Application of Integral Geometry to Minimal Surfaces

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CONTENTS

§1. Introduction.

The theory of minimal higher dimensional surfaces, especially its main branch - the Plateau problem, has been intensively developed since in sixties, E.R.Reifenberg, H.Federer, W.H.Fleming, E.De Giorgi and F.Almgren proved existence and almost regularity theorems for solutions of the higher dimensional Plateau problem (or simply speaking, globally minimal surfaces) in different contexts of geometric measure theory. After that, the other part of the theory, namely, construction, classification and study of geometry of globally minimal surfaces has been developed rapidly . The first non trivial example of globally minimal surfaces was obtained by H.Federer by showing that every Kähler submanifold is a globally (homologically) minimal one in its ambient Kähler manifold [Fe 1]. His method of employing the exterior powers of the Kähler form in Kähler manifolds was been generalized for other Riemannian manifolds in the works of M.Berger, H.B.Lawson, Dao Trong Thi, R.Harvey and H.B.Lawson ([Be], [Ln 2], [D], [H-L]). Now, this method is called calibration method and it has various applications in the study of geometry of globally minimal surfaces as well as of (locally) minimal surfaces ([DGGW 1], [DGGW 2], [Le 1], [Le 2], [LM], [Lr],...). The other interesting examples of globally minimal surfaces was obtained by Fomenko [Fo 1, Le-Fo] using an estimate from below for the volume of globally minimal surfaces in Riemannian manifolds. His idea came from the Nevalinna one of using exhaustion function on algebraic manifolds. His method allows us to construct homological minimal submanifolds when the coefficient group of homologies may be as finite (Z_p) as infinite (Z, R). Note that the calibration method works only for homology groups with coefficients in R. But the Fomenko's method depending on an estimate which includes only the injective radius, riemannian curvature of ambient manifold and dimension of submanifolds, can not give us so much examples of globally minimal surfaces. To our knowledge, up to now, all non-trivial examples of globally minimal surfaces are obtained using the mentioned methods with exeption of some globally minimal hypersurfaces provided with a large symmetry group such that we can reduce the problem of higher dimension to dimension 2 which can be analysed completely. This reduction method was invented by H.B.Lawson [Ln 1].

This paper is an attempt to fill the gap between the calibration method and the Fomenko's method. A new method may be also called an analog of the calibration method for discrete coefficients of homology groups (of Riemannian manifolds). The idea is simple, it also comes from complex geometry. Let us recall the Crofton-type formula (which has originated in the Probability Theory [Sa]).

Theorem. [Ch, p.146] *Let $f : M \longrightarrow P_n(C)$ be a compact holomorphic curve with or without boundary. Then*

$$\int_{P_n} \#(f(M) \bigcap \gamma)\, d\gamma = Area(M) \tag{1.1}$$

where γ is a (complex) hyperplane of $P_n(C)$, and the space of these ones are identified with P_n equiped with the invariant measure, and $\#(X)$ denotes the number of the connected components of X.

A more detailed analysis show that if we replace a holomorphic curve M by any (real) two-dimensional surface M' then the equation (1.1) turns to an inequation, where the right part is greater than the left one. So, the Crofton-type formula gives us a new proof of homological minimality of CP^1 , and moreover, an estimate for measure of all (complex) hyperplanes meeting with a fixed holomorphic curve k times (see Equidistribution Theorem [Ch, p.146] and Theorem 4.1). In fact, some authors have used similar integral formulars in oder to estimate the volume of 2-dimensional analytical sets in C^n, but their formulars concern only the simplest case of dimension 1 (cf.[K-R] and references in that paper). Our idea is a natural generalization of the Crofton-type formula. Namely, we want to estimate the volume of submanifold $N \subset M$ by its intersection number $\#(N \bigcap N^*_\lambda)$ where N^*_λ is a family of submanifolds in M. Since the algebraic intersection number is a homology invariant we hope to get an estimate from below for the volume of submanifolds realizing a given cycle. The use of intersection number as a homology invariant explains the analogy between this method and the calibration method, which essentially employs another homology invariant - the Stokes formula. But in view of the Federer's stability theorem [Fe 2] the relation between these methods proves to be more intimate, in many cases, the effectiveness of one method leads to the effectiveness of the other one (see §4). Applying this intersection method we obtain some old and new examples of globally minimal submanifolds in symmetric spaces. In few cases this give us a classification theorem of globally minimal submanifolds in a certain class (see §3 and §4) and new properties of these ones such as equidistribution in measure of globally minimal surfaces. Other applications of integral geometry to minimal surfaces will appear in our next paper. This note also includes an appendix which presents a complete proof of the Fomenko's and the author's announcement [Le-Fo]. As it is mentioned above those results are closely related to these in the present note.

§2. General construction and examples.

Let us begin with simplest examples. Fisrt two examples are non-compact and compact fibrations over Riemannian manifolds.

Example 2.1. Let M^m be a Riemannian manifold and TM its tangent bundle. Let the Riamanian metric on M be naturally lifted on TM. Then M^m realizes a nontrivial cycle in the homology group $H_m(TM, Z_2)$ and moreover it has the minimal volume in its homology class $[M]$. In order to prove it we consider the compactification $\pi_x^{-1,c}$ of every fiber $\pi^{-1}(x), x \in M$ and denote the resulted bundle by TM^c. So if M' is another submanifold in TM and realizing a cycle $[M] \in H_*(TM, Z)$ then M' has to meet with every submanifold $\pi_x^{-1,c}, x \in M$. Consequently, the projection $\pi : M' \longrightarrow M$ is surjective. It easy to see that the projection π decreases the volume element (in any dimension not exceed $dim(M) = m$). Hence we get the assertion. This example intersesting because if M is not orientable then $H_m(TM, Z) = 0$ and the classical calibration method is not applicable!

Examples 2.2. Consider the group U_n equipped with the Killing metric. Then its subgroup S^1 of all diagonal elements is a homologically minimal submanifold. Indeed, U_n is a fibered space over $S^1 : g \rightarrow det(g)$ whose fibers are congruent with the subgroup SU_n. Moreover, these fibers meet S^1 perpendicularly at only one point. We can argue further as in the first example and conclude that S^1 is a homologically minimal submanifold in SU_n.

Now let us give a *general construction*, which generally does not depend on fibrations (such simple fibrations as the above examples are occured very rarely). Let us consider a Riemannian manifold M^m. Suppose we have a family $\bar{M}$ of n-dimensional submanifolds $N_y \subset M,\ y \in \bar{M}$. Suppose further that $\bar{M}$ is a manifold with an element volume μ_y. For every $X \subset M$ denote by $S_X \subset \bar{M}$ the set of all submanifolds N_y passing through the set X. Now we fixe a point $x \in M$ and a k-dimensional subspace $V_x^k \subset T_xM$. Denote by $B(x, V, r)$ the geodesic ball of radius r in M with its center at x and its tangent space at x equal V. Now define *a deformation coefficient* $cd(x, V)$ as follows:

$$cd(x, V) = \lim_{r \to 0} \frac{vol(S_{B(x,V,r)})}{vol(S_x) \cdot vol(B(x, V, r))} \tag{2.1}$$

Put $\sigma(\bar{M})_k = max\{cd(x, V^k) \mid x \in M, V^k \subset T_x M\}$.

Suppose that $\sigma(\bar{M})_k > 0$. Then we get the following theorem.

Theorem 2.1. *Let W is a compact k-dimensional submanifold in M. Then its volume can be estimate from below:*

$$vol(W) \geq \sigma(\bar{M})_k^{-1} \int_M \#(W \bigcap N_y)\, \mu_y \tag{2.2}$$

Proof. It is easy to find a finite triangulation W_i^ε on W that is $W = \bigcup_i W_i^\varepsilon; W_i \bigcap W_j = 0$ if $i \neq j$, and besides, for every i the number of connection components of W_i with any submanifold N_y is at most one. So we have:

$$vol(W) = \sum_i vol(W_i^\varepsilon) \tag{2.3}$$

$$\int_{\bar{M}} \#(W \bigcap N_y)\, dy = \sum_i \int_{\bar{M}} \#(W_i^\varepsilon \bigcap N_y)\, dy \tag{2.4}$$

With the help of (2.3) and (2.4) Theorem 2.1 can be proved if we show (2.2) for W_i^ε instead of W. Hence, in view of our assumption it suffices to prove:

$$vol(W^\varepsilon) \geq \sigma(\bar{M})_k^{-1} \int_{S_{W^\varepsilon}} \mu_y \tag{2.2.ε}$$

Letting $\varepsilon \to 0$ we get the infinitesmal version of (2.2.ε):

$$\lim_{r \to 0} \frac{vol\, B(x, T_x V, r)}{vol\, S_{B(x,TV,r)}} \geq \sigma(\bar{M})^{-1} \tag{2.2.0}$$

Obiviously, (2.2.0) follows from (2.1). By integration we obtain (2.2.ε). The proof is complete.

