Max-Planck-Institut für Mathematik
Bonn

De Concini-Kac filtration and Gelfand-Tsetlin characters for quantum $\mathfrak{gl}_N$

by

Vyacheslav Futorny
Jonas T. Hartwig

Max-Planck-Institut für Mathematik
Preprint Series 2012 (17)

De Concini-Kac filtration and Gelfand-Tsetlin characters for quantum $\mathfrak{gl}_N$

Vyacheslav Futorny
Jonas T. Hartwig

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
University of São Paulo
São Paulo
Brazil

Department of Mathematics
Stanford University
Stanford, CA
USA

MPIM 12-17

DE CONCINI-KAC FILTRATION AND GELFAND-TSETLIN CHARACTERS FOR QUANTUM $\mathfrak{gl}_N$

VYACHESLAV FUTORNY AND JONAS T. HARTWIG

Abstract. It was shown by the first author and Ovsienko [FO1] that the universal enveloping algebra of $\mathfrak{gl}_N$ is a Galois order, that is, it has a hidden invariant skew group structure. We extend this result to the quantized case and prove that $U_q(\mathfrak{gl}_N)$ is a Galois order over its Gelfand-Tsetlin subalgebra. This leads to a parameterization of finite families of isomorphism classes of irreducible Gelfand-Tsetlin modules for $U_q(\mathfrak{gl}_N)$ by the characters of Gelfand-Tsetlin subalgebra. In particular, any character of the Gelfand-Tsetlin subalgebra extends to an irreducible Gelfand-Tsetlin module over $U_q(\mathfrak{gl}_N)$ and, moreover, extends uniquely when such character is generic. We also obtain a proof of the fact that the Gelfand-Tsetlin subalgebra of $U_q(\mathfrak{gl}_N)$ is maximal commutative, as previously conjectured by Mazorchuk and Turowska.

1. Introduction

An important class of associative algebras, called *Galois orders* was introduced in [FO1]. This class of algebras includes for example Generalized Weyl algebras over integral domains with infinite order automorphisms (e.g. the n-th Weyl algebra A_n, the quantum plane, q-deformed Heisenberg algebra, quantized Weyl algebras, Witten-Woronowicz algebra ([B], [BO])); the universal enveloping algebra of $\mathfrak{gl}_n$ over the Gelfand-Tsetlin subalgebra ([DFO1], [DFO2]), associated shifted Yangians and finite W-algebras ([FMO2], [FMO1]).

These algebras contain a special commutative subalgebra which allows one to embed the algebra into a certain invariant subalgebra of some skew group algebra. In particular, such an embedding enables the computation of the skew field of fractions ([FMO2],[FH]). Representation theory of Galois orders was developed in [FO2]. If U is a Galois order over its commutative subalgebra Γ then one considers a category of Gelfand-Tsetlin U-modules which are direct sums of finite-dimensional Γ-modules parameterized by the maximal ideals of Γ. The set of isomorphism classes of irreducible Gelfand-Tsetlin modules extended from a given maximal ideal $\mathbf{m}$ of Γ is called the *fiber* of $\mathbf{m}$. In the case in which fibers consist of single isomorphism classes, the corresponding irreducible Gelfand-Tsetlin modules are parameterized by the elements of $\operatorname{Specm}\Gamma$ (up to some equivalence).

A natural choice of a commutative subalgebra in many associative algebras is a so-called Gelfand-Tsetlin subalgebra. Classical Gelfand-Tsetlin subalgebras of the universal enveloping algebras of a simple Lie algebras were considered in [FM], [Vi], [KW1], [KW2], [G1], [G2] among the others.

Gelfand-Tsetlin modules were studied in [O1] for $\mathfrak{gl}_n$, in [FMO2] for restricted Yangians of $\mathfrak{gl}_n$ and in [FMO1] for arbitrary finite W-algebras of type A.

In this paper we extend these results to $U_q(\mathfrak{gl}_N)$. This algebra contains a quantum analog of the Gelfand-Tsetlin subalgebra of $U(\mathfrak{gl}_N)$, which we denote by Γ_q.

Based on the properties of so called generic Gelfand-Tsetlin modules obtained in [MT], it was shown in [FH] that $U_q(\mathfrak{gl}_N)$ is a Galois ring with respect to Γ_q. This allowed us to prove the quantum Gelfand-Kirillov conjecture for $U_q(\mathfrak{gl}_N)$ ([FH],[F]).

Note that unlike all the examples listed above, $U_q(\mathfrak{gl}_N)$ is a Galois rings with respect to a subalgebra which not a polynomial algebra. Our first main result is the following.

Theorem I. *$U_q(\mathfrak{gl}_N)$ is a Galois order with respect to the Gelfand-Tsetlin subalgebra.*

The technique used to prove Theorem I is based on the RTT-realization of $U_q(\mathfrak{gl}_N)$ ([J],[KS]) and the De Concini-Kac filtration.

It was conjectured Mazorchuk and Turowska [MT] that Γ_q is a maximal commutative subalgebra of $U_q(\mathfrak{gl}_N)$. As consequence of Theorem I we obtain a proof of this fact.

Theorem II. *The Gelfand-Tsetlin subalgebra of $U_q(\mathfrak{gl}_N)$ is maximal commutative.*

Using the representation theory of Galois orders from [FO2] we obtain our third main result.

Theorem III. *The fiber of any $\mathbf{m} \in \operatorname{Specm} \Gamma_q$ in the category of Gelfand-Tsetlin modules over $U_q(\mathfrak{gl}_N)$ is non-empty and finite.*

Another consequence of [FO2] and Theorem I above is that for a generic $\mathbf{m}$ (i.e. from some dense subset of $\operatorname{Specm} \Gamma_q$), there exists a unique (up to isomorphism) irreducible $U_q(\mathfrak{gl}_N)$-module in the fiber of $\mathbf{m}$. This was established previously in [MT], because all such modules are *generic Gelfand-Tsetlin modules* in the terminology of [MT].

Similarly to the case of finite W-algebras of type A [FMO2], we make the following conjecture about the cardinality of fibers for arbitrary $\mathbf{m}$. We show that the conjecture is valid for $U_q(\mathfrak{gl}_2)$.

Conjecture. *For any $\mathbf{m} \in \operatorname{Specm} \Gamma_q$, the fiber of $\mathbf{m}$ consists of at most*

$$2^{N(N-1)/2}(1!2!\dots(N-1)!)$$

isomorphism classes of irreducible Gelfand-Tsetlin $U_q(\mathfrak{gl}_N)$-modules. The same bound holds for the dimension of the subspace $V(\mathbf{m})$ in any irreducible Gelfand-Tsetlin module V.

Notation. $[\![a, b]\!]$ denotes the set $\{x \in \mathbb{Z} \mid a \le x \le b\}$. The cardinality of a set S is denoted $\#S$. Throughout this paper, the ground field is $\mathbb{C}$ and $q \in \mathbb{C}$ is nonzero and not a root of unity. We put $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

2. The algebra $U_q(\mathfrak{gl}_N)$

In this section we recall some facts about the quantized enveloping algebra $U_q(\mathfrak{gl}_N)$ which will be used.

2.1. **Definition.** For positive integers N we let $U_N = U_q(\mathfrak{gl}_N)$ denote the unital associative $\mathbb{C}$-algebra with generators $E_i^{\pm}$, K_j, K_j^{-1}, $i \in [\![1, N-1]\!]$, $j \in [\![1, N]\!]$ and relations [KS, p.163]

$$\begin{aligned} K_iK_i^{-1} &= K_i^{-1}K_i = 1, \quad [K_i, K_j] = 0, \quad \forall i, j \in [\![1, N]\!], \\ K_iE_j^{\pm}K_i^{-1} &= q^{\pm(\delta_{ij}-\delta_{i,j+1})}E_j^{\pm}, \quad \forall i \in [\![1, N]\!], \forall j \in [\![1, N-1]\!], \\ [E_i^+, E_j^-] &= \delta_{ij}\frac{K_iK_{i+1}^{-1} - K_{i+1}K_i^{-1}}{q - q^{-1}}, \quad \forall i, j \in [\![1, N-1]\!], \\ [E_i^{\pm}, E_j^{\pm}] &= 0, \quad |i - j| > 1, \\ (E_i^{\pm})^2E_j^{\pm} - (q + q^{-1})E_i^{\pm}E_j^{\pm}E_i^{\pm} &+ E_j^{\pm}(E_i^{\pm})^2 = 0, \quad |i - j| = 1. \end{aligned}$$

2.2. **De Concini-Kac filtration.** [BG, Section I.6.11] Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $i \in [\![1, N-1]\!]$ be the standard simple roots of $\mathfrak{gl}_N$ where $\varepsilon_i(\mathrm{diag}(a_1, \dots, a_N)) = a_i$. Fix the following decomposition of the longest Weyl group element:

$$w_0 = s_{i_1} \cdots s_{i_M} = (s_1s_2 \cdots s_{N-1})(s_1s_2 \cdots s_{N-2}) \cdots (s_1s_2)s_1, \tag{2.1}$$

where $s_i = (i\ i+1) \in S_N$, and $M = N(N-1)/2$. Let $\{\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})\}_{j=1}^M$ be the corresponding enumeration of positive roots of $\mathfrak{gl}_N$. One checks that

$$(\beta_1, \beta_2, \dots, \beta_M) = (\beta_{12}, \beta_{13}, \dots, \beta_{1N},\ \beta_{23}, \beta_{24}, \dots, \beta_{2N},\ \dots,\ \beta_{N-1,N}), \tag{2.2}$$

where $\beta_{ij} = \varepsilon_i - \varepsilon_j$ for all $i, j \in [\![1, N]\!]$, $i < j$. Let E_{β_i}, $F_{\beta_i} \in U_q(\mathfrak{gl}_N)$ be the correspodning positive and negative root vectors (see e.g. [BG, Section I.6.8]). The following PBW theorem for $U_q(\mathfrak{gl}_N)$ is well-known:

Theorem 2.1. *The set of ordered monomials*

$$F^rK_\lambda E^k := F_{\beta_1}^{r_1} \cdots F_{\beta_M}^{r_M} \cdot K_1^{\lambda_1} \cdots K_N^{\lambda_N} \cdot E_{\beta_1}^{k_1} \cdots E_{\beta_M}^{k_M} \tag{2.3}$$

where $r, k \in \mathbb{Z}_{\geq 0}^M$ *and* $\lambda \in \mathbb{Z}^N$, *form a basis for* $U_q(\mathfrak{gl}_N)$.

