Gamma}, \mathcal{D}_{\Delta,\Gamma})$ depends *à priori* on the order of blowups in the sequence (14). However, the reader can easily check that different orders satisfying (a) and (b) in 4.1 will result in canonically isomorphic presentation spaces and families over them.

2. The graph Γ does not determine the normalized extended graph Δ as in 4.1, even if we restrict to smooth Gizatullin surfaces. For instance, in the zigzag $[[0, 0, -2, -1, -2]]$ the component C_3 is a $*$-component, while in $[[0, 0, -1, -2, -1]]$ all of them are $+$-components. However, blowing up suitable feathers we can obtain from both zigzags the chain $\Gamma = [[0, 0, -2, -2, -2]]$. In the case of the zigzag $[[0, 0, -2, -1, -2]]$ the resulting surface is an affine pseudo-plane, see [MM] or [FZ], while in the other case the result is a Danilov-Gizatullin surface.

3. In the same way, given a normalized tree, we can construct an associated universal resolved s-standard completion provided that $s \le -2g$. This is easily seen from the proof. If however $s > -2g$, $\mathcal{B}$ is a family of complete curve of genus g and $\mathcal{L}$ is a line bundle on $\mathcal{B}$ of degree s in each fiber, then fiberwise there are non-trivial extensions $0 \to \mathcal{O}_{\mathcal{B}_s} \to \mathcal{G} \to \mathcal{L}_s \to 0$. Even worse, such extensions cannot be organized into a reasonable moduli space, in general.

In the next result we show that the universal family from Definition 4.5 is complete in the sense of deformation theory.

Proposition 4.7. *Let* $\mathcal{V}' \to S'$ *be a family of affine ruled surfaces admitting a resolved* $(-2g)$*-standard completion* $(\tilde{\mathcal{V}}', \mathcal{D}')$*. Then locally in the étale topology of* S' *there is a morphism* $S' \to S = S_{\Delta,\Gamma}$ *such that* $(\tilde{\mathcal{V}}', \mathcal{D}')$ *can be obtained from the universal family* $(\tilde{\mathcal{V}}, \mathcal{D})$ *via a base change* $S' \to S = S_{\Delta,\Gamma}$.

Proof. By Proposition 2.21 there is a family $\mathcal{B}' \to S'$ of curves of genus g and the morphism $\tilde{\Pi}' : \tilde{\mathcal{V}}' \to \tilde{B}$ such that for every $s \in S$ the restriction of $\tilde{\Pi}'$ to the fiber over s is the standard morphism. By the Factorization Lemma 2.24 locally there is a factorization

$$\tilde{\mathcal{V}}' = \mathcal{X}'_n \to \mathcal{X}'_{n-1} \to \ldots \to \mathcal{X}'_1 \to \mathcal{B}' \to S' \,,$$

[10] So $\mathcal{D}_{\rm ext}$ is not, in general, a relative SNC divisor over S.

where $\mathcal{B}' \to S'$ is a smooth family of curves of genus g. In each step we blow up a section $S \to \mathcal{X}_i'$, with the order of blowups corresponding to that in (14).

Locally the family $q' : \mathcal{B}' \to S'$ admits a level l structure. The position of the family of curves $\mathcal{C}_{01}'$ in $\tilde{\mathcal{V}}'$ gives rise to a section $\sigma' : S' \to \mathcal{B}'$ and thus to a marking. Hence it is obtained from the universal family $\mathcal{Z} \to \mathcal{M}$ in 4.2 by a base change $S' \to \mathcal{M}$. The $\mathbb{P}^1$-bundle $\varphi' : \mathcal{X}_1' \to \mathcal{B}'$ possesses a section $\mathcal{C}_1'$ and so $\mathcal{X}_1' \cong \mathbb{P}^1_{\mathcal{B}'}(\mathcal{G}')$ is the projective bundle associated to the 2-bundle $\mathcal{G}' = \varphi_*(\mathcal{O}_{\mathcal{X}_1'}(\mathcal{C}_1'))$. The exact sequence

$$0 \to \mathcal{O}_{\mathcal{X}_1'} \to \mathcal{O}_{\mathcal{X}_1'}(\mathcal{C}_1') \to \mathcal{L}' := \mathcal{O}_{\mathcal{C}_1'}(\mathcal{C}_1') \to 0$$