In the example 2.2 the set of subgroups which are congruent with SU_n is diffeomorphic to $\bar{M} = S^1$ and $\sigma(\bar{M})_1 = 1$. In the example 2.1 if we exhauste TM^m by compact bundles TM_R of tangent vectors of length R over M then we can also get the deformation coefficient $\sigma(\overline{TM})_m$ =1, here the set of tangent spaces $\overline{TM}$ is diffeomorphic to M.

Corollary 2.2. Lower bound of the volume of nontrivial cycles in Riemannian manifolds. *Suppose $N \subset M$ is a k-dimensional submanifold realizing a nontrivial cycle $[N] \in H_k(M^{n+k}, G)$, $G = Z$ or Z_2. Let $\bar{M}$ be a family of submanifolds N_χ^* realizing a nontrivial cycle $[N^*] \in H_n(M^{n+k}, G)$. Let χ be the (algebraic) inresection number of $[N]$ and $[N^*]$. Then we get:*

$$vol(N) \geq \chi \cdot \sigma(\bar{M})_k \cdot vol(\bar{M})$$

We note that Theorem 2.1 is still valid for a compact k-dimensional set W almost everywhere smooth exept singularities of codimension 1. On the other hand it is well known that homological volume-minimizing cycles are such ones [Fe 1]. So Corolary 2.2 yields the following criterion for global minimality.

Corollary 2.3. *Let $N \subset M$ be a k-cycle almost everywhere smooth exept singularities of codimension 1. Suppose that the inequality in Corollary 2.2 turns to an equality for N. Then N is a globally minimal cycle.*

In most of our applications we are interesting in cycles of compact homogeneous Riemannian spaces. We shall denote $(\cdot)$ the group multiplication or the action of a group on homogeneous spaces. Sometime we omit this denotation $(\cdot)$ if one can not misunderstand. Let $M = G/H$ where H is a compact group in a compact group G. Let K be another compact subgroup of G. Denote L the intersection of H and K. We consider the space $\bar{M}$ of all submanifolds $g \cdot K/L \subset G/H$ which are obtained from K/L by the left shift $g, g \in G$. Obviously, G transitively acts on $\bar{M}$ and its isotopy group is isomorphic to the subgroup K. Suppose $M = G/H$ and $\bar{M} = G/K$ are equipped with G-invariant metric. The condition under which submanifold $y \cdot k/L \subset M$ contains a point $x = (g \cdot H)/H \in M$ is the relation $y \in g \cdot H \cdot K$. So we have the following lemma.

Lemma 2.4. *Let $x = \{gH\} \in M = G/H$. Then the set $S_x \in \bar{M} = G/K$ is the submanifold gH/L.*

Let us denote lG the algebra Lie of the group G and so on. Before starting new Theorem we consider the orthogonal decomposition of the following algebras:

$$lG = lH \oplus lH^G = lK \oplus lK^G$$
$$lK = lL \oplus lL^K, \qquad lH = lL \oplus lL^H$$

We identify the tangent space to $M = G/H$ at the point $\{eH\}$ with the space lH^G. So the tangent space to its subspace K/L at the same point can be identified with the orthogonal projection of lL^K on LH^G. For the sake of simplicity we assume that H/L and K/L are totally geodesic submanifolds in G/K and G/H respectively. This means that, $lL^K \subset lH^G$ and $lL^H \subset lK^G$. Denote T the orthogonal complement to $lL \oplus lL^H \oplus lL^K$ in lG. Our purpose now is to compute the deformation coefficient $cd(x, V)$ for $x \in M$. Without loss of generality we can assume that $x = \{eH\} = e$ and then $V \subset lH^G = T_e M$.

Proposition 2.5. *Let $k = codim(K/L)$. Then the k-dimensional deformation coefficient $cd(e, V^k)$ depends only on the H-action orbit passing through the k-dimensional subspace V^k on the space $\bigwedge^k(lH^G)$.*

Proof. Let us denote by exp the exponential map from algebra Lie onto group Lie and by Exp the exponential from tangent space to Riemannian manifold.

According to Lemma 2.4 we have $S_{B(x,V,r)} = expV(r) \cdot H/L$ since in this case $\{(exp\, v)L\}/L = Exp\, v$ for $v \in lL^H$. Hence we get:

$$cd(\{eH\}, V) = \lim_{r \to 0} \frac{vol(expV(r) \cdot H/L)}{vol(H/L) \cdot vol(ExpV(r))} \tag{2.5}$$

We choose an orthonormal basis of vector $\{v_i\}$ in V. Fix a point $\{xL\} \in H/L$ where $x \in H \subset G$. The tangent space to $expV(r) \cdot H/L$ at point x is the sum of the tangent space $T_x(H/L)$ and $T_x(expV(r) \cdot x)$. Consider the map

$$\rho : V \longrightarrow expV(r) \cdot x; \qquad v \to exp\, v \cdot x$$

Its differential $d\rho$ sends the vector v_i to the projection of vector $d/dt_{|t=0} exp\, tv_i \cdot x \in T_xG$ on the tangent space $T_{\{xK\}}G/K$ at the point $\{xH\}$ since G/K is the quotient space of the right K-action on G . Denote $\hat{v}_i(x)$ the resulted vectors. Then we have $T_x(expV(r) \cdot x) = span(\hat{v}_i, i = \overline{1,n})$. So (2.5) can be rewritten as follows :

$$cd(\{eH\}, V) = (vol\, H/L)^{-1} \int_{H/L} vol(\overline{T_x(H/L)} \wedge \hat{V}_x)\, \mu_x \tag{2.6}$$

where $\overline{T_x(H/L)}$ denotes the normed (N-k)-vector associated with $T_x(H/L)$, N-k = dim(H/L), and $\hat{V}_x = \hat{v}_1 \wedge ... \wedge \hat{v}_k$. Our next aim is to compute $\hat{v}_i$. Let us choose an orthonormal basis $f_1 ..., f_N$ of the space $lK^G = T_{\{eK\}}G/K$. The shift $L_x : G/K \longrightarrow G/K$; $\{gK\} \to x \cdot \{gK\}$ sends vector f_i to the vector $f_i(x)$. Obviously, $f_i(x)$ is an orthogonal basis of the tangent space $T_{\{xK\}}G/K$.

Lemma 2.6. *The following relation holds for every i,j:*

$$< \hat{v}_i(x), f_j(x) > = < v_i, Ad_x f_j > \tag{2.7}$$

where $<,>$ *in the right part denotes the Killing metric in the algebra* lG.

Proof.(cf. Lemma 1.1 in [Le 3].) It is easy to see that the left part of (2.7) equals $< x_*^{-1}\hat{v}_i(x), f_j >$. By the definition of $\hat{v}_i$ and taking into account the Ad_x-invariance of the Killing metric we immediately get (2.7).

Let us continue the proof of Proposition 2.5. From (2.7) we obtain that $\hat{v}_i(x)$ is orthogonal to the tangent space at $\{xH\}$ to H/L. This also means that, the polyvector $x_*^{-1}\hat{V}_x$ collinear to the polyvector $\bar{T}$ associated with the space $T \subset lG$. So (2.6) and (2.7) yield

$$cd(\{eH\}, V) = \int_{H/L} | < V, Ad_{\bar{x}}(\bar{T} >)|\, dx \tag{2.8}$$

where $\tilde{x}$ denotes any representative of $\{xL\}$ in H. Clearly, the space T is invariant under the action Ad_L , so $Ad_L\bar{T} = \bar{T}$ and the value under the integral in the part of (2.8) does depend only on x. Now (2.8) yields Proposition 2.4 immediately.

Example 2.3. Let $M = S^n = SO_{n+1}/SO_n$ and $\bar{M} = SO_{n+1}/SO_{k+1}$ the set of great (totally geodesic) k-dimensional spheres in S^n. Here $H = SO_n$ acts on the Grassmanian $G_{n-k}(T_eM) \cong SO_n/S(O_k \times O_{n-k}$ transitively. This means that $cd(x, V)$ is a constant ζ_{n-k}. Taking into account (2.2.ε), (2.2.0) (which turn to equalities in this case) and (2.3), (2.4) we get:

Proposition 2.7 [Sa]. *Let V^{n-k} be a submanifold in S^n. Then its volume can be computed from the following formula:*

$$vol(V) = \zeta_{n-k} \cdot \int_{SO_{n+1}/SO_{k+1}} \#(V \bigcap S^k(x))\, \mu_x \tag{2.9}$$

where $\zeta_{n-k} = 1/2\, vol(S^{n-k}) \cdot vol(SO_{n+1}/SO_{k+1})^{-1}$.

The same formula holds for a submanifold $V \subset RP^n$ but we should replace S^k by RP^k. Further, we note that any projective space RP^k meets almost all projective spaces of complementary dimension at one point (cf. Proposition 3.5). Hence in view of Corollary 2.3 we obtain:

Proposition 2.8. *The projective space RP^k has the minimal volume in its homology class $[RP^k] \in H_k(RP^n, Z_2) = Z_2$.*

This proposition was obtain by Fomenko [Fo 1] using a different method of geodesic defects.

