A NOTE ON EPHRAIM'S HOMOLOGY APPROACH OF THE ZARISKI MULTIPLICITY CONJECTURE

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Abstract. In 1971, Zariski [9] conjectured that if two complex hypersurface singularities have the same (embedded) topological type, then they have the same multiplicity. Partial positive answers have been given regularly but, in general, the conjecture is still open. We show here that the multiplicity is preserved under 'small' homeomorphisms. This is a consequence of an interpretation of the conjecture in terms of the first homology which was given by Ephraim in [3].

1. Introduction

Let $f\colon (U,0)\to(\mathbb{C},0)$ and $g\colon (W,0)\to(\mathbb{C},0)$ be two reduced holomorphic functions defined on open neighbourhoods U and W of the origin in $\mathbb{C}^n$ ($n\geq 2$). Let $V_f := f^{-1}(0)$ and $V_g := g^{-1}(0)$ be the corresponding hypersurfaces in $\mathbb{C}^n$, and ν_f, ν_g the multiplicities at 0 of V_f and V_g respectively. By definition, ν_f is the number of points of intersection near 0 of V_f with a generic (complex) line in $\mathbb{C}^n$ passing arbitrarily close to, but not through, the origin. As f is reduced, ν_f is also the order of f at 0, that is, the lowest degree in the power series expansion of f at 0. We suppose that f and g are topologically equivalent, that is, there exist open neighbourhoods $U'\subset U$ and $W'\subset W$ of the origin in $\mathbb{C}^n$ together with a homeomorphism $\varphi\colon W'\to U'$ such that $\varphi(V_g\cap W')=V_f\cap U'$. In fact, the question being local at 0, we may assume that $W'=W$ and $U'=U$.

Conjecture 1.1 (Zariski's multiplicity conjecture [9]). *Under the hypotheses described above, the multiplicities ν_f and ν_g are equal.*

In fact, to be precise, Zariski 'did not conjecture' but only 'asked' whether the topological equivalence between f and g implies the equality $\nu_f=\nu_g$. Nevertheless, it is common (and convenient) to call Zariski's original question the *Zariski multiplicity conjecture.*

Although partial positive answers have been given regularly, Zariski's multiplicity conjecture is, in general, still unsettled (for a list of the main known

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results we refer to our survey article [4]). One of the most significant and first partial results was given by Ephraim [3] and Trotman [8]. Namely, they proved that the multiplicity is preserved under C^1–diffeomorphisms. In this paper, we are interested in Ephraim's approach which is based on an interpretation of the conjecture in terms of the first homology. This point of view has interesting consequences which are also partial answers to Zariski's conjecture. Especially, we show that if the homeomorphism φ does not move too much the vertices of an appropriate finite simplicial complex which triangulates a small circle around the origin and contained in a generic (complex) line, then the multiplicities ν_f and ν_g are equal (cf. Theorems 3.3 and 3.4).

2. Zariski's conjecture via homology

2.1. **Homology interpretations of the multiplicity.** The following two interpretations of the multiplicity in terms of the first homology are due to Ephraim [3].

Let L be a (complex) line through the origin in $\mathbb{C}^n$ such that L is not contained in the tangent cone $C(V_f)$ of V_f at 0. Then 0 is an isolated point of $V_f \cap L$ and ν_f is equal to the order of $f_{|L,\mathbb{C}}\colon L \to \mathbb{C}$ at 0 (denoted by ord $f_{|L,\mathbb{C}}$).[1] So, if $D \subset L \cap U$ is a closed disc centered at 0 so small that $V_f \cap D = \{0\}$ and if γ is a generator of the first (integral) homology group $H_1(\dot{D})$ ($\dot{D}$ is the boundary of D), then, choosing an isomorphism $H_1(\mathbb{C}^*) \simeq \mathbb{Z}$ (we note $\mathbb{C}^* := \mathbb{C} - \{0\}$), we have

$$(f_{|\dot{D},\mathbb{C}^*})_*(\gamma) = \pm \text{ord } f_{|L,\mathbb{C}} = \pm \nu_f,$$

where $(f_{|\dot{D},\mathbb{C}^*})_*\colon H_1(\dot{D}) \to H_1(\mathbb{C}^*)$ is the homomorphism induced by $f_{|\dot{D},\mathbb{C}^*}$. The sign $\pm$ just depends on the choice of the isomorphism $H_1(\mathbb{C}^*) \simeq \mathbb{Z}$. This gives the first homology interpretation of the multiplicity.