yields a sequence $0 \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{G}' \to \mathcal{L}' \to 0$. In every fiber over $s' \in S'$ the curve $C_1' = \mathcal{C}_1(s')$ satisfies $C_1'^2 = -2g$, see 4.1. Hence $\mathcal{L}'$ is a family of line bundles of degree $-2g$ that is obtained from the universal family $\mathcal{L}$ from 4.2 by a base change $S' \to \mathcal{P}' = S^0$. Furthermore the sequence $0 \to \mathcal{O}_{\mathcal{B}'} \to \mathcal{G}' \to \mathcal{L}' \to 0$ can be regarded as an element in

$$\operatorname{Ext}^1_{\mathcal{O}_{\mathcal{B}'}}(\mathcal{L}', \mathcal{O}_{\mathcal{B}'}) \cong H^1(\mathcal{B}', \mathcal{L}'^{\vee})\,. \tag{20}$$

Since deg $\mathcal{L}'^{\vee} = 2g$ the sheaf $R^1 q'_*(\mathcal{L}'^{\vee})$ vanishes. Hence locally in S' the group (20) vanishes as well. In other words, there is a splitting $\mathcal{G}' \cong \mathcal{O}_{\mathcal{B}'} \oplus \mathcal{L}'$. It follows that $\mathcal{X}_1' \cong \mathcal{X}_1 \times_{S^0} S'$ locally in S'.

Assume now that, for all $j \le i$, $\mathcal{X}_j'$ is already obtained from $\mathcal{X}_j^i$ by a base change $f_i : S' \to S^i$. The morphism $\mathcal{X}_{i+1}' \to \mathcal{X}_i'$ is then the blowup of a section, say, $\sigma : S' \to \mathcal{X}_i'$ with exceptional set $\mathcal{E}_{i+1}' \subseteq \mathcal{X}_{i+1}'$, see Lemma 2.24. Assume that in the blowup $\Delta^{i+1} \to \Delta^i$ a vertex v is created. As in 4.3 we distinguish between the following cases (A) and (B).

In case (A) the vertex v corresponds to a feather component of Δ and is created by an outer blowup at a unique vertex $v' \in \Gamma$. The section σ must be contained in the corresponding family of curves, say, $\mathcal{C}' \subseteq \mathcal{X}_i'$, which is necessarily a component of the image, say, $\mathcal{D}_i'$ of $\mathcal{D}'$ in $\mathcal{X}_i'$. The section σ cannot meet any component of $\mathcal{D}_i'$ different from $\mathcal{C}'$ since otherwise in a fiber over some point $s' \in S'$ a feather is created by an inner blowup of $D_i' = \mathcal{D}_i'(s')$. Thus this newly created feather will divide the zigzag, which is impossible.

Using the construction of S^{i+1} in (A), the section σ induces a morphism $f_{i+1} = (f_i, \sigma) : S' \to S^{i+1}$ such that $\mathcal{X}_{i+1}' \cong \mathcal{X}_{i+1} \times_{S^{i+1}} S'$, as desired. The argument in case (B) of 4.3 (where the newly created vertex v is a vertex of Γ) is similar and so we leave it to the reader. □

Corollary 4.8. *Let V' and V'' be two normal affine ruled surfaces admitting resolved standard completions $(\tilde{V}', D')$ and $(\tilde{V}'', D'')$, respectively, with the same dual graph Γ, the same normalized extended tree Δ, and the same genus assigned to the vertex v_1 of Γ. Then V' and V'' are deformation equivalent.*

Proof. Performing elementary transformations in one of the extremal 0-vertices C_{01}' and C_{01}'' we may suppose that the curves C_1' and C_1'' have the same self-intersection index $-2g$. Then the induced completions $(\bar{V}', D')$ and $(\bar{V}'', D'')$ of V' and V'', respectively, arise as fibers over certain points $s', s'' \in S$ of the corresponding complete family $(\mathcal{X}_m, \mathcal{D})$ over S. Since the moduli space of curves $\mathcal{M}$ is irreducible, also the base S is. In particular, V' and V'' are deformation equivalent, as required. □

4.2. **The map into the configuration space.** We let as before $\varphi : \mathcal{V} \to S$ be a family of affine ruled surfaces, which admits a minimal resolved completion $(\tilde{\mathcal{V}}, \mathcal{D})$ of constant type (Δ, Γ).