Define the *total degree* of a monomial $F^rK_\lambda E^k$ to be

$$d(F^rK_\lambda E^k) = \big(k_M, \dots, k_1, r_1, \dots, r_M, \mathrm{ht}(F^rK_\lambda E^k)\big) \in \mathbb{Z}_{\geq 0}^{2M+1}, \tag{2.4}$$

where

$$\mathrm{ht}(F^rK_\lambda E^k) = \sum_{j=1}^M (k_j + r_j)\mathrm{ht}(\beta_j) \tag{2.5}$$

and $\mathrm{ht}(\beta) = \sum_{i=1}^{N-1} a_i$ if $\beta = \sum_{i=1}^{N-1} a_i\alpha_i$. Equip the monoid $\mathbb{Z}_{\geq 0}^{2M+1}$ with the lexicographical order uniquely determined by the inequalities

$$u_1 < u_2 < \cdots < u_M$$

where $u_i = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 on the i:th position.

Theorem 2.2 (De Concini-Kac)**.** *The total degree function d defined above equips $U = U_q(\mathfrak{gl}_N)$ with a $\mathbb{Z}_{\geq 0}^{2M+1}$-filtration $\{U_{(k)}\}_{k \in \mathbb{Z}_{\geq 0}^{2M+1}}$. The associated graded algebra $\mathrm{gr}\, U$ is the $\mathbb{C}$-algebra on the generators*

$$\bar{E}_{\beta_i}, \bar{F}_{\beta_j}, \bar{K}_\lambda$$

$i = 1, \dots, M$, $\alpha \in \mathbb{Z}^N$ *subject to the following defining relations:*

$$\begin{aligned} &\bar{K}_\alpha\bar{K}_\beta = \bar{K}_{\alpha+\beta} && \bar{K}_0 = 1 \\ &\bar{K}_\alpha\bar{E}_{\beta_i} = q^{(\alpha,\beta_i)}\bar{E}_{\beta_i}\bar{K}_\alpha && \bar{K}_\alpha\bar{F}_{\beta_i} = q^{-(\alpha,\beta_i)}\bar{F}_{\beta_i}\bar{K}_\alpha \\ &\bar{E}_{\beta_i}\bar{F}_{\beta_j} = \bar{F}_{\beta_j}\bar{E}_{\beta_i} && \\ &\bar{E}_{\beta_i}\bar{E}_{\beta_j} = q^{(\beta_i,\beta_j)}\bar{E}_{\beta_j}\bar{E}_{\beta_i} && \bar{F}_{\beta_i}\bar{F}_{\beta_j} = q^{(\beta_i,\beta_j)}\bar{F}_{\beta_j}\bar{F}_{\beta i} \end{aligned} \tag{2.6}$$

for $\alpha, \beta \in Q$ and $1 \le i, j \le M$.

Proof. That d actually defines a filtration follows from the commutation relation known as the *Levendorskiĭ-Soibelman straightening rule* [LS, Proposition 5.5.2]. See [DK, Proposition 1.7] for details. □

Observe that the root vectors E_α, F_α, hence the De Concini-Kac filtration, depend on the choice of decomposition of the longest Weyl group element.

A simple but important corollary which will be used implicitly throughout is that

$$d(ab) = d(a) + d(b) = d(ba) \tag{2.7}$$

for all $a, b \in U_q(\mathfrak{gl}_N)$, where now $d(a)$ denotes the smallest $k \in \mathbb{Z}_{\ge 0}^{2M+1}$ such that $a \in U_{(k)}$. This follows from the fact that the associated graded algebra is a domain.

2.3. **RTT presentation.** $U_q(\mathfrak{gl}_N)$ has an alternative presentation. It is isomorphic to the algebra with generators t_{ij}, $\bar{t}_{ij}$, $i, j \in [\![1, N]\!]$ and relations

$$t_{ij} = 0 = \bar{t}_{ji}, \quad \forall i < j, \tag{2.8a}$$

$$t_{ii}\bar{t}_{ii} = 1 = \bar{t}_{ii}t_{ii}, \quad \forall i, \tag{2.8b}$$

$$q^{\delta_{ij}} t_{ia}t_{jb} - q^{\delta_{ab}} t_{jb}t_{ia} = (q - q^{-1})(\delta_{b<a} - \delta_{i<j})t_{ja}t_{ib}) \tag{2.8c}$$

$$q^{\delta_{ij}} \bar{t}_{ia}\bar{t}_{jb} - q^{\delta_{ab}} \bar{t}_{jb}\bar{t}_{ia} = (q - q^{-1})(\delta_{b<a} - \delta_{i<j})\bar{t}_{ja}\bar{t}_{ib}) \tag{2.8d}$$

$$q^{\delta_{ij}} \bar{t}_{ia}t_{jb} - q^{\delta_{ab}} t_{jb}\bar{t}_{ia} = (q - q^{-1})(\delta_{b<a}t_{ja}\bar{t}_{ib} - \delta_{i<j}\bar{t}_{ja}t_{ib}) \tag{2.8e}$$

for all $i, a, j, b \in [\![1, N]\!]$. An identification of the two sets of generators is given by [KS, Section 8.5.4]:

$$\begin{aligned} \bar{t}_{ii} &= K_i^{-1} & t_{ii} &= K_i \\ \bar{t}_{i,i+1} &= (q - q^{-1})K_i^{-1}E_i & t_{i+1,i} &= -(q - q^{-1})F_iK_i \\ \bar{t}_{ij} &= (q - q^{-1})(-1)^{i-j+1}K_i^{-1}E_{\beta_{ij}} & t_{ji} &= -(q - q^{-1})F_{\beta_{ij}}K_i \end{aligned} \tag{2.9}$$

for $j > i + 1$, where $E_{\beta_{ij}}, F_{\beta_{ij}}$ are the root vectors, defined previously in Section 2.2.

2.4. **Gelfand-Tsetlin subalgebra.** Let $U_q = U_q(\mathfrak{gl}_N)$. It is immediate by the defining relations that, for each $r \in [\![1, N]\!]$, the subalgebra $U_q^{(r)}$ of U_q generated by E_i, F_i, K_j for $i \in [\![1, r-1]\!]$, $j \in [\![1, r]\!]$ (or equivalently, by $t_{ij}, \bar{t}_{ij}$ for $i, j \in [\![1, r]\!]$) can be identified with $U_q(\mathfrak{gl}_r)$. Thus we have a chain of subalgebras

$$U_q^{(1)} \subset U_q^{(2)} \subset \cdots \subset U_q^{(N)} = U_q.$$

Let Z_r denote the center of $U_q^{(r)}$. The subalgebra of U_q generated by $Z_1, \dots, Z_N$ is called the *Gelfand-Tsetlin subalgebra* and will be denoted by Γ_q. It is immediate that Γ_q is commutative.

In [MH, Section 5] it is proved that Z_r is generated by the coefficients of the following polynomial in $U_q^{(r)}[u^{-1}]$:

$$z_r(u) = \sum_{\sigma \in S_r} (-q)^{-l(\sigma)} \prod_{j=1}^{r} \big(t_{\sigma(j)j} - \bar{t}_{\sigma(j)j}q^{2(j-1)}u^{-1}\big). \tag{2.10}$$

It will be useful to rewrite this polynomial in a different way. For this purpose it will be convenient to use the notation

$$t_{ij}^{(k)} = \begin{cases} t_{ij}, & k=0, \\ \bar{t}_{ij}, & k=1. \end{cases} \tag{2.11}$$

A direct computation gives that

$$z_r(u) = \sum_{s=0}^{r} (-1)^r d_{rs} (q^2 u)^{-s}, \tag{2.12}$$

where

$$d_{rs} = \sum_{\sigma\in S_r} (-q)^{-l(\sigma)} \sum_{k\in\{0,1\}^r:\, \sum k_i = s} q^{2(k_1+2k_2+\cdots+rk_r)} t_{\sigma(1)1}^{(k_1)} \cdots t_{\sigma(r)r}^{(k_r)}. \tag{2.13}$$

Observe that $d_{r0} = d_{rr}^{-1}$. Therefore, the (commuting) elements d_{rs}, $1\le s\le r\le N$, generate Γ_q, provided we allow taking negative powers of d_{rr}. In Lemma 2.5 we show that these generators are algebraically independent.

