EULER STRUCTURES, THE VARIETY OF REPRESENTATIONS AND THE MILNOR–TURAEV TORSION

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ABSTRACT. In this paper we extend, and Poincaré dualize, the concept of Euler structures, introduced by Turaev for manifolds with vanishing Euler–Poincaré characteristic, to arbitrary manifolds. We use the Poincaré dual concept, co-Euler structures, to remove all geometric ambiguities from the Ray–Singer torsion by providing a slightly modified object which is a topological invariant. We show that the modified Ray–Singer torsion when regarded as a function on the variety of complex representations of the fundamental group of the manifold is actually the absolute value of a rational function which we call in this paper the Milnor–Turaev torsion. As a consequence one obtains for a closed manifold M and an acyclic unimodular representation of its fundamental group ρ, a computable numerical invariant $S(M,\rho)$ with value in $\mathbb{R}/\pi\mathbb{Z}$ analoguous to the Atiyah–Patodi–Singer eta invariant, cf. section 4.4. Some examples and applications are also discussed, see sections 4.3, 4.5, and 4.6.

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1. INTRODUCTION

This paper was motivated by a question in geometric analysis, whose answer brings to attention a topological invariant which is a rational function on the variety of (some) complex representations of the fundamental group of a closed manifold.

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Question: *Is the Ray–Singer torsion of an odd dimensional manifold,*[1] *when regarded as a positive real valued function on the space of complex representations of its fundamental group the absolute value of a holomorphic/meromorphic function, and if so, what is the meaning of its argument?*[2]

Our answer to the first part of the question goes as follows, cf. section 4.6. With the help of a co-Euler structure, as defined in section 3.2, we decompose the Ray–Singer torsion as a product of two positive real valued functions, and verify that the first one is the absolute value of a rational function while the second is the absolute value of a holomorphic function. The rational function is what we call in this paper *Milnor–Turaev torsion.*[3] The values of the second function are obtained from an integral on a non-compact manifold, possibly divergent, but always regularizable and referred to as the *invariant* R defined in section 2.5.

The rational function is a topological invariant of the manifold and carries significant topological information, see for example Theorem 4 in section 4.5.

To grasp the flavor of this invariant let us briefly describe it. For a collection of non-negative integers $(k_0, k_1, \dots, k_n)$ which satisfy (1): $k_i - k_{i-1} \pm \cdots + (-1)^i k_0 \geq 0$, $i \leq n-1$ and (2): $k_n - k_{n-1} \pm \cdots + (-1)^n k_0 = 0$. We consider the variety $\hat{\mathbb{D}}_{\mathrm{ac}}(k_0, \dots, k_n)$ of generically acyclic complexes, $C \equiv \{0 \to \mathbb{C}^{k_0} \to \cdots \to \mathbb{C}^{k_n} \to 0\}$, and the rational function τ where $\tau(C)$ is the torsion of C when acyclic, cf. section 4.2.[4]

For a finitely presented group Γ we denote by $\mathrm{Rep}(\Gamma; V)$ the complex algebraic set of the complex representations of Γ on the complex vector space V, cf. section 4.1.

For a closed based pointed manifold (M, x_0) with $\Gamma = \pi_1(M, x_0)$ denote by $\mathrm{Rep}^M(\Gamma; V)$ the algebraic closure of $\mathrm{Rep}^M_0(\Gamma; V)$, the Zariski open set of representation $\rho \in \mathrm{Rep}(\Gamma; V)$ so that $H^*(M; \rho) = 0$.

By choosing a triangulation τ of M, a collection of paths $\pi_\tau = \{\pi_{x_\sigma}\}$ from x_0 to the barycenters x_σ of each simplex σ, an ordering o of the set of of simplexes and a base (frame) ϵ of V one obtains a regular map $t_{\pi_\tau, o, \epsilon} : \mathrm{Rep}^M(\Gamma; V) \to \hat{\mathbb{D}}_{\mathrm{ac}}(k_0, \dots, k_n)$ with $k_i = \dim(V) \times \sharp(\mathcal{X}_i)$ where $\mathcal{X}_i$ denotes the set of i–simplexes and $\sharp$ denotes cardinality. One can show that while the map $t_{\pi_\tau, o, \epsilon}$ depends on (π_τ, o, ϵ) the pull-back of τ by $t_{\pi_\tau, o, \epsilon}$ depends only on the Euler structure defined by (τ, π_τ) (more precisely by (X_τ, π_τ) cf. section 3.1) and the cohomology orientation in $H^*(M; \mathbb{R})$ induced by o. If one restricts to unimodular unitary representations, as also noticed by W. Müller, not even on the Euler structure.

The absolute value of the Milnor–Turaev torsion is related to Ray–Singer torsion. To describe this relation one needs again the invariant R. The argument is an unexplored invariant which for manifolds of dimension 3, but not in general, might be related to the Atiyah–Patodi–Singer eta invariant, cf. [APS75], at least when considered for a unimodular representation.

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[1] The question can be formulated and positively answered for all dimensions

[2] A similar question was treated in [Q85]. Instead of the variety of representation Quillen considers the affine space of complex structures in a smooth complex vector bundle over a Riemann surface, and instead of Ray–Singer torsion the square root of the determinant of D^*D where D is the appropriate Cauchy–Riemann operator.

[3] The sign ambiguity in Milnor's definition of torsion was ingeniously fixed by Turaev.

[4] This function is the obvious generalization of $\det : \hat{\mathbb{D}}_{\mathrm{ac}}(r, r) = \mathbb{C}^{r^2} \to \mathbb{C}$.

The main results are contained in Theorems 1, 3, 4, and Corollary 1. The paper contains a number of additional facts/results of independent interest which expand on the concepts described in the paper.

Let us make a more detailed presentation of the contents of this paper.

Section 2 is largely expository. We collect a number of facts about Riemannian metrics and vector fields with isolated zeros and about the Kamber–Tondeur form, cf. sections 2.1–2.3. In section 2.4 we describe the regularization of some possibly divergent integrals which is essential in the subsequent considerations (the invariant R, the comparison of Euler and co-Euler structures, the modifications of torsions). In section 2.5 we define the invariant R. The invariant R was first introduced by Bismut and Zhang in [BZ92].

We regard the invariant R as providing a pairing between a Riemannian metric and a vector field with isolated zeros with values in one currents rel. boundary. This is in analogy with the Chern–Simons form which provides such pairing for two Riemannian metrics and with the Chern–Simons one chain rel. boundary which provides a pairing between two vector fields with isolated zeros.

Section 3 discusses Euler and co-Euler structures and what they are good for. Turaev [T90] introduced the concept of Euler structures for manifolds with vanishing Euler–Poincaré characteristics $\chi(M)$. It was observed in [B99] that Euler structures can be defined for manifolds with arbitrary Euler characteristics at the expense of a base point $x_0 \in M$. The space of Euler structures $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ is an affine version of $H_1(M;\mathbb{Z})$ in the sense that is $H_1(M;\mathbb{Z})$ acts free and transitively on $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$. There is also a real version of Euler structures which is an affine version of $H_1(M;\mathbb{R})$ and will be denoted by $\mathfrak{Eul}_{x_0}(M;\mathbb{R})$. There is a canonic map $\mathfrak{Eul}_{x_0}(M;\mathbb{Z}) \to \mathfrak{Eul}_{x_0}(M;\mathbb{R})$ which is affine over the homomorphism $H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{R})$, see section 3.1.

Given an Euler structure $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$, a cohomology orientation $\mathfrak{o}$, i.e. an orientation of $H^*(M;\mathbb{R})$, and a representation $\rho \in \operatorname{Rep}(\Gamma;V)$ we use the Milnor–Turaev construction [M66] [T90] to provide an element $\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\text{comb}}$ in the determinant line[5]

$$\operatorname{Det}_{x_0}(M;\rho) := \det H^*(M;\rho) \otimes \det V^{-\chi(M)} \tag{1}$$

which will be called *Milnor–Turaev torsion,* see Definition 1 in section 3.3.

For the dependence on $\mathfrak{e}$ and $\mathfrak{o}$ we find

$$\tau^{\rho,\mathfrak{e}+\sigma}_{\text{comb}} = \tau^{\rho,\mathfrak{e}}_{\text{comb}} \cdot [\det\circ\rho](\sigma)^{-1} \quad \text{and} \quad \tau^{\rho,\mathfrak{e},-\mathfrak{o}}_{\text{comb}} = (-1)^{\dim V}\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\text{comb}} \tag{2}$$

for all $\sigma \in H_1(M;\mathbb{Z})$. Here $[\det\circ\rho] : H_1(M;\mathbb{Z}) \to \mathbb{C}^*$ denotes the homomorphism induced from the homomorphism $\det\circ\rho : \Gamma \to \mathbb{C}^*$.

In section 3.2 we introduce the concept of co-Euler structures. The set of co-Euler structures $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ is an affine version of $H^{n-1}(M;\mathcal{O}_M)$ and depends on the choice of a base point $x_0 \in M$ if $\chi(M) \neq 0$. By definition a co-Euler structure is an equivalence class of pairs (g,α), where g is a Riemannian metric on M, and $\alpha \in \Omega^{n-1}(M \setminus \{x_0\};\mathcal{O}_M)$ with $d\alpha = E(g)$ where $E(g) \in \Omega^n(M;\mathcal{O}_M)$ denotes the Euler form of g. Two such pairs (g_1,α_1) and (g_2,α_2) are equivalent if $\alpha_2 - \alpha_1 = \operatorname{cs}(g_1,g_2)$ where $\operatorname{cs}(g_1,g_2) \in \Omega^{n-1}(M;\mathcal{O}_M)/d\Omega^{n-2}(M;\mathcal{O}_M)$ denotes the

[5]Here we use the following notation. For a finite dimensional vector space V we write $\det V := \Lambda^{\dim V} V$ for its determinant line, a one dimensional vector space. For a finite dimensional graded vector space W^* we define the (graded) determinant line by $\det W^* := \det W^{\text{even}} \otimes (\det W^{\text{odd}})^*$. Moreover, for a line L, i.e. one dimensional vector space, and $k \in \mathbb{N}$ we write $L^k := L \otimes \cdots \otimes L$ and $L^{-k} := (L^k)^*$. Then there are natural identifications $L^{k+l} = L^k \otimes L^l$ for all $k, l \in \mathbb{Z}$.

Chern–Simons class. The main properties of Euler and co-Euler structures, and the relationship between them are collected in Theorem 1 in section 3.2. Co-Euler structures are nicely suited to remove the metric ambiguities of the analytic torsion, and finally provide a topological invariant *the modified Ray–Singer metric/torsion*.

More precisely, let $\mathbb{F}_\rho$ denote the flat complex vector bundle over M associated to a representation $\rho \in \operatorname{Rep}(\Gamma; V)$. Recall that the Ray–Singer metric is a Hermitian metric $||\cdot||^{\rho,g,\mu}_{\mathrm{RS}}$ on the determinant line $\det H^*(M;\mathbb{F}_\rho) = \det H^*(M;\rho)$, but depends on a Riemannian metric g on M, and a Hermitian structure μ on $\mathbb{F}_\rho$, see [BZ92]. Using a co-Euler structure $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ permits to define a *modified Ray–Singer metric* $||\cdot||^{\rho,\mathfrak{e}^*}_{\mathrm{an}}$ on the determinant line (1), see Definition 2 in section 3.4. The anomaly formulas for the Ray–Singer torsion imply that this Hermitian metric indeed only depends on the representation ρ and the co-Euler structure $\mathfrak{e}^*$. For the dependence on $\mathfrak{e}^*$ we have

$$||\cdot||^{\rho,\mathfrak{e}^*+\beta}_{\mathrm{an}} = ||\cdot||^{\rho,\mathfrak{e}^*}_{\mathrm{an}} \cdot e^{\langle \log|[\det\circ\rho]|, PD(\beta)\rangle}$$

for all $\beta \in H^{n-1}(M;\mathcal{O}_M)$. Here $\log|[\det\circ\rho]| \in H^1(M;\mathbb{R})$ denotes the class corresponding to the homomorphism $\log|[\det\circ\rho]| : H_1(M;\mathbb{Z}) \to \mathbb{R}$, which is paired with the Poincaré dual of β.

There is an isomorphism $P : \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \to \mathfrak{Eul}_{x_0}(M;\mathbb{R})$ which is affine over the Poincaré duality isomorphism $H^{n-1}(M;\mathcal{O}_M) \to H_1(M;\mathbb{R})$, see section 3.2. This permits to reformulate the Bismut–Zhang theorem [BZ92] as a statement which compares two topological invariants, see Theorem 2 in section 3.4. More precisely, it tells that analysis is able to recover the absolute value of the Milnor–Turaev torsion:

$$||\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}||^{\rho,\mathfrak{e}^*}_{\mathrm{an}} = e^{\langle \log|[\det\circ\rho]|, P(\mathfrak{e}^*)-\mathfrak{e}\rangle}$$

Here $\log|[\det\circ\rho]| \in H^1(M;\mathbb{R})$ denotes the cohomology class corresponding to the homomorphism $\log|[\det\circ\rho]| : H_1(M;\mathbb{Z}) \to \mathbb{R}$. If $H^*(M;\rho) = 0$ then $\operatorname{Det}_{x_0}(M;\rho) = \mathbb{C}$, and $\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ is a non-zero complex number.

Section 4 is dedicated to the Milnor–Turaev torsion viewed as a rational function. Let $\operatorname{Rep}^M_o(\Gamma;V) \subseteq \operatorname{Rep}(\Gamma;V)$ denotes the Zariski open subset of representations ρ for which $H^*(M;\rho) = 0$, and let $\operatorname{Rep}^M(\Gamma;V)$ denote its Zariski closure in $\operatorname{Rep}(\Gamma;V)$, cf. section 4.1. $\operatorname{Rep}^M_o(\Gamma;V)$ and $\operatorname{Rep}^M(\Gamma;V)$ only depend on the homotopy type of M, but might be empty, e.g. if $\chi(M) \neq 0$. The Milnor–Turaev torsion gives rise to a function

$$\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}} : \operatorname{Rep}^M_o(\Gamma;V) \to \mathbb{C}^*$$

which depends on the *simple* homotopy type of M, $\mathfrak{e}$ and $\mathfrak{o}$. In section 4.2 we observe that this function is a rational function on $\operatorname{Rep}^M(\Gamma;V)$ whose zeros and poles are therefore contained in $\operatorname{Rep}^M(\Gamma;V) \setminus \operatorname{Rep}^M_o(\Gamma;V)$, see Theorem 3.

The function $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ contains relevant topological information, even in the simplest possible examples. In section 4.3 we compute this function for a mapping torus in terms of the Lefschetz zeta function of the glueing diffeomorphism.

If M is obtained by surgery on a framed knot, and $\dim V = 1$, the function $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ coincides with the Alexander polynomial of the knot, see [T02]. We hope that passing to higher dimensional representations $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ captures more subtle knot invariants.

The Bismut–Zhang theorem implies that the absolute value of $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\text{comb}}$ is a modified Ray–Singer torsion:

$$|\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\text{comb}}(\rho)| = T_{\text{an}}(\nabla_\rho, g, \mu) \cdot e^{-\int_M \omega(\nabla_\rho,\mu)\wedge\alpha}$$

Here $\mathbb{F}_\rho = (F_\rho, \nabla_\rho)$ denotes the flat complex vector bundle associated to the representation $\rho \in \text{Rep}^M_o(\Gamma; V)$, g is a Riemannian metric on M, μ is a Hermitian structure on F_ρ, $T_{\text{an}}(\nabla_\rho, g, \mu)$ denotes the Ray–Singer torsion, $\omega(\nabla_\rho, \mu)$ is the Kamber–Tondeur form, see section 2.3, and α is such that the pair (g, α) represents the co-Euler structure $P^{-1}(\mathfrak{e})$.

We use this fact to conclude (under some additional hypotheses) that *the Ray–Singer torsion, when viewed as a function on the analytic set of flat connections, is the absolute value of a meromorphic function.* The precise statement is contained in Corollary 1. This partially answers the question formulated at the very beginning of this introduction. This was also known to W. Müller at least in the case when one restricts to unimodular flat connections.

The argument of the complex number $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\text{comb}}(\rho)$ is an apparently unexplored invariant which is analoguous to the Atiyah–Patodi–Singer eta invariant.