Given a point $s \in S$, we consider the completion $(\bar{V}, D) = (\bar{\mathcal{V}}(s), \mathcal{D}(s))$ of the affine surface $V = \mathcal{V}(s)$ and the configuration invariant

$$Q(\bar{V}, D) \in \mathfrak{C} = \prod_{C \subseteq D - C_0 - C_1} \mathfrak{C}_{\delta_C}(C^*)\,,$$

see Definition 3.8. The configuration space $\mathfrak{C}$ does not depend on the choice of $s \in S$ and only depends on the tree $\Delta = N(\Gamma_{\text{ext}})$ and the subtree $\Gamma = \Gamma_D$. Note that δ_C is just the number of feathers of Δ at the vertex C.

Definition 4.9. $\mathfrak{C} = \mathfrak{C}(\Delta, \Gamma)$ is called the *configuration space associated to* (Δ, Γ).

With these notation and assumptions we have the following result.

Proposition 4.10. *The map $S \to \mathfrak{C}$ assigning to $s \in S$ the configuration invariant of the fiber $\mathcal{V}(s)$ is a regular morphism.*

Proof. This follows immediately from the Factorization Lemma 2.24. Indeed, let

$$\bar{\mathcal{V}} = \mathcal{X}_n \to \mathcal{X}_{n-1} \to \dots \mathcal{X}_1 \to \bar{\mathcal{B}} \to S$$

be the factorization as in Lemma 2.24. In each step $\mathcal{X}_{i+1} \to \mathcal{X}_i$ the center of the blowup is a section $S \hookrightarrow \mathcal{X}_i$ of $\mathcal{X}_i \to S$. Thus restricting to the fiber over s the center of blowup varies regularly with s. This readily implies the result. □

5. Moduli spaces of special Gizatullin surfaces

5.1. **Special Gizatullin surfaces.** In this section we discuss the existence of a coarse moduli space of affine ruled surfaces. This does not follow immediately from the construction in section 4. For instance, the moduli space of the Danilov-Gizatullin surfaces V_n consists of one point, while even in simple cases as in Example 2.27, where $n = 3$, our construction leads to a multi-dimensional versal family. However, we show that in certain cases the moduli space does exist and can be cooked out using the deformation family.

Fixing a tree Γ with an extremal 0-vertex as in 4.1 and a normalized extended tree Δ, we consider families $\mathcal{V} \to S$ of normal affine surfaces over an algebraic $\Bbbk$-scheme S with a resolved completion $(\tilde{\mathcal{V}}, \mathcal{D})$ of constant type (Δ, Γ). Two such families $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ are isomorphic if there is an S-isomorphism $\mathcal{X} \xrightarrow{\cong} \mathcal{X}'$.

Definition 5.1. Given an algebraic variety S we let $\mathbf{F}(S)$ be the set of isomorphism classes of families $\mathcal{X} \to S$ which admit a resolved completion of constant type (Δ, Γ). This yields a functor

$$\mathbf{F} = \mathbf{F}_{\Delta,\Gamma} : \mathbf{ASch}_{\Bbbk} \longrightarrow \mathbf{sets}$$

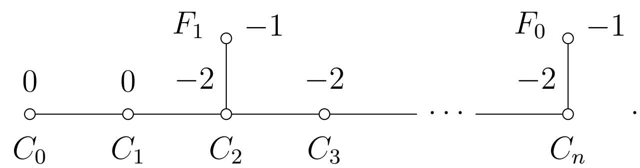
from the category of algebraic $\Bbbk$-schemes into sets. For a morphism $S' \to S$ the corresponding map $\mathbf{F}(S) \to \mathbf{F}(S')$ is given by the fiber product.

Restricting to families of smooth surfaces $\mathcal{X} \to S$ we obtain a functor $\mathbf{F}_s : \mathbf{ASch}_{\Bbbk} \longrightarrow \mathbf{sets}$.

In general $\mathbf{F}$ is not a sheaf. This means that two families $\mathcal{X} \to S$ and $\mathcal{X}' \to S$ that are locally in S isomorphic, do not need to be S-isomorphic globally. On the other hand, a representable functor is always a sheaf. Thus in order to study representability it is necessary to consider the sheaf $\tilde{\mathbf{F}}$ associated to $\mathbf{F}$. We present below concrete classes of surfaces for which $\tilde{\mathbf{F}}$ has a fine moduli space, which we denote by $\mathfrak{M}(\Delta, \Gamma)$. Clearly then $\mathfrak{M}(\Delta, \Gamma)$ will be as well a coarse moduli space for $\mathbf{F}$.