2.5. **Realization of $U_q(\mathfrak{gl}_N)$ as a Galois Γ-ring.** We recall the definition of a *Galois ring* from [FO1]. Let Γ be an integral domain, K be its field of fractions, L be a finite Galois extension of K, and $G = \mathrm{Gal}(L/K)$ be the Galois group. Let G act by conjugation on $\mathrm{Aut}(L)$ and let $\mathcal{M}$ be a G-invariant submonoid of $\mathrm{Aut}(L)$. We require $\mathcal{M}$ to be K*-separating*, meaning $m_1|_K = m_2|_K \Rightarrow m_1 = m_2$ for $m_1, m_2 \in \mathcal{M}$. The action of G on L and on $\mathcal{M}$ (by conjugations) extends uniquely to an action of G on the skew monoid ring $L*\mathcal{M}$ by ring automorphisms. Let $\mathcal{K} = (L*\mathcal{M})^G$ denote the subring of invariants.

Definition 2.3 (Galois ring)**.** A finitely generated Γ-subring U of $\mathcal{K}$ is called a *Galois* Γ*-ring* if $UK = KU = \mathcal{K}$.

Let $U_q = U_q(\mathfrak{gl}_N)$, and q is not a root of unity. We recall the realization of U_q as a Galois ring obtained in [FH]. Let $\Lambda_m = \mathbb{C}[X_{m1}^{\pm1}, \dots, X_{mm}^{\pm1}]$ be a Laurent polynomial algebra in m variables and put $\Lambda = \Lambda_1 \otimes \cdots \otimes \Lambda_N \simeq \mathbb{C}[X_{mi}^{\pm1} \mid 1\le i\le m\le N]$. Let L be the field of fractions of Λ. Let W_m be the Weyl group of type D_m, i.e. $W_m = S_m \ltimes \mathcal{E}_m$ where $\mathcal{E}_m = \{\alpha \in (\mathbb{Z}/2\mathbb{Z})^m \mid \alpha_1 + \cdots + \alpha_m = 0\}$ with the natural S_m-action. Let $G = \prod_{m=1}^N W_m$. Then G acts on L by

$$g(X_{mi}) = (-1)^{\alpha_{mi}} X_{m\zeta_m(i)}, \quad 1\le i\le m\le N, \tag{2.14}$$

for $g = (\zeta_1\alpha_1, \cdots \zeta_N\alpha_N) \in G$ where $\zeta_m \in S_m$, $\alpha_m = (\alpha_{m1}, \dots, \alpha_{mm}) \in \mathcal{E}_m$. Let $\Gamma = \Lambda^G$, and $K = \mathrm{Frac}(\Gamma)$. Let $\mathcal{M}$ be the subgroup of $\mathrm{Aut}(L)$ generated by the set $\{\delta^{mi}\}_{1\le i\le m\le N-1}$, where $\delta^{mi} \in \mathrm{Aut}(L)$ is given by $\delta^{mi} X_{kj} = q^{-\delta_{mk}\delta_{ij}} X_{kj}$ for all $1\le i\le m\le N-1$ and $1\le j\le k\le N$. Clearly $\mathcal{M} \simeq \mathbb{Z}^{N(N-1)/2}$, since q is not a root of unity. One verifies that $\mathcal{M}$ is G-invariant.

Let $\mathcal{K} = (L*\mathcal{M})^G$. The following theorem shows that U_q is isomorphic to a Galois Γ-ring in $\mathcal{K}$.

Theorem 2.4 ([FH])**.** (i) *There exists an injective* $\mathbb{C}$*-algebra homomorphism*

$$\varphi : U_q \longrightarrow \mathcal{K}$$

determined by

$$\varphi(E_m^{\pm}) = \sum_{i=1}^{N} (\pm\delta^{mi}) A_{mi}^{\pm}, \qquad \varphi(K_m) = A_m^0 e \tag{2.15}$$

where $e \in \mathcal{M}$ is the neutral element, and $A^{\pm}_{mi}$, $A^0_m \in L$ are given by

$$A^{\pm}_{mi} = \mp(q-q^{-1})^{-1\mp 1}\frac{\prod_{j=1}^{m\pm 1}\left(X_{m\pm 1,j}X_{mi}^{-1} - X_{m\pm 1,j}^{-1}X_{mi}\right)}{\prod_{j\in\{1,\dots,m\}\setminus\{i\}}\left(X_{mj}X_{mi}^{-1} - X_{mj}^{-1}X_{mi}\right)}, \tag{2.16}$$

$$A^0_m = q^m \prod_{i=1}^{m} X_{mi} \prod_{i=1}^{m-1} X_{m-1,i}^{-1}; \tag{2.17}$$

(ii) *$UK = KU = \mathcal{K}$, where $U = \varphi(U_q)$;*
(iii) *$\mathcal{M}$ is K-separating;*
(iv) *L is a finite Galois extension of K with Galois group $\mathrm{Gal}(L/K) = G$;*
(v) *$\varphi(Z_m) = \Lambda_m^{W_m}$ for each $m \in [\![1,N]\!]$ and $\varphi(\Gamma_q) = \Gamma = \Lambda^G$, where $Z_m = Z(U_q(\mathfrak{gl}_m))$ and Γ_q is the Gelfand-Tsetlin subalgebra of U_q;*
(vi) *The restriction of φ to Z_m can be identified with the quantum Harish-Chandra homomorphism:*

$$\varphi|_{Z_m} = \xi_m^{-1}\circ h_m,$$

where $\xi : \Lambda_m \to U_q(\mathfrak{gl}_m)$, $\xi(X_{mi}) = q^{-i}K_i$ and $h_m : Z_m \to \mathbb{C}[K_1^{\pm 1},\dots,K_m^{\pm 1}]$ is the quantum Harish-Chandra homomorphism.

Proof. See [FH, Propositions 5.9-5.14]. □

We now prove that the generators d_{rs} from (2.13) are algebraically independent.

Lemma 2.5.

$$\Gamma_q \simeq \mathbb{C}[d_{rs} \mid 1\le s\le r\le N][d_{rr}^{-1} \mid 1\le r\le N]. \tag{2.18}$$

Proof. By applying the quantum Harish-Chandra isomorphism $h_r : Z_r \to (U_r^0)^{W_r}$ (see [FH, Lemma 5.3]) to the polynomial $z_r(u)$ from (2.10) (as in [MH, Section 5]) we get

$$\begin{aligned} h_r(z_r(u)) =&(K_1 - K_1^{-1}u^{-1})(K_2 - q^2K_2^{-1}u^{-1})\cdots(K_r - q^{2(r-1)}K_r^{-1}u^{-1})\\ =&q^{r(r+1)}(K_1\cdots K_r)^{-1}\prod_{j=1}^{r}(q^{-2j}K_j^2 - (q^2u)^{-1}) \end{aligned}$$

So

$$h_r(d_{rs}) = q^{r(r+1)/2}(\widetilde{K}_1\cdots\widetilde{K}_r)^{-1}\cdot e_{rs}(\widetilde{K}_1^2,\dots,\widetilde{K}_r^2), \quad r\in[\![1,N]\!], s\in[\![0,r]\!]$$

where $\widetilde{K}_i = q^{-i}K_i$, and e_{rs} is the elementary symmetric polynomial in r variables of degree s. By the proof of [FH, Lemma 5.3], this shows that

$$Z_r \simeq \mathbb{C}[d_{rs} \mid s = 1,2,\dots,r][d_{rr}^{-1}]. \tag{2.19}$$

Recall that $\Lambda^G \simeq \Lambda_1^{W_1}\otimes\cdots\otimes\Lambda_N^{W_N}$. Let $\varphi : U \to \mathcal{K}$ be the map from Theorem 2.4. By parts (i) and (v) of that theorem, φ restricts to an isomorphism $\varphi|_{\Gamma_q} : \Gamma_q \to \Lambda^G$ and $\varphi_i := \varphi|_{Z_m} : Z_m \to \Lambda_m^{W_m}$ for each $m \in [\![1,N]\!]$. Thus we have a commutative diagram

$$\begin{array}{ccc} \Gamma_q & \xrightarrow{\varphi|_{\Gamma_q}} & \Lambda^G \\ \uparrow f & & \uparrow g \\ Z_1\otimes\cdots\otimes Z_N & \xrightarrow{\varphi_1\otimes\cdots\otimes\varphi_N} & \Lambda_1^{W_1}\otimes\cdots\otimes\Lambda_N^{W_N} \end{array}$$

where the vertical arrows are given by multiplication. The horizontal maps and g are isomorphisms. Hence f is an isomorphism. Combining this fact with (2.19) we obtain the required isomorphism. $\square$

2.6. **Harish-Chandra subalgebras.**

Definition 2.6 (Harish-Chandra subalgebra)**.** A subalgebra B of an algebra A is called a *Harish-Chandra subalgebra* provided BaB is finitely generated as a left and right B-module for any $a \in A$.

The following criteron for Γ to be a Harish-Chandra subalgebra of a Galois Γ-ring was given in [FO1].

Proposition 2.7. [FO1, Proposition 5.1] *Let $U \subseteq (L * \mathcal{M})^G$ be a Galois Γ-ring, where Γ is finitely generated as a $\mathbb{C}$-algebra. Then Γ is a Harish-Chandra subalgebra of U if and only if $m \cdot \bar{\Gamma} = \bar{\Gamma}$ for every $m \in \mathcal{M}$, where $\bar{\Gamma}$ denotes the integral closure of Γ in L.*

In [MT, Proposition 1], the following result was stated and a method of proof was suggested. We give a short proof using Galois rings.