As another application we provide a derivation of Marcsik's theorem from the Bismut–Zhang theorem and the computation of $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\text{comb}}$ for mapping tori, see section 4.5.

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2. A few characteristic forms and the invariant R

2.1. **Euler, Chern–Simons, and the global angular form.** Let M be smooth closed manifold of dimension n. Let $\pi : TM \to M$ denote the tangent bundle, and $\mathcal{O}_M$ the orientation bundle, a flat real line bundle over M. For a Riemannian metric g denote by

$$E(g) \in \Omega^n(M; \mathcal{O}_M)$$

its Euler form, and for two Riemannian metrics g_1 and g_2 by

$$\text{cs}(g_1, g_2) \in \Omega^{n-1}(M; \mathcal{O}_M)/d(\Omega^{n-2}(M; \mathcal{O}_M))$$

their Chern–Simons class. The definition of both quantities is implicit in the formulae (6) and (7) below. They have the following properties which follow immediately from (6) and (7) below.

$$d\,\text{cs}(g_1, g_2) = E(g_2) - E(g_1) \tag{3}$$

$$\text{cs}(g_2, g_1) = -\text{cs}(g_1, g_2) \tag{4}$$

$$\text{cs}(g_1, g_3) = \text{cs}(g_1, g_2) + \text{cs}(g_2, g_3) \tag{5}$$

Let ξ denote the Euler vector field on TM which assigns to a point $x \in TM$ the vertical vector $-x \in T_xTM$.

A Riemannian metric g determines the Levi–Civita connection in the bundle $\pi : TM \to M$. There is a canonic $\text{vol}(g) \in \Omega^n(TM; \pi^*\mathcal{O}_M)$, which vanishes when contracted with horizontal vectors and which assigns to an n-tuple of vertical vectors *their volume times their orientation.* The global angular form, see for instance [BT82], is the differential form

$$\Psi(g) := \frac{\Gamma(n/2)}{(2\pi)^{n/2}|\xi|^n} i_\xi \,\text{vol}(g) \in \Omega^{n-1}(TM \setminus M; \pi^*\mathcal{O}_M).$$

This form was also considered by Mathai and Quillen [MQ86], and was referred to as the Mathai–Quillen form in [BZ92].

Note that $\Psi(g)$ is the pull back of a form on $(TM \setminus M)/\mathbb{R}_+$. Moreover, we have the equalities

$$d\Psi(g) = \pi^* E(g). \tag{6}$$
$$\Psi(g_2) - \Psi(g_1) = \pi^* \operatorname{cs}(g_1, g_2) \mod \pi^* d\Omega^{n-2}(M; \mathcal{O}_M) \tag{7}$$

Suppose X is a vector field with isolated zero x, and let $B_\epsilon(x)$ denote the ball of radius ϵ centered at x, with respect to some chart. Then

$$\lim_{\epsilon \to 0} \int_{\partial(M \setminus B_\epsilon(x))} X^* \Psi(g) = \mathrm{IND}_X(x), \tag{8}$$

where $\mathrm{IND}_X(x)$ denotes the Hopf index of X at x. For $M = \mathbb{R}^n$ equipped with the standard Riemannian metric g_0 we have

$$\Psi(g_0) = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \sum_{i=1}^n (-1)^i \frac{\xi_i}{(\sum \xi_i^2)^{n/2}} d\xi_1 \wedge \cdots \wedge \widehat{d\xi_i} \wedge \cdots \wedge d\xi_n.$$

in standard coordinates $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ on TM.

The same definition provides a global angular form on every real vector bundle equipped with a linear connection and a parallel Hermitian (fiber) metric.

2.2. **Euler and Chern–Simons class for vector fields.** Let M be a closed manifold and suppose X is a vector field with isolated set of zeros $\mathcal{X} \subseteq M$. For every $x \in \mathcal{X}$ we have a Hopf index $\mathrm{IND}_X(x)$ of X at x. Define a singular[6] zero chain

$$E(X) := \sum_{x \in \mathcal{X}} \mathrm{IND}_X(x) x \in C_0(M; \mathbb{Z}).$$

Given two vector fields X_1 and X_2 we are going to define a singular one chain, well defined up to boundaries,

$$\operatorname{cs}(X_1, X_2) \in C_1(M; \mathbb{Z}) / \partial C_2(M; \mathbb{Z}) \tag{9}$$

with the properties:

$$\partial \operatorname{cs}(X_1, X_2) = E(X_2) - E(X_1) \tag{10}$$
$$\operatorname{cs}(X_2, X_1) = -\operatorname{cs}(X_1, X_2) \tag{11}$$
$$\operatorname{cs}(X_1, X_3) = \operatorname{cs}(X_1, X_2) + \operatorname{cs}(X_2, X_3) \tag{12}$$

This should be compared with the equalities (3)–(5) in section 2.1.

The construction of (9) is accomplished by first reducing to vector fields with non-degenerate zeros with the help of a small perturbation, and then taking the zero set of a generic homotopy connecting X_1 with X_2.

More precisely, suppose we have two vector fields X_1' and X_2' with non-degenerate zeros. Consider the vector bundle $p^*TM \to I \times M$, where $I := [1, 2]$ and $p : I \times M \to M$ denotes the natural projection. Choose a section $\mathbb{X}$ of p^*TM which is transversal to the zero section and which restricts to X_i' on $\{i\} \times M$, $i = 1, 2$. The zero set of $\mathbb{X}$ is a canonically oriented one dimensional submanifold with boundary. Its fundamental class, when pushed forward via p, gives rise to $c(\mathbb{X}) \in C_1(M; \mathbb{Z}) / \partial C_2(M; \mathbb{Z})$ satisfying (10).

[6] Here, and throughout the paper, singular chain stands for *smooth* singular chain.

Suppose $\mathbb{X}_1$ and $\mathbb{X}_2$ are two non-degenerate homotopies from X_1' to X_2'. Then $c(\mathbb{X}_1) = c(\mathbb{X}_2) \in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z})$. Indeed, consider the vector bundle $q^*TM \to I \times I \times M$, where $q : I \times I \times M \to M$ denotes the natural projection. Choose a section of q^*TM which is transversal to the zero section, restricts to $\mathbb{X}_i$ on $\{i\} \times I \times M$, $i = 1, 2$, and which restricts to X_i' on $\{s\} \times \{i\} \times M$ for all $s \in I$ and $i = 1, 2$. The zero set of such a section then gives rise to σ satisfying $c(\mathbb{X}_2) - c(\mathbb{X}_1) = \partial\sigma$. So we get a well defined $\mathrm{cs}(X_1', X_2') := c(\mathbb{X}) \in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z})$ satisfying (10), provided X_1' and X_2' have non-degenerate zero set.

Now suppose X is a vector field with isolated zeros. For every zero $x \in \mathcal{X}$ we choose a contractible open neighborhood B_x of x, assuming all B_x are disjoint. Set $B := \bigcup_{x\in\mathcal{X}} B_x$. Choose a vector field with non-degenerate zeros X' that coincides with X on $M \setminus B$. Let $\mathcal{X}'$ denote its zero set. For every $x \in \mathcal{X}$ we have $\mathrm{IND}_X(x) = \sum_{y\in\mathcal{X}'\cap B_x} \mathrm{IND}_{X'}(y)$. So we can choose a one chain $c(X, X')$ in B which satisfies $\partial c(X, X') = E(X') - E(X)$. Since $H_1(B;\mathbb{Z})$ vanishes, the one chain $c(X, X')$ is well defined up to a boundary.

Given two vector fields X_1 and X_2 with isolated zeros we choose B, X_1', X_2', $c(X_1, X_1')$ and $c(X_2, X_2')$ as above and set

$$\mathrm{cs}(X_1, X_2) := c(X_1, X_1') + \mathrm{cs}(X_1', X_2') - c(X_2, X_2').$$

Note that for two vector fields which coincide on $M \setminus B$ and have non-degenerate zeros only, one can construct a non-degenerate homotopy between them which is constant on $M \setminus B$. Together with $H_1(B;\mathbb{Z}) = 0$ this immediately implies that $\mathrm{cs}(X_1, X_2) \in C_1(M;\mathbb{Z})/\partial C_2(M;\mathbb{Z})$ is independent of these choices. Clearly (10)–(12) hold too.

2.3. **Kamber–Tondeur one form.** Let E be a real or complex vector bundle over M. For a connection ∇ and a Hermitian structure μ on E we define a real valued one form $\omega(\nabla, \mu) := -\frac{1}{2}\,\mathrm{tr}_\mu(\nabla\mu) \in \Omega^1(M;\mathbb{R})$. More explicitly, for a tangent vector $X \in TM$

$$\omega(\nabla, \mu)(X) := -\frac{1}{2}\,\mathrm{tr}_\mu(\nabla_X\mu).$$

This convention differs from the one in [BZ92] by a factor $-\frac{1}{2}$. Note that if ∇ is flat then $\omega(\nabla, \mu)$ will be a closed one form. This and the following formulas are easily seen by noticing that $\omega(\nabla^{\det E}, \mu^{\det E}) = \omega(\nabla, \mu)$ where $\nabla^{\det E}$ and $\mu^{\det E}$ denote the induced connection and Hermitian structure on the determinant line $\det E := \Lambda^{\mathrm{rank}\, E} E$.

Recall that the space of connections on E is an affine space over $\Omega^1(M; E^* \otimes E)$. For two connections ∇^1 and ∇^2 we find

$$\omega(\nabla^2, \mu) - \omega(\nabla^1, \mu) = \Re(\mathrm{tr}(\nabla^2 - \nabla^1)) \tag{13}$$

where the trace is with respect to the last two variables.

If μ_1 and μ_2 are two Hermitian structures then there exists a unique section $A \in C^\infty(E^* \otimes E)$ such that $\mu_2(e, f) = \mu_1(Ae, f)$. We set $V(\mu_1, \mu_2) := |\det A|$ which is a positive real valued function on M. Its value at $x \in M$ is the volume, with respect to $(\mu_2)_x$, of a parallelepiped obtained from an orthonormal base, with respect to $(\mu_1)_x$. We have:

$$\omega(\nabla, \mu_2) - \omega(\nabla, \mu_1) = -\frac{1}{2} d \log V(\mu_1, \mu_2) \tag{14}$$

Suppose ∇ is flat, and let $U \subseteq M$ be a contractible open set. Then we find a parallel Hermitian structure $\tilde{\mu}$ over U, and can use (14) to obtain a local potential of the closed one form $\omega(\nabla, \mu)|_U = d(-\frac{1}{2}\log V(\tilde{\mu}, \mu))$.

Remark 1. Suppose ∇ is flat, and let x_0 be a base point in M. Parallel transport defines a right action of $\pi_1(M, x_0)$ on E_{x_0}, the fiber over x_0. Composing with the inversion we obtain a representation $\rho : \pi_1(M, x_0) \to \mathrm{GL}(E_{x_0})$. Let $[\det \circ \rho] : H_1(M; \mathbb{Z}) \to \mathbb{C}^*$ denote the homomorphism induced by the homomorphism $\det \circ \rho : \pi_1(M, x_0) \to \mathbb{C}^*$. Let $\log |[\det \circ \rho]| \in H^1(M; \mathbb{R})$ denote the cohomology class corresponding to the homomorphism $\log |[\det \circ \rho]| : H_1(M; \mathbb{Z}) \to \mathbb{R}$. This class is called the *Kamber–Tondeur class* and is represented by the Kamber–Tondeur form $\omega(\nabla, \mu)$ for every Hermitian metric μ. Put another way, the cohomology class of $\omega(\nabla, \mu)$ captures the absolute value of the holonomy in $(\det E, \nabla^{\det E})$.

The following observation will be useful in the proof of Corollary 1.

Proposition 1. *Let E be a complex vector bundle over M equipped with a Hermitian structure μ. Then there exist mappings $\nabla \mapsto \tilde{\omega}(\nabla, \mu)$ which are affine over the map $\mathrm{tr} : \Omega^1(M; E^* \otimes E) \to \Omega^1(M; \mathbb{C})$ such that $\Re(\tilde{\omega}(\nabla, \mu)) = \omega(\nabla, \mu)$, and such that $\tilde{\omega}(\nabla, \mu)$ is closed whenever ∇ is flat. Particularly, the assignment $\nabla \mapsto \tilde{\omega}(\nabla, \mu)$ is holomorphic.*

Proof. Choose a reference connection ∇' on E, and define

$$\tilde{\omega}(\nabla, \mu) := \mathrm{tr}(\nabla - \nabla') + \omega(\nabla', \mu).$$

This is certainly affine over the trace, and its real part coincides with $\omega(\nabla, \mu)$ in view of (13). If E admits flat connections we can choose ∇' to be flat. Then $\tilde{\omega}(\nabla, \mu)$ will be closed provided ∇ was flat. $\square$

2.4. **Regularization of an integral.** Let M be a closed manifold of dimension n. Suppose $\alpha \in \Omega^{n-1}(M \setminus \mathcal{X}; \mathcal{O}_M)$ is a form with isolated set of singularities $\mathcal{X} \subseteq M$, and suppose $d\alpha \in \Omega^n(M; \mathcal{O}_M)$ is globally smooth. Here $\mathcal{O}_M$ denotes the orientation bundle of M, a flat real line bundle over M. For $x \in \mathcal{X}$ define a real number

$$\mathrm{IND}_\alpha(x) := \lim_{\epsilon \to 0} \int_{\partial(M \setminus B_\epsilon(x))} \alpha = \int_{\partial(M \setminus B_{\epsilon'}(x))} \alpha + \int_{B_{\epsilon'}(x)} d\alpha$$

where $B_\epsilon(x)$ denotes the ball of radius ϵ in a chart centered at x. Note that for a Riemannian metric g, and a vector field X with isolated zero x, the form $X^*\Psi(g)$ has an isolated singularity at x, $dX^*\Psi(g) = E(g)$ is globally smooth, see (6), and in view of (8) we have

$$\mathrm{IND}_{X^*\Psi(g)}(x) = \mathrm{IND}_X(x). \tag{15}$$

Let α be as above, and suppose $\omega \in \Omega^1(M)$ is a closed real or complex valued one form. Choose a function f such that f vanishes on $\mathcal{X}$ and such that $\omega' := \omega - df$ vanishes locally around $\mathcal{X}$. Define:

$$S(\omega, \alpha) := S(\omega, \alpha; f) := \int_{M \setminus \mathcal{X}} \omega' \wedge \alpha - \int_M f d\alpha$$

The notation is justified by the following

Lemma 1. *The quantity $S(\omega, \alpha; f)$ does not depend on the choice of f. It is linear in ω, and α. It satisfies:*

(i) $S(dh, \alpha) = \sum_{x \in \mathcal{X}} h(x) \mathrm{IND}_\alpha(x) - \int_M h d\alpha$ *for every smooth function h.*

(ii) $S(\omega,\beta)=\int_M \omega\wedge\beta$ *for all* $\beta\in\Omega^{n-1}(M;\mathcal{O}_M)$. *Particularly* $S(\omega,d\gamma)=0$ *for all* $\gamma\in\Omega^{n-2}(M;\mathcal{O}_M)$.

Proof. Let us first show that $S(\omega,\alpha;f)$ does not depend on f. So suppose f_i are functions which vanish on $\mathcal{X}$ and are such that $\omega_i':=\omega-df_i$ vanishes locally around $\mathcal{X}$. Then $\omega_2'-\omega_1'=-d(f_2-f_1)$ and f_2-f_1 vanishes locally around $\mathcal{X}$. Using Stokes' theorem for forms with compact support on $M\setminus\mathcal{X}$ we get

$$S(\omega,\alpha;f_2)-S(\omega,\alpha;f_1)=-\int_{M\setminus\mathcal{X}} d(f_2-f_1)\wedge\alpha-\int_M (f_2-f_1)d\alpha=0.$$

Now let us turn to (i). Choose a function f so that $dh-df$ vanishes locally around $\mathcal{X}$ and so that f vanishes on $\mathcal{X}$. For every $x\in\mathcal{X}$ choose an embedded ball $B_\epsilon(x)$ around x such that $dh-df$ vanishes on $B_\epsilon(x)$. Then $h-f$ is constant equal to $h(x)$ on $B_\epsilon(x)$, and Stokes' theorem implies

$$S(dh,\alpha)=\int_{M\setminus\mathcal{X}}(dh-df)\wedge\alpha-\int_M fd\alpha=\sum_{x\in\mathcal{X}}h(x)\int_{\partial(M\setminus B_\epsilon(x))}\alpha-\int_M hd\alpha$$

With $\epsilon\to 0$ the statement follows. The remaining assertions are obvious. □

2.5. **The invariant** R. Let M be a closed manifold of dimension n. For a closed one form ω, and two Riemannian metrics g_1 and g_2 we set

$$R(\omega,g_1,g_2):=\int_M \omega\wedge\mathrm{cs}(g_1,g_2). \tag{16}$$

Note that even though $\mathrm{cs}(g_1,g_2)$, see section 2.1, is only defined up to $d\Omega^{n-2}(M;\mathcal{O}_M)$ the quantity (16) is unambiguously defined according to Stokes' theorem.