Example 5.2. Consider a Danilov–Gizatullin surface $V_n = \Sigma_d \backslash C$ of type n, where $C \subseteq \Sigma_d$ is an ample section with $C^2 = n$ in the Hirzebruch surface Σ_d. According to [DG] (see also [CNR, FKZ$_5$]) V_n only depends on n and neither on d nor on the choice of the section C. The normalized extended graph Δ_n of V_n is

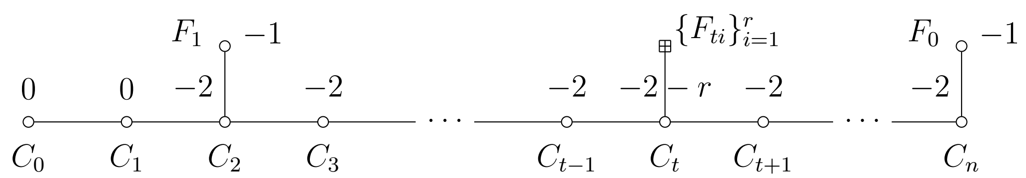
(21) Δ_n :



By the Isomorphism Theorem of Danilov and Gizatullin cited in the Introduction, the coarse moduli space for such surfaces consists of a single reduced point.

Example 5.3. According to [FKZ$_4$, Definition 1.0.4] a *special Gizatullin surface* with invariants (n, r, t) is a smooth Gizatullin surface with normalized extended graph $\Delta = \Delta(n, r, t)$:

(22)



where $n \geq 3$ and $\{F_{ti}\}_{i=1}^r$ is a family of r feathers joined to C_t and consisting each one of a single (-1)-curve. In the case where $t = 2$ or $t = n$ the number of (-1)-feathers attached to C_t is $r + 1$. Thus there are δ_t feathers with mother component C_t, where $\delta_t = r + 1$ for $t \in \{2, n\}$ and $\delta_t = r$ otherwise. The associated configuration space is $\mathfrak{C} = \mathfrak{C}_{\delta_t}(C_t^*)$.

Assigning to S the isomorphism classes of completable families over S of special Gizatullin surfaces with invariants (n, r, t) we obtain as before a moduli functor $\mathbf{F}$. With this notation we can reformulate Corollary 6.1.4 in [FKZ$_4$] as follows.

Theorem 5.4. *$\mathfrak{C} := \mathfrak{C}_{\delta_t}(C_t^*)$ is a coarse moduli space for* $\mathbf{F}$.

Proof. By Proposition 4.10 there is a functorial morphism

$$\alpha_S : \mathbf{F}(S) \to \operatorname{Hom}(S, \mathfrak{C})\,.$$

As follows from Corollary 6.1.4 in [FKZ$_4$], the elements of $\mathbf{F}(0)$ are in one-to-one correspondence with the elements of $\mathfrak{C}$, where 0 denotes the reduced point. It remains to show that for every other space $\mathfrak{M}$ together with a functorial morphism $\beta : \mathbf{F}(S) \to$

$\mathrm{Hom}(S, \mathfrak{M})$ there is a unique morphism $\varphi : \mathfrak{C} \to \mathfrak{M}$ such that the diagram

$$\begin{array}{ccc} & \mathbf{F}(S) & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ \mathrm{Hom}(S, \mathfrak{C}) & \xrightarrow{\ \varphi_*\ } & \mathrm{Hom}(S, \mathfrak{M}) \end{array} \tag{23}$$

commutes. Let $S(\Gamma)$ be the space of presentations as in 4.5 and $\mathcal{V}(\Gamma) \to S(\Gamma)$ be the universal family. Using β this family induces a morphism

$$\psi = \beta([\mathcal{V}(\Gamma) \to S(\Gamma)]) : S(\Gamma) \to \mathfrak{M}.$$

By Corollary 6.1.4 in [FKZ$_4$] this morphism is constant on the fibers of $S(\Gamma) \to \mathfrak{C}$. For any two elements in the fiber define the same element in $\mathbf{F}(0)$. Using the fact that $\mathfrak{C}$ is normal and $S(\Gamma) \to \mathfrak{C}$ is surjective ψ induces a morphism $\varphi : \mathfrak{C} \to \mathfrak{M}$ making the diagram (23) commutative, as required. □