Proposition 2.8 ([MT])**.** *The Gelfand-Tsetlin subalgebra Γ_q of $U_q = U_q(\mathfrak{gl}_N)$ is a Harish-Chandra subalgebra.*

Proof. We will use Proposition 2.7. By Theorem 2.4(v), in the realization of U_q as a Galois algebra, $\Gamma = \Lambda^G$ and $\mathcal{M} = \mathbb{Z}^{N(N-1)/2}$. It is enough to prove that $m \cdot \Gamma \subseteq \bar{\Gamma}, \forall m \in \mathcal{M}$. Since m acts by automorphisms, it is further enough to prove that $m \cdot X \subseteq \bar{\Gamma}$ for some generating set X of Γ, for m in some generating set of $\mathcal{M}$. Since $\Lambda^G \simeq \Lambda_1^{W_1} \otimes \cdots \otimes \Lambda_N^{W_N}$, it follows from [FH, Lemma 5.3] that Λ^G is generated by

$$x_{rs} := e_{rs}(X_{r1}^2, \ldots, X_{rr}^2), \quad 1 \le s < r \le N,$$
$$x_{rr}^{\pm 1} := (X_{r1}X_{r2}\cdots X_{rr})^{\pm 1}, \quad 1 \le r \le N,$$

where e_{rs} is the elementary symmetric polynomial in r variables of degree s. Recall that the action of $\mathcal{M}$ on $L = \mathrm{Frac}(\Lambda)$ is given by $\delta^{ji} \cdot X_{rs} = q^{-\delta_{jr}\delta_{is}} X_{rs}$. We have $\delta^{ji} \cdot x_{rr}^{\pm 1} = q^{\mp\delta_{jr}} x_{rr}^{\pm 1}$ which even belongs to Γ, hence to $\bar{\Gamma}$. For the other generators, first recall the splitting polynomial for L/K [FH], where $K = L^G = \mathrm{Frac}(\Gamma)$:

$$p(x) = \prod_{j=1}^{N} (x^2 - X_{j1}^2)(x^2 - X_{j2}^2)\cdots(x^2 - X_{jj}^2)(x - X_{j1}X_{j2}\cdots X_{jj}).$$

Since $p(x) \in \Gamma[x]$, it is clear that all $X_{jr} \in \bar{\Gamma}$, hence $\Lambda_+ \subseteq \bar{\Gamma}$, where $\Lambda_+ := \mathbb{C}[X_{ji} \mid 1 \le i \le j \le N]$. In particular, it follows immediately that $\delta^{ji} \cdot x_{rs} \in \Lambda_+ \subseteq \bar{\Gamma}$ for $s < r$. $\square$

3. Galois orders

We recall the definition of Galois orders from [FO1].

Definition 3.1 (Galois order)**.** A Galois Γ-ring is a *right (respectively left) Galois Γ-order* if for any finite dimensional right (respectively left) K-subspace $W \subseteq UK$ (respectively $W \subseteq KU$), $W \cap U$ is a finitely generated right (respectively left) Γ-module. A Galois ring is Galois order if it is both right and left Galois order.

Proposition 3.2 ([FO1])**.** *Let U be a Galois Γ-ring. Then U is a Galois Γ-order if and only if the following two conditions hold:*

(i) *Γ is a Harish-Chandra subalgebra of U;*

(ii)
$$\forall u \in U, \gamma \in \Gamma \setminus \{0\} : \big(u\gamma \in \Gamma \vee \gamma u \in \Gamma\big) \Longrightarrow u \in \Gamma. \tag{3.1}$$

The following result shows that under certain circumstances, condition (3.1) may be replaced by the condition that Γ be maximal commutative in U.

Proposition 3.3. *Let $U \subseteq (L * \mathcal{M})^G$ be a Galois Γ-ring where Γ is a Harish-Chandra subalgebra of U. Then the following two statements hold:*

(i) *If Γ is a maximal commutative subalgebra of U, then U is a Galois Γ-order;*

(ii) *If U is a Galois Γ-order, $\mathcal{M}$ is a group and Γ is finitely generated and normal, then Γ is a maximal commutative subalgebra of U.*

Proof. (i) Suppose Γ is maximal commutative in U. By Proposition 3.2, it is enough to show that (3.1) holds. Suppose that $u\gamma \in \Gamma$ for some $u \in U$, $\gamma \in \Gamma \setminus \{0\}$. Since Γ is commutative we get

$$\gamma_1 u\gamma = u\gamma\gamma_1 = u\gamma_1\gamma, \quad \forall \gamma_1 \in \Gamma.$$

Since U is torsion-free as a right Γ-module, this implies that $\gamma_1 u = u\gamma_1$ for all $\gamma_1 \in \Gamma$. This forces $u \in \Gamma$, since Γ is a maximal commutative subalgebra of U. The case $\gamma u \in \Gamma$ is analogous.

(ii) We follow the proof of [FMO2, Corollary 6.7]. By [FO1, Theorem 4.1(3)], $U \cap K$ is a maximal commutative subalgebra of U, so it suffices to show that $U \cap K = \Gamma$. By [FO1, Theorem 5.2(2)], $U \cap Le$ is an integral extension of Γ, where $Le = \{\lambda e \mid \lambda \in L\} \subseteq L * \mathcal{M}$ and $e \in \mathcal{M}$ is the neutral element. Hence $U \cap K$ is an also an integral extension of Γ. Since Γ is normal, $U \cap K = \Gamma$. □

4. $U_q(\mathfrak{gl}_N)$ is a Galois order

In this section we give a proof that $U_q(\mathfrak{gl}_N)$ is a Galois order. The main technical result is the following theorem which determines the leading terms of the generators d_{rs} of Γ_q with respect to the De Concini-Kac filtration.

Theorem 4.1. *The leading term of d_{rs} (see (2.13)), with respect to the De Concini-Kac filtration using (2.1) as decomposition of the longest Weyl group element, is obtained by taking*

$$\sigma = (1\ 2\ \cdots\ r)^s.$$

in the sum (2.13). That is,

$$\mathrm{lt}(d_{rs}) = \lambda \cdot t^{(0)}_{1+s,1} t^{(0)}_{2+s,2} \cdots t^{(0)}_{r,r-s} \cdot t^{(1)}_{1,r-s+1} t^{(1)}_{2,r-s+2} \cdots t^{(1)}_{s,r} \tag{4.1}$$

for some nonzero $\lambda \in \mathbb{C}$.

Example 4.2. As an example, we determine directly the leading term of d_{42}. The most significant component of the total degree (2.4) is the height. Using (4.2)-(4.3), it is easy to see that there are four permutations in S_4 which gives the maximal possible height 8:

$$(13)(24), \quad (14)(23), \quad (1324), \quad (1423).$$

The monomial associated to such a permutation σ is

$$t^{(k_1)}_{\sigma(1)1} t^{(k_2)}_{\sigma(2)2} t^{(k_3)}_{\sigma(3)3} t^{(k_4)}_{\sigma(4)4}$$

where $k_i = 0$ if $\sigma(i) > i$ and $k_i = 1$ if $\sigma(i) < i$. After the height we need to compare the exponent of $F_{\beta_{34}}$ in the four different monomials, because β_{34} is the largest positive root in the ordering

$$\beta_{12} < \beta_{13} < \beta_{14} < \beta_{23} < \beta_{24} < \beta_{34}$$

(see (2.2)). This exponent is the same as the exponent (either 1 or 0) of $t^{(0)}_{43}$ due to the identifications (2.9). But this exponent is 0 in all four cases because none of the permutations map 3 to 4.

So we look at the second largest positive root, which is β_{24}. As in the previous case, we ask if $\sigma(2) = 4$ in any of the four permutations. There are two for which this holds, (13)(24) and (1324). The others do not map 2 to 4 which means their corresponding monomials are of lower total degree.

To compare the two candidates (13)(24) and (1324) we look at the third largest root, β_{23}. But $\sigma(2) \neq 3$ in both. Next is β_{14} but again $\sigma(1) \neq 4$ in both. Next is β_{13} and now $\sigma(1) = 3$ for both $\sigma = (13)(24)$ and $\sigma = (1324)$. Next is β_{12} and $\sigma(1) \neq 2$ in both. So we still don't know which monomial is largest. We have compared the $1 + 6$ biggest components of the total degree, namely the height and the 6 exponents of the negative root vectors F_β.

Thus we turn to comparing the remaining 6 exponents of the positive root vectors E_β. Now care must be taken since, by (2.4), these are ordered in reverse relative to the positive roots themselves. Therefore, the next component to compare is the exponent of $E_{\beta_{12}}$ because β_{12} is the smallest root. By (2.9), this is the same as the exponent of $t^{(1)}_{12}$ so we check if the permutations satisfy $\sigma(2) = 1$. None of them do, so we move on, checking $E_{\beta_{13}}$ which amounts to checking if $\sigma(3) = 1$. Here we finally get a discrepancy, (13)(24) satisfies this, but (1324) does not. Therefore (13)(24) is the permutation that gives the leading term in d_{42}.

Of course, $(13)(24) = (1234)^2$, so this proves Theorem 4.1 in the case $(r, s) = (4, 2)$.

The following notation will be used for a permutation $\sigma \in S_r$:

$$c_<(\sigma) = \#\{i \in [\![1, r]\!] \mid \sigma(i) < i\}, \qquad c_>(\sigma) = \#\{i \in [\![1, r]\!] \mid \sigma(i) > i\}.$$

The following lemma describes which nonzero terms appear in d_{rs}.