Example 2.4. Let $M = CP^n = U_{n+1}/(U_n \times U_1)$. Then $T_eCP^n = C^n = R^{2n}$ and $H = U_n \times U_1$ does not acts on $G_k(R^{2n})$ transitively. But H acts on the complex Grassmannian $G_k(C^n)$ transitively and H also acts on the Lagrangian Grassmannian $GL(C^n) = U_n/O_n$ transitively. Hence we get:

Proposition 2.9. *a) Crofton type formula. Let V^{2k} be a complex manifold in CP^n. Then its volume can be computed from the following formula:*

$$vol(V^{2k}) = \zeta_k^C \cdot \int_{U_{n+1}/U_{n-k+1}} \#(V \bigcap CP^{n-k}(x))\, \mu_x \tag{2.10}$$

where the constant ζ_k^C does not depend on V.

b) Let V^n be a Lagrangian manifold in CP^n. Then its volume can be computed from the following formula:

$$vol(V^n) = \zeta_n^L \cdot \int_{U_{n+1}/O_{n+1}} \#(V \bigcap RP^n(x))\, \mu_x \tag{2.11}$$

where the constant ζ_n^L does not depend on V.

§3. Minimal cycles in Grassmanian manifolds.

We denote G_kR^n the Grassmanian of unoriented k-planes through the origin in R^n and its 2-sheeted covering by $G_k^+R^n$. We denote G_kC^n and G_kH^n the complex Grassmannian and the quaternionic Grassmannian correspondingly. The question of finding out and classification of globally minimal cycles in Grassmannian manifolds has attracted attention of many mathematicians. The first non-trivial result was obtained by A.T.Fomenko in 1972 using his method of geodesic defects [Fo 1, Le-Fo] and by M.Berger in the same year using calibrations method [Be]. In particular Fomenko proved that that the standardly embedded real projective space $RP^l \longrightarrow G_kR^n$, $l \leq n$, is globally minimal cycle, and Berger proved that HP^k is homologically volume-minimyizing in HP^n if $k \leq n$. Recently H.Gluck, F.Morgan and F.Ziller exploying Euler forms and theirs "adjusted powers" as calibration proved that *if* $k =$ *even* ≥ 4, *then each*

$$G_1^+R^{k+1} \subset G_2^+R^{k+2} \subset ... \subset G_l^+R^{k+l}$$

is uniquely volume minimizing in its homolgy class. [G-M-Z]. Tasaki showed that the same proof implies G_kH^{m+k} is uniquely volume minimizing in its homology class in G_nH^{m+n} for all m , even and odd [T]. In this section using our method we prove:

Theorem 3.1. *The standardly embedded Grassmannian submanifold $G_k(R^{k+m})$ in $G_l(R^{l+m})$, $k \geq l$, has the minimal volume in its homology class of homology group $H_*(G_l(R^{l+m}), Z_2)$.*

We will show in §4 that this theorem implies the mentioned G-M-Z Theorem. But the G-M-Z Theorem implies our Theorem only in the case of m = event because when m = odd, each $G_k^+R^{k+m}$ bounds over the reals in $G_l^+R^{l+m}$.

Theorem 3.2. *Classification Theorem. Let M be a volume-minimizing cycle of non-trivial homology class $[G_k(R^{m+k})] \in H_*(G_l(R^{m+l}), G)$ where $G = Z$ or Z_2 respectively. Then M must be one of these subgrassmannians.*

Proof of Theorem 3.1. We consider the family $\bar{M} = SO_{l+m}/SO_{l-k+m}$ of homogeneous subspaces obtained from $G_{l-k}(R^{l-k+m})$ by the action of the group $SO(R^{l+m})$ (see §2). According to Lemma 2.3 and formula (2.8) we get:

$$cd(\{eS(O_l \times O_m)\}, V) = \int_{S(O_l \times O_m)/S(O_k \times O_m)} | < V, Ad_{\bar{x}}T > | \mu_x$$

$$= \int_{SO_l/SO_k} | < V, Ad_{\bar{x}}T > | \mu_x \tag{3.1}$$

where T denotes the normed polyvector associated with the tangent space to the subspace $G_k(R^{k+m})$ at e.

Clearly, the group SO_l acts on the tangent space $T_eG_l(R^{l+m}) = R^l \otimes R^m$ as the sum of m irreducible representations π_1 of dimension l. Namely, in the matrix representation of $T_eG_l(R^{l+m}) \longrightarrow so_{l+m}$ these irreducible spaces can be chosen as m columns. Further, we can choose an orthogonal basis $v_1^i, ..., v_m^i$, $i = \overline{1,k}$ in T such that v_j^i belongs to j^{th}-column $R^l \otimes v_j$. Denote T_j the polyvector $v_j^1 \wedge ... \wedge v_j^k$. Obviously, we have $T = T_1 \wedge ... \wedge T_m$. So, $Ad_{\bar{x}}T = Ad_{\bar{x}}T_1 \wedge ... \wedge Ad_{\bar{x}}T_m$. Observe that the polyvector $Ad_{\bar{x}}T_j$ belongs to the j^{th}-column R_j^k. Straightforward calculation shows that :

$$| < V, Ad_{\bar{x}}T > | = | \prod_{i=1}^{m} < V_i, Ad_{\bar{x}}T_i > | \tag{3.2}$$

where V_i is the orthogonal projection of V in i^{th}-column.

We need the following proposition.

Proposition 3.2. *Let $f_i(x) \geq 0$. Then the following inequality holds:*

$$(\int_M f_1(x) \cdot ... \cdot f_m(x)\, \mu_x)^m \leq \prod_{i=1}^{m} \int_M (f_i(x))^m \mu_x$$

Moreover, the inequality turns to an equality if and only if $f_j(x)/A_j = f_1(x)/A_1$ for every $j \geq 2$ and $x \in M$ and here $A_i = \int_M (f_i)^m$.

Proof. The above inequality probably is well known. For the sake of completeness we give here a proof. Consider the polynom $F(x_1, ..., x_{m-1}) = a_i x_i^m - b x_1 \cdot ... \cdot x_{m-1} + a_m$. Here $x_i \geq 0$ and the coefficients a_i, b are positive. We want to find a relation $R(a_i, b)$ which would be a necessary and sufficient condition for non-negativity of the polynom F. Suppose that F get its minimum at x_i^0. Let $x_1^0, x_2^0, ..., x_k^0$ be positive and the rest be zeros. So we get for every $i = \overline{1,k}$ the equation:

$$(\partial F/\partial x_k)_{|x^0} = m\, a_k (x_k^0)^{m-1} - b x_1^0 \cdot x_2^0 \cdot ...(\bar{k})... \cdot x_{m-1}^0 = 0 \tag{3.3}$$

where $x^0 = (x_1^0, ..., x_{m-1}^0)$. Taking into account $x_1^0 = 0$ from (3.3) we obtain that $x_k^0 = 0$ for every k. This implies that either F get its minimum at $0, ...0$ or F get its minimum at $x^0 \in (R^+)^{m-1}$ for which the equation (3.3) holds. The last one is equivalent to:

$$x_i^0 = (\frac{b}{m a_i})^{1/(m-1)} \prod_{k \neq i} (x_k^0)^{m-1} \tag{3.4.i}$$

Hence we obtain:

$$\prod_{i=1}^{m-1} (x_i^0)^{1/(m-1)} = \frac{b}{n} \prod_{i=1}^{m-1} (a_i)^{-1/(n-1)} \tag{3.5}$$

The equations (3.4.i) and (3.5) yield:

$$(x_i^0)^{m/(m-1)} = (\frac{b}{n})^{m/(m-1)} \prod_{k=1}^{m-1} (a_k)^{-1/(m-1)} \cdot (a_i)^{-1/(m-1)} \tag{3.6}$$

Note that the equation (3.6) has only one solution in $(R^+ \cap \{0\})^{m-1}$. The value of F at this solution is

$$F(x_1^0, ..., x_{m-1}^0) = -(m-1)^2 \cdot (\frac{b}{m})^m \cdot \prod_{i=1}^{m-1} a_i^{-1} + a_m \tag{3.7}$$

Lemma 3.3. *The polynom F takes non-negative values if and only if*

$$(m-1)^2 \cdot b^m \leq m^m \cdot \prod_{i=1}^{m} a_i$$

Proof. Observe that if one of variables x_i is zero or tends to infinity then the polynom F takes only positive values. So F is non-negative if and only if its value at the critical point $(x_1^0, ..., x_{m-1}^0)$ is non-negative. Now Lemma 3.3 immediately follows from (3.7).