For a closed one form ω, and two vector fields X_1 and X_2 with isolated zeros we set

$$R(\omega,X_1,X_2):=\int_{\mathrm{cs}(X_1,X_2)}\omega. \tag{17}$$

Again, even though $\mathrm{cs}(X_1,X_2)$, see section 2.2, is only defined up to a boundary the quantity (17) is well defined in view of Stokes' theorem.

For a Riemannian metric g and a vector field X with isolated zeros $\mathcal{X}\subseteq M$ we have $X^*\Psi(g)\in\Omega^{n-1}(M\setminus\mathcal{X};\mathcal{O}_M)$ and $dX^*\Psi(g)=E(g)$, see section 2.1, especially (6). So we may define, see section 2.4,

$$R(\omega,X,g):=S(\omega,X^*\Psi(g))$$

for every closed one form ω. The invariant R has first appear in Bismut–Zhang work [BZ92].

Proposition 2. *Let ω be a closed one form, h a smooth function, g and g_i Riemannian metrics, and let X and X_i be vector fields with isolated zeros, on M. Then:*

(i) $R(\omega,g_1,g_2)$, $R(\omega,X_1,X_2)$ *and* $R(\omega,X,g)$ *are linear in* ω.
(ii) $R(\omega,g_2,g_1)=-R(\omega,g_1,g_2)$.
(iii) $R(\omega,X_2,X_1)=-R(\omega,X_1,X_2)$.
(iv) $R(dh,X,g)=h(E(X))-\int_M hE(g)$.
(v) $R(dh,g_1,g_2)=\int_M hE(g_1)-\int_M hE(g_2)$.
(vi) $R(dh,X_1,X_2)=h(E(X_2))-h(E(X_1))$.
(vii) $R(\omega,X_1,X_3)=R(\omega,X_1,X_2)+R(\omega,X_2,X_3)$.
(viii) $R(\omega,g_1,g_3)=R(\omega,g_1,g_2)+R(\omega,g_2,g_3)$.

(ix) $R(\omega, X, g_2) - R(\omega, X, g_1) = R(\omega, g_1, g_2)$.
(x) $R(\omega, X_2, g) - R(\omega, X_1, g) = R(\omega, X_1, X_2)$.

Here we wrote $h(E(X)) = \sum_{x\in\mathcal{X}} h(x)\,\mathrm{IND}_X(x)$ *where* $\mathcal{X}$ *denotes the zero set of* X.

Proof. Statements (i)–(iii) and (v)–(viii) follow immediately from Stokes' theorem, Lemma 1, equations (3)–(5) and equations (10)–(12).

Assertion (iv) follows from Lemma 1(i), equation (15), and from $dX^*\Psi(g) = E(g)$, see (6).

Statement (ix) follows from Lemma 1(ii) and $X^*\Psi(g_2) - X^*\Psi(g_1) = \mathrm{cs}(g_1, g_2)$, see (7). Statements (iv) and (ix) can be also found in [BZ92].

Remains to prove (x). In view of (iv) and (vi) we may assume that ω vanishes in a neighborhood of the zero set $\mathcal{X}_1 \cup \mathcal{X}_2$. It is then clear that we may also assume that X_1 and X_2 have non-degenerate zeros only. Choose a non-degenerate homotopy $\mathbb{X}$ from X_1 to X_2. Perturbing the homotopy slightly, cutting it into several pieces and using (vii) we may assume that the zero set $\mathbb{X}^{-1}(0) \subseteq I \times M$ is actually contained in a closed simply connected $I \times V$. Again, we may assume that ω vanishes on V. Then obviously $R(\omega, X_1, X_2) = 0$. Moreover, in this situation Stokes' theorem implies

$$\begin{aligned} R(\omega, X_2, g) - R(\omega, X_1, g) &= \int_{M\setminus V} \omega \wedge X_2^*\Psi(g) - \int_{M\setminus V} \omega \wedge X_1^*\Psi(g) \\ &= \int_{I\times(M\setminus V)} d(p^*\omega \wedge \mathbb{X}^*\tilde{p}^*\Psi(g)) \\ &= -\int_{I\times(M\setminus V)} p^*(\omega \wedge E(g)) = 0 \end{aligned}$$

where $p : I\times M \to M$ denotes the natural projection, and $\tilde{p} : p^*TM \to TM$ denotes the natural vector bundle homomorphism over p. For the last calculation note that $d\mathbb{X}^*\tilde{p}^*\Psi(g) = p^*E(g)$ in view of (6), and that $\omega \wedge E(g) = 0$ because of dimensional reasons. □

Remark 2. A similar definition of $R(X, g, \omega)$ works for any vector field X with arbitrary singularity set $\mathcal{X} := \{x \in M \mid X(x) = 0\}$ provided ω is exact when restricted to a sufficiently small neighborhood of $\mathcal{X}$. This situation might lead to interesting invariants for holomorphic vector fields on a Kähler manifolds or even more general, for symplectic vector fields on a symplectic manifold equipped with an almost complex structure which tames the symplectic structure.

2.6. **Extension of Chern–Simons theory.** The material in this section will not be used in the rest of the paper, but puts the observations in Proposition 2 in a more natural perspective.

Let M be a closed manifold of dimension n. We equip $\Omega^k(M;\mathbb{R})$ with the C^∞–topology. The continuous linear functionals on $\Omega^k(M;\mathbb{R})$ are called k currents. We denote the space of all k currents by $\mathcal{D}_k(M)$. Let $\delta : \mathcal{D}_k(M) \to \mathcal{D}_{k-1}(M)$ be given by

$$(\delta\varphi)(\alpha) := \varphi(d\alpha).$$

Clearly $\delta^2 = 0$. By the deRham theorem for currents the chain complex $(\mathcal{D}_*(M), \delta)$ computes the homology of M with real coefficients.

We have a morphism of chain complexes

$$C_*(M;\mathbb{R}) \to \mathcal{D}_*(M), \quad \sigma \mapsto \hat{\sigma}, \quad \hat{\sigma}(\alpha) := \int_\sigma \alpha.$$

Here $C_*(M;\mathbb{R})$ denotes the space of singular chains with real coefficients. Moreover, we have a morphism of chain complexes

$$\Omega^{n-*}(M;\mathcal{O}_M) \to \mathcal{D}_*(M), \quad \beta \mapsto \hat{\beta}, \quad \hat{\beta}(\alpha) := (-1)^{\frac{1}{2}|\alpha|(|\alpha|+1)} \int_M \alpha \wedge \beta.$$

Here $|\alpha|$ denotes the degree of α.

Note that any zero current represents a degree zero homology class, and $H_0(M;\mathbb{R})$ identifies canonically to $\bigoplus_\alpha \mathbb{R}_\alpha$ where $\mathbb{R}_\alpha$ is a copy of $\mathbb{R}$ corresponding to a connected component of $M = \bigsqcup_\alpha M_\alpha$. One denotes by $\overline{\chi}(M)$ the vector $\{\chi(M_\alpha)\} \in \bigoplus_\alpha \mathbb{R}_\alpha$, and by $\mathcal{E}_0$ the affine subspace of $\mathcal{D}_0(M)$ consisting of zero currents which represent $\overline{\chi}(M)$. Elements in $\mathcal{E}_0$ are called *Euler currents*.

Every vector field X with isolated set of zeros $\mathcal{X}_\tau$ gives rise to a zero chain $E(X)$, see section 2.2. Via the first morphism we get a zero current $\hat{E}(X)$. More explicitly, $\hat{E}(X)(h) = \sum_{x\in\mathcal{X}} \mathrm{IND}_x(X)h(x)$ for all functions $h \in \Omega^0(M;\mathbb{R})$. By Hopf's theorem this is an Euler current.

A Riemannian metric g has an Euler form $E(g) \in \Omega^n(M;\mathcal{O}_M)$ see section 2.1. Via the second morphism we get a zero current $\hat{E}(g)$ which, by Gauss–Bonnet, is an Euler current. More explicitly, $\hat{E}(g)(h) = \int_M hE(g)$ for all functions $h \in \Omega^0(M;\mathbb{R})$.

Let $\mathcal{Z}^k(M;\mathbb{R}) \subseteq \Omega^k(M;\mathbb{R})$ denote the space of closed k forms on M equipped with the C^∞–topology. The continuous linear functionals on $\mathcal{Z}^k(M;\mathbb{R})$ are referred to as *k currents rel. boundary* and identify to $\mathcal{D}_k(M)/\delta(\mathcal{D}_{k+1}(M))$. The two chain morphisms provide mappings

$$C_k(M;\mathbb{R})/\partial(C_{k+1}(M;\mathbb{R})) \to \mathcal{D}_k(M)/\delta(\mathcal{D}_{k+1}(M)) \tag{18}$$

and

$$\Omega^{n-k}(M;\mathcal{O}_M)/d(\Omega^{n-k-1}(M;\mathcal{O}_M)) \to \mathcal{D}_k(M)/\delta(\mathcal{D}_{k+1}(M)) \tag{19}$$

For two vector fields, X_1 and X_2, with isolated zeros we have constructed $\mathrm{cs}(X_1,X_2) \in C_1(M;\mathbb{Z})/\partial(C_2(M;\mathbb{Z}))$, cf. section 2.2, which gives rise to $\mathrm{cs}(X_1,X_2) \in C_1(M;\mathbb{R})/\partial(C_2(M;\mathbb{R}))$ and via (18) we get a one current rel. boundary which we will denote by $\hat{\mathrm{cs}}(X_1,X_2)$. More precisely, $\hat{\mathrm{cs}}(X_1,X_2)(\omega) = \int_{\mathrm{cs}(X_1,X_2)} \omega = R(\omega,X_1,X_2)$ for all closed one forms $\omega \in \mathcal{Z}^1(M;\mathbb{R})$.

For two Riemannian metrics, g_1 and g_2, we have the Chern–Simons form $\mathrm{cs}(g_1,g_2) \in \Omega^{n-1}(M;\mathcal{O}_M)/d(\Omega^{n-2}(M;\mathcal{O}_M))$, see section 2.1. Via (19) we get a one current rel. boundary which we will denote by $\hat{\mathrm{cs}}(g_1,g_2)$. More precisely, $\hat{\mathrm{cs}}(g_1,g_2)(\omega) = -\int_M \omega \wedge \mathrm{cs}(g_1,g_2) = -R(\omega,g_1,g_2)$ for all closed one forms $\omega \in \mathcal{Z}^1(M;\mathbb{R})$.

Suppose X is a vector field with isolated zeros and g is a Riemannian metric. We define a one current rel. boundary by $\hat{\mathrm{cs}}(X,g)(\omega) := -R(\omega,X,g)$ for all closed one forms $\omega \in \mathcal{Z}^1(M;\mathbb{R})$, see section 2.5. Moreover set $\hat{\mathrm{cs}}(g,X) := -\hat{\mathrm{cs}}(X,g)$.

Proposition 2 can now be reformulated as

Proposition 3. *Let any of the symbols x,y,z denote either a Riemannian metric or a vector field with isolated zeros. Then one has:*

(i) $\delta\hat{\mathrm{cs}}(x,y) = \hat{E}(y) - \hat{E}(x)$.
(ii) $\hat{\mathrm{cs}}(y,x) = -\hat{\mathrm{cs}}(x,y)$
(iii) $\hat{\mathrm{cs}}(x,z) = \hat{\mathrm{cs}}(x,y) + \hat{\mathrm{cs}}(y,z)$

2.7. **Smooth triangulations.** Smooth triangulations provide a remarkable source of vector fields with isolated zeros. To any smooth triangulation τ of a smooth manifold M one can associate a smooth vector field X_τ called *Euler vector field*, with the following properties:

- P1: The zeros of X_τ are all non-degenerate and are exactly the barycenters x_σ of the simplexes σ of τ.
- P2: The flow of $-X_\tau$ is defined for all $t \in \mathbb{R}$.[7]
- P3: For each zero x_σ the unstable set with respect to $-X_\tau$ is exactly the open simplex σ, consequently the zeros are hyperbolic. The Morse index of $-X_\tau$ at x_σ equals $\dim(\sigma)$ and the (Hopf) index of X_τ at x_σ equals $(-1)^{\dim(\sigma)}$.
- P4: The piecewise differentiable function $f_\tau : M \to \mathbb{R}$ defined by $f_\tau(x_\sigma) = \dim(\sigma)$, extended to M by linearity on the barycentric subdivision of τ, is a Lyapunov function for $-X_\tau$, i.e. strictly decreasing on non-constant trajectories of $-X_\tau$.

The space of vector fields satisfying P1–P4 is a contractible space. The convex combination provides a contraction.

To construct such a vector field we begin with a standard simplex Δ_n of vectors $(t_0, \dots, t_n) \in \mathbb{R}^{n+1}$ satisfying $0 \le t_i \le 1$ and $\sum t_i = 1$.

- (i) Let E_n denote the Euler vector field of the corresponding affine space $(\sum t_i = 1)$ centered at the barycenter O (of coordinates $(1/(n+1), \dots, 1/(n+1))$ and restricted to Δ_n.
- (ii) Let $e : \Delta_n \to [0, 1]$ denote the function which is 1 on the barycenter O and zero on all vertices.
- (iii) Let $r : \Delta_n \setminus \{O\} \to \partial\Delta_n$ denote the radial retraction to the boundary.

Set $X'_n := e \cdot E_n$, which is a vector field on Δ_n.

By induction we will construct a canonical vector field X_n on Δ_n which at any point $x \in \Delta_n$ is tangent to the open face the point belongs to and vanishes only at the barycenter of each face. We proceed as follows:

Suppose we have constructed such canonical vector fields on all $\Delta(k)$, $k \le n-1$. Using the canonical vector fields X_{n-1} we define the vector field X_n on the boundary $\partial\Delta_n$ and extend it to the vector field X''_n by taking at each point $x \in \Delta_n$ the vector parallel to $X_n(r(x))$ multiplied by the function $(1 - e)$ and at O the vector zero. Clearly such vector field vanishes on the radii $\overline{OP}$ (P the barycenter of any face). We finally put

$$X_n := X'_n + X''_n.$$

The vector field X_n is continuous and piecewise differential (actually Lipschitz) and has a well defined continuous flow.

Putting together the vector fields X_n on all simplexes (cells) we provide a piecewise differential (and Lipschitz) vector field X on any simplicial (cellular) complex or polyhedron and in particular on any smoothly triangulated manifold. The vector field X has a flow and f_τ is a Lyapunov function for $-X$. The vector field X is not necessary smooth but by a small (Lipschitz) perturbation we can approximate it by a smooth vector field which satisfies P1–P4. Any of the resulting vector fields is referred to as the Euler vector field of a smooth triangulation τ and denoted by X_τ. It was pointed out to us that the vector field X_τ has first appeared in [S99].

[7] This is always the case if M is closed.

Note that the above construction extends to smooth cell structures. Recall that a smooth cell structure on a smooth manifold M is a CW-complex structure provided by cells whose characteristic map $\chi : D^k \to M$ are smooth embeddings (one to one and of maximal rank) from the compact manifolds with corners (homeomorphic to the unit disc in the Euclidean space). One denotes by $M(r)$ the r–skeleton and for any cell one calls barycenter, the image by the characteristic map of the center of D^k. By an entirely similar procedure one can construct piecewise differential vector fields X on M which are tangent to $M(r)$ and have the barycenters as the only zeros. Moreover one can construct piecewise differential functions which are Lyapunov for X and when composed with the characteristic maps are Morse functions on the interior of any cell with only one critical point, the center of the cell. One can show that the space of all these vector fields with the Lipschitz topology is contractible.