Lemma 4.3. *Let $s \in [\![1, r]\!]$ and let $\sigma \in S_r$. Then the following two statements are equivalent.*

(i) $t^{(k_1)}_{\sigma(1)1} t^{(k_2)}_{\sigma(2)2} \cdots t^{(k_r)}_{\sigma(r)r} \neq 0$ *for some* $k \in \{0, 1\}^r$ *with* $\sum_{i=1}^r k_i = s$;

(ii) $c_<(\sigma) \leq s$ *and* $c_>(\sigma) \leq r - s$.

Proof. This follows from the fact that $t^{(1)}_{ij} \neq 0$ iff $i \leq j$ and $t^{(0)}_{ij} \neq 0$ iff $i \geq j$. □

Define the *height* of a permutation $\sigma \in S_r$ by

$$\mathrm{ht}(\sigma) := \sum_{i=1}^{r} |\sigma(i) - i|. \tag{4.2}$$

The motivation for this terminology comes from the fact that

$$\mathrm{ht}(\sigma) = \mathrm{ht}(t^{(k_1)}_{\sigma(1)1} t^{(k_2)}_{\sigma(2)2} \cdots t^{(k_r)}_{\sigma(r)r}) \tag{4.3}$$

where the right hand side is given by (2.5) and the identification (2.9).

As the next step towards proving Theorem 4.1, we show that the permutation σ which gives the leading term of d_{rs} has to be a derangement (i.e. $\sigma(i) \neq i \ \forall i \in [\![1, r]\!]$).

Lemma 4.4. *Let $s \in [\![1, r]\!]$ and let $\sigma \in S_r$ be a permutation such that*

$$t_{\sigma(1)1}^{(k_1)} t_{\sigma(2)2}^{(k_2)} \cdots t_{\sigma(r)r}^{(k_r)} \neq 0$$

for some $k \in \{0,1\}^r$ with $\sum_i k_i = s$. Then there exists a $\widetilde{\sigma} \in S_r$ such that

(i) $t_{\widetilde{\sigma}(1)1}^{(l_1)} \cdots t_{\widetilde{\sigma}(r)r}^{(l_r)} \neq 0$ *for some $l \in \{0,1\}^r$ with $\sum_i l_i = s$;*
(ii) $t_{\widetilde{\sigma}(1)1}^{(l_1)} \cdots t_{\widetilde{\sigma}(r)r}^{(l_r)} \geq t_{\sigma(1)1}^{(k_1)} \cdots t_{\sigma(r)r}^{(k_r)}$;
(iii) $\widetilde{\sigma}$ *is a derangement.*

In particular, the permutation σ such that (4.1) *holds (for some $\lambda \in \mathbb{C}^\times$ and $k \in \{0,1\}^r$ with $\sum_i k_i = s$) must be a derangement.*

Proof. If σ already is a derangement, there is nothing to prove (take $\widetilde{\sigma} = \sigma$). So suppose $f := \#\{i \in S_r \mid \sigma(i) = i\} > 0$. It is enough to construct $\widetilde{\sigma}$ satisfying properties (i)-(ii) with $\#\{i \in S_r \mid \widetilde{\sigma}(i) = i\} = f - 1$ because then we can iterate this construction to arrive at a permutation satisfying all three conditions (i)-(iii).

We introduce some terminology. An element $(i_1, i_2) \in [\![1, r]\!]^2$ is called a *σ-drop* (respectively *σ-jump*) provided $\sigma(i_1) = i_2$ and $i_2 < i_1$ (respectively $i_2 > i_1$). As a visual support we will draw parts of permutations as graphs with vertices on a square lattice, vertices (a, b) and $(a+1, d)$ connected iff $\sigma(b) = d$. See Figure 1 for an example. Then drops and jumps are simply as in Figure 2.

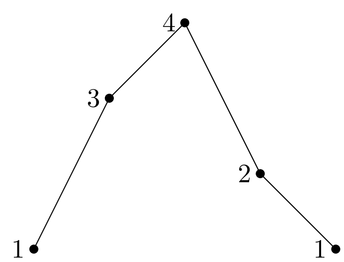


FIGURE 1. Pictorial representation of the cyclic permutation (1432).

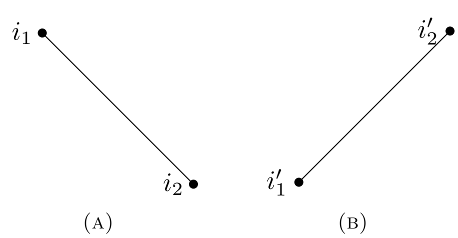


FIGURE 2. A σ-drop (A) and a σ-jump (B). The diagrams mean $i_2 = \sigma(i_1)$, $i_1 > i_2$ and $i'_2 = \sigma(i'_1)$, $i'_1 < i'_2$.

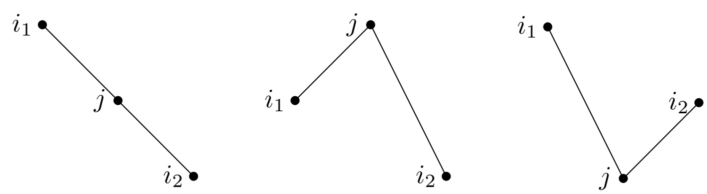
A σ-drop (i_1, i_2) will be called *drop-admissible* if we can "add another drop between i_1 and i_2", that is, if there exists $j \in [\![1, r]\!]$ with $\sigma(j) = j$ and $i_2 < j < i_1$. Then we can put $\widetilde{\sigma} = \sigma \circ (i_1\ j)$. With this $\widetilde{\sigma}$ we have

$$c_<(\widetilde{\sigma}) = c_<(\sigma) + 1, \qquad c_>(\widetilde{\sigma}) = c_>(\sigma).$$

Similarly, a σ-drop (i_1, i_2) is *jump-admissible* if there exists $j \in [\![1, r]\!]$ with $\sigma(j) = j$ and $j \notin [\![i_2, i_1]\!]$. Then $\widetilde{\sigma} = \sigma \circ (i_1\ j)$ satisfies

$$c_<(\widetilde{\sigma}) = c_<(\sigma), \qquad c_>(\widetilde{\sigma}) = c_>(\sigma) + 1.$$

See Figure 3 for an illustration of the possible scenarios in the case of a σ-drop.



(A) (B) (C)

FIGURE 3. The three possible ways the i_1, j, i_2 piece of $\widetilde{\sigma} = \sigma \circ (i_1\ j)$ can look like, when (i_1, i_2) is a σ-drop: $i_1 < j < i_2$ (A), $j > i_1, i_2$ (B), and $j < i_1, i_2$ (C). The σ-drop (i_1, i_2) is drop-admissible in case (A), and jump-admissible in (B) and (C).

Analogously, a σ-jump (i_1, i_2) is *jump-admissible* if $\exists j \in [\![1, r]\!]$ with $\sigma(j) = j$ and $i_1 < j < i_2$. A σ-jump (i_1, i_2) is *drop-admissible* if $\exists j \in [\![1, r]\!]$ with $\sigma(j) = j$ and $j \notin [\![i_1, i_2]\!]$.

We will now show that there always exists a jump-admissible σ-drop or σ-jump.

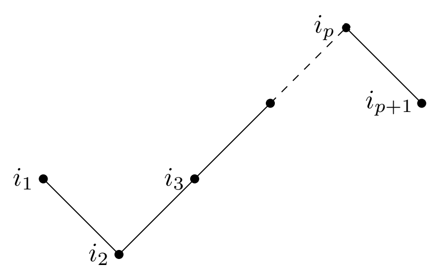


FIGURE 4. Illustration of a permutation σ satisfying conditions (a)-(d).

We know that σ is not the identity permutation since $\sum_i k_i = s \geq 1$. Thus there exists a tuple $(i_1, i_2, \ldots, i_p, i_{p+1}) \in [\![1, r]\!]^{p+1}$, where $p > 2$, such that (see Figure 4)

(a) $i_{j+1} = \sigma(i_j)$ for $j \in [\![1, p]\!]$;
(b) $i_1 > i_2$;
(c) $i_j < i_{j+1}$ for $j \in [\![2, p-1]\!]$;
(d) $i_p > i_{p+1}$.

Note that we do not exclude the possibility that $(i_p, i_{p+1}) = (i_1, i_2)$. Also, since σ is not a derangement, there is some $j \in [\![1, r]\!] \setminus \{i_1, \ldots, i_{p+1}\}$ fixed by σ.

If $j \notin [\![i_2, i_1]\!]$, then (i_1, i_2) is a jump-admissible σ-drop (as in case (B) or (C) in Figure 3). So suppose $i_1 > j > i_2$. If $j < i_p$ then (i_a, i_{a+1}) is a jump-admissible σ-jump for the $a \in [\![2, p-1]\!]$ with $i_a < p < i_{a+1}$. So suppose $j > i_p$. Then (i_p, i_{p+1}) is a jump-admissible σ-drop. This proves that, provided $\sigma(j) = j$ for some j, there always exists a jump-admissible σ-drop or σ-jump.

Similarly one proves there always exists a drop-admissible σ-drop or σ-jump.