A disc D as above will be called a *good disc for* f.

Now let B_ε (respectively $\bar{B}_\varepsilon$) be the open (respectively closed) ball in $\mathbb{C}^n$ with centre 0 and radius ε, and let S_ε be the boundary of $\bar{B}_\varepsilon$. The local conic structure lemma (cf. Milnor [5], Burghelea–Verona [1], Ephraim [3]), applied to V_f, says that for any sufficiently small $\varepsilon > 0$ there is a homeomorphism of pairs

$$(\bar{B}_\varepsilon, \bar{B}_\varepsilon \cap V_f) \simeq (\bar{B}_\varepsilon, c(S_\varepsilon \cap V_f)), \tag{2.1}$$

where $c(S_\varepsilon \cap V_f) := \{s\, z \in \bar{B}_\varepsilon\, ;\, z \in S_\varepsilon \cap V_f,\ s \in [0,1]\}$ is the cone over $S_\varepsilon \cap V_f$ (with vertex the origin).

Next, consider a positive number $\varepsilon_0 > 0$ for which (2.1) is satisfied for any $0 < \varepsilon \le \varepsilon_0$ and such that $B_{\varepsilon_0} \subset U$. Pick a good disc D for f which is contained

[1] As a general notation, if (X, A) and (Y, B) are pairs of topological spaces with $A \subset X$ and $B \subset Y$ and if $u\colon X \to Y$ is a continuous map such that $u(A) \subset B$, then we denote by $u_{|A,B}\colon A \to B$ the restriction of u to A (at the source) and B (at the target).

in some B_ε with $0 < \varepsilon < \varepsilon_0$. Denote by $j\colon \dot{D} \hookrightarrow B_\varepsilon \setminus V_f$ the inclusion map. By the first homology interpretation of ν_f, we have

$$(f_{|B_\varepsilon \setminus V_f, \mathbb{C}^*})_* \circ j_*(\gamma) = \pm\nu_f.$$

Now suppose f is *irreducible*. Then $H_1(B_\varepsilon \setminus V_f) \simeq \mathbb{Z}$ (cf. [3, Theorem 2.6]) and, by [3, Theorem 2.7], $(f_{|B_\varepsilon \setminus V_f, \mathbb{C}^*})_*\colon H_1(B_\varepsilon \setminus V_f) \to H_1(\mathbb{C}^*)$ is an isomorphism. It follows that, choosing an isomorphism $H_1(B_\varepsilon \setminus V_f) \simeq \mathbb{Z}$,

$$j_*(\gamma) = \pm\nu_f.$$

This gives the second homology interpretation of the multiplicity ν_f which is valid only in the case where the function f is irreducible.

2.2. **Homology formulation of Zariski's conjecture.** *From now on we suppose that f and g are irreducible.* The following homology formulation of Zariski's conjecture also appears in Ephraim's paper (cf. [3, p. 802]).

By shrinking ε_0, we can assume that (2.1) also holds, for any $0 < \varepsilon \leq \varepsilon_0$, with f replaced by g, and that $B_{\varepsilon_0} \subset W$. Choose ε and ε' satisfying $0 < \varepsilon' \leq \varepsilon < \varepsilon_0$ and so that $\bar{B}_\varepsilon \subset \varphi(B_{\varepsilon_0})$ and $\varphi(\bar{B}_{\varepsilon'}) \subset B_\varepsilon$, and consider a disc $D \subset B_{\varepsilon'} \subset B_\varepsilon$ which is good for both f and g.

Theorem 2.1 (Ephraim)**.** *Choose an isomorphism $H_1(B_\varepsilon \setminus V_f) \simeq \mathbb{Z}$. The multiplicities ν_f and ν_g are equal if and only if the embedding*

$$\varphi_{|\dot{D}, B_\varepsilon \setminus V_f}\colon \dot{D} \to B_\varepsilon \setminus V_f$$

(induced by the homeomorphism φ) and the inclusion map

$$j\colon \dot{D} \hookrightarrow B_\varepsilon \setminus V_f$$

induce (up to sign) the same homomorphism $H_1(\dot{D}) \to H_1(B_\varepsilon \setminus V_f)$.