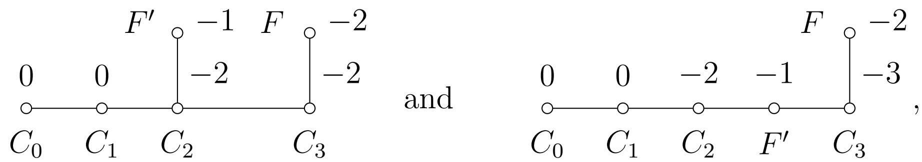
5.2. **Appendix: An example.** Let us recall from Corollary 2.10 that for a completable family of affine ruled surfaces $\mathcal{V} \to S$ the total space $\mathcal{V}$ is affine if so is S.

We emphasize that this does not remain true if we allow degenerations of $\mathcal{D}$ in our completable families. Namely, allowing such degenerations we show below the existence of a smooth family of affine surfaces $\mathcal{V} \to S$ over $S = \mathbb{A}^1$ with a completion $(\bar{\mathcal{V}}, \mathcal{D})$ of $\mathcal{V}$ such that the total space $\mathcal{V}$ is not affine and even not quasi-affine. We write in the sequel $\mathbb{A}^n_{x_1,\ldots,x_n} = \mathrm{Spec}\, \Bbbk[x_1, \ldots, x_n]$.

Example 5.5. Similarly as in Example 2.27 we consider the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ with the zigzag $C_0 + C_1 + C_2$, where

$$C_0 = \{\infty\} \times \mathbb{P}^1, \qquad C_1 = \mathbb{P}^1 \times \{\infty\}, \quad \text{and} \quad C_2 = \{0\} \times \mathbb{P}^1 .$$

We let $V_1 \to Q$ be the blowup in $(0, 0)$ to create a feather F, and $V_2 \to V_1$ the blowup of $F \cap C_2$ to create a boundary curve C_3 on V_2. After choosing an isomorphism $j : S \cong C_2 \backslash C_1$ we can blow up the image of the diagonal embedding $(1, j) : S \hookrightarrow S \times C_2$ on $S \times V_2$. Thus we obtain a surface $\bar{\mathcal{V}}$ with a relative (-1)-curve $\mathcal{F}'$ on it and a divisor $\mathcal{D} = \sum_{i=0}^{3} \mathcal{C}_i$, where $\mathcal{C}_i$ is the proper transform of $S \times C_i$ in $\bar{\mathcal{V}}$. We let $\mathcal{V}$ be the complement $\bar{\mathcal{V}} \backslash \mathcal{D}$. The extended graphs of $(\bar{\mathcal{V}}(s), \mathcal{D}(s))$ over $s \neq 0$ and $s = 0$ are



respectively, where F' stands for the fiber of $\mathcal{F}'$ over s.

Proposition 5.6. *(a) Letting $z = y/x$ we obtain*

$$H^0(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) \cong \Bbbk[s, zx, z^2x - sz, x] \subseteq \Bbbk[s, x, z]\,,$$

while $H^0(\mathcal{V}(0), \mathcal{O}_{\mathcal{V}(0)}) = \Bbbk[x, z]$.

(b) The variety $\mathcal{V}$ is not quasi-affine.

Proof. We only give a rough sketch of the argument; details will be given elsewhere.

(a) implies (b), since the global functions on $\mathcal{V}$ all vanish on the line $x = 0$ of $\mathcal{V}(0)$.

To deduce (a) let us simplify the setup by considering $V_i' := V_i \backslash (C_0 \cup C_1)$, $i = 1, 2$, so that the restricted maps $V_2' \to V_1' \to \mathbb{A}^2$ are blowups. In coordinates $V_1' \to \mathbb{A}^2$ can be described by

$$(x_1,\ y_1) = (x/y,\ y)\ \text{, where } C_2 \text{ corresponds to the } y_1\text{-axis and } F \text{ to the } x_1\text{-axis.}$$

Clearly $V_1' \backslash C_2 \cong \mathbb{A}^2_{x,z}$ with $z = y/x$ since the other chart of the blowup V_1' is given by $(x, z = y/x)$. Blowing up $C_2 \cap F$ with coordinates $(0, 0)$ we obtain V_2' with coordinates

$$(x_2,\ y_2) = (x_1/y_1,\ y_1) = (x/y^2,\ y)\,,$$

where

$$C_2 \text{ corresponds to the } y_2\text{-axis and the exceptional set } C_3 \text{ to the } x_2\text{-axis.}$$