Let us continue the proof of Proposition 3.2. Obviously it suffices to prove the inequality for $f_i(x) > 0$. We correspond every $x \in M$ the function

$$F_x(z_1, ..., z_{m-1}) = \sum_{i=1}^{m-1} f_i(x)\,(z_i)^m - \alpha(\prod_{i=1}^{m}(f_i(x))^{1/m}) z_1 \cdot z_2 \cdot ... \cdot z_m + f_m(x)$$

where $\alpha = m \cdot (m-1)^{-2/m}$. According to Lemma 3.3 the polynom F_x takes non-negative values on $(R^+)^{m-1}$. Hence the polynom

$$F_M(z_1, ..., z_{m-1}) = \int_M F_x(z_1, ..., z_{m-1})\, \mu_x$$

also takes non-negative values. Applying Lemma 3.3 again we get the inequality in Proposition 3.2. Observe that the inequality turns to an equality if and only if the critical point of F_M is also the critical point of F_x for every $x \in M$. This yields the last assertion of Propostion 3.2.

Let us continue the proof of Theorem 3.1. Applying Proposition 3.2 to (3.2) we get:

$$\int_{SO_l/SO_k} | < V, Ad_{\bar{x}} T > | \mu_x \leq (\prod_{i=1}^{m} \int_{SO_l/SO_k} | < V_i, Ad_{\bar{x}} T_i >^m |)^{1/m} \tag{3.8}$$

Observe that the group SO_l acts transitively on the Grassmannian of k-planes in i-column $R^l \otimes v_j$. Consequently we get

$$\int_{SO_l/SO_k} | < V_i, Ad_{\tilde{x}}T_i >^m | \mu_x = (\|V_i\|)^m \int_{SO_l/SO_k} (| < T_i, Ad_{\tilde{x}}T_i > |)^m \mu_x \tag{3.9}$$

From (3.1) (3.8) and (3.9) we imply:

$$cd(e,V) \leq \prod_{i=1}^{m} \|V_i\| \cdot \int_{SO_l/SO_k} | < T_1, Ad_{\tilde{x}}T_1 > | \mu_x \tag{3.10}$$

Obviously $\|V_i\| \leq 1$ and the equality holds if and only if $V = V_1 \wedge \ldots \wedge V_m$. Once again applying Proposition 3.2 to (3.8) we obtain

Proposition 3.4. *The deformation coefficient $cd(e,V)$ get its maximum at V_0 if and only if there exists $x \in SO_l$ such that $V_0 = Ad_x T$.*

Now we study the intersection between Grassmannian submanifolds in $G_l(R^{l+m})$.

Proposition 3.5. *For almost (in dimension sense) $y \in \bar{M} = SO_{l+m}/SO_{l-k+m}$ space $N_y = \tilde{y}(G_{l-k}(R^{l+m})$ meets $G_k(R^{k+m})$ at only one point.*

Proof. Geometrically, the embedding $G_k(R^{k+m}) \longrightarrow G_l(R^{l+m})$ can be described as follows:

$$G_k(R^{k+m}) \ni x \to x \wedge v_{l-k} \in G_l(R^{l+m})$$

where v_{l-k} denotes the subspace orthogonal to R^k in R^l. So the intersection $I(y)$ of the considered Grassmanians consists of those l-subspaces W_l such that:

$$W_l \in (G_k(R^{k+m}) \wedge v_{l-k}) \bigcap (G_{l-k}(\tilde{y} \cdot R^{l-k+m}) \wedge \tilde{y} \cdot v_k) \tag{3.11}$$

Clearly, the following lemmas yield Proposition 3.5.

Lemma 3.6. *The set of all elements $y \in \bar{M}$ such that the dimension of $\tilde{y} \cdot R^k \bigcap R^{l-k}$ is greater or equal 1 has codimension 1.*

Lemma 3.7. *If $\tilde{y} \cdot R^k \bigcap R^{l-k}$ contains only the origin in R^{l+m} then $I(y)$ contains only one element.*

Proof of Lemma 3.6. It suffices to prove that the set of $\tilde{y} \in SO_{l+m}$ of the above property has codimension greater or equal 1 in SO_{l+m}. Let $\tilde{y}$ belong this set, then its entries (we consider $\tilde{y}$ as a matrix) satisfy the equation:

$$vol(\tilde{y} \cdot v_k \wedge v_{l-k}) = 0 \tag{3.12}$$

The solution to (3.12) is an algebraic hypersurface in SO_{l+m}. This completes the proof.

Proof of Lemma 3.7. Let $W_l \in I(y)$. According to (3.11) W_l contains both R^{l-k} and $y \cdot R^k$. By the assumption W_l must be their spanning. This yields the assertion.

Let us complete the proof of Theorem 3.1. Suppose V is a submanifold in the same homology class of $G_k(R^{k+m})$. So V meets every submanifold $N_y = \tilde{y} \cdot G_{l-k}(R^{l-k+m})$ at least one time. With the help of Theorem 2.1, Proposition 3.4, Proposition 3.5 we immediately get

$$vol(V) \geq vol(G_k(R^{k+m}))$$

that completes the proof.

Proof of Theorem 3.2. Let N be a volume-minimizing cycle in the homology class $[G_k R^{m+k}]$. First we observe that N is almost everywhere smooth and then we can apply Corollary 2.2 to N. On the other hand , since $G_k R^{m+k}$ satisfies the condition in Corollary 2.3, we conclude that the cycle N satisfies this condition, too. In particular, we obtain that for almost all $x \in N$ (in dimension sense) the tangent space $T_x N$ to N satisfies the condition of maximal deformation coefficient : $cd(x, T_x N) = \sigma(\bar{M})$. In view of Proposition 3.4 we obtain that the tangent space $T_x N$ is tangent to some subgrassmanian $g \cdot G_k R^{k+m}$. Then we can apply Proposition 3.2 [G-M-Z], which states that such a submanifold must be one of subgrassmannian $g \cdot G_k R^{k+m}$. Indeed, Proposition 3.2 [G-M-Z] states for case of grassmanian of oriented planes $G_k^+ R^{k+m}$, but their grassmannian and our one are locally isometric, so their Proposition is still valid in our case. This completes the proof of Theorem 3.2.

§4. Properties of M^*-minimal cycles.

Let N be a k-cycle in Riemanian manifold M^m provided with a family M^* of submanifolds N_λ^* in M realizing a cycle $[N]^*$ as in Corollary 2.2. If the inequality in this corollary for the volume of N turns to an equality we will call N a M^*-minimal cycle. Corollary 2.3 states that a M^*-minimal cycle is homologically volume-minimizing. Homological class $[N] \in H_*(M)$ of such a cycle will be called a M^*-class. First we show that there is an anolog of *Equidistribution Theorem* for homologically volume-minimizing cycles in a M^*-homological class.

Theorem 4.1. *Equidistribution Theorem. Let N' be a homological volume-minimizing cycle in a M^*-homological class. Then the set of $N_\lambda^* \subset M^*$ such that $\#(N_\lambda^* \cap N') \neq \chi$ is of measure zero. Here χ equals the intersection number of cycles $[N]$ and $[N^*]$.*

Theorem 4.1 is a trivial consequence of Corollary 2.2. Applying this Theorem to complex submanifolds in the complex projective manifolds CP^n we obtain the following corollary. Recall that $H_{2k}(CP^n, Z) = Z$ is generated by the element $[CP^k]$.

Corollary 4.2. *Let N^{2k} be a complex submanifold realizing element $r[CP^k] \in H_{2k}(CP^n, Z)$. Then the set of $(2n-2k)$-dimensional projective space $CP_\lambda^{n-k} \in CP^n$ such that $\#(CP_\lambda^{n-k} \cap N^{2k}) \neq r$ is of measure zero (in the set of all CP_λ^{n-k} which is diffeomorphic to SU^n/SU^{n-k} provide with the invariant measure).*

Proof. Applying Proposition 2.8.a to the cycle rCP^k we get that all homological classes in $H_*(CP^n, Z)$ are M^*-homological classes. The Federer's Theorem says that the complex submanifold N^{2k} is volume minimizing in its homological class. Hence we infer Corollary 4.2 from Theorem 4.1.

Volume-minimizing cycles in a M^*-homological class possess some properties similar to those of ϕ-currents, where ϕ is some calibration on M. First we note that the cycles under consideration are also M^*-minimal. Further the tangent space to a M^*-minimal cycle belongs to a certain distribution of $k-$planes in TM. Namely at every point $x \in M$ we put

$$I(x) = \{v \in G_k(T_xM) |\ cd(x, v) = \sigma(M^*)_k\}$$

Then M^*-minimal cycles are integral submanifolds of the distribution $I(x)$ almost everywehere. Recall that the tangent spaces to a ϕ-submanifold belong to the distribution $G_\phi(M) = \{v \in TM |\ \phi(v) = 1\}$.

The similarity between M^*-cycle and ϕ-currents also appears in the following theorem.