3. Euler and co-Euler structures

3.1. **Euler structures.** Euler structures have been introduced by Turaev [T90] for manifolds with vanishing Euler Poincaré characteristic. In the presentation below we remove the vanishing hypothesis at the expense of a base point.

Let (M, x_0) be a base pointed closed connected manifold of dimension n. Let X be a vector field and let $\mathcal{X}$ denote its zero set. Suppose the zeros of X are isolated and recall the singular zero chain $E(X) = \sum_{x\in\mathcal{X}} \mathrm{IND}_X(x)x$ from section 2.2. An *Euler chain* for X is a singular one chain $c \in C_1(M;\mathbb{Z})$ so that

$$\partial c = E(X) - \chi(M)x_0.$$

Since $\sum_{x\in\mathcal{X}} \mathrm{IND}_X(x) = \chi(M)$ every vector field with isolated zeros admits Euler chains.

Consider pairs (X, c) where X is a vector field with isolated zeros, and c is an Euler chain for X. We call two such pairs (X_1, c_1) and (X_2, c_2) equivalent if

$$c_2 = c_1 + \mathrm{cs}(X_1, X_2) \in C_1(M;\mathbb{Z})/\partial(C_2(M;\mathbb{Z})).$$

For the definition of $\mathrm{cs}(X_1, X_2)$ see section 2.2. We will write $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ for the set of equivalence classes as above and $[X, c] \in \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ for the element represented by the pair (X, c). Elements of $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ are called *Euler structures of M based at x_0*. There is an obvious $H_1(M;\mathbb{Z})$ action on $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ defined by

$$[X, c] + [\sigma] := [X, c + \sigma],$$

for $[\sigma] \in H_1(M;\mathbb{Z})$ and $[X, c] \in \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$. Obviously this action is free and transitive. In this sense $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ is an affine version of $H_1(M;\mathbb{Z})$.

Considering Euler chains with real coefficients one obtains, in exactly the same way, an affine version of $H_1(M;\mathbb{R})$ which we will denote by $\mathfrak{Eul}_{x_0}(M;\mathbb{R})$. Moreover, there is a natural map $\mathfrak{Eul}_{x_0}(M;\mathbb{Z}) \to \mathfrak{Eul}_{x_0}(M;\mathbb{R})$ which is affine over the homomorphism $H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{R})$.

Remark 3. There is another way of understanding the $H_1(M;\mathbb{Z})$ action on $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$. Suppose $n > 2$ and represent $[\sigma] \in H_1(M;\mathbb{Z})$ by a simple closed curve σ. Choose a tubular neighborhood N of $\sigma(S^1)$ considered as vector bundle $N \to S^1$. Choose a fiber metric and a linear connection on N. Choose a representative of $[X, c] \in \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ such that $X|_N = \frac{\partial}{\partial\theta}$, the horizontal lift of the canonic vector field on S^1. Choose a function $\lambda : [0,\infty) \to [-1, 1]$, which satisfies $\lambda(r) = -1$ for $r \leq \frac{1}{3}$ and $\lambda(r) = 1$ for $r \geq \frac{2}{3}$. Finally choose a function $\mu : [0,\infty) \to \mathbb{R}$ satisfying $\mu(r) = r$

for $r \leq \frac{1}{3}$, $\mu(r) = 0$ for $r \geq \frac{2}{3}$ and $\mu(r) > 0$ for all $r \in (\frac{1}{3}, \frac{2}{3})$. Now construct a new smooth vector field $\tilde{X}$ on M by setting

$$\tilde{X} := \begin{cases} X & \text{on } M \setminus N \\ \lambda(r)\frac{\partial}{\partial\theta} + \mu(r)\frac{\partial}{\partial r} & \text{on } N, \end{cases}$$

where $r : N \to [0, \infty)$ denotes the radius function determined by the fiber metric on N and $-r\frac{\partial}{\partial r}$ is the Euler vector field of N. This construction is known as Reeb surgery, see e.g. [N03]. If the zeros of X are all non-degenerate the homotopy $X_t := (1-t)X + t\tilde{X}$ is a non-degenerate homotopy from $X_0 = X$ to $X_1 = \tilde{X}$ from which one easily deduces that $[\tilde{X}, c] = [X, c] + [\sigma]$. Particularly all the choices that entered the Reeb surgery do not affect the out-coming Euler structure $[\tilde{X}, c]$.

Let us consider a change of base point. Let $x_0, x_1 \in M$ and choose a path σ from x_0 to x_1. Define

$$\mathfrak{Eul}_{x_0}(M;\mathbb{Z}) \to \mathfrak{Eul}_{x_1}(M;\mathbb{Z}), \quad [X, c] \mapsto [X, c - \chi(M)\sigma]. \tag{20}$$

This is an $H_1(M;\mathbb{Z})$ equivariant bijection but depends on the homology class of σ. However, if $\chi(M) = 0$, then $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ does not depend on the base point, cf. [T90].

Let $\mathfrak{X}(M, x_0)$ denote the space of vector fields which vanish at x_0 and are non-zero elsewhere. We equip this space with the C^∞–topology. Let $\pi_0(\mathfrak{X}(M, x_0))$ denote the set of homotopy classes of such vector fields. If $X \in \mathfrak{X}(M, x_0)$ we will write $[X]$ for the corresponding class in $\pi_0(\mathfrak{X}(M, x_0))$. The following proposition is due to Turaev in the case $\chi(M) = 0$.

Proposition 4. *Suppose $n > 2$. Then there exists a natural bijection*

$$\pi_0(\mathfrak{X}(M, x_0)) = \mathfrak{Eul}_{x_0}(M;\mathbb{Z}), \qquad [X] \mapsto [X, 0]. \tag{21}$$

Proof. Clearly (21) is well defined. Let us prove that it is onto. So let $[X, c]$ represent an Euler structure. Choose an embedded disk $D \subseteq M$ centered at x_0 which contains all zeros of X and the Euler chain c. For this we may have to change c, but without changing the Euler structure $[X, c]$. Choose a vector field X' which equals X on $M \setminus D$ and vanishes just at x_0. Since $H_1(D;\mathbb{Z}) = 0$ we clearly have $[X', 0] = [X, c] \in \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ and thus (21) is onto.

Let us prove injectivity of (21). Let $X_1, X_2 \in \mathfrak{X}(M, x_0)$ and suppose $\mathrm{cs}(X_1, X_2) = 0 \in H_1(M;\mathbb{Z})$. Let $D \subseteq M$ denote an embedded open disk centered at x_0. Consider the vector bundle $p^*TM \to I \times M$ and consider the two vector fields as a nowhere vanishing section of p^*TM defined over the set $\partial I \times \dot{M}$, where $\dot{M} := M \setminus D$. We would like to extend it to a nowhere vanishing section over $I \times \dot{M}$. The first obstruction we meet is an element in

$$\begin{aligned} H^n(I \times \dot{M}, \partial I \times \dot{M}; \{\pi_{n-1}\}) &= H_1(I \times \dot{M}, I \times \partial D; \mathbb{Z}) \\ &= H_1(M, \bar{D}; \mathbb{Z}) \\ &= H_1(M; \mathbb{Z}) \end{aligned}$$

which corresponds to $\mathrm{cs}(X_1, X_2) = 0$. Here $\{\pi_{n-1}\}$ denotes the system of local coefficients determined by the sphere bundle of p^*TM with $\pi_{n-1} = \pi_{n-1}(S^{n-1})$. Since this obstruction vanishes by hypothesis the next obstruction is defined and is an element in:

$$\begin{aligned} H^{n+1}(I\times\dot{M},\partial I\times\dot{M};\{\pi_n\}) &= H_0(I\times\dot{M},I\times\partial D;\pi_n(S^{n-1})) \\ &= H_0(M,\bar{D};\pi_n(S^{n-1})) \\ &= 0 \end{aligned}$$

So obstruction theory, see e.g. [W78], tells us that we find a nowhere vanishing section of p^*TM defined over $I\times\dot{M}$, which restricts to X_i on $\{i\}\times\dot{M}$, $i=1,2$. Such a section can easily be extended to a globally defined section of $p^*TM\to I\times M$, which restricts to X_i on $\{i\}\times M$, $i=1,2$, and whose zero set is precisely $I\times\{x_0\}$.[8] Such a section can be considered as homotopy from X_1 to X_2 showing $[X_1]=[X_2]$. Hence (21) is injective. □

Remark 4. If $n>2$, Reeb surgery defines an $H_1(M;\mathbb{Z})$ action on $\pi_0(\mathfrak{X}(M,x_0))$ which via (21) corresponds to the $H_1(M;\mathbb{Z})$ action on $\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$, cf. Remark 3.

Let $\mathfrak{X}_0(M)$ denote the space of nowhere vanishing vector fields on M equipped with the C^∞–topology. Let $\pi_0(\mathfrak{X}_0(M))$ denote the set of its connected components.

Proposition 5. *Suppose $\chi(M)=0$ and $n>2$. Then we have a surjection:*

$$\pi_0(\mathfrak{X}_0(M))\to\mathfrak{Eul}_{x_0}(M;\mathbb{Z}),\qquad [X]\mapsto[X,0]. \tag{22}$$

Proof. The assignment (22) is certainly well defined. Let us prove surjectivity. Let $[X,c]$ be an Euler structure. Choose an embedded disk $D\subseteq M$ which contains all zeros of X and its Euler chain c, cf. the proof of Proposition 4. Since $\chi(M)=0$ the degree of $X:\partial D\to TD\setminus D$ vanishes. Modifying X only on D we get a nowhere vanishing X' which coincides with X on $M\setminus D$. Certainly X' has an Euler chain c' which is also contained in D and satisfies $[X,c]=[X',c']$. Since X' has no zeros we get $\partial c'=0$ and since $H_1(D;\mathbb{Z})=0$ we arrive at $[X,c]=[X',c']=[X',0]$ which proves that (22) is onto. □

3.2. **Co-Euler structures.** We will now describe another approach to Euler structures which is in some sense Poincaré dual to the one in section 3.1. We still consider a closed connected base pointed manifold (M,x_0) of dimension n. Consider pairs (g,α) where g is a Riemannian metric on M and $\alpha\in\Omega^{n-1}(M\setminus x_0;\mathcal{O}_M)$ with $d\alpha=E(g)$ where $E(g)\in\Omega^n(M;\mathcal{O}_M)$ denotes the Euler class of g, see section 2.1. We call two pairs (g_1,α_1) and (g_2,α_2) equivalent if

$$\mathrm{cs}(g_1,g_2)=\alpha_2-\alpha_1\in\Omega^{n-1}(M\setminus x_0;\mathcal{O}_M)/d\Omega^{n-2}(M\setminus x_0;\mathcal{O}_M).$$

We will write $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ for the set of equivalence classes and $[g,\alpha]$ for the equivalence class represented by the pair (g,α). Elements of $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ are called *co-Euler structures based at* x_0. There is a natural $H^{n-1}(M;\mathcal{O}_M)$ action on $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ given by

$$[g,\alpha]+[\beta]:=[g,\alpha-\beta]$$

for $[\beta]\in H^{n-1}(M;\mathcal{O}_M)$. Since $H^{n-1}(M;\mathcal{O}_M)=H^{n-1}(M\setminus x_0;\mathcal{O}_M)$ this action is obviously free and transitive. In this sense $\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ is an affine version of $H^{n-1}(M;\mathcal{O}_M)$.

[8]For example trivialize the bundle p^*TM over $I\times\bar{D}$ and extend the section on $\partial(I\times\bar{D})$ to $I\times\bar{D}$ by scaling radially centered at $(\frac{1}{2},x_0)\in I\times\bar{D}$.

Suppose $[X, c] \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$ and $[g, \alpha] \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$. In view of Lemma 1 and Proposition 2, the quantity

$$R(\omega, X, g) - S(\omega, \alpha) - \int_c \omega \tag{23}$$

does only depend on the Euler structure $[X, c] \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$, the co-Euler structure $[g, \alpha] \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$, and the cohomology class $[\omega] \in H^1(M;\mathbb{R})$. Thus (23) defines a coupling

$$\mathbb{T} : \mathfrak{Eul}_{x_0}(M;\mathbb{R}) \times \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \to H_1(M;\mathbb{R}).$$

From the very definition we have

$$\big\langle[\omega], \mathbb{T}([X, c], [g, \alpha])\big\rangle = \int_M \omega \wedge (X^*\Psi(g) - \alpha) - \int_c \omega, \tag{24}$$

where ω is any representative of $[\omega]$ which vanishes locally around the zeros of X and vanishes locally around the base point x_0. Moreover, we have

$$\mathbb{T}(\mathfrak{e} + \sigma, \mathfrak{e}^* + \beta) = \mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) - \sigma + \mathrm{PD}(\beta) \tag{25}$$

for all $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$, $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$, $\sigma \in H_1(M;\mathbb{R})$ and $\beta \in H^{n-1}(M;\mathcal{O}_M)$. Here PD is the Poincaré duality isomorphism $\mathrm{PD} : H^{n-1}(M;\mathcal{O}_M) \to H_1(M;\mathbb{R})$.

We have the following affine version of Poincaré duality:

Proposition 6. *There is a natural isomorphism*

$$P : \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \to \mathfrak{Eul}_{x_0}(M;\mathbb{R})$$

affine over the Poincaré duality isomorphism $\mathrm{PD} : H^{n-1}(M;\mathcal{O}_M) \to H_1(M;\mathbb{R})$. *More precisely, for* $\beta \in H^{n-1}(M;\mathcal{O}_M)$ *and* $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ *we have* $P(\mathfrak{e}^* + \beta) = P(\mathfrak{e}^*) + \mathrm{PD}(\beta)$. *Moreover,* $\mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) = P(\mathfrak{e}^*) - \mathfrak{e}$.

Proof. In view of (25) we may define P by $\mathbb{T}(P(\mathfrak{e}^*), \mathfrak{e}^*) = 0$. The equivariance property and the last equation follow at once. □

Let us collect the observations from the last two sections in the following

Theorem 1. *Let* (M, x_0) *be a closed connected base pointed manifold.*

(i) *Let* $\pi_0(\mathfrak{X}(M, x_0))$ *denote the set of connected components of the space of vector fields which just vanish in* x_0, *equipped with the* C^∞*–topology. If* $\dim M > 2$ *then we have a bijection:*

$$\pi_0(\mathfrak{X}(M, x_0)) \to \mathfrak{Eul}_{x_0}(M;\mathbb{Z}), \qquad [X] \mapsto [X, 0].$$

(ii) *Let* $\pi_0(\mathfrak{X}_0(M))$ *denote the set of connected components of the space of nowhere vanishing vector fields, equipped with the* C^∞*–topology. If* $\chi(M) = 0$ *and* $\dim M > 2$ *we have a surjection:*

$$\pi_0(\mathfrak{X}_0(M)) \to \mathfrak{Eul}_{x_0}(M;\mathbb{Z}), \qquad [X] \mapsto [X, 0].$$

(iii) *There exists an isomorphism*

$$P : \mathfrak{Eul}^*_{x_0}(M;\mathbb{R}) \to \mathfrak{Eul}_{x_0}(M;\mathbb{R})$$

affine over the Poincaré duality isomorphism $\mathrm{PD} : H^{n-1}(M;\mathcal{O}_M) \to H_1(M;\mathbb{R})$, *i.e.* $P(\mathfrak{e}^* + \beta) = P(\mathfrak{e}^*) + \mathrm{PD}(\beta)$, *for all* $\beta \in H^{n-1}(M;\mathcal{O}_M)$.

(iv) *The assignment* $\mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) := P(\mathfrak{e}^*) - \mathfrak{e}$
$$\mathbb{T} : \mathfrak{Eul}_{x_0}(M; \mathbb{R}) \times \mathfrak{Eul}^*_{x_0}(M) \to H_1(M; \mathbb{R})$$
is a corrected version of the invariant R. *More precisely, suppose* $\mathfrak{e} = [X, c]$, $\mathfrak{e}^* = [g, \alpha]$. *Then for every* $[\omega] \in H^1(M; \mathbb{R})$ *we have*
$$\langle [\omega], \mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) \rangle = \int_M \omega \wedge (X^*\Psi(g) - \alpha) - \int_c \omega$$
where $\omega \in \Omega^1(M)$ *is any representative of* $[\omega]$ *which vanishes locally around* x_0 *and vanishes locally around the zeros of* X.