If $c_<(\sigma) < s$ then we add a drop by putting $\widetilde{\sigma} = \sigma \circ (i\ j)$ where $(i, \sigma(i))$ is a drop-admissible σ-drop or σ-jump. Then $\widetilde{\sigma}$ will have one more drop than σ but the same number of jumps. That is, $c_<(\widetilde{\sigma}) = c_<(\sigma) + 1 \le s$ and $c_>(\widetilde{\sigma}) = c_>(\sigma) \le r - s$ which by Lemma 4.3 ensures that property (i) is satisfied.

Analogously, if instead $c_>(\sigma) < r - s$ we add a jump by putting $\widetilde{\sigma} = \sigma \circ (i\ j)$ for appropriate i.

Clearly $\widetilde{\sigma}$ has one less fixpoint than σ.

It remains to verify that property (ii) holds. The change from σ to $\widetilde{\sigma}$ has the following effect on monomials:

$$t_{jj}^{(k_j)} t_{\sigma(i)i}^{(k_i)} \longmapsto t_{\widetilde{\sigma}(j)j}^{(k_j)} t_{\widetilde{\sigma}(i)i}^{(k_i)} = t_{\sigma(i)j}^{(k_j)} t_{ji}^{(k_i)}$$

(unchanged factors omitted).

If j is not between i and $\sigma(i)$, then by definition of the height (4.2) one checks that $\mathrm{ht}(\widetilde{\sigma}) > \mathrm{ht}(\sigma)$ so (ii) holds by just looking at the height, which is the most significant part of the total degree (see (2.4)).

If j is between i and $\sigma(i)$, then $\mathrm{ht}(\widetilde{\sigma}) = \mathrm{ht}(\sigma)$ so we must compare roots in order to establish property (ii).

Suppose $i < j < \sigma(i)$. Then the change from σ to $\widetilde{\sigma}$ corresponds to

$$t_{\sigma(i)i}^{(0)} t_{jj}^{(k_j)} \longmapsto t_{\sigma(i)j}^{(0)} t_{ji}^{(0)}$$

The change in total degrees is

$$d(F_{\beta_{i,\sigma(i)}}) \longmapsto d(F_{\beta_{j,\sigma(i)}} F_{\beta_{ij}})$$

Since $\beta_{j,\sigma(i)} > \beta_{i,\sigma(i)}, \beta_{i,j}$ (recall the ordering (2.2)) it follows that property (ii) holds in this case. The case $i > j > \sigma(i)$ is analogous, keeping in mind that E_β are ordered in reverse. The proof is finished. □

The following result describes the height of the permutation giving rise to the leading term.

Lemma 4.5. *Fix $r \in \mathbb{Z}_{>0}$ and let $s \in [\![1, r]\!]$. Let $\sigma \in S_r$ be the permutation which gives rise to the leading term of d_{rs}. That is,*

$$\mathrm{lt}(d_{rs}) = \lambda t_{\sigma(1)1}^{(k_1)} t_{\sigma(2)2}^{(k_2)} \cdots t_{\sigma(r)r}^{(k_r)} \tag{4.4}$$

for some nonzero $\lambda \in \mathbb{C}$ and some $k \in \{0,1\}^r$ with $\sum_i k_i = s$. Then

$$\mathrm{ht}(\sigma) = 2s(r - s). \tag{4.5}$$

Proof. First we prove that $\mathrm{ht}(\sigma) \ge 2s(r-s)$. Let $\tau = (1\ 2\ \cdots\ r)^s$. We show that $\mathrm{ht}(\tau) = 2s(r-s)$. Since

$$\tau(i) = \begin{cases} i + s, & i + s \le r \\ i + s - r, & i + s > r \end{cases}$$

we have by definition of $\mathrm{ht}(\tau)$

$$\mathrm{ht}(\tau)=\sum_{i=1}^{r-s}(i+s-i)+\sum_{i=r-s+1}^{r}(i-(i+s-r))=2s(r-s).$$

Since (4.4) is the leading term of d_{rs}, we in particular have $\mathrm{ht}(\sigma)\geq \mathrm{ht}(\tau)=2s(r-s)$ by definition of total degree of a monomial (2.4).

It remains to show that $\mathrm{ht}(\sigma)\leq 2s(r-s)$. By Lemma 4.4, σ is a derangement. Thus

$$\mathrm{ht}(\sigma)=\sum_{i=1}^{r}|\sigma(i)-i|=\sum_{i:\,\sigma(i)<i}(i-\sigma(i))+\sum_{i:\,\sigma(i)>i}(\sigma(i)-i),$$

where the first sum has s terms and the second has $r-s$ terms. Clearly we have the estimate

$$\begin{aligned}\sum_{i:\,\sigma(i)<i}(i-\sigma(i))+\sum_{i:\,\sigma(i)>i}(\sigma(i)-i)&\\ \leq (r+(r-1)+\cdots+(r-s+1))-(1+2+\cdots+s)&\\ +(r+(r-1)+\cdots(s+1))-(1+2+\cdots+(r-s))=2s(r-s).&\end{aligned}$$

This proves the claim. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. The case $r=s$ is trivial: By (2.13), $d_{rr}=\lambda\cdot t_{11}^{(1)}\cdots t_{rr}^{(1)}$, where $\lambda\in\mathbb{C}^\times$. Thus d_{rr} has only one term, corresponding to the identity permutation (1). Thus the conjecture holds in this case because $(1\ 2\ \cdots\ r)^r=(1)$. So we may assume $s<r$.

Let $\sigma\in S_r$ be the permutation which gives rise to the leading term of d_{rs}. That is,

$$\mathrm{lt}(d_{rs})=\lambda t_{\sigma(1)1}^{(k_1)}t_{\sigma(2)2}^{(k_2)}\cdots t_{\sigma(r)r}^{(k_r)}$$

for some nonzero $\lambda\in\mathbb{C}$ and some $k\in\{0,1\}^r$ with $\sum_i k_i=s$. By Lemma 4.4, σ is a derangement. In particular, k is uniquely determined: $k_i=0$ iff $\sigma(i)>i$ and $k_i=1$ iff $\sigma(i)<i$. Moreover, since σ is a derangement, Lemma 4.3 implies that

$$s=\#\{i\in[\![1,r]\!]\mid \sigma(i)<i\}. \tag{4.6}$$

We will now show that

$$\sigma^{-1}(r)=r-s. \tag{4.7}$$

This is equivalent to that $t_{r,r-s}^{(0)}$ occurs in $\mathrm{lt}(d_{rs})$. By (2.9) and that the K_i don't contribute to the total degree, we have $d(t_{r,r-s}^{(0)})=d(F_{\beta_{r-s,r}})$. To show (4.7), note that $t_{r,r-s}^{(0)}$ occurs in the monomial corresponding to $\tau=(1\ 2\ \cdots\ r)^s$. Thus it is enough to prove that if $t_{ji}^{(0)}$ occurs in the leading monomial of d_{rs} then $\beta_{ij}\leq\beta_{r-s,r}$.

Suppose the opposite is true, i.e. that $\sigma^{-1}(j_0)=i_0\in[\![r-s+1,j_0-1]\!]$ for some j_0 with $i_0<j_0\leq r$. We show that this leads to a contradiction in the height of σ. We have

$$\mathrm{ht}(\sigma)=\sum_{i=1}^{r}|\sigma(i)-i|=\sum_{i:\,\sigma(i)<i}(i-\sigma(i))+\sum_{i:\,\sigma(i)>i}(\sigma(i)-i). \tag{4.8}$$

The first sum has s elements, by (4.6), and the second one has $r-s$ terms, since σ is a derangement. Since $\sigma(i_0)=j_0>i_0$, we may estimate the first sum from above by assuming that i runs through the s largest elements of $[\![1,r]\!]\setminus\{i_0\}$, and $\sigma(i)$ just runs through the s smallest elements of $[\![1,r]\!]$. That is,

$$\sum_{i:\sigma(i)<i}(i-\sigma(i))\leq(r+(r-1)+\cdots+(r-s)-i_0)-(1+2+\cdots+s)$$
$$=r-i_0+s(r-s-1). \quad (4.9)$$

On the other hand, i_0 does belong to the summation range of the other sum and therefore

$$\sum_{i:\,\sigma(i)>i}(\sigma(i)-i)\leq(r+(r-1)+\cdots+(s+1))-(1+2+\cdots+(r-s-1)+i_0)$$
$$=(r-s-1)s+r-i_0, \quad (4.10)$$

i.e. the sum of the $r-s$ largest elements of $[\![1,r]\!]$ minus the smallest sum of $r-s$ elements of $[\![1,r]\!]$ requiring that one of them is i_0. Combining (4.8)-(4.10) we obtain

$$\mathrm{ht}(\sigma)\leq 2(r-s-i_0)+2s(r-s)<2s(r-s) \quad (4.11)$$

since $i_0>r-s$ by assumption. This contradicts Lemma 4.5 and finishes the proof of (4.7).

Then, since $\beta_{r-s-1,r-1}$ is the largest positive root of the form $\beta_{r-s-1,j}$ where $j<r$, $\beta_{r-s-2,r-2}$ is the largest positive root of the form $\beta_{r-s-2,j}$ with $j<r-1$, and so on, we conclude that the leading term of d_{rs} must have the form

$$\lambda\cdot t^{(0)}_{1+s,1}t^{(0)}_{2+s,2}\cdots t^{(0)}_{r,r-s}\cdot t^{(k_1)}_{\sigma(r-s+1),r-s+1}\cdots t^{(k_s)}_{\sigma(r)r}.$$

But $\sum k_i=s$ which forces $k_i=1$ for $i\in[\![1,s]\!]$. So $\sigma(i)<i$ for $i\in[\![r-s+1,r]\!]$. Since $d(t^{(1)}_{ij})=d(E_{\beta_{ij}})$ for $i<j$ and by definition (2.4) of the total degree, the E_β are ordered in *reverse* with respect to the order of the positive roots β, we are led to the question: What is the smallest possible root β_{ij} $(i<j)$ which may still occur in the monomial?