Theorem 4.3. *Let N be a M^*-minimal cycle realizing an element in homology group $H_k(M, Z)$. If M is a compact manifold then N is a ϕ-current for some calibration ϕ on M and the homology class $[N]$ is stable.*

Proof. Let us recall the Federer's Theorem on stability of integral homology classes.

Theorem. [Fe 2]. *For every $\alpha \in H_k(M, G)$ we put $mass(\alpha) = min\{volX^k \subset M |\ [X^k] = \alpha\}$. Then the following equality holds for $\alpha \in H_k(M, Z)$.*

$$\lim_{n\to\infty} \frac{mass(n\alpha)}{n} = mass(\alpha_R)$$

where α_R denotes the image of α under the map $H_k(M, Z) \to H_k(M, R)$.

If for some $n \in Z^+$ we have $mass(n\alpha)/n = mass(\alpha_R)$ we say that the homology class α is stable.

Now assume N be as in Theorem 4.3. We observe that the cycle pN is also a M^*-cycle for all $p \in Z^+$. So we get

$$mass(p[N])/p = mass([N])$$

Therefore, according to the Federer's Theorem $[N]$ must be a stable class and N is a volume-minimizing cycle in the class $[N]_R \in H(M,R)$. It is well known, that there is a calibration ϕ on M which calibrates N (cf. [D-F], [Le 4]).

Applying Theorem 4.3 to Theorem 3.1 we obtain the following corollary .

Corollary 4.4.[G-M-Z]. *If the grassmannian of oriented planes $G_k^+(R^{k+m})$ realizes a non-trivial element in the homology group $H_{km}(G_L^+(R^{l+m}),R)$ with real coefficients, then $G_k^+R^{k+m}$ is a volume-minimizing cycle in its homology class with real ceffiicients.*

Proof. Obviously, G_kR^{k+m} and its 2-sheeted covering $G_k^+R^{k+m}$ has the same homolgy groups with real coefficients. By Theorem 4.3 G_kR^{k+m} is a volume-minimizing real current. Its is well known that in this case there exists an invariants calibration ϕ on G_lR^{l+m} such that ϕ calibrates G_kR^{k+m}. It is easy to see that the lifted calibration ϕ^* on $G_l^+R^{l+m}$ must calibrate $G_k^+R^{k+m}$ too. This means that $G_k^+R^{k+m}$ is a globally minimal submanifold.

§5. Appendix. Curvature estimate for the growth of globally minimal submanifolds.

In this Appendix we give a complete proof of the Fomenko's and author's announcement [Le-Fo]. In particular, we obtain an estimate for the growth of the volume of globally minimal surfaces in Riemannian manifolds, new isoperimetric inequality for globally minimal surfaces, an explicite formula of the minimal volumes of closed surfaces in symmetric spaces and as a result, new examples of globally minimal surfaces in these spaces. The technique of the Fomenko's method of geodesic defects employed in our proof is very close to our technique in §2.

5.1. Defect of Riemannian manifolds and the volume of globally minimal submanifolds.

a) Let $B_r(x)$ be the ball of radius r in a tangent space T_xM. Recall that the injective radius $R(x)$ of a Riemannian manifold M at a point x is defined as follows $R(x) = sup\{r|\ exp : B_r(x) \longrightarrow M$ is a diffeomorphism $\}$. The injective radius $R(M)$ of M is defined as: $R(M) = \inf_{x\in M} R(x)$. Now we fixe a point $x_0 \in M$. We define *k-dimensional deformation coefficient* $\chi_k(x > x_0)$ as follows (cf.[Fo 2]). Suppose that Π_x^{k-1} is a (k-1)-plane through x in the tangent space T_xM. Denote D_ε^{k-1} the disk of radius ε in Π_x^{k-1}, and by S_ε the disk $exp(D_\varepsilon^{k-1})$. We consider the cone CS_ε formed by geodesics joining the vertex x_0 and the base S_ε. We put

$$\chi(x > x_0, \Pi^{k-1}) = \lim_{\varepsilon\to 0} \frac{vol_kCS_\varepsilon}{vol_{k-1}S_\varepsilon},$$

$$\chi(x > x_0) = \max_{\Pi^{k-1}\subset T_xM} \chi_k(x, \Pi^{k-1}).$$

b)Let $f(x)$ be the function which measures the distance between point $x \in M$ and the fixed point x_0. We set

$$\Omega_k(x_0) = \lambda_k q(x_0, R(x_0)),$$

where λ_k is the volume of the ball of radius 1 in R^k and

$$q(x_0, r) = exp(\int_0^r (\max_{x \in \{f=t\}} \chi_k(x > x_0))^{-1}\, dt).$$

We put

$$\Omega_k = \inf_{x_0 \in M} \Omega_k(x_0).$$

The defined value is called *the k-th geodesic defect* of Riemannian manifold M. The following theorem was obtained by Fomenko in 1972 [Fo 2].

Theorem 5.1.1. *Let $X^k \subset M^n$ is a globally minimal surface. Then the following inequality holds*

$$vol_k(X^k) \geq \Omega_k \geq 0.$$

§*5.2. Lower bound for geodesic defects of Riemannian manifolds. New isoperimetric inequalities.*

Let the section curvature of manifold M in any 2-plane is not greater then a^2 ($a \in R$ or $a \in \sqrt{-1} \otimes R$).

Theorem 5.2.1 [Le-Fo]. *Lower bound of geodesic defects.*

a) If $a^2 \geq 0$ and $Ra \leq \pi$ then we have:

$$\Omega_k(M) \geq k\, \lambda_k\, a^{1-k} \int_0^R (sin\, at)^{k-1}\, dt$$

b) If $a^2 > 0$ and $Ra > \pi$ then we have:

$$\Omega_k(M) > vol(S^k(r = 1/a))$$

c) If $a = 0$ then we have $\Omega_k(M) \geq \lambda_k R^k$.

d) If $a^2 \leq 0$ then we have:

$$\Omega_k(M) \geq k\, \lambda_k\, |a|^{1-k} \int_0^R (sh|a|t)^{k-1}\, dt$$

Theorem 5.2.2 [Le-Fo]. *Upper bound of the deformation coefficient. Let r be the distance between x and x_0.*

a) If $a^2 \geq 0$ and $r \leq \pi/a$ then we have:

$$\chi_k(x > x_0) \leq \frac{\int_0^r (\sin at)^{k-1} dt}{(\sin ar)^{k-1}}$$

b) If $a = 0$ then we have:

$$\chi_k(x > x_0) \leq \frac{r}{k}$$

c) If $a^2 \leq 0$ then we have

$$\chi_k(x > x_0) \leq \frac{\int_0^r (sh\,|a|)^{k-1}\, dt}{(sh|a|r)^{k-1}}.$$

Theorem 5.2.3 [Le-Fo]. *Isoperimetric inequality. Assume that X^k is a globally minimal surfaces through a point $x \in M$. Let $B_x(r)$ be the geodesic ball of radius r and centre at x. Denote A_r^{k-1} the boundary of the intersection $X^k \bigcap B_x(r) = X_r^k$.*

a) If $a^2 > 0$ and $r \leq \min(R, \pi/a)$ then we have:

$$\frac{vol(A_r^{k-1})}{vol(X_r^k)} \geq \frac{\sin (ar)^{k-1}}{\int_0^r (\sin at)^{k-1} dt}$$

Consequently, the following inequality holds

$$vol(A_r^{k-1}) \geq k\, \lambda_k\, a^{1-k} \sin^{k-1}(ar)$$

b) If $a = 0$ and $r \leq R$ then we have:

$vol(A_r^{k-1}) \geq k\, \lambda_k\, r^{k-1}$ = the volume of the standard k-dimensional sphere S^k of radius r.

Hence follow the following inequalities:

$$vol(A_r^{k-1}) \geq (kr)^{-1}\, vol(X_r^K) \tag{b.1}$$

$$vol(X_r^k) \leq (k)^{\frac{k}{1-k}}\, (\lambda_k)^{\frac{1}{1-k}} (vol_{k-1} A_r)^{\frac{k}{k-1}} \tag{b.2}$$

c) If $a^2 < 0$ then we have:

$$\frac{vol(A_r^{k-1})}{vol(X_r^k)} \geq \frac{(sh|a|r)^{k-1}}{\int_0^r (sh|a|t)^{k-1}\, dt}$$

Hence we get

$$vol(A_r^{k-1}) \geq k\, \lambda_k\, sh^{k-1}(|a|t)/|a|^{k-1}$$

The estimates in Theorems 5.2.1 and 5.2.2 are sharp, that is, in many cases they turn to equalities. Roughly speaking, these theorems tell us that globally minimal surfaces tend to a position of "maximal cuvature" in their ambient manifold. Now we show some consequences of Theorem 5.2.1.