3.3. **Milnor–Turaev torsion.** Let M be a closed connected manifold of dimension n. Fix a base point $x_0 \in M$ and let $\Gamma = \pi_1(M, x_0)$ denote the fundamental group. Let $\mathbb{K}$ be a field and let V be a finite dimensional vector vector space over $\mathbb{K}$. Let $\rho : \Gamma \to \mathrm{GL}(V)$ be a representation. Consider the graded $\mathbb{K}$–vector space $H^*(M; \rho)$, its determinant line $\det H^*(M; \rho)$, and the one dimensional $\mathbb{K}$–vector space[9]
$$\mathrm{Det}_{x_0}(M; \rho) := \det H^*(M; \rho) \otimes (\det V)^{-\chi(M)}. \tag{26}$$
Remark 5. If $\chi(M) = 0$ then $\mathrm{Det}_{x_0}(M; \rho) = \det H^*(M; \rho)$. If $H^*(M; \rho) = 0$ then $\mathrm{Det}_{x_0}(M; \rho)$ is canonically isomorphic to $\mathbb{K}$.[10]

Suppose we have a smooth triangulation τ of M. It gives rise to a cochain complex of $\mathbb{K}$–vector spaces $C^*_\tau(M; \rho)$ which computes the cohomology $H^*(M; \rho)$. By standard linear algebra we get a canonical isomorphism of $\mathbb{K}$–vector spaces:
$$\varphi^\rho_\tau : \det C^*_\tau(M; \rho) \to \det H(C^*_\tau(M; \rho)) = \det H^*(M; \rho) \tag{27}$$
This isomorphism is described in details in [FT00, section 2, formula (2.2)] where a sign correction $(-1)^{N(C)}$ of previously used definitions is added, cf. the appendix of this paper.

Recall the Euler vector field X_τ of τ from section 2.7. Its zero set $\mathcal{X}_\tau$ coincides with the set of barycenters of τ. For a cell σ of τ we let $x_\sigma \in \mathcal{X}_\tau$ denote its barycenter. We have $\mathrm{IND}_{X_\tau}(x_\sigma) = (-1)^{\dim \sigma}$ for every cell σ of τ. Suppose we have given an Euler structure $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M; \mathbb{Z})$. Choose a collection of paths π_x, from x_0 to $x \in \mathcal{X}_\tau$, so that that with $c := \sum_{x \in \mathcal{X}_\tau} \mathrm{IND}_{X_\tau}(x)\pi_x$ we have $\mathfrak{e} = [X_\tau, c]$.[11]

Let v be a non-zero element in $\det V$.[12] Using parallel transport along π_x we get a non-zero element in $\det C^*_\tau(M; \rho)$ with a sign ambiguity. If the set of barycenters $\mathcal{X}_\tau$ was ordered we would get a well defined non-zero element in $\det C^*_\tau(M; \rho)$.

Suppose $\mathfrak{o}$ is a cohomology orientation of M, i.e. an orientation of $\det H^*(M; \mathbb{R})$. We say an ordering of the zeros is compatible with $\mathfrak{o}$ if the non-zero element in $\det C^*_\tau(M; \mathbb{R})$ provided by this ordered base is compatible with the orientation $\mathfrak{o}$ via the canonic isomorphism
$$\varphi^{\mathbb{R}}_\tau : \det C^*_\tau(M; \mathbb{R}) \to \det H(C^*_\tau(M; \mathbb{R})) = \det H^*(M; \mathbb{R}).$$

So given a collection of paths π_x with $\mathfrak{e} = [X_\tau, c]$, an ordering of $\mathcal{X}_\tau$ compatible with $\mathfrak{o}$, and a non-zero element $v \in \det V$ one obtains (using Milnor's construction

[9] For the definition of (graded) determinant lines see the footnote in the introduction.

[10] Note that the convention that the determinant line of a zero dimensional vector space is canonically isomorphic to the field $\mathbb{K}$ is consistent with the calculus with determinant lines.

[11] Such a representative for the Euler structure is called spray or Turaev spider.

[12] Note that a frame (basis) in V determines such an element. Note that if instead of v we pick up a frame ϵ of V, we will obtain a base in $C^*_\tau(M; \rho)$.

modified by Turaev) a non-zero element in $\det C^*_\tau(M;\rho)$ which corresponds to a non-zero element in $\det H^*(M;\rho)$ via (27). We thus get a mapping

$$\det V \setminus 0 \to \det H^*(M;\rho) \setminus 0. \tag{28}$$

This mapping is obviously homogeneous of degree $\chi(M)$. A straight forward calculation shows that it does not depend on the choice of π_x or the ordering on $\mathcal{X}_\tau$. As a matter of fact this mapping does not depend on τ either, only on the Euler structure $\mathfrak{e}$ and the cohomology orientation $\mathfrak{o}$.[13] This is a non-trivial fact, and its proof is contained in [M66] and [T86] for the acyclic case, and implicit in the existing literature cf. [FT00], but see also Corollary 2 in the appendix.

Definition 1 (Milnor–Turaev torsion)**.** The element $\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\text{comb}}$ in $\operatorname{Det}_{x_0}(M;\rho)$, see (26), corresponding to the homogeneous mapping (28) is referred to as the *Milnor–Turaev torsion.*

We obviously have, but see Remark 1 in section 2.3 for the sign,

$$\tau^{\rho,\mathfrak{e}+\sigma,\mathfrak{o}}_{\text{comb}} = \tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\text{comb}} \cdot [\det\circ\rho](\sigma)^{-1}$$

for all $\sigma \in H_1(M;\mathbb{Z})$. Here $[\det\circ\rho] : H_1(M;\mathbb{Z}) \to \mathbb{K}^*$ denotes the homomorphism induced from the homomorphism $\det\circ\rho : \Gamma \to \mathbb{K}^*$. In particular, if ρ is an unimodular representation then $\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\text{comb}}$ is independent of $\mathfrak{e}$. Moreover

$$\tau^{\rho,\mathfrak{e},-\mathfrak{o}}_{\text{comb}} = (-1)^{\dim V}\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\text{comb}}. \tag{29}$$

Remark 6. Note that when $H^*(M;\rho) = 0$ then, in view of the canonical isomorphism $\operatorname{Det}_{x_0}(M;\rho) = \mathbb{K}$, see Remark 5, the Milnor–Turaev torsion is an element in $\mathbb{K}^*$. We will denote the inverse of this number by $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\text{comb}}(\rho) := (\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\text{comb}})^{-1}$, and later consider it as a function on the space of acyclic representations. The sign is added in order to stay with traditional notation.

3.4. **Modified Ray–Singer metric.** Let us briefly recall the definition of the Ray–Singer torsion. Suppose $\mathbb{F} = (F,\nabla)$ is a real or complex vector bundle F equipped with a flat connection ∇. Let g be a Riemannian metric on M and let μ be a Hermitian metric on F. The Ray–Singer torsion $T_{\text{an}}(\nabla,g,\mu)$ is the positive real number given by:

$$\log T_{\text{an}}(\nabla,g,\mu) := \frac{1}{2}\sum_k (-1)^{k+1} k \log\det{}'\Delta^k$$

Here Δ^k denotes the Laplacian in degree k of the elliptic complex $(\Omega^*(M;F), d_\nabla)$ when equipped with the scalar product induced from the Riemannian metric g and the Hermitian metric μ, and $\det{}'\Delta^k$ denotes its zeta regularized determinant, see [RS71].

Recall that $H^*(M;\mathbb{F})$ canonically identifies with the harmonic forms and thus inherits a Hermitian metric from g and μ. Let us write $||\cdot||^{\mathbb{F},g,\mu}_{\text{Hodge}}$ for the induced Hermitian metric on $\det H^*(M;\mathbb{F})$. The Ray–Singer metric, see [BZ92], is the Hermitian metric

$$||\cdot||^{\mathbb{F},g,\mu}_{\text{RS}} := T_{\text{an}}(\nabla,g,\mu)\cdot||\cdot||^{\mathbb{F},g,\mu}_{\text{Hodge}}$$

on $\det H^*(M;\mathbb{F})$.

Suppose $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Let V be a finite dimensional vector space over $\mathbb{K}$. Every representation $\rho : \Gamma \to \operatorname{GL}(V)$ gives rise to an associated flat vector bundle $\mathbb{F}_\rho =$

[13] In fact it depends on $\mathfrak{o}$ only when $\dim V$ is odd, cf. (29) below.

(F_ρ, ∇_ρ) over M. Moreover the fiber over x_0 is canonically identified with V. Note that holonomy at x_0 provides a right Γ–action on the fiber over x_0 which, when composed with the inversion in $\mathrm{GL}(V)$, reproduces ρ.

Choose a Riemannian metric on g on M, and a Hermitian metric on F_ρ. This induces a Hermitian metric $||\cdot||^{\mu}_{x_0}$ on $(\det V)^{-\chi(M)}$. Moreover, we have the Ray–Singer metric $||\cdot||^{\mathbb{F}_\rho,g,\mu}_{\mathrm{RS}}$ on $\det H^*(M;\mathbb{F}_\rho) = \det H^*(M;\rho)$. Suppose $\mathfrak{e}^* \in \mathfrak{Eul}_{x_0}(M;\mathbb{R})$ is a co-Euler structure. Choose α so that $\mathfrak{e}^* = [g,\alpha]$, see section 3.2. Recall the Kamber–Tondeur form $\omega(\nabla_\rho,\mu) \in \Omega^1(M;\mathbb{R})$ from section 2.3. Consider the Hermitian metric

$$||\cdot||^{\rho,\mathfrak{e}^*}_{\mathrm{an}} := ||\cdot||^{\mathbb{F}_\rho,g,\mu}_{\mathrm{RS}} \otimes ||\cdot||^{\mu}_{x_0} \cdot e^{-S(\omega(\nabla_\rho,\mu),\alpha)} \tag{30}$$

on $\mathrm{Det}_{x_0}(M;\rho)$, see (26) and Lemma 1. The known anomaly formulas for the Ray–Singer metric, see [BZ92, Theorem 0.1], imply that this is indeed independent of μ and only depends on the co-Euler structure $\mathfrak{e}^*$. For $\beta \in H^{n-1}(M;\mathcal{O}_M)$ we obviously have

$$||\cdot||^{\rho,\mathfrak{e}^*+\beta}_{\mathrm{an}} = ||\cdot||^{\rho,\mathfrak{e}^*}_{\mathrm{an}} \cdot e^{\langle \log|[\det\circ\rho]|,\mathrm{PD}(\beta)\rangle}$$

where $\log|[\det\circ\rho]| \in H^1(M;\mathbb{R})$ is the Kamber–Tondeur class corresponding to the homomorphism $\log|[\det\circ\rho]| : H_1(M;\mathbb{R}) \to \mathbb{R}$, see Remark 1 in section 2.3.

Definition 2 (Modified Ray–Singer metric)**.** The Hermitian metric $||\cdot||^{\rho,\mathfrak{e}^*}_{\mathrm{an}}$ on $\mathrm{Det}_{x_0}(M;\rho)$, see (26) and (30), will be referred to as the *modified Ray–Singer metric.*

The Bismut–Zhang theorem [BZ92, Theorem 0.2] can now be reformulated as a statement comparing two topological invariants, one defined with the help of analysis, and one defined with the help of a triangulation.

Theorem 2 (Bismut–Zhang)**.** *Let (M, x_0) be a closed connected manifold with base point, and let $\rho : \pi_1(M, x_0) \to \mathrm{GL}(V)$ be a finite dimensional real or complex representation. Suppose $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ is an Euler structure, and $\mathfrak{e}^* \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$ a co-Euler structure. Then one has:*

$$||\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}||^{\rho,\mathfrak{e}^*}_{\mathrm{an}} = e^{\langle \log|[\det\circ\rho]|,\mathbb{T}(\mathfrak{e},\mathfrak{e}^*)\rangle}$$

Particularly, if $\mathbb{T}(\mathfrak{e},\mathfrak{e}^*) = 0$ then $||\tau^{\rho,\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}||^{\rho,\mathfrak{e}^*}_{\mathrm{an}} = 1$.

For an alternative proof of the (original) Bismut–Zhang theorem see also [BFK01].

Remark 7. If $H^*(M;\rho) = 0$ then $\mathrm{Det}_{x_0}(M;\rho) = \mathbb{K}$, see Remark 5. In this case the modified Ray–Singer metric gives rise to a positive real number $\mathcal{T}^{\mathfrak{e}^*}_{\mathrm{an}}(\rho) := ||1||^{\rho,\mathfrak{e}^*}_{\mathrm{an}} > 0$. From the very definition we have

$$\mathcal{T}^{\mathfrak{e}^*}_{\mathrm{an}}(\rho) = T_{\mathrm{an}}(\nabla_\rho, g, \mu) \cdot e^{-S(\omega(\nabla_\rho,\mu),\alpha)}$$

where ∇_ρ denotes the flat connection on the associated bundle F_ρ, g is a Riemannian metric on M, μ is a Hermitian structure on F_ρ, and α is such that $[g,\alpha] = \mathfrak{e}^*$. Recall the number $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho) \in \mathbb{K}^*$ from Remark 6. Provided $\mathbb{T}(\mathfrak{e},\mathfrak{e}^*) = 0$, the Bismut–Zhang theorem implies $|\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho)| = \mathcal{T}^{\mathfrak{e}^*}_{\mathrm{an}}(\rho)$.

Remark 8. Note that given a vector bundle F over M, equipped with a flat connection ∇, a Hermitian metric μ, a Riemannian metric g on M, and a co-Euler structure $\mathfrak{e}^* = [g,\alpha] \in \mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$, we can consider the quantity:

$$T_{\mathrm{an}}(\nabla, g, \mu, \mathfrak{e}^*) := T_{\mathrm{an}}(\nabla, g, \mu) \cdot e^{-S(\omega(\nabla,\mu),\alpha)}$$

Then $T_{\mathrm{an}}(\nabla, g, \mu, \mathfrak{e}^*)$ is independent of μ. If $H^*(M; \rho_\nabla) = 0$, then $T_{\mathrm{an}}(\nabla, g, \mu, \mathfrak{e}^*)$ coincides with $\mathcal{T}_{\mathrm{an}}^{\mathfrak{e}^*}(\rho_\nabla)$, is gauge invariant in ∇, and independent of g.

4. Torsion and representations

Warning: Following standard conventions in algebraic geometry cf. [H77] we reserve the name *affine algebraic variety* for an irreducible variety in the affine space, and the term *algebraic set* for any finite union of varieties. By a rational function on an algebraic set we simply mean a rational function on each irreducible component.

4.1. **Complex representations.** Let Γ be a finitely presented group with generators $g_1, \dots, g_r$ and relations $R_i(g_1, g_2, \dots, g_r) = e$, $i = 1, \dots, p$, and V be a complex vector space of dimension N. Let $\mathrm{Rep}(\Gamma; V)$ be the set of linear representations of Γ on V, i.e. group homomorphisms $\rho : \Gamma \to \mathrm{GL}_{\mathbb{C}}(V)$. By identifying V to $\mathbb{C}^N$ this set is in a natural way an algebraic set inside the space $\mathbb{C}^{rN^2+1}$ given by $pN^2 + 1$ equations. Precisely if $A_1, \dots, A_r, z$ represent the coordinates in $\mathbb{C}^{rN^2+1}$ with $A := (a^{ij})$, $a^{ij} \in \mathbb{C}$, so $A \in \mathbb{C}^{N^2}$ and $z \in \mathbb{C}$, then the equations defining $\mathrm{Rep}(\Gamma; V)$ are

$$\begin{aligned} z \cdot \det(A_1) \cdot \det(A_2) \cdots \det(A_r) &= 1 \\ R_i(A_1, \dots, A_r) &= \mathrm{id}, \qquad i = 1, \dots, p \end{aligned}$$

with each of the equalities R_i representing N^2 polynomial equations.