Let us recall briefly the idea of the proof. Consider the inclusion map $l\colon \dot{D} \to B_{\varepsilon'} \setminus V_g$, and choose an isomorphism $H_1(B_{\varepsilon'} \setminus V_g) \simeq \mathbb{Z}$. By the second homology interpretation of the multiplicity, if γ is a generator of $H_1(\dot{D})$, then $l_*(\gamma) = \pm\nu_g$. On the other hand, by [3, Theorem 2.8], $(\varphi_{|B_{\varepsilon'} \setminus V_g, B_\varepsilon \setminus V_f})_*$ is an isomorphism. It follows that $(\varphi_{|\dot{D}, B_\varepsilon \setminus V_f})_*(\gamma) = (\varphi_{|B_{\varepsilon'} \setminus V_g, B_\varepsilon \setminus V_f})_* \circ l_*(\gamma) = \pm\nu_g$. But by the second homology interpretation of the multiplicity again we know that $j_*(\gamma) = \pm\nu_f$.

Remark 2.2. Theorem 2.1 still holds even if we consider different good discs for the functions f and g. More precisely, if $D_g \subset B_{\varepsilon'}$ and $D_f \subset B_\varepsilon$ are good discs (not necessarily contained in the same line) for g and f respectively, then we have: 'The multiplicities ν_f and ν_g coincide if and only if for any generators γ_g and γ_f of $H_1(\dot{D}_g)$ and $H_1(\dot{D}_f)$, respectively,

$$(\varphi_{|\dot{D}_g, B_\varepsilon \setminus V_f})_*(\gamma_g) = \pm j_f(\gamma_f),$$

where $j_f\colon \dot{D}_f \hookrightarrow B_\varepsilon \setminus V_f$ is the inclusion map'.

2.3. **The knots $\dot{D}$ and $\varphi(\dot{D})$.** Theorem 2.1 shows that if the knots $\dot{D}$ and $\varphi(\dot{D})$ are *cobordant* in the smooth manifold $B_\varepsilon \setminus V_f$, that is, if there is an embedding

$$\phi\colon \mathbb{S}^1 \times [0,1] \to (B_\varepsilon \setminus V_f) \times [0,1]$$

such that $\phi(\mathbb{S}^1 \times \{0\}) = \dot{D} \times \{0\}$ and $\phi(\mathbb{S}^1 \times \{1\}) = \varphi(\dot{D}) \times \{1\}$, then the multiplicities ν_f and ν_g are the same. Indeed, since ϕ is an embedding, the sets $\varphi(\dot{D}) \times \{1\}$ and $\dot{D} \times \{0\}$ are both strong deformation retracts of $\phi(\mathbb{S}^1 \times [0,1])$. Then the natural homomorphisms

$$H_1(\varphi(\dot{D}) \times \{1\}) \to H_1(\phi(\mathbb{S}^1 \times [0,1])) \quad \text{and} \quad H_1(\dot{D} \times \{0\}) \to H_1(\phi(\mathbb{S}^1 \times [0,1])),$$

induced by inclusion, are automorphisms of $\mathbb{Z}$. This implies that the maps

$$z \in \dot{D} \mapsto (\varphi(z), 1) \in \phi(\mathbb{S}^1 \times [0,1]) \quad \text{and} \quad z \in \dot{D} \mapsto (z, 0) \in \phi(\mathbb{S}^1 \times [0,1])$$

induce (up to sign) the same homomorphism between the first homology groups. It follows easily that $(\varphi_{|\dot{D}, B_\varepsilon \setminus V_f})_* = j_*$. Now apply Theorem 2.1.

Remark 2.3. In fact, if D_g and D_f are discs as in Remark 2.2 and if the knot $\varphi(\dot{D}_g)$ is cobordant to the knot $\dot{D}_f$ in the ambient space $B_\varepsilon \setminus V_f$, then again we have $\nu_f = \nu_g$. The same proof as above applies with Theorem 2.1 replaced with Remark 2.2.