Corollary 5.2.4. *If M is a compact simply-connected symmetric space of sectional curvature not greater than a then the volume of any non-trivial cycle is not less than the volume of k-dimensional sphere of curvature a.*

Corollary 5.2.5. *The length of a homologically non-trivial loop in a manifold M is not less then the double injective radius of M.*

Corollary 5.2.6 *Lower bound for the volume of a manifold.*

a) If $a^2 > 0$ then we get:

$$vol(M^n) \geq n\,\lambda_n\, a^{1-n} \int_0^R (sin\, at)^{n-1}\, dt$$

b) If $a = 0$, then we get: $vol(M^n) \geq n\,\lambda_n R^n$

c) If $a^2 < 0$ then we get

$$vol(M^n) \geq n\,\lambda_n |a|^{1-n} \int_0^R (sh|a|t)^{n-1}\, dt$$

Remark. The estimate in Corollary 5.2.6 coincides with that one, which is obtained from Bishop's theorem [B-C].

Now we infer from Theorems 5.2.2 and 5.2.3 the following consequence on the growth of the volume of globally minimal surfaces.

Corollary 5.2.8. *Let X^k be a globally minimal surface in a complete non-compact Riemannian manifold M of non-positive curvature. Then the function $V(r) = vol_k B_X(r)$ grows at least as a polynom of r of degree k, where $B_X(r)$ is a geodesic ball of radius r in X^k. If the curvature of M has an upper bound strictly less than zero then the function $V(r)$ grows at least as the exponent of r.*

Proof of Theorems and Corollaries. Let us write down an explicite formula for the coefficient $\chi_k(x > x_0, \Pi_x^{k-1})$. Suppose $\lambda(t)$ is the shortest geodesic curve joining the points $x_0 = \lambda(0)$ and $x = \lambda(r)$. So, for $0 < t < r$ point $\lambda(t)$ is not conjugated with x_0. We now consider the case if $x = \lambda(r)$ is not conjugated with x_0 (otherwise, we easely get that $\chi_k(x > x_0) = \infty$). Choose an orthonormal basis of vectors $Y_1(r), ..., Y_{k-1}(r)$ in the plane $\Pi^{k-1} \subset T_x M$. (Let us recall that by definition Π^{k-1} has to be orthogonal to $\dot\lambda(r)$). We denote K_ρ the (k-1)-dimensional cubic in Π_x^{k-1} with the edges $\rho Y(r)$. Then the formula for deformation coefficient $\chi_k(x > x_0)$ can be rewritten as follows:

$$\chi_k(x > x_0, \Pi_x^{k-1}) = \lim_{\rho \to 0} \frac{vol_k(C\tilde K_\rho)}{vol_{k-1}\tilde K_\rho}$$

here we set $\tilde{K}_\rho = exp_x K_\rho$.

We denote λ^j_{st} the s-geodesic, joining points x_0 and $exp_x(sY_j(r))$. Put

$$Y_j(t) = \frac{d}{ds}_{|s=0} \lambda^j_{st}.$$

Then $Y_j(t)$ is an Jacobian vector field with the data $Y_j(0) = 0$: $Y_j(r)$ - the chosen vector in Π^{k-1}, and besides, for every t we have $Y_j(t) \perp \dot{\lambda}(t)$. We note that the tangent plane to the orthogonal section $\tilde{K}_{t\rho}$ of the cone $C\tilde{K}_\rho$ at the point $\lambda(t)$ possesses the basis of vector $Y_1(t), .., Y_{k-1}(t)$. Hence,

$$vol_{k-1}(\tilde{K}_{t\rho}) = \rho^{k-1}(|Y_1(t) \wedge ... \wedge Y_{k-1}(t)|) + o(\rho^{k-1}).$$

This yields

$$\chi_k(x > x_0, \Pi^{k-1}) = \lim_{\rho \to 0} \frac{vol(C\tilde{K}_\rho)}{vol_{k-1}(\tilde{K}_\rho)} =$$

$$= \lim_{\rho \to 0} \frac{\int_0^r vol_{k-1}\tilde{K}_{t\rho}\, dt}{vol_{k-1}\tilde{K}_{r\rho}} = \frac{\int_0^r |Y_1(t) \wedge ... \wedge Y_{k-1}(t)|\, dt}{|Y_1(r) \wedge ... \wedge Y_{k-1}(r)|} \tag{5.1}$$

Proof of theorem 5.2.2. Put $F(t) = |Y_1(t)| \cdot ... \cdot |Y_{k-1}(t)|$. Since $|Y_1(t) \wedge ... \wedge Y_{k-1}(t)| \leq F(t)$ and this inequality turns to an equality at $t = r$, the formula (5.1) yields

$$\chi_k(x > x_0, \Pi^{k-1}) \leq \frac{\int_0^r F(t)\, dt}{F(r)} \tag{5.2}$$

We need the following lemmas.

Lemma 5.2.9. *Suppose $F(t)$ be in (5.2). If for all t and Y_j the section curvature $S(\dot{\lambda}(t), Y_j(t)) \leq a^2$, where $a > 0$, then the function $F(t)/G(t)$ increases on the interval $[0, r]$. Here $G(t) = (\sin at)^{k-1}/(\sin ar)^{k-1}$.*

Lemma 5.2.10. *Suppose the function $F(t)$ and $G(t)$ be in the Lemma 5.2.9. Then the following inequality holds*

$$\frac{\int_0^r F(t)\, dt}{F(r)} \leq \frac{\int_0^r G(t)\, dt}{G(r)}.$$

Proof of Lemma 5.2.9. The Rauch's comparision Theorem [B-C] states that the function $f_j(t) = |Y_j(t)|/\sin at$ increases on $[0, r]$. Hence the function $F(t)/G(t) = \prod f_j$ is such a function.

Proof of Lemma 5.2.10. Since the function $F(t)/G(t)$ increases on the interval $[0, r]$, we get $F(x_i)G(r) \leq G(x_i)F(r)$ for every $0 \leq x_i \leq r$. Hence we obtain

$$\sum_{k=0}^{n} F(kr/n)G(r) \leq \sum_{k=0}^{n} G(kr/n)F(r).$$

Letting $n \to \infty$ we easely infer Lemma 5.2.10 from the above inequality.

Let us continue the proof of Theorem 5.2.2.

Taking into account (5.2) and lemmas 5.2.9, 5.2.10 we get

$$\chi_k(x, \Pi^{k-1}) \leq \frac{\int_0^r F(t)\, dt}{F(r)} \leq \frac{\int_0^r (sin\, at)^{k-1}\, dt}{(sin\, ar)^{k-1}}$$

The proof of the first part in Theorem 5.2.2 is completed. In the same way we can prove the rest parts (b) and (c).

Proof of Theorem 5.2.1. Let us recall the definition

$$\Omega_k(x_0, r) = \lambda_k\, exp \int_0^r (\max_{x \in \{f=t\}} \chi_k(x > x_0))^{-1}\, dt.$$

Theorem 5.2.1 (a) yields

$$\Omega_k(x_0, r) \geq \lambda_k\, exp \int_0^r \frac{sin\, at)^{k-1}\, dt}{\int_0^t (sin\, a\tau)^{k-1}\, d\tau}$$

Put

$$\Phi_k(r) = \lambda_k\, exp \int_0^r \frac{(sin\, at)^{k-1}\, dt}{\int_0^t (sin\, a\tau)^{k-1}\, d\tau}.$$

Clearly , we can infer Theorem 5.2.1(a) from the following identity

$$\Phi_k(r) = k\, \lambda_k\, a^{1-k} \int_0^r (sin\, at)^{k-1}\, dt \tag{5.3}$$

Proof of Formula (5.3). Put $\Phi_k^*(r)$ equal the right part in (5.3). We observe that the functions $\Phi_k(r)$ and $\Phi_k^*(r)$ satisfy the same differential equation:

$$\frac{\Phi_k(r)}{(\partial/\partial r)\Phi_k(r)} = \frac{\Phi_k^*(r)}{(\partial/\partial r)\Phi_k^*(r)} = \frac{\int_0^r (sin\, at)^{k-1}\, dt}{(sin\, ar)^{k-1}} \tag{5.4}$$

Let us consider the limit

$$\lim_{r \to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} = \lim_{r \to 0} \frac{\lambda_k\, exp \int_0^r (\int_0^t (sin\, a\tau)^{k-1}\, d\tau)^{-1} (sin\, at)^{k-1}\, dt}{k\, \lambda_k a^{1-k} \int_0^r (sin\, at)^{k-1}\, dt} \tag{5.5}$$

Taking into account the increaseness of the function $(a\tau/sin\, a\tau)^{k-1}$ on the interval $[0, t]$ where $0 \leq t \leq \pi/a$ and using Lemma 5.2.10 we obtain

$$\frac{(sin\, at)^{k-1}}{\int_0^t (sin\, a\tau)^{k-1}\, d\tau} < \frac{t^{k-1}}{\int_0^t \tau^{k-1}\, d\tau} = \frac{k}{t} \tag{5.6}$$

Combining (5.5) and (5.6) yields the following inequality

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} \le \lim_{r\to 0}\frac{\lambda_k\, exp \int_0^r k\, t^{-1}\, dt}{k\,\lambda_k\, a^{1-k}\int_0^r (sin\, at)^{k-1}\, dt}$$