Suppose $\Gamma = \pi_1(M, x_0)$, M a closed manifold. Denote by $\mathrm{Rep}_0^M(\Gamma; V)$ the set of representations $\rho \in$ with $H^*(M; \rho) = 0$ and notice that they form a Zariski open set in $\mathrm{Rep}(\Gamma, V)$ and denote the closure of this set by $\mathrm{Rep}^M(\Gamma; V)$. This is again an algebraic set which depends only on the homotopy type of M, can be empty (for example if $\chi(M) \neq 0$) but in some interesting cases can be the full space $\mathrm{Rep}(\Gamma; V)$, cf. section 4.3.

Definition 3 (V–acyclicity)**.** Given a complex vector space V we say that the closed manifold M is *V–acyclic* if there exist representations $\rho \in \mathrm{Rep}(\Gamma; V)$ with $H^*(M; \rho) = 0$. Equivalently $\mathrm{Rep}^M(\Gamma; V) \neq \emptyset$.

Recall that every representation $\rho \in \mathrm{Rep}(\Gamma; V)$ induces a canonical vector bundle F_ρ equipped with a canonical flat connection ∇_ρ. They are obtained from the trivial bundle $\tilde{M} \times V \to \tilde{M}$ and the trivial connection by passing to the Γ quotient spaces. Here $\tilde{M}$ is the canonical universal covering provided by the base point x_0. The Γ–action is the diagonal action of deck transformations on $\tilde{M}$ and the action ρ on V. The fiber of F_ρ over x_0 identifies canonically with V. The holonomy representation determines a right Γ–action on the fiber of F_ρ over x_0, i.e. an anti homomorphism $\Gamma \to \mathrm{GL}(V)$. When composed with the inversion in $\mathrm{GL}(V)$ we get back the representation ρ. The pair (F_ρ, ∇_ρ) will be denoted by $\mathbb{F}_\rho$.

If ρ_0 is a representation in the connected component $\mathrm{Rep}_\alpha(\Gamma; V)$ one can identify $\mathrm{Rep}_\alpha(\Gamma; V)$ to the connected component of ∇_{ρ_0} in the complex analytic space of flat connections of the bundle F_{ρ_0} modulo the group of bundle isomorphisms of F_{ρ_0} which restrict to the identity on the fiber above x_0.

4.2. **The space of cochain complexes and Turaev functions.** Let $(k_0, k_1, \dots, k_n)$ be a string of non-negative integers which satisfy the following requirements:

$$k_0 - k_1 + k_2 \mp \cdots + (-1)^n k_n = 0 \tag{31}$$
$$k_i - k_{i-1} + k_{i-2} \mp \cdots + (-1)^i k_0 \geq 0 \quad \text{for any } i \leq n-1. \tag{32}$$

Denote by $\mathbb{D}(k_0, \dots, k_n)$ the collection of cochain complexes of the form

$$C = (C^*, d^*) : 0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \xrightarrow{d^{n-2}} C^{n-1} \xrightarrow{d^{n-1}} C^n \to 0$$

with $C^i := \mathbb{C}^{k_i}$, and by $\mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n) \subseteq \mathbb{D}(k_0, \dots, k_n)$ the subset of acyclic complexes. The cochain complex C is determined by the collection $\{d^i\}$ of linear maps $d^i : \mathbb{C}^{k_i} \to \mathbb{C}^{k_{i+1}}$. If regarded as the subset of $\{d^i\} \in \mathbb{L}(k_0, \dots, k_n) = \bigoplus_{i=0}^{n-1} L(\mathbb{C}^{k_i}, \mathbb{C}^{k_{i+1}})$[14] which satisfy the quadratic equations

$$d^{i+1} \cdot d^i = 0,$$

the set $\mathbb{D}(k_0, \dots, k_n)$ is an affine algebraic set given by degree two homogeneous polynomials and $\mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$ is a Zariski open set.

The map

$$\pi_0 : \mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n) \to \mathrm{Emb}(C^0, C^1)$$

which associates to $C \in \mathbb{D}(k_0, \dots, k_n)$ the linear map d^0, is a bundle whose fiber is isomorphic to $\mathbb{D}_{\mathrm{ac}}(k_1 - k_0, k_2, \dots, k_n)$. Note also that $\mathrm{Emb}(C^0, C^1)$ is the Stiefel manifold of k_0–frames in C^1, and is the total space of a $\mathrm{GL}(k_0)$ principal bundle over the Grassmann manifold $\mathrm{Gr}_{k_0}(k_1)$.

As a consequence we have

Proposition 7.

(i) $\mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$ *is a connected smooth quasi algebraic set of dimension*

$$k_0 \cdot k_1 + (k_1 - k_0) \cdot k_2 + \cdots + (k_{n-1} - k_{n-2} + \cdots \pm k_0) \cdot k_n.$$

(ii) *The closure* $\hat{\mathbb{D}}_{\mathrm{ac}}(k_0, \dots, k_n)$ *is an irreducible algebraic set, hence an affine algebraic variety*

For any cochain complex in $C \in \mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$ denote by

$$B^i := \mathrm{img}(d^{i-1}) \subseteq C^i = \mathbb{C}^{k_i}$$

and consider the short exact sequence

$$0 \to B^i \xrightarrow{\mathrm{inc}} C^i \xrightarrow{d^i} B^{i+1} \to 0.$$

Choose a bases b_i for each B_i, and choose lifts $\overline{b}_{i+1}$ of b_{i+1} in C^i using d^i, i.e. $d^i(\overline{b}_{i+1}) = b_{i+1}$. Clearly $\{b_i, \overline{b}_{i+1}\}$ is a base of C^i.

Consider the base $\{b_i, \overline{b}_{i+1}\}$ as a collection of vectors in $C^i = \mathbb{C}^{k_i}$ and write them as a columns of a matrix $[b_i, \overline{b}_{i+1}]$. Define the torsion of the acyclic complex C, by

$$\tau(C) := (-1)^N \prod_{i=0}^{n} \det[b_i, \overline{b}_{i+1}]^{(-1)^i}$$

where $(-1)^N$ is Turaev's sign, see [FT00]. The result is independent of the choice of the bases b_i and of the lifts $\overline{b}_i$ cf. [M66] [FT00], and leads to the function

$$\tau : \mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n) \to \mathbb{C} \setminus 0.$$

[14]We denote by $L(V, W)$ the space of linear maps from V to W.

Turaev provided a simple formula for this function, cf. [T01], which permits to recognize τ as the restriction of a rational function on $\hat{\mathbb{D}}_{\mathrm{ac}}(k_0, k_1, \dots, k_n)$.

Following Turaev [T01], one considers symbols $\alpha \equiv (\alpha_0, \alpha_1, \dots, \alpha_n)$ with α_i subsets the set of integers $\{1, 2, \dots, k_i\}$ so that:

$$\begin{aligned} \alpha_n &= \emptyset \\ \sharp(\alpha_i) &= k_{i+1} - \sharp(\alpha_{i+1}). \end{aligned}$$

Such α is called a τ–chain. There are, of course, only finitely many such τ–chains for a given collection $(k_0, \dots, k_n)$ as above.

For any cochain complex $C \in \mathbb{D}(k_0, \dots, k_n)$, and in fact for any element $d \equiv \{d^i\} \in \mathbb{L}$, a τ–chain defines the collection of of square matrices $A_i(d^i)$ representing the $\sharp(\alpha_i) \times (k_{i+1} - \sharp(\alpha_{i+1}))$–minor of the matrix d^i consisting of the columns whose indices are in α_i and the rows not in α_{i+1}.

For example if $k_i = 2$, $k_{i+1} = 4$, $\alpha_i = \{2\}$, $\alpha_{i+1} = \{1, 3, 4\}$, and d^i is the matrix whose entries are

$$\begin{pmatrix} d^i_{1,1} & d^i_{1,2} \\ d^i_{2,1} & d^i_{2,2} \\ d^i_{3,1} & d^i_{3,2} \\ d^i_{4,1} & d^i_{4,2} \end{pmatrix}$$

with $A_i(d^i)$ the 1×1 matrix $(d^i_{2,2})$.

One considers the expression

$$F_\alpha(d) = \epsilon(\alpha) \prod_{i=0}^{n} (\det A_i(d^i))^{(-1)^{i+1}}$$

where $\epsilon(\alpha) = \pm 1$ and is chosen according to the strategy explained in [T01, Remark 2.4, page 9]. Here $d \equiv \{d^i\} \in \mathbb{L}$. Clearly F_α is a rational function on $\mathbb{L}$.

One says that the cochain complex is α–non-degenerate if all quantities $\det A_i(d^i)$ are non-zero. Turaev has shown that:

(i) For an α–non-degenerate cochain complex $C \in \mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$, $\tau(C) = F_\alpha(d)$ where $d = \{d_i\}$ is collection of the differentials in the complex C, cf. [T01, Theorem 2.2].

(ii) If U_α denotes the subsets of $\mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$ consisting α–non-degenerate complexes then U_α is open, and $\bigcup_\alpha U_\alpha = \mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$.

(iii) For any τ–chain α there exists a cochain complex $C \in \mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$ which is α–non-degenerate.

In view of the connectivity of $\mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$ and (i)–(iii) above, F_α, are rational functions on $\mathbb{L}$ which agree on the connected set $\mathbb{D}_{\mathrm{ac}}(k_0, \dots, k_n)$. Therefore they define a rational function τ on the irreducible variety $\hat{\mathbb{D}}_{\mathrm{ac}}(k_0, \dots, k_n)$ whose zero and poles are in Σ, the algebraic subset of non-acyclic cochain complexes.

Consider now a smooth triangulation τ of M whose set of simplexes of dimension q is denoted by $\mathcal{X}_q$ and the collection of integers $k_i = \sharp(\mathcal{X}_i) \cdot \dim V$ where V is a fixed complex vector space. Let $\mathfrak{e} \in \mathfrak{Eul}_{x_0}(M; \mathbb{Z})$ be an Euler structure, $\mathfrak{o}$ a cohomology orientation, and suppose that M is V–acyclic. Then the integers $(k_0, \dots, k_n)$ satisfy (31) and (32). As in section 3.3, choose a collection of paths $\pi_{\mathfrak{e}} \equiv \{\pi_x\}$ from x_0 to the barycenters x, so that $\mathfrak{e} = [X_\tau, c]$ where c is the Euler chain defined by $\{\pi_x\}$. Choose also an ordering o of the barycenters (zeros of X_τ) compatible with $\mathfrak{o}$, and a framing ϵ of V.

Consider the chain complex $(C^*_\tau(M;\rho), d_\tau(\rho))$ associated with the triangulation τ which computes the cohomology $H^*(M;\rho)$. Using π_x, o and ϵ one can identify $C^q_\tau(M;\rho)$ with $\mathbb{C}^{k_q}$. We obtain in this way a map

$$t_{\pi_\mathfrak{e},o,\epsilon}:\mathrm{Rep}(\Gamma;V)\to\mathbb{D}(k_0,\dots,k_n)$$

which sends $\mathrm{Rep}^M(\Gamma;V)\setminus\Sigma(M)$ to $\mathbb{D}_{\mathrm{ac}}(k_0,\dots,k_n)$. A look at the explicit definition of $d_\tau(\rho)$ implies that $t_{\pi_\mathfrak{e},o,\epsilon}$ is actually a regular map between two algebraic sets. Change of ϵ, $\pi_\mathfrak{e}$ and o changes the map $t_{\pi_\mathfrak{e},o,\epsilon}$. Since we are in the case $H^*(M;\rho)=0$ we easily recognize that the composition $\tau\cdot t_{\pi_\mathfrak{e},o,\epsilon}$, when restricted to $\mathrm{Rep}^M(\Gamma;V)\setminus\Sigma(M)$, is exactly the Milnor–Turaev torsion $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$, see Remark 6 in section 3.3, hence independent of all these choices $\pi_\mathfrak{e},o,\epsilon$.

We summarize these observations in the following

Theorem 3. *Let M be a closed manifold, x_0 a base point, $\Gamma:=\pi_1(M,x_0)$ the fundamental group, $\mathfrak{e}\in\mathfrak{Eul}_{x_0}(M;\mathbb{Z})$ an Euler structure, $\mathfrak{o}$ a cohomology orientation, and let V be a finite dimensional complex vector space. Consider the Milnor–Turaev torsion $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}:\mathrm{Rep}^M(\Gamma;V)\setminus\Sigma(M)\to\mathbb{C}\setminus 0$.*

- (i) *$\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ is a rational function on the algebraic set $\mathrm{Rep}^M(\Gamma;V)$ whose poles and zeros are contained in $\Sigma(M)$.*
- (ii) *The absolute value of $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}$ is the modified Ray–Singer torsion $\mathcal{T}^{\mathfrak{e}^*}_{\mathrm{an}}$, where $\mathfrak{e}^*=P^{-1}(\mathfrak{e})\in\mathfrak{Eul}^*_{x_0}(M;\mathbb{R})$, see Remark 7. More precisely, for $\rho\in\mathrm{Rep}^M(\Gamma;V)\setminus\Sigma(M)$ one has*

$$|\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho)|=\mathcal{T}^{\mathfrak{e}^*}_{\mathrm{an}}(\rho)=T_{\mathrm{an}}(\nabla_\rho,g,\mu,\mathfrak{e}^*)=T_{\mathrm{an}}(\nabla_\rho,g,\mu)\cdot e^{-S(\omega(\nabla_\rho,\mu),\alpha)}$$

  where (F_ρ,∇_ρ) denotes the flat complex vector bundle associated to the representation ρ, g is a Riemannian metric on M, μ is a Hermitian structure on F_ρ, $\omega(\nabla_\rho,\mu)$ the Kamber–Tondeur form, and α is such that the pair (g,α) represents the co-Euler structure $P^{-1}(\mathfrak{e})$.
- (iii) *For $\sigma\in H_1(M;\mathbb{Z})$, and $\rho\in\mathrm{Rep}^M(\Gamma;V)\setminus\Sigma(M)$ we have*

$$\mathcal{T}^{\mathfrak{e}+\sigma,\mathfrak{o}}_{\mathrm{comb}}(\rho)=\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho)\cdot[\det\circ\rho](\sigma)\quad\textit{and}\quad\mathcal{T}^{\mathfrak{e},-\mathfrak{o}}_{\mathrm{comb}}(\rho)=(-1)^{\dim V}\cdot\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(\rho)$$

  where $[\det\circ\rho]:H_1(M;\mathbb{Z})\to\mathbb{C}\setminus 0$ is the homomorphism induced from $\det\circ\rho:\Gamma\to\mathbb{C}\setminus 0$.

4.3. **Milnor–Turaev torsion for mapping tori.** Let N be a closed connected manifold and $\varphi:N\to N$ a diffeomorphism. Define the mapping torus $M=N_\varphi$ by glueing the boundaries of $N\times I$ with the help of φ, more precisely identifying $(x,1)$ with $(\varphi(x),0)$. The manifold M comes with a projection $p:M\to S^1$, and an inclusion $N\to M$, $x\mapsto(x,0)$. Set $\Gamma:=\pi_1(S^1,0)$. Let $\mathbb{K}$ be a field of characteristics zero, and suppose V is a finite dimensional vector space over $\mathbb{K}$. For $\rho\in\mathrm{Rep}(\Gamma;V)=\mathrm{GL}(V)$ define

$$P^k_\varphi(\rho):=\det\Big(\varphi^*\circ\rho_*-\mathrm{id}:H^k(N;V)\to H^k(N;V)\Big)$$

and the Lefschetz zeta function:

$$\zeta_\varphi(\rho):=\frac{\prod_{k\ \mathrm{even}}P^k_\varphi(\rho)}{\prod_{k\ \mathrm{odd}}P^k_\varphi(\rho)}$$

This is a rational function $\zeta_\varphi:\mathrm{GL}(V)\to\mathbb{K}$.