The affine surfaces $V_2 = V_2' \backslash (C_2 \cup C_3)$ and $V_1 = \mathbb{A}^2_{x,z}$ are clearly isomorphic. Next we have to blow up the product $S \times V_2'$ along the curve $\{(s, 0, s) \mid s \in S\}$, where the last two coordinates are the (x_2, y_2)-coordinates as above. The ideal I of this curve is given by $I = (x_2\,, y_2 - s)$. Blowing it up yields a 3-fold $\mathcal{V}'$ with a coordinate chart $U_3 \cong \mathbb{A}^3$ and coordinates

$$(s, x_3,\ y_3) = (s, x_2,\ (y_2 - s)/x_2) = (s, x/y^2, (y^3 - sy^2)/x)\,, \tag{24}$$

where the new exceptional set $\mathcal{F}'$ corresponds to $\{x_3 = 0\}$. By construction $\mathcal{V} = \mathcal{V}' \backslash (\mathcal{C}_2 \cup \mathcal{C}_3)$. Moreover $\mathcal{C}_3$ is given on U_3 by $y_2 = 0$, or in (x_3, y_3) coordinates as $x_3 y_3 + s = 0$. The threefold $\mathcal{V}$ is equal to the union of the two coordinate charts

$$S \times \mathbb{A}^2_{xz} \qquad \text{and} \qquad D(x_3 y_3 + s) \subseteq S \times \mathbb{A}^2_{x_3, y_3}\,.$$

Accordingly

$$A := H^0(\mathcal{V}, \mathcal{O}_{\mathcal{V}}) = \Bbbk[s, x, z] \cap \Bbbk[s, x_3, y_3, (x_3 y_3 + s)^{-1}]\,.$$

Using (24) it is easy to see that A contains the functions

$$s, \quad zx = x_3 y_3 + s, \quad x = \frac{(zx)^2}{z^2 x}, \quad z^2 x - sz = \frac{z^3 x^2 - sz^2 x}{zx}\,.$$

It can be shown that A is actually generated by them over $\Bbbk$. Thus the result follows. □

References

[Ar] M. Artin: *Algebraic construction of Brieskorn's resolutions.* J. Algebra 29 (1974), 330–348.

[BHPV] W. P. Barth; K. Hulek; C. A. M. Peters; A. Van de Ven: *Compact complex surfaces.* Second edition. Springer-Verlag, Berlin, 2004.

[CNR] P. Cassou-Noguès, P. Russell: *Birational morphisms $\mathbb{C}^2 \to \mathbb{C}^2$ and affine ruled surfaces*, in: Affine algebraic geometry. In honor of Prof. M. Miyanishi, World Sci. 2007, 57–106.

[Ca] F. Catanese: *Deformation in the large of some complex manifolds. I.* Ann. Mat. Pura Appl. (4) 183 (2004), 261–289.

[DG] V. I. Danilov, M. H. Gizatullin: *Automorphisms of affine surfaces.* I. Math. USSR Izv. 9 (1975), 493–534; II. *ibid.* 11 (1977), 51–98.

[DM] P. Deligne, D. Mumford: *The irreducibility of the space of curves of given genus.* Inst. Hautes Études Sci. Publl. Math. No. 36 (1969), 75–109.

[Du_1] A. Dubouloz: *Completions of normal affine surfaces with a trivial Makar-Limanov invariant.* Michigan Math. J. 52 (2004), 289–308.

[Du_2] A. Dubouloz: *Danielewski-Fieseler surfaces.* Transfor. Groups 10 (2005), 139-162.

[Ei] D. Eisenbud: *Commutative algebra. With a view toward algebraic geometry.* Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995. xvi+785 pp.

[FZ] H. Flenner, M. Zaidenberg: *On a result of Miyanishi-Masuda.* Arch. Math. (Basel) 87 (2006) 15-18.

[FKZ_1] H. Flenner, S. Kaliman, M. Zaidenberg: *Birational transformations of weighted graphs*, in: Affine algebraic geometry. In honor of Prof. M. Miyanishi, World Sci. 2007, 107-147. *Corrigendum*, in: Peter Russell's Festschrift, Centre de Recherches Mathmatiques, 123–163. CRM Proceedings and Lecture Notes 54, 2011.