Fix $\varepsilon > 0$. Since $\lim_{y\to 0}(sin\, at/\, at) = 1 > 1-\varepsilon$ we get the following inequality.

$$\lim \frac{\Phi_k(r)}{\Phi_k^*(r)} \le \lim_{r\to 0}\frac{exp \int_0^r (k/t)\, dt}{k\int_0^r (1-\varepsilon)^{k-1}(at)^{k-1}\, a^{1-k}\, dt} =$$

$$= \lim_{r\to 0}\frac{r^k}{r^k(1-\varepsilon)^{k-1}} = (1-\varepsilon)^{1-k} \tag{5.7}$$

Since the inequality (5.7) holds for all $\varepsilon > 0$ we have

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} \le \lim_{\varepsilon\to 0}(1-\varepsilon)^{1-k} = 1 \tag{5.8}$$

On the other hand, applying the inequality $sin\, at < at$ to (5.5) we get

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} \ge \lim_{r\to 0}\frac{\lambda_k\, exp(\int_0^r (sin\, ay)^{k-1}\, a^{1-k}\, k\, y^{-k}\, dy}{k\,\lambda_k\int_0^r t^{k-1}\, dt}$$

Fixed ε as above we have

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} \ge \lim_{r\to 0} exp(\int_0^r \frac{(1-\varepsilon)(ay)^{k-1}\, dy}{a^{k-1}\cdot y^k\cdot k^{-1}})\, r^{-k} =$$

$$= \lim_{r\to 0} r^{-k}\, exp(\int_0^r \frac{(1-\varepsilon)^{k-1}\, k\, dy}{y}) = r^{k((1-\varepsilon)^{k-1}-1)} \tag{5.9}$$

Letting $\varepsilon \to 0$ we infer from (5.9)

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} \ge \lim_{\varepsilon\to 0} r^{k((1-\varepsilon)^{k-1}-1)} = 1 \tag{5.10}$$

Now we obtain from (5.8) and (5.10)

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} = 1 \tag{5.11}$$

The differential equation (5.4) for $\Phi_k(r)$ and $\Phi_k^*(r)$ has the same initial data (5.11). So we get the identity $\Phi_k^* = \Phi_k$ that completes the proof of Theorem 5.2.2 (a).

The rest parts (c), (d) can be proved in the same way. The part (b) follows from that fact if $R > \pi/a$ then we have $\Omega_k(M) > \Omega_k(x_0, \pi/a) \ge vol(S^k, 1/a)$. This completes the proof of Theorem 5.2.2.

Proof of Theorem 5.2.3. We denote CA_r^{k-1} the geodesic cone of base A_r^{k-1} and vertex at point x. Since X_r^k is a globally minimal surface and the cone CA_r^{k-1} is homological to X_r^k we have $vol(X_r^k) \leq vol(CA_r^{k-1})$. Hence we conclude

$$\frac{vol(A_r^{k-1})}{vol(X_r^k)} \geq \frac{vol(A_r^{k-1})}{vol(CA_r^{k-1})} \geq$$
$$\geq (\max_{y \in A_r} \chi_k(y > x))^{-1} \geq \frac{(sin\, ar)^{k-1}}{\int_0^r (sin\, at)^{k-1}\, dt} \tag{5.12}$$

(The second inequality in (5.12) is infered from the following formula

$$vol(CA_r^{k-1}) = \int_{A_r^{k-1}} \chi_k(y > x, \Pi_y^{k-1})\, dy$$

here Π_y^{k-1} denotes the tangent space to A_y^{k-1} at y. The third inequality in (5.12) is a consequence of Theorem 5.2.2(a).)

We infer from (5.12) the following inequality

$$vol(CA_r^{k-1}) \geq vol(X_r^k) \frac{(sin\, ar)^{k-1}}{\int_0^r (sin\, at)^{k-1}\, dt} \geq \Omega_k(r) \frac{(sin\, ar)^{k-1}}{\int_0^r (sin\, at)^{k-1}\, dt} \tag{5.13}$$

Combining (5.13) and Theorem 5.1.1(a) yields

$$vol\, A_r^{k-1} \geq k\, \lambda_k\, a^{1-k}\, (sin\, ar)^{k-1}$$

that completes the proof of Theorem 5.2.3(a). The rest parts of Theorem 5.2.3 can be proved in the same way.

Proof of Corollary 5.2.4. It is well known that a compact symmetric simply-connected space satisfies the relation: $Ra = \pi$. So we get Corollary 5.2.4 from Theorem 5.2.1.

Proof of Corollary 5.2.5.. Clearly, $\lambda_1 = 2$. So we obtain $\Omega_1(M) \geq 2 \int_0^R 1\, dt = 2R$.

Proof of Corollary 5.2.6.a. Let $dim(M) = m$. Then $vol(M) \geq \Omega_m(M)$. With the help of (5.3) we obtain $\Omega_m(M) \geq \Phi_m(R) = k\, \lambda_k a^{1-k} \int_0^r (sin\, at)^{k-1}\, dt$. This completes the proof Corollary 5.2.6.a. The rest assertions can be proved in the same way.

5.3. Explicit formula for geodesic defect of symmetric spaces. List of globally minimal Hegalson's spheres.

Suppose M is a compact symmetric space. Let us compute the deformation coefficient associated with fixed point $e \in M$. Without loss of generality we compute this coefficient at point $Exp\, tx \in M$ where x is a vector in the Cartan space B of the tangent space lM to M at e. We shall redenote $\chi_k(Exp\, rx) = \chi_k(Exp\, rx > e)$.

Theorem 5.3.1. *Let $\{\alpha_i\}$ be the roots systems of symmetric space M with respect to B. Suppose x is a vector of unite length in B . Without loss of generality we assume that $\alpha_1(x) \geq ... \geq \alpha_p(x) = 0 = \alpha_{p+1}(x) = ...$*

a) If $k < p$ then the following equality holds

$$\chi_k(Exp\,rx) = \frac{\int_0^r sin(\alpha_1(x)t) \cdot ... \cdot sin(\alpha_k(x)t)\, dt}{sin\,\alpha_1(x)r \cdot ... \cdot sin\,\alpha_k(x)r}.$$

b) If $k \geq p$ then the following inequality holds

$$\chi_k(Exp\,rx) = \frac{\int_0^r sin(\alpha_1(x)t) \cdot ... \cdot sin(\alpha_{p-1}(x)t)t^{k-p}\, dt}{sin(\alpha_1(x)r) \cdot ... \cdot sin(\alpha_{p-1}(x)r)r^{k-p}}.|$$

Lemma 5.3.2. *Let $\{v_1, ..., v_k\} \in M$ be an orthonormal frame which consisting of the eigenvectors of eigenvalues $\alpha_1^2(x), ..., \alpha_p^2(x), , , , 0, ...0$ of the operator ad_x^2. Denote $V_i(t)$ the prallel vector field along the geodesics $Exp\,tx$ such that $V_i(0) = v_i$ and denote $W_i(t)$ the Jacobian vector field along $Exp\,tx$ such that $W_i(0) = v_i$. Then we have the following relation*

- if $i < p$ then $W_i(t) = \alpha_i(x)^{-1} sin(\alpha_i(x)t)V(t)$,

- if $i \geq p$ then $W_i(t) = tV_i(t)$.

Proof of Lemma 5.3.2. In the tangent space lM the vector field tv_i is a Jacobian field along the ray tx. It is well-known that the vector field $dExp_{|tx}(tv_i)$ is also a Jacobian vector field along the geodesic $Exp\,tx \subset M$ [He]. Let us write an explicit formula for the differential of the exponential mapping at point x (cf. [He]). We will identify M with the quotient G/H, moreover, the tangent space lM with the orthogonal complement to the algebra lH in the algebra lG. We denote exp the expnential mapping from the algebra to the group. Then $exp\,tx$ is an element in G acting on M and we denote $d\tau(exp\,tx)$ the differential of this action. We have

$$dExp_{|tx}(tv) = d\tau(exp\,tx) \sum_{n=0}^{\infty} \frac{t^2\, ad_x^2(tv_i)}{(2n+1)!} =$$

$$= d\tau\,(exp\,tx) \sum_{n=0}^{\infty} \frac{(t^2\alpha_i^2(x))^n(-1)^n}{(2n+1)!}(tv_i) =$$

$$= d\tau(exp\,tx)\frac{sin\,\alpha_i(x)t}{t\,\alpha_i(x)}tv_i = \frac{sin(\alpha_i(x)t)}{\alpha_i(x)}(d\tau(exp\,tx)v_i) \qquad (5.13)$$

Now we observe that the parallel vector field V_i is obtained from the vector v_i by the shift $d\tau(exp)$ along the geodesic $Exp\,tx$, that is, $V_i(t) = d\tau(exp\,tx)\, v_i$. Hence we get Lemma 5.3.2 from (5.13).