Choose a base point $x_0\in M$ with $p(x_0)=0\in S^1$. Every representation $\rho\in\mathrm{Rep}(\Gamma;V)$ gives rise to a representation $p^*\rho$ of $\pi_1(M,x_0)$ on V. It is not hard to see that $H^*(M;p^*\rho)=0$ iff $P^k_\varphi(\rho)\neq 0$ for all k. Indeed, we have a long exact sequence, the Wang sequence:

$$\cdots\to H^*(M;p^*\rho)\to H^*(N;V)\xrightarrow{\varphi^*\circ\rho_*-\mathrm{id}} H^*(N;V)\to H^{*+1}(M;p^*\rho)\to\cdots \quad (33)$$

Recall that this sequence is derived from the long exact sequence of the pair (M,N), and the isomorphism $H^*(M,N;p^*\rho)=H^*(N\times I,N\times\partial I;V)=H^{*-1}(N;V)$. The latter is the Thom isomorphism with the standard orientation on I, and the first comes from a trivialization of the coefficient system on $N\times I$ from $N\times\{0\}$ upwards.

Let $\omega:=p^*dt\in\Omega^1(M)$. The space of vector fields X on M which satisfy $\omega(X)<0$ is contractible and hence defines an Euler structure $\mathfrak{e}\in\mathfrak{Eul}(M;\mathbb{Z})$, cf. Theorem 1(ii). Note that we do not have to specify a base point since the Euler characteristics of M vanishes.

Recall that a finite dimensional long exact sequence of $\mathbb{K}$ vector spaces

$$\cdots\to A^n\to B^n\to C^n\to A^{n+1}\to\cdots$$

induces a canonical isomorphism of graded determinant lines[15]

$$u:\det(A^*)\otimes\det(C^*)\to\det(B^*),\qquad a\otimes b\to u(a\otimes b).$$

If in addition $B^*=C^*$ we obtain a canonical isomorphism $\det(A^*)\to\mathbb{K}$, $a\to\mathrm{tr}(u(a\otimes\cdot))$. Applying this to the Wang exact sequence for the trivial one dimensional real representation

$$\cdots\to H^*(M;\mathbb{R})\to H^*(N;\mathbb{R})\to H^*(N;\mathbb{R})\to H^{*+1}(M;\mathbb{R})\to\cdots$$

we obtain a canonic isomorphism $\det H^*(M;\mathbb{R})=\mathbb{R}$. Thus M has a canonic cohomology orientation $\mathfrak{o}$, the one corresponding to the standard orientation of $\mathbb{R}$ via this isomorphism.

The rest of this section is dedicated to the proof of

Proposition 8. *With this notations, if $H^*(M;p^*\rho)=0$, then $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(p^*\rho)=(-1)^{\dim(V)\cdot z_\varphi}\zeta_\varphi(\rho)$ where*

$$z_\varphi=\sum_q\dim H^q(N;\mathbb{R})\cdot\dim\ker\bigl(\varphi^*-\mathrm{id}:H^q(N;\mathbb{R})\to H^q(N;\mathbb{R})\bigr)$$

To get a hold on the Euler structure $\mathfrak{e}$ we start with the following

Lemma 2. *Suppose $\tilde\tau$ is a triangulation of M such that N is a subcomplex. For every simplex $\sigma\in\tilde\tau$ let π_σ be a path in $N\times[0,1)$ considered as path in M from the base point $x_0\in N$ to the barycenter x_σ. Consider the Euler structure $\tilde{\mathfrak{e}}=[X_{\tilde\tau},c_{\tilde\tau}]$ where $c_{\tilde\tau}:=\sum_{\sigma\in\tilde\tau}(-1)^{\dim\sigma}\pi_\sigma$. Then $[\det\circ p^*\rho](\tilde{\mathfrak{e}}-\mathfrak{e})=1$. Particularly $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(p^*\rho)=\mathcal{T}^{\tilde{\mathfrak{e}},\mathfrak{o}}_{\mathrm{comb}}(p^*\rho)$.*

Proof. Consider $N_\epsilon\subseteq M$, $x\mapsto(1-\epsilon,x)$, where $\epsilon>0$ is sufficiently small such that $\omega(X_\tau)<0$ on N_ϵ and such that no zeros of X_τ are contained in $(1-\epsilon,1)\times N$. This is possible since N is a sub-complex, hence the Euler vector field X_τ is tangential to N and points away from N in a small neighborhood of N. We thus find a homotopy $\mathbb{X}$ from X to X_τ which has the property that $\omega(\mathbb{X}_t)<0$ on N_ϵ for all $t\in I$. We conclude that $\mathrm{cs}(X,X_\tau)$ has a representative contained in $M\setminus N_\epsilon$. Since $\tilde{\mathfrak{e}}-\mathfrak{e}\in H_1(M;\mathbb{Z})$ is represented by $c_\tau-\mathrm{cs}(X,X_\tau)$ we conclude $p_*(\tilde{\mathfrak{e}}-\mathfrak{e})=0\in H_1(S^1;\mathbb{Z})$, and hence $[\det\circ p^*\rho](\tilde{\mathfrak{e}}-\mathfrak{e})=1$. The last assertion follows from Theorem 3(iii). □

[15] For more details consult the appendix.

Note that the statement remains the same if one replaces *triangulation* by *cell structure.*

Lemma 3. *Let $\varphi_i : N \to N$ be diffeomorphisms, and suppose $H^*(N_{\varphi_i}; p^*\rho) = 0$, $i = a, b$. Then $\mathcal{T}_{\mathrm{comb}}^{N_{\varphi_b},\mathfrak{e},\mathfrak{o}}(p^*\rho) \cdot \mathcal{T}_{\mathrm{comb}}^{N_{\varphi_a},\mathfrak{e},\mathfrak{o}}(p^*\rho)^{-1} = (-1)^{\dim(V)\cdot(z_{\varphi_b} - z_{\varphi_a})}\zeta_{\varphi_b}(\rho) \cdot \zeta_{\varphi_a}(\rho)^{-1}$.*

Proof. We may assume, by changing φ_i up to isotopy, that there are smooth triangulations τ_0 and τ_1 of N such that both φ_a and φ_b are cellular, i.e. they send the r–skeleton of τ_0 to the r–skeleton of τ_1. Let τ be a smooth triangulation on $N \times [0, 1]$ such that $N \times \{0\}$ and $N \times \{1\}$ are subcomplexes and the triangulation induced by τ on $N \times \{0\}$ resp. $N \times \{1\}$ is exactly τ_0 resp. τ_1. Let $\tilde\tau_a$ and $\tilde\tau_b$ denote the *cell structures* on N_{φ_a} and N_{φ_b} induced by τ. Note that $\tilde\tau_a$ and $\tilde\tau_b$ are smooth cell structures but not smooth triangulations. The characteristic maps are smooth embeddings of compact affine simplexes which satisfy the axioms of a CW–complex but not of a triangulation. Also recall that the torsion $\mathcal{T}_{\mathrm{comb}}^{N_{\varphi_i},\mathfrak{e},\mathfrak{o}}(p^*\rho)$ can be computed using any cell structure [M66].

Recall that we denote by $\mathcal{X}$, $\mathcal{X}_0$ and $\mathcal{X}_1$ the set of simplexes (cells) of τ, τ_0 and τ_1. The cells of both $\tilde\tau_a$ and $\tilde\tau_b$ are indexed by $\mathcal{X} \setminus \mathcal{X}_0$. Hence the graded vector spaces $C^*_{\tilde\tau_a}(N_{\varphi_a}; p^*\rho)$ and $C^*_{\tilde\tau_b}(N_{\varphi_b}; p^*\rho)$ are tautologically the same. Moreover the relative cochain complexes $C^*_{\tilde\tau_a}(N_{\varphi_a}, N; p^*\rho)$, $C^*_{\tilde\tau_b}(N_{\varphi_b}, N; p^*\rho)$ and $C^*_\tau(N \times I, N \times \partial I; p^*\rho)$ are tautologically the same; their cells are indexed by $\mathcal{X} \setminus \{\mathcal{X}_0 \cup \mathcal{X}_1\}$ and the differentials coincide.

The two short exact sequences of cochain complexes

$$C^*_\tau(N \times I, N \times \partial I; p^*\rho) = C^*_{\tilde\tau_i}(N_{\varphi_i}, N; p^*\rho) \to C^*_{\tilde\tau_i}(N_{\varphi_i}; p^*\rho) \to C^*_{\tau_1}(N; V)$$

$i = a, b$, give rise to a commutative diagram:

$$\begin{tikzcd}
\det C^*_{\tilde\tau_a}(N_{\varphi_a}; p^*\rho) \arrow[d, "\varphi_{\tilde\tau_a}"'] & \det C^*_\tau(N \times I, N \times \partial I; p^*\rho) \otimes \det C^*_{\tau_1}(N; V) \arrow[l, "(-1)^{y_a}\psi_a"'] \arrow[r, "(-1)^{y_b}\psi_b"] \arrow[d, "\varphi\otimes\varphi_{\tau_1}"] & \det C^*_{\tilde\tau_b}(N_{\varphi_b}; p^*\rho) \arrow[d, "\varphi_{\tilde\tau_b}"] \\
\mathbb{R} = \det H^*(N_{\varphi_a}; p^*\rho) & \det H^*(N \times I, N \times \partial I; p^*\rho) \otimes \det H^*(N; V) \arrow[l, "\varphi_{\mathcal{H}^*_a}"'] \arrow[r, "\varphi_{\mathcal{H}^*_b}"] & \det H^*(N_{\varphi_b}; p^*\rho) = \mathbb{R}
\end{tikzcd}$$

Here each square is a fusion diagram as in Lemma 5 in the appendix with the middle complex C^*_1 acyclic. Note that $(-1)^{y_a} = (-1)^{y_b}$ in view of Lemma 5. It follows from (33) that $\varphi_{\mathcal{H}^*_b} \circ \varphi^{-1}_{\mathcal{H}^*_a} = \zeta_{\varphi_a}(\rho) \circ \zeta_{\varphi_b}(\rho)^{-1}$.

Consider now the diagram

$$\begin{tikzcd}
\det C^*_{\tilde\tau_b}(N_{\varphi_b}; p^*\rho) & \det C^*_\tau(N \times I, N \times \partial I; p^*\rho) \otimes \det C^*_{\tau_1}(N; V) \arrow[l, "(-1)^{y_b}\psi_b"'] \arrow[d, "(-1)^{y_a}\psi_a"] \\
\mathbb{R} = \det V^\chi \arrow[u, "\beta_{\varphi_b}"] \arrow[r, "\beta_{\varphi_a}"] & \det C^*_{\tilde\tau_a}(N_{\varphi_a}; p^*\rho)
\end{tikzcd} \tag{34}$$

where β_{φ_a} and β_{φ_b} are induced from the homogeneous maps of degree $\chi = \chi(N_{\varphi_a}) = \chi(N_{\varphi_b}) = 0$ which associates to a frame of V the equivalence class of bases induced from the frame, a collection of paths representing the canonical Euler structure of N_{φ_a} resp. N_{φ_b} and the ordering of simplexes of τ representing the canonical cohomology orientation of N_{φ_a} resp. N_{φ_b}. Observe that the diagram is commutative up to sign because the canonical Euler structures on N_{φ_i} can be realized by the same

Euler chain in $N \times I$ cf. Lemma 2. We conclude $\mathcal{T}_{\text{comb}}^{N_{\varphi_b},\mathfrak{e},\mathfrak{o}}(p^*\rho) \cdot \mathcal{T}_{\text{comb}}^{N_{\varphi_a},\mathfrak{e},\mathfrak{o}}(p^*\rho)^{-1} = \pm\zeta_{\varphi_b}(\rho) \cdot \zeta_{\varphi_a}(\rho)^{-1}$.

To decide about the sign we have to compare the orderings of $\mathcal{X} \setminus \mathcal{X}_0$ induced from the canonical cohomology orientation on N_{φ_a} and on N_{φ_b}. To do this we have to consider the fusion diagram for the trivial one dimensional real representation:

$$\begin{tikzcd}
\det C^*_{\tilde\tau_a}(N_{\varphi_a};\mathbb{R}) \arrow[d, "\varphi^{\mathbb{R}}_{\tilde\tau_a}"'] & \det C^*_\tau(N \times I, N \times \partial I;\mathbb{R}) \otimes \det C^*_{\tau_1}(N;\mathbb{R}) \arrow[l, "(-1)^{y_a^{\mathbb{R}}}\psi_a^{\mathbb{R}}"'] \arrow[r, "(-1)^{y_b^{\mathbb{R}}}\psi_b^{\mathbb{R}}"] \arrow[d, "\varphi^{\mathbb{R}}\otimes\varphi^{\mathbb{R}}_{\tau_1}"] & \det C^*_{\tilde\tau_b}(N_{\varphi_b};\mathbb{R}) \arrow[d, "\varphi^{\mathbb{R}}_{\tilde\tau_b}"] \\
\det H^*(N_{\varphi_a};\mathbb{R}) & \det H^*(N \times I, N \times \partial I;\mathbb{R}) \otimes \det H^*(N;\mathbb{R}) \arrow[l, "\varphi^{\mathbb{R}}_{\mathcal{H}^*_a}"'] \arrow[r, "\varphi^{\mathbb{R}}_{\mathcal{H}^*_b}"] & \det H^*(N_{\varphi_b};\mathbb{R})
\end{tikzcd}$$

Using Lemma 5 one can show that $y_b^{\mathbb{R}} - y_a^{\mathbb{R}} = z_b - z_a \mod 2$. Together with $y_b - y_a = 0 \mod 2$ this implies that the digram (34) commutes up to the sign $(-1)^{\dim(V)\cdot(z_b - z_a)}$. This completes the proof. $\square$

Lemma 4. *We have* $\mathcal{T}_{\text{comb}}^{N_{\text{id}},\mathfrak{e},\mathfrak{o}}(p^*\rho) = (-1)^{\dim(V)\cdot z_{\text{id}}}\zeta_{\text{id}}(\rho)$ *whenever* $H^*(N_{\text{id}}; p^*\rho) = 0$.

Proof. Equip the interval I with the three cell structure, let τ_1 be a smooth triangulation of N, and let τ denote the product cell structure on $N \times I$. Let $\tilde\tau$ denote the induced (product) cell structure on $N_{\text{id}} = N \times S^1$. Then the complex $C^*_\tau(N \times I, N \times \partial I; p^*\rho)$ is the suspension of $C^*_{\tau_1}(N; p^*\rho)$. Using Lemma 5 and Lemma 2 it is easy to see that $\mathcal{T}_{\text{comb}}^{N_{\text{id}},\mathfrak{e},\mathfrak{o}}(p^*\rho) = \pm\zeta_{\text{id}}(\rho)$.

As in the proof of Lemma 3 consider the fusion diagram of the short exact sequence $C^*_\tau(N \times I, N \times \partial I; p^*\rho) \to C^*_{\tilde\tau}(N_{\text{id}}; p^*\rho) \to C^*_{\tau_1}(N; V)$ for an acyclic ρ and for the trivial one dimensional real representation. Let $(-1)^{y^\rho}$ and $(-1)^{y^{\mathbb{R}}}$ denote their signs. Using Lemma 5 one can show $\dim(V) \cdot y^{\mathbb{R}} - y^\rho = \dim(V) \cdot z_{\text{id}} \mod 2$. This completes the proof. $\square$

Proposition 8 follows from the previous two lemmas, and the simple observation that $H^*(N_\varphi; p^*\rho) = 0$ implies $H^*(N_{\text{id}}; p^*\rho) = 0$. For the last statement consider cohomology in degree zero where $\varphi^* = \text{id}^*$ for all φ.

4.4. **An invariant with values in $\mathbb{R}/\pi\mathbb{Z}$.** If M is a closed manifold with $\Gamma = \pi_1(M, x_0)$ and ρ an acyclic (i.e. $H^*(M,\rho) = 0$) unimodular representation on a finite dimensional complex Hilbert space then $\mathcal{T}_{\text{comb}}^{\mathfrak{e},\mathfrak{o}}$ is by Theorem 3(iii) independent on $\mathfrak{e}$ and its argument mod π independent on $\mathfrak{o}$. Denote by

$$S(M;\rho) := \arg(\mathcal{T}_{\text{comb}}^{\mathfrak{e},\mathfrak{o}}) \in \mathbb{R}/\pi\mathbb{Z}$$

This is a topological invariant computable with the help of a triangulation. Proposition 8 gives a simple formula for the case M is a mapping torus and ρ is the pullback by p of a unimodular representation of $\mathbb{Z}$ and implicitly establishes its non-triviality.

In a forthcoming paper an analytic construction of this invariant will be given; such construction justifies the name of *eta invariant of the DeRham complex*.