Also it is easy to see that if the knots $\dot{D}$ and $\varphi(\dot{D})$ (respectively, and more generally, the knots $\dot{D}_f$ and $\varphi(\dot{D}_g)$ of Remarks 2.2 and 2.3) are *equivalent*, that is, if there exists a homeomorphism

$$\psi\colon B_\varepsilon \setminus V_f \to B_\varepsilon \setminus V_f$$

sending $\dot{D}$ onto $\varphi(\dot{D})$ (respectively sending $\dot{D}_f$ onto $\varphi(\dot{D}_g)$), then the multiplicities ν_f and ν_g are the same.

3. Zariski's conjecture and small homeomorphisms

We are still working under the hypotheses of §2.2.

In this section, we point out interesting consequences of Theorem 2.1 which are also partial answers to Zariski's conjecture. Namely, we show that if the homeomorphism φ, when restricted to a certain subset E, is small enough (i.e., if $\varphi_{|E}$ is sufficiently close to the inclusion map $E \hookrightarrow \mathbb{C}^n$), then the multiplicities ν_f and ν_g are the same. We do not require the homeomorphism φ to be close to the identity on the whole open set $W \subset \mathbb{C}^n$. The meaning of 'sufficiently close' differs according to the subset E we consider.

We may already notice the following first corollary.

Corollary 3.1. *If $|\varphi(z) - z| < \operatorname{dist}(\dot{D}, V_f)$, for any $z \in \dot{D}$, then the multiplicities ν_f and ν_g are equal. (Here, $\operatorname{dist}(\dot{D}, V_f)$ is the usual distance between $\dot{D}$ and V_f.)*

Indeed, by Theorem 2.1, we know that if the embeddings j and $\varphi_{|\dot{D},B_\varepsilon\setminus V_f}$ are homotopic, then the multiplicities ν_f and ν_g are equal. On the other hand, under the hypothesis of the corollary, it is clear that the image of the straight homotopy $(z,t)\in\dot{D}\times[0,1]\mapsto(1-t)\,z+t\,\varphi(z)$ from j to $\varphi_{|\dot{D},B_\varepsilon\setminus V_f}$ is contained in $B_\varepsilon\setminus V_f$.

Remark 3.2. Corollary 3.1 still holds if we replace $\operatorname{dist}(\dot{D},V_f)$ with $\operatorname{dist}(\varphi(\dot{D}),V_f)$.

In this corollary, the subset E where the homeomorphism φ is required to be not too far from the identity is equal to the circle $\dot{D}$ and the expression 'sufficiently close' has a precise meaning described in terms of the distance $\operatorname{dist}(\dot{D},V_f)$ (respectively $\operatorname{dist}(\varphi(\dot{D}),V_f)$). However, the most interesting case is when E is just a *finite* set. This is the object of Theorems 3.3 and 3.4 below in which the terms 'sufficiently close' have a different meaning. Informally, these theorems assert that if the homeomorphism φ does not move too much the vertices of an appropriate finite simplicial complex triangulating $\dot{D}$, then the multiplicities ν_f and ν_g are equal.

More precisely, let $\mathcal{O}=\{O_k\}$ be an open cover of $B_\varepsilon\setminus V_f$ such that any two $\mathcal{O}$–close continuous maps $u,v\colon\dot{D}\to B_\varepsilon\setminus V_f$ are always homotopic (we recall that u and v are said to be $\mathcal{O}$–close if, for any $z\in\dot{D}$, $u(z)$ and $v(z)$ belong to the same O_k). By Cauty [2, Lemme 1.4], such a cover always exists as $B_\varepsilon\setminus V_f$ is an absolute neighbourhood retract (stratifiable). Notice that if the inclusion map $j\colon\dot{D}\hookrightarrow B_\varepsilon\setminus V_f$ and the embedding $\varphi_{|\dot{D},B_\varepsilon\setminus V_f}$ are $\mathcal{O}$–close, then the multiplicities ν_f and ν_g are equal (this follows immediately from Theorem 2.1). Next, consider a triangulation $((K,L),\phi)$ of the smooth pair $(B_\varepsilon\setminus V_f,\dot{D})$, that is, a simplicial pair (K,L) together with a homeomorphism