Proof of Theorem 5.3.1. Now we compute the coefficient $\chi_k(Exprx, \Pi^{k-1})$. We observe that the tangent space $\Pi^{k-1}(t)$ to the normal section of the cone CD_ε^{k-1} at the point $Exptx$ can be respresented as the sum $\sum a_i \Pi_i^{k-1}(t)$ where a_i are constants and $\Pi_i^{k-1}(t)$ is the basis in the space $\Lambda_{k-1}(T_{Exptx}M)$ such that $\Pi_i^{k-1}(t)$ is generated by the orthonormal frame of vectors $W_i(t) \in T_{Exptx}M$. Using formula (5.1) we get

$$\chi_k(Exprx, \Pi^{k-1}) = \frac{\int_0^r |\Pi^{k-1}(t)|\, dt}{|\Pi^{k-1}(r)|} = \frac{\sum_i \int_0^r a_i\, |\Pi_{k-1}^i(t)|\, dt}{\sum_i a_i\, |\Pi_{k-1}^i(r)|}$$

Hence we obtain

$$\chi_k(Exprx) = \max_i \frac{\int_0^r |\Pi_{k-1}^i(t)|\, dt}{\Pi_{k-1}^i(r)}$$

Combining Lemma 5.3.2 , Lemma 5.2.9 and Lemma 5.1.10 we get

$$max_i \frac{\int_0^r |\Pi_i^{k-1}(t)| dt}{|\Pi_{k-1}^i(r)|} = \frac{\int_0^r sin(\alpha_1(x)t) \cdot \ldots \cdot sin(\alpha_k(x)t)\, dt}{sin(\alpha_1(x)r) \cdot \ldots \cdot sin(\alpha_k(x)r)}$$

if $k < p$. In the same way we can prove the theorem in the case $k \geq p$. The proof of Theorem 5.3.1 is complete.

Corollary 5.3.3. *If M is a symmetric space of rank $= 1$,that is, $dim\, B = 1$, then the deformation coefficient $\chi_k(Exprx)$ depends only on r.*

a) For $M = S^n$ (or RP^n) we have $\chi_k(r) = \int_0^r (sin\, t)^{k-1}\, dt / (sin\, r)^{k-1}$.

b) For $M = CP^n$ we have

$$\chi_k(r) = \frac{\int_0^r (sin\, \sqrt{2}t)(sin\, t)^{2k-2}\, dt}{sin\, \sqrt{2}r(sin\, r)^{2k-2}}$$

c) For $M = HP^n$ we have

$$\chi_k(r) = \frac{\int_0^r (sin\sqrt{2}t)^3 (sin\, t)^{4k-4}\, dt}{(sin\sqrt{2}r)^3 (sin\, r)^{4k-4}}$$

We immediately obtain the following consequence.

Corollary 5.3.4. [Fo]. *For any $k \leq n$ the standardly embedded space RP^n (and CP^n, HP^n resp.) has the volume $= \Omega_k(RP^n)$ (and $\Omega_{2k}(CP^n), \Omega_{4k}(HP^n)$ resp.) and therefore is a globally minimal submanifold.*

Let us now compute geodesic defects $\Omega_k(RP^n), \Omega_{2k}CP^n, \Omega_k(HP^n)$. Clearly, $\Omega_k(RP^n) = \frac{1}{2} vol(S^k(1))$ can be computed from the following formulas. First we take integration over parallel section of the unit ball

$$\lambda_k = 2\lambda_{k-1} \int_0^{\pi/2} cos^k \alpha\, d\alpha$$

Taking into account (5.3) we get

$$vol\, S^k(1) = (k+1)\lambda_{k+1} = 2\lambda_k k \int_0^{\pi/2} sin^{k-1}\alpha\, d\alpha$$

Hence we obtain the following identity

$$k+1 = \frac{2k\lambda_k \int_0^{\pi/2} sin^{k-1}\alpha\, d\alpha}{2\lambda_k \int_0^{\pi/2} cos^{k+1}\alpha\, d\alpha} \tag{5.14}$$

We infer from (5.14) the following equation

$$\int_0^{\pi/2} sin^{k+1}\alpha\, d\alpha = \frac{k}{k+1}\int_0^{\pi/2} sin^{k-1}\alpha\, d\alpha \tag{5.15}$$

Using (5.15) we easely get

$$\lambda_{2k} = \frac{\pi^k}{k!} \qquad \lambda_{2k+1} = \frac{\pi^k 2^{k+1}}{(2k+1)!!}$$

Let us compute $\Omega_{2k}(CP^n) = vol(CP^k)$. Using Corollary 5.3.3 and taking into account $R(CP^n) = \pi/sqrt2$ we get

$$\begin{aligned}\Omega_k(CP^n) &= \lambda_{2k} exp \int_0^{\pi/sqrt2} \frac{dt}{(sin\sqrt{2}t)(sin\, t)^{2k-1}} = \\ &= 2k\lambda_{2k} \int_0^{\pi/sqrt2} (sin\sqrt{2}t/\sqrt{2})(sin\, t)^{2k-2}\, dt = \\ &= 2^k 2k\, \lambda_{2k} \int_0^1 x^{2k-1}\, dx = \pi^{2k}/k!\end{aligned}$$

In the same way we compute $\Omega_{4k}(HP^n) = vol(HP^k)$. We have

$$\begin{aligned}\Omega_{4k}(HP^n) &= \lambda_{4k} exp \int_0^{\pi/\sqrt{2}} \frac{dt}{(sin\sqrt{2}t)^3(sin\, t)^{4k-4}} = \\ &= 4k\lambda_{4k} \int_0^{\pi/\sqrt{2}} (sin\sqrt{2}t/\sqrt{2})^3(sin\, t)^{4k-4}\, dt = \\ &= 2^{2k} 4k\lambda_{4k} \int_0^1 y^{4k-4}(1-y^2)\, dy = \\ &= \pi^{2k}\, 2^{2k}/(2k+1)1\end{aligned}$$

Remark. Operator ad_x^2 coincides with the Ricci transformation $R_x : y \to R_{xy}x$ in the tangent space lM. Therefore, the deformation coefficient $\chi_k(Exp\, rx)$ get the maximal value if and only if the plane Π^{k-1} is an eigenspace with the maximal eigenvalue of the induced Ricci transformation in the space $\Lambda_{k-1}T_{Exprx}M$. Roughly speaking, the curvature at point $Exprx$ in direction (rx, Π^{k-1}) get the maximal value.

It is well known that in a symply connected irreducible compact symmetric space M there are totally geodesic spheres of curvature a^2 where a^2 is the upper bound of section curvature on M. Further, any such sphere lies in some totally geodesic Hegalson's sphere of maximal dimension $i(M)$. All Hegalson's spheres are equivalent under the action of group $Iso(M)$. Moreover they are of the same curvature a^2. Now we immediately get from Corollary 5.2.9 the following Proposition.

Proposition 5.3.5. *If a Hegalson's sphere $S(M)$ realizes a non-trivial cycle in a homology group of space M then it is a globally minimal submanifold in M.*

Now we write the list of homologically nontrivial Hegalson's spheres. This list can be obtained by analysis the Fomeko's list of totally geodesic spheres realizing a non-trivial cycles in symmetric spaces [Fo 2].

1) If M is a symple compact group then $i(M) = 3$ and $S(M)$ is a group generated by a root of maximal length.
2) $M = SU_{l+m}/SU_l \times SU_m$, $i(M) = 2$
3) $M = SO_{l+2}/S(O_l \times O_2)$, $i(M) = 2$
4)$M = SU_{2n}/Sp_n$, $i(M) = 5$
5)$M = Sp_{m+n}/Sp_m \times Sp_n$, $i(M) = 4$ (this sphere is the quaternion projective space HP^1).
6) $M = SO_{2n}/U_n$, $i(M) = 2$
7)$M = F_4/Spin_9$, $i(M) = 8$. The global minimality of this sphere was proved by Fomenko [Fo 1].
8)$M = Ad\,E_6/T^1\,Spin10$, $i(M) = 2$.
9) $M = Ad\,E_7/T^1\,E_6$, $i(M) = 2$.

Remark. In all cases if the dimension of Hegalson's spheres i(M) =2, the corresponding symmetric space are Kälerian manifolds, so their Helgason's spheres are diifeomorphic to CP^1. It would be interesting to find calibrations which would calibrate the Hegalson's spheres in 4) and 7). We also conjecture that all Hegalson's spheres are M^*-minimal submanifolds. Y.Ohnita shows that any Hegalson's spheres are stable minimal submanifolds [O].

In conclusion we show a consequence of Theorem 5.2.1 for non-compact symmetric spaces. It is well known that the upper bound of section curvature of these spaces is zero [He].

Corollary 5.3.3. *Let X be a flat totally geodesic submanifold in a non-compact symmetric space M. Then X is a globally minimal submanifold.*

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