We know that $\{\sigma(r-s+1),\sigma(r-s+2),\dots,\sigma(r)\}=\{1,2,\dots,s\}$. Thus, the smallest root we can get is $\beta_{1,r-s+1}$, obtained iff $\sigma(r-s+1)=1$. But this happens for the permutation $\tau=(1\ 2\ \cdots\ r)^s$. So, to have any chance of getting a larger monomial we must continue. But at each step we see that the smallest possible root is $\beta_{i,r-s+i}$ for $i=1,2,\dots,s$. This proves that $(1\ 2\ \cdots\ r)^s$ indeed is the permutation that gives the leading term of d_{rs}. □

Define

$$X(r,s)=t^{(1)}_{sr} \quad (4.12)$$

for each $1\leq s\leq r\leq N$. Then, by Theorem 4.1, $X(r,s)$ occurs in the leading term of d_{rs} and does not occur in the leading term of any other d_{ab}, $(a,b)\neq(r,s)$.

For $u\in U_q$ we let $\mathrm{lt}(u)\in\mathrm{gr}\,U_q$ denote the corresponding leading term.

Lemma 4.6. *Let $\gamma\in\Gamma_q$. Then*

$$\mathrm{lt}(\gamma)=\mathrm{lt}\big(\mu\prod_{1\leq s\leq r\leq N}d_{rs}^{k_{rs}}\big)$$

for some $\mu\in\mathbb{C}^\times$, $k_{rs}\in\mathbb{Z}_{\geq 0}$ $\forall s<r$ and $k_{rr}\in\mathbb{Z}$. Moreover k_{rs} is the number of occurrences of $X(r,s)$ in $\mathrm{lt}(\gamma)$.

Proof. By Lemma 2.5, Γ_q is a semi-Laurent polynomial algebra in the d_{rs}:

$$\Gamma_q \simeq \mathbb{C}[d_{rs} \mid 1 \le s \le r \le N][d_{rr}^{-1} \mid 1 \le r \le N].$$

The number of occurrences of $X(r,s)$ in $\prod_{r,s} \mathrm{lt}(d_{rs})^{k_{rs}}$ is equal to k_{rs}. Thus

$$\prod \mathrm{lt}(d_{rs})^{k_{rs}} = \prod \mathrm{lt}(d_{rs})^{l_{rs}} \Longrightarrow k_{rs} = l_{rs}\ \forall r,s.$$

This in turn implies that the set

$$\{\prod_{r,s} d_{rs}^{k_{rs}} \mid k_{rs} \in \mathbb{Z}_{\ge 0} \forall s < r,\ k_{rr} \in \mathbb{Z}\}$$

is totally ordered. Thus, for any $\gamma \in \Gamma_q$ we have $\mathrm{lt}(\gamma) = \mathrm{lt}(\lambda \prod d_{rs}^{k_{rs}})$ where k_{rs} equals the number of occurrences of $X(r,s)$ in $\mathrm{lt}(\gamma)$. This proves the claim. □

An algebra of the form

$$A(Q,m,n) = \mathbb{C}\langle a_1, \dots, a_m, a_{m+1}^{\pm 1}, \dots a_n^{\pm 1} \mid a_i a_j = Q_{ij} a_j a_i \forall i < j,$$
$$a_k a_k^{-1} = 1 = a_k^{-1} a_k,\ k > m\rangle$$

for some $Q_{ij} \in \mathbb{C}^\times$, will be called a *quantum semi-Laurent polynomial algebra.*

Now we can prove Theorem I from Introduction.

Theorem 4.7. *$U_q(\mathfrak{gl}_N)$ is a Galois order with respect to its Gelfand-Tsetlin subalgebra.*

Proof. Suppose $u\gamma = \gamma_1$ for some $u \in U, \gamma, \gamma_1 \in \Gamma \setminus \{0\}$. Consider the leading terms on both sides. Since $\mathrm{gr}\, U_q$ is a quantum semi-Laurent polynomial algebra (by Theorem 2.2), it is in particular a domain. So

$$\mathrm{lt}(u)\mathrm{lt}(\gamma) = \mathrm{lt}(u\gamma) = \mathrm{lt}(\gamma_1).$$

We count the number k_{rs} of occurrences of the distinguished variable $X(r,s)$ in $\mathrm{lt}(\gamma_1)$, for each r,s. Then we count the number l_{rs} of occurrences of $X(r,s)$ in $\mathrm{lt}(\gamma)$. Then we look at

$$\widetilde{u} = u - \lambda \prod_{1 \le s \le r \le N} d_{rs}^{k_{rs} - l_{rs}}, \tag{4.13}$$

where $\lambda \in \mathbb{C}^\times$ is to be determined. We have

$$\widetilde{u}\gamma = \gamma_1 - \lambda \prod_{r,s} d_{rs}^{k_{rs} - l_{rs}} \cdot \gamma.$$

By Lemma 4.6,

$$\mathrm{lt}(\gamma_1) = \mathrm{lt}\big(\mu \prod d_{rs}^{k_{rs}}\big), \qquad \mathrm{lt}(\gamma) = \mathrm{lt}\big(\xi \prod d_{rs}^{l_{rs}}\big)$$

for some $\mu, \xi \in \mathbb{C}^\times$. Thus

$$\mathrm{lt}\big(\lambda \prod d_{rs}^{k_{rs} - l_{rs}} \cdot \gamma\big) = \lambda \cdot \mathrm{lt}\big(\prod d_{rs}^{k_{rs} - l_{rs}}\big) \cdot \mathrm{lt}(\gamma) = \lambda\xi \prod d_{rs}^{k_{rs}} = \mathrm{lt}(\gamma_1)$$

provided we choose $\lambda = \mu/\xi$. Then $\mathrm{lt}(\widetilde{u}\gamma) < \mathrm{lt}(u\gamma)$. By induction we are reduced to the case when the total degree $d(u\gamma) = d(\gamma)$ which implies that $\mathrm{lt}(u)$, hence u has degree $(0,0,\dots,0) \in \mathbb{Z}_{\ge 0}^{2M+1}$, which means that $u \in \mathbb{C}[K_1^{\pm 1}, \dots, K_N^{\pm 1}] \subseteq \Gamma_q$. This completes the proof. □

5. Maximal commutativity of Gelfand-Tsetlin subalgebras

It is well known that the Gelfand-Tsetlin subalgebra of $U(\mathfrak{gl}_N)$ is maximal commutative (see for example [O2]). It is also known that the Gelfand-Tsetlin subalgebra is maximal commutative in $Y_p(\mathfrak{gl}_N)$ and in any finite W-algebra ([FMO2, Corollary 6.7]). It is natural to ask if the analogous statement holds for $U_q(\mathfrak{gl}_N)$. This was explicitly conjectured to be the case by Mazorchuk and Turowska in [MT]. Using Theorem 4.7, we can now prove this conjecture, establishing our second main theorem.

Theorem 5.1. *The Gelfand-Tsetlin subalgebra of $U_q(\mathfrak{gl}_N)$ is maximal commutative.*

Proof. By Theorem 2.4, $U_q(\mathfrak{gl}_N)$ is a Galois ring with respect to Γ_q. In that realization, $\mathcal{M}$ is a group. By Lemma 2.5, Γ_q is a finitely generated normal integral domain and by Proposition 2.8, Γ_q is a Harish-Chandra subalgebra. Thus, combining Theorem 4.7 and Proposition 3.3, it follows that Γ_q is a maximal commutative subalgebra of $U_q(\mathfrak{gl}_N)$. □

6. Application to Gelfand-Tsetlin characters

6.1. **Gelfand-Tsetlin modules over Galois orders.** We recall main results on the representations of Galois orders obtained in [FO2]. Let U be a Galois order over commutative noetherian subring Γ. All rings in this section are assumed to be algebras over an algebraically closed field.

Denote by $\operatorname{Specm}\Gamma$ the set of maximal ideals of Γ. A finitely generated module M over U is called a *Gelfand-Tsetlin module* with respect to Γ if

$$M = \bigoplus_{\mathbf{m}\in\operatorname{Specm}\Gamma} M(\mathbf{m}),$$

where

$$M(\mathbf{m}) \;=\; \{x \in M \mid \mathbf{m}^k x = 0 \,\text{for some}\; k \geqslant 0\}.$$

Given $\mathbf{m} \in \operatorname{Specm}\Gamma$, let $F(\mathbf{m})$ be the fiber of $\mathbf{m}$ consisting of isomorphism classes of irreducible Gelfand-Tsetlin U-modules M with respect to Γ such that $M(\mathbf{m}) \neq 0$. Equivalently, this is the set of left maximal ideals of U containing $\mathbf{m}$ (up to some equivalence). If M is such irreducible module with $M(\mathbf{m}) \neq 0$ then we say that a character $\mathbf{m}$ *extends* to M. If any $\mathbf{m}$ has a finite fiber then one can use $\operatorname{Specm}\Gamma$ to get a "rough" classification (up to some finiteness) of irreducible Gelfand-Tsetlin U-modules.