[FKZ_2] H. Flenner, S. Kaliman, M. Zaidenberg: *Completions of $\mathbb{C}^*$-surfaces*, in: Affine algebraic geometry. In honor of Prof. M. Miyanishi, World Sci. 2007, 149–200.

[FKZ_3] H. Flenner, S. Kaliman, M. Zaidenberg: *Uniqueness of $\mathbb{C}^*$- and $\mathbb{C}_+$-actions on Gizatullin surfaces.* Transformation groups 13 (2008), 305–354.

[FKZ_4] H. Flenner, S. Kaliman, M. Zaidenberg: *Smooth Affine Surfaces with Non-Unique $\mathbb{C}^*$-Actions.* J. Algebraic Geom. 20 (2011), 329–398.

[FKZ_5] H. Flenner, S. Kaliman, M. Zaidenberg: *On the Danilov-Gizatullin Isomorphism Theorem.* Enseignement Mathématiques (2) 55 (2009), 275–283.

[Gi] M. H. Gizatullin: *Quasihomogeneous affine surfaces.* Math. USSR Izv. 35, 1971, 1057–1081.

[GS] R. V. Gurjar, A. R. Shastri: *A topological characterization of $\mathbb{C}^2/G$.* J. Math. Kyoto Univ. 25, 1985, no. 4, 767–773.

[Gro_1] A. Grothendieck, *Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert.* Séminaire Bourbaki, Vol. 6, Exp. 221, 249–276, Soc. Math. France, Paris, 1995.

[Gro_2] A. Grothendieck: *Technique de descente et théorèmes d'existence en géométrie algébrique. V. Les schmas de Picard : Théorèmes d'existence.* Séminaire Bourbaki, Vol. 7, Exp. 232, 143161, Soc. Math. France, Paris, 1995.

[Gro_3] A. Grothendieck: *Technique de descente et théorèmes d'existence en géométrie algébrique. VI. Les schmas de Picard : Propriétés générales.* Séminaire Bourbaki, Vol. 7, Exp. 236, 221–243, Soc. Math. France, Paris, 1995.

[Gro_4] A. Grothendieck: *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents.* I. Inst. Hautes tudes Sci. Publ. Math. No. 11 (1961).

[Ha_1] R. Hartshorne:*Connectedness of the Hilbert scheme.* Inst. Hautes Études Sci. Publ. Math. 29 (1966) 5–48.

[Ha_2] R. Hartshorne: *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Ha_3] R. Hartshorne: *On the connectedness of the Hilbert scheme of curves in $\mathbb{P}^3$.* Special issue in honor of Robin Hartshorne. Comm. Algebra 28 (2000), 6059–6077.

[Ka] Y. Kawamata: *On deformations of compactifiable complex manifolds.* Math. Ann. 235 (1978), 247265.

[Ko] K. Kodaira: *On stability of compact submanifolds of complex manifolds.* Amer. J. Math. 85 (1963), 79–94.

[KS] J. Kollár, N. I. Shepherd-Barron: *Threefolds and deformations of surface singularities*, Invent. Math. 91 (1988), 299–338.

[Mi] M. Miyanishi: *Open algebraic surfaces.* CRM Monograph Series, 12. American Mathematical Society, Providence, RI, 2001.

[MM] M. Miyanishi, K. Masuda: *Affine pseudo-planes with torus actions.* Transform. Groups 11 (2006), 249-267.

[Po] H. Popp: *Moduli theory and classification theory of algebraic varieties.* Lecture Notes in Math. 620, Springer 1977.

[Ra] C. P. Ramanujam: *A topological characterisation of the affine plane as an algebraic variety.* Ann. of Math. (2) 94 (1971) 69–88.

[Se] E. Sernesi: *Deformations of algebraic schemes.* Grundlehren der Mathematischen Wissenschaften 334, Springer-Verlag, Berlin, 2006.

Fakultät für Mathematik, Ruhr Universität Bochum, Geb. NA 2/72, Universitätsstr. 150, 44780 Bochum, Germany
E-mail address: Hubert.Flenner@ruhr-uni-bochum.de

Department of Mathematics, University of Miami, Coral Gables, FL 33124, U.S.A.
E-mail address: kaliman@math.miami.edu

Université Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, BP 74, 38402 St. Martin d'Hères cédex, France
E-mail address: zaidenbe@ujf-grenoble.fr