$$\phi\colon(|K|,|L|)\to(B_\varepsilon\setminus V_f,\dot{D}).$$

As usual $(|K|,|L|)$ is the pair of underlying spaces (polytopes) of the simplicial pair (K,L). Such a triangulation always exists by Munkres [6, Problem 10.14]. Notice that L is a finite subcomplex (cf. Munkres [7, Lemma 2.5]). Hereafter we identify the pairs $(B_\varepsilon\setminus V_f,\dot{D})$ and $(|K|,|L|)$ by means of the homeomorphism ϕ. By subdividing we can assume that the simplicial pair (K,L) is so that, for any vertex v of L, there exists an open set $O_{k(v)}\in\mathcal{O}$ such that

$$\operatorname{st}(v,K)\subset O_{k(v)},\tag{3.1}$$

where $\operatorname{st}(v,K)$ is the closed star of v in K (cf. [7, Theorem 16.4]). Notice that $\operatorname{st}(v,K)$ is the underlying space of a subcomplex $\mathit{st}(v,K)$ of K which is *not* necessarily a *full* subcomplex of K. However if we consider the first barycentric subdivision (denoted by $\operatorname{sd}^1(\cdot)$), then $\operatorname{sd}^1(\mathit{st}(v,K))$ is always a full subcomplex of sd^1K (cf. [7, §70]). Of course $|\operatorname{sd}^1(\mathit{st}(v,K))|=|\mathit{st}(v,K)|=\operatorname{st}(v,K)$ as topological spaces (cf. [7, §15]). In addition, by [7, Theorem 16.1], there exists an integer m

such that the map $\varphi_{|\dot{D},B_\varepsilon\setminus V_f}$ has a simplicial approximation

$$\psi\colon \mathrm{sd}^m L \to \mathrm{sd}^1 K,$$

where $\mathrm{sd}^m L$ is the m–th iterated barycentric subdivision of L. Notice that $\mathrm{sd}^m L$ is also a finite complex.

Theorem 3.3. *Suppose that, for any vertex v of $\mathrm{sd}^m L$, one has $\varphi(v) \in \mathrm{st}(v, \mathrm{sd}^m K)$. Then the multiplicities ν_f and ν_g are equal. (Of course $\mathrm{sd}^m K$ is the m–th iterated barycentric subdivision of K.)*

Let v be a vertex of $\mathrm{sd}^m L$. If v is also a vertex of L, then $\mathrm{st}(v, \mathrm{sd}^m K) \subset \mathrm{st}(v, K)$. If v is not a vertex of L, let $\sigma(v)$ be the carrier of v in L, that is, the unique 1–simplex $\sigma(v)$ of L such that v belongs to $\mathrm{int}\,\sigma(v)$ (we denote by $\mathrm{int}\,\sigma(v)$ the interior of $\sigma(v)$). Let $w_1(v)$ and $w_2(v)$ be the vertices of $\sigma(v)$. By Munkres [7, Lemma 15.1], $\mathrm{st}(v, \mathrm{sd}^m K) \subset \mathrm{st}(w_i(v), K)$ $(1 \leq i \leq 2)$. Then Theorem 3.3 follows from the next result.

Theorem 3.4. *Suppose that, for any vertex v of $\mathrm{sd}^m L$, one has:*

$$\begin{cases} \varphi(v) \in \mathrm{st}(v, K) \text{ if } v \text{ is a vertex of } L, \\ \varphi(v) \in \mathrm{st}(w_1(v), K) \cup \mathrm{st}(w_2(v), K) \text{ otherwise,} \end{cases}$$

where $w_1(v)$ and $w_2(v)$ are the vertices of the carrier $\sigma(v)$ of v in L. Then the multiplicities ν_f and ν_g are equal.

Notice that, although we will prove that the simplicial map $\psi\colon \dot{D} \to B_\varepsilon \setminus V_f$ is $\mathcal{O}$–close to the inclusion map $j\colon \dot{D} \hookrightarrow B_\varepsilon \setminus V_f$, we do *not* assume in Theorems 3.3 and 3.4 that the embeddings $\varphi_{|\dot{D},B_\varepsilon\setminus V_f}$ and j are $\mathcal{O}$–close. Indeed, the hypotheses of these theorems do not imply *a priori* that, given a point $z \in \dot{D}$, its image $\varphi(z)$ belongs to the closed star in K of a vertex of $\mathrm{sd}^m L$. Hence the points z and $\varphi(z)$ do not necessarily belong to the same open set $O_k \in \mathcal{O}$.