Let Λ be the integral extension of Γ such that $\Gamma = \Lambda^G$ and $\varphi : \operatorname{Specm}\Lambda \to \operatorname{Specm}\Gamma$. Then $\varphi^{-1}(\mathbf{m})$ is finite for any $\mathbf{m} \in \operatorname{Specm}\Gamma$. Fix any $l_{\mathbf{m}} \in \varphi^{-1}(\mathbf{m})$. Set

$$\operatorname{St}_{\mathcal{M}}(\mathbf{m}) = \{x \in \mathcal{M} | x \cdot l_{\mathbf{m}} = l_{\mathbf{m}}\}.$$

The set $\operatorname{St}_{\mathcal{M}}(\mathbf{m})$ does not depend on the choice of $l_{\mathbf{m}}$.

Theorem 6.1. (i) [FO2, Theorem A] *Let U be a Galois order over a finitely generated Γ, $\mathbf{m} \in \operatorname{Specm}\Gamma$. If the set $\operatorname{St}_{\mathcal{M}}(\mathbf{m})$ is finite, then the fiber $F(\mathbf{m})$ is non-trivial and finite.*

(ii) [FO2, Theorem B] *There exists a massive subset $X \subset \operatorname{Specm}\Gamma$ such that any $\mathbf{m} \in X$ extends uniquely to an irreducible Gelfand-Tsetlin module (up to an isomorphism).*

6.2. **Extension of characters for $U_q(\mathfrak{gl}_N)$.** For any $\mathbf{m} \in \operatorname{Specm}\Gamma_q$ the set $\mathrm{St}_{\mathcal{M}}(\mathbf{m})$ is finite. Since $U_q(\mathfrak{gl}_N)$ is a Galois order over the semi-Laurent polynomial Gelfand-Tsetlin subalgebra, then Theorem III follows immediately from Theorem 6.1. Hence, we obtain a classification of irreducible Gelfand-Tsetlin modules by the maximal ideals of Γ_q up to some finiteness which corresponds to the finite fibers of maximal ideals of Γ_q and up to some equivalence between maximal ideals (when they give isomorphic Gelfand-Tsetlin modules).

For a *generic* $\mathbf{m} \in X$ from some dense subset $X \subset \operatorname{Specm}\Gamma_q$, $\mathcal{M}$ acts freely on X and $\mathcal{M}\cdot\mathbf{m} \cap G\cdot\mathbf{m} = \{\mathbf{m}\}$. Therefore, if $U = U_q(\mathfrak{gl}_N)$, then $U/U\mathbf{m}$ is an irreducible $U_q(\mathfrak{gl}_N)$-module for any $\mathbf{m} \in X$.

6.3. **Cardinality of the fibers for $\mathfrak{gl}_2$.** We show that the conjecture about the size of the fibers from the introduction holds for $\mathfrak{gl}_2$.

It is easy to check that $U_q(\mathfrak{gl}_2)$ is isomorphic to the generalized Weyl algebra $R(\sigma, t)$ where $R = \mathbb{C}[K_1, K_1^{-1}, K_2, K_2^{-1}][t]$ where $\sigma(t) = t + (K_1K_2^{-1} - K_1^{-1}K_2)/(q - q^{-1})$, $\sigma(K_i) = q^{\delta_{i2}-\delta_{i1}}K_i$. Under this isomorphism, the Gelfand-Tsetlin subalgebra is identified with R. Since any generalized Weyl algebra is free over its distinguished subalgebra R, it follows that $U_q(\mathfrak{gl}_2)$ is free as a right (and left) module over the Gelfand-Tsetlin subalgebra. Now using [FO2, Theorem 5.2(iii)] and [FO2, Lemma 3.7], analogously to the proof of [FO2, Corollary 6.1], we obtain the desired bound from the conjecture in this case.

7. Acknowledgment

The authors are grateful to A.Molev for helpful comments. The first author is grateful to Max Planck Institute in Bonn for support and hospitality during his visit. The first author is supported in part by the CNPq grant (301743/2007-0) and by the Fapesp grant (2010/50347-9).

References

[B] Bavula V., Generalized Weyl algebras and their representations, Algebra i Analiz **4** (1992), 75–97. (English translation: St. Petersburg Math. J. **4** (1993), 71–92.

[BO] Bavula V., Oystaeyen F., Simple Modules of the Witten-Woronowicz algebra, Journal of Algebra **271** (2004), 827–845.

[DFO1] Drozd Yu.A., Ovsienko S.A., Futorny V.M. *On Gelfand–Zetlin modules*, Suppl. Rend. Circ. Mat. Palermo, **26** (1991), 143-147.

[DFO2] Drozd Yu.A., Ovsienko S.A., Futorny V.M., *Harish-Chandra subalgebras and Gelfand–Zetlin modules,* in: "Finite dimensional algebras and related topics", NATO ASI Ser. C., Math. and Phys. Sci., **424**, (1994), 79-93.

[DK] De Concini C., Kac V.G., *Representations of quantum groups at roots of 1* in "Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory" (Paris 1989), Birkhäuser, Boston, 1990, pp. 471-506.

[BG] Brown K.A., Goodearl K.R., *Lectures on algebraic quantum groups*, Advance course in Math. CRM Barcelona, vol 2., Birkhauser Verlag, Basel, 2002

[F] Fauquant-Millet F., *Quantification de la localisation de de Dixmier de $U(sl_{n+1}(\mathbb{C}))$*, J. Algebra **218** (1999), 93-116.

[FM] Fomenko T., Mischenko A., *Euler equation on finite-dimensional Lie groups*, Izv. Akad. Nauk SSSR, Ser. Mat. **42** (1978), 396-415.

[FH] Futorny V., Hartwig J.T., *Solution of a q-difference Noether problem and the quantum Gelfand-Kirillov conjecture for $\mathfrak{gl}_N$.*, arXiv:1111.6044v2 [math.RA].

[FMO1] Futorny V., Molev A. and Ovsienko S., *Harish-Chandra modules for Yangians*, Represent. Theory, **9** (2005), 426–454.

[FMO2] Futorny V., Molev A., Ovsienko S., *The Gelfand-Kirillov Conjecture and Gelfand-Tsetlin modules for finite W-algebras*, Advances in Mathematics, **223** (2010), 773-796.

[FO1] Futorny V., Ovsienko S., *Galois orders in skew monoid rings*, J.Algebra, **324** (2010), 598-630.

[FO2] Futorny V., Ovsienko S., *Fibers of characters in Harish-Chandra categories*, arXiv:math/0610071.

[G1] Graev M.I., *Infinite-dimensional representations of the Lie algebra* $gl(n,\mathbb{C})$ *related to complex analogs of the Gelfand-Tsetlin patterns and general hupergeometric functions on the Lie group* $GL(n,\mathbb{C})$, Acta Appl. Mathematicae **81** (2004), 93-120.

[G2] Graev, M.I., *A continuous analogue of Gelfand-Tsetlin schemes and a realization of the principal series of irreducible unitary representations of the group GL(n,C) in the space of functions on the manifold of these schemes*. Dokl. Akad. Nauk **412** (2007), no.2, 154-158.

[J] Jimbo M., *A q-analogue of* $U_q(\mathfrak{gl}(N+1))$*, Hecke algebra, and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), 247-252.

[KS] Klimyk A., Schmudgen K., *Quantum groups and their representations*, Springer-Verlag, Berlin Heidelberg, 1997.

[KW1] Kostant B., Wallach N.: *Gelfand-Zeitlin theory from the perspective of classical mechanics I.* In Studies in Lie Theory Dedicated to A. Joseph on his Sixtieth Birthday, Progress in Mathematics, Vol. **243**, (2006), 319-364.

[KW2] Kostant B., Wallach N.: *Gelfand-Zeitlin theory from the perspective of classical mechanics II.* In The Unity of Mathematics In Honor of the Ninetieth Birthday of I.M. Gelfand, Progress in Mathematics , Vol. **244**, (2006), 387–420.

[LS] Levendorskiĭ, Soibelman, *Algebras of Functions on Compact Quantum Groups, Schubert Cells and Quantum Tori*, Commun. Math. Phys. **139** (1991), 141-170.

[MH] Molev, A., Hopkins M., *A q-Analogue of the Centralizer Construction and Skew Representations of the Quantum Affine Algebra*, SIGMA, 2 (2006), 092, 29 pp.

[MT] Mazorchuk V., Turowska L., *On Gelfand-Zetlin modules over* $U_q(\mathfrak{gl}_n)$, Czechoslovak J. Physics, **50** (2000), 139-144.

[O1] Ovsienko S., *Strongly nilpotent matrices and Gelfand-Tsetlin modules*, Linear Algebra and Its Appl., **365** (2003), 349-367.

[O2] Ovsienko S., *Finiteness statements for Gelfand-Zetlin modules*, in: "Algebraic Structures and Their Applications", Inst. of Math. Acad.Sci. of Ukraine, (2002), 323-328.

[Vi] Vinberg E., *On certain commutative subalgebras of a universal enveloping algebra*, Math. USSR Izvestiya **36** (1991), 1-22.

Department of Mathematics, University of São Paulo, São Paulo, Brazil and Max Planck Institute for Mathematics, Bonn, Germany
E-mail address: `futorny@ime.usp.br`

Department of Mathematics, Stanford University, Stanford, CA, USA
E-mail address: `jonas.hartwig@gmail.com`