It is interesting to compare this invariant with the one defined using the spectral flow in [APS75] for an odd dimensional manifold and a unitary representation. More will be said in the forthcoming paper.

4.5. **Marcsik's theorem.** In this section we will use the Bismut–Zhang theorem and Proposition 8 to give a proof of a theorem due to J. Marcsik. We continue to use the notations from section 4.3.

Let g be a Riemannian metric on M. For $z \in \mathbb{C}$ denote by $d_\omega(z)\alpha := d\alpha + z\omega \wedge \alpha$ the Witten deformed differential on $\Omega^*(M;\mathbb{C})$ and by $\delta_\omega(z)$ its adjoint. Denote by

$$\Delta_\omega(z) := d_\omega(z)\delta_\omega(z) + \delta_\omega(z)d_\omega(z)$$

the Laplacian, and by $\Delta^k_\omega(z)$ the Laplacian on k forms. Introduce the Ray–Singer torsion:

$$T_{\mathrm{an}}(\omega, g)(z) := \exp\Big(\frac{1}{2}\sum_k (-1)^{k+1} k \log\det{}' \Delta^k_\omega(z)\Big)$$

Here $\det'\Delta$ denotes the regularized determinant of the non-negative self-adjoint elliptic operator Δ obtained by ignoring the zero modes.

Identifying $\mathbb{C}^* = \mathrm{GL}(\mathbb{C})$ we have $P^k_\varphi(w)$ and $\zeta_\varphi(w)$ defined for all $w \in \mathbb{C}^*$, see section 4.3.

Theorem 4 (Marcsik, [M98])**.** *Let N be a closed manifold, $\varphi : N \to N$ a diffeomorphism, M the mapping torus of φ, $p : M \to S^1$ the canonic projection, and ω the pull back of the canonic volume form on S^1. Let g be a Riemannian metric on M, and suppose X is a vector field on M with $\omega(X) < 0$. Then*

$$\log T_{\mathrm{an}}(\omega, g)(z) = \log|\zeta_\varphi(e^z)| + \Re(z)\int_M \omega \wedge X^*\Psi(g)$$

for all $z \in \mathbb{C}$ which satisfy $P^k_\varphi(e^z) \neq 0$ for all k.[16]

Proof. Consider the trivial line bundle $M \times \mathbb{C}$ over M. Let ∇^0 denote the trivial flat connection, and let μ denote the standard Hermitian structure on $M \times \mathbb{C}$. For $z \in \mathbb{C}$ define a connection $\nabla^z := \nabla + z\omega$ on $M \times \mathbb{C}$. This connection is flat for ω is closed. Clearly $d^{\nabla^z} = d_\omega(z)$, and hence

$$T_{\mathrm{an}}(\omega, g)(z) = T_{\mathrm{an}}(\nabla^z, g, \mu).$$

Define a co-Euler structure $\mathfrak{e}^* := [g, X^*\Psi(g)] \in \mathfrak{Eul}^*(M;\mathbb{R})$. Recall the Euler structure $\mathfrak{e} = [X, 0] \in \mathfrak{Eul}(M;\mathbb{Z})$. From Theorem 1(iv) we see that $\mathbb{T}(\mathfrak{e}, \mathfrak{e}^*) = 0$. Assume $P^k_\varphi(e^z) \neq 0$ for all k. Then the Bismut–Zhang theorem implies, see Remark 7:

$$|\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(e^z)| = T_{\mathrm{an}}(\nabla^z, g, \mu) \cdot e^{-S(\omega(\nabla^z,\mu), X^*\Psi(g))}$$

Note that $\nabla^z\mu = -2\Re(z\omega)\mu$, hence $\omega(\nabla^z, \mu) = \Re(z)\omega$, and thus

$$S(\omega(\nabla^z, \mu), X^*\Psi(g)) = \Re(z)\int_M \omega \wedge X^*\Psi(g).$$

From Proposition 8 we get $\mathcal{T}^{\mathfrak{e},\mathfrak{o}}_{\mathrm{comb}}(e^z) = \pm\zeta_\varphi(e^z)$. Putting everything together and taking the logarithm completes the proof. ∎

[16] This statement is slightly more precise than the one formulated in [M98].

4.6. **An application.** Suppose that $p : F \to M$ is a smooth complex vector bundle which admits flat connections equipped with a fiber metric μ over a closed manifold M. Let g be a Riemannian metric on M. Denote by $\mathcal{C}(F)$ the space of all connections on F. This is an affine Fréchet space over $\mathbb{C}$ when equipped with the C^∞–topology, and consider $\mathcal{FC}(F)$ the subset of all flat connections. This is a closed complex analytic subspace of $\mathcal{C}(F)$. Let $\Sigma(F)$ denote the subset of flat connections ∇ which are not a local minimum for the function $d : \mathcal{FC}(F) \to \mathbb{Z}$ defined by $d(\nabla) := \sum_{i=0}^n \dim H^i(M; \rho_\nabla)$. Here ρ_∇ denotes the holonomy representation of ∇. The set $\Sigma(F)$ is a closed analytic subspace of $\mathcal{FC}(F)$.

Denote by $\mathcal{FC}^M(F)$ the flat connections whose holonomy representations are in $\mathrm{Rep}^M(\Gamma; F_{x_0})$ and by $\mathcal{G}_{x_0}(F)$ the gauge group which fixes the fiber over the base point x_0. Observe that $\mathcal{FC}^M(F)$ is a closed analytic subspace of $\mathcal{FC}(F)$ and that $\mathcal{G}_{x_0}(F)$ acts freely and holomorphically on $\mathcal{FC}(F)$ and $\mathcal{FC}^M(F)$. The quotient space in both cases is a complex analytic space which identifies to a union of connected components of $\mathrm{Rep}^M(\Gamma; F_{x_0})$, resp. of irreducible components of $\mathrm{Rep}^M(\Gamma; F_{x_0})$.

The Ray–Singer torsion $T_{\mathrm{an}}(\nabla, g, \mu)$ defines a positive real valued function $T_{\mathrm{an}}^{g,\mu} : \mathcal{FC}(F) \to \mathbb{R}$, and if the dimension of M is odd it also defines a positive real valued function $T_{\mathrm{an}}^g : \mathrm{Rep}(\Gamma; V) \to \mathbb{R}$ by the formula

$$T_{\mathrm{an}}^g(\rho) := T_{\mathrm{an}}(\nabla_\rho, g, \mu)$$

where μ is any fiber metric in F_ρ. Indeed, in this case in view of the Hermitian anomaly [BZ92, Theorem 0.1] and because the Euler form $E(g)$ is identically zero, $T_{\mathrm{an}}(\nabla_\rho, g, \mu)$ is independent on μ. Both functions have jumps on $\Sigma(F)$ resp. $\Sigma(M)$. One can ask however if they are the absolute values of meromorphic functions with poles and zeros located on $\Sigma(F)$ resp. $\Sigma(M)$. It is not hard to verify that the polyharmonicity property is satisfied away from the jumps, so these functions are at least locally and away from the jumps absolute values of holomorphic maps. The following is a straightforward consequence of Theorem 3 and Proposition 1.

Corollary 1.

(i) *There exists a meromorphic function on $\mathcal{FC}^M(F)$ whose restriction to $\mathcal{FC}^M(F) \setminus \Sigma(F)$ has $T_{\mathrm{an}}^{g,\mu}$ as the absolute value the function.*
(ii) *If M has odd dimension then there exists a meromorphic function on $\mathrm{Rep}^M(\Gamma; V)$ whose restriction to $\mathrm{Rep}^M(\Gamma; V) \setminus \Sigma(M)$ has T_{an}^g as the absolute value function.*

Proof. To prove the result we fix a co-Euler structure $\mathfrak{e}^* = [g, \alpha] \in \mathfrak{Eul}_{x_0}^*(M; \mathbb{R})$, and write

$$T_{\mathrm{an}}^{g,\mu}(\nabla) = T_{\mathrm{an}}(\nabla, g, \mu) = T_{\mathrm{an}}(\nabla, g, \mu, \mathfrak{e}^*) \cdot e^{S(\omega(\nabla,\mu),\alpha)}$$

resp.

$$T_{\mathrm{an}}^g(\rho) = T_{\mathrm{an}}(\nabla_\rho, g, \mu) = T_{\mathrm{an}}(\nabla_\rho, g, \mu, \mathfrak{e}^*) \cdot e^{S(\omega(\nabla,\mu),\alpha)}$$

if M is of odd dimension. Theorem 3 shows that the first factor is the absolute value of a function which, in the first case factors via the gauge group by, in the second case is a rational function, hence is meromorphic. The second factor is holomorphic in ∇ and in second case factors via gauge group by a holomorphic function in view of Proposition 1. Indeed in this second case the second factor is gauge invariant given the fact that $E(g)$ is identically zero. □

Appendix A. Some homological algebra

Let $\mathbb{K}$ be a field of characteristics zero. Suppose C^* is a finite dimensional $\mathbb{Z}$–graded complex over $\mathbb{K}$, with differential of degree one, $d^q : C^q \to C^{q+1}$. We will always assume $C^q = 0$ for $q < 0$. Let us write $Z^q = \ker d^q \subseteq C^q$ for the cocycles, $B^q := \operatorname{img} d^{q-1} \subseteq C^q$ for the coboundaries, and $H^q := Z^q/B^q$ for the cohomology.

Let $\mathfrak{c}^q$ be a base of C^q, let $\mathfrak{h}^q$ be a base of H^q, and let $\mathfrak{b}^q$ be a base of B^q. Recall the short exact sequences

$$0 \to Z^q \to C^q \xrightarrow{d} B^{q+1} \to 0 \quad \text{and} \quad 0 \to B^q \to Z^q \to H^q \to 0.$$

Choose lifts $\tilde{\mathfrak{h}}^q$ in Z^q of $\mathfrak{h}^q$. Then $\mathfrak{b}^q\tilde{\mathfrak{h}}^q$ is a base of Z^q. Choose lifts $\tilde{\mathfrak{b}}^{q+1}$ in C^q of $\mathfrak{b}^{q+1}$. Then $(\mathfrak{b}^q\tilde{\mathfrak{h}}^q)\tilde{\mathfrak{b}}^{q+1}$ is a base of C^q. For two bases $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of a common vector space we write $[\mathfrak{a}_1/\mathfrak{a}_2]$ for the determinant of the matrix expressing $\mathfrak{a}_1$ in terms of $\mathfrak{a}_2$. The quantity $\prod_q [\mathfrak{c}^q/(\mathfrak{b}^q\tilde{\mathfrak{h}}^q)\tilde{\mathfrak{b}}^{q+1}]^{(-1)^q}$ does not depend on the choice of $\mathfrak{b}^q$ and does not depend on the choice of lifts $\tilde{\mathfrak{b}}^q$ and $\tilde{\mathfrak{h}}^q$, see [M66].

Recall that the determinant line of a finite dimensional vector space V is $\det V := \Lambda^{\dim V} V$. Moreover, we will write $L^1 := L$ and $L^{-1} := \operatorname{Hom}(L; \mathbb{K})$ for a line, i.e. one dimensional vector space, L. Consider the graded determinant line of C^*

$$\det C^* := \bigotimes_q (\det C^q)^{(-1)^q}$$

and let $\mathfrak{c}$ denote the base of $\det C^*$ induced from the bases $\mathfrak{c}^q$. Similarly, let $\mathfrak{h}$ denote the base of $\det H^*$ induced from the bases $\mathfrak{h}^q$. Consider the isomorphism

$$\varphi_{C^*} : \det C^* \to \det H^*, \qquad \mathfrak{c} \mapsto (-1)^N \prod_q [\mathfrak{c}^q/(\mathfrak{b}^q\tilde{\mathfrak{h}}^q)\tilde{\mathfrak{b}}^{q+1}]^{(-1)^q}\mathfrak{h}$$

where $N := \sum_{q\geq 0} \alpha^q\beta^q$ with $\alpha^q := \sum_{j\geq q} \dim C^j$ and $\beta^q := \sum_{j\geq q} \dim H^j$, see [FT00] and [T01]. The isomorphism $\varphi_{C^*} : \det C^* \to \det H^*$ does not depend on the choice of bases $\mathfrak{c}^q$ or $\mathfrak{h}^q$. The sign $(-1)^N$ is referred to as Turaev's sign refinement. Note that $(-1)^N = 1$ if C^* is acyclic.

Suppose

$$0 \to C_0^* \to C_1^* \to C_2^* \to 0 \tag{35}$$

is a short exact sequence of finite dimensional $\mathbb{Z}$–graded cochain complexes over $\mathbb{K}$. Then we have isomorphisms $\varphi_{C_i^*} : \det C_i^* \to \det H_i^*$, $i = 0, 1, 2$. Let $\mathcal{H}^*$ denote the corresponding long exact sequence

$$0 \to H_0^0 \to H_1^0 \to H_2^0 \xrightarrow{\delta^0} H_0^1 \to \cdots \xrightarrow{\delta^{q-1}} H_0^q \to H_1^q \to H_2^q \xrightarrow{\delta^q} H_0^{q+1} \to \cdots \tag{36}$$

considered as (acyclic) $\mathbb{Z}$–graded cochain complex, the grading so that $\mathcal{H}^{3q+i} = H_i^q$. As an acyclic complex it provides an isomorphism $\varphi_{\mathcal{H}^*} : \det \mathcal{H}^* \to \mathbb{K}$ which we regard as an isomorphism $\varphi_{\mathcal{H}^*} : \det H_0^* \otimes \det H_2^* \to \det H_1^*$.

The short exact sequence (35) provides isomorphisms $\det C_0^q \otimes \det C_2^q = \det C_1^q$ for all q, and then an isomorphism $\psi : \det C_0^* \otimes \det C_2^* \to \det C_1^*$. For the proof of the following lemma look carefully at the proof of Theorem 3.2 in [M66].

Lemma 5. *The following diagram commutes:*

$$\begin{tikzcd}
\det C_0^* \otimes \det C_2^* \arrow[r, "(-1)^y\psi"] \arrow[d, "\varphi_{C_0^*}\otimes\varphi_{C_2^*}"'] & \det C_1^* \arrow[d, "\varphi_{C_1^*}"] \\
\det H_0^* \otimes \det H_2^* \arrow[r, "\varphi_{\mathcal{H}^*}"] & \det H_1^*
\end{tikzcd}$$

Here

$$y = N(C_0^*) + N(C_1^*) + N(C_2^*) + \sum_q f_1^q \cdot b_2^q + b_2^q \cdot b_0^{q+1} + f_2^q \cdot b_0^{q+1}$$

with $b_i^q := \dim B^q(C_i^*)$*, and* $f_i^q := \dim B^{3q+i}(\mathcal{H}^*)$.

Following Farber–Turaev [FT00] we call the above diagram *fusion diagram*, and $(-1)^y\psi$ the *fusion isomorphism*. Note however, that the fusion homomorphism in general depends on the differentials of the complexes. If the short exact sequence of complexes splits is does not, see [FT00].

Lemma 6. *In the situation of Lemma 5 suppose in addition that* C_2^* *is acyclic and two dimensional, concentrated in degree* k *and* $k+1$*. Then the sign* $(-1)^y$ *does not depend on the differentials on* C_0^* *and* C_1^**.*

Proof. Indeed we have

$$\begin{aligned}
N(C_0^*) + N(C_1^*) &= \beta^{k+1}(C_0^*) \mod 2 \\
N(C_2^*) &= 0 \\
f_1^q &= \dim H_1^q = \dim H_0^q \\
f_2^q &= 0 \\
b_2^q &= 0 \quad \text{for } q \neq k+1 \\
b_2^{k+1} &= 1
\end{aligned}$$

Using the general fact $\alpha^q(C_0^*) + \beta^q(C_0^*) + b_0^q = 0 \mod 2$ this implies

$$y = \beta^{k+1}(C_0^*) + \dim H_0^{k+1} + b_0^{k+2} = \beta^{k+2}(C_0^*) + b_0^{k+2} = \alpha^{k+2}(C_0^*) \mod 2$$

Hence $(-)^y$ does not depend on the differentials. □

Since elementary expansions give rise to short exact sequences as in Lemma 6, we have

Corollary 2. *The Milnor–Turaev torsion as defined in section 3.3, Definition 1 does indeed not depend on the triangulation.*

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