Proof of Theorem 3.4. By Theorem 2.1, it suffices to show that $\varphi_{|\dot{D},B_\varepsilon\setminus V_f}$ and j are homotopic maps. In addition, by Munkres [7, Theorem 19.4], we know that $\varphi_{|\dot{D},B_\varepsilon\setminus V_f}$ is homotopic to ψ. Therefore it is enough to prove that ψ and j are homotopic. To do that it suffices to show that ψ and j are $\mathcal{O}$–close, that is, for any $z \in \dot{D}$ the points z and $\psi(z)$ belong to the same open set $O_k \in \mathcal{O}$.

First suppose that $z \in \dot{D}$ is a vertex v of both $\mathrm{sd}^m L$ and L. By (3.1), there exists an open set $O_{k(v)} \in \mathcal{O}$ such that

$$v \in \mathrm{st}(v, K) \subset O_{k(v)}.$$

By the hypothesis, $\varphi(v) \in \mathrm{st}(v, K) = |\mathrm{sd}^1(st(v, K))|$. In particular, there is an unique simplex s of $\mathrm{sd}^1 K$, contained in $\mathrm{st}(v, K)$, such that $\varphi(v) \in \mathrm{int}\, s$. Since ψ is a simplicial approximation of $\varphi_{|\dot{D},B_\varepsilon\setminus V_f}$, it follows that $\psi(v) \in s$ (cf. [7, Lemma 14.2]). Hence $\psi(v) \in O_{k(v)}$.

Now suppose that $z \in \dot{D}$ is a vertex v of $\mathrm{sd}^m L$ but not a vertex of L. Let $\sigma(v)$ be the carrier of v in L, and $w_1(v), w_2(v)$ its vertices. By (3.1), there exist two open sets $O_{k(w_1(v))}$ and $O_{k(w_2(v))}$ in $\mathcal{O}$ such that

$$\mathrm{st}(w_i(v), K) \subset O_{k(w_i(v))} \quad (1 \leq i \leq 2).$$

Combined with [7, Lemma 15.1], this gives:

$$v \in \mathrm{st}(v, \mathrm{sd}^m K) \subset \mathrm{st}(w_i(v), K) \subset O_{k(w_i(v))} \quad (1 \leq i \leq 2).$$

By the hypothesis, $\varphi(v)$ belongs to $\mathrm{st}(w_1(v), K) \cup \mathrm{st}(w_2(v), K)$ which is nothing but the underlying space of the full subcomplex

$$\mathrm{sd}^1(\boldsymbol{st}(w_1(v), K) \cup \boldsymbol{st}(w_2(v), K))$$

of $\mathrm{sd}^1 K$. In particular, there is an unique simplex s of $\mathrm{sd}^1 K$, contained in $\mathrm{st}(w_1(v), K) \cup \mathrm{st}(w_2(v), K)$, such that $\varphi(v) \in \mathrm{int}\, s$. Now, as above, since ψ is a simplicial approximation of $\varphi_{|\dot{D}, B_\varepsilon \setminus V_f}$, it follows that $\psi(v) \in s$. Hence

$$\psi(v) \in O_{k(w_1(v))} \cup O_{k(w_2(v))}.$$

Finally suppose that $z \in \dot{D}$ is not a vertex of $\mathrm{sd}^m L$. Let $\varsigma(z)$ be the carrier of z in $\mathrm{sd}^m L$, that is, the unique simplex $\varsigma(z)$ of $\mathrm{sd}^m L$ such that $z \in \mathrm{int}\, \varsigma(z)$. Let v_1, v_2 be the vertices of $\varsigma(z)$. There are three cases:

(1) v_1 and v_2 are vertices of L;
(2) v_1 is a vertex of L but v_2 is not a vertex of L;[2]
(3) v_1 and v_2 are not vertices of L.

In case (1), there exist open sets $O_{k(v_1)}$ and $O_{k(v_2)}$ in $\mathcal{O}$ such that

$$z \in \mathrm{st}(v_i, K) \subset O_{k(v_i)} \quad (1 \leq i \leq 2).$$

By the hypothesis, $\varphi(v_i) \in \mathrm{st}(v_i, K)$. As above, this implies $\psi(v_i) \in \mathrm{st}(v_i, K)$. In addition, the simpliciality of ψ also implies that $\psi(\varsigma(z))$ is a simplex of $\mathrm{sd}^1 K$. Since its vertices $\psi(v_i)$ are contained in the full subcomplex

$$\mathrm{sd}^1(\boldsymbol{st}(v_1, K) \cup \boldsymbol{st}(v_2, K)),$$

$\psi(\varsigma(z))$ is also a simplex of it. Then

$$\psi(\varsigma(z)) \subset |\mathrm{sd}^1(\boldsymbol{st}(v_1, K) \cup \boldsymbol{st}(v_2, K))| = \mathrm{st}(v_1, K) \cup \mathrm{st}(v_2, K).$$

Therefore the point $\psi(z)$ (which lies in $\psi(\varsigma(z))$) belongs to

$$O_{k(v_1)} \cup O_{k(v_2)}.$$

In case (2), consider the carrier $\sigma(v_2)$ of v_2 in L. Necessarily one of its vertices $w_1(v_2)$ or $w_2(v_2)$ is equal to v_1. Suppose for example $w_1(v_2) = v_1$. There exist

[2] The case where v_2 is a vertex of L and v_1 is not a vertex of L is similar.

open sets $O_{k(w_1(v_2))} = O_{k(v_1)}$ and $O_{k(w_2(v_2))}$ in $\mathcal{O}$ such that

$$\begin{cases} z \in \mathrm{st}(v_1, K) = \mathrm{st}(w_1(v_2), K) \subset O_{k(w_1(v_2))} \quad \text{and} \\ z \in \mathrm{st}(v_2, \mathrm{sd}^m K) \subset \mathrm{st}(w_i(v_2), K) \subset O_{k(w_i(v_2))} \quad (1 \leq i \leq 2). \end{cases}$$

By the hypothesis, $\varphi(v_1) = \varphi(w_1(v_2)) \in \mathrm{st}(w_1(v_2), K)$ and $\varphi(v_2) \in \mathrm{st}(w_1(v_2), K) \cup \mathrm{st}(w_2(v_2), K)$. The same argument as before shows that $\psi(v_1)$ and $\psi(v_2)$ are vertices of the full subcomplex $\mathrm{sd}^1(\mathit{st}(w_1(v_2), K) \cup \mathit{st}(w_2(v_2), K))$ of $\mathrm{sd}^1 K$, and $\psi(\varsigma(z))$ is a simplex of it, that is,

$$\psi(\varsigma(z)) \subset \mathrm{st}(w_1(v_2), K) \cup \mathrm{st}(w_2(v_2), K).$$

One deduces

$$\psi(z) \in O_{k(w_1(v_2))} \cup O_{k(w_2(v_2))}.$$

As for case (3), we have $\sigma(v_1) = \sigma(v_2) = \sigma(\varsigma(z))$, where $\sigma(v_i)$ (respectively $\sigma(\varsigma(z))$) is the carrier of v_i in L (respectively the carrier of $\varsigma(z)$ in L, that is, the unique simplex of L containing $\varsigma(z)$ in its interior). Let $w_1(\varsigma(z)), w_2(\varsigma(z))$ be the vertices of $\sigma(\varsigma(z))$. There exist open sets $O_{k(w_1(\varsigma(z)))}$ and $O_{k(w_2(\varsigma(z)))}$ in $\mathcal{O}$ such that

$$z \in \mathrm{st}(v_i, \mathrm{sd}^m K) \subset \mathrm{st}(w_{i'}(\varsigma(z)), K) \subset O_{k(w_{i'}(\varsigma(z)))} \quad (1 \leq i, i' \leq 2).$$

By the hypothesis, $\varphi(v_1)$ and $\varphi(v_2)$ belong to $\mathrm{st}(w_1(\varsigma(z)), K) \cup \mathrm{st}(w_2(\varsigma(z)), K)$. One concludes, by the same argument as above, that

$$\psi(z) \in O_{k(w_1(\varsigma(z)))} \cup O_{k(w_2(\varsigma(z)))}.$$

□

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